A counterexample to Premet’s and Joseph’s conjectures

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INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra of rank $l$ over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let $G$ be the adjoint group of $\mathfrak{g}$. Given $x \in \mathfrak{g}$, we denote by $\mathfrak{g}_x$ the centraliser of $x$ in $\mathfrak{g}$.

**Conjecture 1** (Premet). For any $x \in \mathfrak{g}$ the algebra $S(\mathfrak{g}_x)^{\mathfrak{g}_x}$ of $\mathfrak{g}_x$-invariants is a graded polynomial algebra in $l$ variables.

In some particular cases the problem is simple. For example, for regular nilpotent elements the algebra $S(\mathfrak{g}_x)^{\mathfrak{g}_x}$ is known to be free. In [5], Conjecture 1 is shown to be true in types $A$ and $C$. It is also verified for some nilpotent elements of orthogonal Lie algebra and for the minimal nilpotent orbits in simple Lie algebras except of type $E_8$. Later, by a different method, Brown and Brundan [1] proved that Conjecture 1 holds in type $A$.

Suppose that $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are opposite parabolic subalgebras of $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{p}_-$. Then the intersection $\mathfrak{q} := \mathfrak{p}_+ \cap \mathfrak{p}_-$ is called a biparabolic or, in other terminology, seaweed subalgebra. Since $\mathfrak{g}$ itself is a parabolic subalgebra, we see that parabolics are particular cases of seaweeds.

For any Lie algebra $\mathfrak{q}$ let $\mathfrak{q}^\prime := [\mathfrak{q}, \mathfrak{q}]$ denote its derived algebra. In [3, Section 7.7], the following conjecture was made.

**Conjecture 2** (Joseph). For any seaweed subalgebra $\mathfrak{q} \subset \mathfrak{g}$ the semi-invariants $S(\mathfrak{q})^{\mathfrak{q}^\prime}$ form a polynomial algebra.

A formula for $\text{tr.deg} S(\mathfrak{q})^{\mathfrak{q}^\prime}$ is given in [3]. It is rather complicated and we are not going to use it in full generality. In [2] and [3], it is proved that Conjecture 2 holds for all

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parabolics and seaweeds in simple Lie algebras of types $A$ and $C$. As was noticed in [5, Section 4.9], minimal nilpotent orbits provide a testing site for Joseph’s conjecture as well as Premet’s one. If $\mathfrak{g}$ is simple, then for each minimal nilpotent element $e \in \mathfrak{g}$ there exists a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that $S(\mathfrak{g}_e)^{\mathfrak{p}} \cong S(\mathfrak{p})^{\mathfrak{p}'}$. The detailed explanation of this construction is given below. We note only that naturally $\text{tr.deg} \ S(\mathfrak{p})^{\mathfrak{p}'} = \text{tr.deg} \ S(\mathfrak{g}_e)^{\mathfrak{p}e}$ and $\text{tr.deg} \ S(\mathfrak{g}_e)^{\mathfrak{p}e} = l$ by [4].

In this note, we show that Conjecture 1 does not hold for the minimal nilpotent orbit in the simple Lie algebra of type $E_8$. As a consequence, a conjecture of Joseph on the semi-invariants of (bi)parabolics is not true either.

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1. **Theory**

Let us say a few words about the general method of [5], which, unfortunately, does not work for the minimal nilpotent orbit in $E_8$. Let $\mathfrak{g}$ be a simple Lie algebra and $e \in \mathfrak{g}$ a nilpotent element. Suppose that $\langle e, h, f \rangle \subset \mathfrak{g}$ is an $\mathfrak{sl}_2$-triple containing $e$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ by means of the Killing form. For each $F \in S(\mathfrak{g})^G$ let $^eF$ stand for the minimal degree component of the restriction $F|_{e+\mathfrak{g}_f}$. As was shown in [5], $^eF \in S(\mathfrak{g}_e)^{\mathfrak{g}e}$. A set of homogeneous generators $\{F_1, \ldots, F_l\} \subset S(\mathfrak{g})^G$ is said to be *good* if the $^eF_i$’s are algebraically independent.

