Fractal structure of the block-complexity function

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Abstract

We demonstrate that the block-complexity function for words from 3-letter and 4-letter alphabets exhibits a fractal structure. The resulting fractals have dimensions approximately equal to 1.892 and 1.953 respectively. We visualize approximations of the corresponding fractals using sequences of length 6 and 5 respectively. We note that a similar fractal structure has been established recently for the block-complexity function for words from a 2-letter alphabet, using a different terminology. In this case, the resulting fractal has dimension approximately equal to 1.584.

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I. INTRODUCTION

In the modern theory of dynamical systems, there is a standard arsenal of concepts and techniques for studying the complexity of arbitrary symbolic sequences with elements from a finite alphabet.

In the recent paper [4], the authors demonstrate how a seemingly simple combinatorial construction leads to a fractal reminiscent of the Sierpinski Gasket. They call their fractal $K2$ and they show that its dimension is equal to the dimension of the Sierpinski Gasket, namely $\frac{\ln 3}{\ln 2}$. Their combinatorial construction employs binary strings of length m, arranged in $2 \times m$ matrices and for each pair of sequences A and B they define the *complementary pairing number* as being equal to $\tilde{p}(A,B) = 4 - p(A,B)$, where $p(A,B)$ is the *pairing number* associated to A and B. Note that the pairing number $p(A,B)$ is the number of distinct columns in the $2 \times m$ with A as its first row and B as its second row. Since $\tilde{p}(A,B)$ can take on the values $0, 1, 2, 3$, the authors use a color-coding of the complementary pairing number to draw their fractal $K2$, in fact they use sequences of length $m = 9$, which already give a quite accurate approximation of the limit. The authors generalize their construction to higher dimensions, by considering K binary strings and they calculate the dimensions of these higher dimensional fractals to be equal to $\frac{\ln(2^K - 1)}{\ln 2}$.

Upon examining more closely the combinatorial construction leading to the $K2$ fractal, one sees that this construction essentially embodies the definition of the block-complexity function, see [1] and [2] for instance.

II. CONSTRUCTION OF THE K3 AND K4 FRACTALS

In this paper we generalize the construction of [4] in another direction, namely we use ternary and quaternary sequences, instead of binary sequences. For simplicity, we use the 3-letter alphabet $\{-1, 0, +1\}$ and the 4-letter alphabet $\{-1, 0, +1, +2\}$. Since there are now $3^2 = 9$ and $4^2 = 16$ possible columns respectively, the *ternary and quaternary complementary pairing numbers* are now defined as

$$\tilde{p}_3(A,B) = 9 - p_3(A,B) \text{ and } \tilde{p}_4(A,B) = 16 - p_4(A,B)$$

where $p_3(A,B)$ and $p_4(A,B)$ are the *ternary and quaternary pairing numbers* associated to the sequences A and B (which are defined according to the prototypical binary pairing numbers in [4]). Note that the ternary complementary pairing number takes on the nine values $0, \ldots, 8$ and the quaternary complementary pairing number takes on the sixteen values $0, \ldots, 15$.

Following [4] we call K3 and K4 the sets that arise via the ternary and quaternary complementary pairing numbers, in the same way that K2 arises via the binary complementary pairing numbers.

We will now use the box-counting method to compute the fractal dimension of the freedom regions of the fractals K3 and K4. These computations use the Stirling numbers of the second kind, see [3], for instance.

The dimension D_3 of the freedom region of the K3 fractal is calculated via the Stirling numbers of the second kind

$$9!S(m,9) = 9 + 9^m - 9\,8^m + 36\,7^m - 84\,6^m + 126\,5^m - 126\,4^m + 84\,3^m - 36\,2^m$$

and it is found to be equal to

$$D_3 = \lim_{m\to\infty} \frac{3^{2m} - 9!S(m,9)}{3^{2m}} = \frac{3\ln 2}{\ln 3} \equiv 1.892789260.$$

The dimension D_4 of the freedom region of the K4 fractal is calculated via the Stirling numbers of the second kind $16!S(m,16) =$

$$-16 + 16^m - 16\,15^m + 120\,14^m - 560\,13^m + 1820\,12^m - 4368\,11^m + 8008\,10^m - 11440\,9^m +$$

$$+12870\,8^m - 11440\,7^m + 8008\,6^m - 4368\,5^m + 1820\,4^m - 560\,3^m + 120\,2^m$$

and it is found to be equal to

$$D_4 = \lim_{m\to\infty} \frac{4^{2m} - 16!S(m,16)}{4^{2m}} = \frac{\ln 3 + \ln 5}{2\ln 2} \equiv 1.953445298.$$

III. PICTURES OF $K3$ AND $K4$

1. $K3$

We produced a picture of $K3$ (see appendix) using ternary sequences of length $m = 6$, so we had to compute $3^6 \times 3^6 = 531,441$ complementary pairing numbers. The distribution of these ternary complementary pairing numbers for $m = 6$ is as follows:

value of $\tilde{p}_3(A,B)$	0	1	2	3	4	5	6	7	8
# elements for $m = 6$	0	0	0	60,480	226,800	196,560	45,360	2,232	9

.

