SELF-SIMILAR P-ADIC FRACTAL STRINGS AND THEIR COMPLEX DIMENSIONS

Michel LAPIDUS and Lu HUNG

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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MICHEL L. LAPIDUS AND LŨ' HÙNG

Abstract. We develop a geometric theory of self-similar p-adic fractal strings and their complex dimensions. We obtain a closed-form formula for the geometric zeta functions and show that these zeta functions are rational functions in an appropriate variable. We also prove that every self-similar p-adic fractal string is lattice. Finally, we define the notion of a nonarchimedean self-similar set and discuss its relationship with that of a self-similar p-adic fractal string. We illustrate the general theory by two simple examples, the nonarchimedean Cantor and Fibonacci strings.

1. Introduction

In this paper, we lay out the foundation of a theory of self-similar p-adic (or nonarchimedean) fractal strings, that is, bounded open subsets of the p-adic line $\mathbb{Q}_p$ having a self-similar subset of $\mathbb{Q}_p$ for "boundary". This theory extends in a natural way the theory of real (or archimedean) self-similar fractal strings and their complex dimensions developed in [20, 21]. We also introduce certain geometric zeta functions, the poles of which play the role for self-similar p-adic fractal strings of the complex dimensions for the standard real self-similar fractal strings. Furthermore, we discuss the analogies and the differences with the usual theory.

In recent years, p-adic analysis [10, 29, 30] has been used in various areas of mathematics as well as in aspects of quantum physics and string theory. In particular, p-adic models of quantum mechanics and string theory have been developed as a possible way to understand the elusive geometry of spacetime at extremely high energies and very small scales (typically, below Planck scale); see, e.g., [5, 31, 32]. Furthermore, several physicists have suggested that the small scale structure of spacetime may be "fractal"; see, e.g., [7, 8, 26, 33].

On the other hand, in the recent book [12], it has been suggested that fractal strings and their quantization, fractal membranes, may be related to aspects of string theory. The present theory may be helpful in developing some of these ideas further and eventually providing a framework for unifying the archimedean and nonarchimedean fractal strings and membranes.

1.1. **p-adic numbers.** Let $p \in \mathbb{N}$ be a fixed prime number. Then for any nonzero $x \in \mathbb{Q}$, we can always write $x = p^v \cdot a/b$, for a pair of coprimes $a, b \in \mathbb{Z}$ and a unique $v \in \mathbb{Z}$ so that p does not divide ab. The *p-adic norm* is a function $|\cdot|_p : \mathbb{Q} \longrightarrow [0, \infty)$

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given by

$$|x|_p = p^{-v} \quad \text{and} \quad |0|_p = 0.$$

One can verify that $|\cdot|_p$ is indeed a norm on $\mathbb{Q}$. Furthermore, it satisfies a *strong triangle inequality*: for any $x, y \in \mathbb{Q}$, we have $|x+y|_p \leq \max\{|x|_p, |y|_p\}$. The induced metric, $d_p(x,y) = |x-y|_p$, is therefore called an ultrametric. Relative to the p-adic norm, $\mathbb{Q}$ satisfies the *nonarchimedean property* because for each $x \in \mathbb{Q}$, $|nx|_p$ will never exceed $|x|_p$ for any $n \in \mathbb{N}$. The completion of $\mathbb{Q}$ with respect to the ultrametric d_p is the field of p-adic numbers $\mathbb{Q}_p$. More concretely, there is a unique representation of every $z \in \mathbb{Q}_p$: $z = a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots$, for some $v \in \mathbb{Z}$ and $a_i \in \{0, 1, \ldots, p-1\}$ for all $i \geq v$. An important subspace of $\mathbb{Q}_p$ is the unit ball, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$, which can also be represented as follows:

$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}, \ \forall i \geq 0\}.$$

Using this *p-adic expansion*, we can see that

$$\mathbb{Z}_p = \bigcup_{a=0}^{p-1} (a + p\mathbb{Z}_p), \tag{1.1}$$

where $a + p\mathbb{Z}_p = \{y \in \mathbb{Q}_p \mid |y-a|_p \leq 1/p\}$. Moreover, by the strong triangle inequality of the p-adic norm, $\mathbb{Z}_p$ is closed under addition and hence is a ring. It is called the ring of *p-adic integers.* Note that $\mathbb{Z}_p$ is compact and thus complete. Furthermore, $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. Finally, $\mathbb{Q}_p$ is a locally compact group, so there is a unique (translation invariant) Haar measure μ_H, normalized so that $\mu_H(\mathbb{Z}_p) = 1$; hence $\mu_H(a + p^n\mathbb{Z}_p) = p^{-n}$ for any $n \in \mathbb{Z}$. For general references on p-adic analysis, we point out, e.g., [10, 29, 30].

1.2. **Real fractal strings.** A real (or archimedean) fractal string $\mathcal{L}$ is a bounded open subset Ω of the real line $\mathbb{R}$ with fractal boundary. The boundary $\partial\mathcal{L}$ of a fractal string is defined to be the topological boundary $\partial\Omega$ of the bounded open set Ω, i.e., $\partial\mathcal{L} = \partial\Omega$. It is well known that every bounded open subset of $\mathbb{R}$ can be written as a countable union of disjoint open intervals I_j. Therefore, we write

$$\Omega = \bigcup_{j=1}^{\infty} I_j.$$

Furthermore, let us label the length (one-dimensional Lebesgue measure) of each interval I_j by l_j, then $\mathcal{L}$ can be identified with the sequence $\{l_j\}_{j=1}^{\infty}$, arranged in a descending order $l_1 \geq l_2 \geq l_3 \geq \cdots > 0$.

Important geometric information about a fractal string $\mathcal{L}$ is contained in its *geometric zeta function*

$$\zeta_{\mathcal{L}}(s) = \sum_{j=1}^{\infty} l_j^s \qquad \text{for } \Re(s) > D_{\mathcal{L}},$$

where $D_{\mathcal{L}}$ is the dimension of the fractal string. It is the inner Minkowski dimension of the topological boundary of the fractal string $\partial\mathcal{L}$ and is defined as follows. For any $\varepsilon > 0$, let $V(\varepsilon)$ be the volume of the inner ε-neighborhood of $\partial\mathcal{L}$:

$$V(\varepsilon) = \mu_L\{x \in \Omega \mid d(x, \partial\Omega) < \varepsilon\},$$

where μ_L is the one-dimensional Lebesgue measure on $\mathbb{R}$ and $d(x, \partial\Omega)$ is the distance of $x \in \mathbb{R}$ to the set $\partial\Omega \subset \mathbb{R}$. Then the inner Minkowski dimension of $\partial\Omega$ is

$$D_{\mathcal{L}} = D = \inf\{\alpha \geq 0 \mid V(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\}.$$

Precious information is encoded in the set $\mathcal{D}$ of poles of the geometric zeta function, which are called the *complex dimensions* of a fractal string. The general philosophy of the research monograph *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings* by Michel L. Lapidus and Machiel van Frankenhuijsen [21] is that the complex dimensions describe the oscillations in the geometry (and the spectrum) of a fractal string.

