# Dirichlet Duality and the Nonlinear Dirichlet Problem

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**ABSTRACT**

We study the Dirichlet problem for fully nonlinear, degenerate elliptic equations of the form $\mathbf{F}(\mathrm{Hess}\, u) = 0$ on a smoothly bounded domain $\Omega \subset\subset \mathbf{R}^n$. In our approach the equation is replaced by a subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ of the symmetric $n \times n$ matrices with $\partial F \subseteq \{\mathbf{F} = 0\}$. We establish the existence and uniqueness of continuous solutions under an explicit geometric "$F$-convexity" assumption on the boundary $\partial\Omega$. The topological structure of $F$-convex domains is also studied and a theorem of Andreotti-Frankel type is proved for them. Two key ingredients in the analysis are the use of subaffine functions and Dirichlet duality, both introduced here. Associated to $F$ is a Dirichlet dual set $\widetilde{F}$ which gives a dual Dirichlet problem. This pairing is a true duality in that the dual of $\widetilde{F}$ is $F$ and in the analysis the roles of $F$ and $\widetilde{F}$ are interchangeable. The duality also clarifies many features of the problem including the appropriate conditions on the boundary. Many interesting examples are covered by these results including: All branches of the homogeneous Monge-Ampère equation over $\mathbf{R}$, $\mathbf{C}$ and $\mathbf{H}$; equations appearing naturally in calibrated geometry, Lagrangian geometry and $p$-convex riemannian geometry, and all branches of the Special Lagrangian potential equation.

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## 1. Introduction.

The point of this paper is to study the Dirichlet problem for certain fully nonlinear, degenerate elliptic, second order differential equations which appear naturally in geometry. The class of problems we consider has a rich structure and covers a wide variety of interesting cases. To be more specific, we suppose that $\Omega$ is a bounded domain in $\mathbf{R}^n$ with smooth boundary and that the nonlinear operator $\mathbf{F}$ depends only on the second derivatives of the unknown function. We then consider the homogeneous Dirichlet problem: to show

$$(DP) \qquad \text{Given } \varphi \in C(\partial\Omega), \ \ \exists!\ u \in C(\overline{\Omega}) \ \text{ with } \ \mathbf{F}(\operatorname{Hess} u) = 0 \text{ on } \Omega \ \text{ and } \ u\big|_{\partial\Omega} = \varphi.$$

To our surprise, uniqueness does not seem to be included in the celebrated theory of viscosity solutions unless $\mathbf{F}$ is either uniformly elliptic or proper with respect to the variable $u$. Moreover, a local geometric condition on $\partial\Omega$ needed for existence only seems to be available in certain cases (cf. the inspiring paper [CNS]). We shall give answers to these two questions.

We take a geometric approach to the equation (in the spirit of Krylov [Kr]) which eliminates the operator $\mathbf{F}$ and replaces it with a closed subset $F$ of the space $\mathrm{Sym}^2(\mathbf{R}^n)$ of real symmetric $n \times n$ matrices, with the property that $\partial F$ is contained in $\{\mathbf{F} = 0\}$. In this approach we formulate the notion of solution as a *dual notion*. Although the fact is not needed in this paper, we show at the end of Section 4 that our solutions are the standard viscosity solutions. We feel our duality makes all the basic properties and the comparison theorems more transparent. Furthermore, this duality is a true duality in that every equation has a well defined dual equation, and their roles are interchangeable in the theory.

The geometric approach to the problem also leads naturally to a pointwise convexity condition on the boundary $\partial\Omega$ needed for the existence question. This condition generalizes the usual convexity and pseudoconvexity for the classical Monge-Ampere equation in the real and complex case, as well as the $\phi$-convexity introduced for domains in a calibrated manifold $(M, \phi)$ in [$\mathrm{HL}_2$].

Interestingly, this convexity condition for $\partial\Omega$ gives explicit restrictions on the topology of the domain $\Omega$. In particular, there is an integer $D$, depending only on the subset $F$, such that if $\partial\Omega$ satisfies the convexity condition at each point, then $\Omega$ has the homotopy type of a CW-complex of dimension $\leq D$.

An important aspect of this theory is that it applies to a wide spectrum of interesting cases. For example, suppose $K = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ (the real, complex and quaternionic number fields respectively), and consider $K^n = \mathbf{R}^N$ where $N = n, 2n$ or $4n$. Every real symmetric $N \times N$-matrix $A$ has a $K$-hermitian symmetric part $A_K$ with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$. The associated determinant $\det_K A_K = \lambda_1 \cdots \lambda_n$ is a polynomial of degree $n$ in the entries of $A$ and there is an associated Monge-Ampère equation

$$\det{}_K\{\operatorname{Hess} u\}_K \ = \ 0.$$

Solutions to the Dirichlet problem for this equation are understood in the case where $\{\operatorname{Hess} u\}_K \geq 0$, i.e., $\lambda_1 = 0$ (see, for example, [$\mathrm{Alex}_2$], [Br], [BT], [Al]). However, our theory gives unique solutions of (DP) for the other branches of the equation, namely

$$\lambda_q \ = \ 0$$

for any fixed $q$. This important result is due to Hunt and Murray [HM] and Slodkowski [S] in the complex case. The work of Slodkowski was an inspiration for this paper. His result on the largest eigenvalue of a convex function is the deepest ingredient in our uniqueness proof.

We note incidentally that the problem dual to $\lambda_q = 0$ in our sense is $\lambda_{n-q+1} = 0$.

One can also treat the equation

$$\lambda_p + \lambda_{p+1} + \cdots + \lambda_{p+q} \;=\; 0$$

for fixed $p$ and $q$ by these methods.

A large and important class of examples are those which are *geometrically defined*. In particular every calibration on $\mathbf{R}^n$ gives rise to an equation of our type. More details are given just below.

Yet another interesting case is the equation

$$\mathrm{Im}\left\{e^{i\theta}\det(I + i\mathrm{Hess}\, u)\right\} \;=\; 0,$$

(for fixed $\theta$) which governs the potential functions in the theory of Special Lagrangian submanifolds. The locus of this equation, considered as a subset of $\mathrm{Sym}^2(\mathbf{R}^n)$, has $n$ distinct connected components or *branches*, unless $\theta = \pi/2$ when $n$ is odd or $\theta = 0$ when $n$ is even. In these exceptional cases there are $n-1$ branches. For the two outermost branches and with $\theta = 0$, the Dirichlet problem was treated in depth in [CNS]. Furthermore, they conjectured that there exist the same number of solutions as there are branches. Our results show that indeed the Dirichlet problem is uniquely solvable in continuous functions for every branch and for every $\theta$ in each dimension. In particular the $n$ (or $n-1$) distinct solutions for a given boundary function exist and are uniquely determined by the distinct branches. They are also nested, i.e., $u_1 \leq u_2 \leq \cdots$.

Our general set-up here is the following. We start with a given closed subset $F$ of the space real symmetric matrices $\mathrm{Sym}^2(\mathbf{R}^n)$. We are interested in formulating and solving the Dirichlet problem for the equation

$$\mathrm{Hess}_x u \in \partial F \quad \text{for all } x \in \Omega. \tag{1.1}$$

using the functions of "type $F$", i.e., which satisfy

$$\mathrm{Hess}_x u \in F \quad \text{for all } x. \tag{1.2}$$

*Apriori* these conditions make sense only for $C^2$ functions $u$. We shall extend the notion to functions which are only upper semicontinuous.

This extension requires two ingredients. First we introduce the class of **subaffine functions**. These are upper semicontinuous functions $u$ defined locally by the condition:

*For each affine function $a$, if $u \leq a$ on the boundary of a ball $B$, then $u \leq a$ on $B$.*

These locally subaffine functions are globally subaffine and hence satisfy the maximum principle on any compact set. A $C^2$-function is subaffine if and only if $\mathrm{Hess}\, u$ has at least one eigenvalue $\geq 0$ at each point.

The second step is to consider the **Dirichlet dual set**

$$\widetilde{F} \equiv -(\sim \mathrm{Int} F). \tag{1.3}$$

and define an upper semicontinuous function $u$ to be **of type** $F$ if

$$u + v \text{ is subaffine for all } C^2\text{-functions } v \text{ of type } \widetilde{F}.$$

In other words, $u \in$ USC is type $F$ if for any "test function" $v \in C^2$ of dual type $\widetilde{F}$, the sum $u + v$ satisfies the maximum principle.

Note that $\widetilde{\widetilde{F}} = F$, and so our condition above has an inherent symmetry.

The key requirement on $F$ for solving the Dirichlet problem is that the maximum of two functions of type $F$ be again of type $F$. This is, in effect, equivalent to the following positivity condition on our set. We say that $F$ is a **Dirichlet set** if it satisfies the condition

$$F + \mathcal{P} \subset F. \tag{1.4}$$

where

$$\mathcal{P} = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0\}$$

is the subset of non-negative matrices. This condition corresponds to degenerate ellipticity in modern fully nonlinear theory.

The simplest case, where $F = \mathcal{P}$, is classical. Here the functions of type $\mathcal{P}$ are the convex functions, and strict $\mathcal{P}$-convexity of the boundary is the conventional notion.

In the dual case where $F = \widetilde{\mathcal{P}} = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \not< 0\}$ we shall prove that an upper semicontinuous function $u$ is of type $\widetilde{\mathcal{P}}$ if and only if it is subaffine.

It is easy to see that $F$ is a Dirichlet set if and only if $\widetilde{F}$ is a Dirichlet set.

Dirichlet sets can be quite general in structure. Translates, unions (when closed) and intersections of Dirichlet sets are Dirichlet sets. However there are quite interesting ones coming from geometry as follows. Let $G(p, \mathbf{R}^n)$ denote the grassmannian of $p$-planes in $\mathbf{R}^n$ and fix any compact subset $G \subset G(p, \mathbf{R}^n)$. Let

$$\mathcal{P}(G) = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{trace}(A\big|_\xi) \geq 0 \ \text{ for all } \xi \in G\}$$

Then $\mathcal{P}(G)$ is a Dirichlet set with Dirichlet dual

$$\widetilde{\mathcal{P}}(G) = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{trace}(A\big|_\xi) \geq 0 \ \text{ for some } \xi \in G\}$$

The $C^2$-functions of type $\mathcal{P}(G)$ are characterized by being subharmonic on all $G$-planes. In fact they are subharmonic on all minimal $G$-submanifolds (those whose tangent planes are all $G$-planes). Every calibration $\phi$ gives a set $G = G(\phi)$ of this type where $G$-submanifolds are automatically minimal. As special cases one considers the complex and quaternionic grassmannians. Another interesting case, not coming from a calibration, is given by the set $G = LAG$ of all Lagrangian $n$-planes in $\mathbf{C}^n$.

Further interesting examples arise from restriction. If $F_W \subset \mathrm{Sym}^2(W)$ is a Dirichlet set, where $W \subset \mathbf{R}^n$ is a linear subspace, then $F = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A\big|_W \in F_W\}$ is also a Dirichlet set. Since arbitrary intersections of Dirichlet sets are Dirichlet sets, this yields a new Dirichlet set for each family of Dirichlet sets on subspaces of $\mathbf{R}^n$

In addition, many of the interesting examples can be introduced in terms of Gårding polynomials on $\mathrm{Sym}^2(\mathbf{R}^n)$ with the identity $I$ a hyperbolic direction. These in turn can generate more examples by taking directional derivatives in the direction $I$.

The very general nature of Dirichlet sets complicates the question: "What geometric conditions on $\partial\Omega$ are necessary to solve the Dirichlet problem for a given $F$?" Associated to each $F$ is an asymptotic cone or "ray set" $\overrightarrow{F}$. This is a closed cone with vertex at the origin and consisting essentially of the rays which lie inside $F$ after some point.

Suppose now that $\Omega \subset\subset \mathbf{R}^n$ is a domain with smooth boundary. Denote by $II$ the second second fundamental form of the boundary with respect to the inward-pointing unit normal. Then $\partial\Omega$ is said to be **strictly $\overrightarrow{F}$-convex** at $x \in \partial\Omega$ if

$$II_x \;=\; B\big|_T \quad \text{for some } B \in \mathrm{Int}\overrightarrow{F}.$$

where $T = T_x(\partial\Omega)$. This is equivalent to the condition that $II_x + tP_n \in \mathrm{Int}\overrightarrow{F}$ for all $t \geq$ some $t_0$ where $P_n$ is projection onto the line normal to $\partial\Omega$ at $x$.

By a *global defining function* for $\partial\Omega$ we mean a function $\rho \in C^\infty(\overline{\Omega})$ with $\rho < 0$ on $\Omega$ and with $\rho = 0$ and $\nabla\rho \neq 0$ on $\partial\Omega$. We prove the following result.

**THEOREM 5.12.** *Suppose $F$ is a Dirichlet set. If the boundary $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at each point, then there exists a global defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$ which is strictly of type $\overrightarrow{F}$ on $\overline{\Omega}$. Moreover,*

$$\exists\ \epsilon > 0 \text{ and } R > 0 \text{ such that } \; C\left(\rho - \epsilon\tfrac{1}{2}|x|^2\right) \;\in\; F(\overline{\Omega}) \;\; \text{for all } C \geq R$$

We are now in a position to discuss the main theorem. A function $u$ on a domain $\Omega$ is said to be *$F$-Dirichlet* if $u$ is of type $F$ and $-u$ is of type $\widetilde{F}$. Such a function $u$ is automatically continuous, and at any point $x$ where $u$ is $C^2$, it satisfies the condition (1.2) above.

**THEOREM 6.2. (The Dirichlet Problem).** *Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and let $F$ be a Dirichlet set. Suppose that $\partial\Omega$ is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex at each point. Then for each $\varphi \in C(\partial\Omega)$, there exists a unique $u \in C(\overline{\Omega})$ which is an $F$-Dirichlet function on $\Omega$ and equals $\varphi$ on $\partial\Omega$.*

**Note.** The requirement of both $\overrightarrow{F}$-convexity and $\overrightarrow{\widetilde{F}}$-convexity for $\partial\Omega$ is necessary. In fact this explains the restriction $2q < n$ which appears in the work of Hunt and Murray [HM].

**Note.** Well known uniqueness results (cf. [J], [I], [IL], [$SO_1$] for example) require either uniform ellipticity or properness of the equation with respect to the variable $u$. See [CIL], and [$SO_*$] for a fuller account and references.

The uniqueness part of Theorem 6.2 is immediate from the following comparison result and the maximum principle for subaffine functions. For an open set $X \subset \mathbf{R}^n$, let $F(X)$

denote the set of (u.s.c.) functions of type $F$ on $X$ and let $\mathrm{SA}(X)$ denote the subaffine functions on $X$.

**THEOREM 6.5. (The Subaffine Theorem).** *Assume that $F$ is a Dirichlet set. If $u \in F(X)$ and $v \in \widetilde{F}(X)$, then $u + v \in \mathrm{SA}(X)$.*

The proof of this result is given in Sections 7 and 8. In Section 7 we use a breakthrough technique of Slodkowski to prove the Subaffine Theorem when $u$ and $v$ are quasi-convex. Slodkowski's work enables one to pass from an estimate which holds almost everywhere to one which holds at all points, and can therefore be used to establish the maximum principle. Then, in Section 8, sup-convolution is used to approximate arbitrary $u$ and $v$, of type $F$ and $\widetilde{F}$ respectively, by quasi-convex functions of the same type.

**Remark.** We note that if $F_1 \subset F_2$ are Dirichlet sets, and if $u_1, u_2$ are the corresponding solutions to the Dirichlet problem above (for the same boundary function $\varphi$), then

$$u_1 \ \leq \ u_2 \quad \text{on } \overline{\Omega}. \tag{1.5}$$

Thus the entire lattice of Dirichlet sets, ordered by inclusion, maps in an order preserving way to the set of solutions. If, for example, one restricts to Dirichlet sets which are cones with vertex at the origin, then our ordered family has an initial object $\mathcal{P}$ and final object $\widetilde{\mathcal{P}}$. For any continuous function given on the boundary of a convex domain we obtain a huge family of solutions all lying above the convex solution and below the concave one. They serve as "barriers" for each other in that (1.5) holds whenever $F_1 \subset F_2$. Of course, somewhere in there lies the harmonic solution corresponding to $F_{\mathrm{harm}} = \{A : \mathrm{tr}A \geq 0\}$. Even in two variables it is interesting to contemplate this family. Within it, for example, lie the Dirichlet sets $F = \{A : a_{11} \geq 0\}$ and $F = \{A : a_{11} \geq 0 \text{ and } a_{22} \geq 0\}$ whose associated Dirichlet functions are weak solutions of $u_{xx} = 0$ and $u_{xx}u_{yy} = 0$ respectively.

**Remark .** The case $F = \{A : a_{11} \geq 0\}$, corresponding to $u_{xx} = 0$, demonstrates the utter lack of regularity (beyond continuity) for general solutions obtained here. The $F$-convexity required for a domain $\Omega \subset \mathbf{R}^2$ is that it be horizontal-slice convex (i.e., horizontal slices are connected) and the unique solution for a given boundary function $\varphi$ is the linear interpolation on these slices.

The paper is organized as follows.

In Section 2 we introduce the notion of a subaffine function. This is a class of functions which satisfy the maximum principle and are determined by local properties.

In Section 3 the "positivity condition" $F + \mathcal{P} \subset F$ is discussed in some detail. For convenience, and to avoid the overused word *elliptic*, these sets are called Dirichlet sets. This is exactly the natural condition to ensure that $u, v \in F(X) \ \Rightarrow \ \max\{u, v\} \in F(X)$.

In Section 4 the dual set $\widetilde{F}$ is investigated. This duality clarifies our weak definition of type $F$ and leads to a natural discussion of uniqueness via the Subaffine Theorem.

In Section 5 the associated ray set $\overrightarrow{F}$ is introduced, $\overrightarrow{F}$-convexity of the boundary is discussed, and Theorem 5.12 is proved.

In Sections 6, 7 and 8 the Dirichlet problem is solved. Existence follows from the Perron method and the classical "barrier" argument, combined with a regularity argument of Walsh. Uniqueness is reduced to the Subaffine Theorem, which is proved in Sections 7 and 8.

In Section 9 we show that the natural domains $\Omega$ for which the $F$-Dirichlet problem can be solved are topologically restricted. If $D$ is the "free dimension of $F$", then $\Omega$ has homotopy dimension $D$ and $H_k(\partial\Omega; \mathbf{Z}) \cong H_k(\Omega; \mathbf{Z})$ for all $k < n - D - 1$.

In Section 10 we discuss numerous examples of Dirichlet sets, as well as general principles for constructing them. This section shows that there are many interesting applications of the main results.

In Appendix A we show that for Dirichlet sets $F$ which can be defined using fewer of the variables in $\mathbf{R}^n$ ( i.e., in terms of a Dirichlet set $F_0$ associated to a proper subspace $\mathbf{R}^p \subset \mathbf{R}^n$) that an u.s.c. function $u$ is of type $F$ if and only if the restriction of $u$ to each horizontal $\mathbf{R}^p$ is of type $F_0$.

In Appendix B a distributional definition of type $F$ is given when $F$ is convex.