We remark that since there are only six non-zero values in the above table, we only need six colors to color-code the $K3$ fractal.

2. $K4$

We produced a picture of $K4$ (see appendix) using quaternary sequences of length $m = 4$, so we had to compute $4^4 \times 4^4 = 65,536$ quaternary complementary pairing numbers. The distribution of these complementary pairing numbers for $m = 4$ is as follows:

value of $\tilde{p}_4(A, B)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
# elements for $m = 4$	0	0	0	0	0	0	0	0	0	0	0	0	43,680	20,160	1,680	16

We remark that since there are only four non-zero values in the above table, we only need four colors to color-code the $K4$ fractal.

IV. CONCLUSION

We demonstrate that the block-complexity function for words from 3-letter and 4-letter alphabets exhibits a fractal structure. We compute explicitly the associated fractal dimensions and we furnish visual approximations of the corresponding fractals using sequences of length 6 and 4 respectively.

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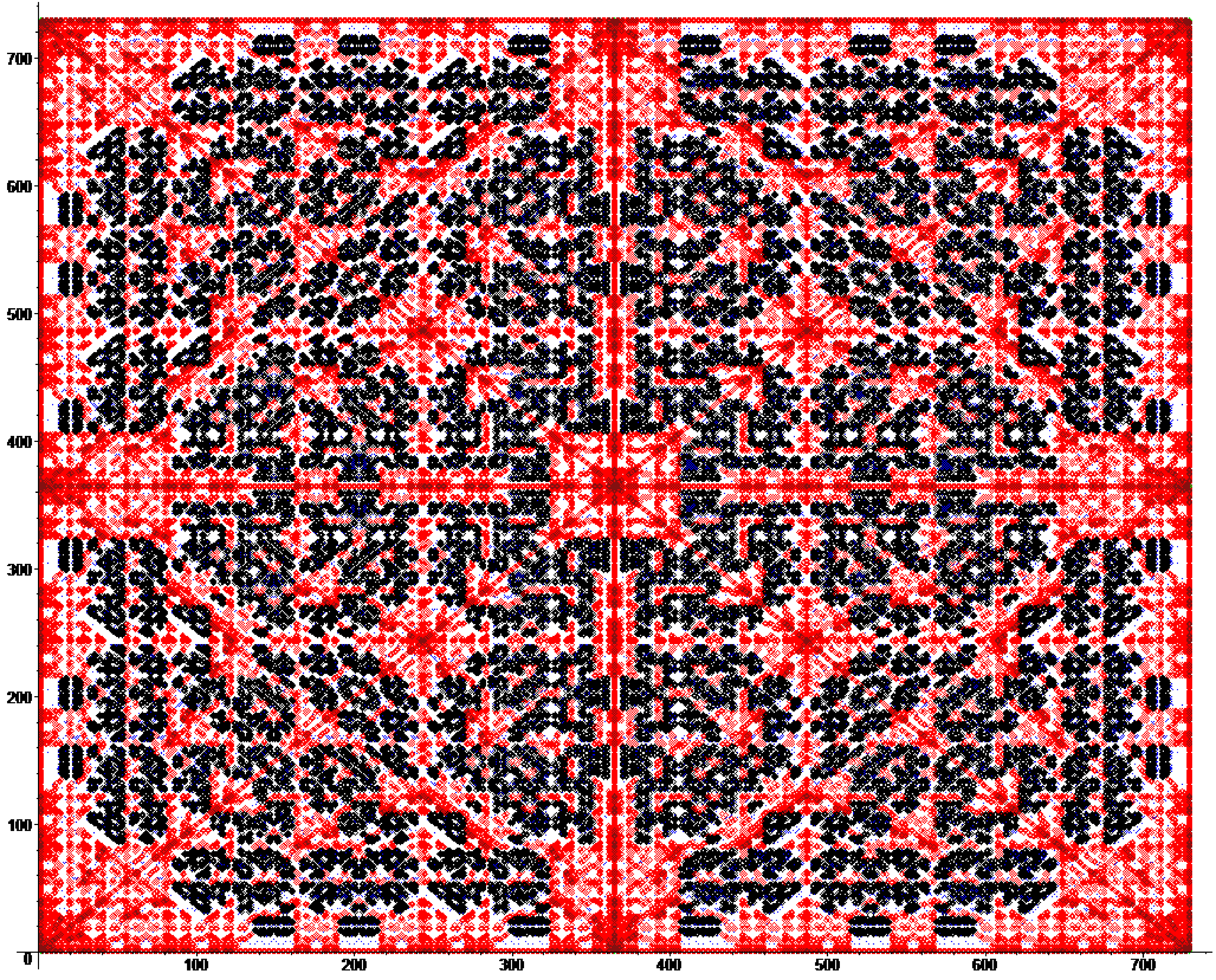