1.3. **Self-similar fractal strings.** An important class of real fractal strings is composed of the self-similar strings $\mathcal{L}_S$, which are generated by $J \geq 2$ scaling ratios

$$1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0$$

by a procedure reminiscent of the construction of the classical Cantor set. These scaling ratios possess an important dichotomy. Let G be a multiplicative subgroup of the positive real line $\mathbb{R}^+$ generated by J scaling ratios $r_1, r_2, \ldots, r_J$. Then it is well known that G is either discrete or dense in $\mathbb{R}^+$. The self-similar string $\mathcal{L}_S$ is said to be lattice if G is discrete in $\mathbb{R}^+$ and nonlattice if G is dense in $\mathbb{R}^+$. In the lattice case, there is a unique positive real number $r < 1$ and J positive integers $k_1, k_2, \ldots, k_J$ without common divisor such that $1 \leq k_1 \leq k_2 \leq \cdots \leq k_J$ and $r_j = r^{k_j}$ for all $j = 1, 2, \ldots, J$. Lattice strings are simpler than nonlattice strings in the sense that their complex dimensions are periodically distributed on finitely many vertical lines rather than quasiperiodically distributed within a certain horizontally bounded strip as is the case with nonlattice strings [21, Thm. 2.17]. Nevertheless, every nonlattice string can be approximated by a sequence of lattice strings via a Diophantine approximation procedure [21, Thm. 3.6]. Moreover, a self-similar string is Minskowski measurable if and only if it is nonlattice [21, Thms. 8.23 and 8.36].

2. p-adic fractal strings

Let Ω be a bounded open subset of $\mathbb{Q}_p$. Then it can be decomposed into a countable union of disjoint open balls[1] with radius p^{-n_j} centered at $a_j \in \mathbb{Q}_p$,

$$a_j + p^{n_j}\mathbb{Z}_p = B(a_j, p^{-n_j}) = \{x \in \mathbb{Q}_p \mid |x - a_j|_p \leq p^{-n_j}\},$$

where $n_j \in \mathbb{Z}$ and $j \in \mathbb{N}$. There may be many different decompositions since each ball can always be decomposed into smaller disjoint balls [10]; see Eq. (1.1). However, there is a canonical decomposition of Ω into disjoint balls with respect to a certain equivalence relation, as we now explain.

Definition 2.1. Let U be an open subset of $\mathbb{Q}_p$. Given $x, y \in U$, we write that $x \sim y$ if and only if there is an open ball $B \subseteq U$ such that $x, y \in B$.

It is clear from the definition that the relation $\sim$ is reflexive and symmetric. To prove the transitivity, let $x \sim y$ and $y \sim z$. Then there are open balls B_1 containing x, y and B_2 containing y, z. Thus $y \in B_1 \cap B_2$, so either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ [30]. In any case, x and z are contained in the same open ball, so $x \sim z$. Hence, the above relation $\sim$ is indeed an equivalence relation on the open set U.

[1] We shall sometimes call the ball an *interval*.

Definition 2.2. A p-adic (or nonarchimedean) fractal string $\mathcal{L}_p$ is a bounded open subset Ω of $\mathbb{Q}_p$.

Thus it can be written, relative to the above equivalence relation, canonically as a disjoint union of intervals or balls:

$$\mathcal{L}_p = \bigcup_{j=1}^{\infty}(a_j + p^{n_j}\mathbb{Z}_p) = \bigcup_{j=1}^{\infty} B(a_j, p^{-n_j}).$$

Hence, $B(a_j, p^{-n_j})$ is the largest ball centered at a_j and contained in Ω. We may assume that the lengths (i.e., Haar measure) of the intervals $a_j + p^{n_j}\mathbb{Z}_p$ are nonincreasing, by reindexing if necessary. That is,

$$p^{-n_1} \geq p^{-n_2} \geq p^{-n_3} \geq \cdots > 0.$$

Definition 2.3. The *geometric zeta function* of a p-adic fractal string $\mathcal{L}_p$ is defined as

$$\zeta_{\mathcal{L}_p}(s) = \sum_{j=1}^{\infty}(\mu_H(a_j + p^{n_j}\mathbb{Z}_p))^s, \qquad \text{for } \Re(s) > D_{\mathcal{L}_p}. \tag{2.1}$$

(See Definition 2.9 and Theorem 2.10 below for the definition of the number $D = D_{\mathcal{L}_p}$.)

Remark 2.4. The geometric zeta function $\zeta_{\mathcal{L}_p}$ is well defined since the decomposition of $\mathcal{L}_p$ into open intervals $a_j + p^{n_j}\mathbb{Z}_p$ is unique.

Definition 2.5. The set of *complex dimensions* of $\mathcal{L}_p$ is defined to be

$$\mathcal{D}_{\mathcal{L}_p} = \{\text{poles of } \zeta_{\mathcal{L}_p}\}.$$

Definition 2.6. Given a point $a \in \mathbb{Q}_p$ and a positive real number $r > 0$, let $B(a,r) = \{x \in \mathbb{Q}_p \mid |x-a|_p \leq r\}$ be a *metrically closed* ball in $\mathbb{Q}_p$, as above. We call $S(a,r) = \{x \in \mathbb{Q}_p \mid |x-a|_p = r\}$ the *sphere* of B.

Let $\mathcal{L}_p = \bigcup_{j=1}^{\infty} B(a_j, r_j)$ be a p-adic fractal string, we then define the *metric boundary* $\beta\mathcal{L}_p$ of $\mathcal{L}_p$ to be the union of the corresponding spheres, i.e.,

$$\beta\mathcal{L}_p = \bigcup_{j=1}^{\infty} S(a_j, r_j).$$

Given a real number $\varepsilon > 0$, let us define the *"inner ε-neighborhood"* of $\mathcal{L}_p$ to be

$$N_\varepsilon = \{x \in \mathcal{L}_p \mid d_p(x, \beta\mathcal{L}_p) < \varepsilon\},$$

where $d_p(x, \beta\mathcal{L}_p) = \inf\{|x-y|_p \mid y \in \beta\mathcal{L}_p\}$ is the p-adic distance of $x \in \mathbb{Q}_p$ to a subset $\beta\mathcal{L}_p \subset \mathbb{Q}_p$. Then the volume $V(\varepsilon)$ of the inner ε-neighborhood of $\mathcal{L}_p$ is defined to be the Haar measure of N_ε, i.e., $V(\varepsilon) = \mu_H(N_\varepsilon)$.

