Piecewise principal comodule algebras

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PIECEWISE PRINCIPAL COMODULE ALGEBRAS

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ABSTRACT. A comodule algebra P over a Hopf algebra H with bijective antipode is called principal if the coaction of H is Galois and P is H-equivariantly projective (faithfully flat) over the coaction-invariant subalgebra $P^{\mathrm{co}H}$. We prove that principality is a piecewise property: given N comodule-algebra surjections $P \to P_i$ whose kernels intersect to zero, P is principal if and only if all P_i's are principal. Furthermore, assuming the principality of P, we show that the lattice these kernels generate is distributive if and only if so is the lattice obtained by intersection with $P^{\mathrm{co}H}$. Finally, assuming the above distributivity property, we obtain a flabby sheaf of principal comodule algebras over a certain space that is universal for all such N-families of surjections $P \to P_i$ and such that the comodule algebra of global sections is P.

1. INTRODUCTION

Comodule algebras provide a natural noncommutative-geometry generalisation of spaces equipped with group actions. Less evidently, principal extensions [BH04] appear to be a proper analogue of principal bundles in this context. (See Section 2 for precise definitions.) Principal comodule algebras can be considered as functors from the category of finite-dimensional corepresentations of the Hopf algebra (replacing the structure group) to the category of finitely generated projective modules over the coaction-invariant subalgebra (playing the role of the base space). Thus we have a noncommutative version of assigning to a group representation an associated vector bundle. (See [BHMS, Theorem 3.3] for a precise statement in the setting of topological vector bundles.)

The aim of this article is to establish a viable concept of locality of comodule algebras, and to analyse its relationship with principality. The notion of locality we use herein results from decomposing algebras into "pieces", meaning expressing them as multiple fibre products (called multirestricted direct sums in [P-GK99, p. 264]). If X is a compact Hausdorff space and $X_1, \ldots, X_N$ form a finite closed covering, then $C(X)$ can be expressed as such a multiple fibre product of its quotient C^*-algebras $C(X_i)$. This leads to a C^*-algebraic notion of a "covering of a quantum space" given by a finite family of algebra surjections $\pi_i : P \to P_i$ with $\bigcap_i \ker \pi_i = 0$. (See [BK96, CM00], cf. [D-M97].)

Recall that not all properties of group actions are local in nature: there is a natural example of a locally proper action of $\mathbb{R}$ on $\mathbb{R}^2$ that is not proper. (See [P-RS61, p.298], cf. [BHMS, Example 1.14] for more details.) On the other hand, a group action is free if and only if it is locally free. Therefore, since for compact groups all actions are proper, the principal (i.e., free and proper) actions of compact groups are local in nature. Our main result is a noncommutative analogue of this statement:

Theorem 3.3. *A comodule algebra P which is "covered by pieces" P_i is principal if and only if so are the pieces.*

In particular, a smash product of an H-module algebra B with the Hopf algebra H (with bijective antipode) is principal (see Section 2.3), so that gluing together smash products is a way of constructing principal comodule algebras. This is a major application of our result, and is presented in the example section. Another application of Theorem 3.3 was found very recently in algebraic topology. Notably, it allows one to prove a compact-group version of the well-known equivalence of discrete-group principal bundles and Galois covering maps [BH].

More precisely, first one extends the notion of the algebra of regular functions (spanned by the matrix coefficients of the irreducible unitary corepresentations) from compact quantum groups to unital C^*-algebras on which they act [BHMS, (3.1.4)]. Such a subalgebra of regular functions is called the Peter-Weyl comodule algebra. It is a comodule algebra over the Hopf algebra or regular functions on a compact quantum group. In general, it is not a C^*-algebra, although its coaction-invariant (the base space) subalgebra is always a C^*-algebra. Now, one can show that the Peter-Weyl comodule algebra of functions on a compact Hausdorff space with an action of a compact group is principal if and only if the action is free [BH]. In other words, the Galois condition of Hopf-Galois theory holds if and only if we have a compact principal bundle.

Concerning coverings of a quantum space in the earlier described sense, it was pointed out in [CM00, p.369] that they feature a certain incompleteness problem when going beyond the C^*-setting. This is related to the fact that the lattice of ideals generated by the $\ker \pi_i$'s is in general not distributive. (This problem does not arise for C^*-algebras.) Hence we analyse a stronger notion of covering that includes the assumption of distributivity as part of the definition (see Definition 3.7). This allows us to consider well-behaved coverings beyond the C^*-setting, which is useful for Peter-Weyl comodule algebras. Furthermore, if all covering comodule algebras are smash products, we arrive at a concept of piecewise trivial comodule algebras. They appear to be a good noncommutative replacement of locally trivial compact principal bundles.

The data of a covering by N pieces can be equivalently encoded into a flabby sheaf of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. This is the 2-element field $(N-1)$-projective space whose topology subbasis is its usual covering by N affine spaces. It is a finite space encoding the "combinatorics" of an N-covering, and is non-Hausdorff unless $N = 1$. The family of all non-empty open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ turns out to be a lattice isomorphic to the free distributive lattice on N generators.[1] Combining this with the Chinese Remainder Theorem for distributive lattices of ideals in an arbitrary ring proves that distributive lattices generated by N ideals are equivalent to flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ (cf. *Lattice of Ideals* in [R-GC97]).

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[1]Note that it is extremely difficult to determine the number of elements $FD(N)$ of this finite lattice. The asymptotic behaviour of FD was analysed in [Y-K54], but the problem of what is a general formula for $FD(N)$, originally posed by R. Dedekind [D-R97, p.147], remains open since 1897. As of 1999, the biggest known value of FD was for $N = 8$.

For example, consider a compact Hausdorff space X with a covering by N closed subsets $X_1, \ldots, X_N$. Then we have the soft sheaf of continuous functions with N distinguished C^*-algebras $C(X_1), \ldots, C(X_N)$. However, the soft sheaf of complex-valued continuous functions on X is not a sheaf of C^*-algebras. Therefore, there seems to be no evident way to use soft sheaves in the noncommutative setting. To overcome this difficulty, we declare the closed sets open and consider X with the new topology generated by these open sets. This leads us to flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

Thus we obtain a covering version of the commutative Gelfand-Neumark Theorem: there is an equivalence between the category of compact Hausdorff spaces with ordered closed coverings by N subsets and the opposite category of flabby sheaves of unital commutative C^*-algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. In the noncommutative setting, this sheaf-theoretic reformulation of coverings allows us to view a piecewise trivial comodule algebra as a geometrically desired refinement of what is called an "$\mathbf{A}$-quantum principal bundle" in [P-MJ94]. (For instance, our framework allows us to conclude that the comodule algebra of global sections is principal.)

Summarising, much as the gluings of topological spaces, the pullbacks of C^*-algebras are fundamental for constructing new algebras and applying Mayer–Vietoris arguments. We are motivated by the study of pullbacks of C^*-algebras equipped with a quantum group action. These pullbacks can be exemplified by a join construction of compact quantum groups. Insisting on the freeness of these actions leads to Galois-type extensions of C^*-algebras by Hopf algebras. This allows us to use at the same time both algebraic techniques of Hopf-Galois theory and analytic tools coming with C^*-algebras. Generalising pullbacks from two to finitely many morphisms brings up distributive lattices as a fundamental language. As a by-product of our considerations, we obtain an equivalence between the distributive lattices generated by N ideals intersecting to zero and flabby sheaves of algebras over a projective space $P^{N-1}(\mathbb{Z}/2)$.

2. Background

Throughout, we work over a field k. All considered algebras, coalgebras etc., are over k. An unadorned $\otimes$ denotes the tensor product of k-vector spaces. For coproducts and coactions we adopt the Heyneman-Sweedler notation with the summation sign suppressed: $\Delta(h) = h_{(1)} \otimes h_{(2)} \in H \otimes H$, $\Delta_P(p) = p_{(0)} \otimes p_{(1)} \in P \otimes H$.

2.1. **Fibre products.** We recall here elementary facts concerning pullback diagrams that will be used in what follows. To focus attention, we consider the category of vector spaces. Since all our algebras and comodules are over a field, this suffices for our applications.