**Note .** Throughout this paper $X$ will denote an open connected subset of $\mathbf{R}^n$.

## 2. The Maximum Principle and Subaffine Functions.

For a discussion of the maximum principle it is natural to consider the space USC($X$) of upper semi-continuous functions on $X$ with values in $[-\infty, \infty)$. A function $u \in \mathrm{USC}(X)$ satisfies the *maximum principle* if for each compact subset $K \subset X$

$$\sup_K u \ \le\ \sup_{\partial K} u. \tag{2.1}$$

A function $u$ may locally satisfy the maximum principle without satisfying the maximum principle on all of $X$. (Consider, for example, a function $u$ on $\mathbf{R}$ with compact support, $0 \le u \le 1$, $u \equiv 1$ on a neighborhood of the origin and otherwise monotone.) However, this situation is easily remedied. First note that (2.1) is equivalent to the condition that:

$$u \ \le\ c \ \text{ on } \partial K \qquad \Rightarrow \qquad u \ \le\ c \ \text{ on } K \qquad \text{for all constants } c, \tag{2.1$'$}$$

i.e., $u$ is *sub-constants*. Replacing the constant functions by the affine functions, consider the condition:

$$u \ \le\ a \ \text{ on } \partial K \qquad \Rightarrow \qquad u \ \le\ a \ \text{ on } K \qquad \text{for all affine functions } a \tag{2.2}$$

**Definition 2.1.** A function $u \in \mathrm{USC}(X)$ satisfying (2.2) for all compact subsets $K \subset X$ will be called *subaffine on* $X$. Let $\mathrm{SA}(X)$ denote the space of all $u \in \mathrm{USC}(X)$ that are locally subaffine on $X$, i.e., for all $x \in X$ there exists a neighborhood $B$ of $x$ with $u\big|_B$ subaffine on $B$.

Note that if $u$ is subaffine on $X$, then the restriction to any open subset is also subaffine.

**Lemma 2.2.** *If $u$ is locally subaffine on $X$, then $u$ is subaffine on $X$.*
*Moreover, $u \notin \mathrm{SA}(X)$ if and only if*

$$\begin{aligned} &\textit{there exist } \ x_0 \in X, \ \ a \textit{ affine}, \ \textit{and } \epsilon > 0 \textit{ such that} \\ &\qquad (u-a)(x) \ \le\ -\epsilon|x-x_0|^2 \ \ \textit{near } x_0, \ \ \textit{and} \\ &\qquad (u-a)(x_0) \ =\ 0 \end{aligned} \tag{2.3}$$

**Proof.** Subaffine implies locally subaffine, which implies (2.3) is impossible. Hence, it remains to show that if (2.3) is false, then $u$ is subaffine, or equivalently, if $u$ is not subaffine on $X$, then (2.3) is true. If $u$ is not subaffine on $X$, then for some compact set $K \subset X$ and some affine function $b$, the difference $w = u - b$ has a strict interior maximum point for $K$. For $\epsilon > 0$ sufficiently small, the same is true for $w = u + \epsilon|x|^2 - b$. Choose a maximum point $x_0 \in \mathrm{Int}K$ for $w$ and let $M = w(x_0)$ denote the maximum value on $K$. Then $u+\epsilon|x|^2-b-M \le 0$ on $K$ and equals zero at $x_0$. Since $\epsilon|x|^2$ and $\epsilon|x-x_0|^2$ differ by an affine function, this proves that there is an affine function $a$ such that $u+\epsilon|x-x_0|^2-a \le 0$ on $K$ and is equal to zero at $x_0$, i.e., (2.3) is true. ∎

**PROPOSITION 2.3.** **(Maximum Principle).** *Suppose $K \subset \mathbf{R}^n$ is compact and $u \in \mathrm{USC}(K)$. If $u \in \mathrm{SA}(\mathrm{Int}K)$, then*

$$\sup_K u \;\le\; \sup_{\partial K} u.$$

**Proof.** Exhaust Int$K$ by compact sets $K_\epsilon$. Since $u \in \mathrm{SA}(\mathrm{Int}K)$, $\sup_{K_\epsilon} u \le \sup_{\partial K_\epsilon} u$. Since $u \in \mathrm{USC}(K)$, each $U_\delta = \{x \in K : u(x) < \sup_{\partial K} u + \delta\}$, for $\delta > 0$, is an open neighborhood of $\partial K$ in $K$. Therefore, there exits $\epsilon > 0$ with $\partial K_\epsilon \subset U_\delta$ which implies that $\sup_{\partial K_\epsilon} u \le \sup_{\partial K} u + \delta$. ■

For functions which are $C^2$ (twice continuously differentiable), the subaffine condition is a condition on the hessian of $u$ at each point.

**PROPOSITION 2.4.** *Suppose $u \in C^2(X)$. Then*

$$u \in \mathrm{SA}(X) \quad\iff\quad \mathrm{Hess}_x u \ \textit{ has at least one eigenvalue } \ge 0 \textit{ at each point } x \in X.$$

**Proof.** Suppose $\mathrm{Hess}_{x_0} u < 0$ (negative definite) at some point $x_0 \in X$. Then the Taylor expansion of $u$ about $x_0$ implies (2.3). Therefore, since $u(x_0) = 0$, $u \notin \mathrm{SA}(X)$.

Conversely, if $u \notin \mathrm{SA}(X)$, then (2.3) is valid for some point $x_0 \in X$ which implies that $\mathrm{Hess}_{x_0} u + \epsilon I \le 0$. So $\mathrm{Hess}_x u < 0$ is negative definite. ■

**Example (n=1).** Suppose $I$ is an open interval in $\mathbf{R}$. Then

$$u \in \mathrm{SA}(I) \quad\iff\quad \text{either } \ u \in \mathrm{Convex}(I) \text{ or } u \equiv -\infty.$$

**Proof.** Suppose $u \in \mathrm{SA}(I)$ equals $-\infty$ at one point $\alpha \in I$ but $u$ is finite at another point $\beta \in I$ with $\alpha < \beta$. Choose $a$ to be the affine function with $a(\alpha) = -N$ and $a(\beta) = u(\beta)$. By (2.2), we have $u \le a$ on $[\alpha, \beta]$, which implies (by letting $N \to \infty$) that $u \equiv -\infty$ on $[\alpha, \beta)$. The case $\beta < \alpha$ is identical. Hence $u$ is either $\equiv -\infty$ or it is finite-valued on all of $I$ (and therefore convex). The converse is immediate. ■

Next we give a characterization of subaffine functions and convex functions which is the prototype of Dirichlet duality.

**PROPOSITION 2.5.** *Suppose $v \in \mathrm{USC}(X)$. Then $v \in \mathrm{SA}(X) \iff u + v \in \mathrm{SA}(X)$ for all $u \in \mathrm{Convex}(X)$.*

**Proof.** Since $u = 0$ is convex, we need only prove that if $u \in \mathrm{Convex}(X)$ and $v \in \mathrm{SA}(X)$, then $u + v \in \mathrm{SA}(X)$. Equivalently, we must show that if $v \in \mathrm{SA}(X)$, then:

$$\text{For all } w \in \mathrm{Concave}(X), \quad v \;\le\; w \ \text{ on } \partial B \quad\Rightarrow\quad v \;\le\; w \ \text{ on } B \tag{2.4}$$

for an arbitrary closed ball $B$ contained in $X$. That is,

$$v \ \text{ is subaffine} \qquad\Rightarrow\qquad v \ \text{ is “subconcave''} \tag{2.5}$$

To prove (2.4), choose $a$ affine with $w \le a$ on $B$. Then $v \le w \le a$ on $\partial B$ implies $v \le a$ on $B$ since $v$ is subaffine. Now any concave function $w$ is the infimum over the family of affine functions $a$ with $w \le a$. (To see this, apply the finite-dimensional Hahn-Banach Theorem to the graph of $w$.) It follows that that $v \le w$ on $B$. ■

Let $\overline{\text{C}}\text{onvex}(X)$ denote the set of functions on $X$ that locally are either convex or $\equiv -\infty$. It is easy to see by the example above, that:

$$u \in \overline{\text{C}}\text{onvex}(X) \quad \iff \quad \text{the restriction of } u \text{ to each line } L \text{ is in } \overline{\text{C}}\text{onvex}(L \cap X) \tag{2.6}$$

**PROPOSITION 2.6.** *Suppose* $u \in \text{USC}(X)$. *Then:*

$$u \in \overline{\text{C}}\text{onvex}(X) \textit{ if and only if } u + v \in \text{SA}(X) \textit{ for all } v \in \text{SA}(X).$$

**Proof.** If $u \in \text{Convex}(X)$ and $v \in \text{SA}(X)$, then by Proposition 2.5, $u + v \in \text{SA}(X)$. Furthermore, the extra case $u \equiv -\infty$ is obvious.

It remains to show that if $u + v \in \text{SA}(X)$ for all $v \in \text{SA}(X)$, then $u \in \overline{\text{C}}\text{onvex}(X)$. It will suffice to show that:

$$u \notin \overline{\text{C}}\text{onvex}(X) \;\Rightarrow\; \exists \text{ a subaffine quadratic function } B \text{ with } u + B \notin \text{SA}(X). \tag{2.7}$$

Since $u \notin \overline{\text{C}}\text{onvex}(X)$, we know that for the restriction $\overline{u}$ of $u$ to some line $L$, we have $\overline{u} \notin \overline{\text{C}}\text{onvex}(L)$. For $n = 1$, $\text{SA} = \overline{\text{C}}\text{onvex}$, so that (2.3) applies to $\overline{u}$. Assume that the line is the $x_1$-axis and that the point on the line $L$ in (2.3) is the origin in $\mathbf{R}^n$. Also, change $\overline{u}$ by the affine function in (2.3). Then there exists $\delta > 0$ so that $\overline{u}(t) \le -\epsilon t^2$ for $|t| \le \delta$ and $\overline{u}(0) = 0$. Hence, by the upper semicontinuity of $u$, there exists $r > 0$ small with

$$u(t, y) + \frac{\epsilon}{2} t^2 < 0 \quad \text{for} \quad t = \pm\delta, \quad |y| \le r.$$

Now choose $\lambda >> 0$ so that

$$u(t, y) + \tfrac{\epsilon}{2} t^2 - \lambda |y|^2 \;<\; 0 \quad \text{for} \quad |t| \le \delta, \quad |y| = r.$$

The quadratic function $B \equiv \frac{\epsilon}{2} t^2 - \lambda |y|^2$ is subaffine by Proposition 2.4, but the sum $u + v$ is zero at the origin and strictly less than zero on the boundary of the cylinder $|t| \le \delta$, $|y| \le r$ about the origin. Hence, $u + B$ is not subaffine. ■

## 3. Dirichlet Sets – The Maximum of Two Functions.

Each subset $F$ of $\mathrm{Sym}^2(\mathbf{R}^n)$ defines a class of $C^2$-functions $u$ by requiring that $\mathrm{Hess}_x u \in F$ at each point $x$. An important property that we want functions of this type $F$ to have is:

**The Maximum Property:** If $u, v$ are of type $F$, then $\max\{u, v\}$ is of type $F$.

Of course, we must first extend the definition of type $F$ functions to include non $C^2$-functions such as $\max\{u, v\}$. The appropriate condition on $F$ which insures this maximum property is the standard positivity (or elliptic) condition given in the next definition. See Remark 3.3 at the end of this section for more detail.

**Definition 3.1.** A proper non-empty closed subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ will be called a *Dirichlet set* if it satisfies the *Positivity Condition:*

$$F + \mathcal{P} \ \subset \ F \tag{3.1}$$

where

$$\mathcal{P} \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0\} \tag{3.2}$$

denotes the set of non-negative quadratic forms on $\mathbf{R}^n$.

We first introduce the notion of $F$-plurisubharmonicity for $C^2$-functions. The definition will be substantially generalized in the next section.

**Definition 3.2.** Suppose $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a Dirichlet set. If $u \in C^2(X)$ has $\mathrm{Hess}_x u \in F$ for all $x \in X$, then $u$ is of *type $F$* or *$F$-plurisubharmonic*. If $\mathrm{Hess}_x u \in \mathrm{Int}F$ for some $x \in X$, then $u$ is called *strict of type $F$ at $x$*.

**Elementary Properties of Dirichlet Sets $F$:**

(1) $F + \mathrm{Int}\mathcal{P} \ \subset \ \mathrm{Int}F$

(2) $F \ = \ \overline{\mathrm{Int}F}$

(3) $\mathrm{Int}F + \mathcal{P} \ \subset \ \mathrm{Int}F$

(4) For each $B \in \mathrm{Sym}^2(\mathbf{R}^n)$ the set $\{t \in \mathbf{R} : B + tI \in F\} = [b, \infty)$ for some $b \in \mathbf{R}$.

(5) ($F$ is "Asymptotically convex") Given $A, B \in F$, $\exists\, t > 0$ such that $A + tI$ and $B + tI$ both belong to the convex subset $(A + \mathcal{P}) \cap (B + \mathcal{P})$ of $F$.

(6) $F$ is Dirichlet $\Rightarrow \lambda F + A$ is Dirichlet for $\lambda > 0$ and $A \in \mathrm{Sym}^2(\mathbf{R}^n)$.

(7) $F$ is Dirichlet $\iff$ $gF$ is Dirichlet with $g \in GL_n(\mathbf{R})$ acting on $\mathrm{Sym}^2(\mathbf{R}^n)$ by the standard action $g(A) = g^t \circ A \circ g$.

**Proofs:**

(1) For each $A \in F$ the open set $A + \mathrm{Int}\mathcal{P}$ is contained in $F$.

(2) Use (1) and $A = \lim_{\epsilon \to 0}(A + \epsilon I)$.

(3) For each $P \in \mathcal{P}$ the open set $\mathrm{Int}F + P$ is contained in $F$.

(4) Since $F$ Dirichlet implies that $F - B$ is Dirichlet, we may assume that $B = 0$. We must show that the set $\Lambda_F \equiv \{t \in \mathbf{R} : tI \in F\}$ is connected, proper, and non-empty.

If $t_0 \in \Lambda_F$, then by the Positivity Condition $t \geq t_0$ implies $t \in \Lambda_F$. Hence, $\Lambda_F$ is connected. If $\Lambda_F = \mathbf{R}$, then $-tI \in F$ for all $t > 0$. Hence, $-tI + \mathcal{P} \subset F$ for all $t > 0$. This implies that $F$ equals $\mathrm{Sym}^2(\mathbf{R}^n)$ which is not allowed. Therefore, $\Lambda_F \neq \mathbf{R}$. This implies $\Lambda_F \neq \emptyset$ by duality. (See Remark 4.2 in the next section.)

(5) Pick $t >> 0$ so large that $A + tI \in B + \mathcal{P}$ and $B + tI \in A + \mathcal{P}$.

(6) and (7) are straightforward.

**Remark 3.3.** Motivation for the Positivity Condition is provided by

**The Hessian Lemma:** *Suppose $u, v \in C^2(X)$ and $\nabla(u-v) \neq 0$ on $\{u = v\}$. Then taking the distributional hessian, we have*

$$\mathrm{Hess}(\max\{u, v\}) \;=\; \chi_{\{u \geq v\}}\mathrm{Hess}u + \chi_{\{v \geq u\}}\mathrm{Hess}v + \mu\nabla(u-v) \circ \nabla(u-v)$$

*where $\mu$ is a non-negative measure supported on $\{u = v\}$.*

This formula strongly suggests that one should require:

$$A + \xi \circ \xi \in F \qquad \text{for all } A \in F,\ \xi \in \mathbf{R}^n.$$

Since each $P \geq 0$ can be written as $P = \sum_j \lambda_j e_j \circ e_j$, this condition is equivalent to the Positivity Condition (3.1) that $F + \mathcal{P} \subset F$. We omit the proof of this lemma.

## 4. Dirichlet Duality.

As noted in Definition 3.2, each subset $F$ of $\mathrm{Sym}^2(\mathbf{R}^n)$ defines a class of $C^2$-functions $u$ by requiring that $\mathrm{Hess}_x u \in F$ at each point $x$. In this section we will give a dual characterization of this condition, which will enable us to define functions of type $F$ which are not necessarily of class $C^2$. This nonlinear duality can be used in a fashion which has some similarity to the use of distribution theory in linear problems.

Throughout this section we assume that $F$ is a Dirichlet set. Let

$$\widetilde{\mathcal{P}} \ = \sim(-\mathrm{Int}\mathcal{P}) \ = \ -(\sim \mathrm{Int}\mathcal{P}).$$

denote the set of all quadratic forms except those that are negative definite, i.e., $A \in \widetilde{\mathcal{P}}$ iff $A$ has at least one eigenvlaue $\geq 0$. Note that for $u \in C^2(X)$,

$$u \text{ is convex iff } u \text{ is of type } \mathcal{P} \qquad \text{and} \qquad u \text{ is is subaffine iff } u \text{ is of type } \widetilde{\mathcal{P}}.$$

The second statement is just Proposition 2.4

The key to the dual characterization of functions of type $F$ is the existence of a dual subset $\widetilde{F}$. This is made precise in Lemma 4.3 below.

**Definition 4.1.** Suppose $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a Dirichlet set. The *Dirichlet dual* of $F$ is the set

$$\widetilde{F} \ = \sim(-\mathrm{Int}F) \ = \ -(\sim \mathrm{Int}F).$$

**Elementary Properties of the Dirichlet Dual.**

(1) $\widetilde{\widetilde{F}} \ = \ F.$

(2) $F_1 \ \subset \ F_2 \quad \Rightarrow \quad \widetilde{F}_2 \subset \widetilde{F}_1.$

(3) $\widetilde{F_1 \cap F_2} \ = \ \widetilde{F}_1 \cup \widetilde{F}_2$

(4) $\widetilde{F_1 \cup F_2} \ = \ \widetilde{F}_1 \cap \widetilde{F}_2$

(5) $\widetilde{F+A} \ = \ \widetilde{F} - A.$

(6) $F$ is a Dirichlet set $\iff$ $\widetilde{F}$ is a Dirichlet set.

**Proofs.**

(1) follows from $F = \overline{\mathrm{Int}F}$.

(2) (3) and (4) are obvious.

(5) Note that $B \in \widetilde{F+A} \iff -B \notin \mathrm{Int}(F+A) = \mathrm{Int}F + A \iff -(B+A) \notin \mathrm{Int}F \iff B + A \in \widetilde{F} \iff B \in \widetilde{F} - A$.

(6) Suppose $P \in \mathcal{P}$. Then $F + P \subset F$ or equivalently $F \subset F - P$. By (2) this implies that $\widetilde{F-P} \subset \widetilde{F}$. By (5) we have $\widetilde{F-P} = \widetilde{F} + P$ so that $\widetilde{F} + P \subset \widetilde{F}$.

**Remark 4.2.** Define $\widetilde{\Lambda}_F = \sim(-\mathrm{Int}\Lambda_F)$ and note that $\widetilde{\Lambda}_F = \Lambda_{\widetilde{F}}$. Hence $\Lambda_F = \emptyset \Rightarrow \Lambda_{\widetilde{F}} = \mathbf{R} \Rightarrow \widetilde{F} = \mathrm{Sym}^2(\mathbf{R}^n) \Rightarrow F = \emptyset$, completing the proof of Property (4) in Section 3.