Remark 2.7. The above definition of $V(\varepsilon)$ is a natural extension of the standard definition for the volume of the inner ε-neighborhood; cf. [21, §1.1]. Indeed, when Definition 2.6 is applied to a real fractal string, we recover the usual definition of $V(\varepsilon)$ recalled in §1.2.

Remark 2.8. D. V. Chistyakov [2] showed that the normalized one-dimensional Hausdorff measure in the ultrametric space $(\mathbb{Q}_p, |\cdot|_p)$ coincides with the standard normalized Haar measure μ_H in $\mathbb{Q}_p$.

2.1. Minkowski dimension.

Definition 2.9. The *dimension* of a p-adic fractal string $\mathcal{L}_p$ is defined as the inner Minkowski dimension of $\beta\mathcal{L}_p$:

$$D_{\mathcal{L}_p} = D = \inf\{\alpha \geq 0 \mid V(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+\} \tag{2.2}$$

The p-adic fractal string $\mathcal{L}_p$ is *Minkowski measurable*, with *Minkowski content*

$$\mathcal{M} = \lim_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{D-1}$$

if this limit exists in $(0, \infty)$. Otherwise, the *upper Minkowski content* and *lower Minkowski content* are respectively defined by

$$\begin{aligned} \mathcal{M}^* &= \limsup_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{D-1} \\ \mathcal{M}_* &= \liminf_{\varepsilon \to 0^+} V(\varepsilon)\varepsilon^{D-1} \end{aligned}$$

Thus $0 \leq \mathcal{M}_* \leq \mathcal{M}^* \leq \infty$ and a p-adic fractal string $\mathcal{L}_p$ is Minkowski measurable if and only if $\mathcal{M}_* = \mathcal{M}^* = \mathcal{M}$ is a positive real number.

Theorem 2.10. *Let $\mathcal{L}_p$ be a p-adic fractal string and suppose that it has infinitely many intervals of nonzero length. Then the abscissa of convergence σ of the geometric zeta function $\zeta_{\mathcal{L}_p}$ coincides with the dimension D of $\mathcal{L}_p$. That is, $D = \sigma$.*

Remark 2.11. Theorem 2.10 is the exact counterpart of Theorem 1.10 in [21], first noted in [11]. For a proof, we refer to a forthcoming paper [14] or to [22].

3. Self-similar p-adic fractal strings

Definition 3.1. A map $\Phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ is called a *similarity contraction mapping* of $\mathbb{Q}_p$ if there is a real number $r \in (0, 1)$ such that

$$|\Phi(x) - \Phi(y)|_p = r \cdot |x - y|_p$$

for all $x, y \in \mathbb{Z}_p$.

For any prime number $p > 2$, it follows from [24, Thm. 2.2 and Rmk. 2.3] that Φ is an affine map given by $\Phi(x) = ax + b$, where $0 \neq a \in p\mathbb{Z}_p$ and $b \in \mathbb{Z}_p$. (For $p = 2$, we will only consider affine similarity contraction mappings.) Then it is well known that every nonzero element $a \in \mathbb{Z}_p$ can be written as $a = u \cdot p^n$ for some unit $u \in \mathbb{Z}_p$ (i.e., $|u|_p = 1$) and $n \in \mathbb{N}$ [25]. Thus $r = |a|_p = p^{-n}$ and we record this fact in the following lemma:

Lemma 3.2. *Let $\Phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be a similarity contraction mapping of $\mathbb{Q}_p$ with the scaling ratio r, then $r = p^{-n}$ for some $n \in \mathbb{N}$. Moreover, if $p > 2$, then $\Phi(x) = ax + b$, with $a, b \in \mathbb{Z}_p$ and $a = u \cdot p^n$, for some unit u and $n \in \mathbb{N}$.*

For simplicity, let us take the unit interval (or ball) $\mathbb{Z}_p$ in $\mathbb{Q}_p$ and construct a *self-similar p-adic fractal string* $\mathcal{L}_p$ as follows: Let $J \geq 2$ be an integer and $\Phi_1, \ldots, \Phi_J : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be J similarity contraction mappings with the respective scaling ratios $r_1, \ldots, r_J \in (0, 1)$ satisfying

$$1 > r_1 \geq r_2 \geq \cdots \geq r_J > 0. \tag{3.1}$$

(See Figure 1.) Assume that

$$\sum_{j=1}^{J} r_j < 1, \tag{3.2}$$

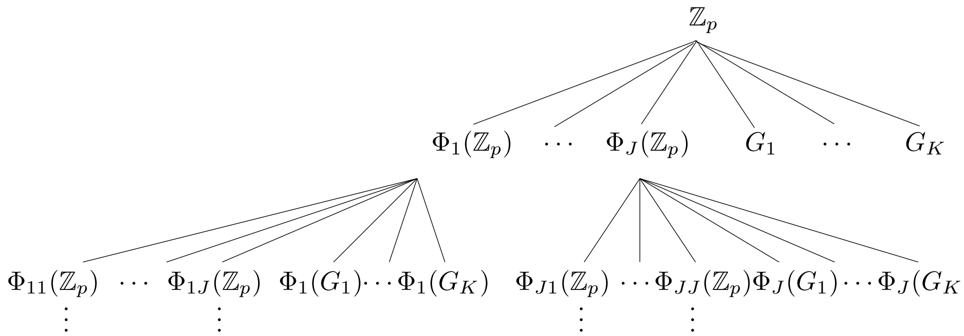

FIGURE 1. Construction of a p-adic self-similar fractal string.

and the images $\Phi_j(\mathbb{Z}_p)$ of $\mathbb{Z}_p$ do not overlap, i.e., $\Phi_j(\mathbb{Z}_p) \cap \Phi_l(\mathbb{Z}_p) = \emptyset$ for all $j \neq l$ ($j, l \in \{1, 2, \dots, J\}$). Note that it follows from Equation (3.2) that $\bigcup_{j=1}^{J} \Phi_j(\mathbb{Z}_p)$ is not all of $\mathbb{Z}_p$. We therefore have the following (nontrivial) decomposition of $\mathbb{Z}_p$ into disjoint intervals:

$$\mathbb{Z}_p = \bigcup_{j=1}^{N} \Phi_j(\mathbb{Z}_p) \cup \bigcup_{k=1}^{K} G_k, \tag{3.3}$$

where G_k is defined below.