Let $\pi_1 : V_1 \to V_{12}$ and $\pi_2 : V_2 \to V_{12}$ be linear maps of vector spaces. The pullback (fibre product) $V_1 \times_{\pi_1, \pi_2} V_2$ of π_1 and π_2 is defined by a universal property, and turns out to be isomorphic to $\ker(\pi_1 - \pi_2 : V_1 \times V_2 \to V_{12}) = \{(p, q) \in V_1 \times V_2 \,|\, \pi_1(p) = \pi_2(q)\}$. As a consequence of this description, we obtain:

$$(1) \qquad (V_1 \underset{\pi_1, \pi_2}{\times} V_2)^{\otimes 2} = \ker((\pi_1 - \pi_2) \otimes \mathrm{id}) \cap \ker(\mathrm{id} \otimes (\pi_1 - \pi_2)).$$

Next, let us consider the following commutative diagram of linear maps

(2)

and show:

Lemma 2.1. *Assume that the ϕ_i's and π_i's in the diagram (2) are surjective. Then η is surjective if and only if $\ker(\pi_i \circ \phi_i) = \ker \phi_1 + \ker \phi_2$.*

Proof. Assume first that η is surjective, and $v \in \ker(\pi_i \circ \phi_i)$. Then both $(\phi_1(v), 0)$ and $(0, \phi_2(v))$ belong to $V_1 \times_{\pi_1, \pi_2} V_2$, and there exist v_1 and v_2 such that $\eta(v_1) = (\phi_1(v), 0)$ and $\eta(v_2) = (0, \phi_2(v))$. Clearly, $v - (v_1 + v_2) \in \ker \eta$. Therefore, as $v_1 \in \ker \phi_2$, $v_2 \in \ker \phi_1$, and

(3)
$$\ker \eta = \ker \phi_1 \cap \ker \phi_2\,,$$

we conclude that $v \in \ker \phi_1 + \ker \phi_2$.

Conversely, assume that $\ker(\pi_i \circ \phi_i) = \ker \phi_1 + \ker \phi_2$. Let $(\phi_1(v_1), \phi_2(v_2))$ be any element of the fibre product. Then $v_1 - v_2 \in \ker \phi_1 + \ker \phi_2$, so that $v_1 - v_2 = k_1 + k_2$ for some $k_1 \in \ker \phi_1$ and $k_2 \in \ker \phi_2$. Hence, for $v := v_1 - k_1 = v_2 + k_2$, we have $(\phi_1(v_1), \phi_2(v_2)) = (\phi_1(v), \phi_2(v)) = \eta(v)$. $\qquad\square$

To end with, let us remark that an appropriate version of this lemma becomes a theorem characterising the exact Maltsev categories among the regular categories [CKP93].

2.2. **Distributive lattices.** To reformulate coverings in terms of flabby sheaves, we will need a method yielding a presentation of the elements of finitely generated distributive lattices. To this end, we consider first the lattice of upper sets. We call a non-empty set α of non-empty subsets of $\{1, \dots, N\}$ an *upper set* (or a dual order ideal) if

(4)
$$a \in \alpha \quad \text{and} \quad a \subseteq b \quad \implies \quad b \in \alpha.$$

We denote by $\Upsilon_N \subseteq 2^{2^{\{1, \dots, N\}}}$ the set of all upper sets in the set $2^{\{1, \dots, N\}}$ of all subsets of $\{1, \dots, N\}$. The set union and intersection make Υ_N a distributive lattice. Furthermore, the upper sets are nonempty open sets of $2^{\{1, \dots, N\}}$ in the Alexandrov topology. One can prove that Υ_N is generated as a distributive lattice by the sets

(5)
$$S_i := \{a \subseteq 2^{\{1, \dots, N\}} \mid i \in a\}, \quad i \in \{1, \dots, N\}.$$

We want to show that Υ_N is isomorphic to the free distributive lattice generated by N elements (see [B-G67, par. III.4]). This means verifying that Υ_N enjoys the

universal property defining the free distributive lattice with N-generators. For any lattice $(\Lambda, \vee, \wedge)$ generated by $\lambda_1, \ldots, \lambda_N$ we can define a map

$$(6) \qquad R^{(\lambda_i)_i} : \Upsilon_N \ni \alpha \longmapsto \bigvee_{\{i_1,\ldots,i_k\}\in\alpha} (\lambda_{i_1} \wedge \ldots \wedge \lambda_{i_k}) \in \Lambda.$$

It is is a homomorphism of lattices if, for all $\alpha_1, \alpha_2 \in \Upsilon_N$,

$$(7) \quad R^{(\lambda_i)_i}(\alpha_1 \cap \alpha_2) = R^{(\lambda_i)_i}(\alpha_1) \wedge R^{(\lambda_i)_i}(\alpha_2), \quad R^{(\lambda_i)_i}(\alpha_1 \cup \alpha_2) = R^{(\lambda_i)_i}(\alpha_1) \vee R^{(\lambda_i)_i}(\alpha_2).$$

The second equality of (7) always holds. To prove the first one, we need to assume that the lattice Λ is distributive. First, observe that for all $\alpha_1, \alpha_2 \in \Upsilon_N$:

$$(8) \qquad \alpha_1 \cap \alpha_2 = \{a_1 \cup a_2 \mid a_1 \in \alpha_1, \ a_2 \in \alpha_2\}.$$

Indeed, since $a_1 \subseteq a_1 \cup a_2$ and $a_2 \subseteq a_1 \cup a_2$, the left hand side contains the right hand side. The other inclusion follows from $\alpha_1 \cap \alpha_2 \subseteq \alpha_1$, $\alpha_1 \cap \alpha_2 \subseteq \alpha_2$ and $a = a \cup a$. Now, by the distributivity of Λ, formula (8) implies the first equation of (7).

Next, let P be an algebra, and let $(I_i)_{i\in\{1,\ldots,N\}}$ be an ordered family of ideals of P intersecting to zero and generating a distributive lattice Ξ_N with $+$ and $\cap$ as meet and join operations, respectively. The pairs $(P, (I_i)_{i\in\{1,\ldots,N\}})$ form a category $\mathbf{I}_N$ of distributive lattices of ideals. A morphism between $(P, (I_i)_{i\in\{1,\ldots,N\}})$ and $(Q, (J_i)_{i\in\{1,\ldots,N\}})$ is, by definition, an algebra map $f : P \to Q$ such that $f(I_i) \subseteq J_i$, for all $i \in \{1, \ldots, N\}$. On the other hand, we consider the category $\mathbf{F}_N$ of flabby sheaves of algebras over the space $\mathbb{S}_N := 2^{\{1,\ldots,N\}} \setminus \{\emptyset\}$ with the Alexandrov topology (open sets are upper sets). Since every sheaf is a functor, we define as morphisms between sheaves natural transformations of functors.

Proposition 2.2. *Let $\mathbf{I}_N$ be the category of distributive lattices of ideals generated by N elements, and $\mathbf{F}_N$ be the category of flabby sheaves of algebras over $\mathbb{S}_N$. Then the following assignments*

$$(9) \qquad \mathbf{I}_N \ni (P, (I_i)_{i\in\{1,\ldots,N\}}) \quad \overset{F}{\longmapsto} \quad \left\{\mathcal{P} : U \mapsto P/R^{(I_i)_i}(U)\right\}_U \in \mathbf{F}_N,$$

$$(10) \qquad \mathbf{F}_N \ni \mathcal{P} \quad \overset{G}{\longmapsto} \quad \left(\mathcal{P}(\mathbb{S}_N), (\ker(\mathcal{P}(\mathbb{S}_N) \to \mathcal{P}(S_i)))_i\right) \in \mathbf{I}_N,$$

are functors establishing an equivalence of categories.

