The Geometry of Small Causal Diamonds

G. W. GIBBONS and S. N. SOLODUKHIN

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Mars 2007

IHES/P/07/11

The Geometry of Small Causal Diamonds

G W Gibbons
D.A.M.T.P.,
Cambridge University,
Wilberforce Road, Cambridge CB3 0WA, U.K.

and

S Solodukhin
School of Engineering and Science,
International University Bremen,
P.O. Box 750561, Bremen 28759, Germany and
Institut des Hautes Études Scientifiques,
35, route de Chartres, 91440 Bures-sur-Yvette, France

March 9, 2007

Abstract

The geometry of causal diamonds or Alexandrov open sets whose initial and final events p and q respectively have a proper-time separation τ small compared with the curvature scale is a universal. The corrections from flat space are given as a power series in τ whose coefficients involve the curvature at the centre of the diamond. We give formulae for the total 4-volume V of the diamond, the area A of the intersection the future light cone of p with the past light cone of q and the 3-volume of the hyper-surface of largest 3-volume bounded by this intersection valid to $\mathcal{O}(\tau^4)$. The formula for the 4-volume agrees with a previous result of Myrheim. Remarkably, the iso-perimetric ratio $\frac{3V_3}{4\pi}/(\frac{A}{4\pi})^{\frac{3}{2}}$ depends only on the energy density at the centre and is bigger than unity if the energy density is positive. These results are also shown to hold in all spacetime dimensions. Formulae are also given, valid to next non-trvial order, for causal domains in two spacetime dimensions.

We suggest a number of applications, for instance, the directional dependence of the volume allows one to regard the volumes of causal diamonds as an observable providing a measurement of the Ricci tensor.

1 Introduction

Causal Diamonds, or Alexandrov open sets, play an increasingly important role in quantum gravity, for example in the approach via casual sets [1], in discussions of 'holography', and also of the probability of various observations in eternal inflation models (see [2] for a recent example and references to earlier work). Curiously, however, not very much of a quantitative nature appears to be known about them. The purpose of this note is to embark on a remedy of that situation at least in the case that the diamond is small compared with the curvature scale of the ambient spacetime. In fact the causal structure and volume measure are sufficient to fix the spacetime topology, differential structure, and metric completely [4, 5]. This information can be encoded in a knowledge of the set of small causal diamonds and their *volumes*. In particular, it allows one to extract not only the metric, but the Ricci tensor as well allowing one to formulate the vacuum Einstein equations in a simple way [6]. The volume of causal diamonds plays an important role in studies of eternal inflation where one introduces a single connected *Meta-Universe* [1] and takes the probability for the occurrence of a 'pocket' in the Meta-Universe to be the volume of an appropriate causal diamond.

2 Causal Diamonds

A *causal diamond* or *Alexandrov open set* is a subset of a Lorentzian spacetime $\{M, g\}$ of the form

$$I^+(p) \cap I^-(q)\,, \tag{1}$$

where I^+, I^- denotes chronological future or past respectively. The causal diamond depends only on the conformal class of the Lorentzian metric g but the 4-volume for example

$$V(p,q) = \mathrm{Vol}(I^+(p) \cap I^-(q)) = \int_{I^+(p)\cap I^-(q)} \sqrt{|g|} d^4 x\,, \tag{2}$$

depends upon the metric itself. In Minkowski spacetime $\mathbb{E}^{3,1}$, [2] as long as p is in the chronological past of q, there exists a unique straight line, i.e. time-like geodesic, joining them, and if its proper lengths is τ, then, as pointed out by 't Hooft [3]

$$V(p,q) = \frac{\pi}{24}\tau^4\,. \tag{3}$$

In a general curved spacetime, if p and q are sufficiently close, there will still be a unique time-like geodesic $\gamma(t)$, parametrised by proper time $t \in [-\frac{\tau}{2}, \frac{\tau}{2}]$. joining them as long as p remains in the chronological past of q. 't Hooft's

[1] sometimes called a *Multiverse*

[2] we use signature $-+++$ and MTW/HE curvature conventions throughout

formula (3) will be approximately true but there are corrections involving the curvature evaluated at the mid point $0 = \gamma(0)$.

One way of calculating them is to use Riemann normal coordinates x^μ centred on 0, in which the metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\sigma\nu\tau}(0) x^\sigma x^\tau + \dots \quad . \tag{4}$$

This was done by Myrheim [6] who obtained the result

$$V(p,q) = \frac{\pi}{24} \tau^4 \Big(1 + a\tau^2 R(0) + b R_{\mu\nu}(0) T^\mu T^\nu + \dots \Big) , \tag{5}$$

where T^μ is the time-like vector tangent to the geodesic at the origin γ and with magnitude

$$g_{\mu\nu}(0) T^\mu T^\nu = -\tau^2 . \tag{6}$$

The expansion in τ is thus also expansion in T^μ.

Thus (5) may also be written as

$$V(p,q) = \frac{\pi}{24} \tau^4 \Big(1 + a\tau^2 R(0) + b\tau^2 R_{\hat{0}\hat{0}}(0) + \dots \Big) , \tag{7}$$

where $R_{\hat{0}\hat{0}}(0)$ are the time-time components of the Ricci tensor at the origin evaluated in an orthonormal frame at the origin whose zero leg is aligned with the tangent vector of the geodesic γ.