In a procedure reminiscent of the construction of the classic Cantor set, subdivide the interval $\mathbb{Z}_p$ by means of the subintervals $\Phi_j(\mathbb{Z}_p)$. Then the convex [2] components of

$$\mathbb{Z}_p \setminus \bigcup_{j=1}^{J} \Phi_j(\mathbb{Z}_p)$$

are the first *substrings* of the self-similar p-adic fractal string $\mathcal{L}_p$, say $G_1, G_2, \dots, G_K$, with $K \geq 1$. These intervals G_k are called the *generators* [3] of $\mathcal{L}_p$. The length of each G_k is denoted by $g_k = \mu_H(G_k)$. We assume that the lengths [4] $g_1, g_2, \dots, g_K$ of the first substrings satisfy

$$1 > g_1 \geq g_2 \geq \cdots \geq g_K > 0. \tag{3.4}$$

It follows from Equation (3.3) that

$$\sum_{j=1}^{J} r_j + \sum_{k=1}^{K} g_k = 1. \tag{3.5}$$

We then repeat this process with each of the remaining subintervals $\Phi_j(\mathbb{Z}_p)$, $j = 1, 2, \dots, J$. As a result, we obtain a *self-similar p-adic fractal string* $\mathcal{L}_p = l_1, l_2, l_3, \dots,$ consisting of intervals of length l_n given by

$$r_{\nu_1} r_{\nu_2} \cdots r_{\nu_q} g_k, \tag{3.6}$$

for $k = 1, \dots, K$ and all choices of $q \in \mathbb{N} \cup \{0\}$ and $\nu_1, \dots, \nu_q \in \{1, \dots, J\}$. The lengths are of the form $r_1^{e_1} \dots r_J^{e_J} g_k$ with $e_1, \dots, e_J \in \mathbb{N} \cup \{0\}$.

[2] We choose the convex components instead of the connected components because $\mathbb{Z}_p$ is totally disconnected. Naturally, no such distinction is necessary in the archimedean case; cf. [21, p.36].

[3] These are the "deleted" intervals in the first generation of the construction of $\mathcal{L}_p$.

[4] They are called 'gaps' in [21].

Remark 3.3. In general, we may start with any interval $a+p^n\mathbb{Z}_p$ instead of the unit interval $\mathbb{Z}_p$ in the above construction of a self-similar p-adic string. Then the lengths of the first substrings would be $l_k = g_k\mu_H(a+p^n\mathbb{Z}_p) = g_k/p^n$ and everything else should be adjusted accordingly.

Example 3.4. *Nonarchimedean Cantor string.*

Let $\Phi_1, \Phi_2 : \mathbb{Z}_3 \longrightarrow \mathbb{Z}_3$ be two similarity contraction mappings of $\mathbb{Q}_p$ given by

$$\Phi_1(x) = 3x \quad \text{and} \quad \Phi_2(x) = 2+3x, \tag{3.7}$$

with the same scaling ratio $r = 1/3$ (i.e., $r_1 = r_2 = 1/3$). By analogy with the construction of the standard Cantor string, subdivide the interval $\mathbb{Z}_3$ into subintervals

$$\Phi_1(\mathbb{Z}_3) = 0+3\mathbb{Z}_3 \text{ and } \Phi_2(\mathbb{Z}_3) = 2+3\mathbb{Z}_3.$$

The remaining convex component

$$\mathbb{Z}_3 \backslash \bigcup_{j=1}^{2} \Phi_j(\mathbb{Z}_3) = 1+3\mathbb{Z}_3 = G$$

is the first substring of a self-similar 3-adic fractal string, called the *nonarchimedean* (or 3-adic) *Cantor string* and denoted by $\mathcal{CS}_3$ [13]. The length of G is

$$l_1 = \mu_H(1+3\mathbb{Z}_p) = 1/3.$$

By repeating this process with the remaining subintervals $\Phi_j(\mathbb{Z}_3)$, for $j = 1, 2$, we obtain a sequence $\mathcal{CS}_3 = l_1, l_2, l_3, \ldots,$ associated with the open set consisting of intervals of lengths $l_n = 3^{-n}$ with multiplicities $m_n = 2^{n-1}$. The nonarchimedean Cantor string $\mathcal{CS}_3$ can also be written as follows (see Figure 2):

$$\mathcal{CS}_3 = (1+3\mathbb{Z}_3) \cup (3+9\mathbb{Z}_3) \cup (5+9\mathbb{Z}_3) \cup \cdots, \tag{3.8}$$

which is an enumeration of the following set

$$\mathcal{CS}_3 = \bigcup_{\alpha=0}^{\infty} \bigcup_{w\in W_\alpha} \Phi_w(1+3\mathbb{Z}_3),$$

where $W_\alpha = \{1,2\}^\alpha$ is the set of all finite words, on 2 symbols, of a given length $\alpha \geq 0$ and $\Phi_w := \Phi_{w_\alpha} \circ \cdots \circ \Phi_{w_1}$ for $w = (w_1, \ldots, w_\alpha) \in W_\alpha$, where the maps Φ_1 and Φ_2 are given as in Equation (3.7). See §4 and [13] for more details.

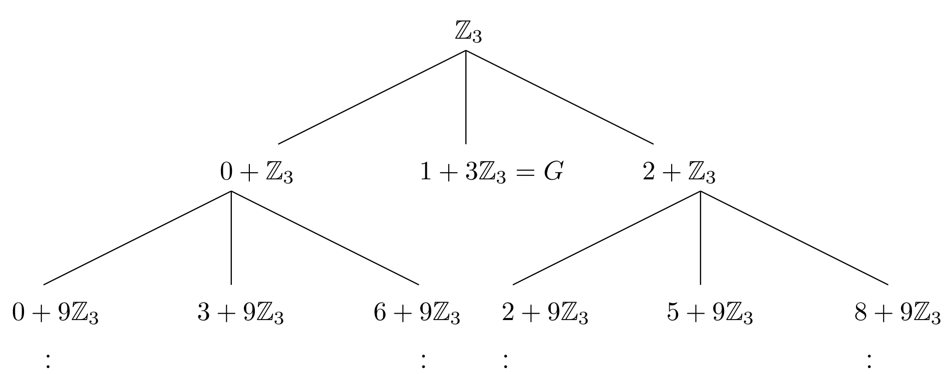

FIGURE 2. Construction of the nonarchimedean (or 3-adic) Cantor string.