Proof. Assume that we are given an object $(P, (I_i)_{i\in\{1,\ldots,N\}}) \in \mathbf{I}_N$. We want to show that (9) assigns to it a flabby sheaf. For brevity, we put $R^{(I_i)_i} = R$. Let U, U' be open subsets of $\mathbb{S}_N$ such that $U \subseteq U'$. It follows from (7) that $R(U') \subseteq R(U)$. Therefore, one can define the restriction maps

$$(11) \qquad \pi_{U',U} : \mathcal{P}(U') \longrightarrow \mathcal{P}(U), \quad p + R(U') \longmapsto p + R(U),$$

so that $\mathcal{P}$ is a presheaf. Next, let U be an open subset of $\mathbb{S}_N$ and let $\{U_k\}_k$ be an open covering of U (i.e. $\bigcup_k U_k = U$). Consider elements $p_k \in \mathcal{P}(U_k)$ such that $\pi_{U_k, U_k \cap U_l}(p_k) = \pi_{U_l, U_k \cap U_l}(p_l)$, for all k, l. By the distributivity of the lattice Ξ_N and the generalised Chinese Remainder Theorem (e.g., see [SS58], Theorem 18 on p. 280), there exists an element $p_U \in \mathcal{P}(U)$ such that $\pi_{U,U_k}(p_U) = p_k$, $\forall k$. To prove the uniqueness of $p_U =: p + R(U)$, suppose that $\pi_{U,U_k}(p_U) := p + R(U_k) = 0$, $\forall k$. This means that $p \in \bigcap_k R(U_k)$. On the other hand, it follows from (7) that $R(U \cup U') = R(U) \cap R(U')$, for any open subsets U, U' of $\mathbb{S}_N$. Consequently, $p \in R(\bigcup_k U_k) = R(U)$,

that is $p_U = 0$. This establishes the desired uniqueness. Hence $\mathcal{P}$ is a flabby sheaf, and the assignment F is well defined.

Now suppose that we are given a flabby sheaf $\mathcal{P}$ of algebras over $\mathbb{S}_N$. Let $\pi_{V,U} : \mathcal{P}(V) \to \mathcal{P}(U)$ denote the restriction map for any open U, V, $U \subseteq V$. For brevity, we write π_U instead of $\pi_{V,U}$, if $V = \mathbb{S}_N$. By the flabbiness of $\mathcal{P}$, the morphisms π_{S_i} (see (5)) are surjective. The property $\bigcap_{i=1}^N \ker \pi_{S_i} = \{0\}$ follows from the sheaf condition. Furthermore, as lattices of sets are always distributive, to prove the distributivity of the lattice Ξ_N it is enough to show that the assignment $U \mapsto \ker \pi_U$ defines a surjective morphism from Υ_N onto Ξ_N transforming the union and intersection of open subsets to the intersection and sum of ideals, respectively. To show this, let U', U'' be open subsets of $\mathbb{S}_N$. Since $\mathcal{P}$ is a sheaf, we know that $\mathcal{P}(U' \cup U'')$ is the fibre product of $\mathcal{P}(U')$ and $\mathcal{P}(U'')$. Hence (3) implies that

$$(12) \qquad\qquad \ker \pi_{U' \cup U''} = \ker \pi_{U'} \cap \ker \pi_{U''} ,$$

as needed. Similarly, since the sheaf $\mathcal{P}$ is flabby, Lemma 2.1 implies that

$$(13) \qquad\qquad \ker \pi_{U' \cap U''} = \ker \pi_{U'} + \ker \pi_{U''} .$$

Thus we have shown that (10) assigns to flabby sheaves objects of $\mathbf{I}_N$, so that the assignment G is well defined.

The functoriality of G is immediate. Indeed, let $\Phi : \mathcal{P} \to \mathcal{Q}$ be a natural transformation of flabby sheaves of algebras over $\mathbb{S}_N$. Denote by η the restriction morphisms of $\mathcal{Q}$. Then, since $\eta_{S_i} \circ \Phi_{\mathbb{S}_N} = \Phi_{S_i} \circ \pi_{S_i}$, we have $\Phi_{\mathbb{S}_N}(\ker \pi_{S_i}) \subseteq \ker \eta_{S_i}$. To prove that F is a functor, consider an algebra homomorphism $f : P \to Q$ such that $f(I_i) \subseteq J_i$ for all i. (It is a morphism in $\mathbf{I}_N$.) For any open subset U, we obtain an analogous inclusion $f(R^{(I_i)_i}(U)) \subseteq R^{(J_i)_i}(U)$. Hence we have algebra homomorphisms

$$(14) \qquad P/R^{(I_i)_i}(U) \longrightarrow Q/R^{(J_i)_i}(U), \quad p + R^{(I_i)_i}(U) \longmapsto f(p) + R^{(J_i)_i}(U),$$

defining the desired morphism of sheaves.

It is clear that $G \circ F$ is a functor naturally isomorphic to the identity functor on the category $\mathbf{I}_N$ because $R^{(I_i)_i}(S_j) = I_j$ for any index j. On the other hand, consider an object $\mathcal{P}$ in the category $\mathbf{F}_N$ of flabby sheaves. Combining (12) and (13) with the fact that $R^{(I_i)_i}$ is a morphism of lattices, we can conclude that there is always a natural isomorphism $\mathcal{P}(U) \cong \mathcal{P}(\mathbb{S}_N)/R^{(I_i)_i}(U)$. Hence $F \circ G$ is naturally isomorphic to the identity functor on $\mathbf{F}_N$. $\qquad\square$

2.3. **Principal comodule algebras.** Let $(H, \Delta, \varepsilon, S)$ be a Hopf algebra with bijective antipode. A right H-comodule algebra P is a unital associative algebra equipped with an H-coaction $\Delta_P : P \to P \otimes H$ that is an algebra map. For a comodule algebra P, we call

$$(15) \qquad\qquad P^{\mathrm{co}H} := \{p \in P \,|\, \Delta_P(p) = p \otimes 1\}$$

the subalgebra of coaction-invariant elements in P. The assumed existence of the inverse of the antipode allows us to define a left coaction $_P\Delta : P \to H \otimes P$ by the formula $p \mapsto S^{-1}(p_{(1)}) \otimes p_{(0)}$. This makes P a left H-comodule, and a left H^{op}-comodule algebra.

Definition 2.3. *Let P be a right comodule algebra over a Hopf algebra H with bijective antipode, and let $B := P^{\mathrm{co}H}$ be the coaction-invariant subalgebra. The comodule algebra P is called* principal *if the following conditions are satisfied:*

(1) *the coaction of H is Galois, that is, the map*
$$\mathsf{can} : P \underset{B}{\otimes} P \longrightarrow P \otimes H, \quad p \otimes q \longmapsto pq_{(0)} \otimes q_{(1)},$$
(called the canonical map) is bijective,

(2) *the comodule algebra P is right H-equivariantly projective as a left B-module, i.e., there exists a right H-colinear and left B-linear splitting of the multiplication map $B \otimes P \to P$.*

This splitting can always be chosen to be unital [BH04, BHMS+]. Also, in this setting, one can show that the H-equivariant projectivity of P over B is equivalent to the faithful flatness of P as a B-module [SS05, SS]. If P is a principal comodule algebra, then the extension of algebras $B \subseteq P$ is a special case of principal extensions defined in [BH04].

A cleft Hopf-Galois extension (e.g., a smash product $B\#H$ of an H-module algebra B by H) is always principal. Indeed, by [BM89, p.42], cleft Hopf-Galois extensions always enjoy the normal basis property, and the latter can be viewed as equivariant freeness, a special case of equivariant projectivity. For more details and an introduction to Hopf-Galois theory, see, e.g., [M-S93, S-HJ94].

