Tube Formulas and complex Dimensions of Self-Similar tilings

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Mai 2008

IHES/M/08/27

TUBE FORMULAS AND COMPLEX DIMENSIONS OF SELF-SIMILAR TILINGS

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Abstract. We use the self-similar tilings constructed in [32] to define a generating function for the geometry of a self-similar set in Euclidean space. This geometric zeta function encodes scaling and curvature properties related to the complement of the fractal set, and the associated system of mappings. This allows one to obtain the complex dimensions of the self-similar tiling as the poles of the geometric zeta function and hence develop a tube formula for self-similar tilings in $\mathbb{R}^d$. The resulting power series in ε is a fractal extension of Steiner's classical tube formula for convex bodies $K \subseteq \mathbb{R}^d$. Our sum has coefficients related to the curvatures of the tiling, and contains terms for each integer $i = 0, 1, \dots, d-1$, just as Steiner's does. However, our formula also contains a term for each complex dimension. This provides further justification for the term "complex dimension". It also extends several aspects of the theory of fractal strings to higher dimensions and sheds new light on the tube formula for fractals strings obtained in [30].

1. Introduction

In [32], the second author has shown that a self-similar tiling $\mathcal{T}$ is canonically associated with any self-similar system, i.e., any finite collection $\Phi = \{\Phi_j\}$ of contractive similarity transformations. Such a tiling $\mathcal{T}$ is essentially a decomposition of the complement of the unique self-similar set associated with Φ, and is reviewed in greater detail in §2. The main result of this paper is a tube formula for $\mathcal{T}$, where by a tube formula for $A \subseteq \mathbb{R}^d$, we mean an explicit expression for the d-dimensional volume of the inner ε-neighbourhood of A, i.e.,

$$V_A(\varepsilon) = \mathrm{vol}_d\{x \in A : dist(x, \partial A) \leq \varepsilon\}. \tag{1.1}$$

At the heart of this paper is the geometric zeta function $\zeta_{\mathcal{T}}(s)$ of a self-similar tiling $\mathcal{T}$. It will take some work before we are able to describe this meromorphic distribution-valued function precisely in §7. The function $\zeta_{\mathcal{T}}$ is a generating function for the geometry of a self-similar tiling: it encodes the density of geometric states of a tiling, including curvature and scaling properties. The poles of $\zeta_{\mathcal{T}}$ are the complex dimensions $\mathcal{D}_{\mathcal{T}}$ of the tiling, and we obtain a tube formula for $\mathcal{T}$ given as a sum over $\mathcal{D}_{\mathcal{T}}$ of the residues of $\zeta_{\mathcal{T}}$, taken at the complex dimensions.

The first ingredient of $\zeta_{\mathcal{T}}$ is a *scaling zeta function* $\zeta_{\mathfrak{s}}(s)$ which encodes the scaling properties of the tiling and is discussed in §4.2. This comparatively simple zeta function is the Mellin transform of a discrete *scaling measure* $\eta_{\mathfrak{s}}$ which encodes

The work of MLL was partially supported by the US National Science Foundation under the research grants DMS-0070497 and DMS-0707524.

Keywords: Complex dimensions, zeta functions, tube formula, fractal Steiner formula, inradius, self-similar tiling, curvature matrix, generating function for the geometry, distributional explicit formula, fractal string.

the combinatorics of the scaling ratios of a self-similar tiling. More precisely, if one considers a composition of similarity mappings Φ_j, each with scaling ratio r_j, then

$$\Phi_w = \Phi_{w_1 w_2 \dots w_n} = \Phi_{w_n} \circ \dots \circ \Phi_{w_2} \circ \Phi_{w_1}$$

has scaling ratio $r_w = r_{w_1} r_{w_2} \dots r_{w_n}$. The measure $\eta_\mathfrak{s}$ is a sum of Dirac masses, where each mass is located at a reciprocal scaling ratio r_w^{-1}. The total mass of any point in the support of $\eta_\mathfrak{s}$ corresponds to the multiplicity with which such a scaling ratio can occur. The scaling zeta function $\zeta_\mathfrak{s}$ is formally identical to the zeta functions studied in [30]. The function $\zeta_\mathfrak{s}$ also allows us to define the complex dimensions of a self-similar set in $\mathbb{R}^d$ (as the poles of $\zeta_\mathfrak{s}$), and we find these dimensions to have the same structure as in the 1-dimensional case. The definition and properties of the scaling measure $\eta_\mathfrak{s}$ and zeta function $\zeta_\mathfrak{s}$ is the subject of §4.1.

The next ingredient of $\zeta_\mathcal{T}$ is a *generator tube formula* γ_G. In [32], it is shown that certain tiles $G_1, \dots, G_Q$ of $\mathcal{T}$ are generators in the sense that any tile R_n of $\mathcal{T}$ is the image of some G_q under some composition of the mappings Φ_j, i.e.,

$$R_n \in \mathcal{T} \quad \Longrightarrow \quad R_n = \Phi_w(G_q),$$

for some G_q and some $w = w_1 w_2 \dots w_m$. In §5, we discuss the role of the generators and introduce the function γ_G which gives the inner tube formula for a generator in the sense of Def. 1.1. Moreover, appropriately parameterizing γ_G yields the inner tube formula for a scaled generator. Therefore, by integrating γ_G against $\eta_\mathfrak{s}$, one obtains the total contribution of G_q (and its images under the maps Φ_w) to the final tube formula $V_\mathcal{T}$. This is elaborated upon in §5.3.

At last, the geometric zeta function of the tiling $\zeta_\mathcal{T}$ is assembled from the scaling zeta function, the tiling, and the terms appearing in γ_G. In some precise sense, $\zeta_\mathcal{T}$ is a generating function for the geometry of the self-similar tiling. Using $\zeta_\mathcal{T}$, and following the distributional techniques and explicit formulas of [30], we are able to obtain an explicit distributional tube formula for self-similar tilings.

Theorem 1.1. *The d-dimensional volume of the inner tubular neighbourhood of $\mathcal{T}$ is given by the following distributional explicit formula:*

$$V_\mathcal{T}(\varepsilon) = \sum_{\omega \in \mathcal{D}_\mathcal{T}} \operatorname{res}\left(\zeta_\mathcal{T}(\varepsilon, s); \omega\right). \tag{1.2}$$

This tube formula extends previous results in two ways. On one hand, it provides a fractal analogue of the classical Steiner formula of convex geometry. Steiner's formula for the exterior ε-neighbourhood of a compact convex set is a polynomial in ε with coefficients given by curvature measures. This is discussed further in §8.4. On the other hand, the tube formula (1.2) also provides a natural higher-dimensional analogue of the tube formula for fractal strings obtained in [30] and recalled in (6.17). The present work can be considered as a further step towards a higher-dimensional theory of fractal strings, especially in the self-similar case, following upon [27], §10.2 and §10.3, and our earlier paper [21]. This is discussed further in §8.3 and in Rem. 9.1.

The primary object of study in [30] is a *fractal string*, a countable collection $L = \{L_n\}_{n=1}^\infty$ of disjoint open intervals which form a bounded open subset of $\mathbb{R}$. Due to the trivial geometry of such intervals, this reduces to studying the lengths of these intervals $\mathcal{L} = \{\ell_n\}_{n=1}^\infty$, and the sequence $\mathcal{L}$ is also referred to as a fractal string. The tube formula for a fractal string $\mathcal{L}$ (and in particular, for a self-similar tiling in $\mathbb{R}$) is defined to be $V_{\mathcal{L}}(\varepsilon) := V_L(\varepsilon)$ and is shown to be essentially given by a sum of the form

$$V_{\mathcal{L}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{L}} \cup \{0\}} c_\omega \varepsilon^{1-\omega} \tag{1.3}$$

in [30], Thm. 8.1. Here, the sum is taken over the set of complex dimensions $\mathcal{D}_{\mathcal{L}} = \{\text{poles of } \zeta_{\mathcal{L}}\}$, and c_w is defined in terms of the residue at $s = \omega$ of $\zeta_{\mathcal{L}}(s)$, the geometric zeta function of $\mathcal{L}$. The definition $V_{\mathcal{L}}(\varepsilon) := V_L(\varepsilon)$ is justified because, as is shown in [25], V_L depends exclusively on $\mathcal{L}$.

In §1.4 of [30] (following [26]), a *fractal spray* is defined to be given by a nonempty bounded open set $B \subseteq \mathbb{R}^d$ (called the *basic shape* or *generator*), scaled by a fractal string η. That is, a fractal spray is a bounded open subset of $\mathbb{R}^d$ which is the disjoint union of open sets Ω_n for $n = 1, 2, \ldots$, where Ω_n is congruent to $\ell_n B$ (the homothetic of Ω by ℓ_n) for each ℓ_n. Thus, a fractal string is a fractal spray on the basic shape $B = (0, 1)$, the unit interval. In the context of the current paper, a self-similar tiling is a union of fractal sprays on the basic shapes $G_1, \ldots, G_Q$, each scaled by a fixed *self-similar* string. In fact, we first prove Thm. 1.1 for the more general case of fractal sprays, and then refine it to obtain the formula for self-similar tilings.

The rest of this paper is organized as follows. §2 contains a quick overview of the background material concerning self-similar tilings. §3 discusses how the notion of inradius describes the different scales of the tiling. §4 defines the scaling and geometric measures, the scaling zeta function, and complex dimensions of a self-similar tiling. §5 develops the tube formula for the generators of a self-similar tiling, and establishes the general form of $V_{\mathcal{T}}(\varepsilon)$ in terms of this. §6 reviews the explicit formulas for fractal strings which will be used in the proof of the main results. §7 defines the geometric zeta function of the tiling, and states and proves the tube formula for fractal sprays (a generalization of a tiling) given in Thm. 7.5, from which the tube formula for self-similar tilings follows readily, and §9 discusses several examples illustrating the theory. Appendix A verifies the validity of the definition of the geometric zeta function $\zeta_{\mathcal{T}}$. Appendix B verifies the distributional error term and its estimate, from Thm. 7.5.

Remark 1.2 (A note on the references). The primary references for this paper are [32] and the research monograph “Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and spectra of fractal strings” by Lapidus and van Frankenhuijsen [30]. This volume is essentially a revised and much expanded version of [27], by the same authors. The present paper cites [30] almost exclusively, so we provide the following partial correspondence between chapters for the aid of the reader:

[27]	Ch. 2	Ch. 3	Ch. 4	Ch. 6	Ch. 10
[30]	Ch. 2–3	Ch. 4	Ch. 5	Ch. 8	Ch. 12

Remark 1.3. Throughout, we reserve the symbol $\mathbb{i} = \sqrt{-1}$ for the imaginary number.

1.1. **Acknowledgements.** The authors wish to thank Martina Zähle for several helpful discussions on geometric measure theory and for bringing the reference [13] to our attention. Additionally, the authors would like to thank Steffen Winter for

many helpful discussions and suggestions, and for finding mistakes in an earlier version of this paper.

2. The Self-Similar Tiling

This section provides an overview of the necessary background material concerning self-similar tilings. Further details may be found in [32].

Definition 2.1. A *self-similar system* is a family $\{\Phi_j\}_{j=1}^J$ (with $J \geq 2$) of contraction similitudes

$$\Phi_j(x) := r_j A_j x + a_j, \quad j = 1, \dots, J.$$

For $j = 1, \dots, J$, we have $0 < r_j < 1, a_j \in \mathbb{R}^d$, and $A_j \in O(d)$, the orthogonal group of rigid rotations in d-dimensional Euclidean space $\mathbb{R}^d$. The number r_j is the *scaling ratio* of Φ_j. For convenience, assume that

$$1 > r_1 \geq r_2 \geq \dots \geq r_J > 0. \tag{2.1}$$

It is well known that there is a unique nonempty compact subset $F \subseteq \mathbb{R}^d$ satisfying the fixed-point equation

$$F = \Phi(F) := \bigcup_{j=1}^J \Phi_j(F). \tag{2.2}$$

This (self-similar) set F is called the *attractor* of Φ. We abuse notation and let Φ denote both an operator on compacta (as in (2.2)) and the family $\{\Phi_j\}$. Different self-similar systems may give rise to the same self-similar set; therefore we emphasize the self-similar system and its corresponding dynamics.

It is shown in [32] that for a self-similar system satisfying the tileset condition (see Def. 2.2), there exists a natural decomposition of $C \setminus F$ which is produced by the system Φ. The construction of this tiling is illustrated for a well-known example, the Koch curve, in Fig. 1. It may help the reader to look at this example before diving into the next paragraph and the thicket of definitions therein. Further examples are depicted in §9.

Let $C := [F]$ be the convex hull of F, and let $T := \operatorname{relint} C$ be the relative interior of C. Iterates of the hull C under Φ are denoted

$$C_k := \Phi^k(C) = \bigcup_{w \in \mathcal{W}_k} \Phi_w(C), \tag{2.3}$$

where $w = w_1 \dots w_k$ is a *word* in $\mathcal{W}_k := \{1, 2, \dots, J\}^k$ and $\Phi_w := \Phi_{w_k} \circ \dots \circ \Phi_{w_2} \circ \Phi_{w_1}$. For future reference, let $\mathcal{W} := \bigcup_{k=1}^\infty \mathcal{W}_k$ be the set of all *finite* words w over the alphabet $\{1, 2, \dots, J\}$.

Definition 2.2. The system satisfies the *tileset condition* iff $T \nsubseteq \Phi(C)$ and

$$\operatorname{int} \Phi_j(C) \cap \operatorname{int} \Phi_\ell(C) = \varnothing, \qquad j \neq \ell. \tag{2.4}$$

This is a restriction on the overlap of the images of the mappings and implies (but is not equivalent to) the open set condition. For any system satisfying the tileset condition,

$$T_1 := T \setminus C_1 \tag{2.5}$$

is well defined and nonempty, and hence so is $T_k := \Phi^k(T_1)$. As an open set, T_1 is a disjoint union of connected open sets:

$$T_1 = G_1 \cup G_2 \cup \dots \cup G_Q, \quad G_p \cap G_q = \varnothing, p \neq q. \tag{2.6}$$

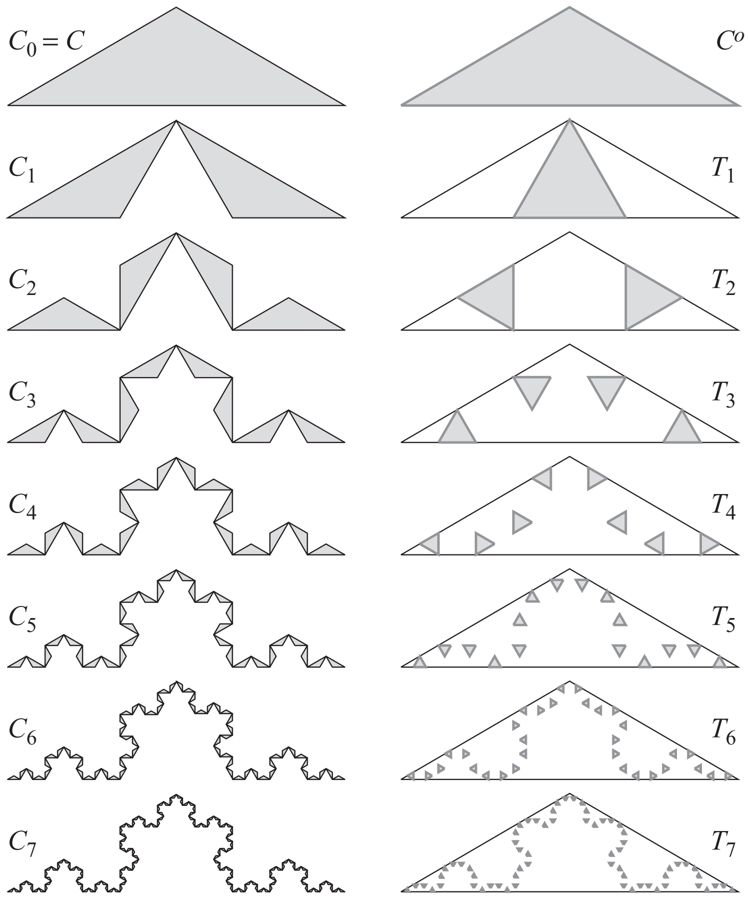

FIGURE 1. Construction of the Koch tiling $\mathcal{K}$. This example is discussed further in §9. The tiling $\mathcal{K}$ has the single generator $T_1 = G_1$, an equilateral triangle.

Definition 2.3. The *generators* of the tiling are the connected components of T_1, i.e., the disjoint open sets $\{G_q\}$ in (2.6).

The number Q of generators depends on the system Φ, not just on F. In general, the number of connected components of an open subset of $\mathbb{R}^d$ may be countable; however, in this paper we assume $Q < \infty$.

Definition 2.4. The *self-similar tiling* of Φ is

$$\mathcal{T} = \{R_n\}_{n=1}^{\infty} = \{\Phi_w(G_q) : w \in \mathcal{W}, q = 1, \ldots, Q\}. \tag{2.7}$$

In (2.7), the sequence $\{R_n\}$ is an enumeration of the sets $\{\Phi_w(G_q)\}$, and Φ_w is as in (2.3). We say $\mathcal{T}$ is a *tiling of* $C \setminus F$ because the tiles R_n have disjoint interiors and F does not intersect the interior of any R_n (see Fig. 3):

$$C = \bigcup_n \overline{R_n}, \quad F \subseteq \bigcup_n \partial R_n, \quad \text{and} \quad R_{n_1} \cap R_{n_2} = \partial R_{n_1} \cap \partial R_{n_2}.$$

3. The Inradius

As alluded to in (1.1), we are interested in that portion of a set which lies within ε of its boundary.

Definition 3.1. Given $\varepsilon > 0$, the *inner ε-neighbourhood* of a set $A \subseteq \mathbb{R}^d$, $d \geq 1$, is

$$A_\varepsilon := \{x \in A : dist(x, \partial A) \leq \varepsilon\}, \tag{3.1}$$

where ∂A is the boundary of A. We are primarily interested in the d-dimensional Lebesgue measure of A_ε, denoted $V_A(\varepsilon) := \mathrm{vol}_d(A_\varepsilon)$.

