Scalar curvature on lightlike hypersurfaces.

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Octobre 2007

IHES/M/07/35

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Abstract

In [3], the concept of induced scalar curvature of lightlike hypersurfaces is introduced, restricting on a special class of the latter. This paper removes some of these constraints and construct this scalar quantity by an approach that is consistent with the well-known non degenerate theory. Basic calculation examples are provided.

MSC(2000): 53B30; 53B50.

Key words: Lightlike hypersurface; screen distributon; extrinsic scalar curvature.

1 Introduction

Physically, the scalar curvature has the following interpretation. Start a point in a D-dimensional space and move a geodesic distance ε in all directions. In essence you would form the equivalent of a generalized sphere in this space. The area of this sphere can be calculated in flat space. But in curved space the area will deviate from the one we calculated by an amount proportional to the scalar curvature. Precisely,

$$R = \lim_{\varepsilon \to 0} \frac{6D}{\varepsilon^2}\Big[1 - \frac{A_{curved}(\varepsilon)}{A_{flat}(\varepsilon)}\Big].$$

From a geometric point of view, this is just the contraction of the Ricci tensor Ric,

$$R = g^{ij} Ric_{ij}.$$

In [3], the problem of inducing scalar curvature on lightlike manifolds is considered. Such a problem arises, mainly due to two difficulies: since the induced connection is not a Levi-Civita connection (unless M be totally geodesic) the $(0, 2)$ induced Ricci tensor is not symmetric in general. Also, as the induced metric is degenerate, its inverse does not exist and it is not

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possible to proceed in the usual way by contracting the Ricci tensor to get a scalar quantity.

To overcome this difficulties, Duggal considered in [3] a class of lightlike hypersurfaces in ambiant lorentzian signature, called lightlike hypersurfaces of genus zero. Elements of such a class are subject to the following contraints:

To admit a canonical screen distribution that induces a canonical transversal vector bundle;

To admit an induced symmetric Ricci tensor.

Although the above two conditions are interesting to compensate lacking due to the above quoted difficulties, to admit symmetric induced Ricci tensor in lightlike setting is very restrictive. Also, the problem in contracting with respect to the noninvertible induced metric is still unsolved for the general setting. In this paper, we drop and solve the induced symmetric Ricci condition and propose a solution for the contraction with respect to the degenerate metric in a consistent way to the known nondegenerate approach.

By the approach developed in the book [4] the extrinsic geometry of lightlike hypersurfaces depends on an additional structure, the screen distribution. For this, we start in this paper with a given normalization. We first introduce a symmetrized induced Ricci tensor Ric^{sym}. Thanks to the concept of pseudo-inversion of degenerate metric we introduced in [1], we overcome the above quoted problem in contracting with respect to the degenerate metric. Then we state a definition of the extrinsic scalar curvature which generalizes in a consistent way the one in [3]. Some Physical and mathematical relevant models are then discussed. To know to what extent a change in normalization influences our scalar quantity, we investigate relationship between induced geometric objects and operators involved in the curvature expression with their analogous with a change in screen distribution.

2 Facts about lightlike hypersurfaces

Let (M, g) be a hypersurface of an $(n+2)-$dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ of constant index $0 < \nu < n+2$. In the theory of nondegenerate hypersurfaces, the normal bundle has trivial intersection $\{0\}$ with the tangent one and plays an important role in the introduction of main induced geometric objects of M. In case of lightlike hypersurfaces, the situation is totally different. The normal bundle $TM^{\perp}$ is a rank-one distribution over M: $TM^{\perp} \subset TM$ and then coincides with the radical distribution $RadTM = TM \cap TM^{\perp}$. Hence, the induced metric tensor g is degenerate with constant rank n.

A complementary bundle of $RadTM$ in TM is a rank n nondegenerate distribution over M, called a *screen distribution* of M, denoted by $S(TM)$. Existence of $S(TM)$ is secured provided M be paracompact. A lightlike hypersurface with a specific screen distribution is denoted by $(M, g, S(TM))$.