2.4. **Strong connections.** The inverse of the canonical map defines a monomorphism $H \to P \otimes_B P$, $h \mapsto \mathsf{can}^{-1}(1 \otimes h)$, called the translation map. It turns out that lifts of this map to $P \otimes P$ that are both right and left H-colinear yield an equivalent approach to principality [BH04]:

Definition 2.4. *Let H be a Hopf algebra with bijective antipode, and $\widetilde{\mathsf{can}} : P \otimes P \to P \otimes H$ be the lift of can to $P \otimes P$. Then a strong connection (cf. [H-PM96, DGH01]) on a right H-comodule algebra P is a unital linear map $\ell : H \to P \otimes P$ satisfying*

$$(\mathrm{id}_P \otimes \Delta_P) \circ \ell = (\ell \otimes \mathrm{id}_H) \circ \Delta, \quad ({}_P\Delta \otimes \mathrm{id}_P) \circ \ell = (\mathrm{id}_H \otimes \ell) \circ \Delta, \quad \widetilde{\mathsf{can}} \circ \ell = 1 \otimes \mathrm{id}.$$

The last property of the strong connection (splitting of $\widetilde{\mathsf{can}}$) gives rise to the commutative diagram

(16)
$$
\begin{array}{ccc}
H & \xrightarrow{\quad \ell \quad} & P \otimes P \\
{\scriptstyle 1\otimes\mathrm{id}} \downarrow & {\scriptstyle \widetilde{\mathsf{can}}} \swarrow & \downarrow {\scriptstyle \text{canonical surjection}} \\
P \otimes H & \xleftarrow[\mathsf{can}]{} & P \otimes_B P.
\end{array}
$$

Using the Heyneman–Sweedler-type notation $h \mapsto \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}$ (summation suppressed), we can write the bicolinearity and splitting property of a strong connection as follows:

(17) $\qquad \ell(h)^{\langle 1 \rangle} \otimes \ell(h)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}{}_{(1)} = \ell(h_{(1)})^{\langle 1 \rangle} \otimes \ell(h_{(1)})^{\langle 2 \rangle} \otimes h_{(2)},$

(18) $\qquad \ell(h)^{\langle 1 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 1 \rangle}{}_{(1)} \otimes \ell(h)^{\langle 2 \rangle} = \ell(h_{(2)})^{\langle 1 \rangle} \otimes S(h_{(1)}) \otimes \ell(h_{(2)})^{\langle 2 \rangle},$

(19) $\qquad \ell(h)^{\langle 1 \rangle} \ell(h)^{\langle 2 \rangle}{}_{(0)} \otimes \ell(h)^{\langle 2 \rangle}{}_{(1)} = 1 \otimes h.$

Applying $\mathrm{id}\otimes\varepsilon$ to the last equation yields a very useful formula:

$$(20) \qquad\qquad \ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \varepsilon(h).$$

One can prove that an H-comodule algebra P is principal if and only if it admits a strong connection [BH04, BHMS+, BB05]. Given a strong connection ℓ, one can show that the formula

$$(21) \qquad\qquad P\otimes H \longrightarrow P\underset{B}{\otimes} P, \quad p\otimes h \longmapsto p\ell(h)^{\langle 1\rangle}\underset{B}{\otimes}\ell(h)^{\langle 2\rangle},$$

defines the inverse of the canonical map can, so that the coaction of H is Galois. Next, one can also show that

$$(22) \qquad\qquad s: P \ni p \longmapsto p_{(0)}\ell(p_{(1)})^{\langle 1\rangle}\otimes\ell(p_{(1)})^{\langle 2\rangle} \in B\otimes P$$

is a splitting whose existence proves the equivariant projectivity. Much as above, one argues that the formula

$$(23) \qquad\qquad s': P \ni p \longmapsto \ell(S^{-1}(p_{(1)}))^{\langle 1\rangle}\otimes\ell(S^{-1}(p_{(1)}))^{\langle 2\rangle}p_{(0)} \in P\otimes B$$

provides a left H-colinear and right B-linear splitting of the multiplication map $P\otimes B \to P$.

2.5. Actions of compact quantum groups.

Let $\bar{H}$ be the C^*-algebra of a compact quantum group in the sense of Woronowicz [W-SL87, W-SL98] and H its dense Hopf $*$-subalgebra spanned by the matrix coefficients of the irreducible unitary corepresentations. Let $\bar{P}$ be a unital C^*-algebra and let $\delta: \bar{P} \to \bar{P}\otimes_{\min}\bar{H}$ be a C^*-algebraic right coaction of $\bar{H}$ on $\bar{P}$. (See [BS93, Definition 0.2] for a general definition and [B-FP95, Definition 1] for the special case of compact quantum groups.) Then the subalgebra $P \subseteq \bar{P}$ of elements for which the coaction lands in $\bar{P}\otimes H$ (algebraic tensor product),

$$(24) \qquad\qquad P := \{p \in \bar{P}\,|\,\delta(p) \in \bar{P}\otimes H\},$$

is an H-comodule algebra. It follows from the results of [B-FP95] and [P-P95] that P is dense in $\bar{P}$. We call P the Peter-Weyl comodule algebra associated to the C^*-algebraic coaction δ. It is straightforward to verify that the operation $\bar{P}\mapsto P$ is a functor commuting with taking fibre products. Note also that $\bar{P}^{co\bar{H}} = P^{coH}$.

3. Piecewise principality

To show the piecewise nature of principality, we begin by proving lemmas concerning quotients and fibre products of principal comodule algebras.

Lemma 3.1. *Let $\pi: P \to Q$ be a surjection of right H-comodule algebras (bijective antipode assumed). If P is principal, then:*

(1) *The induced map $\pi|_{P^{coH}}: P^{coH} \to Q^{coH}$ is surjective.*

(2) *There exists a unital H-colinear splitting of π.*

Proof. It follows from the colinearity of π that $\pi(P^{coH}) \subseteq Q^{coH}$. To prove the converse inclusion, we take advantage of the left P^{coH}-linear retraction of the inclusion $P^{coH} \subseteq P$ that was used to prove [BH04, Theorem 2.5(3)]:

$$(25) \qquad\qquad \sigma: P \longrightarrow P^{coH}, \quad \sigma(p) := p_{(0)}\ell(p_{(1)})^{\langle 1\rangle}\varphi(\ell(p_{(1)})^{\langle 2\rangle}).$$

Here ℓ is a strong connection on P and φ is any unital linear functional on P. If $\pi(p) \in Q^{\mathrm{co}H}$, then $\sigma(p)$ is a desired element of $P^{\mathrm{co}H}$ that is mapped by π to $\pi(p)$. Indeed, since $\pi(p_{(0)}) \otimes p_{(1)} = \pi(p) \otimes 1$, using the unitality of ℓ, π and φ, we compute

$$(26) \qquad \pi(\sigma(p)) = \pi(p_{(0)})\pi(\ell(p_{(1)})^{\langle 1 \rangle})\varphi(\ell(p_{(1)})^{\langle 2 \rangle}) = \pi(p).$$

To prove the second assertion, let us choose any k-linear unital splitting of $\pi|_{P^{\mathrm{co}H}}$ and denote it by $\alpha^{\mathrm{co}H}$. One can readily verify that the formula

$$(27) \qquad \alpha(q) := \alpha^{\mathrm{co}H}(q_{(0)}\pi(\ell(q_{(1)})^{\langle 1 \rangle}))\ell(q_{(1)})^{\langle 2 \rangle}$$

defines a unital colinear splitting of π. (Note also that $\alpha^{\mathrm{co}H} = \alpha|_{Q^{\mathrm{co}H}}$.) $\qquad\square$

Lemma 3.2. *Let P be a fibre product in the category of right H-comodule algebras:*

Then, if P_1 and P_2 are principal and π_2^1 and π_1^2 are surjective, P is a principal comodule algebra.

Proof. Given strong connections ℓ_1 and ℓ_2 on P_1 and P_2, respectively, we want to construct a strong connection on P. A first approximation for such a strong connection is as follows:

$$(28) \qquad \lambda : H \longrightarrow P \otimes P, \quad \lambda(h) := (\ell_1(h)^{\langle 1 \rangle}, f_2^1(\ell_1(h)^{\langle 1 \rangle})) \otimes (\ell_1(h)^{\langle 2 \rangle}, f_2^1(\ell_1(h)^{\langle 2 \rangle})).$$

Here $f_2^1 := \sigma_2 \circ \pi_2^1$, and σ_2 is a unital colinear splitting of π_1^2, which exists by Lemma 3.1(2). (Note that our Heyneman-Sweedler-type summation convention for strong connections $x^{\langle 1 \rangle} \otimes x^{\langle 2 \rangle} = \sum_i x_i \otimes y_i$ is extended to the direct products by the formula $(x^{\langle 1 \rangle}, x^{\langle 1 \rangle}) \otimes (x^{\langle 2 \rangle}, x^{\langle 2 \rangle}) := \sum_i (x_i, x_i) \otimes (y_i, y_i)$.) The map λ is unital and bicolinear, but it does not split the lifted canonical map:

$$(1,1) \otimes h - \widetilde{\mathrm{can}}(\lambda(h)) = (0,1) \otimes h - (0, f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle})f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})) \otimes h_{(2)} \in P_2 \otimes H.$$

