

# Nonarchimedean Cantor Set and String

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# NONARCHIMEDEAN CANTOR SET AND STRING

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*Dedicated to Professor Vladimir Arnold, on the occasion of his jubilee*

ABSTRACT. We construct a nonarchimedean (or  $p$ -adic) analogue of the classical ternary Cantor set  $\mathcal{C}$ . In particular, we show that this nonarchimedean Cantor set  $\mathcal{C}_3$  is self-similar. Furthermore, we characterize  $\mathcal{C}_3$  as the subset of 3-adic integers whose elements contain only 0's and 2's in their 3-adic expansions and prove that  $\mathcal{C}_3$  is naturally homeomorphic to  $\mathcal{C}$ . Finally, from the point of view of the theory of fractal strings and their complex fractal dimensions [7, 8], the corresponding nonarchimedean Cantor string resembles the standard archimedean (or *real*) Cantor string perfectly.

## 1. INTRODUCTION

Our goal in this article is to provide a good *nonarchimedean* (or  $p$ -adic) analogue of the classic Cantor ternary set  $\mathcal{C}$  and to show that it satisfies a counterpart of some of the key properties of  $\mathcal{C}$  in this nonarchimedean context. We also show that the corresponding  $p$ -adic fractal string, called the nonarchimedean Cantor string and denoted by  $\mathcal{CS}_3$ , is an exact analogue of the ordinary archimedean Cantor string, a central example in the theory of real (or archimedean) fractal strings and their complex dimensions [7, 8]. Furthermore, we compute the geometric zeta function of  $\mathcal{CS}_3$  and the associated complex fractal dimensions.

In a forthcoming paper [9], we will develop a general framework for formulating a theory of self-similar  $p$ -adic (or nonarchimedean) strings and their complex fractal dimensions. Besides answering a natural mathematical question, these results may be useful in various aspects of mathematical physics, including  $p$ -adic quantum mechanics and string theory, where extensions from the archimedean to the nonarchimedean setting have been extensively explored [12].

**1.1.  $p$ -adic numbers.** Let  $p \in \mathbb{N}$  be a fixed prime number. For any nonzero  $x \in \mathbb{Q}$ , we can always write  $x = p^v \cdot a/b$ , with  $a, b \in \mathbb{Z}$  and for some unique  $v \in \mathbb{Z}$  so that  $p$  does not divide  $ab$ . The  $p$ -adic norm is a function  $|\cdot|_p : \mathbb{Q} \rightarrow [0, \infty)$  given by

$$|x|_p = p^{-v} \quad \text{and} \quad |0|_p = 0.$$

One can verify that  $|\cdot|_p$  is indeed a norm on  $\mathbb{Q}$ . Furthermore, it satisfies a *strong triangle inequality*: for any  $x, y \in \mathbb{Q}$ , we have  $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ ; the induced metric is therefore called an ultrametric. This inequality is called the *nonarchimedean property* because for each  $x \in \mathbb{Q}$ ,  $|nx|_p$  will never exceed  $|x|_p$  for any  $n \in \mathbb{N}$ . The metric completion of  $\mathbb{Q}$  with respect to the  $p$ -adic norm is

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the field of  $p$ -adic numbers  $\mathbb{Q}_p$ . More concretely, there is a unique representation of every  $z \in \mathbb{Q}_p$ :  $z = a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$ , for some  $v \in \mathbb{Z}$  and  $a_i \in \{0, 1, \dots, p-1\}$  for all  $i \geq v$ . An important subspace of  $\mathbb{Q}_p$  is the unit ball,  $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ , which can also be represented as follows:

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \dots, p-1\}, \forall i \geq 0\}.$$