One may also express T^μ in terms of Synge's *World Function* $\Omega(p,q)$ giving the distance squared between two events p, q and which satisfies

$$g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega = -4\Omega . \tag{8}$$

One has

$$T^\mu = \frac{1}{2} g^{\mu\nu} \partial_\nu \Omega . \tag{9}$$

If p is in the past of q one may define $\tau(p,q) = \sqrt{\Omega(p,q)}$ and (8) becomes the Hamilton-Jacobi equation

$$g^{\mu\nu} \partial_\mu \tau \partial_\nu \tau = 1 . \tag{10}$$

Myrheim did not give the full details of his calculation in Riemann normal coordinates. Indeed the derivation is complicated by the need to find the curvature induced deviation of the light cones from their flat space positions. To our knowledge no other derivation has been given since. Because the coefficients a and b in Myrheim's formula are universal, i.e. valid for *any* spacetime, one should be able to avoid the use of Riemann normal coordinates altogether, but rather to determine the coefficients by considering two special cases. This is what we shall now do. Using the same technique we shall also obtain new results for the area $A(p,q)$ of the intersection of the past and future light cones, and for maximal three-volume $V_3(p,q)$ of any hypersurface which it bounds.

2.1 The Einstein Static Universe

The metric is

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi\big(d\theta^2 + \sin^2\theta\phi^2\big)\,. \tag{11}$$

We take p as $(-\frac{1}{2}\tau, 0, 0, 0)$ and q as $(\frac{1}{2}\tau, 0, 0, 0)$. Since the metric is an unwarped (i.e ultrastatic) product of the unit round three-sphere with time, $R_{\hat{0}\hat{0}}(0) = 0$ and $R = 6$, the value for the unit three-sphere.

The past light cone of q is given by $t = \frac{\tau}{2} - \chi$ and the future light cone of p by $t = -\frac{\tau}{2} + \chi$.

The volume is easily seen to be

$$\begin{aligned} V(p,q) = V(\tau) &= 8\pi \int_0^{\frac{1}{2}\tau} dt \Big(\int_0^{\frac{1}{2}\tau - t} \sin^2\chi d\chi \Big) \\ &= \frac{\pi}{2}(\cos^2\frac{\tau}{2} + \frac{\tau^2}{4} - 1)\,. \end{aligned} \tag{12}$$

The small τ expansion of $V(\tau)$ then can be obtained to arbitrary order,

$$V(\tau) = \frac{\pi\tau^4}{24}\big(1 - \frac{\tau^2}{30} + \ldots\big)\,. \tag{13}$$

This fixes, in our conventions, the constant a to be

$$a = -\frac{1}{180}\,. \tag{14}$$

and agrees up to a sign with the expression given by Myrheim [6].

2.2 de-Sitter spacetime

The metric is

$$ds^2 = -dt^2 + \cosh^2 t\Big(d\chi^2 + \sin^2\chi\big(d\theta^2 + \sin^2\theta\phi^2\big)\Big)\,. \tag{15}$$

We take p as $(t = -\frac{1}{2}\tau, 0, 0, 0)$ and q as $(t = \frac{1}{2}\tau, 0, 0, 0)$ The Ricci scalar $R = 12$ and since $R_{\mu\nu} = 3g_{\mu\nu}$, then $R_{\hat{0}\hat{0}}(0) = -3$.

To obtain the past light cone we introduce conformal time η by

$$d\eta = \frac{dt}{\cosh t}\,, \tag{16}$$

choosing the constant of integration so that at $t = 0$, $\eta = 0$, one finds that

$$\tan(\frac{\pi}{4} + \frac{\eta}{2}) = e^t\,. \tag{17}$$

Further useful relations are

$$\sinh t = \tan\eta\,, \qquad \cosh t = \frac{1}{\cos\eta}\,. \tag{18}$$

The metric is now

$$ds^2 = \frac{1}{\cos^2\eta}\Big(-d\eta^2 + d\chi^2 + \sin^2\chi\big(d\theta^2 + \sin^2\theta\phi^2\big)\Big)\,. \tag{19}$$

Define N by

$$\cosh\frac{\tau}{2} = \frac{1}{\cos\frac{N}{2}}\,, \tag{20}$$

then p is at $(\eta = -\frac{1}{2}N, 0, 0, 0)$ and q is at $(\eta = \frac{1}{2}N, 0, 0, 0)$ The past light cone of q is given by $\eta = \frac{N}{2} - \chi$ and the future light cone of p by $\eta = -\frac{N}{2} + \chi$.