By definition, the geometric zeta function of $\mathcal{CS}_3$ is given by

$$\begin{aligned}\zeta_{\mathcal{CS}_3}(s) &= (\mu_H(1+3\mathbb{Z}_3))^s + (\mu_H(3+9\mathbb{Z}))^s + (\mu_H(5+9\mathbb{Z}_3))^s + \cdots \\ &= \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{ns}} = \frac{3^{-s}}{1-2\cdot 3^{-s}} \qquad \text{for } \Re(s) > \log 2/\log 3.\end{aligned}$$

The meromorphic extension of $\zeta_{\mathcal{CS}_3}$ to the entire complex plane $\mathbb{C}$ is given by

$$\zeta_{\mathcal{CS}_3}(s) = \frac{3^{-s}}{1-2\cdot 3^{-s}}, \tag{3.9}$$

with poles at

$$\omega = \frac{\log 2}{\log 3} + i\nu\frac{2\pi}{\log 3}, \qquad \nu \in \mathbb{Z}.$$

Therefore, the set of complex dimensions of $\mathcal{CS}_3$ is given by

$$\mathcal{D}_{\mathcal{CS}_3} = \{D + i\nu\mathbf{p} \mid \nu \in \mathbb{Z}\}, \tag{3.10}$$

where $D = \log 2/\log 3$ is the dimension of $\mathcal{CS}_3$ and $\mathbf{p} = 2\pi/\log 3$ is its oscillatory period. (See Definition 3.11 below.) Finally, note that $\zeta_{\mathcal{CS}_3}$ is a rational function of $z = 3^{-s}$, i.e.,

$$\zeta_{\mathcal{CS}_3}(s) = \frac{z}{1-2z}. \tag{3.11}$$

We refer the interested reader to [13] for additional information concerning the nonarchimedean Cantor string and the associated 3-adic Cantor set.

3.1. **Geometric zeta function.**

Theorem 3.5. *Let $\mathcal{L}_p$ be a self-similar p-adic fractal string. Then the geometric zeta function of $\mathcal{L}_p$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ and is given by*

$$\zeta_{\mathcal{L}_p}(s) = \frac{\sum_{k=1}^{K} g_k^s}{1-\sum_{j=1}^{J} r_j^s}, \qquad \text{for } s \in \mathbb{C}. \tag{3.12}$$

Proof. By Definition 2.1 and Equation (3.6),

$$\begin{aligned}\zeta_{\mathcal{L}_p}(s) &= \sum_{n=1}^{\infty} l_n^s = \sum_{k=1}^{K}\sum_{q=0}^{\infty}\left(\sum_{\nu_1=1}^{J}\cdots\sum_{\nu_q=1}^{J}(r_{\nu_1}\cdots r_{\nu_q}g_k)^s\right) \\ &= \sum_{k=1}^{K} g_k^s \sum_{q=0}^{\infty}\left(\sum_{\nu_1=1}^{J} r_{\nu_1}^s \cdots \sum_{\nu_q=1}^{J} r_{\nu_q}^s\right) \\ &= \sum_{k=1}^{K} g_k^s \sum_{q=0}^{\infty}\left(\sum_{j=1}^{J} r_j^s\right)^q.\end{aligned}$$

It is immediate to check (as in [23]) that there is a unique real solution D to the equation $\sum_{j=1}^{J} r_j^s = 1$. For $\Re(s) > D$, we have $|\sum_{j=1}^{J} r_j^s| < 1$, so the above geometric series converges. Therefore, we obtain

$$\zeta_{\mathcal{L}_p}(s) = \frac{\sum_{k=1}^{K} g_k^s}{1-\sum_{j=1}^{J} r_j^s}.$$

This computation is valid for $\Re(s) > D$, but by the analytic continuation principle, the meromorphic extension of $\zeta_{\mathcal{L}_p}$ to all of $\mathbb{C}$ exists and is given by the last expression. $\square$

An immediate consequence of Theorem 3.5 is the following result:

Corollary 3.6. *The set of complex dimensions of a self-similar p-adic fractal string $\mathcal{L}_p$ is contained in the set of complex solutions ω of the equation $\sum_{j=1}^{J} r_j^{\omega} = 1$.* [5]

3.2. **Lattice versus nonlattice string.** The following definition is the exact counterpart in this context of [21, Def. 2.14]:

Definition 3.7. A self-similar p-adic fractal string $\mathcal{L}_p$ is said to be lattice (or nonlattice) if the multiplicative group generated by the scaling ratios $r_1, r_2, \ldots, r_J$ is discrete (or dense) in $\mathbb{R}^+$.

Theorem 3.8. *Every self-similar p-adic fractal string is lattice.*

Proof. Let $\mathcal{L}_p$ be a self-similar p-adic fractal string generated by a family of similarity contraction mappings $\mathbf{\Phi} = \{\Phi_1, \ldots, \Phi_J\}$, with scaling ratios $r_1, \ldots, r_J$. By Lemma 3.2, for each $j = 1, \ldots, J$, we have $r_j = p^{-n_j}$ for some $n_j \in \mathbb{N}$. Let $n := \gcd\{n_1, \ldots, n_J\}$; then $r := p^{-n}$ is the generator of the multiplicative group $G = \prod_{j=1}^{J} r_j^{\mathbb{Z}}$ generated by $r_1, \ldots, r_J$. Hence G is a discrete subgroup of $\mathbb{R}^+$ and $\mathcal{L}_p$ is therefore lattice. $\square$

Remark 3.9. Definition 3.7 can be slightly refined as in [21, §3.1.1]. More specifically, $\mathcal{L}_p$ is said to be lattice if the multiplicative group generated by its *distinct* scaling ratios is of rank 1. Theorem 3.8 then remains valid without change.

Remark 3.10. Theorem 3.8 is in sharp contrast with the usual theory of real self-similar strings developed in [21, Chs. 2, 3]. Indeed, in the real case, there are both lattice and nonlattice strings; see §1.3.

3.3. **Periodicity of the poles and the zeros.** Actually, a small modification of the above argument (in §3.2) enables us to show that every self-similar p-adic fractal string is "lattice" in a much stronger sense, as we now explain. It will follow (see Theorem 3.13) that not only the poles (i.e., the complex dimensions of $\mathcal{L}_p$) but also the zeros of $\zeta_{\mathcal{L}_p}$ are periodically distributed.

We introduce some useful notation. First, by Lemma 3.2, we write

$$r_j = p^{-n_j}, \text{ with } n_j \in \mathbb{N} \text{ for } j = 1, 2, \ldots, J. \tag{3.13}$$

Second, we write

$$g_k = \mu_H(G_k) = p^{-m_k}, \text{ with } m_k \in \mathbb{N} \text{ for } k = 1, 2, \ldots, K. \tag{3.14}$$

Third, let

$$d = \gcd\{n_1, \ldots, n_J, m_1, \ldots, m_K\}. \tag{3.15}$$

Then there exist positive integers n'_j and m'_k such that

$$n_j = dn'_j \text{ and } m_k = dm'_k \text{ for } j = 1, 2, \ldots, J \text{ and } k = 1, \ldots, K. \tag{3.16}$$

Finally, we set

$$q = p^d. \tag{3.17}$$

[5] Of course, if the string has a single generator (i.e., if $K = 1$), then this inclusion is an equality; see Examples 3.4, 3.16 and Theorem 3.12.