Now, let $\widetilde{\mathrm{can}}_2$ be the lifted canonical map on $P_2 \otimes P_2$. Applying the splitting of $\widetilde{\mathrm{can}}_2$ given by ℓ_2 (cf. (21)) to the right hand side of the above equation, gives a correction term for λ:

$$(29) \qquad T(h) := \ell_2(h) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle})f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})\ell_2(h_{(2)})^{\langle 1 \rangle} \otimes \ell_2(h_{(2)})^{\langle 2 \rangle}$$
$$= \left(\varepsilon(h_{(1)}) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle})f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})\right)\ell_2(h_{(2)})^{\langle 1 \rangle} \otimes \ell_2(h_{(2)})^{\langle 2 \rangle}.$$

This defines a bicolinear map into $P_2 \otimes P_2$ which annihilates 1. Considering λ as a map into $(P_1 \oplus P_2)^{\otimes 2}$, we can add these two maps. The map $\lambda + T$ is still unital and bicolinear and splits the lifted canonical map on $(P_1 \oplus P_2)^{\otimes 2}$. Remembering (1) and (20), it is clear from the formula for T that to make it take values in $P \otimes P$ we only need to add the term

$$(30) \qquad T'(h) := \left(\varepsilon(h_{(1)}) - f_2^1(\ell_1(h_{(1)})^{\langle 1 \rangle})f_2^1(\ell_1(h_{(1)})^{\langle 2 \rangle})\right)\ell_2(h_{(2)})^{\langle 1 \rangle} \otimes f_1^2(\ell_2(h_{(2)})^{\langle 2 \rangle}).$$

Much as above, here $f_1^2 := \sigma_1 \circ \pi_1^2$ and σ_1 is a unital colinear splitting of π_2^1, which exists by Lemma 3.1(2). The formula (30) defines a bicolinear map into $P_2 \otimes P_1$ which annihilates 1. Since the lifted canonical map on $(P_1 \oplus P_2)^{\otimes 2}$ vanishes on $P_2 \otimes P_1$, the sum $\lambda + T + T'$ splits the lifted canonical map, takes values in $P \otimes P$ and is unital and bicolinear. Thus it is, as desired, a strong connection on P. $\square$

Let us now consider a family $\pi_i : P \to P_i$, $i \in \{1, \ldots, N\}$, of surjections of right H-comodule algebras with $\bigcap_{i=1}^N \ker \pi_i = 0$. Denote $J_i := \ker \pi_i$. By Lemma 2.1 and formula (3), for any $k = 1, \ldots, N-1$ there is a fibre-product diagram of right H-comodule algebras

(31)
$$P/(J_1 \cap \ldots \cap J_{k+1})$$

$$P/(J_1 \cap \ldots \cap J_k) \qquad\qquad P/J_{k+1}$$

$$\pi_2^1 \qquad\qquad\qquad \pi_1^2$$

$$P/((J_1 \cap \ldots \cap J_k) + J_{k+1}) \qquad .$$

Assume that all the $P_i \cong P/J_i$ are principal. Then Lemma 3.2 implies by an obvious induction that $P/(J_1 \cap \ldots \cap J_k)$ is principal for all $k = 1, \ldots, N$. In particular ($k = N$), P is principal. On the other hand, if P is principal then all the P_i's are principal: If $\ell : H \to P \otimes P$ is a strong connection on P, then

(32)
$$(\pi_i \otimes \pi_i) \circ \ell : H \longrightarrow P_i \otimes P_i$$

is a strong connection on P_i. Thus we have proved the following:

Theorem 3.3. *Let $\pi_i : P \to P_i$, $i \in \{1, \ldots, N\}$, be surjections of right H-comodule algebras such that $\bigcap_{i=1}^N \ker \pi_i = 0$. Then P is principal if and only if all the P_i's are principal.*

Remark 3.4. In [HW], Lemma 3.2 has been generalised to one-surjective pullbacks, i.e., to pullbacks with only one of the defining morphisms π_1^2 and π_2^1 assumed to be surjective. Furthermore, it seems straightforward to replace principal comodule algebras by principal extensions in the sense of [BH04]. $\diamond$

Our next step is a statement about a relation between the ideals of a principal comodule algebra P that are also subcomodules, and ideals in the subalgebra B of coaction-invariant elements. Both sets are obviously lattices with respect to the operations $+$ and $\cap$.

Proposition 3.5. *Let P be a principal right H-comodule algebra and $B := P^{\mathrm{co}H}$ the coaction-invariant subalgebra. Denote by Ξ_B the lattice of all ideals in B and by Ξ_P the lattice of all ideals in P that are simultaneously subcomodules. Then the map*

$$\mathcal{L} : \Xi_P \longrightarrow \Xi_B, \quad \mathcal{L}(J) := J \cap B$$

is a monomorphism of lattices.

Proof. The only non-trivial step in proving that $\mathcal{L}$ is a homomorphism of lattices is establishing the inclusion $(B \cap J) + (B \cap J') \supseteq B \cap (J + J')$. To this end, we proceed

along the lines of the proof of Lemma 3.1(1). Since J is a comodule and an ideal, from the formula (22) we obtain:

$$(33) \qquad p \in J \implies s(p) = p_{(0)}\ell(p_{(1)})^{\langle 1 \rangle} \otimes \ell(p_{(1)})^{\langle 2 \rangle} \in (J \cap B) \otimes P.$$

Now, let $p \in J$, $q \in J'$, $p + q \in B$. Then (33) implies that

$$(34) \qquad (p + q) \otimes 1 = s(p) + s(q) \in (B \cap J) \otimes P + (B \cap J') \otimes P.$$

Applying any unital linear functional $P \to k$ to the second tensor component implies that $p + q \in (B \cap J) + (B \cap J')$. Finally, since s is a splitting of the multiplication map, it follows from (33) that $J = (J \cap B)P$. This, in turn, proves the injectivity of $\mathcal{L}$. □

Note that, much as the formula (22) implies (33), the formula (23) implies

$$(35) \qquad p \in J \implies s'(p) = \ell(S^{-1}(p_{(1)}))^{\langle 1 \rangle} \otimes \ell(S^{-1}(p_{(1)}))^{\langle 2 \rangle} p_{(0)} \in P \otimes (J \cap B).$$

Therefore, since s' is a splitting of the right multiplication map, $J = P(J \cap B)$. Combining this with the above discussed left-sided version proves that J is a two-sided ideal:

$$(36) \qquad P(J \cap B) = J = (J \cap B)P.$$

Remark 3.6. *The homomorphism $\mathcal{L}$ is not surjective in general.* A counterexample is given by a smash product (trivial principal comodule algebra) of the Laurent polynomials $B = k[u, u^{-1}]$ with the Hopf algebra $H = k[v, v^{-1}]$ of Laurent polynomials ($\Delta(v) = v \otimes v$). The action is defined by $v \triangleright u = qu$, $q \in k \setminus \{0, 1\}$. Viewing u and v as generators of P, it is clear that $vu = quv$. It is straightforward to verify that, if I is the two-sided ideal in B generated by $u - 1$, then the right ideal IP is not a two-sided ideal of P. Hence the map $\mathcal{L}$ cannot be surjective by (36). ◇

In [CM00], families $\pi_i : P \to P_i$ of algebra homomorphisms as in Theorem 3.3 were called coverings. However, it was explained therein (see Proposition 2) that such coverings are well-behaved when the kernels $\ker \pi_i$ generate a distributive lattice of ideals. Therefore, we adopt in the present paper the following terminology:

Definition 3.7. *A finite family $\{\pi_i : P \to P_i\}_{i \in \{1,\ldots,N\}}$ of surjective algebra homomorphisms is called a* weak covering *if $\cap_{i=1,\ldots,N} \ker \pi_i = \{0\}$. Denote by Ξ_N the lattice of ideals generated by $I_i := \ker \pi_i$, $i = 1, \ldots, N$, with $\cap$ and $+$ as the join and meet operations, respectively. A weak covering is called a* covering *if the lattice Ξ_N is distributive. Finally, an ordered family $(\pi_i : P \to P_i)_{i \in \{1,\ldots,N\}}$ is called an* ordered covering *if the set $\{\pi_i : P \to P_i\}_{i \in \{1,\ldots,N\}}$ is a covering.*

The above definition can obviously be extended to the case when the π_i's are algebra and H-comodule morphisms. Then the $\ker \pi_i$'s are ideals and H-subcomodules. Note also that the reason for distinguishing between coverings and ordered coverings comes from the categorical considerations of the following section. To compare coverings with appropriate sheaves of algebras, it is more convenient to use the category of ordered coverings.