Using this  $p$ -adic expansion, we can see that

$$(1) \quad \mathbb{Z}_p = \bigcup_{c=0}^{p-1} (c + p\mathbb{Z}_p),$$

where  $c + p\mathbb{Z}_p := \{y \in \mathbb{Q}_p : |y - c|_p \leq 1/p\}$ . Moreover, by the nonarchimedean property of the  $p$ -adic norm,  $\mathbb{Z}_p$  is closed under addition and hence is a ring. It is called the ring of  $p$ -adic integers and  $\mathbb{Z}$  is dense in  $\mathbb{Z}_p$ . Note that  $\mathbb{Z}_p$  is compact and thus complete. (For general references on  $p$ -adic analysis, see, e.g., [4, 11].) It is also known that there are topological models of  $\mathbb{Z}_p$  in the Euclidean space  $\mathbb{R}^d$  as fractal spaces such as the Cantor set and the Sierpiński gasket [11, §I.2.5]. In fact,  $\mathbb{Z}_p$  is homeomorphic to the ternary Cantor set. It is thus natural to wonder what exactly is the *nonarchimedean* (or  $p$ -adic) analogue of the ternary Cantor set. We will answer this question in §2.

**1.2. Ternary Cantor set.** The classical ternary Cantor set, denoted by  $\mathcal{C}$ , is the set that remains after iteratively removing the open middle third subinterval(s) from the closed unit interval  $C_0 = [0, 1]$ . The construction is illustrated in Figure 1. Hence, the ternary (or archimedean) Cantor set  $\mathcal{C}$  is equal to  $\bigcap_{n=0}^{\infty} C_n$ .

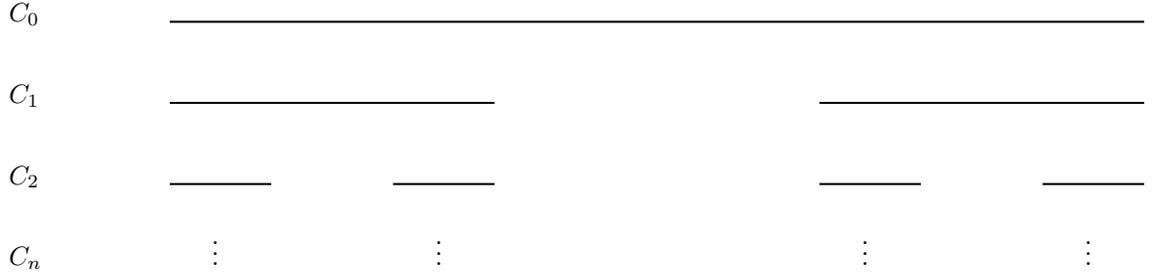


FIGURE 1. Construction of the archimedean Cantor set  $\mathcal{C} = \bigcap_{n=0}^{\infty} C_n$ .

For comparison with our results in the nonarchimedean case, we state without proof the following well-known results (see, e.g., [1, Ch. 9] and [2, p. 50]):

**Theorem 1.1.** *The ternary Cantor set  $\mathcal{C}$  is self-similar. More specifically, it is the unique nonempty, compact invariant set in  $\mathbb{R}$  generated by the family  $\{\Phi_1, \Phi_2\}$  of similarity contraction mappings of  $[0, 1]$  into itself, where  $\Phi_1(x) = x/3$  and  $\Phi_2(x) = x/3 + 2/3$ . That is,*

$$\mathcal{C} = \Phi_1(\mathcal{C}) \cup \Phi_2(\mathcal{C}).$$

**Theorem 1.2.** *The Cantor set is characterized by the ternary expansion of its elements as*

$$\mathcal{C} = \left\{ c \in [0, 1] : c = a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots, a_i \in \{0, 2\}, \forall i \geq 0 \right\}.$$

We note that, as usual, we choose the nonrepeating ternary expansion here. Such a precaution will not be needed in §2 for the elements of  $\mathbb{Q}_3$  because the 3-adic expansion is unique.