The following duality result is stated in several forms: first for the special case of points $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ (i.e., quadratic functions), and then for functions $u \in C^2(X)$.

**Lemma 4.3.** *Suppose $F$ is a Dirichlet set. Then*

*(1)* $$A \in F \quad \Leftrightarrow \quad A + B \in \widetilde{\mathcal{P}} \quad \text{for all } B \in \widetilde{F}.$$

*(2)* $u \in C^2(X)$ *is of type* $F$ $\Leftrightarrow$ $u + B \in \mathrm{SA}(X)$ *for all quadratic* $B \in \widetilde{F}$.

*(3)* $u \in C^2(X)$ *is of type* $F$ $\Leftrightarrow$ $u + v \in \mathrm{SA}(X)$ *for all* $v \in C^2(X)$ *of type* $\widetilde{F}$.

**Proof.** Statement (3) follows from the special case (1) by setting $A = \mathrm{Hess}_x u$, $B = \mathrm{Hess}_x v$, and using Definition 3.2 along with Proposition 2.4. Thus the three conditions are equivalent.

To prove (1), first note that:

$(1)'$ $\quad A \in F \iff A + \mathcal{P} \subset F$

is obviously true because of the positivity condition (3.1).

Now $A + \mathcal{P} \subset F \iff \widetilde{F} \subset \widetilde{A + \mathcal{P}}$ (which equals $\widetilde{\mathcal{P}} - A$) $\iff A + \widetilde{F} \subset \widetilde{\mathcal{P}}$. Thus $(1)'$ is equivalent to:

(1) $\quad A \in F \iff A + \widetilde{F} \subset \widetilde{\mathcal{P}}$. ■

Because of this Lemma we can extend our Definition 3.2 of type $F$ from $C^2$-functions to upper semi-continuous functions.This extension is another central concept of the paper.

**Definition 4.4.** A function $u \in \mathrm{USC}(X)$ is said to be of *type $F$* or *$F$-plurisubharmonic* if

$$u + v \in \mathrm{SA}(X) \quad \text{for all } v \in C^2(X) \quad \text{of type } \widetilde{F}. \tag{4.1}$$

Let $F(X)$ denote the set of all $u \in \mathrm{USC}(X)$ of type $F$.

**PROPOSITION 4.5.** *Suppose $u \in \mathrm{USC}(X)$. Then (for $X$ connected)*

$$u \text{ is convex or } u \equiv -\infty \quad \iff \quad u \text{ is of type } \mathcal{P} \qquad \text{and}$$
$$u \text{ is subaffine} \quad \iff \quad u \text{ is of type } \widetilde{\mathcal{P}}.$$

*Moreover, for any $u$ of type $\mathcal{P}$ and any $v$ of type $\widetilde{\mathcal{P}}$, the sum $u + v \in \mathrm{SA}(X)$.*

**Proof.** This is just a restatement of Propositions 2.5 and 2.6. ■

Note that for two Dirichlet sets $F_1$ and $F_2$,

$$F_1(X) \subset F_2(X) \qquad \iff \qquad F_1 \subset F_2 \tag{4.2}$$

It is important to have some equivalent formulations of the definition of functions of type $F$. For example, as it stands it is not clear that if $u$ is of type $F$ on $X$, then the restriction of $u$ to a smaller open subset is also of type $F$. This however is true and is easily seen from other equivalent definitions.

In making these reformulations we first reduce the space of test functions from $C^2(X) \cap \widetilde{F}(X)$ to just $\widetilde{F}$, the space of quadratic functions of type $F$. The second formulation says that if $u \notin F(X)$, then near some point $x_0 \in X$, the condition for type $F$ is strongly violated.

**Lemma 4.6.** *A function $u$ is in $F(X)$ if and only if*

$$u + B \in \mathrm{SA}(X) \quad \textit{for all quadratic functions } B \in \widetilde{F}. \tag{4.3}$$

*Moreover, $u \notin F(X)$ if and only if*

$$\begin{aligned} &\textit{there exist } B \in \mathrm{Int}\widetilde{F}, x_0 \in X, a \textit{ affine, and } \epsilon > 0 \textit{ such that} \\ &u + B - a \ \leq \ -\epsilon|x - x_0|^2 \ \textit{ near } x_0 \ \textit{ and} \\ &\qquad\qquad = 0 \qquad\qquad\quad \text{at } x_0. \end{aligned} \tag{4.4}$$

**Proof.** If $u \in F(X)$, then, taking $v = B$, we see that (4.1) implies (4.3). Furthermore, (4.3) obviously implies that (4.4) is false. It remains to show that if (4.1) is false, then (4.4) is true. If (4.1) is false, then there exists $v \in C^2(X) \cap \widetilde{F}(X)$ such that $u+v \notin \mathrm{SA}(X)$. Applying Lemma 2.2, there exist $x_0 \in X, \epsilon > 0$ and and affine function $a$ with $u + v - a \leq -2\epsilon|x-x_0|^2$ near $x_0$ and equal to zero at $x_0$. Since $v \in C^2(X)$,replacing $v$ by the quadratic part $B$ of $v$ at $x_0$ yields: $u + B - a \leq -2\epsilon|x - x_0|^2$ near $x_0$ and $u + B - a = 0$ at $x_0$. Finally, since $B \in \widetilde{F}$, we have $B + \epsilon I \in \mathrm{Int}\widetilde{F}$, proving (4.4). ∎

**Properties of the class $F(X)$ for Dirichlet Sets $F$.**

(1) (Local Property). A function $u$ is locally of type $F$ if and only if $u$ is (globally) of type $F$.

(2) (Affine Property). $F(X) + \mathrm{Aff}(X) \subset F(X)$, i.e., if $u \in F(X)$ and $a$ is affine, then $u + a \in F(X)$.

(3) (Translation Property). If $u \in F(X)$, then $v(x) \equiv u(x - y) \in F(X + y)$.

As anticipated, the Positivity Condition insures the maximum property.

(4) (Maximum Property). If $u, v \in F(X)$, then $\max\{u, v\} \in F(X)$.

(5) (Decreasing Limits) If $\{u_j\}_{j=0}^{\infty}$ is a decreasing (i.e., $u_j \geq u_{j+1}$) sequence of functions in $F(X)$, then $\lim_j u_j \in F(X)$.

(5)′ (Uniform Limits) If $\{u_j\}_{j=0}^{\infty}$ is a sequence of functions in $F(X)$ which converges uniformly to $u$ on compact subsets, then $u \in F(X)$.

(6) (Families Locally Bounded Above) Suppose $\mathcal{F} \subset F(X)$ is locally uniformly bounded above. Then the upper envelope $u = \sup_{f\in\mathcal{F}} f$ has u.s.c. regularization $u^* \in F(X)$.

(7) If $u$ is twice differentiable at $x \in X$, then $\mathrm{Hess}_x u \in F$.

**Proofs.**

(1) By Definition 2.2, subaffine functions restrict to subaffine functions, and by Lemma 2.2, locally subaffine implies subaffine. Condition (4.3) now ensures that functions of type $F$ are locally of type $F$ (since the quadratic functions are "universal", i.e.,

defined on all of $\mathbf{R}^n$). Conversely, if $u$ is locally of type $F$, then (4.4) is false, and hence $u$ is globally of type $F$.

(2) , (3) and (4) follow from the definitions.

(5) This standard proof uses the fact that $\{x \in \partial K : u_j(x) + v(x) \geq a(x) + \epsilon\}$ is compact for $u_j \in \mathrm{USC}(X)$ and hence empty for $j$ large.

(5)′ This is standard from (5). Given $\epsilon_j \searrow 0$, $\epsilon_{j+1} < \epsilon_j$, pick $j$ large so that $|u_j - u| < \frac{1}{2}(\epsilon_{j+1} - \epsilon_j)$ and set $u'_j = u_j + \frac{1}{2}(\epsilon_{j+1} + \epsilon_j)$.

(6) If $u^* + v \leq a$ on $\partial K$, then $f + v \leq a$ on $\partial K$ for all $f \in \mathcal{F}$. Hence, $f + v \leq a$ on $K$ for all $f \in \mathcal{F}$, and so $u + v \leq a$ on $K$. Since $v \in C^2$, $u^* + v = (u+v)^* \leq a^* = a$ on $K$.

(7) Let $H = \mathrm{Hess}_x u$. Then the quadratic function $H(y)$ is the uniform limit as $\epsilon \to 0$ of the approximate hessians

$$H_\epsilon(y) \;=\; \epsilon^{-2}\{u(x+\epsilon y) - u(x) - \nabla u(x)\cdot y\}.$$

By (5)′ it suffices to prove that:

$$\text{The approximate hessians } H_\epsilon \text{ are of type } F \tag{4.5}$$

Since $\frac{1}{\epsilon^2}(u(x) + \epsilon \nabla u(x)\cdot y)$ is affine, we must show that $(Lu)(y) = \frac{1}{\epsilon^2}u(x+\epsilon y)$ is of type $F$. Now $L$ has inverse given by

$$(L^{-1}v)(y) \;=\; \epsilon^2 v\left(\frac{y-x}{\epsilon}\right).$$

Note that if $v$ is $C^2$, then $\mathrm{Hess}\, L^{-1}v = \mathrm{Hess}\, v$ at corresponding points. Consequently, if $v$ is $C^2$ and of type $\widetilde{F}$, then $L^{-1}v$ is of type $\widetilde{F}$. Therefore, $u + L^{-1}v \in \mathrm{SA}(X)$ because $u \in F(X)$. Finally, since $L$ maps subaffine to subaffine (This is Property (7) for $\widetilde{\mathcal{P}}$ and can be verified directly), we conclude that $Lu + v = L(u + L^{-1}v)$ is subaffine. Hence, $Lu$ is of type $F$ as desired.

**Remark 4.7. (The Maximum Principle).** This principle is not always true and (perhaps surprisingly) not necessary for the Dirichlet problem. However, the Maximum Principle for all functions in $F(X)$ is true if and only if $F(X) \subset \mathrm{SA}(X)$ because of (2) above. By (4.2) this is equivalent to $F \subset \widetilde{\mathcal{P}}$. Note that

$$F \subset \widetilde{\mathcal{P}} \qquad \iff \qquad 0 \notin \mathrm{Int}F \tag{4.6}$$

If $0 \in \mathrm{Int}F$, then $F$ contains a negative definite quadratic form so that $F \subset \widetilde{\mathcal{P}}$ is impossible. Conversely, if $F \not\subset \widetilde{\mathcal{P}}$, then $F$ contains a negative definite quadratic form $A = -P_0$, $P_0 > 0$. The open set $\{A + P : P > 0\} \subset F$ contains the origin. This proves (4.6) and hence

**PROPOSITION 4.8.** *Suppose $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is a Dirichlet set. The maximum principle holds for each $u \in F(X)$ if and only if $0 \notin \mathrm{Int}F$.*

In the cases where $0 \in \mathrm{Int}F$, (i.e., when the maximum principle does not hold), the functions $u \in F(X)$ will be called *$F$-quasi-plurisubharmonic*.

**Remark 4.9. (Viscosity Subsolutions).** The condition (4.4) above is equivalent to:

$$\begin{gathered} \exists\, x_0 \in X \quad \text{and} \quad \psi \in C^2(X) \quad \text{which is strict of type } \widetilde{F} \text{ at } x_0 \\ \textit{such that} \quad u + \psi \quad \text{has a local maximum at } x_0. \end{gathered} \tag{4.4$'$}$$

**Proof.** That (4.4) ⇒ (4.4)′ is obvious. For the converse set $-a = \langle (\nabla\psi)(x_0), x - x_0 \rangle$ and $B = \frac{1}{2}(\mathrm{Hess}_{x_0}\psi)(x - x_0) - 2\epsilon I \in \mathrm{Int}\widetilde{F}$. ■

Since $\mathrm{Int}\widetilde{F} = -(\sim F)$, if we set $\varphi = -\psi$, then (4.4)′ is equivalent to the condition:

$$\begin{gathered} \exists\, x_0 \in X \quad \text{and} \quad \varphi \in C^2(X) \quad \text{with} \quad \mathrm{Hess}_{x_0}\varphi \notin F \\ \text{but} \quad u - \varphi \quad \text{has a local maximum at } x_0. \end{gathered} \tag{$\sim V$}$$

Finally the negation of (∼ V) is:

$$\begin{gathered} \forall\, x_0 \in X \quad \text{and} \quad \varphi \in C^2(X) \\ \textit{if} \quad u - \varphi \quad \text{has a local maximum at } x_0, \text{ then } \mathrm{Hess}_{x_0}\varphi \in F \end{gathered} \tag{$V$}$$

Condition (V) is the standard viscosity definition of subsolution.

## 5. Boundary Convexity.

We assume throughout this section that $\Omega$ is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. It turns out that the natural boundary-convexity condition associated to a Dirichlet set $F$ is expressed in terms of another Dirichlet set $\overrightarrow{F}$, the *ray set associated with* $F$, which will be defined in a moment. Our main result (Theorem 5.12 below) asserts that strict local $\overrightarrow{F}$-convexity of the boundary implies the existence of a global defining function which is strictly $\overrightarrow{F}$-plurisubharmonic. This global function will play a key role in our solution to the Dirichlet problem in §6.

The key property of this associated ray set $\overrightarrow{F}$ is the

**Ray Property:**

$$A \in \overrightarrow{F} \quad \iff \quad tA \in \overrightarrow{F} \ \text{ for all } t \geq 0. \tag{5.1}$$

Moreover, $\overrightarrow{F} = F$ if and only if $F$ itself has the ray property.

A Dirichlet set $\overrightarrow{F}$ with property (5.1) will be called a **Dirichlet-Ray set** or **D-Ray set**. We assume for the moment that $\overrightarrow{F}$ is any Dirichlet-Ray set (not necessarily the one associated with $F$). Note that since scalar multiplication by $t > 0$ is a homeomorphism of $\text{Sym}^2(\mathbf{R}^n)$, (5.1) implies

$$A \in \text{Int}\overrightarrow{F} \quad \iff \quad tA \in \text{Int}\overrightarrow{F} \ \text{ for all } t > 0. \tag{5.1$'$}$$

A smooth function $\rho$ defined near a point $x \in \partial\Omega$ is said to be a *local defining function for* $\partial\Omega$ *near* $x$ if on some neighborhood of $x$, we have $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$. At the boundary point $x$, let $T = T_x\partial\Omega$ denote the tangent space and $n$ a unit normal vector.

**Definition 5.1.** The boundary $\partial\Omega$ is *strictly* $\overrightarrow{F}$*-convex at a point* $x \in \partial\Omega$ if

$$\text{Hess}_x\rho\big|_T \ = \ B\big|_T \ \text{ for some } B \in \text{Int}\overrightarrow{F} \tag{5.2}$$

**Lemma 5.2.** *The condition of strict* $\overrightarrow{F}$*-convexity for* $\partial\Omega$ *is independent of the defining function* $\rho$.

**Proof.** Any other defining function $\widetilde{\rho}$ is of the form $\widetilde{\rho} = u\rho$ with $u > 0$. At $x \in \partial\Omega$, $\text{Hess}_x\widetilde{\rho} = u\text{Hess}_x\rho + \nabla u \circ \nabla\rho$. Since $\nabla u \circ \nabla\rho$ restricts to be zero on $T$, we have $\widetilde{H}\big|_T = uH\big|_T$. By the ray property (5.1)$'$ for $\text{Int}\overrightarrow{F}$ the proof is complete. ∎

The notion of strict $\overrightarrow{F}$-convexity has other useful formulations. Let $P_n = n \circ n \in \text{Sym}^2(\mathbf{R}^n)$ denote orthogonal projection onto the line in the normal direction $n$

**Lemma 5.3.** *The following conditions on a local defining function* $\rho$ *for* $\partial\Omega$ *are equivalent.*

*(1)* $\text{Hess}_x\rho\big|_T \ = \ B\big|_T$ *for some* $B \in \text{Int}\overrightarrow{F}$ *(i.e.* $\partial\Omega$ *is strictly* $\overrightarrow{F}$*-convex at* $x$*).*

*(2)* $\text{Hess}_x\rho\big|_T + tP_n \ \in \ \text{Int}\overrightarrow{F}$ *for all* $t \geq$ *some* $t_0$*.*

*(2)$'$* $\text{Hess}_x\rho + tP_n \ \in \ \text{Int}\overrightarrow{F}$ *for all* $t \geq$ *some* $t_0$*.*

**Proof.** Let $H = \mathrm{Hess}_x \rho$. Statements (2) and (2)$'$ each imply (1) since in both cases the restriction to $T$ equals $H\big|_T$. Suppose now that (1) is true. Then, in terms of the $2\times 2$ blocking induced by $\mathbf{R}^n = \mathrm{span}\, n \oplus T$, we have $H - B = \begin{pmatrix} a & \alpha \\ \alpha & 0 \end{pmatrix}$. Therefore, $H + tP_n = B - \epsilon I + tP_n + H - B + \epsilon I$. If $\epsilon > 0$ is chosen small, then $B - \epsilon I \in \mathrm{Int}\overrightarrow{F}$, while $tP_n + H - B + \epsilon I = \begin{pmatrix} t + a + \epsilon & \alpha \\ \alpha & \epsilon I \end{pmatrix}$ which is positive definite for $t >> 0$. By Property (3) for Dirichlet sets, this implies that $H + tP_n \in \mathrm{Int}\overrightarrow{F}$ for $t >> 0$. ∎

**COROLLARY 5.4.** *Let $II$ denote the second fundamental form of $\partial\Omega$ with respect to the inward-pointing unit normal $n$. Then $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at $x \in \partial\Omega$ if and only if*

*(1)* $II_x = B\big|_T$ *for some* $B \in \mathrm{Int}\overrightarrow{F}$, *or equivalently*

*(2)* $II_x + tP_n \in \mathrm{Int}\overrightarrow{F}$ *for all* $t \geq$ *some* $t_0$.

**Proof.** By Lemma 5.2 we may choose $\rho$ to be the signed distance function in a neighborhood of $\partial\Omega$, i.e., for $x$ near $\partial\Omega$

$$\rho(x) = \delta(x) = \begin{cases} -\mathrm{dist}(x, \partial\Omega) & \text{if } x \in \Omega \\ +\mathrm{dist}(x, \partial\Omega) & \text{if } x \notin \Omega \end{cases}.$$

Then it is a standard calculation (cf. [HL$_2$, (4.7)]) that

$$\mathrm{Hess}\,\delta = \begin{pmatrix} 0 & 0 \\ 0 & II \end{pmatrix}.$$

with respect to the splitting $\mathbf{R}^n = (\mathbf{R}\cdot\nabla\delta) \oplus (\nabla\delta)^\perp$. We now apply Lemma 5.3. ∎

**The Associated Ray Set $\overrightarrow{F}$ of $F$.**

**Definition 5.5.** Suppose $F$ is a Dirichlet set and $B \in \mathrm{Sym}^2(\mathbf{R}^n)$ is fixed. The *ray set with vertex $B$ associated to $F$*, denoted by $\overrightarrow{F_B}$, is defined by

$$\overrightarrow{F_B} = \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, t_0 \text{ such that } B + tA \in F \text{ for all } t \geq t_0\}$$

**Example 5.6.** The set $\overrightarrow{F_B}$ may not be closed. Take $B = 0$ and $F = \mathcal{P} \cap \{\det \geq 1\}$.