The next claim is concerned with the distributivity condition from Definition 3.7 for coverings of principal comodule algebras. It follows from Proposition 3.5 and Lemma 3.1(1).

Corollary 3.8. *Let $\{\pi_i : P \to P_i\}_{i\in\{1,\dots,N\}}$ be a family of surjective homomorphisms of right H-comodule algebras. Assume that P is principal. Then $\{\pi_i : P \to P_i\}_i$ is a covering of P if and only if $\{\pi_i|_{P^{\mathrm{co}H}} : P^{\mathrm{co}H} \to P_i^{\mathrm{co}H}\}_i$ is a covering of $P^{\mathrm{co}H}$.*

The above corollary is particularly helpful when $P^{\mathrm{co}H}$ is a C^*-algebra because lattices of closed ideals in a C^*-algebra are always distributive (due to the property $I\cap J = IJ$).

We are now ready to propose a noncommutative-geometric replacement for the concept of local triviality of principal bundles. Recall that the subsets that admit a natural translation into the language of C^*-algebras are the closed (not the open) subsets of a compact Hausdorff space. Therefore, we use finite closed rather than open coverings to trivialise bundles. As is explained in [BHMS, Example 1.24], there is a difference between these two approaches. We reserve the term "locally trivial" for bundles trivializable over an open cover, and call bundles trivializable over a finite closed cover "piecewise trivial". It is the latter (slightly more general) property that we generalise to the noncommutative setting.

Definition 3.9. *An H-comodule algebra P is called* piecewise principal *(respectively,* trivial*) if there exists a weak covering $\{\pi_i : P \to P_i\}_{i\in\{1,\dots,N\}}$ by H-colinear maps such that:*

(1) *The restrictions $\pi_i|_{P^{\mathrm{co}H}} : P^{\mathrm{co}H} \to P_i^{\mathrm{co}H}$ form a covering.*
(2) *The P_i's are principal (respectively, the P_i's are smash products, i.e., for all i, $P_i \cong P_i^{\mathrm{co}H}\#_i H$ as H-comodule algebras).*

While not every compact principal bundle is piecewise (or locally) trivial ([BHMS, Example 1.22]), every piecewise principal compact G-space (i.e., covered by finitely many compact principal G-bundles) is clearly a compact principal G-bundle. The second statement becomes non-trivial when we replace compact G-spaces by comodule algebras. However, it is an immediate consequence of Theorem 3.3 and Corollary 3.8:

Corollary 3.10. *Let H be a Hopf algebra with bijective antipode and P be an H-comodule algebra that is piecewise principal with respect to $\{\pi_i : P \to P_i\}_i$. Then P is principal and $\{\pi_i : P \to P_i\}_i$ is a covering of P.*

Finally, let us consider the relationship between piecewise triviality and a similar concept referred to as "local triviality" in [P-MJ94]. Therein, sheaves $\mathcal{P}$ of comodule algebras were viewed as quantum analogues of principal bundles. They were called locally trivial provided that the space X on which $\mathcal{P}$ is defined admits an open covering $\{U_i\}_i$ such that all $\mathcal{P}(U_i)$'s are smash products. If we assume such a sheaf to be flabby (that is, for all open subsets of $V, U, V \subseteq U$, the restriction maps $\pi_{U,V} : \mathcal{P}(U) \to \mathcal{P}(V)$ are surjective), then we can use Theorem 3.3 to deduce the principality of all $\mathcal{P}(U)$'s:

Corollary 3.11. *Let H be a Hopf algebra with bijective antipode and $\mathcal{P}$ be a flabby sheaf of H-comodule algebras over a topological space X. If $\{U_i\}_i$ is a finite open covering such that all $\mathcal{P}(U_i)$'s are principal, then $\mathcal{P}(U)$ is principal for any open subset $U \subseteq X$.*

4. Coverings and flabby sheaves

In this section, we focus entirely on a flabby-sheaf interpretation of distributive lattices of ideals defining coverings of algebras (see Definition 3.7). We will explain that for a flabby sheaf in Corollary 3.11, the underlying topological space plays only a secondary role and can be replaced by a certain space that is universal for all N-element coverings. This space is the 2-element field $(N-1)$-projective space

$$(37) \qquad \mathbb{P}^{N-1}(\mathbb{Z}/2) := \{0,1\}^N \setminus \{(0,\ldots,0)\}$$

whose topology subbasis is its affine covering, i.e., whose topology is generated by the subsets

$$(38) \qquad A_i := \{(z_1,\ldots,z_N) \in \mathbb{P}^{N-1}(\mathbb{Z}/2) \mid z_i \neq 0\}.$$

Consider now an arbitrary space X with a finite covering $\{U_1,\ldots,U_N\}$. Define on X the topology generated by the U_i's (considered as open sets) and pass to the quotient by the equivalence relation

$$(39) \qquad x \sim y \iff (\forall i : x \in U_i \iff y \in U_i).$$

Obviously, $X/\sim$ depends on the specific features of the covering $\{U_i\}_i$. However, for a fixed N, it can always be embedded into $\mathbb{P}^{N-1}(\mathbb{Z}/2)$:

Proposition 4.1. *Let $X = U_1 \cup \ldots \cup U_N$ be any set equipped with the topology generated by the U_i's. Let $p : X \to X/\sim$ be the quotient map defined by the equivalence relation (39). Then*

$$\xi : X/\sim \longrightarrow \mathbb{P}^{N-1}(\mathbb{Z}/2), \quad p(x) \longmapsto (z_1,\ldots,z_N), \quad z_i = 1 \Leftrightarrow x \in U_i, \ \forall i,$$

is an embedding of topological spaces.

Proof. It is immediate that ξ is well defined and injective. Next, since $p^{-1}(\xi^{-1}(A_i)) = U_i$ is open for all i, each $\xi^{-1}(A_i)$ is open in the quotient topology on $X/\sim$. Now the continuity of ξ follows from the fact that the A_i's form a subbasis of the topology of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.

The key step is to show that images of open sets in $X/\sim$ are open in $\xi(X/\sim)$. First note that by the definition of the relation (39),

$$(40) \qquad p^{-1}(p(U_{i_1} \cap \ldots \cap U_{i_n})) = U_{i_1} \cap \ldots \cap U_{i_n}.$$

Therefore, as preimages and images preserve unions and any open set in X is the union of intersections of U_i's, p is an open map. On the other hand, by the surjectivity of p, we have $p(p^{-1}(V)) = V$ for any subset $V \subseteq X/\sim$. Hence it follows that a set in $X/\sim$ is open if and only if it is an image under p of an open set in X. Finally, by the definition of ξ,

$$(41) \qquad \xi(p(U_{i_1} \cap \ldots \cap U_{i_n})) = A_{i_1} \cap \ldots \cap A_{i_n} \cap \mathrm{Im}(\xi),$$

and the claim follows from the distributivity of $\cap$ with respect to $\cup$. $\qquad \square$

Note that the map ξ is a homeomorphism precisely when the U_i's are in a generic position, that is, when all intersections $U_{i_1} \cap \ldots \cap U_{i_k} \cap (X \setminus U_{j_1}) \cap \ldots \cap (X \setminus U_{j_l})$ such that $\{i_1,\ldots,i_k\} \cap \{j_1,\ldots,j_l\} = \emptyset$ are non-empty.

Thus we have shown that if we consider X and the U_i's as in Corollary 3.11, then the composition $\xi \circ p : X \to \mathbb{P}^{N-1}(\mathbb{Z}/2)$ is continuous. Hence we can produce flabby sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ by taking the direct images of flabby sheaves over X. They will have the same sections globally and on the covering sets. In this sense, they carry an essential part of the data encoded in the original sheaf.