It is well-known [4] that for such a triplet, there exists a unique vector sub bundle $tr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying

$$\bar{g}(N, \xi) = 1, \quad \bar{g}(N, N) = \bar{g}(N, W) = 0, \qquad \forall\ W \in \Gamma(ST(M)|_{\mathcal{U}}). \qquad (1)$$

TM and $T\bar{M}$ are decomposed as follows:

$$\begin{aligned} TM &= S(TM) \overset{\perp}{\oplus} TM^{\perp}, & (2)\\ T\bar{M}|_M &= TM \oplus tr(TM). & (3) \end{aligned}$$

We denote by $\Gamma(E)$ the $\mathcal{F}(M)-$module of smooth sections of a vector bundle E over M, $\mathcal{F}(M)$ being the algebra of smooth functions on M. Also, all manifolds are supposed to be smooth, paracompact and connected.

The induced connection, say ∇, on M is defined by

$$\nabla_X Y = Q(\bar{\nabla}_X Y), \qquad \forall\ X, Y \in \Gamma(TM),$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $(\bar{M}, \bar{g})$ and Q the projection morphism on TM with respect to the decomposition (2). Notice that ∇ depends on both g and a screen distribution $S(TM)$ of M.

Let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2). Consider a normalizing pair $\{\xi, N\}$ satisfying (1). Then, the local Gauss and Weingarten type formulas are given by

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + B(X, Y)N,\\ \bar{\nabla}_X N &= -A_N X + \tau(X)N,\\ \nabla_X PY &= \overset{\star}{\nabla}_X\ PY + C(X, PY)\xi, & (4)\\ \nabla_X \xi &= -\ \overset{\star}{A}_\xi\ X - \tau(X)\xi, \qquad \forall X, Y\ \in \Gamma(TM|_{\mathcal{U}}), \end{aligned}$$

where B and C are the local second fundamental forms on $\Gamma(TM)$ and $\Gamma(S(TM))$, respectively, $\overset{\star}{\nabla}$ is a metric connection on $\Gamma(S(TM))$, $\overset{\star}{A}_\xi$ the local shape operator on $S(TM)$ and τ a $1-$form on TM defined by

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi).$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$ [5]. As per [4, page 83], the second fundamental form B of M is independent of the choice of a screen distribution and satisfies for all $X, Y \in\ \Gamma(TM)$

$$B(X, \xi) = 0, \quad \text{and} \quad B(X, Y) = g(\overset{\star}{A}_\xi\ X, Y).$$

Denote by $\bar{R}$ and R the Riemann curvature tensors of $\bar{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations [4, p. 93]

$$\begin{aligned}
\langle \bar{R}(X,Y)Z, \xi\rangle &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\
&\quad + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \qquad (5)\\
\langle \bar{R}(X,Y)Z, PW\rangle &= \langle R(X,Y)Z, PW\rangle + B(X,Z)C(Y,PW) \\
&\quad - B(Y,Z)C(X,PW), \qquad (6)\\
\langle \bar{R}(X,Y)\xi, N\rangle &= \langle R(X,Y)\xi, N\rangle = C(Y, \overset{\star}{A}_\xi X) - C(X, \overset{\star}{A}_\xi Y) \\
&\quad - 2d\tau(X,Y), \quad \forall X,Y,Z,W \in \Gamma(TM|_{\mathcal{U}}). \qquad (7)
\end{aligned}$$

Finally, we recall from [1] the following results. Consider on M a normalizing pair $\{\xi, N\}$ satisfying (1) and define the one-form

$$\eta(\bullet) \;=\; \bar{g}(\,N\,,\;\bullet\;).$$

For all $X \in \Gamma(TM)$, $X = PX + \eta(X)\xi$ and $\eta(X) = 0$ if and only if $X \in \Gamma(S(TM))$. Now, we define $\flat$ by

$$\begin{aligned}
\flat : \Gamma(TM) &\longrightarrow \Gamma(T^*M) \\
X &\longmapsto X^\flat = g(\,X\,,\;\bullet\;) + \eta(X)\eta(\;\bullet\;). \qquad (8)
\end{aligned}$$