**Lemma 5.7.** *The closure of $\overrightarrow{F_B}$ is independent of the vertex $B$.*

**Proof.** This is Property (6) proven below.

**Definition 5.8.** Suppose $F$ is a Dirichlet set. The *ray set associated to $F$*, denoted $\overrightarrow{F}$, is defined to be the closure of $\overrightarrow{F_B}$ for any vertex $B$.

**Elementary Properties of $\overrightarrow{F_B}$:**

(1) $\overrightarrow{F_B} + \mathcal{P} \subset \overrightarrow{F_B}$

(2) $\overrightarrow{F_B} + \mathrm{Int}\mathcal{P} \subset \mathrm{Int}\overrightarrow{F_B}$

(3) $\overrightarrow{F_B} \ \subset\ \text{Closure}(\text{Int}\overrightarrow{F_B})$

(4) $\text{Int}\overrightarrow{F_B} \ \subset\ \overrightarrow{F_{B'}}$ for all $B'$

(5) $\text{Int}\overrightarrow{F_B} \ =\ \text{Int}\overrightarrow{F_{B'}}$ for all $B'$

(6) $\text{Closure}(\overrightarrow{F_B}) \ =\ \text{Closure}(\overrightarrow{F_{B'}})$ for all $B'$

**Proofs.**

(1) $B + tA \in F$ for $t \geq t_0 \Rightarrow\ B + t(A + P) = B + tA + tP \in F$ for $t \geq t_0 \geq 0$.

(2) If $A \in \overrightarrow{F_B}$, then by (1) the open set $A + \text{Int}\mathcal{P} \subset \overrightarrow{F_B}$, and hence $A + \text{Int}\mathcal{P} \subset \text{Int}\overrightarrow{F_B}$.

(3) If $A \in \overrightarrow{F_B}$, then by (2) we have that for $\epsilon > 0$, $A + \epsilon I \in \text{Int}\overrightarrow{F_B}$. Hence, $A = \lim_{\epsilon\to 0}(A + \epsilon I) \in \overline{\text{Int}\overrightarrow{F_B}}$. Note that for Example 1 equality in (3) does not hold.

(4) If $A \in \text{Int}\overrightarrow{F_B}$, then $A - \epsilon I \in \overrightarrow{F_B}$ for some $\epsilon > 0$. This means that there exists a $t_0$ so that $B + t(A - \epsilon I) \in F$ for all $t \geq t_0$. Now $B' + tA = B + t(A - \epsilon I) + t\epsilon I - (B - B')$. Choose $\lambda > 0$ so that $\lambda I - (B - B') > 0$ is positive definite. If $t \geq t_0$ and $t \geq \frac{\lambda}{\epsilon}$, then $B' + tA \in F + \mathcal{P} \subset F$, proving that $A \in \overrightarrow{F_{B'}}$.

(5) follows from (4)

(6) follows from (5) and (3).

■

Since boundary convexity involves $\text{Int}\overrightarrow{F}$, some additional facts about $\text{Int}\overrightarrow{F}$ are useful. The associated ray set $\overrightarrow{F}$ for $F$ was defined to be as large as possible. The smallest set of rays associated with $F$ is $\text{Int}\overrightarrow{(\text{Int}F)_B}$ where

$$\overrightarrow{(\text{Int}F)_B} \ \equiv\ \{A \in \text{Sym}^2(\mathbf{R}^n) : \exists\ t_0 \text{ so that } B + tA \in \text{Int}F \ \ \forall\ t \geq t_0\}.$$

**Example .** The set $\overrightarrow{(\text{Int}F)_B}$ may not be open. Take $F = \mathcal{P}$ and $B = I$.

**Lemma 5.9.**

$$\text{Int}\overrightarrow{(\text{Int}F)_B} \ =\ \text{Int}\overrightarrow{F}.$$

**Proof.** Since $\overrightarrow{(\text{Int}F)_B} = \overrightarrow{(\text{Int}(F - B))_0}$ and $\overrightarrow{F - B} = \overrightarrow{F} - B$, we may assume $B = 0$ Because $\overrightarrow{(\text{Int}F)_0} \subset \overrightarrow{F}$, it suffices to show $\text{Int}\overrightarrow{F} \subset \overrightarrow{(\text{Int}F)_0}$. Suppose $A \in \text{Int}\overrightarrow{F}$. Then there exists $\epsilon > 0$ with $A - \epsilon I \in \overrightarrow{F} = \text{Closure}(\overrightarrow{F_0})$. Therefore for all $\delta > 0$ there exists $B \in \overrightarrow{F_0}$ with $|A - \epsilon I - B| < \delta$, which implies that $\delta I + A - \epsilon I - B > 0$. Take $\delta = \frac{\epsilon}{2}$. Then $A - \frac{\epsilon}{2}I - B > 0$ and $B \in \overrightarrow{F_0}$. Hence, there exists $t_0 > 0$ so that $t \geq t_0 \ \Rightarrow\ tB \in F$. Therefore, $t(B + \frac{\epsilon}{2}I) = tB + \frac{\epsilon t}{2}I \in \text{Int}F$ if $t \geq t_0$. This proves that $B + \frac{\epsilon}{2}I \in \overrightarrow{(\text{Int}F)_0}$. Finally, $A = B + \frac{\epsilon}{2}I + A - B - \frac{\epsilon}{2}I \in \overrightarrow{(\text{Int}F)_0} + \text{Int}\mathcal{P} \subset \overrightarrow{(\text{Int}F)_0}$. ■

**COROLLARY 5.10.** *One has $A \in \text{Int}\overrightarrow{F}$ if and only if*

$$\exists\ \epsilon > 0 \text{ and } R > 0 \text{ such that } C(A - \epsilon I) \in F \ \text{ for all } C \geq R \tag{5.3}$$

**Proof.** It follows easily from the previous lemma that $\text{Int}\overrightarrow{F} = \text{Int}\overrightarrow{F_0}$. Therefore, $A \in \text{Int}\overrightarrow{F}$, then for some $\epsilon > 0$, $A - \epsilon I \in \overrightarrow{F_0}$, i.e., there exists $R > 0$ such that $C \geq R$ implies $C(A - \epsilon I) \in F$.

Conversely, assume (5.3) is true. Suppose $B + \epsilon I > 0$. This condition defines a neighborhood of the origin in $\text{Sym}^2(\mathbf{R}^n)$. It suffices to show that $A + B \in \overrightarrow{F}$ for all such $B$. Now $A + B = A - \epsilon I + B + \epsilon I$ and hence $C(A + B) = C(A - \epsilon I) + C(B + \epsilon I)$ which belongs to $F + \mathcal{P} \subset F$ if $C \geq R$. Hence $A + B \in \overrightarrow{F_0} \subset \overrightarrow{F}$. ∎

**PROPOSITION 5.11.** *If $F$ is a Dirichlet set, then the associated ray set $\overrightarrow{F}$ is also a Dirichlet set. Moreover, $\overrightarrow{F}$ has the ray property.*

**Proof.** Take the closure in (1) above. ∎

**Remark .** Since the associated ray set $\overrightarrow{F}$ of $F$ is a Dirichlet-Ray set, the definition of strict $\overrightarrow{F}$-convexity at a boundary point $x \in \partial\Omega$ is independent of the defining function $\rho$ (Lemma 5.2).

A smooth function $\rho \in C^\infty(\overline{\Omega})$ is called a *global defining function for* $\partial\Omega$ if $\Omega = \{\rho < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$.

**THEOREM 5.12.** *Suppose $\overrightarrow{F}$ is a Dirichlet-Ray set. If the boundary $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at each point, then there exists a global defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$ which is strict of type $\overrightarrow{F}$ on $\overline{\Omega}$. Moreover, if $\overrightarrow{F}$ is the ray set associated with a Dirichlet set $F$, then*

$$\exists\ \epsilon > 0 \text{ and } R > 0 \text{ such that } \ C\left(\rho - \epsilon \tfrac{1}{2}|x|^2\right) \ \in \ F(\overline{\Omega}) \ \text{ for all } C \geq R \tag{5.4}$$

The existence of the function $\rho$ in this theorem is the only part of this section needed to solve the Dirichlet Problem in the §6.

**Proof.** Pick any smooth defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$. Let $\widetilde{\rho} = \rho + C\rho^2$. Since $\partial\Omega$ is strictly $\overrightarrow{F}$-convex at each point, we have by Lemma 5.3, part $(2)'$, that on $\partial\Omega$, $\text{Hess}\widetilde{\rho} = (1 + 2C\rho)\text{Hess}\rho + C\nabla\rho \circ \nabla\rho = \text{Hess}\rho + C\nabla\rho \circ \nabla\rho \in \text{Int}\overrightarrow{F}$ for all $C >> 0$. That is, for large $C$, the defining function $\widetilde{\rho}$ is strictly $\overrightarrow{F}$-plurisubharmonic at each boundary point. This proves that we may assume the defining function $\rho$ is strict of type $\overrightarrow{F}$ in a neighborhood of $\partial\Omega$. Choose $r > 0$ so that the set $\{-r < \rho < r\}$ is contained in this neighborhood of $\partial\Omega$ where $\rho$ is strict. Choose $\delta > 0$ small enough so that $-r + \delta|x|^2 < \rho$ in a neighborhood $U$ of $\partial\Omega$. We extend $\rho$ to $\overline{\Omega}$ by setting

$$\widehat{\rho} \ \equiv \ \max\{\rho, -r + \delta|x|^2\}.$$

On the open set $\Omega_{-r} = \{\rho < -r\}$ we have $\widehat{\rho} = -r + \delta|x|^2$, while on the neighborhood $U$ of $\partial\Omega$, $\widehat{\rho} = \rho$. Therefore, by the Maximum Property (4) of Section 4, $\rho$ is strict of type $\overrightarrow{F}$ on $\overline{\Omega}$.

To complete the proof we smooth the maximum $\widehat{\rho} \equiv M(u_1, u_2) \equiv \max\{u_1, u_2\}$ of $u_1 = \rho$ and $u_2 = -r + \delta|x|^2$, without changing $M(u_1, u_2)$ on the set where $|u_1 - u_2| \geq \epsilon$. Then choosing $\epsilon > 0$ small enough, the smoothing $\widehat{\rho}_\epsilon$ will equal $\widehat{\rho}$ in a neighborhood of $\partial\Omega$.

Let $M_\epsilon(t_1, t_2)$ denote the smoothing of $M(t_1, t_2) = \max\{t_1, t_2\}$ on $\mathbf{R}^2$ (see [HL$_2$, Remark 1.6] for more details). This can be done so that:

(1) $M_\epsilon(t_1, t_2) = M(t_1, t_2) \quad \text{if } |t_1 - t_2| \geq \epsilon.$

(2) $\frac{\partial M_\epsilon}{\partial t_1} + \frac{\partial M_\epsilon}{\partial t_2} \ = \ 1, \ \frac{\partial M_\epsilon}{\partial t_1} \ \geq \ 0, \ \frac{\partial M_\epsilon}{\partial t_2} \ \geq \ 0.$

(3) $M_\epsilon(t_1, t_2)$ converges uniformly to $M(t_1, t_2)$ as $\epsilon \to 0$.

It remains to show that $\widehat{\rho}_\epsilon = M_\epsilon(u_1, u_2)$ is strict of type $\overrightarrow{F}$ at each point $x \in \overline{\Omega}$. By the chain rule

$$\text{Hess} M_\epsilon(u_1, u_2) \ = \ \frac{\partial M_\epsilon}{\partial t_1} \text{Hess}\, u_1 + \frac{\partial M_\epsilon}{\partial t_2} \text{Hess}\, u_2 + \sum_{i,j=1}^{2} \frac{\partial^2 M_\epsilon}{\partial t_i \partial t_j} \nabla u_i \circ \nabla u_j.$$

One can show that the third term is $\geq 0$. Hence, by (2) above it suffices to show that

$$A_s \ = \ s\text{Hess}_x \rho + (1 - s) 2\delta I \ \in \ \text{Int} \overrightarrow{F}.$$

At all points in the neighborhood of $\partial\Omega$ where $s = \frac{\partial M}{\partial t_1} \neq 0$, we have $\text{Hess}_x \rho \in \text{Int} \overrightarrow{F}$ and hence, $A_s \in \text{Int} \overrightarrow{F}$. At points $x$ where $s = 0$, $\delta I \in \text{Int} \overrightarrow{F}$. ∎

We conclude this section by listing some of the properties of Dirichlet-Ray sets and their corresponding plurisubharmonic functions.

**Elementary Properties of Dirichlet-Ray Sets $F$:**

(1) $\{t \in \mathbf{R} : tI \in F\} = [0, \infty)$.

(2) $0 \in \partial F$ $\qquad$ (2)$'$ $0 \in \partial \widetilde{F}$

(3) $\mathcal{P} \subset F$ $\qquad$ (3)$'$ $\widetilde{F} \subset \widetilde{\mathcal{P}}$

(4) $A \in \text{Int} F \quad \iff \quad tA \in \text{Int} F$ for all $t \geq 0$.

(5) $F$ is a D-ray set $\iff$ $\widetilde{F}$ is a D-ray set

(6) $F \subset \widetilde{\mathcal{P}}$ $\qquad$ (6)$'$ $\mathcal{P} \subset \widetilde{F}$

**Proofs:** Since $0 \in F$ and $F + \mathcal{P} \subset F$, we have $tI \in F$ for all $t \geq 0$. If $-tI \in F$ for some $t > 0$, then $-tI + \mathcal{P} \subset F$ for all $t > 0$ and $F = \text{Sym}^2(\mathbf{R}^n)$ contrary to hypothesis. This proves (1). (2) follows from (1). For any Dirichlet set $F$, (2) and (2)$'$ are equivalent since $\partial \widetilde{F} = -\partial F$. (3) follows from (2) because of the positivity condition. (3)$'$ is the Dirichlet dual of (3). For (4) note that if $N \subset F$ is a neighborhood of $A \in \text{Int} F$ and $t > 0$, then $tA \in \text{Int} F$ since $tN$ is a neighborhood of $tA$. (5) follows from (4). (6) follows from (3)$'$ and (5). (6)$'$ is the Dirichlet dual of (6). ∎

**Properties of the Class $F(X)$ for Dirichlet-Ray Sets $F$:**

(8) Affine functions $u$ satisfy $\text{Hess}_x u \in \partial F$.

(9) Convex functions are $F$-plurisubharmonic.

(10) $F$-plurisubharmonic are subaffine.

(11) $F$-plurisubharmonic satisfy the maximum principle.

**Proofs:** (2) $\Rightarrow$ (8), (3) $\Rightarrow$ (9), (6) $\Rightarrow$ (10). and finally, (8) and (10) $\Rightarrow$ (11). ∎

## 6. The Dirichlet Problem.

In this section we state the main result, the existence and uniqueness of solutions of the Dirichlet problem. We then discuss how uniqueness follows from a local result – the Subaffine Theorem, whose proof is postponed to Sections 7 and 8. We conclude the section with the proof of existence.

Given a Dirichlet set $F$ note that $\partial F = F \cap (\sim \mathrm{Int} F) = F \cap (-\widetilde{F})$, i.e., $A \in \partial F$ if and only if $A \in F$ and $-A \in \widetilde{F}$. Also note that $\partial \widetilde{F} = -\partial F$.

**Definition 6.1.** A function $u$ with both

$$u \in F(X) \qquad \text{and} \qquad -u \in \widetilde{F}(X)$$

will be called an *$F$-Dirichlet function on $X$* or an *$F$-Dirichlet solution on $X$*.

In particular, a $C^2$-function $u$ is $F$-Dirichlet if and only if

$$\mathrm{Hess}_x u \in \partial F \qquad \text{for all } \ x \in X.$$

**THEOREM 6.2. (The Dirichlet Problem).** *Let $\Omega$ be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, and let $F$ be a Dirichlet set. Suppose that $\partial\Omega$ is both $\overrightarrow{F}$ and $\overrightarrow{\widetilde{F}}$ strictly convex. Then for each $\varphi \in C(\partial\Omega)$, there exists a unique $u \in C(\overline{\Omega})$ which is an $F$-Dirichlet function on $\Omega$ and equals $\varphi$ on $\partial\Omega$.*

**Remark.** In most interesting cases either $F \subset \widetilde{F}$ or $\widetilde{F} \subset F$, and then only one boundary hypothesis is required.

### Uniqueness and the Subaffine Theorem.

No boundary regularity is required for uniqueness, so we replace $\overline{\Omega}$ by an arbitrary compact subset $K \subset \mathbf{R}^n$

**THEOREM 6.3. (Uniqueness).** *Suppose that $F$ is a Dirichlet set. If $u, v \in C(K)$ are both $F$-Dirichlet on* Int$K$ *and $u = v$ on $\partial K$, then $u = v$ on $K$.*

This uniqueness theorem follows immediately from the next result.

**THEOREM 6.4 (The Comparison Principle).** *Suppose $F$ is a Dirichlet set and that $u, -v \in \mathrm{USC}(K)$. If*

$$u \in F(\mathrm{Int}K) \qquad \text{and} \qquad -v \in \widetilde{F}(\mathrm{Int}K),$$

*then*

$$u \ \le\ v \quad \text{on} \ \ \partial K \quad \Rightarrow \quad u \ \le\ v \quad \text{on} \ \ K$$

**Proof.** Because of the Maximum Principle (Proposition 2.3) for subaffine functions, the Comparison Principle is an immediate consequence of the next, purely local result. ∎

**THEOREM 6.5. (The Subaffine Theorem).** *Assume that $F$ is a Dirichlet set. If $u \in F(X)$ and $v \in \widetilde{F}(X)$, then $u + v \in \mathrm{SA}(X)$.*

The proof of this result is given in Sections 7 and 8.

Proof of Existence (The Perron Solution). Let

$$\mathcal{F}(\varphi) \equiv \left\{v \in \mathrm{USC}(\overline{\Omega}) : v\big|_{\Omega} \in F(\Omega) \text{ and } v\big|_{\partial\Omega} \leq \varphi\right\}$$

denote the Perron family for the boundary function $\varphi \in C(\partial\Omega)$. While this family $\mathcal{F}(\varphi)$ may not necessarily satisfy the maximum principle, there is a translate $\mathcal{F}(\varphi) + \lambda|x|^2$ of the family which does.