The volume is easily seen to be

$$\begin{aligned} V(p,q) &= V(\tau) = 8\pi\int_0^{\frac{1}{2}N}\frac{d\eta}{\cos^4\eta}\Big(\int_0^{\frac{1}{2}N-\eta}\sin^2\chi d\chi\Big) \\ &= \frac{4}{3}\pi(\cos^2\frac{N}{2} - 2\ln\cos\frac{N}{2} - 1) \\ &= \frac{4}{3}\pi(\frac{1}{\cosh^2\frac{\tau}{2}} + 2\ln\cosh\frac{\tau}{2} - 1)\,. \end{aligned} \tag{21}$$

This function can be expanded to any order in τ. We are however interested in the first few terms,

$$V(\tau) = \frac{\pi}{24}\tau^4\big(1 - \frac{\tau^2}{6} + \ldots\big)\,. \tag{22}$$

This gives the value

$$b = \frac{1}{30} \tag{23}$$

for the coefficient b, which agrees, up to a sign convention, with the result of Myrheim who uses the opposite signature to us. His metric is thus minus our metric and hence, although his Ricci tensor is the same as ours, his Ricci scalar has the opposite sign.

On the other hand, in both cases the volume has an interesting behavior for large τ. One has that

$$V(\tau) = \frac{4}{3}\pi\tau + O(1) \tag{24}$$

for de Sitter space-time and

$$V(\tau) = \frac{\pi}{2}\tau^2 + O(1) \tag{25}$$

for the Einstein Static Universe. Notice that in both cases the leading term grows much slower than τ^4.This presumably reflects the fact that both space-times satisfy the null convergence condition, $R_{\mu\nu}l^\mu l^\nu \geq 0$ for all light-like vectors l^μ.

3 Area

The area $A(p,q)$ of the intersection $\dot{I}^+(p) \cap \dot{I}^-(q)$ of the future light cone of p, $\dot{I}^+(q) = \partial I^+(p)$ with the past light cone of q, $\dot{I}^+(q) = \partial I^+(p)$ is also given by a universal formula which to next to lowest order might be expected to depend on both the Ricci scalar R and $R_{\hat{0}\hat{0}}$, the time-time component of the Ricci tensor.

In both our examples the intersection is on the surface $t = 0$ and we have

$$A(p,q) = 4\pi \sin^2(\frac{1}{2}N) . \tag{26}$$

For the Einstein Static Universe one has $N = \tau$ and hence

$$\begin{aligned} A(p,q) &= 4\pi \sin^2 \frac{\tau}{2} \\ &= \tau^2 \big(1 - \frac{1}{12}\tau^2 + \ldots\big) . \end{aligned} \tag{27}$$

For de-Sitter spacetime we have a relation $\cos(N/2) = 1/\cosh(\tau/2)$ and thus

$$\begin{aligned} A(p,q) &= 4\pi \tanh^2(\frac{\tau}{2}) \\ &= \pi\tau^2 \big(1 - \frac{1}{6}\tau^2 + \ldots\big) \end{aligned} \tag{28}$$

These imply that in general

$$A(p,q) = \text{Area}(\dot{I}^+(p) \cap \dot{I}^-(q)) = \pi\tau^2 \big(1 - \frac{1}{72}R\tau^2 + \ldots\big) . \tag{29}$$

Thus the area $A(p,q)$, unlike the 4-volume $V(p,q)$, contains no directional information. Notice, however, that the area is given by different functions (26) and (28) of τ in these two cases. This is due to the presence of powers of the Ricci tensor in the τ expansion of the area which show up in the higher order terms.

4 Three-volume

There are infinitely many space-like hypersurfaces having the intersection $\dot{I}^+(p) \cap \dot{I}^-(q)$ of the future light cone of p with the past light cone of q as their boundary. Among them, provided p and q are sufficiently close, there is one with maximal volume [3]. The maximal value of this 3- volume $V_3(p,q)$ should also be given by a universal formula.

Our two examples are time symmetric and therefore the hypersurface of maximal volume has $t = 0$. Thus

$$\begin{aligned} V_3(p,q) &= 4\pi \int_0^{\frac{N}{2}} d\chi \sin^2 \chi \\ &= \pi(N - \sin N) . \end{aligned} \tag{30}$$

[3] we remind the reader that spacelike hypersurfaces in Lorentzain spacetimes can have arbitrarily small volumes and hence the concept of a minimal spacelike hypersurface is not well defined

Using the already established relations between N and τ we have that

$$V_3(\tau) = \pi(\tau - \sin\tau) \tag{31}$$

in the case of the Einstein Universe and

$$V_3(\tau) = 2\pi\left(\arctan(\sinh\frac{\tau}{2}) - \frac{\sinh\frac{\tau}{2}}{\cosh^2\frac{\tau}{2}}\right) \tag{32}$$

in the case of de-Sitter space-time. Therefore for the Einstein Static Universe,

$$V_3(p,q) = \frac{\pi}{6}\tau^3\big(1 - \frac{1}{20}\tau^2 + \ldots\big)\,, \tag{33}$$

while for de-Sitter spacetime

$$V_3(p,q) = \frac{\pi}{6}\tau^3\big(1 - \frac{7}{40}\tau^2 + \ldots\big)\,. \tag{34}$$