Without loss of generality, we can assume that the scaling ratios r_j (respectively, the gaps g_k) are written in nondecreasing order as in Equations (3.1) and (3.4), so that

$$0 < n'_1 \leq n'_2 \leq \cdots \leq n'_J \text{ and } 0 < m'_1 \leq m'_2 \leq \cdots \leq m'_K. \tag{3.18}$$

Definition 3.11. Let $\mathbf{p} = 2\pi/\log q = 2\pi/d \log p$. Then $\mathbf{p}$ is called the *oscillatory period* of $\mathcal{L}_p$.

Theorem 3.12. *Let $\mathcal{L}_p$ be a self-similar p-adic fractal string and $z = q^{-s}$, with $q = p^d$ as in Equation (3.17). Then the geometric zeta function $\zeta_{\mathcal{L}_p}$ of $\mathcal{L}_p$ is a rational function in z. Specifically,*

$$\zeta_{\mathcal{L}_p}(s) = \frac{\sum_{k=1}^{K} z^{m'_k}}{1 - \sum_{j=1}^{J} z^{n'_j}}, \tag{3.19}$$

where $m'_k, n'_j \in \mathbb{N}$ are given by Equation (3.16).

Proof. In view of Theorem 3.5 and Equations (3.13)–(3.17), we have

$$\zeta_{\mathcal{L}_p}(s) = \frac{\sum_{k=1}^{K} g_k^s}{1 - \sum_{j=1}^{J} r_j^s} = \frac{\sum_{k=1}^{K} p^{-m_k s}}{1 - \sum_{j=1}^{J} p^{-n_j s}} = \frac{\sum_{k=1}^{K} z^{m'_k}}{1 - \sum_{j=1}^{J} z^{n'_j}} \tag{3.20}$$

for $z = q^{-s}$. The final expression for $\zeta_{\mathcal{L}_p}$ is a rational function of z. □

To avoid any confusion, we stress that in the statement of the next theorem, $\zeta_{\mathcal{L}_p}$ is viewed as a function of the original complex variable s.

Theorem 3.13. *Let $\mathcal{L}_p$ be a self-similar p-adic fractal string. Then*

(i) The complex dimensions of $\mathcal{L}_p$ and the zeros of $\zeta_{\mathcal{L}_p}$ are periodically distributed along finitely many vertical lines, with period $\mathbf{p}$, the oscillatory period of $\mathcal{L}_p$.

(ii) Furthermore, along a given vertical line, each pole (respectively, each zero) of $\zeta_{\mathcal{L}_p}$ has the same multiplicity.

(iii) Finally, the dimension D of $\mathcal{L}_p$ is the only complex dimension that is located on the real axis [6]*. Moreover, D is simple* [7] *and is located on the right most vertical line. That is, D is equal to the maximum of the real parts of the complex dimensions.*

Proof. By Theorem 3.12, $\zeta_{\mathcal{L}_p}$ is a rational function of $z = q^{-s}$, hence is a periodic function of s with period $i\mathbf{p}$. Moreover, the numerator and denominator of $\zeta_{\mathcal{L}_p}$ (viewed as functions of z) are polynomials of degree m'_K and n'_J, respectively. Therefore, each has, respectively, m'_K and n'_J complex zeros, counted with multiplicity. Parts (i) and (ii) now follow easily. See the proof of Theorems 2.17 and 3.6 in [21] for more detail, as well as for the proof of Part (iii). □

Remark 3.14. Again, this result is in sharp contrast with the case of real (or archimedean) self-similar strings, where the zeros or the poles of the geometric zeta function need not be periodically distributed along finitely many vertical lines, let alone with the same period; cf. [21, Thm. 2.17]. Indeed, the complex dimensions of nonlattice archimedean strings are quasiperiodically distributed within a certain horizontally bounded strip, as is shown in [21, Ch. 3]. Moreover, even for real *lattice* strings, only the poles, but not in general the zeros, are periodically

[6] By contrast, it is immediate to check that there are no real zeros (still in the s variable).

[7] i.e., D is a simple pole of $\zeta_{\mathcal{L}_p}$.

distributed. Indeed, in the archimedean case, the gaps g_k can take any positive real values (less than one) rather than values in $p^{-\mathbb{N}}$.

Remark 3.15. As will be apparent to the expert reader, the situation described above—specifically, the rationality of the zeta function in the variable $z = q^{-s}$, with $q = p^d$, and the ensuing periodicity of the poles and the zeros—is analogous to the one encountered for a curve (or more generally, a variety) over a finite field $\mathbb{F}_q$; see, e.g., Chapter 3 of [27]. In this analogy, the prime number p corresponds to the characteristic of the finite field.

Example 3.16. *Nonarchimedean Fibonacci String.*
Let $\Phi_1, \Phi_2 : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ be two similarity contraction mappings of $\mathbb{Q}_2$ given by

$$\Phi_1(x) = 2x \quad \text{and} \quad \Phi_2(x) = 1 + 4x, \tag{3.21}$$

with the respective scaling ratios $r_1 = 1/2$ and $r_2 = 1/4$. The self-similar 2-adic fractal string with generator $G = 3 + 4\mathbb{Z}_2$ is called the *nonarchimedean* (or 2-adic) *Fibonacci string* and denoted by $\mathcal{FS}_2$; compare with the archimedean counterpart discussed in [21, §2.3.2]. It is given by the sequence $\mathcal{FS}_2 = l_1, l_2, l_3, \ldots$ and consists of intervals of lengths $l_n = 2^{-(n+1)}$ with multiplicities f_{n+1}, the Fibonacci numbers. [8] More specifically, the nonarchimedean Fibonacci string can also be written as follows:

$$\mathcal{FS}_2 = (3 + 4\mathbb{Z}_2) \cup (6 + 8\mathbb{Z}_2) \cup (12 + 16\mathbb{Z}_2) \cup (13 + 16\mathbb{Z}_2) \cup \cdots,$$

which is an enumeration of the following set

$$\mathcal{FS}_2 = \bigcup_{\alpha=0}^{\infty} \bigcup_{w \in W_\alpha} \Phi_w(3 + 4\mathbb{Z}_2),$$

where Φ_1 and Φ_2 are given as in Equation (3.21). See §4 for more detail.
By Theorem 3.5, the geometric zeta function of $\mathcal{FS}_2$ is given by

$$\zeta_{\mathcal{FS}_2}(s) = \frac{4^{-s}}{1 - 2^{-s} - 4^{-s}}. \tag{3.22}$$

The set of complex dimensions of $\mathcal{FS}_2$ is given by

$$\mathcal{D}_{\mathcal{FS}_2} = \{D + i\nu\mathbf{p} \mid \nu \in \mathbb{Z}\} \cup \{-D + i(\nu + 1/2)\mathbf{p} \mid \nu \in \mathbb{Z}\}, \tag{3.23}$$