Given a linear function $\gamma$ on $\mathfrak{g}_e$ we denote by $(\mathfrak{g}_e)_\gamma$ the stabiliser of $\gamma$ in $\mathfrak{g}_e$ and set

$$(\mathfrak{g}_e^*)_{\text{sing}} := \{\gamma \in \mathfrak{g}_e^* \mid \dim(\mathfrak{g}_e) \gamma > l\}.$$

**Theorem 1.** [5] Suppose $e$ admits a good generating system $F_1, \ldots, F_l$ in $S(\mathfrak{g})^G$ and assume further that $(\mathfrak{g}_e^*)_{\text{sing}}$ has codimension $\geq 2$ in $\mathfrak{g}_e^*$. Then $S(\mathfrak{g}_e)^{\mathfrak{g}e}$ is a polynomial algebra in $^eF_1, \ldots, ^eF_l$.

Suppose now that $e$ is a minimal nilpotent element. Then $\dim(\mathfrak{g}_e)_\gamma = l$ for generic $\gamma \in \mathfrak{g}_e^*$, see [4]; and $(\mathfrak{g}_e^*)_{\text{sing}}$ is of codimension $\geq 2$, see [5, Section 3.10.]. If $\mathfrak{g}$ is of type $E_8$, then there is no good generating system, [5, Remark 4.2.]. For that reason in Section 4.8 of [5] another approach was developed. As was proved there, Conjecture 1 holds if and only if there is a certain system of generating invariants in $E_7$. 
Since \( e \) is a minimal nilpotent element, the \( \mathbb{Z} \)-grading defined by \( h \) is
\[
g = g(-2) \oplus g(-1) \oplus g(0) \oplus g(1) \oplus g(2),
\]
with \( g(2) = \mathbb{K}e \) and \( g(1) \oplus g(2) \) being a Heisenberg Lie algebra. Set \( I := g(0)_e = g_e \cap g(0). \)
Then \( g(0) = I \oplus \mathbb{K}h. \) Clearly \( p := g(0) \oplus g(1) \oplus g(2) \) is a parabolic subalgebra of \( g \) and \( p' = g(0)' \oplus g(1) \oplus g(2). \) Since \( p = \mathbb{K}h \oplus g_e \) and \([h, e] = 2e, \) we have
\[
S(p)^p \subset S(p)^p \subset S(g_e).
\]
If \( g \) is not of type \( A, \) then \( l = g(0)' \) and \( p' = g_e. \) Hence \( S(p)^p = S(g_e)^{g_e}. \)

Remark 1. If \( g \) is of type \( A, \) then, so far, we can only say that \( S(p)^p = S(g_e)^{g_e}. \) Set \( n := g(1) \oplus g(2). \) Then there is an isomorphism of \( l \)-modules \((S(g_e)[1/e])^n \cong S(l)[e, 1/e], \) see [5, Section 4.8.] or [7, Lemma 3.]. Therefore the centre of \( l \) acts on \( n \)-invariants trivially and \( S(p)^p = S(g_e)^{g_e} = S(g_e)^{g_e}. \)

From now on assume that \( g \) is of type \( E_8. \) Then \( l \) is of type \( E_7. \) For generic \( v \in g(1) \) the stabiliser \( l_v \) is a simple Lie algebra of type \( E_6. \) Fix such \( v \in g(1). \) Let \( t \subset l \) and \( l \subset l_v \) be maximal tori such that \( l \subset t. \) Then there is a unique orthogonal decomposition \( t = l \oplus \mathbb{K}h_0. \) Let \( W \) and \( W' \) denote the Weyl groups of \( l \) and \( l_v, \) respectively. Each \( \varphi \in S(t)^W \) can be presented uniquely as
\[
\varphi = \sum_{j=0}^\nu \varphi_i^{(j)} h_0 j \quad \hfill \left( \varphi_i^{(j)} \in S(t)^{W'}, \, \varphi_i^{(\nu)} \neq 0, \, \nu = \nu(i) \right).
\]

Theorem 2. [5, Theorem 4.14.] The algebra \( S(g_e)^{g_e} \) is free if and only if there is a homogeneous generating system \( \varphi_1, \ldots, \varphi_7 \) in \( S(t)^W \) such that the elements \( \varphi_1^{(\nu)} h_0^{(1)}, \ldots, \varphi_7^{(\nu)} h_0^{(7)} \) are algebraically independent.