It is clear that if A is a bounded set, $A \subseteq A_\varepsilon$ for sufficiently large ε. Alternatively, it is apparent that for a fixed $\varepsilon > 0$, any sufficiently small set will be entirely contained within its ε-neighbourhood. The notion of *inradius* allows us to see when this phenomenon occurs.

Definition 3.2. The *inradius* ρ of a set A is

$$\rho = \rho(A) := \sup\{\varepsilon > 0 : \exists x \text{ with } B(x, \varepsilon) \subseteq A\}. \tag{3.2}$$

Note that the supremum is taken over $\varepsilon > 0$, because $A_0 = \overline{A}$. The inradii $\rho_n = \rho(R_n)$ replace the lengths $\ell_n = 2\rho(L_n)$ of the 1-dimensional theory; furthermore, the inradius is characterized by the following theorem.[1]

Theorem 3.3. *In $\mathbb{R}^d$, the inradius is the furthest distance from a point of A to ∂A, or the radius of the largest ball contained in A, i.e.,*

$$\rho(A) = \sup\{\varepsilon > 0 : V(A_\varepsilon) < V(A)\}. \tag{3.3}$$

Proof. Continuity of the distance and volume functionals gives $V(A_\delta) < V(A)$ if and only if there is a set U of positive d-dimensional measure contained in the interior of A which is further than δ from any point of ∂A. For any $x \in U$, $B(x, \delta) \subseteq A$. Conversely, for δ strictly less than the right-hand side of (3.3), the same reasons imply the existence of the set U of positive measure. □

Remark 3.4. The proof of Thm. 3.3 shows that the inradius may also be defined by $\rho(A) = \sup\{d(x, \partial A) : x \in A\}$.

The utility of the inradius in the present paper arises primarily from the equality (3.3) and the fact that the inradius behaves well under the action of the self-similar system:

$$\rho_n = \rho(R_n) = \rho(\Phi_w(G_q)) = r_1^{e_1} \dots r_J^{e_J} g_q, \tag{3.4}$$

where r_j is the scaling ratio of Φ_j, and the exponent $e_j \in \mathbb{N}$ indicates the multiplicity of the letter j in the finite word $w \in \mathcal{W}$.

Definition 3.5. For $q = 1, \dots, Q$, the q^{th} *generating inradius* is the inradius of the q^{th} generator of the tiling $\mathcal{T}$ and denoted

$$g_q := \rho(G_q). \tag{3.5}$$

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[1] While this result is probably folklore in some circles, we were unable to find it in the literature and so have provided a proof.

Throughout the remainder of this paper,[2] we assume that $\mathcal{T}$ is a self-similar tiling associated with a given self-similar system (and having inradii g_q), as described in §2–§3 and in further detail in [32]. For convenience, we may take the generators in nonincreasing order, i.e., index the generators so that

$$g_1 \geq g_2 \geq \cdots \geq g_Q > 0. \tag{3.6}$$

4. Measures and zeta functions

In this section and the rest of the paper, any zeta function is understood to be the meromorphic extension of its defining expression.

4.1. **The geometric zeta function of a fractal string.** In [30],[3] a *fractal string* is defined to be a bounded open subset of $\mathbb{R}$, that is, a countable collection of disjoint open intervals, $L = \bigcup_{n=1}^{\infty} L_n$, with lengths $\mathcal{L} = \{\ell_n\}_{n=1}^{\infty}$. The geometric zeta function of such an object is

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \ell_n^s, \tag{4.1}$$

and can be used to study the geometry of $\mathcal{L}$ and of its (presumably fractal) boundary $\partial\mathcal{L} := \partial L$. Observe that $\zeta_{\mathcal{L}}(s)$ is the Mellin transform of the measure

$$\eta_{\mathcal{L}} = \sum_{n=1}^{\infty} \delta_{1/\ell_n}, \tag{4.2}$$

where δ_x denotes the Dirac mass (or Dirac measure) at x. Thus,

$$\zeta_{\mathcal{L}}(s) = \int_0^{\infty} x^s \, d\eta_{\mathcal{L}}(x). \tag{4.3}$$

4.2. **The scaling zeta function of a self-similar tiling.** We now extend the ideas of §4.1 to higher dimensions. In 1 dimension, the length of an interval is just twice its inradius, and the distinction between the scale of a set and its volume is blurred. For the higher-dimensional case the scaling zeta function $\zeta_{\mathfrak{s}}$ is separate from the geometric zeta function of the tiling $\zeta_{\mathcal{T}}$; the tiling zeta function $\zeta_{\mathcal{T}}$ encodes the density of geometric states of Φ and acts as a generating function for the geometry of the entire tiling. The scaling zeta function encodes only scaling data. In [30], both of these roles are essentially played by $\zeta_{\mathcal{L}}$. Discussion of $\zeta_{\mathcal{T}}$ is postponed to Def. 7.2, as it takes some work to give a precise description.

Definition 4.1. The *scaling measure* encodes the scaling factors of Φ as a sum of Dirac masses:

$$\eta_{\mathfrak{s}}(x) := \sum_{w \in \mathcal{W}} \delta_{1/r_w}(x). \tag{4.4}$$

Definition 4.2. The *scaling zeta function* is defined by the Mellin transform of the scaling measure $\eta_{\mathfrak{s}}$:

$$\zeta_{\mathfrak{s}}(s) := \int_0^{\infty} x^{-s} \, d\eta = \sum_{w \in \mathcal{W}} r_w^s = \sum_{k=0}^{\infty} \sum_{w \in \mathcal{W}_k} r_w^s. \tag{4.5}$$

[2]Except for the discussion surrounding Thm. 7.5, wherein $\mathcal{T}$ is taken to be a fractal spray. See the introduction to §7.

[3]See also [25–2,20,18–3,27–3,10], along with [17], Exm. 5.1 and App. C.

The scaling zeta function $\zeta_{\mathfrak{s}}$ encodes the combinatorics of the scaling ratios r_j of Φ and is thus a generating function for the scaling properties of Φ.

Theorem 4.3. *The scaling zeta function of a self-similar system is*

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - \sum_{j=1}^{J} r_j^s}. \tag{4.6}$$

This remains valid for the meromorphic extension of $\zeta_{\mathfrak{s}}$ to all of $\mathbb{C}$.

This theorem is the higher-dimensional counterpart of [30], Thm. 2.4, and can, in fact, be viewed as a corollary of it; see §4.3. Indeed, it is proved in precisely the same way.

Definition 4.4. We can now define the *scaling (complex) dimensions* of a tiling $\mathcal{T}$ as the poles of the scaling zeta function:

$$\mathcal{D}_{\mathfrak{s}} := \{\omega \in \mathbb{C} : \zeta_{\mathfrak{s}}(s) \text{ has a pole at } \omega\}. \tag{4.7}$$

4.3. **Comparison with [30].** Although the measures and zeta function introduced in Def. 4.1 and Def. 4.2 above correspond to fractal subsets of $\mathbb{R}^d$, it is crucial to note that they are also formally identical to the objects η and ζ_η studied in [30]. To be precise, the scaling measure is a self-similar string of the sort studied in Chap. 2–3 of [30], and a generalized fractal string of the kind introduced in Chap. 4 of [30]. In that context, $\zeta_{\mathfrak{s}}$ is just the geometric zeta function of a self-similar string with scaling ratios $\{r_j\}_{j=1}^J$ and a single *gap*[4], which has been normalized so as to have $\ell_1 = 1$, where ℓ_1 is the first length in the string. Consequently, all of the explicit formulas developed in [30] are applicable to the measures and zeta functions described in the present paper. This is key to the proof of Thm. 7.5.

Let D be the unique real number satisfying $\sum_{j=1}^J r_j^D = 1$. One can check (as in [30], §5.1) that for some real constant $c > D$,

$$\eta_{\mathfrak{s}}(x) = \frac{1}{2\pi \mathbb{i}} \int_{c-\mathbb{i}\infty}^{c+\mathbb{i}\infty} x^{s-1} \zeta_{\mathfrak{s}}(s)\, ds, \quad \text{and} \quad \zeta_{\mathfrak{s}}(s) = \int_0^\infty x^{-s}\, \eta_{\mathfrak{s}}(dx). \tag{4.8}$$

Additionally, the Structure Theorem for complex dimensions [30], Thm. 3.6, holds for the set of scaling complex dimensions of Def. 4.4. By (4.6), $\mathcal{D}_{\mathfrak{s}}$ consists of the set of complex solutions of the complexified Moran equation $\sum_{j=1}^J r_j^s = 1$ which is studied in detail in Chap. 2–3 of [30]. In particular, the complex dimensions lie in a horizontally bounded strip of the form $C \leq \operatorname{Re} s \leq D$, where D is as just above and $C < D$ is some other (finite, possibly negative) constant. The positive number D is called the *similarity dimension* of Φ (or of its attractor F) and coincides with the abscissa of convergence of $\zeta_{\mathfrak{s}}$ [30], Thm. 1.10.[5] Furthermore, the following dichotomy prevails:

- *Lattice case.* When the logarithms of the scaling ratios r_j are each an integer power of some common positive real number, the complex dimensions lie periodically on finitely many vertical lines, including the line $\operatorname{Re} s = D$. In this case, there are infinitely many complex dimensions with real part D

[4] In this paper, we use the term "generator" in place of "gap".

[5] If the self-similar system defining F satisfies the 'open set condition' (see [14], as described in [2] or [15]), as is the case when the tileset condition is satisfied, then D coincides with the Hausdorff and Minkowski dimensions of F.

- *Nonlattice case.* Otherwise, the complex dimensions are quasiperiodically distributed and $s = D$ is the only complex dimension with real part D. However, there exists an infinite sequence of complex dimensions approaching the line $\operatorname{Re} s = D$ from the left. In this case, the set $\{\operatorname{Re} s : s \in \mathcal{D}\}$ appears to be dense in $[D_l, D]$.

It has been proven in [30] that for $d = 1$, the attractor of Φ fails to be Minkowski measurable if and only if ζ_η has nonreal complex dimensions with real part D, and in [30], Conj. 12.18, this is conjectured to hold also in higher dimensions. See also Remark 8.7.

5. The Generators

The inner tube formula for the tiling will consist of the sum of the inner tube formulas for each tile, and each of these can be expressed as a rescaled version of the tube formula for a generator. That is, if $R = \Phi_w(G)$ for some $w \in \mathcal{W}$, then the inradius of such a tile is

$$\rho = \rho(R) = \rho(\Phi_w(G)) = r_w g = r_1^{e_1} \dots r_J^{e_J} g,$$

and invariance of Lebesgue measure under rigid motions gives

$$V_R(\varepsilon) = V_{\Phi_w(G)}(\varepsilon) = V_{r_w G}(\varepsilon). \tag{5.1}$$

Thus, it behooves us to find an expression for

$$\gamma_G(x, \varepsilon) := V_{(1/x)G}(\varepsilon), \tag{5.2}$$

where $\frac{1}{x}G$ is a homothetic image of G, scaled by some factor $\frac{1}{x} > 0$. Then $\gamma_G(1, \varepsilon)$ gives the inner tube formula for G, and $\gamma_G(x, \varepsilon)$ is the volume of a tile which is similar to G but which has been scaled by $1/x$. The motivation for defining γ_G in terms of $1/x$ (rather than x) appears in (5.21).

For the moment, the following definition is left intentionally vague; motivation for the definition (and its name) can be found in §8.4.

Definition 5.1. A generator G is said to be *Steiner-like* iff its inner tube formula admits an expansion of the form

$$V_G(\varepsilon) = \sum_{i=0}^{d-1} \kappa_i(G, \varepsilon)\varepsilon^{d-i}, \tag{5.3}$$

for $\varepsilon < g$, where each $\kappa_i(G, \varepsilon)$ is some reasonably nice (e.g., bounded and locally integrable) function for $\varepsilon \in [0, \infty)$. In particular, we require that

(i) each $\kappa_i(G, \varepsilon)$ is homogeneous of degree i, so that for $\lambda > 0$,

$$\kappa_i(\lambda G, \lambda\varepsilon) = \kappa_i(G, \varepsilon)\,\lambda^i, \quad \text{and} \tag{5.4}$$

(ii) each $\kappa_i(G, \varepsilon)$ is rigid motion invariant, so that

$$\kappa_i(T(G), \varepsilon) = \kappa_i(G, \varepsilon), \tag{5.5}$$

for any (affine) isometry T of $\mathbb{R}^d$.

(iii) for each $\kappa_i(G, \varepsilon)$, $i = 0, 1, \dots, d-1$, the limit $\lim_{\varepsilon \to 0^+} \kappa_i(G, \varepsilon)$ exists in $\mathbb{R}$.

5.1. **The tube formula for a diphase generator.** In this paper, we treat only the special case of a Steiner-like generator G where the coefficient functions $\kappa_i(G,\varepsilon)$ are piecewise constant functions of ε, in which case G is called *pluriphase* as in Def. 5.4. In the special case when each $\kappa_i(G,\varepsilon)$ takes on only two values, G is called *diphase* as in Def. 5.2. We will treat the general case in the forthcoming collaboration with Steffen Winter [24].

Definition 5.2. A Steiner-like generator G is said to be a *diphase generator*, or to have a *diphase tube formula*, iff

$$V_G(\varepsilon) = \gamma_G(1,\varepsilon) = \sum_{i=0}^{d-1} \kappa_i(G)\varepsilon^{d-i}, \qquad \text{for } \varepsilon < g, \tag{5.6}$$

for some $\kappa_i(G) \in \mathbb{R}$, $i = 0, 1, \dots, d-1$.

Not every polyhedral G is diphase; the more general pluriphase case is discussed in §5.2. In general, the computation of $\gamma_G(x,\varepsilon)$ may be nontrivial. We define $\kappa_d(G)$ to be the negative of the d-dimensional Lebesgue measure of G:

$$\kappa_d(G) = -\lambda_d(G). \tag{5.7}$$

Theorem 5.3. *If G is diphase, then for any tile congruent to the homothetic image $\frac{1}{x}G$, the inner tube formula is given by*

$$\gamma_G(x,\varepsilon) = \begin{cases} \sum_{i=0}^{d-1} \kappa_i(G)x^{-i}\varepsilon^{d-i}, & \varepsilon \le g/x, \\ -\kappa_d(G)x^{-d}, & \varepsilon \ge g/x. \end{cases} \tag{5.8}$$

Proof. So far, we have only defined $V_G(\varepsilon)$ for $\varepsilon < g$. To extend it to all of $\mathbb{R}^+$, note that $V_G(\varepsilon)$ is just the Lebesgue measure of G for $\varepsilon \ge g$. Therefore, define

$$\begin{aligned} \kappa_i(G;\varepsilon) &:= \kappa_i(G)\chi_{[0,g)}(\varepsilon), && i = 0, 1, \dots, d-1 \\ \kappa_d(G;\varepsilon) &:= -\kappa_d(G)\chi_{[g,\infty)}(\varepsilon), \end{aligned}$$

where $\kappa_i(G)$ is as in (5.6), χ_A is the usual characteristic function of the set A, and μ_d is Lebesgue measure on $\mathbb{R}^d$. Now we have

$$V_G(\varepsilon) = \gamma_G(1,\varepsilon) = \sum_{i=0}^{d} \kappa_i(G;\varepsilon)\varepsilon^{d-i}, \qquad \text{for } \varepsilon \ge 0. \tag{5.9}$$

Next, we would like to adapt this formula so as to obtain a tube formula valid for a tile of any size. Note that $V_{rG}(r\varepsilon) = r^d V_G(\varepsilon)$, as both expressions are measuring congruent regions in $\mathbb{R}^d$. Hence for $\varepsilon < g$, one has

$$\sum_{i=0}^{d-1} r^i \kappa_i(G;\varepsilon)(r\varepsilon)^{d-i} = r^d V_G(\varepsilon) = V_{rG}(r\varepsilon) = \sum_{i=0}^{d-1} \kappa_i(rG;r\varepsilon)(r\varepsilon)^{d-i},$$

and thus for $\varepsilon < g/x$, one has

$$\gamma_G(x,\varepsilon) = V_{\frac{1}{x}G}(\varepsilon) = \sum_{i=0}^{d-1} \left(\tfrac{1}{x}\right)^i \kappa_i(G;x\varepsilon)\varepsilon^{d-i} = \sum_{i=0}^{d-1} \kappa_i(G;x\varepsilon)x^{-i}\varepsilon^{d-i}.$$

Since $\kappa_i(G;x\varepsilon) = \kappa_i(G)\chi_{[0,g)}(x\varepsilon) = \kappa_i(G)\chi_{[0,g/x)}(\varepsilon)$ for $i = 0, 1, \dots, d-1$, it is clear that (5.2) may be expressed as

$$\gamma_G(x,\varepsilon) = \sum_{i=0}^{d} \kappa_i(\tfrac{1}{x}G;\varepsilon)\varepsilon^{d-i} = \begin{cases} \sum_{i=0}^{d-1} \kappa_i(G)x^{-i}\varepsilon^{d-i}, & \varepsilon \le g/x, \\ -\kappa_d(G)x^{-d}, & \varepsilon \ge g/x, \end{cases} \tag{5.10}$$

where the constants $\kappa_i(G)$ are as defined in (5.6) for $i = 0, 1, \ldots, d-1$, and in (5.7) for $i = d$. $\square$

The function $\gamma_G(x, \varepsilon)$ gives the volume of the ε-neighbourhood of a tile which is congruent to a generator scaled by $1/x$. The value $\varepsilon = g/x$ corresponds to the value of ε at which the inner ε-neighbourhood of the tile becomes equal to the tile itself. Thus, γ_G is continuous (but generally not differentiable) at $\varepsilon = g/x$.

As was mentioned before, not every G is diphase; the more complicated pluriphase case is discussed in §5.2. In fact, even if G is polyhedral or convex, it still may not be diphase; see Example 5.7 for an example of the latter. We expect that all convex generators are pluriphase, but this has not yet been proved. For situations even more general, it is an interior version of Federer's notion of *reach* (see [4]) that is required. For such cases, the inner tube formula will be obtained in [22] via the more general methods of [13] and others.