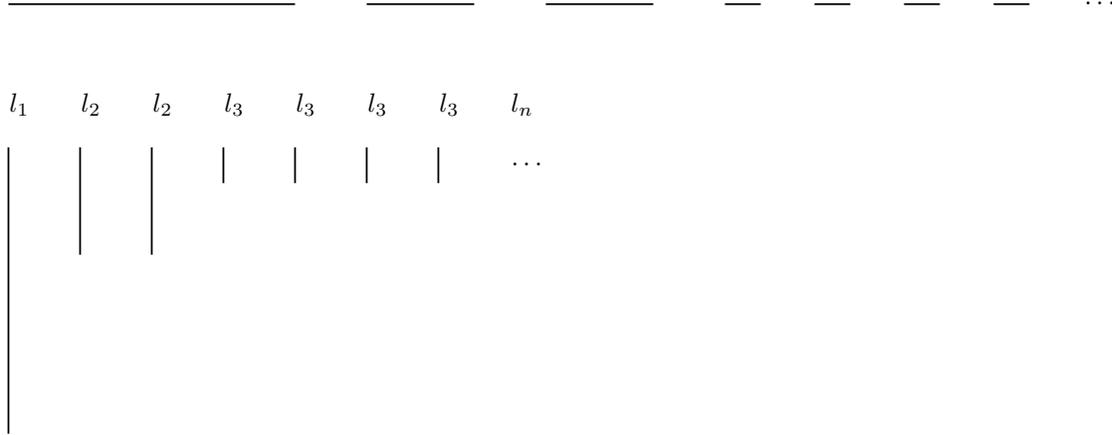


FIGURE 2. Cantor string (above); Cantor string viewed as a fractal harp (below).

**1.3. Cantor fractal string.** The *archimedean* (or *real*) *Cantor string*  $\mathcal{CS}$  is defined as the complement of the ternary Cantor set in the closed unit interval  $[0, 1]$ . By construction, the topological boundary of  $\mathcal{CS}$  is the ternary Cantor set  $\mathcal{C}$ . The Cantor string is one of the simplest and most important examples in the research monographs [7, 8] by Lapidus and van Frankenhuijsen. Indeed, it is used throughout those books to illustrate and motivate the general theory; see also, e.g., [5] and [6]. From the point of view of the theory of fractal strings and their complex dimensions [7, 8], it suffices to consider the sequence  $\{l_n\}_{n \in \mathbb{N}}$  of lengths associated to  $\mathcal{CS}$ . More specifically, these are the lengths of the intervals of which the bounded open set  $\mathcal{CS} \subset \mathbb{R}$  is composed. Accordingly, the Cantor string consists of  $1 = m_1$  interval of length  $l_1 = 1/3$ ,  $2 = m_2$  intervals of length  $l_2 = 1/9$ ,  $4 = m_3$  intervals of length  $l_3 = 1/27$ , and so on; see Figure 2.

Important information about the geometry of  $\mathcal{CS}$ , e.g., the Minkowski dimension and the Minkowski measurability ([5–8]), is contained in its *geometric zeta function*

$$(2) \quad \zeta_{\mathcal{CS}}(s) := \sum_{n=1}^{\infty} m_n \cdot l_n^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \quad \text{for } \Re(s) > D,$$

where  $D = \log 2 / \log 3$  is the *Minkowski dimension* of the ternary Cantor set. In addition,  $\zeta_{\mathcal{CS}}$  can be extended to a meromorphic function on the entire complex plane  $\mathbb{C}$ , as given by the last expression in (2). The corresponding set of poles of  $\zeta_{\mathcal{CS}}$  is then given by

$$(3) \quad \mathcal{D}_{\mathcal{CS}} = \{D + w\mathbf{p} \mid \nu \in \mathbb{Z}\},$$

where  $\mathbf{p} = 2\pi / \log 3$  is the *oscillatory period* of  $\mathcal{CS}$ . Here and henceforth, we let  $\iota := \sqrt{-1}$ . The set  $\mathcal{D}_{\mathcal{CS}}$  is called the set of *complex dimensions* of the Cantor string; see Figure 5 in §3.

The general theme of the monographs [7, 8] is that *the complex dimensions describe oscillations in the geometry and the spectrum of a fractal string*. In particular, there are oscillations of order  $D$  in the geometry of  $\mathcal{CS}$  and therefore its boundary, the Cantor set, is *not* Minkowski measurable; see [6], [8, §1.1.2].

In §3, we will obtain a nonarchimedean (or *p*-adic) analogue of the Cantor string and establish its main properties.