Therefore, in general,

$$\begin{aligned} V_3(p,q) &= \frac{\pi}{6}\tau^3\big(1 - \frac{1}{120}R\tau^2 + \frac{1}{40}R_{\hat{0}\hat{0}}\tau^2 + \ldots\big) \\ &= \frac{\pi}{6}\tau^3\big(1 - \frac{1}{120}R\tau^2 + \frac{1}{40}R_{\mu\nu}T^\mu T^\nu \ldots\big)\,. \end{aligned} \tag{35}$$

5 Energy Conditions

We have shown that

$$\begin{aligned} V(p,q) &= \frac{\pi}{24}\tau^4\Big(1 + \frac{1}{180}(Rg_{\mu\nu} + 6R_{\mu\nu})T^\mu T^\nu + \ldots\Big) \\ &= \frac{\pi}{24}\tau^4\Big(1 + \frac{4\pi G}{45}(3T_{\mu\nu} - 2Tg_{\mu\nu})T^\mu T^\nu + \ldots\Big)\,. \end{aligned} \tag{36}$$

In the last line we have used the Einstein equations

$$R_{\mu\nu} = 8\pi G\big(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\big)\,, \tag{37}$$

where $T_{\mu\nu}$ is the energy-momentum tensor.

As an illustration rather than a check, one may substitute the energy momentum tensor of an inflating universe

$$T_{\mu\nu} = -V(\phi)g_{\mu\nu}\,, \tag{38}$$

where $V(\phi)$ is the potential energy of an inflation field ϕ to get

$$V(p,q) = \frac{\pi}{24}\tau^4\Big(1 - \frac{4\pi GV(\phi)}{9}\tau^2 + \ldots\Big) \tag{39}$$

which of course agrees with the de-Sitter results if one uses the relations

$$\Lambda = 8\pi G V(\phi) = 3\,. \tag{40}$$

In the presence of a perfect fluid whose velocity is aligned with the time-like tangent vector of the geodesic γ , one finds

$$V(p,q) = \frac{\pi}{24}\tau^4\Big(1 + \frac{4\pi G}{45}(\rho + 6P)\tau^2 + \dots\Big)\,, \tag{41}$$

where ρ is the energy density and P the pressure of the fluid. If both are positive, the causal diamond has a larger volume, for fixed proper time duration τ than it would have in flat spacetime. By contrast for large negative pressures, as during inflation, the volume is smaller than it would be in flat spacetime.

6 Isometric Inequalities

In flat Euclidean space $\mathbb{E}^3$ the 3-ball maximises volume enclosed for fixed surface area. It is thus of interest to examine the iso-perimetric ratio

$$\frac{3V}{4\pi}/(\frac{A}{4\pi})^{\frac{3}{2}}\,. \tag{42}$$

One has

$$\begin{aligned} \frac{3V_3(p,q)}{4\pi}/(\frac{A(p,q)}{4\pi})^{\frac{3}{2}} &= \Big(1 + \frac{1}{80}R\tau^2 + \frac{1}{40}R_{\hat{0}\hat{0}}\tau^2 + \dots\Big) \\ &= \Big(1 + \frac{1}{40}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})T^\mu T^\nu + \dots\Big) \\ &= \Big(1 + \frac{\pi G}{5}T_{\mu\nu}T^\mu T^\nu + \dots\Big)\,, \end{aligned} \tag{43}$$

where in the last line we have used the Einstein equations. Remarkably, the iso-perimetric ratio involves just the energy density at the origin O of the diamond and exceeds unity if the Weak Positive Energy Condition holds.

Relation of a similar type which involves the 4-volume $V(p,q)$ is

$$\frac{24}{\pi}V(p,q)/(\frac{6V_3(p,q)}{\pi})^{\frac{4}{3}} = (1 + \frac{1}{180}R\tau^2 + \dots)\,. \tag{44}$$

As we see the directional part cancels in (44) so that the iso-perimetric ratio (44), at least at the given order in τ, is direction independent. Whether this ratio exceeds unity depends only on the sign of the Ricci scalar, similarly to the iso-perimetric ratio for Euclidean manifolds.

7 Measuring the Ricci and the Riemann tensors

The formula for $V(p,q)$ should be compared with that for the volume $V(r)$ of a 4-ball of radius r in a Riemannian manifold

$$V(r) = \frac{\pi^2}{2} r^4 \Big(1 - \frac{1}{36} R r^2 + \dots\Big) . \tag{45}$$

There is an important difference in that the formula for the volume of a causal diamond is directional in character since it involves not only the Ricci scalar R, but the Ricci tensor $R_{\mu\nu}$. By varying the points p and q and hence varying T^μ one could, if one could measure $V(p,q)$, determine the whole Ricci tensor for vectors within the light cone. Since it is a bilinear function on tangent vectors, its value for space-like vectors is then given by continuity and, assuming continuity, it is thus uniquely determined. be unique.

Indeed, given a causal structure and some measure of volume and proper time, one could use the formula to *define* the Ricci tensor. This might be useful in approaches to quantum gravity based on Causal Sets or Directed Graphs.

The same remarks apply to the three-volume $V_3(p,q)$. However, as remarked earlier, the area $A(p,q)$ would only allow one to measure the Ricci scalar R.