Clearly, such a $\flat$ is an isomorphism of $\Gamma(TM)$ onto $\Gamma(T^*M)$, and generalize the usual nondegenerate theory. In the latter case, $\Gamma(S(TM))$ coincides with $\Gamma(TM)$, and as a consequence the $1-$form η vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. We let $\sharp$ denote the inverse of the isomorphism $\flat$ given by (8). For $X \in \Gamma(TM)$ (resp. $\omega \in T^*M$), $X^\flat$ (resp. $\omega^\sharp$) is called the dual $1-$form of X (resp. the dual vector field of ω) with respect to the degenerate metric g. It follows from (8) that if ω is a 1-form on M, we have for $X \in \Gamma(TM)$,

$$\omega(X) \;=\; g(\omega^\sharp, X) \;+\; \omega(\xi)\eta(X).$$

Define a $(0,2)$-tensor $\tilde{g}$ by

$$\tilde{g}(X,Y) \;=\; X^\flat(Y), \quad \forall X,Y \in \Gamma(TM).$$

Clearly, $\tilde{g}$ defines a non-degenerate metric on M which plays an important role in defining the usual differential operators *gradient*, *divergence*, *Laplacian* with respect to degenerate metric g on lightlike hypersurfaces ([1] for details). Also, observe that $\tilde{g}$ coincides with g if the latter is non-degenerate. The $(0,2)-$tensor $g^{[\,\cdot\,,\,\cdot\,]}$, inverse of $\tilde{g}$ is called *the pseudo-inverse of* g. With respect to the quasi orthonormal local frame field $\{\partial_0 := \xi, \partial_1, \cdots, \partial_n, N\}$ adapted to the decompositions (2) and (3) we have

$$\tilde{g}(\xi,\xi) = 1, \quad \tilde{g}(\xi, X) = \eta(X), \qquad (9)$$
$$\tilde{g}(X,Y) = g(X,Y) \quad \forall X,Y \in \Gamma(S(TM)),$$

and the following is proved [1].

Proposition 2.1 *(α) For any smooth function $f\ :\ \mathcal{U}\subset M\rightarrow\mathbb{R}$ we have*

$$grad^g f = g^{[\alpha\beta]} f_\alpha \partial_\beta \qquad where \quad f_\alpha = \frac{\partial f}{\partial x^\alpha} \quad \partial_\beta = \frac{\partial}{\partial x^\beta} \quad \alpha,\beta = 0,\dots n$$

(β) For any vector field X on $\mathcal{U}\subset M$

$$div^g X = \sum_{\alpha=0}^{n} \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} X, X_\alpha)\ \ ;\ \varepsilon_0 = 1$$

(γ) for a smooth function f defined on $\mathcal{U}\subset M$ we have

$$\Delta^g f = \sum_{\alpha=0}^{n} \varepsilon_\alpha \tilde{g}(\nabla_{X_\alpha} grad^g f, X_\alpha)$$

In particular, ρ being an endomorphism (resp. a symmetric bilinear form) on $(M, g, S(TM))$, we have

$$tr\rho = trace_g \rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \tilde{g}(\rho(\partial_\alpha), \partial_\beta)$$

$$(\text{resp. } trace_g \rho = \sum_{\alpha,\beta=0}^{n} g^{[\alpha\beta]} \rho_{\alpha\beta} \quad).$$

3 Extrinsic scalar curvature

Consider a lightlike hypersurface $(M, g, S(TM))$ of a $(n+2)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$, with induced Ricci tensor Ric. Then we define the *symmetrized induced Ricci* tensor to be the $(0,2)$-tensor Ric^{sym} on M such that for X, Y tangents to M,

$$Ric^{sym}(X,Y) = \frac{1}{2}[Ric(X,Y) + Ric(Y,X)], \tag{10}$$

where $Ric(X,Y) = trace\{Z \longmapsto R(Z,X)Y\}$.

In index notation,

$$Ric^{sym}_{\alpha\beta} = \frac{1}{2}[R_{\alpha\beta} + R_{\beta\alpha}] \tag{11}$$

where we brief $R_{\alpha\beta} := Ric_{\alpha\beta}$. Now, $g^{[\ ]}$ being the pseudo-inverse of g, contract (Eq.11) and obtain the scalar quantity

$$R = g^{[\alpha\beta]} Ric^{sym}_{\alpha\beta}. \tag{12}$$

It is immediate to check that for a fixed triplet $(M, g, S(TM))$, whereas the paire (ξ, N) in (Eq.1) is not uniquely determined being subject to the scalling $\xi \longmapsto \xi' = \alpha\xi$ and $N \longmapsto N' = \frac{1}{\alpha}N$ (α smooth nonvanishing function), the right hand side of (Eq.12) is independent of the choice of the paire (ξ, N). We define R to be the *extrinsic scalar curvature* of the structure $(M, g, S(TM))$.