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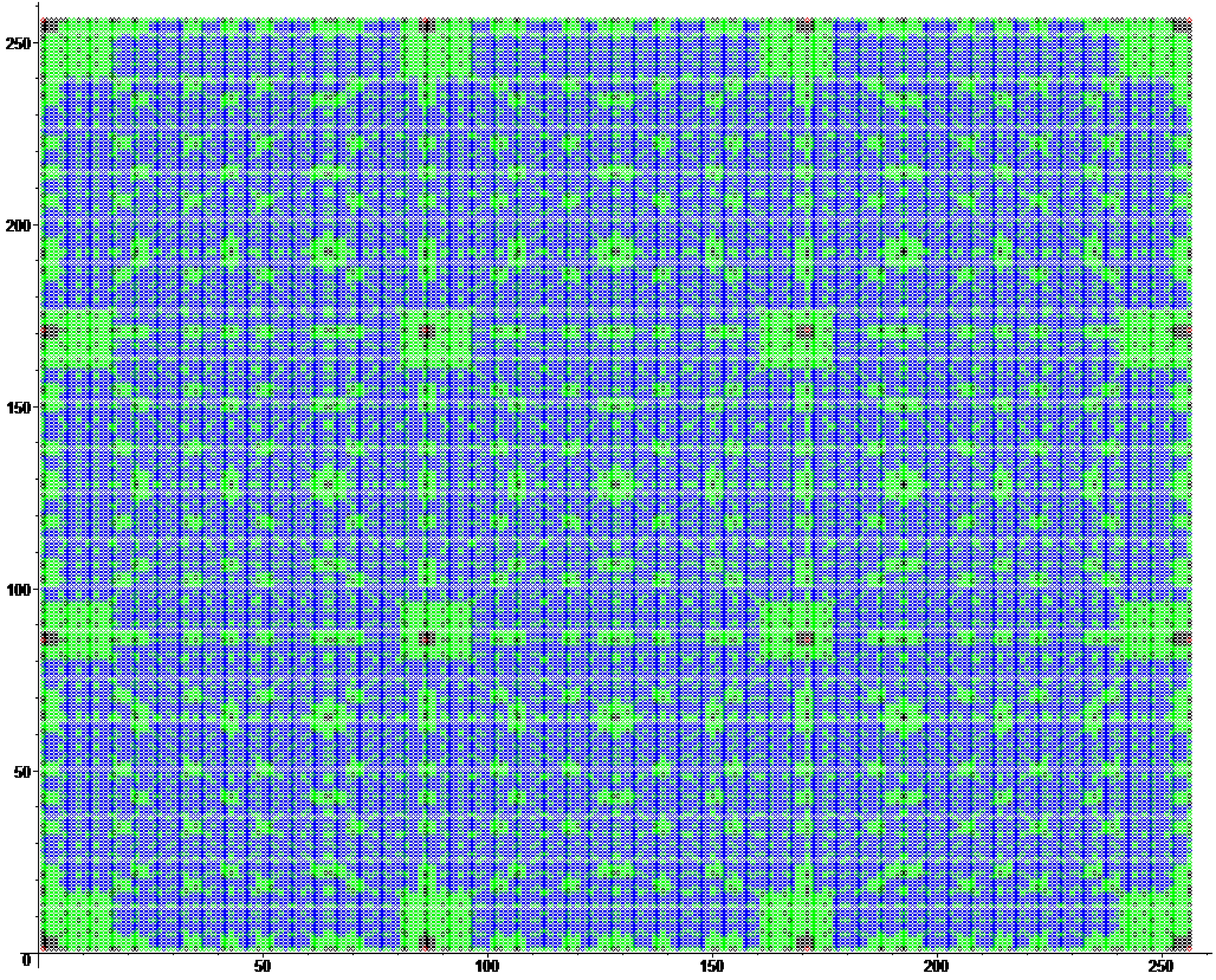

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