The main technical result of this paper is the following:

Proposition 1. Suppose that \( \varphi_1, \ldots, \varphi_7 \) is a system of homogeneous generators of \( S(t)^W \) with \( \deg \varphi_i < \deg \varphi_j \) for \( i < j. \) Then the elements \( \varphi_1^{(\nu)} h_0^{(1)}, \varphi_2^{(\nu)} h_0^{(2)}, \varphi_3^{(\nu)} h_0^{(3)} \) are algebraically dependent.

Combining Theorem 2 and Proposition 1, we conclude that Conjectures 1 and 2 are false.
2. Calculations

Since it is difficult to deal with $E_7$ directly, we first consider a regular subalgebra $\mathfrak{sl}_8 \subset E_7$ such that $\mathfrak{t} \subset \mathfrak{sl}_8$. Let $\varpi_i$ and $\varpi'_i$ denote the fundamental weights of $E_7$ and $SL_8$, respectively. We use the Vinberg–Onishchik numbering of simple roots and fundamental weights, see [6, Tables]. We may (and will) assume that the simple roots of $\mathfrak{sl}_8$ are the first six simple roots of $E_7$ and the lowest root $\delta$. On the extended Dynkin diagram of $E_7$, which is given below, the simple roots of $\mathfrak{sl}_8$ form the upper line. Recall that $\mathfrak{t}$ is a maximal torus in a regular subalgebra $E_6 \subset E_7$. Without loss of generality, we may assume that $\mathfrak{t}$ coincides with the annihilator of the weight $\varpi_1$. Expressing $\delta$ as a linear combination of the simple roots one can see that $\varpi_1(\delta) = -1$. Hence the subtorus $\mathfrak{t}$ is also the annihilator of $\varpi'_1 - \varpi'_7$.

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \delta \\
& & & & & & & 67
\end{array}
\]

Without loss of generality, we may assume that $\mathfrak{t}$ is the subspace of diagonal matrices of $\mathfrak{sl}_8$. The dual space $\mathfrak{t}^*$ is spanned by $\epsilon_1, \epsilon_2, \ldots, \epsilon_8$ subject to the relation $\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8 = 0$ and the Weyl group of $SL_8$ permutes the $\epsilon_i$'s. Since the fundamental weights $\varpi'_1, \varpi'_7$ can be expressed as $\varpi_1 = \epsilon_1$ and $\varpi_7 = -\epsilon_8$, we conclude that $\mathfrak{t}$ is the annihilator of $\epsilon_1 + \epsilon_8$. Therefore $\mathfrak{t}$ can be presented as a linear space of diagonal matrices:

\[
(1) \quad \mathfrak{t} = \{ \text{diag}(b, b_1, b_2, \ldots, b_6, -b) \mid \sum_{i=1}^{6} b_i = 0 \}.
\]

Then

\[
(2) \quad \mathbb{K}h_0 = \{ \text{diag}(a, -a/3, -a/3, -a/3, -a/3, -a/3, -a/3, a) \mid a \in \mathbb{K} \}.
\]

Let us identify $\mathfrak{t}$ with $\mathfrak{t}^*$ by means of the Killing form. Then Weyl group invariants of $SL_8$ can be expressed in terms of variables $a, b, b_1, \ldots, b_6$; the $\varphi_i^{(\nu)}$'s will be polynomials in $b, b_1, \ldots, b_6$ and $h_0$ proportional to $a$. Since $\mathfrak{sl}_8$ is a maximal rank subalgebra of $E_7$, one can write invariants of $E_7$ as polynomials in invariants of $SL_8$ with unknown coefficients. This will give us certain constrains on $\varphi_1^{(\nu)} h_0^{\nu(1)}, \varphi_2^{(\nu)} h_0^{\nu(2)}$, and $\varphi_3^{(\nu)} h_0^{\nu(3)}$. 