**Lemma 6.6.** *Suppose $F$ is a Dirichlet set. Then there exists $\lambda > 0$ with*

$$F + \lambda I \subset \widetilde{\mathcal{P}}.$$

*Hence, the maximum principle applies to $u + \lambda\frac{1}{2}|x|^2$ for all $u \in F(X)$.*

**Proof.** Property (6) of Section 4 says that $\widetilde{F}$ is also a Dirichlet set. Applying Property (4) of Section 3 to $\widetilde{F}$, pick $\lambda I \in \widetilde{F}$. This implies that $\lambda I + \mathcal{P} \subset \widetilde{F}$. Taking duals via Property (5) of Section 4 yields $F \subset \widetilde{\mathcal{P}} - \lambda I$. ∎

Lemma 6.6 implies that the family $\mathcal{F}(\varphi)$ is bounded above on $\overline{\Omega}$. Let

$$u(x) \equiv \sup_{v \in \mathcal{F}(\varphi)} v(x)$$

denote the upper envelope of $\mathcal{F}(\varphi)$.

**PROPOSITION 6.7.** The function $u$ belongs to $\mathcal{F}(\varphi)$, that is,

$$u \in \mathrm{USC}(X), \quad u\big|_{\Omega} \in F(\Omega) \quad \textit{and} \quad u\big|_{\partial\Omega} \leq \varphi$$

**Proof.** By Property (6) in Section 4, $u$ has upper semi-continuous regularization $u^*$ which satisfies

$$u^*\big|_{\Omega} \in F(\Omega) \tag{6.1}$$

**Lemma 6.8.** *If $\Omega$ has a strictly $\overrightarrow{F}$-convex boundary, then*

$$u^*\big|_{\partial\Omega} \leq \varphi \tag{6.2}$$

Now (6.1) and (6.2) imply that $u^* \in \mathcal{F}(\varphi)$. Therefore, $u^* \leq u$, which is the same as

$$u^* = u \quad \text{on } \overline{\Omega} \tag{6.3}$$

This completes the proof of Proposition 6.7 once Lemma 6.8 is established. ∎

**Proof of Lemma 6.8.** By Theorem 5.12 applied to $\widetilde{F}$, there exists a global defining function $\rho$ which is strictly $\overrightarrow{\widetilde{F}}$-convex on $\overline{\Omega}$. Pick $x_0 \in \partial\Omega$. It follows from (5.4) and the affine property (2) in §4 that there exist $\epsilon > 0$ and $R > 0$ so that $C(\rho - \epsilon|x - x_0|^2) \in \widetilde{F}(\overline{\Omega})$ if $C \geq R$. Given $\delta > 0$, pick $C >> 0$ so that

$$\text{on } \partial\Omega: \qquad \varphi + C(\rho - \epsilon|x - x_0|^2) \;=\; \varphi - C\epsilon|x - x_0|^2 \;\leq \varphi(x_0) + \delta. \tag{6.4}$$

Then for each $v \in \mathcal{F}(\varphi)$

$$w \;\equiv\; v + C(\rho - \epsilon|x - x_0|^2) \;\in\; \mathrm{SA}(\Omega) \cap \mathrm{USC}(\overline{\Omega}).$$

By the Maximum Principle we have

$$\sup_{\overline{\Omega}} w \;=\; \sup_{\partial\Omega} w.$$

Now $\sup_{\partial\Omega} w \leq \varphi(x_0) + \delta$ since

$$w\big|_{\partial\Omega} \;=\; v\big|_{\partial\Omega} + C(\rho - \epsilon|x - x_0|^2) \;\leq \varphi - C\epsilon|x - x_0|^2 \;\leq \varphi(x_0) + \delta.$$

This proves that for all $v \in \mathcal{F}(\varphi)$

$$w(x) \;=\; v(x) + C(\rho - \epsilon|x - x_0|^2) \;\leq \varphi(x_0) + \delta \qquad \text{for all } x \in \overline{\Omega}.$$

Hence, the upper envelope $u$ satisfies

$$u(x) + C(\rho - \epsilon|x - x_0|^2) \;\leq \varphi(x_0) + \delta \qquad \text{for all } x \in \overline{\Omega}.$$

Therefore $u^*$ also satisfies

$$u^*(x) + C(\rho - \epsilon|x - x_0|^2) \;\leq \varphi(x_0) + \delta \qquad \text{for all } x \in \overline{\Omega}.$$

Evaluating at $x = x_0$ yields

$$u^*(x_0) \;\leq \varphi(x_0) + \delta.$$

■

**Lemma 6.9.** *If $\Omega$ has a strictly $\overrightarrow{F}$-convex boundary, then*

$$\liminf_{x \to x_0} u(x) \;\geq\; \varphi(x_0) \qquad \textit{for all } \; x_0 \in \partial\Omega.$$

**Proof.** By Theorem 5.12 there exists $\epsilon > 0$ so that for $C >> 0$ the function $C(\rho - \epsilon|x - x_0|^2)$ is of type $F$. Given $\delta > 0$, pick $C >> 0$ so that (cf. (6.4))

$$\text{on } \partial\Omega: \qquad \varphi(x_0) + C(\rho - \epsilon|x - x_0|^2) \;\leq \varphi(x) + \delta. \tag{6.4$'$}$$

Set

$$v(x) = \varphi(x_0) - \delta + C(\rho - \epsilon|x - x_0|^2) \quad \text{on } \; \overline{\Omega}.$$

Then $v \in \mathcal{F}(\varphi)$. Consequently, $v \leq u$ on $\overline{\Omega}$, and so

$$\liminf_{x \to x_0} u(x) \;\geq\; \lim_{x \to x_0} v(x) \;=\; \varphi(x_0) - \epsilon.$$

■

**COROLLARY 6.10.** *If $\partial\Omega$ is both strictly $\overrightarrow{F}$-convex and strictly $\overrightarrow{\widetilde{F}}$-convex, then the function $u$ is continuous at each point of $\partial\Omega$ and $u\big|_{\partial\Omega} = \varphi$.*

We now apply an argument of Walsh [W] to prove interior continuity.

**PROPOSITION 6.11.** $u \in C(\overline{\Omega})$.

**Proof.** Let $\Omega_\delta \equiv \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$ and let $C_\delta \equiv \{x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) < \delta\}$. Suppose $\epsilon > 0$ is given. By the continuity of $u$ at points of $\partial\Omega$ and the compactness of $\partial\Omega$, it follows easily that there exists a $\delta > 0$ such that:

$$\text{If } \;|y| \;\leq\; \delta, \;\text{ then }\; u_y \;<\; u + \epsilon \quad \text{on } C_{2\delta}. \tag{6.5}$$

where $u_y(x) \equiv u(x+y)$ is the $y$-translate of $u$ and where we define $u$ to be $-\infty$ on $\mathbf{R}^n - \overline{\Omega}$. We claim that:

$$\text{If } \;|y| \;\leq\; \delta, \;\text{ then }\; u_y \;\leq\; u + \epsilon \quad \text{on } \overline{\Omega}. \tag{6.6}$$

This implies, by a change of variables, that $u \leq u_y + \epsilon$ also holds, i.e., that:

$$\text{If } \;|y| \;\leq\; \delta, \;\text{ then }\; |u_y - u| \leq \epsilon \quad \text{on } \overline{\Omega},$$

which completes the proof once (6.6) is established.

To establish (6.6) note first that $u_y \in F(\Omega_\delta)$ for each $|y| < \delta$ by Property (3) in Section 4. Since $u_y < u + \epsilon$ on the collar $C_{2\delta}$, one has

$$g_y \;\equiv\; \max\{u_y, u + \epsilon\} \;\in\; F(\Omega)$$

by Property (4) in Section 4. Hence, $g_y - \epsilon \in F(\Omega)$ by Property (2) in Section 4. Now (6.5) implies that $g_y - \epsilon = u$ on $C_{2\delta}$. Therefore,

$$g_y - \epsilon \;\in\; \mathcal{F}(\varphi)$$

and hence $g_y - \epsilon \leq u$ on $\overline{\Omega}$. This proves

$$u_y \;\leq\; g_y \;\leq\; u + \epsilon \quad \text{on } \;\Omega_\delta.$$

Combined with (6.5), this proves (6.6). ■

This proves that $u \in C(\overline{\Omega})$, $u\big|_{\Omega} \in F(\Omega)$ and $u\big|_{\partial\Omega} = \varphi$. To complete the proof of existence for the (DP) we show

**Lemma 6.12.**

$$-u\big|_{\Omega} \in \widetilde{F}(\Omega)$$

**Proof.** If $-u \notin \widetilde{F}(\Omega)$, then since $\widetilde{\widetilde{F}} = F$, Lemma 4.6 implies that there exist $x_0 \in \Omega$, $a$ affine, $\epsilon > 0$ and $A \in \mathrm{Int}F$ such that

$$\begin{aligned} A - u - a \ &\leq \ -\epsilon|x - x_0|^2 \quad \text{near } x_0 \quad \text{and} \\ &= \ 0 \qquad\qquad\qquad \text{at } x_0. \end{aligned}$$

Now $v = A - a + \epsilon|x - x_0|^2$ is of type $F$. Furthermore, $v < u$ on $\partial B_r(x_0)$ for a small $r > 0$ but $v(x_0) = u(x_0)$. Set $v' = v + \delta$ with $\delta > 0$ small so that $v' < u$ remains true on $\partial B_r(x_0)$ but $u(x_0) < v'(x_0)$. Then

$$w \ = \ \begin{cases} u & \text{on } \overline{\Omega} - B_r(x_0) \\ \max\{u, v'\} & \text{on } \overline{B_r(x_0)} \end{cases}$$

defines a function $w \in \mathcal{F}(\varphi)$, the Perron family for the boundary function $\varphi$. This is because the upper semicontinuity of $v' - u$ (by Proposition 4.8) implies that $\{v' < u\} = \{v' - u < 0\}$ is an open neighborhood of $\partial B_r(x_0)$. However, $w(x_0) = v'(x_0) > u(x_0)$, contradicting the definition of $u$ as the upper envelope of $\mathcal{F}(\varphi)$. ∎

## 7. Quasiconvex Functions.

In some sense the nicest class of $F$-plurisubharmonic functions is the one where $F = \mathcal{P}$. If $X$ is connected, then

$$v \ \in \ \mathcal{P}(X) \qquad \Longleftrightarrow \qquad v \in \text{Convex}(X) \ \text{ or } \ v \equiv -\infty. \tag{7.1}$$

(See Proposition 2.5 and its restatement as Proposition 4.5.) The important Property (6) in Section 4, for "families locally bounded above", can be strengthened in this case.

$$\begin{gathered}\text{If } \mathcal{F} \text{ is a family of convex functions which is locally bounded above,}\\ \text{then the upper envelope } v = \sup_{f\in\mathcal{F}} f \text{ is also convex.}\end{gathered} \tag{7.2}$$

The point is that in this case there is no need to regularize from $v$ to $v^*$.

Another useful improvement is that:

$$\begin{gathered}\text{If } \{v_j\} \text{ is a sequence of functions in } \mathcal{P}(X) \ \text{ which converges}\\ \text{pointwise to a function } v, \text{ then } v \in \mathcal{P}(X).\end{gathered} \tag{7.3}$$

These properties are easily established. Another important property of convex functions is due to Alexandrov.

$$\text{If } u \text{ is a convex function, then } u \text{ is twice differentiable a.e.} \tag{7.4}$$

All these properties carry over directly to the quasi-convex case. Their analogues, which are listed below, will be used to prove the Subaffine Theorem 6.5.

**Definition 7.1.** A function $u$ on $X$ is *$\lambda$-quasiconvex* if $v = u + \lambda\frac{1}{2}|x|^2$ is convex.

Set $\mathcal{P}_\lambda \equiv \mathcal{P} - \lambda I$. Then for $X$ connected, we have:

$$u \ \in \ \mathcal{P}_\lambda(X) \qquad \Longleftrightarrow \qquad u \ \text{ is } \lambda\text{−quasiconvex or } \ u \equiv -\infty. \tag{7.1$'$}$$

$$\begin{gathered}\text{If } \mathcal{F} \text{ is a family of } \lambda\text{−quasiconvex functions which is locally bounded above,}\\ \text{then the upper envelope } u = \sup_{f\in\mathcal{F}} f \text{ is also } \lambda\text{−quasiconvex.}\end{gathered} \tag{7.2$'$}$$

Since $\sup_{f\in\mathcal{F}}(f+\lambda\frac{1}{2}|x|^2) = (\sup_{f\in\mathcal{F}} f)+\lambda\frac{1}{2}|x|^2$, the extension of property (7.3) is obvious.

$$\begin{gathered}\text{If } \{u_j\} \text{ is a sequence of functions in } \mathcal{P}_\lambda(X) \ \text{ which converges}\\ \text{pointwise to a function } u, \text{ then } u \in \mathcal{P}_\lambda(X).\end{gathered} \tag{7.3$'$}$$

The extension of Alexandrov's Theorem is also obvious.

$$\text{If } u \text{ is a locally quasiconvex function, then } u \text{ is twice differentiable a.e.} \tag{7.4$'$}$$

Note that if $\varphi$ is smooth, then in any relatively compact subdomain there exists $\lambda > 0$ such that $\varphi$ is $\lambda$-quasiconvex on the subdomain.

We will also need a final property of quasiconvex functions – *Differentiability at Maximum Points*. Since this property is essentially vacuous for purely convex (non-constant) functions, we include a proof.

**(DMP):** Suppose $u$ is quasiconvex and that $x$ is a local maximum point of $u$. Then $u$ is differentiable at $x$ and $(\nabla u)(x) = 0$.

**Proof.** We may assume that the maximum point is the origin and the maximum value is zero. Then $v(x) \equiv u(x) + \lambda\frac{1}{2}|x|^2 \leq \lambda\frac{1}{2}|x|^2$ near $x = 0$ and $v(0) = 0$. Therefore, by convexity of $v$, $0 = 2v(0) \leq v(x) + v(-x) \leq v(x) + \lambda\frac{1}{2}|x|^2$. Thus,

$$-\lambda\tfrac{1}{2}|x|^2 \;\leq\; v(x) \;\leq\; \lambda\tfrac{1}{2}|x|^2,$$

which proves that $v$ is differentiable at the origin and that $(\nabla v)(0) = 0$. Therefore the same conclusion holds for $u$. ∎

Suppose now that $v$ is a convex function. If $v$ is differentiable at a point $x$, then define

$$K(v,x) \;\equiv\; \overline{\lim_{\epsilon\to 0}}\; 2\epsilon^{-2} \sup_{|y|=1} \left\{ v(x+\epsilon y) - v(x) - \epsilon\nabla v(x)\cdot y \right\}. \tag{7.5}$$

Otherwise define $K(v,x) = \infty$. If $v$ is twice differentiable at $x$, then $K(v,x)$ is the largest eigenvalue of $\mathrm{Hess}_x v$.

The next result is a key to our development.

**THEOREM 7.2. (Slodkowski [S]).** *Suppose $v$ is a convex function on $X$. If $K(v,x) \geq \Lambda$ a.e., then $K(v,x) \geq \Lambda$ everywhere.*

This theorem provides a nice test for when a quasi-convex function is subaffine. Recall by Proposition 3.7 that $\widetilde{\mathcal{P}}(X) = \mathrm{SA}(X)$ is the space of subaffine functions.

**THEOREM 7.3.** *Suppose $u$ is locally quasiconvex on $X$. Then*

$$\mathrm{Hess}\, u \in \widetilde{\mathcal{P}} \text{ a.e.} \quad \Rightarrow \quad u \in \widetilde{\mathcal{P}}(X).$$

**Proof.** Set $v(x) \equiv u(x) + \Lambda\frac{1}{2}|x - x_0|^2$. At a point $x$ where $u$ is twice differentiable,

$$\mathrm{Hess}_x u \in \widetilde{\mathcal{P}} \quad \iff \quad \mathrm{Hess}_x v \in (\widetilde{\mathcal{P}} + \Lambda\cdot I) \quad \iff \quad K(v,x) \geq \Lambda. \tag{7.6}$$

Thus the hypothesis $\mathrm{Hess}_x u \in \widetilde{\mathcal{P}}$ a.e. is equivalent to

$$K(v,x) \;\geq\; \Lambda \;\; \text{a.e. on } X. \tag{7.7}$$

By Slodkowski's Largest Eigenvalue Theorem 7.2 this is equivalent to

$$K(v,x) \;\geq\; \Lambda \;\; \text{everywhere on } X. \tag{7.8}$$

We now suppose that $u \notin \widetilde{\mathcal{P}}(X)$ and derive a contradiction. By Lemma 2.2 there exists $x_0 \in X$, $a$ affine and $\epsilon > 0$ such that

$$\begin{aligned} u(x) - a(x) &\leq -\epsilon \tfrac{1}{2}|x - x_0|^2 \qquad \text{near } x_0, \quad \text{and} \\ &= 0 \qquad\qquad\qquad\quad \text{at} \quad x_0 \end{aligned} \tag{7.9}$$

Pick $\Lambda$ so that $v(x) \equiv u(x) + \Lambda \frac{1}{2}|x - x_0|^2$ is convex near $x_0$. By (7.9) the (DMP) implies that $u$, and hence $v$, is differentiable at $x_0$. Thus $K(v, x_0)$ is defined by (7.5). Since

$$\begin{aligned} v(x) - a(x) &\leq (\Lambda - \epsilon) \tfrac{1}{2}|x - x_0|^2 \qquad \text{near } x_0, \quad \text{and} \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{at} \quad x_0 \end{aligned}$$

it follows that $K(v, x_0) \leq \Lambda - \epsilon$, a contradiction. ■

**REMARK.** Suppose $v = u + \Lambda \frac{1}{2}|x|^2$ is convex. Theorem 7.3 states that:

*If $K(v, x) \geq \Lambda$ a.e., then $v - \Lambda \frac{1}{2}|x|^2$ is subaffine*
*(or equivalently that $v$ is of type $\widetilde{\mathcal{P}} + \Lambda \cdot I$).*

**COROLLARY 7.4. (The Subaffine Theorem for Quasiconvex Functions).** *Suppose $F$ is a Dirichlet set. If $u$ and $v$ are $\lambda$-quasiconvex with $u \in F(X)$ and $v \in \widetilde{F}(X)$, then $u + v \in \mathrm{SA}(X)$.*

**Proof.** By Alexandrov's Theorem, $u, v$ and $u + v$ are twice differentiable a.e., and the a.e. hessians satisfy

$$\mathrm{Hess}_x(u + v) = \mathrm{Hess}_x u + \mathrm{Hess}_x v.$$

By Property (7) in Section 4, $\mathrm{Hess}_x u \in F$ a.e. and $\mathrm{Hess}_x v \in \widetilde{F}$ a.e.. Therefore,

$$\mathrm{Hess}_x(u + v) \in F + \widetilde{F} = \widetilde{\mathcal{P}} \quad \text{a.e..} \tag{7.10}$$

Since $u + v$ is $2\lambda$-quasiconvex, Theorem 7.3 implies that $u + v \in \widetilde{\mathcal{P}}(X)$, i.e., $u + v$ is subaffine. ■

Theorem 7.3 extends from $\widetilde{\mathcal{P}}$ to an arbitrary Dirichlet set $F$.