Example 4.2. We want to show that a closed-set version of the affine covering of $\mathbb{P}^{N-1}(\mathbb{C})$ yields ξ which is a homeomorphism. Denote by $[x_1 : \ldots : x_N]$ the class of $(x_1, \ldots, x_N) \in \mathbb{C}^N$ in $\mathbb{P}^{N-1}(\mathbb{C})$. Then the affine subsets

$$(42) \qquad V_i = \left\{ [x_1 : \ldots : x_N] \in \mathbb{P}^{N-1}(\mathbb{C}) \mid x_i \neq 0 \right\}, \ i = 1, \ldots, N,$$

form a covering of $\mathbb{P}^{N-1}(\mathbb{C})$. Next, define the following refinement of this covering:

$$(43) \quad X_i = \left\{ [x_1 : \ldots : x_N] \in \mathbb{P}^{N-1}(\mathbb{C}) \mid |x_i| = \max\{|x_1|, \ldots, |x_N|\} \right\}, \ i = 1, \ldots, N.$$

It is clear that, under the homeomorphisms

$$(44) \qquad \Psi_i : V_i \longrightarrow \mathbb{C}^{N-1}, \quad [x_1 : \ldots : x_N] \longmapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_N}{x_i} \right),$$

defining the manifold structure of $\mathbb{P}^{N-1}(\mathbb{C})$, the sets X_i become closed $(N-1)$-cubes:

$$(45) \qquad \Psi_i(X_i) = \left\{ (y_1, \ldots, y_{N-1}) \in \mathbb{C}^{N-1} \mid |y_j| \leq 1, \ \forall_j \right\}.$$

Thus the X_i's form a closed covering of $\mathbb{P}^{N-1}(\mathbb{C})$. Finally, to show that the embedding ξ induced by this covering is surjective, consider an arbitrary element $(z_1, \ldots, z_N) \in \mathbb{P}^{N-1}(\mathbb{Z}/2)$ that is a non-zero sequence of zeros and ones. Then

$$(46) \qquad (\xi \circ p)([z_1 : \ldots : z_N]) = (z_1, \ldots, z_N),$$

as needed. $\qquad \qquad \Diamond$

Our next aim is to demonstrate that flabby sheaves of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ are just a reformulation of the notion of an ordered covering introduced in Definition 3.7. It turns out that the distributivity condition discussed in the previous section is the key property needed to reconcile the results from [BK96, CM02] with those from [P-MJ94].

To this end, let us call an ordered covering by N surjections an ordered N-covering, and let us define a morphism between ordered N-coverings $(\pi_i : P \to P_i)_i$ and $(\eta_i : Q \to Q_i)_i$ as a family of algebra homomorphisms $f : P \to Q$ and $f_i : P_i \to Q_i$, $i \in \{1, \ldots, N\}$, such that $\eta_i \circ f = f_i \circ \pi_i$ for any index i. This category is obviously equivalent to the category $\mathbf{I}_N$ of distributive lattices of ideals (see Proposition 2.2).

On the other hand, to transform a category of sheaves over the topological space $\mathbb{S}_N$ (i.e., over $2^{\{1, \ldots, N\}} \setminus \{\emptyset\}$ with the Alexandrov topology) to an equivalent category of sheaves over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$, all we need is a homeomorphism between these spaces. This is provided by the obvious bijection

$$(47) \qquad \chi : \quad \mathbb{P}^{N-1}(\mathbb{Z}/2) \ni (z_i)_{i \in \{1, \ldots, N\}} \quad \longmapsto \quad \{j \in \{1, \ldots, N\} \mid z_j = 1\} \in \mathbb{S}_N$$

mapping the subbasis of topology of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ to the subbasis of topology of $\mathbb{S}_N$, i.e., $\chi(A_i) = S_i$, $i \in \{1, \ldots, N\}$ (see (38) and (5)).

Hence Proposition 2.2 enjoys the following corollary:

Corollary 4.3. *Let $\mathbf{C}_N$ be the category of ordered N-coverings of algebras, and $\mathbf{P}_N$ be the category of flabby sheaves of algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$. Then the following assignments*

$$\text{(48)} \qquad \mathbf{C}_N \ni (\pi_i : P \to P_i)_i \quad \longmapsto \quad \left\{ \mathcal{S} : Y \mapsto P/(R^{(I_i)_i} \circ \chi)(Y) \right\}_Y \in \mathbf{P}_N \,,$$

$$\text{(49)} \qquad \mathbf{P}_N \ni \mathcal{S} \quad \longmapsto \quad \left(\mathcal{S}(\mathbb{P}^{N-1}(\mathbb{Z}/2)) \to \mathcal{S}(A_i) \right)_i \in \mathbf{C}_N \,,$$

are functors establishing an equivalence of categories.

Furthermore, since both lattices Γ_N and Υ_N are the lattices of non-empty open subsets of $\mathbb{P}^{N-1}(\mathbb{Z}/2)$ and $\mathbb{S}_N$, respectively, the homeomorphism χ induces an isomorphism of lattices (cf. [B-G67, Theorem 9, p.34]):

Corollary 4.4. *For any N, the homomorphism of lattices $R^{(A_i)_i} : \Upsilon_N \to \Gamma_N$ defined by (6) is bijective. Its inverse is given by the formula $(R^{(A_j)_j})^{-1}(Y) = \chi(Y)$.*

In Corollary 4.3, one would like to replace the category of ordered N-coverings by the category of all finite coverings, which is more natural from the classical topology point of view. For this purpose, one needs to consider appropriate isomorphism classes of flabby sheaves over the infinite projective space $\mathbb{P}^{\infty}(\mathbb{Z}/2) := \varinjlim \mathbb{P}^N(\mathbb{Z}/2)$. This is currently under way in [PZ].

Let us end this section by showing how Corollary 4.3 works in the classical setting. Since the intersection of closed ideals in a C^*-algebra equals their product, the lattices of closed ideals in C^*-algebras are always distributive. On the other hand, the category of compact Hausdorff spaces is equivalent to the opposite category of unital commutative C^*-algebras (see [GN43, Lemma 1] for the original statement). Therefore, remembering that the epimorphisms of commutative unital C^*-algebras can be equivalently presented as the pullbacks of embeddings of compact Hausdorff spaces, we immediately obtain:

Corollary 4.5. *The category of compact Hausdorff spaces X with fixed ordered closed covering $(C_i \subseteq X)_{i \in \{1,\dots,N\}}$ is equivalent to the opposite category of flabby sheaves of commutative unital C^*-algebras over $\mathbb{P}^{N-1}(\mathbb{Z}/2)$.*

Here, the zero algebra is allowed as a unital C^*-algebra. It is "the algebra of all continuous complex-valued functions on the empty set". This is needed if the closed sets are not in generic position. The set of unital morphisms from the zero C^*-algebra to any other one is understood to be empty.

5. Examples

To end with, we recall from [DHH, HMS06a, BHMS05, HMS06b] the constructions of examples of the fibre products of principal comodule algebras. They indicate possible areas of applications of Section 3.

5.1. **A noncommutative join construction.** If G is a compact group, then the join $G * G$ becomes a G-principal fibre bundle over the unreduced suspension ΣG of G, see e.g. [B-GE93], Proposition VII.8.8 or [BHMS]. For example, one can obtain in this way the Hopf fibrations $S^7 \to S^4$ and $S^3 \to S^2$ using $G = SU(2)$ and $G = U(1)$, respectively. Recall that $G * G$ is constructed from $[0, 1] \times G \times G$ by shrinking to a point one factor G at 0 and the other one at 1. Alternatively, at one of the endpoints of the unit interval, one can shrink $G \times G$ to the diagonal. The latter picture is generalised in [DHH] to the noncommutative setting.
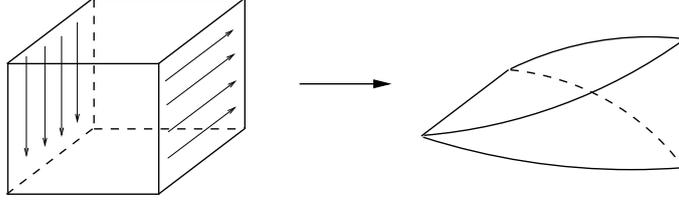
Our first aim is to describe a noncommutative analogue of this construction that nicely fits into our general concepts and is under detailed investigation in [DHH]. To this end, let H be the Hopf algebra underlying a compact quantum group $\bar{H}$ (see [W-SL87, W-SL98] or Chapter 11 of [KS97] for details). We define

$$P_1 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(0) \in \Delta(H)\},$$
(50)
$$P_2 := \{f \in C([0,1], \bar{H}) \otimes H \mid f(1) \in \mathbb{C} \otimes H\}.$$