First we concentrate on $SL_8$-invariants. According to formulas (1) and (2), any diagonal matrix in $s_{l8}$ is of the form:

$$(a + b, -a/3 + b_1, -a/3 + b_2, -a/3 + b_3, -a/3 + b_4, -a/3 + b_5, -a/3 + b_6, a - b)$$

with $\sum_{i=1}^{6} b_i = 0$. Set $\tau_k := \sum_{i=1}^{6} b_i^k$, and let $S_i$ be the trace of the $i + 1$ power of a diagonal matrix. Then

$$S_2 = 2a^2 + 2b^2 + \frac{6}{9}a^2 + \sum_{i=1}^{6} \left( -\frac{2}{3}ab_i + b_i^2 \right) = \frac{8}{3}a^2 + 2b^2 + \tau_2.$$ 

In the same way

$$S_3 = \frac{16}{9}a^3 + 6ab^2 - a\tau_2 + \tau_3;$$

$$S_4 = \left(2 + \frac{2}{27}\right)a^4 + 12a^2b^2 + \frac{2}{3}a^2\tau_2 - \frac{4}{3}a\tau_3 + 2b^4 + \tau_4;$$

$$S_5 = \left(2 - \frac{2}{81}\right)a^5 + 20a^3b^2 - \frac{10}{27}a^3\tau_2 + \frac{10}{9}a^2\tau_3 + 10ab^4 - \frac{5}{3}a\tau_4 + \tau_5;$$

$$S_6 = \left(2 + \frac{2}{27}\right)a^6 + 30a^4b^2 + \frac{5}{27}a^4\tau_2 - \frac{20}{27}a^3\tau_3 + 30a^2b^4 + \frac{5}{3}a^2\tau_4 + \ldots ;$$

$$S_8 = \left(2 + \frac{2}{27}\right)a^8 + 56a^6b^2 + \frac{28}{27}a^6\tau_2 - \frac{56}{27}a^5\tau_3 + 140a^4b^4 + \frac{70}{81}a^4\tau_4 + \ldots .$$

In principle, it is possible to calculate all of them either by hand or using computer. The expression for $S_7$ is of no importance for us. Also, the coefficients of smaller degrees of $a$ play no rôle in the following calculations. Therefore they are not written down in $S_6$ and $S_8$.

Let $\varphi_1$, $\varphi_2$, and $\varphi_3$ be Weyl group invariants of $E_7$ of degrees 2, 6, and 8, respectively. Then $\deg(\varphi_1^{(\nu)}) = 0$. Since $s_{l8}$ is a maximal rank subalgebra of $E_7$, the $E_7$-invariants are polynomials in $SL_8$-invariants. Using this, we will show that $\deg(\varphi_2^{(\nu)}) \leq 2$ and $\deg(\varphi_3^{(\nu)}) \leq 4$. Since both these polynomials are invariants of the Weyl group of $E_6$, they must be algebraically dependent (recall that the degrees of $E_6$-invariants are 2, 5, 6, 8, 9, 12).

**Lemma 1.** For the Weyl group invariant $\varphi_2$ with $\deg \varphi_2 = 6$, we have $\deg(\varphi_2^{(\nu)}) \leq 2$.