**COROLLARY 7.5.** *Suppose $u$ is locally quasi-convex on $X$. Then*

$$\mathrm{Hess}\, u \in F \quad a.e. \quad \Rightarrow \quad u \in F(X)$$

**Proof.** Suppose $\mathrm{Hess}_x u \in F$ a.e. Given $B \in \widetilde{F}$, $\mathrm{Hess}_x(u + B) = \mathrm{Hess}_x u + B \in F + \widetilde{F} \subset \widetilde{\mathcal{P}}$ a.e. By Theorem 7.3 this implies that $u + B \in \mathrm{SA}(X)$, and hence by Definition 4.4 that $u \in F(X)$. ■

Note that the converse is true as well, that is, $u \in F(X) \Rightarrow \mathrm{Hess}_x u \in F$ a.e.. In fact, if $u \in F(X)$ and the second derivatives of $u$ exist at $x$, then, as in the proof of Property (7) in Section 4 (the case where $u$ is $C^2$ at $x$), it follows that $\mathrm{Hess}_x u \in F$.

## 8. Sup-Convolution Approximation.

Suppose that $X$ is an open subset of $\mathbf{R}^n$.

**Definition 8.1. (Sup-Convolution).** Suppose that $u$ is a bounded function on $X$. For each $\epsilon > 0$, define

$$u^\epsilon(x) \;=\; \sup_{y\in X}\left\{u(y) - \frac{1}{\epsilon}|x-y|^2\right\} \qquad \forall x \in X. \tag{8.1}$$

Note that $u \le u^\epsilon$ on $X$. Set $\delta \equiv \sqrt{\epsilon 2N}$ where $|u| \le N$ on $X$, and define $X_\delta = \{x \in X : \mathrm{dist}(x, \partial X) > \delta\}$. The following equivalent formulas for $u^\epsilon$ are useful.

$$u^\epsilon(x) \;=\; \sup_{|x-y|\le\delta}\left\{u(y) - \frac{1}{\epsilon}|x-y|^2\right\} \qquad \forall x \in X_\delta. \tag{8.2}$$

**Proof.** If $x, y \in X$ and $|x-y| > \delta$, then $u(y) - u(x) - \frac{1}{\epsilon}|x-y|^2 \le 2N - \frac{\delta^2}{\epsilon} = 0$. Therefore, $u(y) - \frac{1}{\epsilon}|x-y|^2 \le u(x)$ if $|x-y| > \delta$. Since $u(x) \le u^\epsilon(x)$, this proves that

$$\sup_{|x-y|>\delta,\ y\in X}\left\{u(y) - \frac{1}{\epsilon}|x-y|^2\right\} \;\le\; u^\epsilon(x) \quad \text{if } x \in X$$

which gives (8.2). ■

Making the change of variables $z = x - y$ in (8.2) yields:

$$u^\epsilon(x) \;=\; \sup_{|z|\le\delta}\left\{u(x-z) - \frac{1}{\epsilon}|z|^2\right\} \qquad \forall x \in X_\delta. \tag{8.3}$$

**THEOREM 8.2. (Approximation).** *Suppose $u \in F(X)$ with $|u(x)| \le N$ on $X$. Given $\epsilon > 0$, define $\delta = \sqrt{2\epsilon N}$. Then*

*1) $u^\epsilon$ decreases to $u$ as $\epsilon \to 0$.*

*2) $u^\epsilon$ is $\frac{1}{\epsilon}$-quasiconvex.*

*3) $u^\epsilon \in F(X_\delta)$.*

**Proof.** For 1) note that $\epsilon_1 < \epsilon_2 \iff -\frac{1}{\epsilon_1} < -\frac{1}{\epsilon_2}$. Now any of (8.1), (8.2) or (8.3) imply that $u^\epsilon$ is monotone decreasing as $\epsilon \to 0$. By (8.2)

$$u^\epsilon(x) \;\le\; \sup_{|x-y|\le\delta} u(y) \quad \forall\, x \in X_\delta.$$

As noted above, $u \le u^\epsilon$. Hence,

$$u(x) \;\le\; u^\epsilon(x) \;\le\; \sup_{|x-y|\le\delta} u(y).$$

Since $u \in \mathrm{USC}(X)$, the functions $\sup_{|x-y|\leq\delta} u(y)$ decrease to $u(x)$. This proves 1).

To prove 2) we first note that for $y \in X$ fixed, the function $u(y) - \frac{1}{\epsilon}|x-y|^2 + \frac{1}{\epsilon}|x|^2$ is affine and hence convex. That is, $u(y) - \frac{1}{\epsilon}|x-y|^2$ is a $\frac{1}{\epsilon}$-quasiconvex function of $x$. Now applying $(7.2)'$ to (8.1) (Note that (8.2) does not work here) proves that $u^\epsilon$ is $\frac{1}{\epsilon}$-quasiconvex.

To prove 3) we make use of (8.3). Each function $u_z(x) = u(x-z) \in F(X_\delta)$ if $|z| \leq \delta$ by the translation property (3) in Section 4. Therefore, by the "families locally bounded above" property (6) in Section 4, the upper envelope $u^\epsilon$ of the family

$$\mathcal{F} \;=\; \left\{u(x-z) - \tfrac{1}{\epsilon}|z|^2 : |z| \leq \delta\right\}$$

has upper-semicontinuous regularization in $F(X_\delta)$. However, $u^\epsilon$ is continuous since it is quasiconvex. Hence $u^\epsilon$ equals its u.s.c. regularization. ■

The Subaffine Theorem 6.5 follows easily from the quasi-convex case (Corollary 7.4) because of the Approximation Theorem 8.2.

**Proof of The Subaffine Theorem 6.5.** The result is local, so by upper semicontinuity we may assume $u$ and $v$ are bounded above. We may also assume they are bounded below by replacing them with $u_m = \max\{u, -m\}$ and $u_m = \max\{u, -m\}$, and then taking the decreasing limit of $u_m + v_m$ as $m \to \infty$.

We now apply Theorem 8.2 to $u$ and $v$ to obtain sequences $\{u_j\}$ and $\{v_j\}$ which are quasi-convex for each $j$ and converge monotonically downward to $u$ and $v$ respectively as $j \to \infty$. By Corollary 7.4, the sum $u_j + v_j \in \mathrm{SA}(X)$ for all $j$. Since $u_j + v_j$ decreases to $u+v$, Property (5) in Section 4, applied to the subaffine case, implies that $u+v \in \mathrm{SA}(X)$. ■

## 9. Topological Restrictions on Domains with Strictly $\overrightarrow{F}$-Convex Boundaries.

In this section we show that the strict $\overrightarrow{F}$-convexity of $\partial\Omega$, which was assumed in our Main Theorem 6.2, often places strong restrictions on the topology of $\Omega$. A typical example is that of a domain in $\mathbf{C}^n$ with pseudoconvex boundary (a Stein domain) which has the homotopy-type of a complex of dimension $\leq n$. Theorem 9.5 greatly generalizes this fact. For its statement we need to introduce the following ideas.

Suppose $\mathbf{R}^n = N \oplus W$ is an orthogonal decomposition of $\mathbf{R}^n$. Let $\pi_W : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathrm{Sym}^2(W)$ denote restriction of quadratic functions.

**Definition 9.1.** Suppose that $F$ is a Dirichlet-Ray set, and that $\mathbf{R}^n = N \oplus W$.

(1) $W$ is *$F$-free* if $\pi_W(F) = \mathrm{Sym}^2(W)$.

(2) $W$ is *$F$-Morse* if there exists $A \in F$ with $\pi_W(A) < 0$.

(3) $N$ is *$F$-strict* if $P_N \in \mathrm{Int}F$.

**PROPOSITION 9.2.** *Suppose $F$ is a Dirichlet-Ray set, and that $\mathbf{R}^n = N \oplus W$. Then the following conditions are equivalent:*

(1) $W$ is *$F$-free,*

(2) $W$ is *$F$-Morse,*

(3) $N$ is *$F$-strict*

The proof is given at the end of this section.

**Definition 9.3.** The *free dimension* of a Dirichlet ray set $F$, denoted by free-dim$(F)$, is the maximal dimension of an $F$-free subspace of $\mathbf{R}^n$. By Proposition 9.2, free-dim$(F)$ is also the maximal dimension of an $F$-Morse subspace of $\mathbf{R}^n$.

**Example 9.4.** Suppose $F = \mathcal{P}(G)$ is defined by a closed subset $G \subset G(p, \mathbf{R}^n)$ of the Grassmannian of $p$ planes, as in (10.10) below. Then a subspace $W \subset \mathbf{R}^n$ is $\mathcal{P}(G)$-free if and only if it contains no $G$-planes, i.e.,

$$\nexists\, \xi \in G \quad \text{with} \quad \xi \subset W$$

(For the proof see [$\mathrm{HL}_3$].) This enables one to easily calculate the free dimension in all the standard calibrated geometries. For example, when $G \subset G(2, \mathbf{R}^{2n})$ is the Grassmannian of complex lines in $\mathbf{C}^n = \mathbf{R}^{2n}$, the free dimension is $n$. This is the Stein case. In associative geometry, the free dimension is 3, and in coassociative geometry it is 4. When $G$ is the space of Lagrangian $n$-planes in $\mathbf{C}^n$, the free dimension is $2n-2$.

The following theorem is the main result of this section. It represents a surprising extension of the Andreotti-Frankel Theorem in complex analysis to this general context.

**THEOREM 9.5.** *Let $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ be a Dirichlet-Ray set with free-dim$(F) = D$. Suppose $\Omega \subset\subset \mathbf{R}^n$ is a domain with a smooth, strictly $F$-convex boundary. Then $\Omega$ has the homotopy-type of a CW-complex of dimension $\leq D$.*

**COROLLARY 9.6.** *Let $\Omega \subset\subset X$ be a domain with a smooth, strictly $F$-convex boundary, and let $D$ be the free dimension of $F$. Then*

$$H_k(\partial\Omega; \mathbf{Z}) \;\cong\; H_k(\Omega; \mathbf{Z}) \qquad \text{for all } k < n - D - 1$$

*and the map $H_{n-D-1}(\partial\Omega; \mathbf{Z}) \to H_{n-D-1}(\Omega; \mathbf{Z})$ is surjective.*

**Proof of Theorem 9.5** By Theorem 5.12 there exists a global defining function $\rho \in C^\infty(\overline{\Omega})$ for $\partial\Omega$ which is strictly $F$-plurisubharmonic on $\overline{\Omega}$. Set $u = -\log(-\rho)$ and note that $u$ is a proper proper exhaustion function for $\Omega$. Direct computation shows that

$$\mathrm{Hess}\, u \;=\; -\frac{1}{\rho}\mathrm{Hess}\,\rho + \frac{1}{\rho^2}(\nabla\rho \circ \nabla\rho).$$

Since $\mathrm{Hess}_x\rho \in \mathrm{Int}F$ and $(\nabla\rho \circ \nabla\rho)_x \in \mathcal{P}$, Property (3) shows that

$$\mathrm{Hess}_x u \in \mathrm{Int}F \tag{9.1}$$

at each $x \in \Omega$, i.e., $u$ is strictly $F$-plurisubharmonic on $\Omega$. By standard approximation theorems (cf. [MS]) we may assume that all critical points of $u$ are non-degenerate. The theorem will follow from Morse Theory if we can show that each critical point $x_0$ of $u$ in $X$ has index $\leq D$.

Suppose $x_0$ were a critical point of index $> D$. Then there would exists a linear subspace $W \subset T_{x_0}\mathbf{R}^n = \mathbf{R}^n$ of dimension $> D$ such that

$$\mathrm{Hess}_{x_0} u\big|_W \;<\; 0.$$

However, by Proposition 9.2 and (9.1) we see that $D$ is the largest dimension of a subspace $W$ for which this can hold. Hence, the index of $\mathrm{Hess}_{x_0} u \leq D$ as desired. ■

**Proof of Corollary 9.6** This follows from the exact sequence

$$H_{k+1}(\Omega, \partial\Omega; \mathbf{Z}) \;\to\; H_k(\partial\Omega; \mathbf{Z}) \;\to\; H_k(\Omega; \mathbf{Z}) \;\to\; H_k(\Omega, \partial\Omega; \mathbf{Z}),$$

Lefschetz Duality: $H_k(\Omega, \partial\Omega; \mathbf{Z}) \cong H^{n-k}(\Omega; \mathbf{Z})$, and Theorem 9.5. ■

**Proof of Proposition 9.2** Let $\rho : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathrm{Sym}^2(N)^\perp$ denote orthogonal projection onto the subspace $\mathrm{Sym}^2(N)^\perp$ of $\mathrm{Sym}^2(\mathbf{R}^n)$. Also consider the conditions:

(1)* $\quad \rho(F) \;=\; \mathrm{Sym}^2(N)^\perp$, and

(3)* $\quad \mathrm{Sym}^2(N) \cap \mathrm{Int}F \;\neq\; \emptyset$.

The implications $(1)^* \Rightarrow (1)$ and $(3)^* \Rightarrow (3)$ are trivial. We will prove that $(1) \Rightarrow (3)$, $(3)^* \Rightarrow (1)^*$ and $(1) \iff (2)$.

**Proof that** $(1) \Rightarrow (3)$**:** By (1) there exists $A \in F$ with $\pi_W(A) = -I_W$ where $I_W$ denotes the identity on $W$. It suffices to show that there exist $t > 0$ and $P > 0$ such that $P_N = tA + P$, because by the ray property $tA \in F$, and $F + \mathrm{Int}\mathcal{P} \subset \mathrm{Int}F$. In terms of the $2 \times 2$ blocking induced by $\mathbf{R}^n = N \oplus W$, we have $A = \begin{pmatrix} a & b \\ b & -I_W \end{pmatrix}$. Therefore

$$\tfrac{1}{t}P \;=\; \tfrac{1}{t}P_N - A \;=\; \begin{pmatrix} \frac{1}{t}I_N - a & -b \\ -b & I_W \end{pmatrix}.$$

For $t > 0$ sufficiently small, we have $\frac{1}{t}P > 0$ and hence $P > 0$. ■

**Exercise**. Show that $\begin{pmatrix} \frac{1}{t}I_N - a & -b \\ -b & I_W \end{pmatrix} > 0$ for all sufficiently small $t > 0$.

**Proof that** $(3)^* \Rightarrow (1)^*$**:** Suppose $A \in \mathrm{Sym}^2(N) \cap \mathrm{Int}F$, i.e., $A = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Given $B \in \mathrm{Sym}^2(N)^\perp$, i.e., $B = \begin{pmatrix} 0 & b \\ b & c \end{pmatrix}$, pick $\epsilon > o$ small enough so that $A + \epsilon B \in F$. By the ray property, $\frac{1}{\epsilon}A + B = \begin{pmatrix} \frac{1}{\epsilon}a & b \\ b & c \end{pmatrix} \in F$. Finally $\rho(\frac{1}{\epsilon}A + B) = B$. ■

**Proof that (1) $\iff$ (2) :** Note that: $W$ is not $F$-Morse $\iff$ $\pi_W(F) \subset \widetilde{\mathcal{P}}_W$ where $\mathcal{P}_W = \{A \in \mathrm{Sym}^2(W) : A \geq 0\}$ $\iff$ $\mathcal{P}_W \subset \widetilde{\pi_W(F)}$. It is easy to show that $\pi_W(F)$ satisfies the positivity condition. Moreover, $\pi_W(F) \neq \emptyset$ since $F \neq \emptyset$. Hence, $\pi_W(F)$ is either a Dirichlet set or $\pi_W(F) = \mathrm{Sym}^2(W)$. In either case $\widetilde{\pi_W(F)}$ satisfies the positivity condition. Hence,

$$\widetilde{\mathcal{P}}_W \subset \widetilde{\pi_W(F)} \quad \iff \quad 0 \in \widetilde{\pi_W(F)}.$$

By definition, $0 \in \widetilde{\pi_W(F)} \iff 0 \notin \mathrm{Int}\, \pi_W(F)$. Since $F$ satisfies the ray condition, so does $\pi_W(F)$. Therefore,

$$\pi_W(F) = \mathrm{Sym}^2(W) \quad \iff \quad 0 \in \mathrm{Int}\, \pi_W(F),$$

or equivalently,

$$0 \notin \mathrm{Int}\, \pi_W(F) \quad \iff \quad \pi_W(F) \neq \mathrm{Sym}^2(W),$$

i.e., $W$ is not $F$-free. ■

## 10. Examples of Dirichlet Sets.

Dirichlet sets $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$, to which our main existence and uniqueness theorem applies, are abundant, interesting and quite varied. They arise in quite different contexts, and we have tried to organize our presentation in that way. There are however, some organizational principles which illuminate the constructions. We shall mention these early on.

In many cases the $C^2$-solutions to the Dirichlet problem associated to $F$ satisfy an explicit nonlinear second-order differential equation. When this is so, the equations will be presented.

As mentioned in the introduction, readers are encouraged to look at examples close to their interests and bear them in mind while reading other parts of the paper.

**Three Fundamental Examples.** The most basic example of a Dirichlet set is the set

$$\mathcal{P} \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : A \geq 0\},$$

of non-negative symmetric matrices, whose Dirichlet dual is the set $\widetilde{\mathcal{P}}$ of matrices with at least one non-negative eigenvalue. These sets have analogues over $\mathbf{C}$ and $\mathbf{H}$.