The formula (45) can be generalised. Let's take an n-dimensional cycle Σ_n and consider a tube $T(\Sigma_n, r)$ of radius r in the direction orthogonal to Σ. The volume of this tube has a small r expansion,

$$V(T) = V(B_{4-n}(r)) \int_{\Sigma_n} \left(1 + c_n \sum_{i,j=1}^{4-n} R_{ijij} r^2 + \dots \right) , \tag{46}$$

where $V(B_{4-n}(r))$ is volume of $(4-n)$-dimensional ball of radius r in flat space and $R_{ijij} = R_{\alpha\mu\beta\nu} n_i^\alpha n_i^\beta n_j^\mu n_j^\nu$, where n_i^μ, $i = 1..4-n$ is a set of orthonormal vectors orthogonal to Σ_n. When $n = 0$ (Σ is just a point) one has that $\sum_{ij} R_{ijij} = R$ and (46) becomes (45).

In the Lorentzian signature it is straightforward to introduce an analogous construction of a 'causal tube' for cycle Σ_n lying entirely in a space-like hypersurface. The causal tube then can be defined as the union of causal diamonds of 'size' τ centered at every point of Σ. The causal structure of the tube is determined by the field of time-like vectors T^μ defined everywhere on Σ_n. To leading order in τ the volume of this causal tube is a product of volume $V_{(4-n)}(\tau)$ of $(4-n)$-dimensional causal diamond and the volume of Σ_n,

$$V(T) = V_{(4-n)}(\tau) \mathrm{Vol}(\Sigma_n) .$$

The higher order in τ corrections then involve components of the Riemann tensor in the directions orthogonal to Σ_n similar to (46),

$$V(T) = V_{(4-n)}(\tau) \int_{\Sigma_n} \left(1 + \sum_{i,j=0}^{3-n} (a_n R_{ijij} + b_n R_{i0i0}) \tau^2 + \dots \right) . \tag{47}$$

The set of vectors n_i, $i = 0..(3-n)$ orthogonal to Σ_n includes the time-like vector $n_0^\mu = \tau^{-1} T^\mu$. The exact values of coefficients a_n and b_n are to be determined.

Obviously, using these tubes one could measure components of the Riemann tensor including those in the space-like directions.

In order to check the formula (46) let us consider the Euclidean Schwarzschild metric and take horizon sphere as the cycle, $n = 2$ in this case. The Schwarzschild metric can be brought to the form

$$ds^2 = d\rho^2 + g(\rho) d\phi^2 + r^2(\rho) d\omega^2 \,, \tag{48}$$

where coordinate ϕ has period 2π and functions $g(\rho)$ and $r(\rho)$ are given by expansion

$$\begin{aligned} g(\rho) &= \rho^2 - \frac{1}{3a^2}\rho^4 + O(\rho^6) \,, \\ r(\rho) &= a + \frac{\rho^2}{4a} + O(\rho^4) \,, \end{aligned} \tag{49}$$

where a is radius of the horizon.

The volume of tube T of radius r (in the direction orthogonal to horizon Σ) is given by expression

$$\begin{aligned} V(T) &= 4\pi a^2 2\pi \int_0^r d\rho \sqrt{g(\rho)} r^2(\rho) \\ &= \pi r^2 \text{Area}(\Sigma)(1 + \frac{r^2}{6a^2} + \ldots) \,. \end{aligned} \tag{50}$$

Taking into account that for the Schwarzschild metric one has $\frac{1}{a^2} = \frac{1}{2}\sum_{i,j=1}^{2} R_{ijij}$, where the curvature components are calculated at the horizon, we find that

$$V(T) = \pi r^2 \int_\Sigma \left(1 + \frac{1}{12} \sum_{i,j} R_{ijij} r^2 + \ldots \right) \,, \tag{51}$$

in agreement with general formula (46). Combining (45) and (51) it seems that value of coefficient $c_n = \frac{2n-1}{36}$ fits nicely both cases.

8 Higher Order Results

In this section we obtain the formula for the volume, or area, of a causal diamond in a two-dimensional spacetime which are valid to order τ^6. We start with a two-dimensional metric in general conformally flat form

$$ds^2 = a^2(x, \eta)(-d\eta^2 + dx^2) \,. \tag{52}$$

The volume of causal diamond in this metric takes the form

$$V(\tau) = \int_0^{N/2} d\eta \int_{\eta - \frac{N}{2}}^{\frac{N}{2} - \eta} dx [a^2(x, \eta) + a^2(x, -\eta)] \,, \tag{53}$$

where N should be related to proper time τ measured along the time-like geodesic.

We consider several simple particular cases.