## 2. NONARCHIMEDEAN CANTOR SET

Let  $\mathbb{Z}_p$  be the set of  $p$ -adic integers. The  $p$ -adic ball with center  $a \in \mathbb{Q}_p$  and radius  $p^{-n}$ ,  $n \in \mathbb{Z}$ , is the set

$$a + p^n \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x - a|_p \leq p^{-n}\}.$$

Two interesting “nonarchimedean phenomena” are that each point of the  $p$ -adic ball is a center and a  $p$ -adic ball is both open and closed. Moreover, every interval<sup>1</sup> in  $\mathbb{Q}_p$  can be canonically decomposed into  $p$  equally long subintervals, as in (1).

Consider the ring of 3-adic integers  $\mathbb{Z}_3$ . In a procedure reminiscent of the construction of the classic Cantor set, we construct the *nonarchimedean Cantor set*. First, we subdivide  $T_0 = \mathbb{Z}_3$  into 3 equally long subintervals. We then remove the “middle” third  $T_1 = 1 + 3\mathbb{Z}_3$  and repeat this process with each of the remaining subintervals. Finally, we define the nonarchimedean Cantor set  $\mathcal{C}_3$  to be  $\bigcap_{n=0}^{\infty} T_n$ ; see Figure 3. The nonarchimedean analogue of Theorem 1.1 is given by Theorem 2.1:

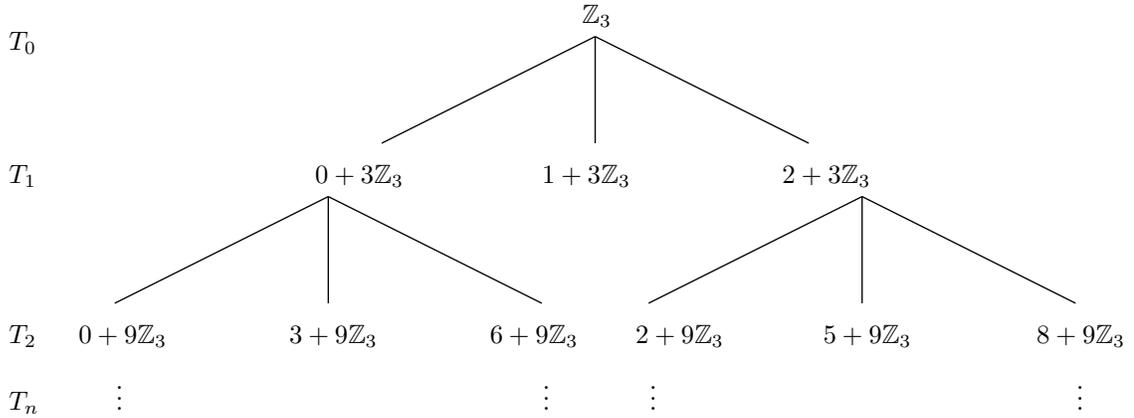


FIGURE 3. Construction of the nonarchimedean Cantor set  $\mathcal{C}_3 = \bigcap_{n=0}^{\infty} T_n$ .

**Theorem 2.1.** *The nonarchimedean Cantor set  $\mathcal{C}_3$  is self-similar. More specifically, it is the unique nonempty, compact invariant set in  $\mathbb{Q}_p$  generated by the family  $\{\Psi_1, \Psi_2\}$  of similarity contraction mappings of  $\mathbb{Z}_3$  into itself, where*

$$(4) \quad \Psi_1(x) = 3x \quad \text{and} \quad \Psi_2(x) = 3x + 2.$$

That is,

$$\mathcal{C}_3 = \Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3).$$

*Proof.* From Figure 4, we can see that  $\Psi_1(T_n) \cup \Psi_2(T_n) = T_{n+1}$  for all  $n \geq 0$ . Since each  $\Psi_i$  is injective ( $i = 1, 2$ ), we have  $\Psi_i(\mathcal{C}_3) = \Psi_i(\bigcap_n T_n) = \bigcap_n \Psi_i(T_n)$ . Therefore,

$$\Psi_1(\mathcal{C}_3) \cup \Psi_2(\mathcal{C}_3) = \bigcap_{n=0}^{\infty} (\Psi_1(T_n) \cup \Psi_2(T_n)) = \bigcap_{n=0}^{\infty} T_{n+1} = \mathcal{C}_3.$$

The Contraction Mapping Principle, applied to the complete metric space of all nonempty compact subsets of  $\mathbb{Z}_3$ ,<sup>2</sup> equipped with the Hausdorff metric induced by the 3-adic norm, shows that there

<sup>1</sup>We shall sometimes call the ball  $a + p^n \mathbb{Z}_p$  an “interval”.