Now, we give expression of symmetrized Ricci and extrinsic scalar curvature R in terms of induced shape operators A_N, $\overset{\star}{A}_\xi$ and ambiant curvatures. Let $\{E_0 = \xi, E_i\}$ be a quasiorthonormal frame for TM induced from a frame $\{E_0 = \xi, E_i, E_{n+1} = N\}$ for $T\bar{M}$ such that $S(TM) = span\{E_1, \cdots, E_n\}$ and $Rad(TM) = span\{\xi\}$. We use the following range of indices. Greck letters $\alpha, \beta ...$ run through $0, ..., n$ and latin letters $i, j, ...$ through $1, ..., n$. By use of (Eq.9) we have

$$\begin{aligned}
Ric(X,Y) &= \sum_{\gamma=0}^{n} \tilde{g}_{\gamma\gamma}\tilde{g}(R(E_\gamma, X)Y, E_\gamma) \\
&= \tilde{g}_{00}\tilde{g}(R(\xi, X)Y, \xi) + \sum_{i=1}^{n} \tilde{g}_{ii}\tilde{g}(R(E_i, X)Y, E_i) \\
&= \bar{g}(R(\xi, X)Y, N) + \sum_{i=1}^{n} g_{ii}g(R(E_i, X)Y, E_i), \qquad \tilde{g}_{00} = 1.
\end{aligned}$$

Then, using Gauss-Codazzi equations (Eqs.5,6,7) provides

$$\begin{aligned}
Ric(X,Y) &= \bar{R}ic(X,Y) - \bar{g}(\bar{R}(N,X)Y,\xi) + B(X,Y)trA_N - g(A_N X, \overset{\star}{A}_\xi\, Y) \\
&= \bar{R}ic(X,Y) + B(X,Y)trA_N - g(A_N X, \overset{\star}{A}_\xi\, Y) - \eta(\bar{R}(\xi, Y)X).
\end{aligned}$$

Thus, we obtain the following expression of the symmetrized induced Ricci,

$$\begin{aligned}
Ric^{sym}(X,Y) &= \bar{R}ic(X,Y) + B(X,Y)trA_N \\
&\quad -\frac{1}{2}[\eta(\bar{R}(\xi,Y)X) + \eta(\bar{R}(\xi,X)Y) \\
&\quad +g(A_N X, \overset{\star}{A}_\xi\, Y) + g(A_N Y, \overset{\star}{A}_\xi\, X)].
\end{aligned} \tag{13}$$

where $\bar{R}ic$ denotes the Ricci curvature of the ambiant manifold $\bar{M}$.

In local coordinates,

$$\begin{aligned}
Ric^{sym}_{\alpha\beta} &= \bar{R}ic_{\alpha\beta} + B_{\alpha\beta}trA_N \\
&\quad -\frac{1}{2}[\eta(\bar{R}(\xi,\partial_\beta)\partial_\alpha) + \eta(\bar{R}(\xi,\partial_\alpha)\partial_\beta) \\
&\quad +g(\overset{\star}{A}_\xi\, A_N\partial_\alpha, \partial_\beta) + g(\overset{\star}{A}_\xi\, A_N\partial_\beta, \partial_\alpha)].
\end{aligned} \tag{14}$$

where we make use of the symmetry of $\overset{\star}{A}_\xi$ with respect to g.

In the sequel, we let

$$\sigma = \frac{1}{\sqrt{2}(n+1)} g^{[\alpha\beta]} B_{\alpha\beta}$$

denote the mean curvature function of M^{n+1} and

$$\bar{\theta} = \bar{R}ic(N, N)$$

represents the transverse energy in null direction N. Now applying (Eq.12) by contracting (Eq.14) with respect to $g^{[\alpha\beta]}$ we get the following expression of the extrinsic scalar curvature on the structure $(M, g, S(TM))$,

$$\begin{aligned} R &= \bar{R} + \sqrt{2}(n+1)\sigma tr A_N - tr(\overset{\star}{A}_\xi\, A_N) \\ &\quad -\bar{\theta} - \frac{1}{2} g^{[\alpha\beta]}[\eta(\bar{R}(\xi, \partial_\alpha)\partial_\beta) + \eta(\bar{R}(\xi, \partial_\beta)\partial_\alpha)]. \end{aligned} \tag{15}$$

For nambiant manifold $\bar{M}$ with constant sectional curvature c, we derive the following.