**Proof.** The invariant $\varphi_2$ is a linear combination of $SL_8$-invariants of degree 6. One can express this as follows:

$$\varphi_2 = x_1S_2^3 + x_2S_3^2 + x_3S_2S_4 + x_4S_6, \quad \text{where } x_i \in \mathbb{K}.$$
Assume that $\deg(\varphi_2^{(\nu)}) > 2$. Since $\varphi_2^{(\nu)}$ is an invariant of $E_6$, it cannot be of degree 3. Hence $\deg(\varphi_2^{(\nu)}) \geq 4$ and the coefficients of $a^6, a^4,$ and $a^3$ in $\varphi_2$ are zeros. This condition gives us four linear equations on $x_i$.

Let us write down the polynomials in question:

\[
S_2^3 = \frac{512}{27}a^6 + \frac{64}{3}a^4(2b^2 + \tau_2) + \ldots; \\
S_3^2 = \frac{256}{81}a^6 + \frac{32}{9}a^4(6b^2 - \tau_2) + \frac{32}{9}a^3\tau_3 + \ldots; \\
S_2S_4 = \frac{448}{81}a^6 + (36 + \frac{4}{27})a^4b^2 + (2 + \frac{50}{27})a^4\tau_2 - \frac{32}{9}a^3\tau_3 + \ldots; \\
S_6 = (2 + \frac{2}{3^5})a^6 + 30a^4b^2 + \frac{5}{27}a^4\tau_2 - \frac{20}{27}a^3\tau_3 + \ldots .
\]

Again we calculate only whose coefficients, which will be used. Since $b$ and all $\tau_i$ are algebraically independent, we indeed obtain four linear equations.

\[
\begin{align*}
\frac{512}{27}x_1 + \frac{256}{81}x_2 + \frac{448}{81}x_3 + (2 + \frac{2}{3^5})x_4 &= 0 \\
\frac{128}{3}x_1 + \frac{64}{3}x_2 + (36 + \frac{4}{27})x_3 + 30x_4 &= 0 \\
\frac{64}{3}x_1 - \frac{32}{9}x_2 + (2 + \frac{50}{27})x_3 + \frac{5}{27}x_4 &= 0 \\
\frac{32}{9}x_2 - \frac{32}{9}x_3 - \frac{20}{27}x_4 &= 0
\end{align*}
\]

The determinant of this system is non-zero. Hence the only solution is trivial. Since $\varphi_2 \neq 0$, we have proved that $\deg(\varphi_2^{(\nu)}) \leq 2$. \hfill $\square$

**Lemma 2.** For the Weyl group invariant $\varphi_3$ with $\deg \varphi_3 = 8$, we have $\deg(\varphi_3^{(\nu)}) \leq 4$.

**Proof.** Argument for this invariant is essentially the same as in Lemma 1, but here calculations are more involved. Again

\[
\varphi_3 = y_1S_2^4 + y_2S_2S_3^2 + y_3S_2^2S_4 + y_4S_2S_6 + y_5S_3S_5 + y_6S_4^2 + y_7S_8, \quad \text{where } y_i \in \mathbb{K}.
\]

We need coefficients of this seven polynomials up to $a^4$. Here they are:

\[
S_2^4 = \frac{4096}{81}a^8 + \frac{4096}{27}a^6b^2 + \frac{2048}{27}a^6\tau_2 + \frac{128}{3}a^4(4b^4 + 4b^2\tau_2 + \tau_2^2) + \ldots; \\
S_3^2 = \frac{2048}{243}a^8 + \frac{5120}{81}a^6b^2 - \frac{512}{81}a^6\tau_2 + \frac{256}{27}a^5\tau_3 + \frac{416}{3}a^4b^4 - \frac{160}{9}a^4b^2\tau_2 - \frac{8}{9}a^4\tau_2 + \ldots.
\]
\[ S_2 S_4 = \frac{3584}{243} a^8 + \frac{8704}{81} a^6 b^2 + \frac{1280}{81} a^6 \tau_2 - \frac{256}{27} a^5 \tau_3 + \]
\[ + \frac{4064}{3} a^4 b^4 - \frac{2144}{27} a^4 b^2 \tau_2 + \frac{152}{27} a^4 \tau_2^2 + \frac{64}{9} a^4 \tau_4 + \ldots; \]
\[ S_2 S_6 = \frac{3904}{729} a^8 + \frac{20416}{243} a^6 b^2 + \frac{488}{243} a^6 \tau_2 - \frac{160}{81} a^5 \tau_3 + \]
\[ + \frac{140}{9} a^4 b^4 + \frac{820}{27} a^4 b^2 \tau_2 + \frac{5}{27} a^4 \tau_2^2 + \frac{40}{9} a^4 \tau_4 + \ldots; \]
\[ S_3 S_5 = \frac{2560}{729} a^8 + \frac{1280}{27} a^6 b^2 - \frac{640}{243} a^6 \tau_2 + \frac{320}{81} a^5 \tau_3 + \]
\[ + \frac{1240}{27} a^4 b^4 - \frac{200}{9} a^4 b^2 \tau_2 + \frac{20}{27} a^4 \tau_2^2 - \frac{80}{27} a^4 \tau_4 + \ldots; \]
\[ S_4^2 = \frac{3136}{729} a^8 + \frac{448}{9} a^6 b^2 + \frac{224}{27} a^6 \tau_2 - \frac{448}{81} a^5 \tau_3 + \frac{4112}{27} a^4 b^4 + 16 a^4 b^2 \tau_2 + \frac{4}{9} a^4 \tau_2^2 + \frac{112}{27} a^4 \tau_4 + \ldots; \]
\[ S_8 = (2 + \frac{2}{3^7}) a^8 + 56 a^6 b^2 + \frac{28}{3^6} a^6 \tau_2 - \frac{56}{3^5} a^5 \tau_3 + 140 a^4 b^4 + \frac{70}{81} a^4 \tau_4 + \ldots. \]

I calculated these expansions on the computer in “Maple”. It is quite possible to check any of the coefficients by hand, but getting them all is rather tiresome.

Assume that \( \deg(\varphi_3^{(\nu)}) > 4 \). Then the coefficients of \( a^8, a^6, a^5, \) and \( a^4 \) in \( \varphi_3 \) are zeros. Therefore there are eight linear equations, corresponding to the summands

\[ a^8, a^6 b^2, a^6 \tau_2, a^5 \tau_3, a^4 b^4, a^4 b^2 \tau_2, a^4 \tau_2^2, a^4 \tau_4, \]

depending on seven variables \( y_i \). Since at least one \( 7 \times 7 \) minor of this matrix is non-zero (it was checked on the computer), the only possible solution is zero. Thus if \( \varphi_3 \neq 0 \), then \( \deg(\varphi_3^{(\nu)}) \leq 4. \)

\[ \square \]

**Proof of Proposition 1.** Suppose that \( \varphi_1, \ldots, \varphi_7 \) is a system of homogeneous generators of \( S(d)^W \) with \( \deg \varphi_i < \deg \varphi_j \) for \( i < j \). Then \( \deg \varphi_1 = 2, \deg \varphi_2 = 6, \) and \( \deg \varphi_3 = 8 \). Clearly \( \varphi_1^{(\nu)} h_0^{(\nu)} \) is proportional to \( h_0^2 \). Hence the polynomials \( \varphi_1^{(\nu)} h_0^{(\nu_1)}, \varphi_2^{(\nu)} h_0^{(\nu_2)}, \) and \( \varphi_3^{(\nu)} h_0^{(\nu_3)} \) are algebraically independent if and only if \( \varphi_2^{(\nu)} \) and \( \varphi_3^{(\nu)} \) are. By Lemmas 1 and 2, we have \( \deg \varphi_2^{(\nu)}, \deg \varphi_3^{(\nu)} \leq 4 \). Recall that \( \varphi_2^{(\nu)} \) and \( \varphi_3^{(\nu)} \) are invariants of \( E_6 \). Since the Weyl group of type \( E_6 \) has no basic invariants of degrees 1, 3, and 4; and only one of degree 2, these polynomials are algebraically dependent.

\[ \square \]
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