5.2. **The tube formula for pluriphase generators.** The most general generators that we consider in this paper are those whose inner tube formula is given by a piecewise polynomial function of ε (see [22] for investigation of the more general case). In this case, each generator has a sequence of values of ε which gives a partition $\{\varepsilon_0 := 0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M := g\}$, of the interval $[0, g]$, with $\varepsilon_{m-1} < \varepsilon_m$ for each m. Then one has the following definition:

Definition 5.4. A Steiner-like generator G is said to be a *pluriphase generator*, or to have a *pluriphase tube formula*, iff one can write V_G for each fixed $m = 1, \ldots, M$ as

$$V_G(\varepsilon) = \gamma_G(1, \varepsilon) = \sum_{i=0}^{d} \kappa_i^m(G)\varepsilon^{d-i}, \qquad \varepsilon_{m-1} \le \varepsilon < \varepsilon_m, \tag{5.11}$$

for some real coefficients $\kappa_i^m(G)$. Then for $i = 0, \ldots d-1$, define a piecewise constant function on $\mathbb{R}^+ = [0, \infty)$ by

$$\kappa_i(G; \varepsilon) := \sum_{m=1}^{M} \kappa_i^m(G)\chi_{[\varepsilon_{m-1}, \varepsilon_m)}(\varepsilon). \tag{5.12}$$

Similarly, let $\kappa_d(G) := -\mu_d(G)$ and define

$$\kappa_d(G; \varepsilon) := \sum_{m=1}^{M} \kappa_d^m(G)\chi_{[\varepsilon_{m-1}, \varepsilon_m)}(\varepsilon) - \kappa_d(G)\chi_{[g,\infty)}(\varepsilon). \tag{5.13}$$

The double negatives for the d^{th} term are regrettable but necessary, as will become clear from the proof of Thm. 7.5 and the examples in §9. The next theorem is proved just like Thm. 5.3.

Theorem 5.5. *If G is pluriphase, then for any tile congruent to the homothetic image $\frac{1}{x}G$, the inner tube formula is given by*

$$\gamma_G(x, \varepsilon) = \sum_{i=0}^{d} \kappa_i^m(\tfrac{1}{x}G; \varepsilon)\varepsilon^{d-i} = \begin{cases} \sum_{i=0}^{d}\sum_{m=1}^{M} \kappa_i^m(G)x^{-i}\varepsilon^{d-i}, & \varepsilon \le g/x, \\ -\kappa_d(G)x^{-d}, & \varepsilon \ge g/x. \end{cases} \tag{5.14}$$

As before, the function $\gamma_G(x, \varepsilon)$ gives the volume of the ε-neighbourhood of a tile which is similar to G but has been scaled by a factor of $1/x$, and γ_G is continuous (but generally not differentiable) at $\varepsilon = \frac{1}{x}$. For the present, we take

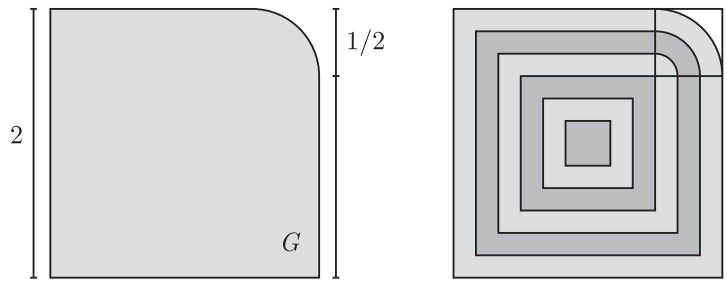

FIGURE 2. A pluriphase generator.

Def. 5.4 as a working definition and note that it is applicable to a wide range of examples, including all of those in §9. In particular, Example 5.7 is pluriphase but not diphase. We will investigate the implications of this definition further in the later paper [22]. To compare with the diphase case, note that if $\varepsilon_0 = 0$ and $\varepsilon_1 = g$, then (5.10) may be rewritten as

$$\gamma_G(x, \varepsilon) = \sum_{i=0}^{d-1} \sum_{m=1}^{1} \kappa_i^m(G) \chi_{\left[\frac{\varepsilon_{m-1}}{x}, \frac{\varepsilon_m}{x}\right)}(\varepsilon) x^{-i} \varepsilon^{d-i}, \qquad \text{for } \varepsilon \leq g/x. \tag{5.15}$$

Definition 5.6. If the generator G is pluriphase and defined on a partition of M subintervals $\{0 = \varepsilon_0, \varepsilon_1, \dots, \varepsilon_M = \rho(G)\}$, then its *curvature matrix* $\boldsymbol{\kappa}$ is the $(M+1) \times (d+1)$ matrix with entries

$$\boldsymbol{\kappa} := [\kappa_i^m(G)] = \begin{bmatrix} \kappa_0^1 & \kappa_1^1 & \dots & \kappa_{d-1}^1 & \kappa_d^1 \\ \kappa_0^2 & \kappa_1^2 & \dots & \kappa_{d-1}^2 & \kappa_d^2 \\ \vdots & \vdots & & \vdots & \\ \kappa_0^M & \kappa_1^M & \dots & \kappa_{d-1}^M & \kappa_d^M \\ 0 & 0 & \dots & 0 & \kappa_d \end{bmatrix}. \tag{5.16}$$

It is worth noting that the continuity of $V_G(\varepsilon)$ at $\varepsilon = 0$ mandates that $\kappa_d^1 = 0$ (recall that this is the coefficient of the constant term). Thus, if the generator G is diphase as in Def. 5.2, then with an abuse of notation, the curvature matrix may be written

$$\boldsymbol{\kappa} := [\kappa_i(G)] = \begin{bmatrix} \kappa_0, & \kappa_1, & \dots, & \kappa_{d-1}, & \kappa_d \end{bmatrix}. \tag{5.17}$$

Example 5.7 (A pluriphase generator)**.** Consider a fractal spray on a generator G consisting of a 2×2 square with one corner replaced by a circular arc, as depicted in Fig. 2. This generator has inradius $g = \rho(G) = 1$ and is pluriphase, but not diphase. Indeed, the relevant partition is

$$\{0 = \varepsilon_0, \varepsilon_1 = 1/2, \varepsilon_2 = 1\}, \tag{5.18}$$

and the tube formula for G is

$$\gamma_G(1, \varepsilon) = \begin{cases} (8 + \frac{\pi}{4})\varepsilon - (5 + \frac{\pi}{4})\varepsilon^2, & \varepsilon_0 \leq \varepsilon \leq \varepsilon_1 \\ \frac{\pi-4}{16} + 8\varepsilon - 4\varepsilon^2, & \varepsilon_1 \leq \varepsilon \leq \varepsilon_2 \\ \frac{\pi-4}{16} + 4, & \varepsilon_2 \leq \varepsilon. \end{cases} \tag{5.19}$$

5.3. **Tilings with one generator.** Suppose we have a tiling $\mathcal{T}$ with just one generator G. Then the inner tube formula of $\mathcal{T}$ is given by

$$V_{\mathcal{T}}(\varepsilon) = \sum_{n=1}^{\infty} V_{R_n}(\varepsilon) = \sum_{\rho_n \geq \varepsilon} V_{R_n}(\varepsilon) + \sum_{\rho_n < \varepsilon} V_{R_n}(\varepsilon), \tag{5.20}$$

much as in [26], Eqn. (3.2). Recall that ρ_n is the inradius of the tile R_n. For $R_n = \Phi_w(G_q)$, invariance under rigid motions allows us to use the equality (5.1) to rewrite the sums in (5.20) as integrals with respect to $\eta_{\mathfrak{s}}$:

$$\begin{aligned} V_{\mathcal{T}}(\varepsilon) &= \sum_{\rho_n^{-1} \leq 1/\varepsilon} V_{R_n}(\varepsilon) + \sum_{\rho_n^{-1} > 1/\varepsilon} V_{R_n}(\varepsilon) \\ &= \int_0^{g/\varepsilon} V_{(1/x)G}(\varepsilon)\, d\eta_{\mathfrak{s}}(x) + \mu_d(G) \int_{g/\varepsilon}^{\infty} x^{-d}\, d\eta_{\mathfrak{s}}(x) \qquad (5.21) \\ &= \int_0^{\infty} \gamma_G(x,\varepsilon)\, d\eta_{\mathfrak{s}}(x) \\ &= \langle \eta_{\mathfrak{s}}, \gamma_G \rangle, \qquad (5.22) \end{aligned}$$

where γ_G is a 'test function' giving the volume of a tile which is similar to G, but which has been scaled by a factor of $1/x$; see (5.2). Although γ_G is not smooth, it fits the criteria given in Thm. 6.5 and is thus amenable to the distributional techniques developed in [30], §5.4.

5.4. **Tilings with multiple generators.** Upon replacing G by G_q, we use the notation $V_q, \gamma_q, \kappa_{qi}$, etc., to refer to the corresponding quantity for the q^{th} generator. For example, $\gamma_G(x,\varepsilon)$ is replaced by $\gamma_q(x,\varepsilon) = \gamma_{G_q}(x,\varepsilon)$, the volume of the ε-neighbourhood of a tile which is similar to G_q but which has been scaled by x.

The contribution to $V_{\mathcal{T}}(\varepsilon)$ resulting from one generator G_q and its successive images is $V_q(\varepsilon) := \langle \eta_{\mathfrak{s}}, \gamma_q \rangle$, so the case of multiple generators can be reduced to a sum of single-generator tilings via the formula

$$V_{\mathcal{T}}(\varepsilon) = \sum_{q=1}^{Q} V_q(\varepsilon) = \sum_{q=1}^{Q} \langle \eta_{\mathfrak{s}}, \gamma_q \rangle. \tag{5.23}$$

Henceforth, we will always assume there is only a single generator, as this simplifying assumption will clarify the exposition. For a concrete example of how this is done, see the example of the pentagasket in §9.4

6. Distributional Explicit Formulas for Fractal Strings

These four definitions and the three theorems that follow them are excerpted from §5.3 of [30]. The technical details described here are used in the proof of Thm. 7.5, especially in Appendix A and Appendix B. The reader can easily skim or skip this section on a first reading.

Definition 6.1. A *generalized fractal string* is defined to be a local positive Borel measure on $(0,\infty)$ and is denoted by η. Here, *local* means locally bounded with support bounded away from 0. The associated scaling zeta function is defined

$$\zeta_\eta(s) := \int_0^{\infty} x^{-s}\, d\eta. \tag{6.1}$$

It is clear that the scaling measure $\eta_{\mathfrak{s}}$ introduced in Def. 4.1 is a generalized fractal string, as discussed in §4.3.

Definition 6.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuous function. Then the *screen* is $S = \{f(t) + \mathbb{i}t : t \in \mathbb{R}\}$, the graph of a function with the axes interchanged. We let

$$\inf S := \inf_t f(t) = \inf\{\operatorname{Re} s : s \in S\}, \text{ and} \tag{6.2}$$

$$\sup S := \sup_t f(t) = \sup\{\operatorname{Re} s : s \in S\}. \tag{6.3}$$

The screen is thus a vertical contour in $\mathbb{C}$. The region to the right of the screen is the set W, called the *window*:

$$W := \{z \in \mathbb{C} : \operatorname{Re} z \geq f(\operatorname{Im} z)\}. \tag{6.4}$$

The poles of ζ_η which lie in the window are called the *visible scaling dimensions* and the set of them is denoted

$$\mathcal{D}_\eta(W) = \{\omega \in W : \lim_{s \to \omega} |\zeta_\eta(s)| = \infty\}. \tag{6.5}$$

Definition 6.3. The generalized fractal string η (as in Def. 6.1) is said to be *languid* if its associated zeta function ζ_η satisfies certain growth conditions relative to the screen.[6] Specifically, let $\{T_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{R}$ such that $T_{-n} < 0 < T_n$ for $n \geq 1$, and

$$\lim_{n\to\infty} T_n = \infty, \lim_{n\to\infty} T_{-n} = -\infty, \text{ and } \lim_{n\to\infty} \frac{T_n}{|T_{-n}|} = 1. \tag{6.6}$$

For η to be languid, there must exist real constants $\varpi, c > 0$ and a sequence $\{T_n\}$ as described in (6.6), such that

L1 For all $n \in \mathbb{Z}$ and all $\sigma \geq f(T_n)$,

$$|\zeta_\eta(\sigma + \mathbb{i}T_n)| \leq c \cdot (|T_n| + 1)^{\varpi}, \text{ and} \tag{6.7}$$

L2 For all $t \in \mathbb{R}$, $|t| \geq 1$,

$$|\zeta_\eta(f(t) + \mathbb{i}t)| \leq c \cdot |t|^{\varpi}. \tag{6.8}$$

In this case, η is said to be *languid of order* ϖ .

Definition 6.4. The generalized fractal string η is said to be *strongly languid* if it satisfies **L1** and the condition **L2'**, which is clearly stronger than **L2**:

L2' There exists a sequence of screens $S_m(t) = f_m(t) + \mathbb{i}t$ for $m \geq 1$, $t \in \mathbb{R}$, with $\sup S_m \to -\infty$ as $m \to \infty$, and with a uniform Lipschitz bound. Additionally, there must exist constants $a, c > 0$ such that

$$|\zeta_\eta(f(t) + \mathbb{i}t)| \leq c \cdot a^{|f_m(t)|}(|t| + 1)^{\varpi}, \tag{6.9}$$

for all $t \in \mathbb{R}$ and $m \geq 1$.

Taking [30], Thm. 5.26 and Thm. 5.30 at level $k = 0$ gives the following distributional explicit formula for the action of a fractal string η on a test function $\psi \in C^\infty(0, \infty)$. Note that ψ may not have compact support; only the decay properties (6.10)–(6.11) are required.

[6] We take ζ_η to be meromorphically continued to an open neighbourhood of W, as in Def. **??**.

Theorem 6.5 (Extended distributional explicit formula)**.** *Let η be a generalized fractal string which is languid of order ϖ. Let $\psi \in C^\infty(0,\infty)$ with n^{th} derivative satisfying, for some $\delta > 0$, and every $n \in \{0,1,\dots,N=[\varpi]+2\}$,*

$$\psi^{(n)}(x) = O\left(x^{-n-D-\delta}\right) \qquad \text{as } x \to \infty, \quad \text{and} \tag{6.10}$$

$$\psi^{(n)}(x) = \sum_\alpha a_\alpha^{(n)} x^{-\alpha-n} + O\left(x^{-n-\inf S+\delta}\right) \qquad \text{as } x \to 0^+. \tag{6.11}$$

Then we have the following distributional explicit formula for η:

$$\langle \eta, \psi \rangle = \sum_{\omega \in \mathcal{D}_\eta} \operatorname{res}\left(\zeta_\eta(s)\widetilde{\psi}(s);\omega\right) + \sum_{\alpha \in W \setminus \mathcal{D}_\eta} a_\alpha^{(0)} \zeta_\eta(\alpha) + \langle \mathcal{R}, \psi \rangle, \tag{6.12}$$

where the error term $\mathcal{R}(x)$ is the distribution given by

$$\langle \mathcal{R}, \psi \rangle = \tfrac{1}{2\pi \mathbb{i}} \int_S \zeta_\eta(s)\widetilde{\psi}(s)\, ds \tag{6.13}$$

and estimated by

$$\mathcal{R}(x) = O\left(x^{\sup S - 1}\right), \qquad \text{as } x \to \infty. \tag{6.14}$$

Here, $\widetilde{\psi}$ is the Mellin transform of the function ψ, defined by

$$\widetilde{\psi}(s) := \int_0^\infty x^{s-1}\psi(x)\, dx. \tag{6.15}$$

Note: the sum in (6.11) is over finitely many complex exponents α with $\operatorname{Re}\alpha > -\sigma_l + \delta$. This condition is described by saying that ψ has an asymptotic expansion of order $-\sigma_l + \delta$ at 0.

Taking [30], Thm. 5.27, at level $k = 0$ gives the following distributional explicit formula for the action of a fractal string η on a test function ψ. Note that in addition to requiring $\psi \in C^\infty(0,\infty)$, we now also require that ψ is a finite linear combination of terms $x^{-\beta}e^{-c_\beta x}$ in a neighbourhood of the interval $(0, A]$, where A is the same constant as in Def 6.4.

Theorem 6.6 (Extended distributional formula, without error term)**.** *Let η be a strongly languid generalized fractal string. Let $q \in \mathbb{N}$ be such that $q > \max\{1, \varpi\}$, where ϖ is as in Def. 6.3. Further, let ψ be a test function that is q times continuously differentiable on $(0,\infty)$. Assume that the j^{th} derivative $\psi^{(j)}(x)$ satisfies* (6.10) *and* (6.11)*, and that there exists a $\delta > 0$ such that*

$$\psi^{(j)}(x) = \sum_\alpha a_\alpha^{(j)} x^{-\alpha} e^{-c_\alpha x}, \qquad \text{for } x \in (0, A+\delta), 0 \le j \le q. \tag{6.16}$$

Then formula (6.12) *holds with $\mathcal{R} \equiv 0$.*

Theorem 6.7 (Tube formula for fractal strings [30], Thm. 8.1)**.** *Let $\eta = \eta_{\mathcal{L}}$ be a languid fractal string with geometric zeta function ζ_η. The volume of the (one-sided) tubular neighbourhood of radius ε of the boundary of η is given by the following distributional explicit formula for test functions $\psi \in C_c^\infty(0,\infty)$, the space of C^∞ functions with compact support contained in $(0,\infty)$:*

$$V_\eta(\varepsilon) = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\frac{\zeta_\eta(s)(2\varepsilon)^{1-s}}{s(1-s)};\omega\right) + \{2\varepsilon\zeta_\eta(0)\} + \mathcal{R}(\varepsilon). \tag{6.17}$$

Here the term in braces is only included if $0 \in W \setminus \mathcal{D}_\eta(W)$, *and* $\mathcal{R}(\varepsilon)$ *is the error term, given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi\mathbb{i}} \int_S \frac{\zeta_\eta(s)(2\varepsilon)^{1-s}}{s(1-s)}\, ds \tag{6.18}$$

and estimated by

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{1-\sup S}), \qquad \textit{as } \varepsilon \to 0^+. \tag{6.19}$$

The meaning of (6.14) and (6.19), the order of the distributional error term, is given in Def. B.6 of Appendix B.