with $D = \log\phi/\log 2$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, and $\mathbf{p} = 2\pi/\log 2$, the oscillatory period of $\mathcal{FS}_2$. (See Figure 3.) Note that $\zeta_{\mathcal{FS}_2}$ does not have any zero (in the variable s) since the equation $4^{-s} = 0$ does not have any solution. Finally, as in Theorem 3.12, $\zeta_{\mathcal{FS}_2}$ is a rational function of $z = 2^{-s}$, i.e.,

$$\zeta_{\mathcal{FS}_2}(s) = \frac{z^2}{1 - z - z^2}. \tag{3.24}$$

4. p-ADIC SELF-SIMILAR SETS AND STRINGS

For $\alpha \in \mathbb{N} \cup \{0\}$, let $W_\alpha = \{1, 2, \ldots, J\}^\alpha$ be the set of all finite words, on J symbols, of length α, and $\Phi_w = \Phi_{w_\alpha} \circ \cdots \circ \Phi_{w_1}$ for $w = (w_1, \ldots, w_\alpha) \in W_\alpha$.

[8] These numbers are defined by the recursive formula: $f_{n+1} = f_n + f_{n-1}, f_0 = 0$, and $f_1 = 1$.

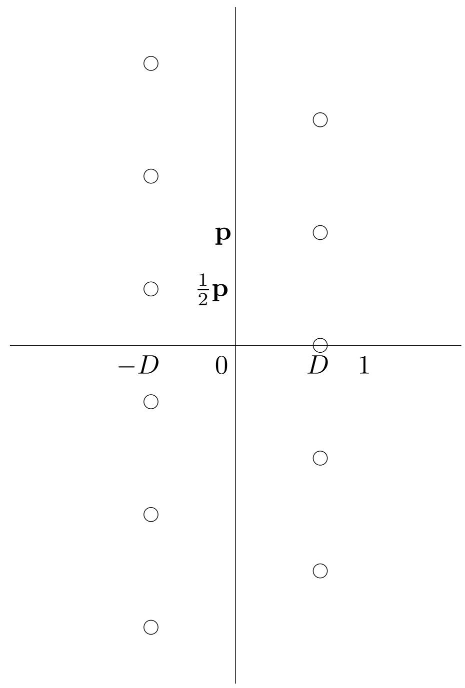

FIGURE 3. The complex dimensions of the nonarchimedean (or 2-adic) Fibonacci string. $D = \log\phi/\log 2$ and $\mathbf{p} = 2\pi/\log 2$.

Definition 4.1. A nonempty compact subset K of $\mathbb{Z}_p$ is said to be a *p-adic* (or nonarchimedean) *self-similar set* generated by the family of contraction mappings $\mathbf{\Phi} = \{\Phi_1, \Phi_2, \ldots, \Phi_J\}$ of $\mathbb{Q}_p$ (leaving $\mathbb{Z}_p$ invariant) [9] if

$$K = \Psi(K) := \bigcup_{j=1}^{J} \Phi_j(K).$$

Throughout the remainder of §4, we assume that $\mathbf{\Phi}$ is a family of similarity contraction mappings of $\mathbb{Q}_p$ leaving $\mathbb{Z}_p$ invariant and such that $\Phi_l(\mathbb{Z}_p) \cap \Phi_j(\mathbb{Z}_p) = \emptyset$ for $l \neq j$. Also, we suppose that $J \geq 2$ and hypothesis (3.2) holds. [10]

Theorem 4.2. *(i) There is a unique nonempty compact subset $\mathcal{S}_p$ of $\mathbb{Z}_p$ such that*

$$\mathcal{S}_p = \Psi(\mathcal{S}_p).$$

In other words, $\mathcal{S}_p$ is the unique p-adic self-similar set associated with the p-adic iterated function system $\mathbf{\Phi} = \{\Phi_1, \ldots, \Phi_J\}$.

(ii) Furthermore, we have

$$\mathcal{S}_p = \bigcap_{\alpha=0}^{\infty} \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_p).$$

[9]That is, such that $\Phi_j(\mathbb{Z}_p) \subset \mathbb{Z}_p$, for $j = 1, \ldots, J$.

[10]When $p = 2$, we assume as in §3 that the maps Φ_j are affine. When $p > 2$, this follows from our hypotheses; see Lemma 3.2.

(iii) Moreover, let $\Psi^1(X) = \Psi(X)$, given as in Definition 4.1, and for $n \geq 2$, let $\Psi^n(X) = \Psi(\Psi^{n-1}(X))$ for every nonempty compact subset X of $\mathbb{Z}_p$. Then the sequence $\Psi^n(X)$ converges to $\mathcal{S}_p$, in the sense of the Hausdorff metric on $\mathbb{Z}_p$.

Proof. Recall from §1.1 that $\mathbb{Z}_p$, equipped with the ultrametric d_p, is a complete metric space. Therefore, the contraction mapping principle, applied to the complete metric space of all nonempty compact subsets of $\mathbb{Z}_p$, equipped with the Hausdorff metric induced by d_p and to the contraction map Ψ defined in part (iii), shows (as in [9]) [11] that there is a unique self-similar (or invariant) set of the family of similarity contraction mappings $\mathbf{\Phi}$. [12] This proves part (i) and part (iii). For part (ii), note that the family of similarity contraction mappings $\mathbf{\Phi}$ satisfies the open set condition with $\mathcal{O} = \mathbb{Z}_p$; see [9, §5.2]. We refer to Hutchinson's paper [9] for a detailed argument in the case of arbitrary complete metric spaces. □

Theorem 4.3. *Let $\mathcal{L}_p$ be the self-similar p-adic fractal string generated by the above family of similarity contraction mappings $\mathbf{\Phi}$ of $\mathbb{Z}_p$ and let $\mathcal{S}_p$ be the p-adic self-similar set associated with this family, as in Theorem 4.2. Then*

$$\mathcal{L}_p = \mathbb{Z}_p \backslash \mathcal{S}_p,$$

the complement of $\mathcal{S}_p$ in $\mathbb{Z}_p$.

Proof. This clearly follows from the construction of the self-similar p-adic fractal string $\mathcal{L}_p$ and, in view of Theorem 4.2, of the definition of the p-adic self-similar set $\mathcal{S}_p$. □

Theorem 4.4. *Under the same assumptions as in Theorem 4.3, we have:*

$$\mathcal{L}_p = \bigcup_{\alpha=0}^{\infty} \bigcup_{w \in W_{\alpha-1}} \bigcup_{k=1}^{K} \Phi_w(G_k),$$

where the generators G_k are defined as in §3 and $\Phi_w(G_k) = \emptyset$ if $w \in W_{-1}$.