<sup>2</sup>Recall from §1.1 that  $\mathbb{Z}_3$  itself is a complete metric space.

is a unique invariant set of the family of similarity contraction mappings  $\{\Psi_1, \Psi_2\}$ . We refer to Hutchinson's paper [2] for a detailed argument in the case of arbitrary complete metric spaces.

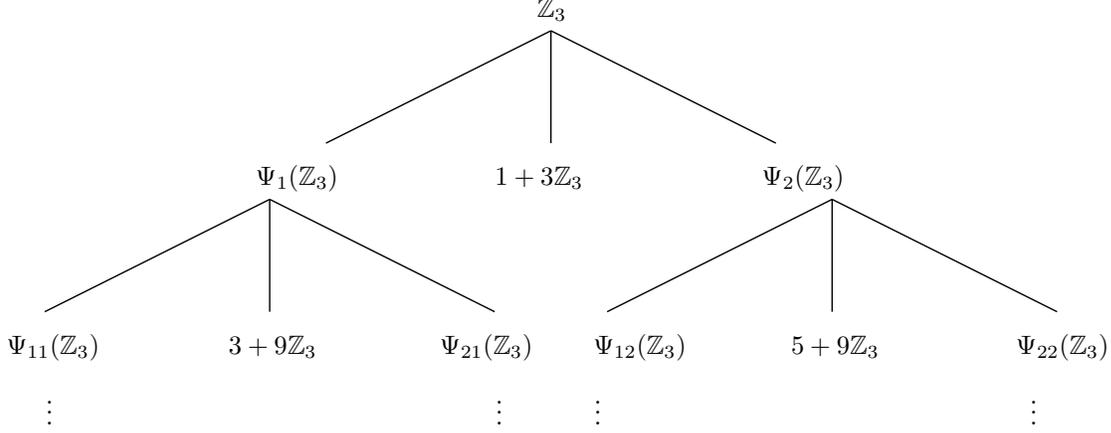


FIGURE 4. Construction of the nonarchimedean Cantor set via an Iterated Function Scheme (IFS).

**Theorem 2.2.** Let  $W_k = \{1, 2\}^k$  be the set of all finite words, on 2 symbols, of a given length  $k \geq 0$ . Then

$$\mathcal{C}_3 = \bigcap_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3),$$

where  $\Psi_w := \Psi_{w_k} \circ \dots \circ \Psi_{w_1}$  for  $w = (w_1, \dots, w_k) \in W_k$  and the maps  $\Psi_{w_i}$  are as in Equation (4).

*Proof.* For each  $k = 0, 1, 2, \dots$ , we have that

$$\bigcup_{w \in W_k} \Psi_w(\mathbb{Z}_3) = T_k.$$

Hence, in light of Theorem 2.1, the result follows at once from the definition of  $\mathcal{C}_3$ ; see Figure 4.

The following result is the nonarchimedean analogue of Theorem 1.2:

**Theorem 2.3.** The nonarchimedean Cantor set is characterized by the 3-adic expansion of its elements. That is,

$$\mathcal{C}_3 = \{\kappa \in \mathbb{Z}_3 \mid \kappa = a_0 + a_1 3 + a_2 3^2 + \dots, a_i \in \{0, 2\}, \forall i \geq 0\}.$$

*Proof.* First of all, observe that the inverses of  $\Psi_1$  and  $\Psi_2$  are, respectively,

$$\Psi_1^{-1}(x) = \frac{x}{3} \quad \text{and} \quad \Psi_2^{-1}(x) = \frac{x-2}{3}.$$