When $\eta = \eta_\mathcal{L}$ is an ordinary fractal string $\mathcal{L}$, as in (4.2) above, then $V_\eta(\varepsilon) = V_L(\varepsilon)$ as in Def. 3.1, if L is the bounded open set defining $\mathcal{L}$. Furthermore, when $\eta_\mathcal{L}$ is a *self-similar* fractal string, the results of Thm. 6.7 may be strengthened as described in §8.2. In particular, one may take $W = \mathbb{C}$ and $\mathcal{R} \equiv 0$.

7. The Tube Formula for Fractal Sprays

In this section, we present the main result of the paper, a higher-dimensional analogue of Thm. 6.7. While the proof is similar in spirit, it is significantly more involved, especially if Appendices A and B are taken into account. This result provides new insight, particularly with regard to the geometric interpretation of the terms of the formula; see Remark 9.6. Also, it introduces the proper conceptual framework and confirms that fractal sprays are clearly the higher-dimensional counterpart of fractal strings. In a similar vein, we will see from Theorem 8.4 (the tube formula for self-similar tilings) that the self-similar tilings are the natural higher-dimensional analogue of self-similar fractal strings.

Although our primary goal in this paper is to obtain a tube formula for self-similar tilings, we state our main result for the more general class of fractal sprays, as we expect it to be useful in the study of other fractal structures and tilings to be investigated in forthcoming work. The important special case of self-similar tilings is stated in Thm. 8.4 of §8.2.

7.1. Statement of the tube formula.

We will prove the tube formula first for the more general case of fractal sprays, and then refine this result to obtain the formula for self-similar tilings. The following definition first appeared in [26].

Definition 7.1. Let $B \subseteq \mathbb{R}^d$ be a nonempty bounded open set, which we will call the *basic shape* or *generator*. Then a *fractal spray* is a bounded open subset of $\mathbb{R}^d$ which is the disjoint union of open sets Ω_n for $n = 1, 2, \ldots$, where each Ω_n is congruent to $\ell_n B$, the homothetic of B by ℓ_n. Here, $\mathcal{L} = \{\ell_n\}$ is a fractal string.

More generally, as in [30], a fractal spray is given by a nonempty bounded open set scaled by a fractal string η.

Thus, any fractal string can be thought of as a fractal spray on the basic shape $B = (0, 1)$, the unit interval. In the context of the current paper, a self-similar tiling is a union of fractal sprays on the basic shapes $G_1, \ldots, G_Q$, each scaled by a fixed *self-similar* string. A general fractal spray may have multiple generators, as long as they are all scaled by the same generalized fractal string η. However, for the remainder of this paper we consider only a single generator G. Indeed, as mentioned in §5.4, the multiple-generator case can readily be reduced to the case of a single generator.

Definition 7.2. The *geometric zeta function of a fractal spray* with a diphase generator is

$$\zeta_{\mathcal{T}}(\varepsilon,s) := \varepsilon^{d-s}\zeta_\eta(s)\sum_{i=0}^{d}\frac{g^{s-i}}{s-i}\kappa_i. \tag{7.1}$$

More generally, the geometric zeta function of a fractal spray with a pluriphase generator is

$$\zeta_{\mathcal{T}}(\varepsilon,s) := \varepsilon^{d-s}\zeta_\eta(s)\sum_{i=0}^{d}\sum_{m=1}^{M}\frac{(\varepsilon_m^{s-i}-\varepsilon_{m-1}^{s-i})}{s-i}\kappa_i^m. \tag{7.2}$$

Here, $\kappa_i = \kappa_i(G)$ or $\kappa_i^m = \kappa_i^m(G)$ as defined in Def. 5.2 or Def. 5.4, respectively.

The *geometric zeta function of a self-similar tiling* or *tiling zeta function* is similarly defined, except that ζ_η is replaced by $\zeta_{\mathfrak{s}}$, the scaling zeta function of a self-similar tiling described in Def. 4.2. It turns out that $\zeta_{\mathcal{T}}$ is a meromorphic distribution-valued function for each fixed $s \in W$, where $W \subseteq \mathbb{C}$ is the *window* defined in Def. 6.2. This verification is given in Def. A.5 and Thm. A.7 of Appendix A. Considered as a distribution, the action of $\zeta_{\mathcal{T}}(s,\cdot)$ on a test function $\psi \in C_c^\infty(0,\infty)$ is given by

$$\langle \zeta_{\mathcal{T}}(\varepsilon,s), \psi(\varepsilon)\rangle = \int_0^\infty \zeta_{\mathcal{T}}(\varepsilon,s)\psi(\varepsilon)\,d\varepsilon. \tag{7.3}$$

Here, $C_c^\infty(0,\infty)$ is the space of smooth functions with compact support contained in $(0,\infty)$. At first glance, it may appear strange that something as concretely geometric as a tube formula is given distributionally. However, the flexibility of the distributional framework allows the proof to proceed; see Rem. 5.20 of [30].

Remark 7.3. The presentation of the geometric zeta function in (7.1) differs from that given in [30], wherein the "geometric zeta function" is actually closer to what we call the scaling zeta function here. The general tube formula [30], Thm. 8.1, does involve the one-dimensional case of $\zeta_{\mathcal{T}}$, but it is not explicitly named as such. For several reasons, it behooves one to think of $\zeta_{\mathcal{T}}$ as the geometric zeta function most naturally associated with the spray (or tiling), especially as pertains to the tube formula:

(i) The function $\zeta_{\mathcal{T}}$ arises naturally in the expression of the tube formula for the tiling, as will be seen in Thm. 7.5 and Thm. 8.4.

(ii) It is the poles of $\zeta_{\mathcal{T}}(\varepsilon,s)$ that naturally index the sum appearing in $V_{\mathcal{T}}$, and the residues of $\zeta_{\mathcal{T}}$ that give the actual volume.

(iii) Using $\zeta_{\mathcal{T}}$ leads to the natural unification of expressions which previously appeared unrelated; compare (8.5) to (8.7) in Cor. 8.8.

Thus, the function $\zeta_{\mathcal{T}}$ encodes all the geometric information of $\mathcal{T}$ as pertains to its tube formula. In Rem. 8.12 we discuss how the unification mentioned in (ii) leads to a geometric interpretation of the term $\{2\varepsilon\zeta_\eta(0)\}$ that appears in (6.17).

Definition 7.4. The set of *visible complex dimensions of a fractal spray* is

$$\mathcal{D}_{\mathcal{T}}(W) := \mathcal{D}_\eta(W) \cup \{0,1,\ldots,d-1\}, \tag{7.4}$$

where $\mathcal{D}_\eta(W)$ is as in §7.1. Thus, $\mathcal{D}_{\mathcal{T}}(W)$ consists of the visible scaling dimensions and the "integral dimensions" of the spray. Furthermore, the poles of $\zeta_{\mathcal{T}}$ are all contained in $\mathcal{D}_\mathcal{T}(W)$. Note that $\mathcal{D}_\eta(W)$ and $\mathcal{D}_\mathcal{T}(W)$ are discrete subsets of $W \subseteq \mathbb{C}$, and hence are at most countable.

Theorem 7.5 (Tube formula for fractal sprays)**.** *Let η be a fractal spray on the diphase generator G, with generating inradius $g = \rho(G) > 0$. Assume that ζ_η is languid on a screen S which avoids the dimensions in $\mathcal{D}_\mathcal{T}(W)$. Then for test functions in $C_c^\infty(0,\infty)$, the d-dimensional volume of the inner tubular neighbourhood of the spray is given by the following distributional explicit formula:*

$$V_\mathcal{T}(\varepsilon) = \sum_{\omega \in \mathcal{D}_\mathcal{T}(W)} \operatorname{res}\left(\zeta_\mathcal{T}(\varepsilon, s); \omega\right) + \mathcal{R}(\varepsilon), \tag{7.5}$$

where the sum ranges over the set (7.4) *of integral and visible complex dimensions dimensions of the spray. Here, the error term $\mathcal{R}(\varepsilon)$ is given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi \mathbb{i}} \int_S \zeta_\mathcal{T}(\varepsilon, s)\, ds, \tag{7.6}$$

and estimated by

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{d-\sup S}), \qquad \text{as } \varepsilon \to 0^+. \tag{7.7}$$

As a distributional formula, (8.4) is valid when applied to test functions $\psi \in C_c^\infty(0,\infty)$. The order of the distributional error term as in (7.7) is defined in Def. B.6. There is a version of this theorem in which the error term vanishes identically; it is presented in Cor. 8.2. Also, the special case of self-similar tilings is presented in Thm. 8.4. The following proof relies heavily on the material in §6; the reader may wish to review this material before proceeding.

7.2. **Proof of the tube formula.** The proof of Theorem 7.5 is given only for a spray with a Steiner-like generator which is diphase; the more general pluriphase case is presented as a corollary, as the proof follows in the same fashion. The reader may now wish to review §6 before proceeding, as the proof uses these explicit formulas and distributional techniques from [30],

Proof of Thm. 7.5. Recall that we view $V_\mathcal{T}(\varepsilon)$ as a distribution,[7] so we understand $V_\mathcal{T}(\varepsilon) = \langle \eta_\mathfrak{s}, \gamma_G \rangle$ by computing its action on a test function ψ:

$$\langle V_\mathcal{T}(\varepsilon), \psi \rangle = \langle \langle \eta_\mathfrak{s}, \gamma_G \rangle, \psi \rangle = \int_0^\infty \left(\int_0^\infty \gamma_G(x,\varepsilon) d\eta_\mathfrak{s}(x) \right) \psi(\varepsilon)\, d\varepsilon \tag{7.8}$$

$$= \int_0^\infty \int_0^\infty \gamma_G(x,\varepsilon)\psi(\varepsilon)\, d\varepsilon\, d\eta_\mathfrak{s}(x) \tag{7.9}$$

$$= \langle \eta_\mathfrak{s}, \langle \gamma_G, \psi \rangle \rangle . \tag{7.10}$$

Now, writing $\kappa_i = \kappa_i(G)$, we use (5.8) to compute

$$\begin{aligned} \langle \gamma_G, \psi \rangle &= \int_0^\infty \gamma_G(x,\varepsilon)\psi(\varepsilon)\, d\varepsilon \\ &= \int_0^\infty \sum_{i=0}^d \kappa_i(\tfrac{1}{x}G; \varepsilon)\varepsilon^{d-i}\psi(\varepsilon)\, d\varepsilon \end{aligned}$$

[7] Indeed, $V_\mathcal{T}(\varepsilon)$ is clearly continuous and bounded (by the total volume of the spray), hence it defines a locally integrable function on $(0,\infty)$.

$$= \sum_{i=0}^{d-1} \int_0^\infty \kappa_i \chi_{[0,g/x)}(\varepsilon) x^{-i} \varepsilon^{d-i} \psi(\varepsilon)\, d\varepsilon - \int_0^\infty \kappa_d \chi_{[g/x,\infty)}(\varepsilon) x^{-d} \psi(\varepsilon)\, d\varepsilon$$
$$= \sum_{i=0}^{d-1} \kappa_i x^{-i} \int_0^{g/x} \varepsilon^{d-i} \psi(\varepsilon)\, d\varepsilon - \kappa_d x^{-d} \int_{g/x}^\infty \psi(\varepsilon)\, d\varepsilon$$
$$= \sum_{i=0}^{d} \varphi_i(x), \tag{7.11}$$

where, for $x > 0$, we have introduced

$$\varphi_i(x) := \begin{cases} \kappa_i x^{-i} \int_0^{g/x} \varepsilon^{d-i} \psi(\varepsilon)\, d\varepsilon, & 0 \le i \le d-1, \\ \kappa_i x^{-i} \int_\infty^{g/x} \psi(\varepsilon)\, d\varepsilon, & i = d, \end{cases} \tag{7.12}$$

in the last line. Caution: φ_i is a function of x, whereas ψ is a function of ε. Putting (7.11) into (7.10), we obtain

$$\langle V_{\mathcal{T}}, \psi \rangle = \left\langle \eta_{\mathfrak{s}}, \sum_{i=0}^{d} \varphi_i \right\rangle = \sum_{i=0}^{d} \langle \eta_{\mathfrak{s}}, \varphi_i \rangle . \tag{7.13}$$

To apply Thm. 6.5, we must first check that the functions φ_i satisfy the hypotheses (6.10)–(6.11). Recall that $\psi \in C_c^\infty(0,\infty)$.

For $i < d$, (6.10) is satisfied because for large x, the corresponding integral in (7.12) is taken over a set outside the (compact) support of ψ. This gives $\varphi_i(x) = 0$ for sufficiently large x, and it is clear that, *a fortiori*, the n^{th} derivative of φ_i satisfies

$$\varphi_i^{(n)}(x) = O(x^{-n-D-\delta}) \quad \text{for } x \to \infty,\ \forall n \ge 0. \tag{7.14}$$

To see that (6.11) is satisfied, note that ψ vanishes for x sufficiently large and thus we have

$$\varphi_i(x) = \kappa_i x^{-i} \int_0^\infty \varepsilon^{d-i} \psi(\varepsilon)\, d\varepsilon \quad \text{for } x \approx 0,$$

i.e., $\varphi_i(x) = a_i x^{-i}$ for all small enough $x > 0$, where a_i is the constant

$$a_i := \kappa_i \int_0^\infty \varepsilon^{d-i} \psi(\varepsilon)\, d\varepsilon = \kappa_i \widetilde{\psi}(d-i+1) = \lim_{x \to 0^+} x^i \varphi_i(x). \tag{7.15}$$

Here $\widetilde{\psi}$ is the Mellin transform of ψ, as in (6.15).

Thus, the expansion (6.11) for the test function φ_i consists of only one term, and for each $n = 0, 1, \ldots, N$,[8]

$$\varphi_i^{(n)}(x) = \tfrac{d^n}{dx^n} \left[a_i x^{-i} \right] = O(x^{-n-i}) \quad \text{for } x \to 0^+,\ \forall n \ge 0. \tag{7.16}$$

A key point is that since ψ is smooth, (7.14) and (7.16) will hold for each $n = 0, 1, \ldots, N$, as required by Thm. 6.5. Since the expansion of φ_i has only one term, the only α in the sum is $\alpha = i$. Thus a_i is the constant corresponding to a_α in (6.11).

Applying Thm. 6.5 in the case when $i < d$, (6.12) becomes

$$\langle \eta_{\mathfrak{s}}, \varphi_i \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\zeta_\eta(s) \widetilde{\varphi}_i(s); \omega \right) \ + \ \{ a_i \zeta_\eta(i) \}_{i \in W \setminus \mathcal{D}_\eta}$$

[8] Recall that η is languid of order ϖ and that $N = [\varpi] + 2$ in the hypotheses of Thm. 6.5.

$$+ \tfrac{1}{2\pi\mathbb{i}} \int_S \zeta_\eta(s)\widetilde{\varphi}_i(s)\,ds, \tag{7.17}$$

where the term in braces is to be included iff $i \in W \setminus \mathcal{D}_\eta$. Here and henceforth, $\widetilde{\varphi}_i$ denotes the Mellin transform of φ_i given by

$$\widetilde{\varphi}_i(s) = \int_0^\infty x^{s-1}\varphi_i(x)\,dx. \tag{7.18}$$

The case when $i = d$ is similar (or antisimilar). The compact support of ψ again gives

$$\varphi_d(x) = \kappa_d x^{-d} \int_\infty^0 \psi(\varepsilon)\,d\varepsilon, \quad \text{for } x \to \infty, \tag{7.19}$$

so that for some positive constant c, and for all sufficiently large x, we have $\kappa_d(x) = cx^{-d}$. Hence

$$\varphi_d^{(n)}(x) = O(x^{-n-d}) \quad \text{for } x \to \infty, \forall n \geq 0, \tag{7.20}$$

and (6.10) is satisfied. For very small x, the integral in the definition of $\kappa_d(x)$ is taken over an interval outside the support of ψ, and hence $\kappa_d(x) = 0$ for $x \approx 0$. Then clearly (6.11) is satisfied:

$$\varphi_d^{(n)}(x) = 0 \quad \text{for } x \to 0^+, \forall n \geq 0. \tag{7.21}$$

An immediate consequence of (7.21) is that for $i = d$ in (7.15), the constant term is

$$a_d = \lim_{x \to 0} x^d \varphi_d(x) = 0, \tag{7.22}$$

and compared with (7.17) we have one term less in

$$\langle \eta_{\mathfrak{s}}, \varphi_d \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\zeta_\eta(s)\widetilde{\varphi}_d(s); \omega\right) + \tfrac{1}{2\pi\mathbb{i}} \int_S \zeta_\eta(s)\widetilde{\varphi}_d(s)\,ds. \tag{7.23}$$

As in (7.18), denote the Mellin transform of the function ψ by $\widetilde{\psi}$ and compute

$$\begin{aligned}
\widetilde{\varphi}_i(s) = \int_0^\infty x^{s-1}\varphi_i(x)\,dx &= \kappa_i \int_0^\infty x^{s-i-1} \int_0^{g/x} \varepsilon^{d-i}\psi(\varepsilon)\,d\varepsilon\,dx \\
&= \kappa_i \int_0^\infty \left(\int_0^{g/\varepsilon} x^{s-i-1}\,dx \right) \varepsilon^{d-i}\psi(\varepsilon)\,d\varepsilon \\
&= \frac{\kappa_i}{s-i} \int_0^\infty g^{s-i}\varepsilon^{i-s}\varepsilon^{d-i}\psi(\varepsilon)\,d\varepsilon \\
&= g^{s-i}\frac{\kappa_i}{s-i}\widetilde{\psi}(d-s+1).
\end{aligned} \tag{7.24}$$

By a similar calculation,

$$\widetilde{\varphi}_d(s) = g^{s-d}\frac{\kappa_d}{s-d}\widetilde{\psi}(d-s+1). \tag{7.25}$$

Note that for $0 \leq i < d-1$, (7.24) is valid for $\operatorname{Re} s > i$, and for $i = d$, (7.25) is valid for $\operatorname{Re} s < i$. Thus both are valid in the strip $d-1 < \operatorname{Re} s < d$, and hence by analytic (meromorphic) continuation, they are valid everywhere in $\mathbb{C}$. Indeed, by Cor. A.4, $\widetilde{\psi}$ is entire.