Case 1. The function $a(x,\eta)$ is function of only time, $a(\eta)$. Then we can change the time variable to $t=\int d\eta\, a(\eta)$ that measures the proper time along the time-like geodesic so that

$$N=\int_{-\tau/2}^{\tau/2}\frac{dt}{a(t)}\,.$$

The volume (53) is symmetric function of τ, i.e. $V(-\tau)=V(\tau)$, so that only even powers of τ appear in the expansion of $V(\tau)$ in powers of τ. The integral over x in (53) may be evaluated to give

$$V(N)=\int_0^{N/2}d\eta(N-2\eta)[a^2(\eta)+a^2(-\eta)]\,. \tag{54}$$

The τ-derivative of volume can be expressed in terms of functions of t only, not involving coordinate η,

$$\partial_\tau V(\tau)=\frac{1}{2}[\frac{1}{a(\frac{\tau}{2})}+\frac{1}{a(-\frac{\tau}{2})}]\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}dt\, a(t)\,. \tag{55}$$

Since only even powers of τ will appear in the expansion of the volume anyway, let us consider $a(t)$ to be even function of t given by a small t expansion

$$a(t)=1+a_1t^2+a_2t^4+\dots$$

Substituting this expansion into equation (55) we find that

$$\partial_\tau V=2(\frac{\tau}{2})-\frac{4}{3}a_1(\frac{\tau}{2})^2+[\frac{4}{3}a_1^2-\frac{8}{5}a_2](\frac{\tau}{2})^3+\dots \tag{56}$$

The scalar curvature of metric (52) has the following expansion

$$R(t)=4a_1+(24a_2-4a_1^2)t^2+\dots \tag{57}$$

The terms in the expansion of the volume then can be expressed in terms of curvature and its derivatives evaluated in the center of the diamond,

$$V(\tau)=2(\frac{\tau}{2})^2-\frac{1}{6}R\,(\frac{\tau}{2})^4+\frac{1}{45}(R^2-\frac{1}{2}R''_{\eta\eta})(\frac{\tau}{2})^6+\dots\,, \tag{58}$$

where we have used the fact that in the center of the diamond $R''_{tt}=R''_{\eta\eta}$.

Case 2. Function $a(\eta,x)$ depends only on coordinate x and does not depend on time η. Normalising $a(x=0)=1$ in the center of the diamond, we have that $N=\tau$ and we find

$$V(\tau)=2\int_0^{\tau/2}d\eta\int_{\eta-\tau/2}^{-\eta+\tau/2}dx\ a^2(x) \tag{59}$$

for the volume. Representing function $a(x)$ in the vicinity of the center of the diamond in terms of expansion

$$a(x) = 1 + b_1 x^2 + b_2 x^4 + \dots$$

we find the following

$$V(\tau) = 2(\frac{\tau}{2})^2 + \frac{2b_1}{3}(\frac{\tau}{2})^4 + \frac{2}{15}(2b_2 + b_1^2)(\frac{\tau}{2})^6 + \dots \tag{60}$$

expansion for the volume. The scalar curvature this time has expansion

$$R(x) = -4b_1 + (-24b_2 + 20b_1^2)x^2 + \dots \tag{61}$$

so that the coefficients in expansion (60) can be expressed in terms of values of the curvature and its derivatives in the center of the diamond,

$$V(\tau) = 2(\frac{\tau}{2})^2 - \frac{1}{6}R(\frac{\tau}{2})^4 + \frac{1}{45}(R^2 - \frac{1}{2}R''_{xx})(\frac{\tau}{2})^6 + \dots \tag{62}$$

In principle, we can not exclude the appearance of the term $R''_{\eta x}$ in the τ^6 term. The presence of this term can not be detected from the above two cases. So that we need to consider one more case.

Case 3. Choose function $a(x, \eta)$ in the form

$$a(x, \eta) = 1 + cx^3\eta\,. \tag{63}$$

Time-like curve $x = 0$ is still a geodesic in this case so that η measures the proper time along this geodesic, and we have that $N = \tau$. The Ricci scalar has expansion

$$R = 12c\,\eta\,x + \dots$$

in this case so that $R''_{\eta x} = 12c$.

We find that volume

$$V(\tau) = 2\int_0^{\tau/2} d\eta \int_{\eta-\tau/2}^{-\eta+\tau/2} dx a^2(x,\eta) = 2(\frac{\tau}{2})^2 + \frac{c^2}{630}(\frac{\tau}{2})^{10} + \dots \tag{64}$$

has expansion with vanishing term τ^6. This indicates that term $R''_{\eta x}$ does not appear in this order.

Combining all these cases the expansion of the volume can be presented in the following covariant form

$$V(\tau) = 2(\frac{\tau}{2})^2 - \frac{1}{6}R(\frac{\tau}{2})^4 + \frac{1}{180}[-g_{\alpha\beta}R^2 + \frac{1}{4}(g_{\alpha\beta}\nabla^2 R - \nabla_\alpha\nabla_\beta R)]T^\alpha T^\beta(\frac{\tau}{2})^4\,, \tag{65}$$

where we have used that $\nabla^2 R = -R''_{\eta\eta} + R''_{xx}$ in the center of the diamond.