Consider the three vector spaces $\mathbf{R}^n, \mathbf{C}^n$, and $\mathbf{H}^n$ with scalar field $\mathrm{K} = \mathbf{R}, \mathbf{C}$ and $\mathbf{H}$ respectively. (In the quaternionic case it is convenient to have the scalars $\mathbf{H}$ act on $\mathbf{H}^n$ from the right.) Let $G(p, \mathrm{K}^n)$ denote the grassmannian of $p$-dimensional K-planes in $\mathrm{K}^n$. For each $\xi \in G(p, \mathrm{K}^n)$ define the $\xi$-trace of $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathrm{K}^n) = \mathrm{Sym}^2(\mathbf{R}^N)$ (with $N = n, 2n$ or $4n$) by

$$\mathrm{tr}_\xi A \;=\; \mathrm{trace}\left\{ A\big|_\xi \right\} \;=\; \langle A, P_\xi \rangle \tag{10.1}$$

where $P_\xi \in \mathrm{Sym}^2(\mathbf{R}^N)$ is orthogonal projection onto $\xi$ and $\langle \cdot\,,\cdot \rangle$ is the standard inner product on $\mathrm{Sym}^2(\mathbf{R}^N)$. Define

$$\mathcal{P}(\mathbf{R}^n) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}_\xi A \geq 0 \ \forall \xi \in G(1, \mathbf{R}^n)\} \tag{10.2}$$

$$\mathcal{P}_{\mathbf{C}}(\mathbf{C}^n) \;=\; \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \mathrm{tr}_\xi A \geq 0 \ \forall \xi \in G(1, \mathbf{C}^n)\} \tag{10.3}$$

$$\mathcal{P}_{\mathbf{H}}(\mathbf{H}^n) \;=\; \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{H}^n) : \mathrm{tr}_\xi A \geq 0 \ \forall \xi \in G(1, \mathbf{H}^n)\} \tag{10.4}$$

These are the three fundamental example of Dirichlet sets. Note that they are convex cones in $\mathrm{Sym}^2(\mathbf{R}^N)$ with vertex at the origin. Their Dirichlet duals are given respectively by

$$\widetilde{\mathcal{P}}(\mathbf{R}^n) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, \xi \in G(1, \mathbf{R}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\}$$
$$\widetilde{\mathcal{P}}_{\mathbf{C}}(\mathbf{C}^n) \;=\; \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \exists\, \xi \in G(1, \mathbf{C}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\}$$
$$\widetilde{\mathcal{P}}_{\mathbf{H}}(\mathbf{H}^n) \;=\; \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{H}^n) : \exists\, \xi \in G(1, \mathbf{H}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\}$$

Given $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathrm{K}^n) = \mathrm{Sym}^2(\mathbf{R}^N)$, consider the hermitian symmetric part of $A$. In the complex case $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ this is just

$$A_{\mathbf{C}} \;=\; \tfrac{1}{2}(A - JAJ) \tag{10.5}$$

while in the quaternionic case $\mathbf{H}^n = (\mathbf{R}^{4n}, I, J, K)$ it is

$$A_{\mathbf{H}} \;=\; \tfrac{1}{4}(A - IAI - JAJ - KAK) \tag{10.6}$$

The hermitian symmetric part is K-linear with $n$ eigenvalues $\lambda_1, ..., \lambda_n$. The Dirichlet sets $\mathcal{P}(\mathbf{R}^n), \mathcal{P}_{\mathbf{C}}(\mathbf{C}^n)$ and $\mathcal{P}_{\mathbf{H}}(\mathbf{H}^n)$ can all be (equivalently) defined as

$$\mathcal{P}_{\rm K}({\rm K}^n) \;=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : A_{\rm K} \geq 0\} \;=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : \lambda_1 \geq 0, ..., \lambda_n \geq 0\} \tag{10.7}$$

**The Monge-Ampère Equation.** In all three cases there is a determinant function on ${\rm Sym}^2(\mathbf{R}^N)$:

$$\det{}_{\rm K} A \;=\; \lambda_1 \cdots \lambda_n. \tag{10.8}$$

Of course , if ${\rm K} = \mathbf{R}$, this is the real determinant of $A \in {\rm Sym}^2(\mathbf{R}^n)$, and if ${\rm K} = \mathbf{C}$, then this is the complex determinant of the hermitian symmetric part of $A \in {\rm Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. If ${\rm K} = \mathbf{H}$, then one can show that $\det_{\mathbf{H}} A$ is also a polynomial of degree $n$ (cf. [DK]). Note that in each of these cases the boundary of the Dirichlet set ($\partial\mathcal{P}$, $\partial\mathcal{P}_{\mathbf{C}}$ or $\partial\mathcal{P}_{\mathbf{H}}$) is contained in the zero locus of the determinant function ($\det_{\mathbf{R}}$, $\det_{\mathbf{C}}$, $\det_{\mathbf{H}}$). Therefore, if $u$ is a $\mathcal{P}_{\rm K}$-Dirichlet function which is $C^2$, then at each point

$$\det{}_{\rm K}({\rm Hess}\, u) \;=\; 0 \tag{10.9}$$

**The Next Tier: Other Branches of Det(Hess u) = 0.** Fix a positive integer $0 \leq q \leq n-1$ and consider the sets

$$\begin{aligned} P_q({\rm K}^n) \;&=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : \exists\, W \in G(n-q, {\rm K}^n) \text{ with } A\big|_W \in \mathcal{P}_{\rm K}(W)\} \\ &=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : A_{\rm K} \text{ has at least } n-q \text{ eigenvalues } \geq 0\} \\ &=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : \forall\, V \in G(q+1, n)\ A\big|_V \in \widetilde{\mathcal{P}}_{\rm K}(V)\}. \end{aligned}$$

It is easy to see that

$$\widetilde{P}_q({\rm K}^n) \;=\; \{A \in {\rm Sym}^2(\mathbf{R}^N) : \forall\, W \in G(n-q, {\rm K}^n)\ A\big|_W \in \widetilde{\mathcal{P}}_{\rm K}(W)\}.$$

In all three cases,

$$\widetilde{P}_q \;=\; P_{n-q-1} \qquad \text{and} \qquad \begin{cases} P_0 \;=\; \mathcal{P} \\ P_{n-1} \;=\; \widetilde{\mathcal{P}} \end{cases}$$

and, therefore, $u$ is a $P_q$-Dirichlet function if $u \in P_q(X)$ and $-u \in P_{n-q-1}(X)$. Thus, if $u$ is $C^2$ and $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of ${\rm Hess}_x u$, then $u$ is $P_q$-Dirichlet iff

$$\lambda_{q+1} \equiv 0.$$

No matter what $q$ $(0 \le q \le n-1)$, one has

$$\partial P_q \ \subset \ \{A : \mathrm{det}_{\mathrm{K}} A_K = 0\}$$

and, in fact, $\partial P_q$ consists of the branch of $\{A : \det_K A_K = 0\}$ where $\lambda_{q+1} = 0$. In particular, a $P_q$-Dirichlet function which is $C^2$ satisfies the Monge-Ampère equation (10.9).

**Dirichlet Sets which are Geometrically Defined.** The three fundamental examples $\mathcal{P}, \mathcal{P}_{\mathbf{C}}$ and $\mathcal{P}_{\mathbf{H}}$ are geometrically defined by the three Grassmannians $G(1, \mathbf{R}^n), G(1, \mathbf{C}^n)$ and $G(1, \mathbf{H}^n)$ respectively. In fact, there exists a vast array of geometrically interesting Dirichlet sets defined in a similar fashion. Let $G \subset G(p, \mathbf{R}^n)$ be a closed subset of the Grassmannian of $p$-planes, and define

$$\mathcal{P}(G) \ = \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, W \in G, \ \mathrm{tr}_W(A) \ \ge \ 0\} \tag{10.10}$$

This is evidently a Dirichlet set. It is also a convex cone with vertex at the origin. Its Dirichlet dual is

$$\widetilde{\mathcal{P}}(G) \ = \ \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, W \in G, \ \mathrm{tr}_W(A) \ \ge \ 0\}$$

In these cases the $\mathcal{P}(G)$-plurisubharmonic functions have the nice property that they are subharmonic on minimal $G$-submanifolds (those whose tangent planes lie in $G$). There are many other important cases coming from calibrated geometry and symplectic geometry. This and other related matters are discussed in detail in [$\mathrm{HL}_{2,4,5}$], and we briefly describe them next.

**The Dirichlet Problem in Calibrated Geometry.** Let $\phi \in \Lambda^p \mathbf{R}^n$ be a (constant coefficient) calibration on $\mathbf{R}^n$, and let $G(\phi) = \{\xi \in G(p, \mathbf{R}^n) : \phi(\xi) = 1\}$ be the Grassmannian of $\phi$-planes (cf. [$\mathrm{HL}_1$]). Then we have a geometrically defined Dirichlet set given by (10.10). The attendant notions of $\phi$-plurisubharmonic functions and $\phi$-convexity are discussed in detail in [$\mathrm{HL}_2$]. Our Main Theorem 6.2 shows that on strictly $\phi$-convex domains $\Omega \subset \mathbf{R}^n$ the Dirichlet problem is uniquely solvable in the class of continuous $\phi$-Dirichlet functions for all continuous boundary data.

We recall that this includes many interesting cases, for example, Special Lagrangian Geometry, Associative, Coassociative and Cayley Geometries, and many others. When a solution to the Dirichlet problem is $C^2$, it is partially $\phi$-pluriharmonic, that is $\mathrm{tr}_\xi \mathrm{Hess}\, u \ge 0$ for all $\phi$-planes $\xi$ and $= 0$ for some $\phi$-plane $\xi$ at each point. The associated differential equations of Monge-Ampère type in these cases have not all been found.

**The Dirichlet Problem in Lagrangian Geometry.** Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ as before, and for $A \in \mathrm{Sym}^2(\mathbf{R}^{2n})$ define its *Lagrangian component* to be

$$\begin{aligned} A_{LAG} &= \tfrac{t}{2}I + \tfrac{1}{2}(A + JAJ) \\ &= \tfrac{t}{2}I + A_{\text{skew}} \end{aligned} \qquad \text{where } t = \mathrm{tr}_{\mathbf{R}} A.$$

The matrix $A_{\text{skew}}$, called the *skew-hermitian part of* $A$, anticommutes with $J$ and therefore has eigenvalues

$$\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, ..., \lambda_n, -\lambda_n$$

with corresponding eigenvectors of the form

$$e_1, Je_1, e_2, Je_2, ..., e_n, Je_n.$$

Following [$\mathrm{HL}_4$] we consider the expression

$$\mathbf{M}_{LAG}(A) \;=\; \prod_{2^n \text{ times}} \left( \frac{t}{2} \pm \lambda_1 \pm \lambda_2 \cdots \pm \lambda_n \right) \tag{10.11}$$

This is a polynomial in $t$ whose coefficients are symmetric functions in $\lambda_1^2, ..., \lambda_n^2$. It follows from the work of Dadok and Katz [DK] that $\mathbf{M}_{LAG}(A)$ is a polynomial in the coefficients of $A$. It is, in fact, one of the factors of $\mathrm{tr}(D_{A_{LAG}})$ on $\Lambda^n \mathbf{R}^{2n}$.

We now consider the set $\mathrm{LAG} \subset G(n, 2n)$ of Lagrangian $n$-planes in $\mathbf{R}^{2n} = \mathbf{C}^n$. This gives us the geometrically defined Dirichlet set

$$\begin{aligned} \mathcal{P}(\mathrm{LAG}) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, \xi \in \mathrm{LAG},\ \mathrm{tr}_\xi A \geq 0\} \\ &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \tfrac{t}{2} - \lambda_1 - \cdots - \lambda_n \geq 0\} \end{aligned}$$

where we assume by convention that $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. The Dirichlet dual is

$$\begin{aligned} \widetilde{\mathcal{P}}(\mathrm{LAG}) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, \xi \in \mathrm{LAG},\ \mathrm{tr}_\xi A \geq 0\} \\ &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \tfrac{t}{2} - \lambda_1 + \cdots + \lambda_n \geq 0\} \end{aligned}$$

Our Dirichlet problem on a strictly Lagrangian-convex domain $\Omega \subset \mathbf{R}^n$ is uniquely solvable for continuous boundary data and gives a Lagrangian plurisubharmonic function $u \in C(\overline{\Omega})$ which, when it is class $C^2$, satisfies the differential equation

$$\mathbf{M}_{LAG}(\mathrm{Hess}\, u) \;=\; 0.$$

We can now elaborate this discussion using the general principle above. Fix a positive integer $p \leq n$ and consider the set

$$\mathrm{ISO_p} \;=\; \{\xi \in G(p, 2n) : \xi \text{ is an isotropic } p \text{ plane}\}.$$

(Recall that a $p$-plane $\xi$ is *isotropic* if $\xi \perp J\xi$, or equivalently, if $\omega\big|_\xi = 0$ where $\omega$ is the standard Kähler form.) Following the gnenral principle we introduce the Dirichlet sets

$$\begin{aligned}
\mathcal{P}^+(\mathrm{ISO_p}) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, W \in G_{\mathbf{C}}(p,n),\ A \in \mathcal{P}^+(\mathrm{LAG})(W)\} \\
&= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, \xi \in \mathrm{ISO_p},\ \mathrm{tr}_\xi A \geq 0\} \\
&= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \tfrac{p}{2n}t - \lambda_{n-p+1} - \cdots - \lambda_n \geq 0\}
\end{aligned}$$

and its Dirichlet dual

$$\begin{aligned}
\widetilde{\mathcal{P}}^+(\mathrm{ISO_p}) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, W \in G_{\mathbf{C}}(p,n),\ A \in \mathcal{P}^+(\mathrm{LAG})(W)\} \\
&= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, \xi \in \mathrm{ISO_p},\ \mathrm{tr}_\xi A \geq 0\} \\
&= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \tfrac{p}{2n}t + \lambda_{n-p+1} + \cdots + \lambda_n \geq 0\}
\end{aligned}$$

Associated to this problem we have the polynomial

$$\mathbf{M}_{\mathrm{ISO_p}}(A) \;=\; \prod_{|I|=p \text{ and } \pm} \left(\tfrac{p}{2n}t \pm \lambda_{i_1} \pm \cdots \pm \lambda_{i_p}\right)$$

which is also a factor of $\mathrm{tr}(D_{A_{LAG}})$ on $\Lambda^n\mathbf{R}^{2n}$. As above we have that any $C^2$ function $u$ which is $\mathrm{ISO_p}$-Dirichlet satisfies the differential equation

$$\mathbf{M}_{\mathrm{ISO_p}}(\mathrm{Hess}\, u) \;=\; 0$$

**The Geometrically $p$-Plurisubharmonic Dirichlet Problem.** There is a second, more geometric, choice for the $p$-plurisubharmonic functions, different from the one made at the beginning of this section. Namely, consider for $1 \leq p \leq n$, the geometrically defined Dirichlet sets

$$\begin{aligned}
\mathcal{P}\left(G(p,\mathbf{R}^n)\right) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{tr}_\xi A \geq 0\ \forall \xi \in G(p,\mathbf{R}^n)\} \\
\mathcal{P}\left(G(p,\mathbf{C}^n)\right) &= \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \mathrm{tr}_\xi A \geq 0\ \forall \xi \in G(p,\mathbf{C}^n)\} \\
\mathcal{P}\left(G(p,\mathbf{H}^n)\right) &= \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{H}^n) : \mathrm{tr}_\xi A \geq 0\ \forall \xi \in G(p,\mathbf{H}^n)\}
\end{aligned}$$

The Dirichlet duals are:

$$\begin{aligned}
\widetilde{\mathcal{P}}\left(G(p,\mathbf{R}^n)\right) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\ \xi \in G(p,\mathbf{R}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\} \\
\widetilde{\mathcal{P}}\left(G(p,\mathbf{C}^n)\right) &= \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \exists\ \xi \in G(p,\mathbf{C}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\} \\
\widetilde{\mathcal{P}}\left(G(p,\mathbf{H}^n)\right) &= \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{H}^n) : \exists\ \xi \in G(p,\mathbf{H}^n) \text{ s.t. } \mathrm{tr}_\xi A \geq 0\}
\end{aligned}$$

In all three of these cases ($\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$) there is a Monge-Ampère polynomial $M_p$. First we consider the real case. For $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, let $D_A : \Lambda^p\mathbf{R}^n \to \Lambda^p\mathbf{R}^n$ be the extension as a derivation. If $A$ has eigenvalues $\lambda_1, ..., \lambda_n$ with eigenvectors $e_1, ..., e_n$, then $D_A$ has eigenvalues $\lambda_I = \lambda_{i_1} + \cdots + \lambda_{i_p}$ with eigenvectors $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ where $I = (i_1, ..., i_p)$ is strictly increasing. One can prove that

$$\begin{aligned}\mathcal{P}(G(p, \mathbf{R}^n)) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : D_A \geq 0\} \\ &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda_I(A) \geq 0 \ \forall\, |I| = p\}\end{aligned}$$

and its Dirichlet dual

$$\begin{aligned}\widetilde{\mathcal{P}}(G(p, \mathbf{R}^n)) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : D_A \text{ has at least one eigenvalue } \geq 0\} \\ &= \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \lambda_I(A) \geq 0 \ \text{ for some } |I| = p\}\end{aligned}$$

If $u$ is $C^2$ and $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of $\mathrm{Hess}_x u$, then $u$ is $\mathcal{P}(G(p, \mathbf{R}^n))$-Dirichlet if and only if

$$\lambda_1 + \cdots + \lambda_p \ \equiv\ 0.$$

Thus $C^2$-solutions to the Dirichlet problem in this case are $p$-plurisubharmonic functions which satisfy the differential equation

$$M_p\,(\mathrm{Hess}\, u) \ =\ \prod_{|I|=p} \lambda_I \ =\ 0. \tag{10.12}$$

The polynomial $M_p(A) = \prod_{|I|=p} \lambda_I$ is of degree $\binom{n}{p}$ and equals $\det(D_A)$. For a domain $\Omega \subset \mathbf{R}^n$, the Dirichlet problem for $\mathcal{P}(G(p, \mathbf{R}^n))$-Dirichlet functions can be solved uniquely provided the boundary is $p$-convex, i.e.,

$$\mathrm{tr}_W\{II_{\partial\Omega}\} \ <\ 0$$

for all $p$-planes tangential to $\partial\Omega$, where $II_{\partial\Omega}$ denotes the second fundamental form of $\partial\Omega$ with respect to the outward-pointing normal. See [HL$_5$] for a more detailed discussion of this case, as well as a discussion of the Levi-problem in this context.

In all three cases ($\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$)

$$\begin{aligned}\mathcal{P}(G(p, \mathrm{K}^n)) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^N) : \lambda_I(A_{\mathrm{K}}) \geq 0 \ \forall\, |I| = p\} \qquad \text{and} \\ \widetilde{\mathcal{P}}(G(p, \mathrm{K}^n)) &= \{A \in \mathrm{Sym}^2(\mathbf{R}^N) : \lambda_I(A_{\mathrm{K}}) \geq 0 \text{ for some } \ |I| = p\}.\end{aligned}$$

The polynomial $M_p$ on $\mathrm{Sym}^2(\mathbf{R}^N)$ defined by $M_p(A) = \prod_{|I|=p} \lambda_I(A_{\mathrm{K}})$ of degree $\binom{n}{p}$ provides the nonlinear differential operator exactly as in the real case.

**The Next Tier for $\mathcal{P}(G(p, \mathbf{R}^n))$.** Fix positive integers $p \leq q \leq n$ and consider the convex Dirichlet set

$$P_q(G(p, \mathbf{R}^n)) \ =\ \left\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, W \in G(n-q, \mathbf{R}^n),\ A\big|_W \in \mathcal{P}(G(p, W))\right\}$$

and its Dirichlet dual

$$\widetilde{P}_q(G(p,\mathbf{R}^n)) \;=\; \left\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, W \in G(n-q,n),\ A\big|_W \in \widetilde{\mathcal{P}}(G(p,W))\right\}$$

Note that $P_q(G(1,\mathbf{R}^n)) \;=\; P_q$ and $\widetilde{P}_q(G(1,\mathbf{R}^n)) \;=\; P_{n-q-1} \;=\; P_{n-q-1}(G(1,\mathbf{R}^n))$.

**Lemma 10.1.** *Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of $A_{\mathrm{K}}$. Then*

$$\begin{aligned} A \;\in\; P_q(G(p,\mathbf{R}^n)) \quad &\iff \quad \lambda_{q+1} + \cdots + \lambda_{q+p} \;\ge\; 0, \qquad \text{and} \\ A \;\in\; \widetilde{P}_q(G(p,\mathbf{R}^n)) \quad &\iff \quad \lambda_{n-q-p+1} + \cdots + \lambda_{n-q} \;\ge\; 0 \end{aligned}$$

The proof is straightforward. One has the following.