They will play the roles of two trivial pieces of a principal comodule algebra. Here we identify elements of $C([0,1], \bar{H}) \otimes H$ with functions $[0,1] \to \bar{H} \otimes H$. The P_i's become H-comodule algebras by applying the coproduct of H to H, $\Delta_{P_i} = \mathrm{id}_{C([0,1],\bar{H})} \otimes \Delta$, and the subalgebras of H-invariants can be identified with

$$B_1 := \{f \in C([0,1], \bar{H}) \mid f(0) \in \mathbb{C}\},$$
(51)
$$B_2 := \{f \in C([0,1], \bar{H}) \mid f(1) \in \mathbb{C}\}.$$

Furthermore, $P_1 \simeq B_1 \# H$, $P_2 \simeq B_2 \otimes H$, where H acts on B_1 via the adjoint action, $(a \triangleright f)(t) = a_{(1)} f(t) S(a_{(2)})$, $a \in H$, $f \in B_1$, $t \in [0, 1]$, see [DHH]. Now one can define P as a gluing of the two pieces along $P_{12} := \bar{H} \otimes H$, that is, as the pull-back

$$(52) \qquad P := \{(p, q) \in P_1 \oplus P_2 \mid \pi_2^1(p) = \pi_1^2(q)\}$$

of the P_i's along the evaluation maps

$$(53) \qquad \pi_2^1 : P_1 \longrightarrow P_{12}, \quad f \longmapsto f(1), \quad \pi_1^2 : P_2 \longrightarrow P_{12}, \quad f \longmapsto f(0).$$

From Theorem 3.3, we can conclude now that P is principal. In particular, taking $\bar{H}$ to be the C^*-algebra of $SU_q(2)$, we can conclude the principality of Pflaum's noncommutative instanton bundle [P-MJ94].

5.2. **The Heegaard-type quantum 3-sphere.** Based on the idea of a Heegaard splitting of S^3 into two solid tori, a noncommutative deformation of S^3 was proposed in [CM02, HMS06a, BHMS05]. On the level of C^*-algebras, it can be presented as a fibre product $C(S^3_{pq\theta})$ of two C^*-algebraic crossed products $\mathcal{T} \rtimes_\theta \mathbb{Z}$ and $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ of the Toeplitz algebra $\mathcal{T}$ by $\mathbb{Z}$.

We denote the isometry generating $\mathcal{T}$ in $\mathcal{T} \rtimes_\theta \mathbb{Z}$ by z_+, and the isometry generating $\mathcal{T}$ in $\mathcal{T} \rtimes_{-\theta} \mathbb{Z}$ by z_-. The $\mathbb{Z}$-actions are implemented by unitaries u_+ and u_-, respectively, in the following way:

$$(54) \qquad 1_\mathbb{Z} \rhd_\theta z_+ := u_+ z_+ u_+^{-1} := e^{2\pi i\theta} z_+, \qquad 1_\mathbb{Z} \rhd_{-\theta} z_- := u_- z_- u_-^{-1} := e^{-2\pi i\theta} z_-.$$

The fibre product is taken over $C(S^1) \rtimes_\theta \mathbb{Z}$ with the action $1_\mathbb{Z} \rhd_\theta Z_+ := U_+ z_+ U_+^{-1} := e^{2\pi i\theta} Z_+$. Here Z_+ is the generator of $C(S^1)$ and U_+ is the unitary implementing the $\mathbb{Z}$-action. The surjections defining the fibre product are:

$$
\begin{aligned}
\pi_2^1 &: \mathcal{T} \rtimes_\theta \mathbb{Z} \longrightarrow C(S^1) \rtimes_\theta \mathbb{Z}, & z_+ &\longmapsto Z_+, & u_+ &\longmapsto U_+, \\
(55) \qquad \pi_1^2 &: \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \longrightarrow C(S^1) \rtimes_\theta \mathbb{Z}, & z_- &\longmapsto U_+, & u_- &\longmapsto Z_+.
\end{aligned}
$$

There is a natural $U(1)$-action on $C(S^3_{pq\theta})$ corresponding classically to the action in the Hopf fibration, see [HMS06a]. Its restriction to the two crossed products is not the canonical action of $U(1)$ viewed as the Pontryagin dual of $\mathbb{Z}$. However, to obtain the canonical actions one can identify $C(S^3_{pq\theta})$ with a fibre product of the same crossed products, but formed with respect to the surjections

$$
\begin{aligned}
\hat{\pi}_2^1 &: \mathcal{T} \rtimes_\theta \mathbb{Z} \longrightarrow C(S^1) \rtimes_\theta \mathbb{Z}, & z_+ &\longmapsto Z_+, & u_+ &\longmapsto U_+, \\
(56) \qquad \hat{\pi}_1^2 &: \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \longrightarrow C(S^1) \rtimes_\theta \mathbb{Z}, & z_- &\longmapsto Z_+^{-1}, & u_- &\longmapsto Z_+ U_+.
\end{aligned}
$$

The identification is given by

$$(57)$$

Here isomorphisms ϕ are given on respective generators by

$$(58) \qquad\qquad\qquad z \longmapsto zu, \qquad u \longmapsto u.$$

The C^*-subalgebra of $U(1)$-invariants is the C^*-algebra of the mirror quantum 2-sphere from [HMS06b]. As mentioned in the introduction, we can pass from $C(S^3_{pq\theta})$ to the associated principal extension, and this procedure always commutes with taking fibre products. (This is unlike the C^*-completion procedure that needs an additional assumption to commute with taking fibre products [HMSSZ, Matsumoto Gluing Theorem].) In this way, we obtain a subalgebra $P \subseteq C(S^3_{pq\theta})$ which is a piecewise trivial $\mathbb{CZ}$-comodule algebra, so that it fits the setting of this paper. The invariant subalgebra $P^{\mathrm{co}H}$ is again the C^*-algebra of the mirror quantum 2-sphere.

There is another natural Hopf-like $U(1)$-action on $C(S^3_{pq\theta})$ described in [HMS06b] (see also [BHMS+]). Again, its restriction to the two crossed products making up the fibre product $C(S^3_{pq\theta})$ is not the canonical action of $U(1)$. To transform this fibre product into an isomorphic one but carrying the canonical $U(1)$-action on the crossed

products, we take the fibre product defined by the morphisms:

$$\check{\pi}_2^1 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \longrightarrow C(S^1) \rtimes_{-\theta} \mathbb{Z}, \qquad z_- \longmapsto Z_-, \qquad u_- \longmapsto U_-,$$

$$(59) \qquad \check{\pi}_1^2 : \mathcal{T} \rtimes_{-\theta} \mathbb{Z} \longrightarrow C(S^1) \rtimes_{-\theta} \mathbb{Z}, \qquad z_- \longmapsto Z_-, \qquad u_- \longmapsto Z_- U_-.$$

Here Z_- and U_- are the generators of $C(S^1) \rtimes_{-\theta} \mathbb{Z}$ corresponding to $C(S^1)$ and $\mathbb{Z}$, respectively. The identifying maps are now given by

$$\tilde{\phi}_1 : \qquad z_+ \longmapsto z_- u_-, \qquad u_+ \longmapsto u_-^{-1},$$

$$\tilde{\phi}_2 : \qquad z_- \longmapsto u_-^{-1} z_-, \qquad u_- \longmapsto u_-,$$

$$(60) \qquad \tilde{\phi}_{12} : \qquad Z_+ \longmapsto Z_- U_-, \qquad U_+ \longmapsto U_-^{-1}.$$

The subalgebra of $U(1)$-invariants is now the C^*-algebra of a generic Podleś quantum 2-sphere [P-P87]. Note that it is not possible to obtain in this way the polynomial algebra of a generic Podleś sphere — replacing $\mathcal{T} = P_i^{\mathrm{co}H}$ by the polynomial algebra of a quantum disc that is generated by x satisfying $x^*x - qxx^* = 1 - q$ would yield a non-isomorphic algebra [CM00]. This is related to the fact that already in the commutative setting the algebra of polynomial functions on a sphere has no covering corresponding to two hemispheres because there are no nontrivial polynomials vanishing on a hemisphere.

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