We return to the evaluation of (7.13), applying Thm. 6.5 to find the action of $\eta_\mathfrak{s}$ on the test function φ_i, for $i = 0 \ldots, d$. Substituting (7.24) and (7.25) into (7.17) gives

$$\langle \eta_\mathfrak{s}, \varphi_i \rangle = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\zeta_\eta(s) \frac{g^{s-i}\kappa_i}{s-i} \widetilde{\psi}(d-s+1); \omega \right) + \{a_i \zeta_\eta(i)\}_{i \in W \setminus \mathcal{D}_\eta} + \langle \mathcal{R}_i, \psi \rangle, \tag{7.26}$$

where $\mathcal{R}_i$ is defined by

$$\langle \mathcal{R}_i, \psi \rangle := \tfrac{1}{2\pi \mathbb{i}} \int_S \zeta_\eta(s) \widetilde{\varphi}_i(s)\, ds. \tag{7.27}$$

Substituting (7.26) into (7.13), we obtain

$$\begin{aligned} \langle V(\varepsilon), \psi \rangle &= \sum_{i=0}^{d} \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\zeta_\eta(s) \frac{g^{s-i}\kappa_i}{s-i} \widetilde{\psi}(d-s+1); \omega \right) \\ &\quad + \sum_{i=0}^{d} \{a_i \zeta_\eta(i)\}_{i \in W \setminus \mathcal{D}_\eta} + \sum_{i=0}^{d} \langle \mathcal{R}_i(\varepsilon), \psi(\varepsilon) \rangle. \end{aligned} \tag{7.28}$$

Recall from (7.22) that the d^{th} term is $a_d = 0$, so the top term of the second sum vanishes. Note that at each such i we have a residue

$$\begin{aligned} \operatorname{res}\left(\zeta_\eta(s) \frac{g^{s-i}\kappa_i}{s-i} \widetilde{\psi}(d-s+1); i \right) &= \kappa_i \lim_{s \to i} \zeta_\eta(s) g^{s-i} \widetilde{\psi}(d-s+1) \\ &= \kappa_i \zeta_\eta(i) \widetilde{\psi}(d-i+1) \\ &= a_i \zeta_\eta(i). \end{aligned} \tag{7.29}$$

Since the terms of the second sum of (7.28) are only included for $i \in W \setminus \mathcal{D}_\eta(W)$, we can use (7.4) and (7.29) to put combine the two sums of (7.28) without losing or duplicating terms:

$$\langle V(\varepsilon), \psi \rangle = \sum_{\omega \in \mathcal{D}_\mathcal{T}(W)} \operatorname{res}\left(\widetilde{\psi}(d-s+1) \zeta_\eta(s) \sum_{i=0}^{d} \frac{g^{s-i}\kappa_i}{s-i}; \omega \right) + \langle \mathcal{R}(\varepsilon), \psi(\varepsilon) \rangle,$$

where $\mathcal{R}(\varepsilon) := \sum_{i=0}^{d} \mathcal{R}_i(\varepsilon)$. This may also be written as the distribution

$$V(\varepsilon) = \sum_{\omega \in \mathcal{D}_\eta(W)} \operatorname{res}\left(\varepsilon^{d-s} \zeta_\eta(s) \sum_{i=0}^{d} \frac{g^{s-i}\kappa_i}{s-i}; \omega \right) + \mathcal{R}(\varepsilon). \tag{7.30}$$

This completes the proof of (8.4). All that remains is the verification of the expression (7.6) for the error term, and error estimate (7.7). Due to their technical and specialized nature, we leave the proofs of (7.6) and (7.7) to Appendix B. □

8. Extensions and Consequences: the Tube Formula for Self-Similar Tilings

8.1. **Generalizations and special cases.** We extend Thm. 7.5 to deal with pluriphase generators. Recall from (5.14) that a pluriphase generator tube formula is given by

$$\gamma_G(x,\varepsilon) = \sum_{i=0}^{d} \sum_{m=1}^{M} \kappa_i^m(G)\chi_{[\varepsilon_{m-1},\varepsilon_m)}(\varepsilon) x^{-i}\varepsilon^{d-i} - \kappa_d(G)\chi_{[\varepsilon_M,\infty)}(\varepsilon) \tag{8.1}$$

and the geometric zeta function of the corresponding fractal spray is

$$\zeta_{\mathcal{T}}(\varepsilon,s) := \varepsilon^{d-s}\zeta_\eta(s) \sum_{i=0}^{d}\sum_{m=1}^{M} \frac{(\varepsilon_m^{s-i} - \varepsilon_{m-1}^{s-i})}{s-i}\kappa_i^m. \tag{8.2}$$

Theorem 8.1 (Pluriphase tube formula for fractal sprays)**.** *Let η be a fractal spray on the pluriphase generator G, where γ_G is defined with respect to the partition $\{0 = \varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}, \varepsilon_M = \infty\}$. Assume that ζ_η is languid on a screen S which avoids the dimensions in $\mathcal{D}_{\mathcal{T}}(W)$. Then just as in Thm. 7.5, the d-dimensional volume of the inner tubular neighbourhood of the spray is given by*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathcal{T}}(W)} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon,s);\omega\right) + \mathcal{R}(\varepsilon), \tag{8.3}$$

where the error term $\mathcal{R}(\varepsilon)$ is as in Thm. 7.5.

Proof. The proof is essentially analogous to that of Thm. 7.5. □

Again, recall from §5.4 that this may easily be extended to multiple generators simply be taking the corresponding finite sum.

The next corollary indicates that when ζ_η is strongly languid, one may take $W = \mathbb{C}$ in the previous theorem and the error term will vanish identically.

Corollary 8.2 (Tube formula for strongly languid fractal sprays)**.** *Let the hypotheses of Thm. 8.1 be satisfied, and additionally assume that ζ_η is* strongly *languid. Then*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathcal{T}}} \operatorname{res}\left(\zeta_{\mathcal{T}}(\varepsilon,s);\omega\right). \tag{8.4}$$

Proof. This is immediate upon combining Thm. 6.6 (the extended distributional formula without error term) with the proof of Thm. 8.1. One finds that $\mathcal{R}_i \equiv 0$ and thus $\mathcal{R} \equiv 0$ in (7.30). □

Remark 8.3 (Reality principle)*.* The nonreal complex dimensions appear in complex conjugate pairs and produce terms with coefficients which are also complex conjugates, in the general tube formula for fractal sprays. This ensures that formulas (8.4) and (8.5) are real-valued.

8.2. **The self-similar case.** Self-similar strings automatically satisfy the more stringent hypothesis of being *strongly languid*, as in Def. 6.4. This automatically entails that Cor. 8.2 holds,[9] so the window may be taken to be all of $\mathbb{C}$ and the error term vanishes identically, i.e., $\mathcal{R}(\varepsilon) \equiv 0$. Hence Thm. 7.5 may be strengthened for self-similar tilings as follows.

[9] This is essentially because Thm. 6.5 and Thm. 6.7 hold without error term. This is discussed further in [30], Thm. 5.27, and the end of [30], Thm. 8.1. A general discussion of the strongly languid case may be found in [30], Def. 5.3, and an argument showing that all self-similar strings are strongly languid is given in §6.4 of [30].

Theorem 8.4 (Tube formula for self-similar tilings)**.**
Let $\mathcal{T} = \{\Phi_w G\}$, be a self-similar tiling with pluriphase generator G and geometric zeta function $\zeta_\mathcal{T}$. Then the d-dimensional volume of the inner tubular neighbourhood of $\mathcal{T}$ is given by the following distributional explicit formula:

$$V_\mathcal{T}(\varepsilon) = \sum_{\omega \in \mathcal{D}_\mathcal{T}} \operatorname{res}\left(\zeta_\mathcal{T}(\varepsilon, s); \omega\right), \tag{8.5}$$

where $\mathcal{D}_\mathcal{T} = \mathcal{D}_\mathcal{T}(\mathbb{C}) = \mathcal{D}_\mathfrak{s}(\mathbb{C}) \cup \{0, 1, \ldots, d-1\}$.

Proof. Note that in this (self-similar) case, one has $\zeta_\eta(s) = \zeta_\mathfrak{s}(s)$ and $\mathcal{D}_\eta(\mathbb{C}) = \mathcal{D}_\mathfrak{s}(\mathbb{C})$, with $\eta = \eta_\mathfrak{s}$ as in (4.4). The proof follows [30], §6.4. According to Thm. 4.3, the scaling zeta function of a self-similar tiling has the form

$$\zeta_\mathfrak{s}(s) = \frac{1}{1 - \sum_{j=1}^J r_j^s}.$$

Let r_J be the smallest scaling ratio. Then from

$$|\zeta_\mathfrak{s}(s)| \ll \left(\frac{1}{r_J}\right)^{-|\sigma|} \qquad \text{as } \sigma = \operatorname{Re}(s) \to -\infty,$$

we deduce that $\zeta_\mathcal{T}$ is strongly languid and therefore apply Cor. 8.2. This argument follows from the analogous ideas regarding self-similar strings, which may be found in Ch. 8 of [30]. □

Remark 8.5. Thm. 8.4 provides a higher-dimensional counterpart of the tube formula obtained for self-similar strings in §8.4 of [30]. It should be noted that Thm. 8.4 applies to a slightly smaller class of test functions than Thm. 7.5. Indeed, the support of the test functions must be bounded away from 0 by $\mu_d(C)g/r_J$, where $C = [F]$ is the hull of the attractor (as in §2), g is the smallest generating inradius (as in (3.6)), and r_J is the smallest scaling ratio of Φ (as in (2.1)). This technicality is discussed further in [30], Def. 5.3 and Thm. 5.27, §6.4, and Thm. 8.1.

Corollary 8.6 (Measurability and the lattice/nonlattice dichotomy)**.** *A self-similar tiling is Minkowski measurable if and only if it is nonlattice.*

Proof. We define a self-similar tiling $\mathcal{T}$ to be Minkowski measurable iff

$$0 < \lim_{\varepsilon \to 0^+} V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)} < \infty, \tag{8.6}$$

i.e., if the limit in (8.6) exists *and* takes a value in $(0, \infty)$. A tiling has infinitely many complex dimensions with real part D iff it is lattice type, as mentioned in §4.3. Furthermore, all the poles with real part D are simple in that case. A glance at (8.8) then shows that $V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)}$ is a sum containing infinitely many purely oscillatory terms $c_\omega \varepsilon^{\mathrm{i}n\mathbf{p}}$, $n \in \mathbb{Z}$, where $\mathbf{p}$ is some fixed period. Thus, the limit (8.6) cannot exist; see also §8.4.2 of [30]. Conversely, the tiling is nonlattice iff D is the only complex dimension with real part D. In this case, D is simple and no term in the sum $V_\mathcal{T}(\varepsilon)\varepsilon^{-(d-D)}$ is purely oscillatory; thus the tiling $\mathcal{T}$ is measurable. See also §8.4.4 of [30]. □

Remark 8.7. In §8.3–8.4 of [30], it is shown that a self-similar fractal string (i.e., a 1-dimensional self-similar tiling) is Minkowski measurable if and only if it is nonlattice. Gatzouras showed in [8] that nonlattice self-similar subsets of $\mathbb{R}^d$ are Minkowski measurable, thereby extending to higher dimensions a result in [19], [3] and partially proving the geometric part of [19], Conj. 3, p. 163. The previous result gives a complete characterization of self-similar tilings in $\mathbb{R}^d$ as nonlattice if and only if they are Minkowski measurable. With the exception of Rem. 9.2, each of the examples discussed in §9 is lattice and hence not Minkowski measurable. Our results, however, apply to nonlattice tilings as well. A more detailed proof of Cor. 8.6 is possible via truncation, by using the screen and window technique of [30], Thm. 5.31 and Thm. 8.36.

The following corollary of Thm. 8.4 will be used in §9.

Corollary 8.8. *If, in addition to the hypotheses of Thm. 8.4, G is diphase and $\zeta_{\mathcal{T}}(s)$ has only simple poles, then*

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \sum_{i=0}^{d} \operatorname{res}\left(\zeta_{\mathfrak{s}}(s); \omega\right) \varepsilon^{d-\omega} \tfrac{g^{\omega-i}}{\omega-i} \kappa_i + \sum_{i=0}^{d-1} \kappa_i \zeta_{\mathfrak{s}}(i) \varepsilon^{d-i}. \tag{8.7}$$

It is not an error that the first sum extends to d in (8.7), while the second stops at $d-1$; see (7.22). Note that in Cor. 8.8, $\mathcal{D}_{\mathfrak{s}}$ does not contain any integer $i = 0, 1, \ldots, d-1$, because this would imply that $\zeta_{\mathcal{T}}$ has a pole of multiplicity at least 2 at such an integer. In general, at most one integer can possibly be a pole of $\zeta_{\mathfrak{s}}$; see §4.3.

Remark 8.9. For self-similar tilings satisfying the hypotheses of Cor. 8.8, it is clear that the general form of the tube formula is

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega \in \mathcal{D}_{\mathcal{T}}} c_\omega \varepsilon^{d-\omega}, \tag{8.8}$$

where for each fixed $\omega \in \mathcal{D}_{\mathfrak{s}}$,

$$c_\omega := \operatorname{res}\left(\zeta_{\mathfrak{s}}(s); \omega\right) \sum_{i=0}^{d} \frac{g^{\omega-i}}{\omega - i} \kappa_i. \tag{8.9}$$

Note that when $\omega = i \in \{0, 1, \ldots, d-1\}$, one has $c_\omega = c_i = \zeta_{\mathfrak{s}}(i)\kappa_i$.

Remark 8.10. The oscillatory nature of the geometry of $\mathcal{T}$ is apparent in (8.8). In particular, the existence of the limit in (8.6) can be determined by examining (8.8) and $\mathcal{D}_{\mathcal{T}}$.

8.3. **Recovering the tube formula for fractal strings.**

Remark 8.11. In the literature regarding the 1-dimensional case [28,30,5], the terms “gaps” and “multiple gaps” have been used where we have used “generators”.

In this section, we discuss the 1-dimensional tube formula of Thm. 6.7 which is true for general (i.e., not necessarily self-similar) fractal strings and which can be recovered from Thm. 7.5. Suppose $\mathcal{L} = \{\ell_n\}_{n=1}^\infty$ is a languid fractal string with associated measure $\eta = \sum_{n=1}^\infty \delta_{1/\ell_n}$, as in (4.2), and geometric zeta function $\zeta_\eta = \sum_{n=1}^\infty \ell_n^s$, as in (4.1). Considering the string now as a tiling, write $\mathcal{L}$ as $L = \{L_n\}_{n=1}^\infty$ to emphasize the fact that we are thinking of it as a spray instead of as a string. If we take the spray L to have as its single generator the interval $G = (0, 2)$, then L has inradii $\rho_n = \frac{1}{2}\ell_n = r_w g$, scaling measure $\eta_{\mathfrak{s}} = \sum \delta_{1/r_w} = \sum_{n=1}^\infty \delta_{2g/\ell_n}$ and geometric zeta function

$$\zeta_{\mathfrak{s}}(s) = \sum_{w \in \mathcal{W}} r_w^s = \sum_{n=1}^\infty \left(\frac{\ell_n}{2g}\right)^s = (2g)^{-s} \zeta_\eta(s).$$

The generator is clearly diphase with $\kappa_0 = 2$ and $\kappa_1 = -2$:

$$\gamma_G(x,\varepsilon) = \begin{cases} 2\varepsilon, & \varepsilon \leq g/x, \\ 2g/x, & \varepsilon \geq g/x. \end{cases} \tag{8.10}$$

One obtains the geometric zeta function of the (1-dimensional) tiling L as

$$\zeta_L(\varepsilon, s) = \varepsilon^{1-s}\zeta_{\mathfrak{s}}(s)\sum_{i=0}^{1}\frac{\kappa_i}{s-i} = \varepsilon^{1-s}(2g)^{-s}\zeta_\eta(s)\left(\frac{2}{s} - \frac{2}{s-1}\right). \tag{8.11}$$

Recall that we chose G so that $g = 1$. Then

$$\zeta_L(\varepsilon, s) = \frac{\zeta_\eta(s)(2\varepsilon)^{1-s}}{s(1-s)}$$

from Thm. 7.5 we exactly recover the tube formula $V_L(\varepsilon) = V_{\mathcal{L}}(\varepsilon)$ (and its error term) as given by Thm. 6.7.

Remark 8.12. In addition to recovering a previously known formula, we also gain a geometric interpretation of the terms appearing in the 1-dimensional tube formula (6.17), in view of the previous computation. In particular, one sees that the linear term $\{2\varepsilon\zeta_\eta(0)\}$ has a geometric interpretation in terms of the inner Steiner formula for an interval, and can be dissected as

$$2\varepsilon\zeta_\eta(0) = \kappa_0(G)\varepsilon^{1-0}\zeta_\eta(0) = (-2)\mu_i(G)\varepsilon^{d-i}\zeta_\eta(i), \tag{8.12}$$

where $i = 0$ and $d = 1$. Note that $\mu_0(G) = -1$ is the Euler characteristic of an open interval. This should be discussed further in [22].

8.4. **The Steiner Formula.** In order to explain the connections between this paper and results from convex geometry, we give a brief encapsulation of Steiner's classical result. Here, we denote the Minkowski sum of two sets in $\mathbb{R}^d$ by

$$A + B = \{x \in \mathbb{R}^d : x = a + b \text{ for } a \in A, b \in B\}.$$

Theorem 8.13. *If B^d is the d-dimensional unit ball and $A \subseteq \mathbb{R}^d$ is convex, then the d-dimensional volume of $A + \varepsilon B^d$ is given by*

$$\mathrm{vol}_d(A + \varepsilon B^d) = \sum_{i=0}^{d}\mu_i(A)\,\mathrm{vol}_{d\text{-}i}(B^{d-i})\varepsilon^{d-i}, \tag{8.13}$$

where μ_i is the renormalized i-dimensional intrinsic volume.