Secondly, it is clear that for  $a'_i \in \{0, 1, 2\}$  and  $i = 0, 1, 2, \dots$ ,

$$(5) \quad a'_0 + a'_1 3 + a'_2 3^2 + \dots \in 1 + 3\mathbb{Z}_3 \Leftrightarrow a'_0 = 1.$$

Now, let  $\kappa = a_0 + a_1 3 + a_2 3^2 + \dots \in \mathbb{Z}_3$  and suppose that some coefficients in its 3-adic expansion are 1's. We will show that  $\kappa$  must then be in the image of  $1 + 3\mathbb{Z}_3$  under some composition of the maps  $\Psi_1$  and  $\Psi_2$ . Let  $l \in \mathbb{N}$  be the first index such that  $a_l = 1$ . Hence,  $a_j = 0$  or  $2$  for all  $j < l$ . If

$a_0 = 0$ , then we apply  $\Psi_1^{-1}$  to  $\kappa$ , and if  $a_0 = 2$ , then we apply  $\Psi_2^{-1}$  to  $\kappa$ . In both cases, we have that

$$(6) \quad \Psi_i^{-1}(\kappa) = a_1 + a_2 3 + \cdots + a_l 3^{l-1} + a_{l+1} 3^l + \cdots .$$

Depending on whether  $a_1 = 0$  or  $2$ , we apply  $\Psi_1^{-1}$  or  $\Psi_2^{-1}$ , respectively, to the above 3-adic expansion (6). Proceeding in this manner, we will get

$$\Psi_w^{-1}(\kappa) = a_l + a_{l+1} 3 + \cdots ,$$

which is in  $1 + 3\mathbb{Z}_3$  for some  $k \in \mathbb{N}$  and  $w \in W_k$ . Thus  $\kappa \in \Psi_w(1 + 3\mathbb{Z}_3)$ . Since  $(1 + 3\mathbb{Z}_3) \cap \mathcal{C}_3 = \emptyset$  and  $\Psi_w$  is injective, we deduce that  $\kappa \notin \mathcal{C}_3$ . Therefore, all of the digits of  $\kappa \in \mathcal{C}_3$  must lie in  $\{0, 2\}$ .

Conversely, suppose that all of the coefficients in  $\kappa = a_0 + a_1 3 + a_2 3^2 + \cdots \in \mathbb{Z}_3$  are 0's or 2's. Then, by the above observation (5),  $\kappa \notin 1 + 3\mathbb{Z}_3$ . Moreover,  $\kappa \notin \Phi_w(1 + 3\mathbb{Z}_3)$  for any  $w \in W_k, k = 0, 1, 2, \dots$ , since none of the coefficients  $a_i$  is equal to 1. That is,

$$\kappa \notin \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Phi_w(1 + 3\mathbb{Z}_3) =: B.$$

But  $\mathcal{C}_3 \cap B = \emptyset$  and  $\mathcal{C}_3 \cup B = \mathbb{Z}_3$ , as can be seen in Equation (8) and Theorem 3.3. Hence,  $\kappa \in \mathcal{C}_3$ , as desired.

**Theorem 2.4.** *The ternary Cantor set  $\mathcal{C}$  and the nonarchimedean Cantor set  $\mathcal{C}_3$  are homeomorphic.*

*Proof.* Let  $\phi : \mathcal{C} \rightarrow \mathcal{C}_3$  be the map sending

$$(7) \quad \sum_{i=0}^{\infty} a_i 3^{-i} \mapsto \sum_{i=0}^{\infty} a_i 3^i,$$

where  $a_i \in \{0, 2\}$  for all  $i \geq 0$ . We note that on the left-hand side of (7), we use the ternary expansion in  $\mathbb{R}$ , whereas on the right-hand side we use the 3-adic expansion in  $\mathbb{Q}_3$ . Then, clearly,  $\phi$  is a continuous bijective map from  $\mathcal{C}$  to  $\mathcal{C}_3$ . Since both  $\mathcal{C}$  and  $\mathcal{C}_3$  are compact spaces in their respective natural metric topologies,  $\phi$  is a homeomorphism.