Proposition 3.1 *Let $(M, g, S(TM))$ be a lightlike hypersurface of a $(n+2)$-dimensional space forme $\bar{M}(c)$. Then*

$$R = n^2 c + \sqrt{2}(n+1)\sigma tr A_N - tr(\overset{\star}{A}_\xi\, A_N). \tag{16}$$

Proof. In case $\bar{M}^{n+2}$ has constant curvature c, we have $\bar{R}ic = (n+1)c\bar{g}$, $\eta(\bar{R}(\xi, Y)X) = cg(X, Y)$, then

$$\begin{aligned} Ric^{sym}(X, Y) &= ncg(X, Y) + B(X, Y) tr A_N \\ &\quad -\frac{1}{2}[g(\overset{\star}{A}_\xi\, A_N X, Y) + g(\overset{\star}{A}_\xi\, A_N Y, X)]. \end{aligned}$$

In local coordinates,

$$\begin{aligned} Ric^{sym}_{\alpha\beta} &= ncg_{\alpha\beta} + B_{\alpha\beta} tr A_N \\ &\quad -\frac{1}{2}[g(\overset{\star}{A}_\xi\, A_N \partial_\alpha, \partial_\beta) + g(\overset{\star}{A}_\xi\, A_N \partial_\beta, \partial_\alpha)], \end{aligned}$$

and

$$R = n^2 c + \sqrt{2}(n+1)\sigma tr A_N - tr(\overset{\star}{A}_\xi\, A_N).$$

4 Some basic examples

(a) The null cone $\wedge_0^{n+1}$ in $\mathbb{R}_1^{n+2}$.

The light cone $\wedge_0^{n+1}$is given in $\mathbb{R}_1^{n+2}$ by the equation $-(x^0)^2 + \sum_{a=1}^{n+1} = 0$, $x \neq 0$. In fact we will be using either $x^0 < 0$ or $x^0 > 0$ since we assumed connexity. It is known that $\wedge_0^{n+1}$ is a lightlike hypersurface with radical distribution spanned by the global position vector field $\xi = \sum_{A=0}^{n+1} x^A \frac{\partial}{\partial x^A}$ on $\wedge_0^{n+1}$. Corresponding null section satisfying (Eq.1) is given by $N = \frac{1}{2(x^0)^2}\{-x^0\partial_0 + \sum_{a=1}^{n+1} x^a \partial_a\}$ and is globally defined. The associated screen distribution $S(T\wedge_0^{n+1})$ is such that $X = \sum_{a=1}^{n+1} X^a \partial_a$ belongs to $S(T\wedge_0^{n+1})$ if and only if $\sum_{a=1}^{n+1} x^a X^a = 0$. Then direct calculations using (Eqs. 4 - 7) leads to the following expressions of the shape operators, $\overset{\star}{A}_\xi X = -PX$ and $A_N X = -\frac{1}{2(x^0)^2} PX$ where P denotes the projection morphism of the tangent bundle $T\wedge_0^{n+1}$ onto $S(T\wedge_0^{n+1})$ with respect to decomposition (Eq.2). Then we get $\sigma = \frac{-n}{\sqrt{2(n+1)}}$, $\overset{\star}{A}_\xi A_N = \frac{1}{2(x^0)^2} \circ P$ and $trA_N = \frac{-n}{2(x^0)^2}$. Finally, since $\mathbb{R}_1^{n+2}$ is flat, the extrinsic scalar curvature of the lightlike cone $\wedge_0^{n+1}$ is given by

$$R = \frac{n^2 - n}{2(x^0)^2}. \tag{17}$$

Remark 4.1

- Observe that this expression is actually independent on the elements defining the screen distribution and depends only upon $\wedge_0^{n+1}$.
- The above screen distribution on $\wedge_0^{n+1}$ is integrable and induces a foliation $\mathcal{F}$. By (Eq.17), the extrinsic scalar curvature R is constant along leaves of $S(T\wedge_0^{n+1})$. Actually, these leaves are n-spheres $(\wedge_0^{n+1} \cap (x^0 = cte))$.
- R vanishes at infinity on $\wedge_0^{n+1}$.