In the two-dimensional case it is straightforward to calculate the volume (length) of the maximal space-like hypersurface that bounds the intersection of the future light cone of p with the past light cone of q. This volume, $V_1(p,q)$ is the two-dimensional analog of $V_3(p,q)$ considered above. We omit the details of this calculation that follows same lines as the above calculation of the volume of the causal domain. Here is the final result valid up τ^5 order,

$$V_1(p,q) = \tau - \frac{1}{6}R(\frac{\tau}{2})^2 + \frac{1}{192}[-g_{\alpha\beta}R^2 + \frac{2}{5}(g_{\alpha\beta}\nabla^2 R - \nabla_\alpha\nabla_\beta R)]T^\alpha T^\beta(\frac{\tau}{2})^3 \,. \quad (66)$$

The formulae (65) and (66) give us an idea on how complicated can be the next to non-trivial order terms in higher dimensions where the number of possible combinations built out of curvature and its derivatives is larger than in two-dimensions. This also allows us to guess the possible structure of such terms in higher dimensions. The direct calculation of the higher order terms in four spacetime dimensions, however, requires more efforts and will be reported elsewhere.

9 Higher Dimensions

In this section we give some results valid in arbitrary dimension d. We follow the strategy outlined in the first part of the paper: consider two particular cases of the d-dimensional Einstein Static Universe and d-dimensional de-Sitter spacetime. These two cases help us to fix the coefficients in the small τ expansion for the volume of the causal daimond just in the same way as it was demonstrated earlier in this paper for dimension $d = 4$.

The metric of d-dimensional Einstein Static Universe is

$$ds^2 = -dt^2 + d\chi^2 + \sin^2\chi d\omega^2_{S_{d-2}} \quad (67)$$

and the metric of d-dimensional de-Sitter spacetime is

$$ds^2 = -dt^2 + \cosh^2 t(d\chi^2 + \sin^2\chi d\omega^2_{S_{d-2}}) \,, \quad (68)$$

where $d\omega^2_{S_{d-2}}$ is standard metric on d-dimensional sphere of unite radius.

The volume of the causal diamond. For metric (67) the volume of the causal diamond is given by

$$V_{\mathrm{Einst}} = 2Vol(S_{d-2})\int_0^{\tau/2} dt(\int_0^{\tau/2-t} \sin^{d-2}\chi d\chi) \,, \quad (69)$$

where $Vol(S_{d-2})$ is volume of d-sphere of unite radius. The volume in the case of de Sitter spacetime is given by expression

$$V_{\mathrm{dS}} = 2Vol(S_{d-2})\int_0^{N/2} \frac{d\eta}{\cos^d\eta}(\int_0^{N/2-\eta} \sin^{d-2}\chi d\chi) \,, \quad (70)$$

the time-like coordinate η is defined by relation $\cos\eta = 1/\cosh t$.

Expanding in both cases the volume in powers of τ and taking into account that $N = \tau(1 - \frac{1}{24}\tau^2 + \ldots)$ in the case of de Sitter spacetime we find that

$$V_{\rm Einst} = V_{\rm flat}(\tau)\left(1 - \frac{d(d-1)(d-2)}{24(d+2)(d+1)}\tau^2 + \ldots\right), \quad (71)$$

$$V_{\rm dS} = V_{\rm flat}(\tau)\left(1 - \frac{d(d-1)}{12(d+2)}\tau^2 + \ldots\right), \quad (72)$$

where

$$V_{\rm flat}(\tau) = Vol(S_{d-2})\frac{1}{(2n+1)(n+1)}(\frac{\tau}{2})^{2n+2}$$

is volume of the causal diamond in d-dimensional flat spacetime.

We have to take into account that $R = (d-1)(d-2)$ and $R_{00} = 0$ in the case of metric (67) and $R = d(d-1)$ and $R_{00} = -(d-1)$ in the case of metric (68). Combined with this the equations (71) and (72) are presented in the form

$$V = V_{\rm flat}(\tau)\left(1 - \frac{d}{24(d+1)(d+2)}R\tau^2 + \frac{d}{24(d+1)}R_{\hat{0}\hat{0}}\tau^2 + \ldots\right). \quad (73)$$

This gives us a formula for the volume of the causal diamond valid for any dimension d.

Volume of the maximal spacelike hypersurface. The volume of the spacelike hypersurface (at $t = 0$) for both spacetimes is

$$V_{d-1}(\tau) = Vol(S_{d-2})\int_0^{\frac{N}{2}} d\chi \sin^{d-2}\chi \quad (74)$$

where $N = \tau$ for the Einstein Static Universe and $N = \tau(1 - \frac{1}{24}\tau^2 + \ldots)$ for de Sitter spacetime. The calculation is straightforward and here is the result

$$V_{\rm Einst} = V^{(d-1)}_{\rm flat}(\tau)\left(1 - \frac{(d-2)(d-1)}{24(d+1)}\tau^2 + ..\right), \quad (75)$$

$$V_{\rm dS} = V^{(d-1)}_{\rm flat}(\tau)\left(1 - \frac{(d-1)(2d-1)}{24(d+1)}\tau^2 + ..\right), \quad (76)$$

where $V^{(d-1)}_{\rm flat}(\tau) = \frac{1}{d-1}Vol(S_{d-2})(\frac{\tau}{2})^{d-1}$.