Up to some normalizing constant, the i-dimensional intrinsic volume is the same thing as the i^{th} total curvature or $(d-i)^{\text{th}}$ Quermassintegral. This valuation μ_i can be defined via integral geometry as the average measure of orthogonal projections to $(d-i)$-dimensional subspaces; see Chap. 7 of [16]. For now, we note that (up to a constant), there is a correspondence

μ_0	$\sim$	Euler characteristic,	μ_{d-1}	$\sim$	surface area,
μ_1	$\sim$	mean width,	μ_d	$\sim$	volume.

We have chosen the term "Steiner-like" for Def. 5.1 because the intrinsic volumes satisfy the following properties

(i) each μ_i is homogeneous of degree i, so that for $x > 0$,

$$\mu_i\,(xA) = \mu_i(A)\,x^i, \text{ and} \tag{8.14}$$

(ii) each $\mu_i(A)$ is rigid motion invariant, so that

$$\mu_i\left(T(A)\right) = \mu_i(A), \tag{8.15}$$

for any (affine) isometry T of $\mathbb{R}^d$.

Caution: the description of $\kappa_i(G)$ given in the conditions of Def. 5.1 is intended to emphasize the resemblance between $\kappa_i(G)$ and μ_i. However, $\kappa_i(G)$ may be signed (even when G is convex and $i = d-1, d$) and is more complicated in general. In contrast, the curvature measures Θ_i are always positive for convex sets.

Note that (8.13) gives the volume of the set of points which are within ε of A, including the points of A. If we denote the *exterior ε-neighbourhood* [10] of A by

$$A_\varepsilon^{ext} := \{x : d(x,A) \leq \varepsilon, x \notin A\},$$

then it is immediately clear that omitting the d^{th} term gives

$$\mathrm{vol}_d(A_\varepsilon^{ext}) = \sum_{i=0}^{d-1} C_i(A)\varepsilon^{d-i} \tag{8.16}$$

with $C_i(A) = \mu_i(A)\,\mathrm{vol}_{d\text{-}i}(B^{d-i})$. The intrinsic volumes μ_i can be localized and understood as the *curvature measures* described in [4] and Ch. 4 of [37]. In this case, for a Borel set $\beta \subseteq \mathbb{R}^d$, one has

$$\mathrm{vol}_d\{x \in A_\varepsilon^{ext} : p(x,A) \in \beta\} = \sum_{i=0}^{d-1} C_i(A,\beta)\varepsilon^{d-i} \tag{8.17}$$

where $p(x,A)$ is the metric projection of x to A, that is, the closest point of A to x. In fact, the curvature measures are obtained axiomatically in [37] as the coefficients of the tube formula, and it is this approach that we hope to emulate in our current work. In other words, we believe that κ_i may also be understood as a (total) curvature, in a suitable sense, and we expect that κ_i can be localized as a curvature measure. A more rigorous formulation of these ideas is currently underway in [23].

In [4], Federer unified the tube formulas of Steiner (for convex bodies, as described in Ch. 4 of [37]) and of Weyl (for smooth submanifolds, as described in [9] and [40]) and extended these results to sets of *positive reach.*[11] It is worth noting that Weyl's tube formula for smooth submanifolds of $\mathbb{R}^d$ is expressed as a polynomial in ε with coefficients defined in terms of curvatures (in the classical sense) that are *intrinsic* to the submanifold [40]. See §6.6–6.9 of [1] and the book [9]. Federer's tube formula has since been extended in various directions by a number of researchers in integral geometry and geometric measure theory, including [36,37], [41,42], [6,7], [38], and most recently (and most generally) in [13]. The books [9] and [37] contain extensive endnotes with further information and many other references.

[10] The primary reason we have worked with the inner ε-neighbourhood instead of the exterior is that it is more intrinsic to the set; it makes the computation independent of the embedding of $\mathcal{T}$ into $\mathbb{R}^d$. At least, this should be the case, provided the 'curvature' terms κ_i of Def. 5.2 are also intrinsic. As a practical bonus, working with the inner ε-neighbourhood allows us to avoid potential issues with the intersections of the ε-neighbourhoods of different components.

[11] A set A has positive reach iff there is some $\delta > 0$ such that any point x within δ of A has a unique metric projection to A, i.e., that there is a unique point A minimizing $dist(x,A)$. Equivalently, every point q on the boundary of A lies on a sphere of radius δ which intersects ∂A only at q.

To emphasize the present analogy, consider that Steiner's formula (8.16) may be rewritten

$$\mathrm{vol}_d(A_\varepsilon^{ext}) = \sum_{i\in\{0,1,\dots,d-1\}} c_i \varepsilon^{d-i}. \tag{8.18}$$

and our result (8.5) may be rewritten

$$V_{\mathcal{T}}(\varepsilon) = \sum_{\omega\in\mathcal{D}_{\mathfrak{s}}\cup\{0,1,\dots,d-1\}} c_\omega \varepsilon^{d-\omega}. \tag{8.19}$$

The obvious similarities between the tube formulas is striking. Our tube formula is a fractal power series in ε, rather than just a polynomial in ε (as in Steiner's formula). Moreover, our series is summed not just over the 'integral dimensions' $\{0, 1, \dots, d-1\}$, but also over the countable set $\mathcal{D}_{\mathfrak{s}}$ of complex dimensions. The coefficients c_ω of the tube formula are expressed in terms of the 'curvatures' and the inradii of the generators of the tiling.

Remark 8.14. The two formulas (8.18) and (8.19) initially appear to be measuring very different things, but this is misleading. If one considers the example of the Sierpinski tiling (discussed in §9.3) then it is immediately apparent that the exterior ε-neighbourhood of the Sierpinski gasket is, in fact, equal to the union of the inner ε-neighbourhood of the tiling and the exterior ε-neighbourhood of the largest triangle. With C_0 as in Figure 4,

$$\mathrm{vol}_2(\mathcal{SG}_\varepsilon^{ext}) = V_{\mathcal{SG}}(\varepsilon) + \mathrm{vol}_2(C_0). \tag{8.20}$$

However, things do not always work out so neatly, as the example of the Koch tiling shows; see §9.2. In the forthcoming paper [34], precise conditions are given for equality to hold as in (8.20). This allows one to compute explicit tube formulas for a large family of self-similar sets.

Remark 8.15 (Comparison of $V_{\mathcal{T}}$ with the Steiner formula). In the trivial situation when the spray consists only of finitely many scaled copies of a diphase generator (so the scaling measure η is supported on a finite set), the geometric zeta function will have no poles in $\mathbb{C}$. Therefore, the tube formula becomes a sum over only the numbers $0, 1, \dots, d-1$ (recall from (7.22) that $a_d = 0$, so the d^{th} summand vanishes), for which the residues simplify greatly as in (7.29). In this case, $\zeta_\eta(i) = \rho_1^i + \dots + \rho_N^i$, so each residue from (7.29) becomes a finite sum

$$\begin{aligned}\zeta_\eta(i)\kappa_i(\varepsilon) &= \rho_1^i \kappa_i \varepsilon^{d-i} + \dots + \rho_N^i \kappa_i \varepsilon^{d-i} \\ &= \kappa_i(r_{w_1}G)\varepsilon^{d-i} + \dots + \kappa_i(r_{w_N}G)\varepsilon^{d-i}\end{aligned}$$

where N is the number of scaled copies of the generator G, and r_{w_n} is the corresponding scaling factor. Thus, for each $n = 1, \dots, N$, we obtain a diphase formula for the scaled basic shape $r_{w_n}G$. The pluriphase case is analogous.

9. Tube Formula Examples

Although Remark 9.2 discusses how one may construct nonlattice examples, the other examples chosen in this section are *lattice* self-similar tilings, in the sense of §4.3. Also, all examples in this section have diphase generators in the sense of Def. 5.2, as is verified in each case.

Moreover, the scaling zeta function $\zeta_{\mathfrak{s}}$ of each example has only simple poles, with a single line of complex dimensions distributed periodically on the line $\operatorname{Re} s = D$. Thus, the tube formula may be substantially simplified via Cor. 8.8.

9.1. **The Cantor tiling.** The Cantor tiling $\mathcal{C}$ (called the Cantor string in [30], §1.1.2 and §2.3.1) is constructed via the self-similar system

$$\Phi_1(x) = \tfrac{x}{3}, \qquad \Phi_2(x) = \tfrac{x+2}{3}.$$

The associated self-similar set F is the classical ternary Cantor set, so $d = 1$ and we have one scaling ratio $r = \frac{1}{3}$, and one generator $G = \left(\frac{1}{3}, \frac{2}{3}\right)$ with generating inradius $g = \frac{1}{6}$. The corresponding self-similar string has inradii $\rho_k = gr^k$ with multiplicity 2^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s}}, \tag{9.1}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_3 2,\ \mathbf{p} = \tfrac{2\pi}{\log 3}. \tag{9.2}$$

We note that $\zeta_{\mathfrak{s}}(0) = -1$ and apply (8.11) from the previous section to recover the following tube formula for $\mathcal{C}$ (as obtained in [30], §1.1.2):

$$V_{\mathcal{C}}(\varepsilon) = \frac{1}{2\log 3} \sum_{n \in \mathbb{Z}} \frac{(2\varepsilon)^{1-D-\mathbb{i}n\mathbf{p}}}{(D + \mathbb{i}n\mathbf{p})(1 - D - \mathbb{i}n\mathbf{p})} - 2\varepsilon. \tag{9.3}$$

Alternatively, this may be written as a series in $\left(\frac{\varepsilon}{g}\right)$ as

$$V_{\mathcal{C}}(\varepsilon) = \frac{1}{3\log 3} \sum_{n \in \mathbb{Z}} \left(\frac{1}{D + \mathbb{i}n\mathbf{p}} - \frac{1}{D - 1 + \mathbb{i}n\mathbf{p}} \right) \left(\frac{\varepsilon}{g}\right)^{1-D-\mathbb{i}n\mathbf{p}} - 2\varepsilon, \tag{9.4}$$

with $g = \frac{1}{6}$, $D = \log_3 2$, and $\mathbf{p} = 2\pi/\log 3$. It is this form of the tube formula which is closer in appearance to the following examples.

9.2. **The Koch tiling.** The standard Koch tiling $\mathcal{K}$ (see Fig. 3, along with Fig. 1 of §2) is constructed via the self-similar system

$$\Phi_1(z) := \xi\overline{z} \quad \text{and} \quad \Phi_2(z) := (1 - \xi)(\overline{z} - 1) + 1, \tag{9.5}$$

with $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$ and $z \in \mathbb{C}$. The attractor of $\{\Phi_1, \Phi_2\}$ is the classical von Koch curve. Thus $\mathcal{K}$ has one scaling ratio $r = |\xi| = 1/\sqrt{3}$, and one generator G: an equilateral triangle of side length $\frac{1}{3}$ and generating inradius $g = \frac{\sqrt{3}}{18}$. This tiling has inradii $\rho_k = gr^k$ with multiplicity 2^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s/2}}, \tag{9.6}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_3 4,\ \mathbf{p} = \tfrac{4\pi}{\log 3}. \tag{9.7}$$

By inspection, a tile with inradius $1/x$ will have tube formula

$$\gamma_G(x, \varepsilon) = \begin{cases} 3^{3/2}\left(-\varepsilon^2 + 2\varepsilon x\right), & \varepsilon \leq 1/x, \\ 3^{3/2}x^2, & \varepsilon \geq 1/x. \end{cases} \tag{9.8}$$

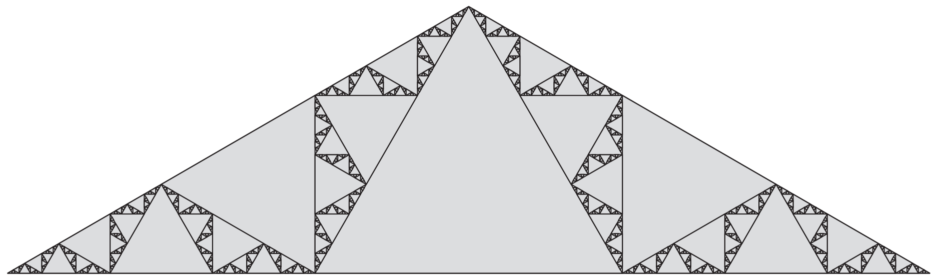

FIGURE 3. The Koch tiling $\mathcal{K}$.

For fixed x, (9.8) is clearly continuous at $\varepsilon = 0^+$. Thus we have

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 2 \cdot 3^{-s/2}} \qquad \text{and}$$
$$\kappa_0 = -3^{3/2}, \kappa_1 = 2 \cdot 3^{3/2}, \kappa_2 = -3^{3/2}.$$

Now applying (8.7), the tube formula for the Koch tiling $\mathcal{K}$ is

$$\begin{aligned} V_{\mathcal{K}}(\varepsilon) &= 3^{3/2} g^2 \sum_{\omega \in \mathcal{D}_{\mathfrak{s}}} \operatorname{res}\left(\frac{1}{1 - 2 \cdot 3^{-s/2}}; \omega\right) \left(-\tfrac{1}{\omega} + \tfrac{2}{\omega-1} - \tfrac{1}{\omega-2}\right) \left(\tfrac{\varepsilon}{g}\right)^{2-\omega} \\ &\qquad + \tfrac{g}{2}\zeta_{\mathfrak{s}}(0) \operatorname{res}\left(-\tfrac{1}{s}; 0\right) \left(\tfrac{\varepsilon}{g}\right)^{2-0} + \tfrac{g}{2}\zeta_{\mathfrak{s}}(1) \operatorname{res}\left(\tfrac{2}{s-1}; 1\right) \left(\tfrac{\varepsilon}{g}\right)^{2-1} \\ &= \frac{g}{\log 3} \sum_{n \in \mathbb{Z}} \left(-\tfrac{1}{D + \mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right) \left(\tfrac{\varepsilon}{g}\right)^{2-D-\mathbb{i}n\mathbf{p}} \\ &\qquad + 3^{3/2}\varepsilon^2 + \tfrac{1}{1-2\cdot 3^{-1/2}}\varepsilon, \end{aligned} \tag{9.9}$$

where $D = \log_3 4$, $g = \frac{\sqrt{3}}{18}$ and $\mathbf{p} = \frac{4\pi}{\log 3}$ as before.

Remark 9.1. In [21], a tube formula was obtained for the ε-neighbourhood of the Koch curve itself (rather than of the tiling associated with it) and the possible complex dimensions of this curve were inferred to be

$$\mathcal{D}_{\mathcal{K}^\star} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \cup \{0 + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\},$$

where $D = \log_3 4$ and $\mathbf{p} = \frac{2\pi}{\log 3}$. The line of poles above D was expected[12], and agrees precisely with the results of this paper. The meaning of the line of poles above 0 is still unclear. A zeta function for the Koch curve was not defined prior to the present paper; all previous reasoning was by analogy with (6.17).

Remark 9.2 (Nonlattice Koch tilings). By replacing $\xi = \frac{1}{2} + \frac{1}{2\sqrt{3}}\mathbb{i}$ in (9.5) with any other complex number satisfying $|\xi|^2 + |1-\xi|^2 < 1$, one obtains a family of examples of nonlattice self-similar tilings. The tube formula computations parallel the lattice case almost identically. The lattice Koch tilings correspond exactly to those $\xi \in B(\frac{1}{2}, \frac{1}{2})$ (the ball of radius $\frac{1}{2}$ centered at $\frac{1}{2} \in \mathbb{C}$) for which $\log_r |\xi|$ and $\log_r |1-\xi|$ are both positive integers, for some fixed $0 < r < 1$. Further discussion (and illustrations) of nonlattice Koch tilings may be found in [32].

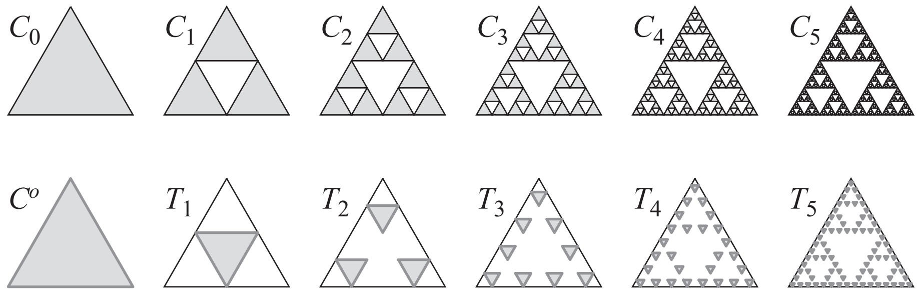

FIGURE 4. The Sierpinski gasket tiling.

9.3. **The Sierpinski gasket tiling.** The Sierpinski gasket tiling $\mathcal{SG}$ (see Fig. 4) is constructed via the system

$$\Phi_1(z) := \tfrac{1}{2}z, \quad \Phi_2(z) := \tfrac{1}{2}z + \tfrac{1}{2}, \quad \Phi_3(z) := \tfrac{1}{2}z + \tfrac{1+\mathbb{i}\sqrt{3}}{4},$$

which has one common scaling ratio $r = 1/2$, and one generator G: an equilateral triangle of side length $\frac{1}{2}$ and inradius $g = \frac{1}{4\sqrt{3}}$. Thus $\mathcal{SG}$ has inradii $\rho_k = gr^k$ with multiplicity 3^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 3 \cdot 2^{-s}}, \tag{9.10}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_2 3,\ \mathbf{p} = \tfrac{2\pi}{\log 2}. \tag{9.11}$$

Aside from $\zeta_{\mathfrak{s}}(s)$, the tube formula calculation for $\mathcal{SG}$ is identical to that for the previous example $\mathcal{K}$:

$$\begin{aligned} V_{\mathcal{SG}}(\varepsilon) = \tfrac{\sqrt{3}}{16 \log 2} \sum_{n \in \mathbb{Z}} \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}} \right) \left(\tfrac{\varepsilon}{g} \right)^{2-D-\mathbb{i}n\mathbf{p}} \\ + \tfrac{3^{3/2}}{2}\varepsilon^2 - 3\varepsilon. \end{aligned} \tag{9.12}$$

Remark 9.3. Suppose that for a tiling $\mathcal{T}$, the boundary of the hull intersects the boundary of a generator in at most a finite set: $|\partial C \cap \partial G_q| < \infty$. In this case, the tube formula for the tiling is almost the (exterior) tube formula for the attractor. This is the case for the Sierpinski gasket, and also for the Sierpinksi carpet (in which case the intersection is empty). In fact, the exterior ε-neighbourhood of the Sierpinski gasket curve is obtained by adding the Steiner's formula for $\overline{C}$:

$$\mathrm{vol}_2((\mathcal{SG})_\varepsilon) = V_{\mathcal{SG}}(\varepsilon) + 3\varepsilon + \pi\varepsilon^2. \tag{9.13}$$

9.4. **The Pentagasket tiling.** The Pentagasket tiling $\mathcal{P}$ (see Fig. 5) is constructed via the self-similar system defined by the five maps

$$\Phi_j(x) = \tfrac{3-\sqrt{5}}{2}x + p_j, \qquad j = 1, \dots, 5,$$

with common scaling ratio $r = \phi^{-2}$, where $\phi = (1+\sqrt{5})/2$ is the golden ratio, and the points $\frac{p_j}{1-r} = c_j$ form the vertices of a regular pentagon of side length 1. The Pentagasket $\mathcal{P}$ is an example of multiple generators G_q: G_1 is a regular pentagon and $G_2, \dots, G_6$ are congruent isosceles triangles, as seen in T_1 of Fig. 5. To make

[12]This set of complex dimensions was predicted in [27], §10.3.