(b) Lightlike Monge hypersurfaces of $\mathbb{R}_q^{n+2}$

Consider a smooth function $F : \Omega \to \mathbb{R}$, where Ω is an open set of $\mathbb{R}^{n+1}$, then

$$M = \{(x^0, \cdots, x^{n+1}) \in \mathbb{R}_q^{n+2}, x^0 = F(x^1, \cdots, x^{n+1})\} \tag{18}$$

is a hypersurface which is called a Monge hypersurface [4]. Such a hypersurface is lightlike if and only if F is a solution of the partial differential equation

$$1 + \sum_{i=1}^{q-1} (F'_{x^i})^2 = \sum_{a=q}^{n+1} (F'_{x^a})^2.$$

The radical distribution is spanned by a global vector field

$$\xi = \frac{\partial}{\partial x^0} - \sum_{i=1}^{q-1} F'_{x^i} \frac{\partial}{\partial x^i} + \sum_{a=q}^{n+1} F'_{x^a} \frac{\partial}{\partial x^a}. \tag{19}$$

Along M consider the constant timelike section $V^* = \frac{\partial}{\partial x^0}$ of $\mathbb{R}_q^{n+2}$. The vector bundle $H^* = span\{V^*, \xi\}$ is nondegenerate on M. The complementary orthogonal vector bundle $S^*(TM)$ to H^* in $T\mathbb{R}_q^{n+2}$ is an integrable screen distribution on M called the natural screen distribution on M. The transversal bundle $tr^*(TM)$ is spanned by $N = -V^* + \frac{1}{2}\xi$ and $\tau(X) = 0 \; \forall X \in \Gamma(TM)$. It follows that $S^\star$(TM) is a globally conformal screen on M with constant positive conformal factor $\frac{1}{2}$, that is $A_N = \frac{1}{2} \overset{\star}{A}_\xi$ [2]. From (Eq.18), the natural parametrization on M is as follows.

$$x^0 = F(v^0, \cdots, v^n), \qquad x^{\alpha+1} = v^\alpha, \;\; \alpha \in \{0, \cdots, n\}.$$

Then the natural frame field on M is globally defined by

$$\frac{\partial}{\partial v^\alpha} = F'_{x^{\alpha+1}} \frac{\partial}{\partial x^0} + \frac{\partial}{\partial x^{\alpha+1}}, \;\; \alpha \in \{0, \cdots, n\}. \tag{20}$$

Now, direct calculations using (Eq.20) and (Eq.19) lead to

$$B_{\alpha\beta} = B(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}) = -\frac{\partial^2 F}{\partial v^\alpha \partial v^\beta}$$

Let $\overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha} = K^\mu_\alpha \frac{\partial}{\partial v^\mu}$. Since $\overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha}$ belongs to $S(TM)$ we have from (Eqs.9),

$$B(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}) = g(\overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}) = \tilde{g}(\overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta})$$

Thus,

$$-\frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} = K^\mu_\alpha \tilde{g}(\frac{\partial}{\partial v^\mu}, \frac{\partial}{\partial v^\beta}) = K^\mu_\alpha \tilde{g}_{\mu\beta}.$$

That is

$$K^\mu_\alpha = -g^{[\mu\beta]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta},$$

and

$$\overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha} = 2A_N \frac{\partial}{\partial v^\alpha} = -g^{[\mu\beta]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \frac{\partial}{\partial v^\mu}. \tag{21}$$

Let $\tilde{F} = (F_{\alpha\beta})$ denote the matrix field with entries $F_{\alpha\beta} = \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta}$.
It follows that

$$\begin{aligned} trA_N = \frac{1}{2} tr \overset{\star}{A}_\xi &= -\frac{1}{2} g^{[\alpha\beta]} g^{[\mu\gamma]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\gamma} \tilde{g}_{\mu\beta} \\ &= -\frac{1}{2} g^{[\alpha\beta]} F^\mu_\alpha \tilde{g}_{\mu\beta} = -\frac{1}{2} trace_g \tilde{F} \end{aligned} \tag{22}$$