**Remark 2.5.** *In view of Theorem 2.4, like its archimedean counterpart  $\mathcal{C}$ , the nonarchimedean Cantor set  $\mathcal{C}_3$  is totally disconnected, uncountably infinite and has no isolated points.*

### 3. NONARCHIMEDEAN CANTOR STRING

The *nonarchimedean* (or *p-adic*) *Cantor string* is defined to be

$$(8) \quad \mathcal{CS}_3 := (1 + 3\mathbb{Z}_3) \cup (3 + 9\mathbb{Z}_3) \cup (5 + 9\mathbb{Z}_3) \cup \cdots = \mathbb{Z}_3 \setminus \mathcal{C}_3,$$

the complement of  $\mathcal{C}_3$  in  $\mathbb{Z}_3$ ; see the “middle” parts of Figure 3. Therefore, by analogy with the relationship between the archimedean Cantor set and Cantor string, the nonarchimedean Cantor set  $\mathcal{C}_3$  can be thought of as some kind of “boundary” of the nonarchimedean Cantor string. Certainly,  $\mathcal{C}_3$  is not the topological boundary of  $\mathcal{CS}_3$  because the latter boundary is empty.

Since  $\mathbb{Q}_p$  is a locally compact group, there is a unique translation invariant Haar measure  $\mu_H$ , normalized so that  $\mu_H(\mathbb{Z}_p) = 1$ , and hence  $\mu_H(a + 3^n \mathbb{Z}_3) = 3^{-n}$ ; see [4], [11]. As in the real case in §1.3, we may identify  $\mathcal{CS}_3$  with the sequence of lengths  $l_n = 3^{-n}$  with multiplicities  $m_n = 2^{n-1}$  for  $n \in \mathbb{N}$ .

The following theorem provides the exact analogue of Equations (2) and (3):

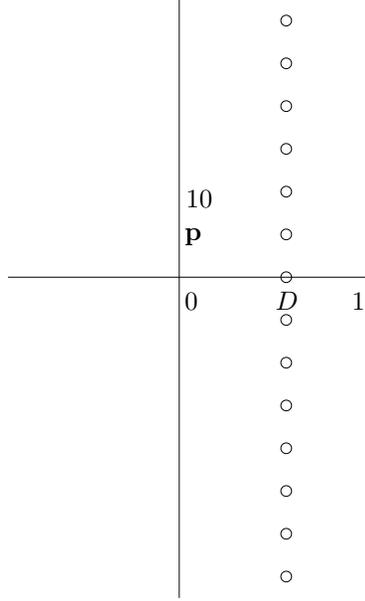


FIGURE 5. The set of complex dimensions,  $\mathcal{D}_{\mathcal{CS}} = \mathcal{D}_{\mathcal{CS}_3}$ , of the archimedean and nonarchimedean Cantor strings,  $\mathcal{CS}$  and  $\mathcal{CS}_3$ .

**Theorem 3.1.** *The geometric zeta function of the nonarchimedean Cantor string is meromorphic in all of  $\mathbb{C}$  and is given by*

$$(9) \quad \zeta_{\mathcal{CS}_3}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}.$$

Hence, the set of complex dimensions of  $\mathcal{CS}_3$  is given by

$$(10) \quad \mathcal{D}_{\mathcal{CS}_3} = \{D + \nu \mathbf{p} \mid \nu \in \mathbb{Z}\},$$

where  $D = \log 2 / \log 3$  is the dimension of  $\mathcal{CS}_3$  and  $\mathbf{p} = 2\pi / \log 3$  is its oscillatory period.