These two expressions help us to determine the coefficients in the expansion of the $(d-1)$-volume

$$V^{(d-1)}(\tau) = V^{(d-1)}_{\rm flat}(\tau)\left(1 - \frac{1}{24(d+1)}R\tau^2 + \frac{(d-1)}{24(d+1)}R_{00}\tau^2 + \ldots\right). \quad (77)$$

The Area. The area $A(p,q)$ of the intersection $\dot{I}^+(p) \cap \dot{I}^-(q)$ is given by expression

$$A = Vol(S_{d-2}) \sin^{d-2} \frac{N}{2}\,, \tag{78}$$

where $N = \tau$ for the Einstein Static Universe and $N = \tau(1 - \frac{1}{24}\tau^2 + \ldots)$ for de Sitter spacetime. Thus, we get the expansion

$$\begin{aligned} A_{\text{Einst}}(\tau) &= A_{\text{flat}}(\tau)\left(1 - \frac{(d-2)}{24}\tau^2 + \ldots\right), &(79)\\ A_{\text{dS}}(\tau) &= A_{\text{flat}}(\tau)\left(1 - \frac{(d-2)}{12}\tau^2 + \ldots\right), &(80)\end{aligned}$$

where $A_{\text{flat}}(\tau) = Vol(S_{d-2})(\frac{\tau}{2})^{d-2}$, so that we have in terms of the curvature

$$A(\tau) = A_{\text{flat}}\left(1 - \frac{1}{24(d-1)}R\tau^2 + \frac{(d-4)}{12(d-1)}R_{00}\tau^2 + \ldots\right). \tag{81}$$

Notice that the directional component in the expansion disappears only in dimension $d = 4$.

The iso-perimetric ratios. The iso-perimetric ratio can be now calculated,

$$\begin{aligned} &V_{(d-1)}/A^{\frac{d-1}{d-2}}\,(A^{\frac{d-1}{d-2}}/V_{(d-1)})_{\text{flat}}\\ &= 1 + \frac{1}{8(d-2)(d+1)}R\tau^2 + \frac{1}{4(d-2)(d+1)}R_{00}\tau^2 + \ldots\\ &= 1 + \frac{1}{4(d-2)(d+1)}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)T^\mu T^\nu + \ldots\\ &= 1 + \frac{2\pi G}{(d-2)(d+1)}T_{\mu\nu}T^\mu T^\nu + \ldots \end{aligned} \tag{82}$$

This extends the results obtained earlier in this paper for $d = 4$ to arbitrary dimension and demonstrates that the iso-perimetric ratio always involves the energy density in the center of the causal diamond. It may be that this dependence is a consequence of the Raychaudhuri equation. It would be interesting to investigate this possibility further [4].

The other iso-perimetric ratio involves the d-volume of the causal diamond and the maximal (d-1)-volume

$$V_d/V_{d-1}^{\frac{d}{d-1}}\,(V_{d-1}^{\frac{d}{d-1}}/V_d)_{\text{flat}} = \left(1 + \frac{d}{8(d+1)(d-1)(d+2)}R\tau^2 + \ldots\right). \tag{83}$$

This indicates that the directional component vanishes in this ratio universally in any dimension d.

[4]The first author thanks Thibault Damour for suggesting that this result on the isoperimetric ratio might be dimension independent.

10 Conclusion

In this paper we have provided universal formulae for the volume and other geometric quantities of small causal diamonds in terms of the local values of the Ricci tensor and its derivatives correcting the 't Hooft's flat space values. In all spacetime dimensions, our corrections are valid to quadratic order in the duration τ of the causal diamond. In two spacetime dimensions we are able to work to quartic order in τ. Going to fourth order and beyond in higher than two spacetime dimensions appears be rather more challenging because of the number of allowed terms that may contribute.

In general the geometry of causal diamonds turns out to be related to the distribution of energy and momentum in a non-obvious and in general directional fashion. However some general trends may be observed. For instance, one striking result was the behaviour of one of the the isoperimetric ratios which depends on just the local energy density in all spacetime dimensions. In the case of the the other isoperimetric ratio there is no directional behaviour in all spacetime dimensions. It is hoped that these results will contribute to a more quantitative understanding of holography and of probabilities in eternal inflationary models.

11 Acknowledgements

The first author would like to thank Jan Myrheim and Raphael Sorkin for helpful discussions at a very early stage of this work. Part of it was carried out at the IHÉS. Both authors would like to thank Thibault Damour and the director Jean Pierre Bourguignon, for their hospitality during our stay at IHÉS.

References

[1] R. D. Sorkin, Causal sets: Discrete gravity, [arXiv:gr-qc/0309009].

[2] R. Bousso, R. Harnik, G. D. Kribs and G. Perez, Predicting the cosmological constant from the causal entropic principle, [arXiv:hep-th/0702115].

[3] G. 't Hooft, Quantum gravity: a fundamental problem and some radical ideas", in *Recent Developments in Gravitation* (Proceedings of the 1978 Cargese Summer Institute) edited by M. Levy and S. Deser (Plenum, 1979).

[4] S W Hawking, A King and P McCarthy, A new topology for curved spacetime which incorporates the causal, differential and conformal structures *J Math Phys* **17** (1976) 174.

[5] D. Malament, The class of continuous timelike paths determines the topology of spacetime *J Math Phys* **18** (1977) 1399.

[6] J Myrheim, Statistical Geometry, CERN preprint TH.25 38-CERN (1978).