Also,

$$\overset{\star}{A}_\xi A_N \frac{\partial}{\partial v^\alpha} = \frac{1}{2} \overset{\star}{A}_\xi \overset{\star}{A}_\xi \frac{\partial}{\partial v^\alpha} \quad = \quad -g^{[\mu\beta]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \overset{\star}{A}_\xi \frac{\partial}{\partial v^\mu}$$
$$= \quad \frac{1}{2} g^{[\mu\beta]} g^{[\gamma\delta]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \frac{\partial^2 F}{\partial v^\mu \partial v^\delta} \frac{\partial}{\partial v^\gamma}.$$

Then,

$$\begin{aligned} tr(\overset{\star}{A}_\xi A_N) &= g^{[\alpha\nu]} \tilde{g}(\overset{\star}{A}_\xi A_N \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\nu}) \\ &= \frac{1}{2} g^{[\alpha\nu]} g^{[\mu\beta]} g^{[\gamma\delta]} \frac{\partial^2 F}{\partial v^\alpha \partial v^\beta} \frac{\partial^2 F}{\partial v^\mu \partial v^\delta} \tilde{g}_{\gamma\nu}. \end{aligned}$$

$$\begin{aligned} tr(\overset{\star}{A}_\xi A_N) &= \frac{1}{2} g^{[\alpha\nu]} F^\mu_\alpha F^\gamma_\mu \tilde{g}_{\gamma\nu} \\ &= \frac{1}{2} trace_g(\tilde{F}^2). \end{aligned} \tag{23}$$

From (Eqs.22,23) and the definition of σ it follows that for lightlike Monge hypersurfaces,

$$R = \frac{1}{2}\Big[[trace_g(\tilde{F})]^2 - trace_g(\tilde{F}^2)\Big]. \tag{24}$$

5 Discussion

Clearly, in case of a canonical polarization of our lightlike hypersurface, we recover results in [3].

We now examine how the operators and induced geometric objects involved in (Eq.15) defining the extrinsic scalar curvature R change with a change in screen distribution. First, note that the local fundamental form B of M is independent of the choice of screen distribution. Hence the mean curvature function σ of M is invariant. Now, starting with a $S(TM)$ with local orthonormal basis $\{W_i\}$, consider another screen distribution $S(TM)'$ with orthonormal basis

$$W_i' = \sum_{j=1}^{n} P_i^j (W_j - \varepsilon_j c_j \xi),$$

where c_i and P_i^j are smooth functions and $\{\varepsilon_j\}$ represents the signature of $\{W_j\}$. Below, we let $W = \sum_{i=1}^n c_i W_i$ and $\rho = \sum_{i=1}^n \varepsilon_i (c_i)^2$ denote characteristic vector and scalar field in respect of this screen change. Then, the following local transformations are derived.

$$\begin{aligned}\eta' &= \eta + \omega,\\ \overset{\star}{A}{}'_\xi &= \overset{\star}{A}_\xi - \mu \otimes \xi\end{aligned}$$

with $\omega = g(W, \cdot)$ and $\mu = B(W, \cdot)$. Also,

$$\begin{aligned}A_{N'}X &= A_N + \sum_i^n \{\varepsilon_i c_i X(c_c) - \tau(X)\varepsilon_i (c_i)^2 + \frac{1}{2}\varepsilon_i (c_i)^2 \mu(X)\\ &\quad - c_i C(X, W_i)\}\xi + \sum_i^n \{c_i(\tau(X) + \mu(X)) - X(c_i)\} W_i\\ &\quad - \sum_i c_i \overset{\star}{\nabla}_X W_i - \frac{1}{2}\rho \overset{\star}{A}_\xi X\end{aligned}$$

for $X \in \Gamma(TM|_{\mathcal{U}})$ and

$$\begin{aligned}\bar{\theta}' &= \bar{\theta} - \frac{1}{2}\rho^2 \bar{R}ic(\xi, \xi) + \bar{R}ic(W, W) - \rho \bar{R}ic(N, \xi)\\ &\quad - \rho \bar{R}ic(\xi, W) + 2\bar{R}ic(W, N).\end{aligned}$$

Finally, the following problem is still open: *Given a triplet* $(M, g, S(TM))$ *with induced extrinsic scalar curvature* R*, how to characterize the set* $\mathcal{S}(R)$ *of screen structures on* M *that preserve* R*?*

Acknowledgment. This work was carried out while I was visiting the IHES (Bures-sur-Yvette, France) within the framework of the program IHES/ Foundation schlumberger. I express my deepest gratitude for hospitality and support.

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