**COROLLARY 10.2.**

$$\widetilde{P}_q(G(p,\mathbf{R}^n)) \;=\; P_{n-q-p}(G(p,\mathbf{R}^n)).$$

It follows that a $C^2$-function $u$ is $P_q(G(p,\mathbf{R}^n))$-Dirichlet if and only if

$$\lambda_{q+1} + \cdots + \lambda_{q+p} \;\equiv\; 0$$

where $\lambda_1 \;\le\; \cdots \;\le\; \lambda_n$ are the eigenvalues of $\mathrm{Hess}\, u$. In particular, $C^2$ solutions of the Dirichlet problem in this case are $p$-plurisubharmonic on $q$-planes and satisfy the equation (10.12). In other words, they are solutions of this equation belonging to other branches of the locus $M_p = 0$.

This discussion holds in perfect analogy in the complex and quaternionic cases.

**The Next Tier Principle.** We have been using the following technique to generate new examples from known ones. Let $\mathcal{W}$ be a family of subspaces of $\mathbf{R}^n$ with a Dirichlet set $F_W \subset \mathrm{Sym}^2(W)$ attached to each $W \in \mathcal{W}$. Define

$$F \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \forall\, W \in \mathcal{W},\ A\big|_W \in F_W\}.$$

One easily verifies that

$$\widetilde{F} \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \exists\, W \in \mathcal{W},\ A\big|_W \in \widetilde{F}_W\}$$

**PROPOSITION 10.3.** *The sets $F$ and $\widetilde{F}$ are Dirichlet sets.*

This is straightforward to verify. The examples we examine here under the heading "the next tier" are of this type. They can be elaborated to more complicated examples by repeatedly applying this principle. For example, let $\mathbf{R}^n = W_1 \oplus \cdots \oplus W_N$ be an orthogonal decomposition and set $\mathcal{W} = \{W_1, ..., W_N\}$, $F_W = \mathcal{P}(G(2,W))$. Then

$$F \;=\; \{A : \mathrm{tr}_\xi(A) \ge 0 \text{ for every 2-plane } \xi \subset W_k \text{ for every } k\}.$$

**Gårding Cones.** Let $M$ be a homogeneous polynomial of degree $m$ on $\mathrm{Sym}^2(\mathbf{R}^n)$, and suppose the identity $I \in \mathrm{Sym}^2(\mathbf{R}^n)$ is a *hyperbolic direction for* $M$ in the sense of Garding [G]. This means that for each $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, the polynomial $p_A(t) = M(tI + A)$ has exactly $m$ real roots, and that $M(I) = 1$. Then the associated differential operator

$$\mathbf{M}(u) \;=\; M(\mathrm{Hess} u)$$

will be called an *$MA$-operator*, and the polynomial $M$ will be called an *$MA$-polynomial.*

Gårding's beautiful theory of hyperbolic polynomials states that the set

$$\Gamma(M) \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : M(tI + A) \neq 0 \ \text{ for } t \geq 0\} \tag{10.13}$$

is an open convex cone in $\mathrm{Sym}^2(\mathbf{R}^n)$ equal to the connected component of $\{M > 0\}$ containing $I$. The closed convex cone

$$F_M \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : M(tI + A) \neq 0 \ \text{ for } t > 0\} \tag{10.14}$$

is the closure of $\Gamma(M)$. Moreover,

$$\partial F_M \;=\; \{A \in \mathrm{Sym}^2(\mathbf{R}^n) : M(A) = 0 \text{ but } M(tI + A) \neq 0 \ \text{ for } t > 0\}.$$

We mention that the Dirichlet condition $F_M + \mathcal{P} \subset F_M$ is equivalent to $\mathcal{P} \subset F_M$ and can be stated in several equivalent ways in terms of $M$:

1) $M(tI + A) \neq 0$ for all $t > 0$ and $A > 0$.

1)′ $M(tI + P_e) \neq 0$ for all $t > 0$ and all unit vectors $e$.

**Symmetric Functions of Hess(u).** A basic example of an $MA$-polynomial on $\mathrm{Sym}^2(\mathbf{R}^n)$ is the determinant. By the principle above we find that each of the elementary symmetric functions

$$\sigma_{n-\ell}(A) \;=\; \frac{1}{\ell!}\frac{d^\ell}{dt^\ell}\det(A + tI)\bigg|_{t=0}$$

is again an $MA$-polynomial whose associated set $F_{\sigma_{n-\ell}}$ is again a Dirichlet set.

**The Special Lagrangian Potential Equation.** Another interesting case to which our general theory applies, comes from the polynomial

$$Q(A) \equiv \operatorname{Im}\{\det(I + iA)\}.$$

for $A \in \mathrm{Sym}^2(\mathbf{R}^n)$. The associated differential equation

$$Q(\mathrm{Hess}\, u) = 0, \tag{10.15}$$

governs the potential functions in the theory of Special Lagrangian submanifolds (cf. [HL$_1$]).

The locus $\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : Q(A) = 0\}$ has $n$ connected components, or *branches*, when $n$ is even, and $n-1$ branches when $n$ is odd. Each branch is a proper analytic submanifold of $\mathrm{Sym}^2(\mathbf{R}^n)$.

The Dirichlet problem for equation (10.15) was treated in [CNS] for the case where the Hess $u$ is required to lie on one of the two outermost branches. Under this assumption, smooth solutions are established for smooth boundary data on appropriately convex domains. In [CNS] the authors asked whether it is possible to treat the other branches of this equation.

We shall show that the answer is yes. In fact we shall study the more general Special Lagrangian potential equation

$$Q_\theta(A) \equiv \operatorname{Im}\left\{e^{-i\theta}\det(I + iA)\right\}.$$

for $\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, with associated differential equation

$$Q_\theta(\mathrm{Hess}\, u) = 0, \tag{10.16}$$

To begin we rewrite the equation $Q_\theta(A) = 0$ in the form

$$\mathrm{Trace}\{\arctan(A)\} = \theta \pm k\pi \qquad \text{for} \quad k \in \mathbf{Z}, |k| < \frac{n}{2} \tag{10.17}$$

**PROPOSITION 10.4.** *Each of the sets*

$$F_c \equiv \left\{A \in \mathrm{Sym}^2(\mathbf{R}^n) : \mathrm{Trace}\{\arctan(A)\} \geq c\right\}$$

*for* $-\frac{n\pi}{2} < c < \frac{n\pi}{2}$ *is a Dirichlet set with Dirichlet dual*

$$\widetilde{F}_c = F_{-c}.$$

**COROLLARY 10.5.** *Let $\Omega \subset\subset \mathbf{R}^n$ be a smoothly bounded domain which is both $\overrightarrow{F}_c$ and $\overrightarrow{F}_{-c}$ strictly convex, with $c$ as above. Then the Dirichlet problem for continuous $F_c$-Dirichlet functions is uniquely solvable for all continuous boundary data on $\partial\Omega$.*

Note that any $C^2$-function $u$, which is $F_c$-Dirichlet , is a solution to equation $Q_c(\mathrm{Hess}\, u) = 0$ which lies on the branch

$$\mathrm{Hess}\, u \ \in \ \partial F_c,$$

that is,

$$\mathrm{Trace}\, \{\arctan\, (\mathrm{Hess}\, u)\} \ = \ c$$

It is an interesting fact that the sets $F_c$ are actually starlike with respect to some point in their interior except for the following finite set of cases: When $n$ is odd, we must assume $\theta \neq \frac{\pi}{2}$, and for $n$ even, we assume $\theta \neq 0$.

The special Lagrangian potential equation is, in fact, *strictly elliptic* in the sense that there is a constant $\kappa > 0$ so that $\mathrm{dist}(A + P, \partial F) \geq \kappa \|P\|$ for $A \in F$ and $P \in \mathcal{P}$.

We note that for $n = 3$ this equation has also been treated by Yuan [Y] who established a $C^{2,\alpha}$-estimate for $C^{1,1}$ viscosity solutions.

# Appendix A.

## Dirichlet Sets Which Can Be Defined Using Fewer of the Variables in $\mathbf{R}^n$.

Suppose $F$ is a subset of $\mathrm{Sym}^2(\mathbf{R}^n)$ which *can be defined using the variables in a subspace* $W \subset \mathbf{R}^n$. That is

$$F \;=\; (F \cap \mathrm{Sym}^2(W)) \oplus \mathrm{Sym}^2(W)^\perp.$$

Let $F_0$ denote the subset $F \cap \mathrm{Sym}^2(W)$ of $\mathrm{Sym}^2(W)$, and let $x = (x', x'') \in W \oplus W^\perp = \mathbf{R}^n$ denote the variables.

**Example A.1.** Let $F_0 = \mathcal{P}(W)$ with $W = \mathbf{R}^p$ and $p < n$. In this case a $C^2$-function $u$ is of type $F$ if

$$\sum_{j=1}^{p} \frac{\partial^2 u}{\partial x_j^2} \;\geq\; 0.$$

That is, for each fixed $x''$ the function $u(x', x'')$ of $x'$ is subharmonic.

**Remark A.2.** It is standard in the fully nonlinear theory to use the word "elliptic" to include Dirichlet sets. Then, in particular, Example A.1 is "elliptic". (See [Kr] for example.) We prefer to reserve the work elliptic for Dirichlet sets which can *not* be defined using fewer of the variables.

**Definition A.3.** Given a function $u(x)$ which is upper semicontinuous with values in $[-\infty, \infty)$, we say that $u$ is *horizontally of type* $F_0$ if for each fixed $x''$ the function $u_{x''}(x') = u(x', x'')$ is of type $F_0$.

**Elementary Properties:**

(1) $F$ is a Dirichlet set $\iff$ $F_0$ is a Dirichlet set.

(2) $F$ is convex $\iff$ $F_0$ is convex.

(3) $\widetilde{F} = \widetilde{F_0} \oplus \mathrm{Sym}^2(W)^\perp$.

(4) $\overrightarrow{F} = \overrightarrow{F}_0 \oplus \mathrm{Sym}^2(W)^\perp$

If $u$ is of class $C^2$, then it is obvious that $u$ is of type $F$ if and only if $u$ is horizontally of type $F_0$.

**THEOREM A.4.** *Suppose $F = F_0 \oplus \mathrm{Sym}^2(W)^\perp$ is a Dirichlet set which can be defined using the variables in $W$. Then $u$ is of type $F$ if and only if $u$ is horizontally of type $F_0$.*

**COROLLARY A.5.** *Let $F$ be as above. Then the Subaffine Theorem is true for $F$ if and only if the Subaffine Theorem is true for $F_0$.*

**Proof.** Suppose $u$ is of type $F$ and $v$ is of type $\widetilde{F}$. Because of Property (3) the Theorem applies to $v$ as well as $u$. If the Subaffine Theorem is true for $F_0$, then $u_{x''} + v_{x''}$ is a subaffine function of $x' \in \mathbf{R}^p$. Finally, we note that if $w_{x''}(x') = w(x', x'')$ is subaffine in $x'$ (horizontally subaffine), then $w$ is subaffine in $x = (x', x'')$. Set $B = B' \times B'' \subset \mathbf{R}^p \times \mathbf{R}^{n-p}$. If $w \leq a$ on $\partial B$, then $s_{x''} \leq a_{x''}$ on $\partial B' \times B'' \subset \partial B$. Hence, $w_{x''} \leq a_{x''}$ on $B' \times B''$. ∎

**Proof of Theorem A.4.** Suppose that $u$ is horizontally of type $F_0$. To show that $u$ is of type $F$ we must show that $u+B$ is subaffine for each $B \in \widetilde{F}$. Since $\widetilde{F} = \widetilde{F}_0 \oplus \mathrm{Sym}^2(W)$, we have $B(x', x'') = b(x') + a(x', x'')$ where $b \in \widetilde{F}_0$ and $a$ is affine. By hypothesis, $u_{s''}(x') + b(x')$ is subaffine in $x'$. Since $a$ is an affine function, $u + B$ is horizontally subaffine. As noted in the proof of the Corollary, this implies that $u + B$ is subaffine.

Suppose $u(x', x'')$ is not of type $F_0$ for some fixed $x_0''$. We may assume $x_0'' = 0$ and that there exist $\epsilon > 0$, $x_0' = 0$, and $b \in \widetilde{F}_0$ such that

$$
\begin{aligned}
u(x', 0) + b(x') &\le -\epsilon |x'|^2 \quad \text{near } x' = 0 \\
&= 0 \qquad\qquad \text{at } x' = 0
\end{aligned}
\tag{1}
$$

after modifying by an affine function of $x'$ and translating so that $x_0' = 0$.

Consider $B(x', x'') = b(x') - \Lambda |x''|^2$ with $\Lambda >> 0$. By (1) and upper semicontinuity,

$$
u(x', x'') + B(x', x'') < 0 \quad \text{on} \quad |x'| = r', |x''| \le r''
$$

for some $r'' > 0$ small. Pick $\Lambda$ large enough so that

$$
u(x', x'') + B(x', x'') < 0 \quad \text{on} \quad |x'| =\le r', |x''| = r''.
$$

Since $u + B$ equals zero at $x = 0$, it is not subaffine and hence $u$ is not of type $F$. ∎

# Appendix B.

## A Distributional Definition of Type F for Convex Dirichlet Sets F.

Suppose $H$ is a closed half space in $\mathrm{Sym}^2(\mathbf{R}^n)$. Then $H$ can be defined by

$$H \;=\; \{B \in \mathrm{Sym}^2(\mathbf{R}^n) : \langle A, B\rangle \geq c\} \tag{B.1}$$

for some non-zero $A \in \mathrm{Sym}^2(\mathbf{R}^n)$ and some $c \in \mathbf{R}$. Note that

$$H \;\text{ is a Dirichlet set} \qquad \Longleftrightarrow \qquad A \in \mathcal{P} \tag{B.2}$$

since, with $B_0 \in \partial H$, one has $c \leq \langle A, B_0 + P\rangle = \langle A, B_0\rangle + \langle A, P\rangle = c + \langle A, P\rangle$ for all $P \geq 0$ if and only if $0 \leq \langle A, P\rangle$ for all $P \geq 0$.

Similarly, one can prove that:

**Lemma B.1.** *If $F$ is a Dirichlet set contained in a closed half-space $H$, then $H$ is a Dirichlet set.*

As a consequence of this Lemma we can state the Hahn-Banach Theorem in the context of Dirichlet sets as follows.

**COROLLARY B.2.** *$F$ is a convex Dirichlet set if and only if $F = \bigcap_\alpha H_\alpha$ over all Dirichlet supporting half-spaces $H_\alpha$ for $F$.*

The Dirichlet dual statement is also true.

**Lemma B.3.** *If $F$ is a convex Dirichlet set, then $\widetilde{F} = \bigcup_\alpha \widetilde{H}_\alpha$ over all Dirichlet supporting half-spaces $H_\alpha$ for $F$.*

**Proof.** If $F \subset H_\alpha$, then $\widetilde{H}_\alpha \subset \widetilde{F}$, so we only need to show that $\widetilde{F} \subset \bigcup_\alpha \widetilde{H}_\alpha$. Suppose $B \in \widetilde{F}$, i.e., $-B \notin \mathrm{Int}F$. We claim there exists $H_\alpha$ with $-B \notin \mathrm{Int}H_\alpha$, i.e., with $B \in \widetilde{H}_\alpha$. There are two cases. If $-B \notin F$ we can pick $H_\alpha$ with $-B \notin H_\alpha$. If $-B \in \partial F$, then we can pick a supporting hyperplane $H_\alpha$ for $F$ at $-B$ by the Hahn-Banach Theorem, so that $-B \in \partial H_\alpha$. ∎

**COROLLARY B.4.** *A function $u$ is of type $F$ if and only if $u$ is of type $H_\alpha$ on $X$ for all Dirichlet supporting half-spaces $H_\alpha$ for $F$.*

**Proof.** Since $F \subset H_\alpha$, type $F$ implies type $H_\alpha$. Conversely, if $u$ is type $H_\alpha$ for all supporting half-spaces $H_\alpha$, then $u + B$ is subaffine for all $B \in \widetilde{H}_\alpha$ and hence by Lemma B.3, $u$ is of type $F$. ∎

**PROPOSITION B.5.** *Suppose $H$ is a half-space through the origin, defined by*

$$H \;=\; \{B : \langle A, B\rangle \geq 0\} \qquad \text{with } A \text{ positive definite.}$$

*Then the following are equivalent:*

1) *$u$ is of type $H$,*
2) *$u$ is sub-$\Delta_A$-harmonic,*
3) *$u$ is $L^1_{\mathrm{loc}}$ and $\Delta_A u \geq 0$ (or $u \equiv -\infty$)*

*where* $\Delta_A u = \sum a_{ij} u_{ij}$.

**Proof.** The equivalence of 2) and 3) is standard. Note that $H$ is self dual, i.e. $\widetilde{H} = H$. Suppose $u$ is of type $H$. Given a $\Delta_A$-harmonic function $h$ with $u \leq h$ on $\partial B$, we have $u \leq h$ on $B$ because $-h$ is of type $\widetilde{H}$ which implies that $u - h$ is subaffine.

Conversely, suppose $u$ is sub-$\Delta_A$-harmonic. Let $v$ be a $C^2$-function of type $\widetilde{H} = H$. We must show that $u + v \leq a$ on $\partial B$ implies $u + v \leq a$ on $B$ for any affine function $a$ and any ball $B$. Replace $v$ by $v - a$ and $a$ by 0. Let $h$ denote the $\Delta_A$-harmonic function with the same boundary values as $v$ on $\partial B$. Now $u + h \leq 0$ on $\partial B$ implies $u + h \leq 0$ on $B$ since $u$ is sub-$\Delta_A$-harmonic, but $v = h$ on $\partial B$ implies $v \leq h$ on $B$ since, as we have shown above, $v$ is sub-$\Delta_A$-harmonic. ■

**COROLLARY B.6.** *Suppose $H$ is a Dirichlet half-space defined by (B.1) with $A > 0$. Pick $B_0 \in \partial H$. Then $u$ is of type $H$ if and only if $u \in L^1_{\rm loc}$ and $\Delta_A(u - B_0) \geq 0$, i.e., $u - B_0$ is $\Delta_A$-subharmonic.*

**Lemma B.7.** *A convex Dirichlet set $F$ cannot be defined using fewer of the variables in $\mathbf{R}^n$ if and only if each $A \in {\rm Int}\mathcal{P}_+(F)$ is positive definite, where $\mathcal{P}_+(F)$ denotes the closure of the cone of directions defining the supporting half-spaces for $F$.*

**Proof.** See Corollary C.4 in Appendix C of [HL$_3$].

Combining Corollary B.4 and Lemma B.7 we have:

**THEOREM B.8.** *Suppose $F$ is a convex Dirichlet set which can not be defined using fewer of the variables in $\mathbf{R}^n$. For each supporting half-space $\{B : \langle A_\alpha, B\rangle \geq c\}$ pick $B_0^\alpha \in \partial H_\alpha$. Then $u$ is of type $F$ if and only if $u - B_0^\alpha$ is $\Delta_{A_\alpha}$-subharmonic for each $A_\alpha$.*

**Remark B.9.** This theorem can be extended to the case where $F$ can be defined using fewer of the variables by applying Theorem A.4. Moreover, one can deduce from this extension that for a Dirichlet set $F$

$$F \ \text{ is convex} \quad \Rightarrow \quad F(X) \ \text{ is convex}$$

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