Proof. By Theorems 4.2 and 4.3, we have

$$\mathcal{L}_p = \mathbb{Z}_p \backslash \mathcal{S}_p = \mathbb{Z}_p \backslash \bigcap_{\alpha=0}^{\infty} \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_p) = \bigcup_{\alpha=0}^{\infty} \left(\mathbb{Z}_p \backslash \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_p) \right). \tag{4.1}$$

Thus, it suffices to prove that

$$\mathbb{Z}_p \backslash \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_p) = \bigcup_{w \in W_{\alpha-1}} \bigcup_{k=1}^{K} \Phi_w(G_k),$$

for every $\alpha \geq 0$. We shall show this by induction on α. For $\alpha = 0$, it is obvious. For $\alpha = 1$, it follows from the first step of the construction of $\mathcal{L}_p$ since

$$\mathbb{Z}_p \backslash \bigcup_{w \in W_1} \Phi_w(\mathbb{Z}_p) = \mathbb{Z}_p \backslash \bigcup_{j=1}^{J} \Phi_j(\mathbb{Z}_p) = \bigcup_{k=1}^{K} G_k = \bigcup_{w \in W_0} \bigcup_{k=1}^{K} \Phi_w(G_k).$$

[11] See also, e.g., [6, §9.2] and [23].

[12] It follows from [9] that Ψ is indeed a contraction.

Suppose that $\mathbb{Z}_p \backslash \bigcup_{w\in W_\alpha} \Phi_w(\mathbb{Z}_p) = \bigcup_{w\in W_{\alpha-1}} \bigcup_{k=1}^K \Phi_w(G_k)$ for some $\alpha \in \mathbb{N}$. Then

$$\begin{aligned}
\mathbb{Z}_p \backslash \bigcup_{w\in W_{\alpha+1}} \Phi_w(\mathbb{Z}_p) &= \mathbb{Z}_p \backslash \bigcup_{j=1}^{J} \Phi_j\left(\bigcup_{w\in W_\alpha} \Phi_w(\mathbb{Z}_p)\right) \\
&= \bigcup_{j=1}^{J} \Phi_j\left(\mathbb{Z}_p \backslash \bigcup_{w\in W_\alpha} \Phi_w(\mathbb{Z}_p)\right) \\
&= \bigcup_{j=1}^{J} \Phi_j\left(\bigcup_{w\in W_{\alpha-1}} \bigcup_{k=1}^{K} \Phi_w(G_k)\right) \\
&= \bigcup_{w\in W_\alpha} \bigcup_{k=1}^{K} \Phi_w(G_k).
\end{aligned}$$

The induction is completed and hence the theorem is established. □

Remark 4.5. We leave it to the interested reader to illustrate the above theorems by means of the 3-adic Cantor string and of the 2-adic Fibonacci string discussed respectively in Example 3.4 and Example 3.16.

5. Concluding Comments

We close this paper with some comments regarding several possible directions for future research in this area. We hope to address these issues in later work.

5.1. **Adèlic self-similar strings and their spectra.** It would be interesting to unify the archimedean and nonarchimedean settings by appropriately defining *adèlic* self-similar fractal strings, and then studying the associated spectral zeta functions (as is done for standard archimedean fractal strings in [11, 15, 19, 20, 21]). To this aim, the spectrum of these adèlic self-similar strings should be suitably defined and its study may benefit from Dragovich's work on adèlic quantum harmonic oscillators [4]. In the process of defining these adèlic fractal strings, we expect to make contact with the notion of a fractal membrane (or "quantized fractal string") introduced in [12, Ch. 3] and rigorously constructed in [16] as a Connes-type noncommutative geometric space; see also [12, §4.2]. The aforementioned spectral zeta function of an adèlic fractal string would then be viewed as the (completed) spectral partition function of the associated fractal membrane, in the sense of [12].

5.2. **p-adic self-similar flows.** We expect that, much as was done in the real case [21, Ch. 7], one could interpret the geometric zeta function of a p-adic self-similar string as the dynamical zeta function of a suitable nonarchimedean dynamical system, to be referred to as a "p-adic self-similar flow". We could then obtain an associated "prime orbit theorem" and explicit formulas for the corresponding counting functions of (weighted) primitive periodic orbits.

5.3. **Nonlattice self-similar p-adic fractal strings and Berkovich's space.** As we have seen in §3.2, there can only exist *lattice* self-similar p-adic fractal strings, because of the discreteness of the valuation group of $\mathbb{Q}_p$. However, in the archimedean setting, there are both lattice and nonlattice self-similar strings. We expect that by suitably extending the notion of self-similar p-adic fractal string to Berkovich's p-adic analytic space [1, 3], it can be shown that self-similar p-adic fractal strings are generically nonlattice in this broader setting. [13] Furthermore, we conjecture that every nonlattice string in Berkovich's p-adic analytic space can be approximated by lattice strings with increasingly large oscillatory periods (much as occurs in the archimedean case [21, Ch. 3]).

5.4. **Tube formulas for p-adic self-similar strings.** It would be interesting to establish suitable "tube formulas" for self-similar p-adic fractal strings, providing a counterpart of the tube formulas obtained for archimedean self-similar strings in [21, §8.4]. By analogy with the usual case, these formulas should express the Haar measure of the appropriately defined inner ε-neighborhoods of the string [14] as power series in ε having exponents the complex (co)dimensions of the string. In view of the discussion given in §5.3 just above, it would be natural to also consider addressing this same question in the setting of Berkovich's p-adic analytic space, an enlarged model of $\mathbb{Q}_p$. Indeed, in that model, the counterpart of $\mathbb{Z}_p$ is no longer totally disconnected but is (uniquely) path connected.

5.5. **Higher-dimensional p-adic self-similar sets and tilings.** Finally, one could consider higher-dimensional analogues of self-similar p-adic fractal strings, associated with suitable p-adic self-similar sets (or iterated function systems) and the corresponding self-similar tilings [28]. One would then obtain higher-dimensional counterparts of the tube formulas discussed in §5.4. We note that in the archimedean case, higher-dimensional counterparts of the usual tube formulas of [21] have been obtained in [17] for self-similar tilings and fractals; see also [18].

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[13] We wish to thank Machiel van Frankenhuijsen for having suggested to work with $\mathbb{C}_p$, the topological completion of the algebraic closure of $\mathbb{Q}_p$, with the same goal.

[14] For the present situation, see the second part of Definition 2.6.

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Department of Mathematics, University of California, Riverside, CA 92521-0135
E-mail address: lapidus@math.ucr.edu

Department of Mathematics, Hawai'i Pacific University, Honolulu, HI 96813-2785
E-mail address: hlu@hpu.edu