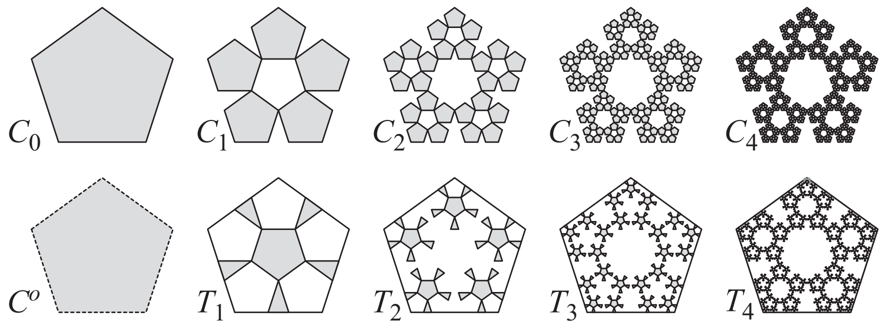

FIGURE 5. The Pentagasket tiling.

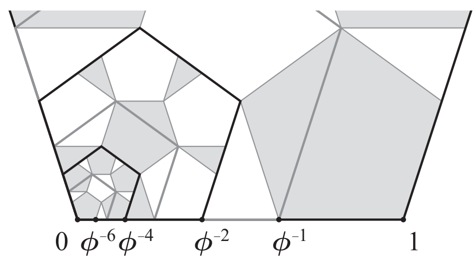

FIGURE 6. The pentagasket and the golden ratio ϕ.

the notation more meaningful, we use the subscripts p, t to indicate a pentagon or triangle, respectively. The generating inradius for the pentagon is $g_p = \frac{\phi^2}{2} \tan \frac{3}{10}\pi$ and the generating inradius for the triangles is $g_t = \frac{\phi^3}{2} \tan \frac{\pi}{5}$. Thus, $\mathcal{P}$ has inradii $\rho_k = g_q r^k$, for $q = p, t$, with multiplicity 5^k, so the scaling zeta function is

$$\zeta_{\mathfrak{s}}(s) = \frac{1}{1 - 5 \cdot r^{-s}}, \tag{9.14}$$

and the scaling complex dimensions are

$$\mathcal{D}_{\mathfrak{s}} = \{D + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}\} \qquad \text{for } D = \log_{1/r} 5,\ \mathbf{p} = \tfrac{2\pi}{\log r^{-1}}. \tag{9.15}$$

We omit the exercise of finding volumes for the pentagonal and triangular generators; the tube formula for a tile of inradius $1/x$ is

$$\gamma_q(x, \varepsilon) = \begin{cases} \kappa_{q0}(\varepsilon)x^0 + \kappa_{q1}(\varepsilon)x^1 = \alpha_q \left(-\varepsilon^2 + 2\varepsilon x\right), & \varepsilon \leq 1/x, \\ \kappa_{q2}(\varepsilon)x^2 = \alpha_q x^2, & \varepsilon \geq 1/x, \end{cases}$$

where $\alpha_p := 5 \cot \frac{3}{10}\pi$ and $\alpha_t := (\cot \frac{\pi}{5}) / \left(1 - \tan^2 \frac{\pi}{5}\right)$. Since $G_2, \ldots, G_6$ are congruent, we will apply Cor. 8.8 to a triangle G_t and multiply by 5 before adding to it to the result of applying Cor. 8.8 to the pentagon G_p. For the pentagon and the triangle, we have $\kappa_0 = -\alpha_q, \kappa_1 = 2\alpha_q$, and $\kappa_2 = -\alpha_q$.

The geometric zeta function of $\mathcal{P}$ is

$$\zeta_{\mathcal{P}}(\varepsilon, s) = \sum_{q=1}^{6} \frac{\alpha_q g_q^s}{1 - 5 \cdot r^{-s}} \left(-\tfrac{1}{s} + \tfrac{2}{s-1} - \tfrac{1}{s-2}\right) \varepsilon^{2-s},$$

and the tube formula for $\mathcal{P}$ is

$$\begin{aligned} V_{\mathcal{P}}(\varepsilon) &= \frac{\alpha_p}{\log r^{-1}} \sum_{n\in\mathbb{Z}} g_p^2 \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right)\left(\tfrac{\varepsilon}{g_p}\right)^{2-D-\mathbb{i}n\mathbf{p}} \\ &+ \frac{5\alpha_t}{\log r^{-1}} \sum_{n\in\mathbb{Z}} g_t^2 \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right)\left(\tfrac{\varepsilon}{g_t}\right)^{2-D-\mathbb{i}n\mathbf{p}} \\ &+ \left[\left(\tfrac{\alpha_p}{4} + \tfrac{5\alpha_t}{4}\right)\varepsilon^2 + \tfrac{(2\alpha_p g_q + 10\alpha_p g_q r)r}{r-5}\varepsilon\right], \end{aligned} \tag{9.16}$$

with $r = \phi^{-2}$, $\alpha_p = 5\cot\frac{3}{10}\pi$, $\alpha_t = (\cot\frac{\pi}{5})/\left(1-\tan^2\frac{\pi}{5}\right)$, $g_p = \frac{\phi^2}{2}\tan\frac{3}{10}\pi$, $g_t = \frac{\phi^3}{2}\tan\frac{\pi}{5}$, $D = \log_{1/r} 5$ and $\mathbf{p} = \frac{2\pi}{\log r^{-1}}$.

Remark 9.4. Much as in the case of fractal strings where $d = 1$ (see [30], §8.4.2), it follows from Thm. 8.4 that for a lattice self-similar tiling $\mathcal{T}$, each line of simple complex dimensions $\beta + \mathbb{i}n\mathbf{p}$ gives rise to a function which consists of a multiplicatively periodic function times $\varepsilon^{d-\beta}$. Here, β is some real constant and $\mathbf{p} = 2\pi/\log r^{-1}$ is the oscillatory period of $\mathcal{T}$. Consequently, the tube formula for each lattice tiling in this section has the form

$$V_{\mathcal{T}}(\varepsilon) = h\left(\log_{r^{-1}}(\varepsilon^{-1})\right)\varepsilon^{d-\beta} + P(\varepsilon), \tag{9.17}$$

where h is an additively periodic function of period 1 and P is a polynomial in ε. For instance, the periodic function appearing in the tube formula (9.9) for the Koch tiling $\mathcal{K}$ of Example 9.2 has the following Fourier expansion:

$$h(u) = \tfrac{g}{\log 3} \sum_{n\in\mathbb{Z}} g^{\mathbb{i}n\mathbf{p}} \left(-\tfrac{1}{D+\mathbb{i}n\mathbf{p}} + \tfrac{2}{D-1+\mathbb{i}n\mathbf{p}} - \tfrac{1}{D-2+\mathbb{i}n\mathbf{p}}\right) e^{2\pi\mathbb{i}nu}, \tag{9.18}$$

where $g = \sqrt{3}/18$, $D = \log_3 4$, $r = 1/\sqrt{3}$, and $\mathbf{p} = 4\pi/\log 3$. We note that multiplicatively periodic terms appear frequently in the mathematics and physics literature. See, for example, the relevant references given in §1.5, §2.7, §6.6, and §12.5 of [30].

9.5. **Some remarks on the results in this paper.**

Remark 9.5. The monograph [30] proposes a new definition of a *fractal* as "an object with nonreal complex dimensions that have a positive real part". With respect to this definition, the present work confirms the fractal nature of all the examples discussed in §9, and more generally, of all self-similar tilings considered in this paper.

Remark 9.6. Our results for tilings shed new light on the (1-dimensional) tube formula for fractal strings (1.3). The origin of the previously mysterious linear term $\{2\varepsilon\zeta_{\mathcal{L}}(0)\}$ (see (6.17))) is now seen to come from a diphase formula for the unit interval, akin to (5.2). This is discussed further in §8.3. In fact, all terms coming from the third sum of the extended distributional formula of Thm. 6.5 are now understood to be related to a pluriphase formula. This reveals a geometric interpretation and allows the two sums to be naturally combined, as seen in (7.29) and (8.12).

Remark 9.7. Many classical fractal curves are attractors of more than one self-similar system. For example, the Koch curve discussed in §9.2 is also the attractor of a system of 4 mappings, each with scaling ratio $r = \frac{1}{3}$. In this particular example, changes in the scaling zeta function produce a different set of complex dimensions. In fact, we obtain a subset of the original complex dimensions: $\{\log_3 4 + \mathbb{i}n\mathbf{p} : n \in \mathbb{Z}, \mathbf{p} = 4\pi/\log 3\}$. This has a natural geometric interpretation which is to be discussed in later work. In particular, it would be desirable to determine precisely which characteristics remain invariant between different tilings which are so related.

Remark 9.8. The tube formula for a self-similar tiling may differ from the tube formula for the corresponding self-similar set, as discussed at the end of Rem. 8.14. Despite this, it gives us valuable information about self-similar geometries (and their associated dynamical systems). Indeed, we can define the complex dimensions of a given self-similar set in $\mathbb{R}^d$ to be those of the self-similar tiling canonically associated to it (as in [32]). This is motivated by focusing on the dynamics of the self-similar system, rather than looking directly at the set. For an example, see §9.2, especially Rem. 9.1. Even so, the inner ε-neighbourhood of certain self-similar tilings is exactly equal to the outer ε-neighbourhood of the attractor, less the outer ε-neighbourhood of the convex hull of the attractor, as discussed in Rem. 8.14. The Sierpinski gasket and carpet are both examples of this phenomenon. For such fractals, our results immediately give additional information, e.g., Minkowski nonmeasurability. Furthermore, we are able to obtain the exact ε-neighbourhood for a large class of self-similar fractal sets; see Rem 9.3.

Appendix A. The Definition and Properties of $\zeta_{\mathcal{T}}$

In this appendix, we confirm some basic properties of the geometric zeta function of a tiling $\zeta_{\mathcal{T}}$. However, we first require some facts about Mellin transformation. If $\psi \in \mathbb{D} = C_c^\infty(0,\infty)$, it is elementary to check that for every $s \in \mathbb{C}$, the Mellin transform $\widetilde{\psi}(s)$ is given by the well-defined integral (6.15) and satisfies $|\widetilde{\psi}(s)| \leq |\widetilde{\psi}|(\operatorname{Re} s) < \infty$. We will need additional estimates in what follows. We also use the forthcoming fact that $\widetilde{\psi}(s)$ is an entire function.

Lemma A.1. *Suppose that $S \subseteq \mathbb{C}$ is horizontally bounded, so $\inf S := \inf_S \operatorname{Re} s$ and $\sup S := \sup_S \operatorname{Re} s$ are finite. Let K be a compact interval containing the support of $\psi \in C_c^\infty(0,\infty)$. Then there is a constant $c_K > 0$ depending only on K such that*

$$\sup_{s \in S} |\widetilde{\psi}(s)| \leq c_K \|\psi\|_\infty. \tag{A.1}$$

In particular, $\widetilde{\psi}(s)$ is always uniformly bounded on any screen S as in Def. 6.2.

Proof. Let K be a compact interval containing the support of ψ. Since

$$|x^{s-1}| = x^{\operatorname{Re} s-1} \leq \begin{cases} x^{\sup S-1}, & x \geq 1, \\ x^{\inf S-1}, & 0 < x < 1, \end{cases} \tag{A.2}$$

one can define a bound

$$b_K := \sup_{x \in K} \max\{x^{\sup S-1}, x^{\inf S-1}\}.$$

Note that b_K is finite because the function $x \mapsto \max\{x^{\sup S-1}, x^{\inf S-1}\}$ is continuous on the compact set K, and hence is bounded. Then we use (A.2) to bound $\widetilde{\psi}$ as follows:

$$\begin{aligned} |\widetilde{\psi}(s)| &\leq \int_0^\infty |x^{s-1}| \cdot |\psi(x)|\, dx \\ &= \int_K x^{\operatorname{Re} s-1} |\psi|(x)\, dx = \widetilde{|\psi|}(\operatorname{Re} s) \\ &\leq b_K \|\psi\|_\infty \cdot \operatorname{vol}_1(K). \end{aligned} \tag{A.3}$$

□

Remark A.2. The exact counterpart of Lemma A.1 holds if $\widetilde{\psi}(s)$ is replaced by a translate $\widetilde{\psi}(s-s_0)$, for any $s_0 \in \mathbb{C}$. Therefore, under the same assumptions as in Lemma A.1, we have

$$\sup_{s\in S} |\widetilde{\psi}(s-s_0)| \leq c_{K,s_0} \|\psi\|_\infty, \tag{A.4}$$

where $c_{K,s_0} := b_{K,s_0} \cdot \operatorname{vol}_1(K)$, and

$$b_{K,s_0} := \sup_{x\in K} \max\{x^{\sup S-\operatorname{Re} s_0-1}, x^{\inf S-\operatorname{Re} s_0-1}\} < \infty. \tag{A.5}$$

In particular, for any compact interval K containing the support of ψ, and for each fixed integer $k \geq 0$,

$$\sup_{s\in S} \left|\widetilde{\psi}(s-d+k+1)\right| \leq c_{K,k} \|\psi\|_\infty, \tag{A.6}$$

where $c_{K,k}$ is a finite and positive constant.

Lemma A.3. *Let (X,μ) be a measure space. Define an integral transform by $F(s) = \int_X f(x,s)\, d\mu(x)$ where*

$$|f(x,s)| \leq G(x), \quad \textit{for } \mu\textit{-a.e. } x \in X,$$

for some $G \in L^1(X,\mu)$, and for all s in some neighbourhood of $s_0 \in \mathbb{C}$. If the function $s \mapsto f(x,s)$ is holomorphic for μ-a.e. $x \in X$, then $F(s)$ is well-defined and holomorphic at s_0.

The proof is a well-known application of Lebesgue's Dominated Convergence Theorem. We use Lemma A.3 to obtain the following corollary, which is used to prove Thm. 7.5 and Thm. A.7.

Corollary A.4. *For $\psi \in C_c^\infty(0,\infty)$, $\widetilde{\psi}(s)$ is entire.*

Proof. Fix $s_0 \in \mathbb{C}$. If s is in a compact neighbourhood of s_0, then $\operatorname{Re} s$ is bounded, say by $\alpha \in \mathbb{R}$. Then for almost every $x > 0$,

$$\left|x^{s-1}\psi(x)\right| \leq x^{\alpha-1} \|\psi\|_\infty \chi_\psi, \tag{A.7}$$

where χ_ψ is the characteristic function of the compact support of ψ. Upon application of Lemma A.3, one deduces that ψ is holomorphic at s_0. □

Caution: note that this does not combine with Lemma A.1 to imply that $\widetilde{\psi}$ is constant; indeed, Liouville's Theorem does not apply because s is restricted to S in these two propositions.

Definition A.5. For $T(\varepsilon, s)$ to be a *weakly meromorphic distribution-valued function* on W, there must exist (i) a discrete set $\mathcal{P}_T \subseteq \mathbb{C}$, and (ii) for each $\omega \in \mathcal{P}_T$, an integer $n_\omega < \infty$, such that $\Psi(s) = \langle T(\varepsilon, s), \psi(\varepsilon)\rangle$ is a meromorphic function of $s \in W$, and each pole ω of Ψ lies in $\mathcal{P}_T$ and has multiplicity at most n_ω.

To say that the distribution-valued function $T : W \to \mathbb{D}'$ given by $s \mapsto T(\varepsilon, s)$ is *(strongly) meromorphic* means that, as a $\mathbb{D}'$-valued function, it is truly a meromorphic function, in the sense of the proof of Lemma A.6. Recall that we are working with the space of distributions $\mathbb{D}'$, defined as the dual of the space of test functions $\mathbb{D} = C_c^\infty(0, \infty)$.

Lemma A.6. *If T is a weakly meromorphic distribution-valued function, then it is a (strongly) meromorphic distribution-valued function.*

Proof. For $\omega \notin \mathcal{P}_T$, note that as $s \to \omega$,

$$\frac{T(\varepsilon, s) - T(\varepsilon, \omega)}{s - \omega} \tag{A.8}$$

converges to a distribution (call it $T'(\varepsilon, \omega)$) in $\mathbb{D}'$, by the Uniform Boundedness Principle for a topological vector space such as $\mathbb{D}$; see [35], Thm. 2.5 and Thm. 2.8. Hence, the $\mathbb{D}'$-valued function T is holomorphic at ω.