*Proof.* By definition (see [9, 10]), the geometric zeta function of  $\mathcal{CS}_3$  is given by

$$\begin{aligned} \zeta_{\mathcal{CS}_3}(s) &:= (\mu_H(1 + 3\mathbb{Z}_3))^s + (\mu_H(3 + 9\mathbb{Z}))^s + (\mu_H(5 + 9\mathbb{Z}_3))^s + \cdots \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \quad \text{for } \Re(s) > \log 2 / \log 3. \end{aligned}$$

Furthermore, the meromorphic extension of  $\zeta_{\mathcal{CS}_3}$  to all of  $\mathbb{C}$  is given by the last expression in the above equation. The *complex dimensions* of  $\mathcal{CS}_3$ , defined as the poles of  $\zeta_{\mathcal{CS}_3}$ , are all the solutions  $\omega$  of the equation  $1 - 2 \cdot 3^{-\omega} = 0$ . These are precisely of the form

$$\omega = \frac{\log 2}{\log 3} + \nu \frac{2\pi}{\log 3}, \quad \nu \in \mathbb{Z}.$$

**Remark 3.2.** *In [9, 10], we prove that  $D$  is the Minkowski dimension of  $\mathcal{CS}_3 \subset \mathbb{Z}_3$ . Clearly,  $D$  is also the abscissa of convergence of the Dirichlet series defining  $\zeta_{\mathcal{CS}_3}$ .*

The following result was used in the second part of the proof of Theorem 2.3:

**Theorem 3.3.** *With the same notation as in Theorem 2.2, we have that*

$$\mathcal{CS}_3 = \bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3).$$

*Proof.* For each  $k = 0, 1, 2, \dots$ , we let  $\widetilde{T}_{k+1} = \mathbb{Z}_3 \setminus T_{k+1}$ , the complement of  $T_{k+1}$  in  $\mathbb{Z}_3$ . Then

$$\widetilde{T}_{k+1} = \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3).$$

Hence, in light of Theorem 2.1, we have that

$$\bigcup_{k=0}^{\infty} \bigcup_{w \in W_k} \Psi_w(1 + 3\mathbb{Z}_3) = \bigcup_{k=0}^{\infty} \widetilde{T}_{k+1} = \widetilde{\bigcap} T_{k+1} = \widetilde{\mathcal{C}}_3 = \mathcal{CS}_3,$$

by the definitions of  $\mathcal{C}_3$  and  $\mathcal{CS}_3$ . See Figure 6.

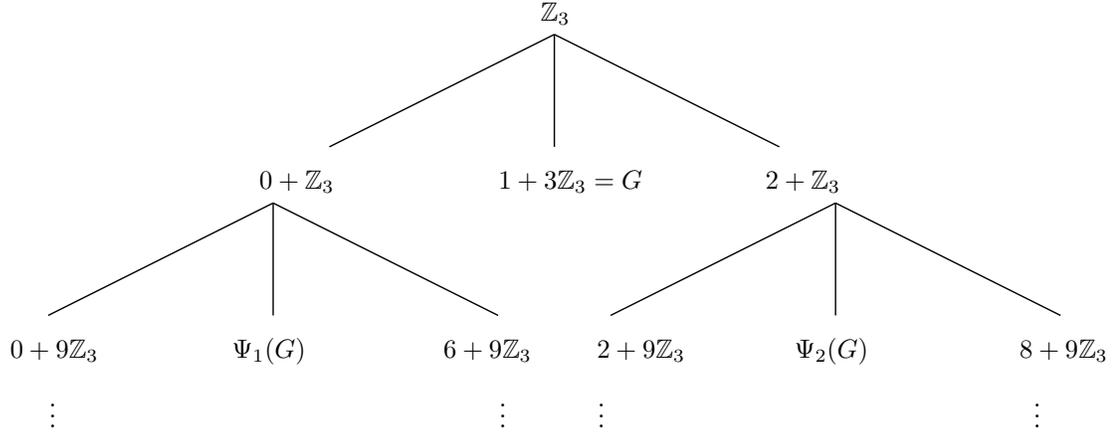


FIGURE 6. Construction of the nonarchimedean Cantor string via an IFS.

**Remark 3.4.** *The above theorem shows that  $G = 1 + 3\mathbb{Z}_3$  is the generator of the nonarchimedean Cantor string. This is a particular case of a more general construction of self-similar  $p$ -adic fractal strings [9, 10]. Moreover,  $\mathcal{CS}_3$  is not Minkowski measurable as a subset of  $\mathbb{Q}_3$ . In fact, in contrast to the archimedean case ([8], Theorems 8.23 and 8.36), self-similar  $p$ -adic fractal strings are never Minkowski measurable.*

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