For $\omega \in \mathcal{P}_T$, apply the same argument to

$$\lim_{s \to \omega} \frac{1}{(n_\omega - 1)!} \left(\frac{d}{ds}\right)^{n_\omega - 1} \Big((s - \omega)^{n_\omega} T(\varepsilon, s)\Big), \tag{A.9}$$

which must therefore define a distribution, i.e., exist as an element of $\mathbb{D}'$. Thus T is truly a meromorphic function with values in $\mathbb{D}'$, and with poles contained in $\mathcal{P}_T$. □

Theorem A.7. *Under the hypothesis of Thm. 7.5 or Thm. 8.4, the geometric zeta function of a fractal spray or tiling*

$$\zeta_{\mathcal{T}}(\varepsilon, s) = \varepsilon^{d-s} \zeta_\eta(s) \sum_{i=0}^{d} \frac{g^{s-i}}{s - i} \kappa_i \tag{A.10}$$

is a distribution-valued (strongly) meromorphic function on W, with poles contained in $\mathcal{D}_{\mathcal{T}}$.

Proof. Let $\mathcal{P}_T = \mathcal{D}_{\mathcal{T}}$ and note that

$$\begin{aligned} \langle \zeta_{\mathcal{T}}(\varepsilon, s), \psi(\varepsilon)\rangle &= \zeta_\eta(s) \sum_{i=0}^{d} \frac{g^{s-i}}{s - i} \kappa_i \int_0^\infty \varepsilon^{d-s} \psi(\varepsilon)\, d\varepsilon \\ &= \zeta_\eta(s) \widetilde{\psi}(d - s + 1) \sum_{i=0}^{d} \frac{g^{s-i}}{s - i} \kappa_i. \end{aligned} \tag{A.11}$$

By Cor. A.4, this is a finite sum of meromorphic functions and hence meromorphic on W, for any test function ψ. Applying Lemma A.6, one sees that $\zeta_{\mathcal{T}}$ is a meromorphic function with values in $\mathbb{D}'$. □

Remark A.8. Note that for each $\psi \in \mathbb{D}$, the poles of the $\mathbb{C}$-valued function

$$s \mapsto \langle \zeta_{\mathcal{T}}(\varepsilon, s), \psi(\varepsilon)\rangle \tag{A.12}$$

are contained in $\mathcal{D}_{\mathcal{T}}$. Further, if m_ω is the multiplicity of $\omega \in \mathcal{D}_{\mathcal{T}}$ as a pole of $\zeta_\eta(s)$, then the multiplicity of ω as a pole of (A.12) is bounded by $m_\omega + 1$.

Corollary A.9. *The residue of $\zeta_{\mathcal{T}}$ at a pole $\omega \in \mathcal{D}_{\mathcal{T}}$ is a well-defined distribution.*

Proof. This follows immediately from the second part of the proof of Lemma A.6, with $\mathcal{P}_{\mathcal{T}} = \mathcal{D}_{\mathcal{T}}$. □

Corollary A.10. *The sum of residues appearing in Thm. 7.5 and Thm. 8.4 is distributionally convergent, and is thus a well-defined distribution.*

Proof. In view of the proof of Thm. A.7, this comes by applying the Uniform Boundedness Principle to an appropriate sequence of partial sums, in a manner similar to the proof of Lemma A.6. Again, see [30], Rem. 5.21. □

Appendix B. The Error Term and Its Estimate

In this appendix, we give the promised proof of the expression for the error term and its estimate, as stated in Thm. 7.5. First, we require a definition.

Definition B.1 (Primitives of distributions)**.** Let T_η be a distribution defined by a measure as $\langle T_\eta, \psi\rangle := \int_0^\infty \psi\, d\eta$. Then the k^{th} *primitive* (or k^{th} antiderivative) of T_η is defined by $\langle T_\eta^{[k]}, \psi\rangle := (-1)^k \langle T_\eta, \psi^{[k]}\rangle$, where $\psi^{[k]}$ is the k^{th} primitive of $\psi \in C_c^\infty(0,\infty)$ that vanishes at ∞ together with all its derivatives. For $k \geq 1$, for example,

$$\langle T_\eta^{[k]}, \psi\rangle = \int_0^\infty \int_y^\infty \frac{(x-y)^{k-1}}{(k-1)!}\psi(x)\, dx\, d\eta(y). \tag{B.1}$$

Theorem B.2. *The Mellin transform of the k^{th} primitive of a test function is given by $\widetilde{\psi^{[k]}}(s) = \tilde{\psi}(s+k)\xi_k(s)$, where ξ_k is the meromorphic function*

$$\xi_k(s) := \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}(-1)^j}{(k-1)!(s+j)}. \tag{B.2}$$

Proof. By direct computation,

$$\begin{aligned}
\widetilde{\psi^{[k]}}(s) &= \int_0^\infty \varepsilon^{s-1} \int_\varepsilon^\infty \frac{(x-\varepsilon)^{k-1}}{(k-1)!}\psi(x)\, dx\, d\varepsilon \\
&= \frac{1}{(k-1)!}\int_0^\infty \int_\varepsilon^\infty \sum_{j=0}^{k-1}\binom{k-1}{j} x^{k-1-j}(-\varepsilon)^j \varepsilon^{s-1}\psi(x)\, dx\, d\varepsilon \\
&= \sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{(k-1)!}\int_0^\infty \int_\varepsilon^\infty x^{k-1-j}\varepsilon^{s+j-1}\psi(x)\, dx\, d\varepsilon \\
&= \sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{(k-1)!}\int_0^\infty x^{k-1-j}\psi(x)\int_0^x \varepsilon^{s+j-1}\, d\varepsilon\, dx \\
&= \sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{(k-1)!(s+j)}\int_0^\infty x^{s+k-1}\psi(x)\, dx \qquad \text{(B.3)} \\
&= \tilde{\psi}(s+k)\xi_k(s).
\end{aligned}$$

Again, the formula (B.2) for ξ_k is valid for $\operatorname{Re} s > k$ by (B.3), but then extends to being valid for $s \in \mathbb{C}$ by meromorphic continuation. □

Corollary B.3. *We also have* $\left|\widetilde{\psi^{[k]}}(s)\right| \leq \left|\widetilde{\psi}(s+k)\xi_k(s)\right|$.

Remark B.4. For $s \in \mathbb{C}, t = \operatorname{Im} s$, and $c_\xi > 0$, we also have

$$|\xi_k(s)| \leq \frac{c_\xi}{|t|^k}. \tag{B.4}$$

We are now in a position to provide the proofs previously promised.

Theorem B.5. *As stated in* (7.6) *of Thm. 7.5, the error term is given by*

$$\mathcal{R}(\varepsilon) = \frac{1}{2\pi\mathbb{i}} \int_S \zeta_{\mathcal{T}}(\varepsilon, s)\, ds, \tag{B.5}$$

and is a well-defined distribution.

Proof. Applying (6.15) to (7.27) for $i = 0, \ldots, d$ gives[13]

$$\langle \mathcal{R}, \varphi \rangle_i = \frac{1}{2\pi\mathbb{i}} \int_S \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \kappa_i \int_0^\infty \varepsilon^{d-s} \psi(\varepsilon)\, d\varepsilon\, ds. \tag{B.6}$$

To see that this gives a well-defined distribution $\mathcal{R}$, we apply the descent method, as described in [30], Rem. 5.20. The first step is to show that $\left\langle \mathcal{R}^{[k]}, \psi \right\rangle_i$ is a well-defined distribution for sufficiently large k; specifically, for any integer $k > \varpi$, where ϖ is the order of languidity, as in Def. 6.3. Note that we can break the integral along the screen S into two pieces and work with each separately:

$$\left\langle \mathcal{R}^{[k]}, \psi \right\rangle_i = \frac{(-1)^k}{2\pi\mathbb{i}} \int_{|\operatorname{Im} s|>1} \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \kappa_i \int_0^\infty \varepsilon^{d-s} \psi^{[k]}(\varepsilon)\, d\varepsilon\, ds \tag{B.7}$$

$$+ \frac{(-1)^k}{2\pi\mathbb{i}} \int_{|\operatorname{Im} s|\leq 1} \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \kappa_i \int_0^\infty \varepsilon^{d-s} \psi^{[k]}(\varepsilon)\, d\varepsilon\, ds. \tag{B.8}$$

Here and throughout the rest of this appendix, it is understood that such integrals (as in (B.7)–(B.8)) are for $s \in S$. Since the screen avoids the integers $0, \ldots, d$ by assumption, the quantity $|s-i|$ is bounded away from 0. Since the screen avoids the poles of $\zeta_{\mathfrak{s}}$ by hypothesis, $\zeta_{\mathfrak{s}}(s)$ is continuous on the compact set $\{s \in S : |\operatorname{Im} s| \leq 1\}$. Therefore, it is clear that (B.8) is a well-defined integral. We focus now on (B.7):

$$\begin{aligned}
&\Big|\frac{\kappa_i}{2\pi\mathbb{i}} \int_{|\operatorname{Im} s|>1} \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \int_0^\infty \varepsilon^{d-s} \psi^{[k]}(\varepsilon)\, d\varepsilon\, ds\Big| \\
&\quad\leq \frac{\kappa_i}{2\pi} \int_{\operatorname{Im} s>1} \left| g^{s-i} \frac{\zeta_{\mathfrak{s}}(s)}{s-i} \right| \cdot \left| \widetilde{\psi^{[k]}}(s-d+1) \right| ds \\
&\quad\leq c_1 \int_1^\infty |t|^{M-1} \cdot \left| \widetilde{\psi}(s-d+k+1) \right| \cdot |\xi_k(s-d+1)|\, dt \\
&\quad\leq c_1 \int_1^\infty c_i |t|^{M-1} \cdot c_K \|\psi\|_\infty \cdot \frac{c_\xi}{|t|^k}\, dt, \\
&\quad= C\|\psi\|_\infty \int_1^\infty |t|^{M-1-k}\, dt,
\end{aligned} \tag{B.9}$$

---

[13]In the proof of Thm. 7.5, the quantity (B.6) was denoted by $\langle \mathcal{R}_i, \psi \rangle$, so that $\mathcal{R}$ could easily be written (formally) as a function in (7.30). Since we work with test functions, this quantity is instead denoted by $\langle \mathcal{R}, \psi \rangle_i$ throughout this proof.

which is clearly convergent for $k > M$. The second inequality in (B.9) comes by condition **L2** of Def. 6.3. Also, recall (from the remark just after the statement of Lemma A.1) that for $s \in S$, the real part of s is given by some function r which is Lipschitz, and hence is almost everywhere differentiable and has bounded derivatives on the support of ψ. The third comes by inequality (A.5) of Rem. A.2, along with Rem. B.4. This establishes the validity of $\langle \mathcal{R}^{[k]}, \psi \rangle_i$ and thus shows that $\mathcal{R}^{[k]}$ defines a linear functional on $\mathbb{D}$.

To check that the action of $\mathcal{R}^{[k]}$ is continuous on $\mathbb{D}$, let $\psi_n \to 0$ in $\mathbb{D}$, so that there is a compact set K which contains the support of every ψ_n, and $\|\psi_n\|_\infty \to 0$. Then

$$\left|\langle \mathcal{R}^{[k]}, \psi_n \rangle\right| \leq C \cdot \left|\widetilde{\psi}_n(s - d + k + 1)\right| \leq c_K \|\psi_n\|_\infty \xrightarrow{n\to\infty} 0, \tag{B.10}$$

by following (B.9) and then applying Lemma A.1, along with its extensions as stated in Rem. A.2. Thus, $\mathcal{R}^{[k]}$ is a well-defined distribution. If we differentiate it distributionally k times, we obtain $\mathcal{R}$. This shows that $\mathcal{R}$ is a well-defined distribution and concludes the proof. ☐

Before finally checking the error estimate, we define what is meant by the expression $T(x) = O(x^\alpha)$ as $x \to \infty$, when T is a distribution.

Definition B.6. When $\mathcal{R}(x) = O(x^\alpha)$ as $x \to \infty$ (as in (6.14)), we say as in [30], §5.4.2, that $\mathcal{R}$ is of *asymptotic order at most* x^α as $x \to \infty$. To understand this expression, first define

$$\psi_a(x) := \tfrac{1}{a}\psi\left(\tfrac{x}{a}\right), \tag{B.11}$$

for $a > 0$ and for any test function ψ. Then "$\mathcal{R}(x) = O(x^\alpha)$ as $x \to \infty$" means that

$$\langle \mathcal{R}, \psi_a \rangle = O(a^\alpha), \qquad \text{as } a \to \infty,$$

for every test function ψ. The implied constant may depend on ψ. Similarly, "$\mathcal{R}(x) = O(x^\alpha)$ as $x \to 0^+$" (as in (6.19) and (7.7)) is defined to mean that

$$\langle \mathcal{R}, \psi_a \rangle = O(a^\alpha), \qquad \text{as } a \to 0^+,$$

for every test function ψ.

Theorem B.7 (Error estimate)**.** *As stated in Thm. 7.5, the error term* $\mathcal{R}(\varepsilon)$ *in* (B.5) *is estimated by*

$$\mathcal{R}(\varepsilon) = O(\varepsilon^{d-\sup S}), \qquad \textit{as } \varepsilon \to 0^+. \tag{B.12}$$

Proof. As in the proof of Thm. B.5, we use the descent method and begin by splitting the integral into two pieces. Since $\langle \mathcal{R}^{[k]}, \psi_a \rangle = (-1)^k \langle \mathcal{R}, (\psi_a)^{[k]} \rangle$, we work with

$$\left\langle \mathcal{R}, (\psi_a)^{[k]} \right\rangle_{qi} = \frac{\kappa_i}{2\pi\mathbb{i}} \int_{|\operatorname{Im} s|>1} \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \int_0^\infty \varepsilon^{d-s} (\psi_a)^{[k]}(\varepsilon)\, d\varepsilon\, ds \tag{B.13}$$

$$+ \frac{\kappa_i}{2\pi\mathbb{i}} \int_{|\operatorname{Im} s|\leq 1} \frac{g^{s-i}}{s-i} \zeta_{\mathfrak{s}}(s) \int_0^\infty \varepsilon^{d-s} (\psi_a)^{[k]}(\varepsilon)\, d\varepsilon\, ds. \tag{B.14}$$

The k^{th} primitive of ψ_a is given by

$$(\psi_a)^{[k]}(\varepsilon) = \int_\varepsilon^\infty \frac{(u-\varepsilon)^{k-1}}{(k-1)!} \tfrac{1}{a}\psi\left(\tfrac{u}{a}\right) du = \int_{\varepsilon/a}^\infty \frac{(au-\varepsilon)^{k-1}}{(k-1)!} \psi(u)\, du. \tag{B.15}$$

By following the same calculations as in Thm. B.2, one observes that

$$\begin{aligned}
&\left|\int_0^\infty \frac{\varepsilon^{d-s}}{s-i}\int_{\varepsilon/a}^\infty \frac{(au-\varepsilon)^{k-1}}{(k-1)!}\psi(u)\,du\,d\varepsilon\right| \\
&= \left|\int_0^\infty \int_0^{au} \frac{}{s-i}\sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{(k-1)!}(au)^{k-1-j}\varepsilon^{d-s+j}\psi(u)\,d\varepsilon\,du\right| \\
&\le \frac{1}{|s-i|}\sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{(k-1)!}\int_0^\infty \left|(au)^{k-1-j}\psi(u)\right|\int_0^{au}\left|\varepsilon^{d-s+j}\,d\varepsilon\right|\,du \\
&\le \frac{c_i}{|s-i|}\xi_k(d-\operatorname{Re}s+1)\int_0^\infty (au)^{k-1-j}(au)^{d-\operatorname{Re}s+j+1}|\psi(u)|\,du
\end{aligned}$$

$$= a^{d-\operatorname{Re}s+k}\frac{c_i}{|s-i|}\xi_k(d-\operatorname{Re}s+1)\widetilde{|\psi|}(d-\operatorname{Re}s+k). \tag{B.16}$$

Using (B.4) for ξ_k and (A.3) for $\widetilde{|\psi|}$ (see Rem. A.2), we bound (B.13) by

$$\frac{c_i}{2\pi}\int_{|\operatorname{Im}s|>1} a^{d-\operatorname{Re}s+k}\cdot\frac{|g^{s-i}\zeta_{\mathfrak{s}}(s)|}{|s-i|}\cdot\frac{c_\psi}{|t|^k}\cdot c_K\|\psi\|_\infty\,ds \tag{B.17}$$

$$\le a^{d-\sup S+k}\left(C\int_1^\infty |t|^{M-1-k}\,dt\right), \tag{B.18}$$

for any $0<a<1$, as in (B.9). Since the integral in (B.18) clearly converges for $k>M$, we have established the estimate for $\mathcal{R}^{[k]}$, along the part of the integral where $|\operatorname{Im}s|>1$. Recall that all our contour integrals are taken along the screen S. The proof for (B.13), where $|\operatorname{Im}s|>1$, readily follows from the corresponding argument in the proof of Thm. B.5. Thus we have established that

$$\left|\langle\mathcal{R}^{[k]}(\varepsilon),\psi_a(\varepsilon)\rangle\right| \le a^{d-\sup S+k}c_k, \quad \text{for all } 0<a<1. \tag{B.19}$$

In (B.19)–(B.21), the constants c_k may depend on the test function ψ.[14]

By iterating the following calculation:

$$\begin{aligned}
\left|\langle\mathcal{R}^{[k-1]}(\varepsilon),\psi_a(\varepsilon)\rangle\right| &= \left|\left\langle\mathcal{R}^{[k]}(\varepsilon),\left(\tfrac{1}{a}\psi\left(\tfrac{\varepsilon}{a}\right)\right)'\right\rangle\right| \\
&= \left|\tfrac{1}{a}\langle\mathcal{R}^{[k]}(\varepsilon),(\psi')_a(\varepsilon)\rangle\right| \\
&\le a^{d-\sup S+k-1}c_{k-1},
\end{aligned} \tag{B.20}$$

one sees that

$$|\langle\mathcal{R}(\varepsilon),\psi_a(\varepsilon)\rangle| \le a^{d-\sup S}c_0, \quad \text{for all } 0<a<1. \tag{B.21}$$

By Def. B.6, this implies that $\mathcal{R}(\varepsilon)=O(\varepsilon^{d-\sup S})$ as $\varepsilon\to 0^+$. □

[14]Note that c_{k-1} does not correspond to c_k when k is replaced by $k-1$; rather, c_{k-1} depends on the support of ψ'. The notation is just used to indicate the analogous roles the constants c_k play.

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