Multi-valued hyperelliptic continued fractions of generalized Halphen type

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Abstract

We introduce and study higher genera generalizations of the Halphen theory of continued fractions. The basic notion we start with is hyperelliptic Halphen (HH) element

$$\frac{\sqrt{X_{2g+2}} - \sqrt{Y_{2g+2}}}{x - y},$$

depending on parameter y, where X_{2g+2} is a polynomial of degree $2g+2$ and $Y_{2g+2} = X_{2g+2}(y)$. We study regular and irregular HH elements, their continued fraction developments and some basic properties of such developments such as: even and odd symmetry and periodicity. There is a $2 \leftrightarrow g+1$ dynamics which lies in the basis of the developed continued fractions theory. We give two geometric realizations of this dynamics. The first one deals with nets of polynomials and with polygons circumscribed about a conic K. The dynamics is realized as a path of polygons of $g+1$ sides inscribed in a curve B of degree $2g$ and circumscribed about the conic K obtained by successive moves, so called – flips along edges. The second geometric realization leads to a new interpretation of generalized Jacobians of hyperelliptic curves.

Contents

1 Introduction

Modern algebraic approximation theory with continued fraction theory was established by Chebyshev (Tchebycheff according to the traditional French transcription) and his Sankt Petersburg school in the second half of the XIX century. Chebyshev's motivation for this studies was his interest in practical problems: in the mechanism theory as an important part of mechanical engineering of that time and ballistics. Steam engines were fundamental tool in technological revolution and their kernel part was *the Watt's complete parallelogram*, a planar mechanism to transform linear motion into circular, shown on Figure 1.

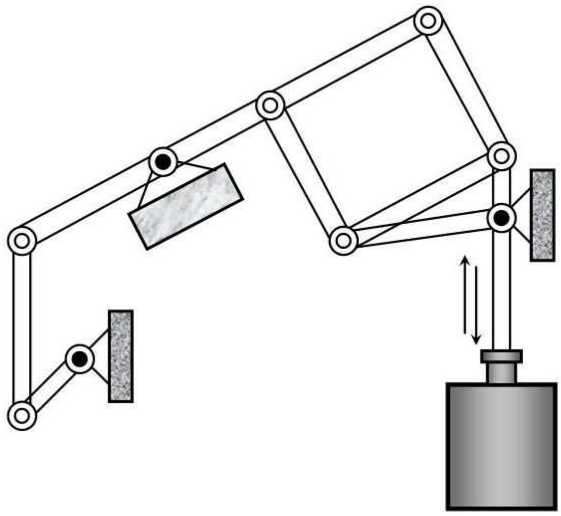

Figure 1: Watt's complete parallelogram

Fundamental problem was to estimate error of the mechanism in execution of that transformation.

Starting point of Chebyshev's investigation ([14]) was work on the theory of mechanisms of French military engineer, professor of mechanics and academician Jean Victor Poncelet [12]. In his study of mistakes of mechanisms, Poncelet came to the question of rational and linear approximation of the function

$$\sqrt{x^2 + 1}.$$

In other words he studied approximation of the functions $\sqrt{X_2(x)}$ of the form of square root of polynomials of the second degree, and he gave two approaches to the posed problems, one based on the analytical arguments and the second one based on geometric consideration.

Although Poncelet was described by Chebyshev as "well-known scientist in practical mechanics" (see [13]), nowadays J. V. Poncelet is known first of all as one of the biggest geometers of the XIX century. The Poncelet Theorem (PT) is considered as one of the nicest and deepest results in projective geometry: Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, PT states that infinitely many such closed polygonal lines exist – every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides. (See Figure 2).

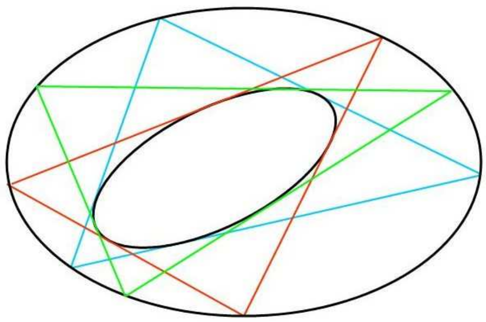

Figure 2: Poncelet theorem

In his *Traité des propriétés projectives des figures* [11], Poncelet proved even a more general result and used only purely geometric, synthetic arguments.

The case when the two ellipses are confocal has clear mechanical interpretation as billiard system within outer ellipse as boundary and having inner ellipse as the caustic of a given billiard trajectory, as shown on Figure 3. PT in this case describes periodic billiard trajectories. For a modern account of PT see [6, 7] and references therein.

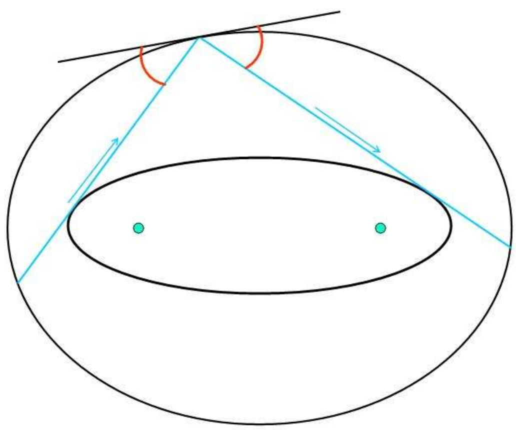

Figure 3: Billiard system and confocal conics

However, nowadays it is almost forgotten that there is an amazing connection between Poncelet Theorem and continued fractions and approximation theory of the functions of the form

$$\sqrt{X_4(x)},$$

where $X_4(x)$ denotes general polynomial of the fourth degree. This connection of continued fractions and approximations of functions of the form of square root of polynomials of fourth degree with the Poncelet configuration and PT was indicated by Halphen [9]. (A Poncelet configuration, i.e. a polygonal line inscribed in one and circumscribed about another conic is shown on Figure 4).

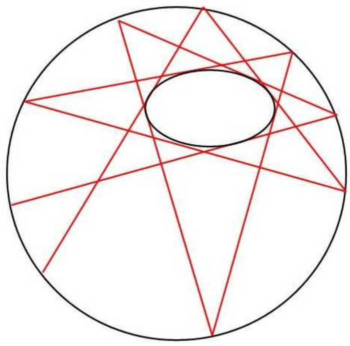

Figure 4: Poncelet configuration

Theory of continued fractions of square roots $\sqrt{X_4(x)}$ of polynomials of degree up to four started with Abel and Jacobi. Faced to problems with very complicated algebraic formulae in algorithm, Jacobi [10] turned to an approach based on elliptic functions theory. Further development of that approach has been done by Halphen [9]. Halphen studied, instead of the square root of a polynomial, the following, more general, *Halphen element*

$$\frac{\sqrt{X_4} - \sqrt{Y_4}}{x - y},$$

where $Y_4 = X_4(y)$ is the value of polynomial X at given point y.

In this paper we are going to study more general theory of continued fractions of *hyperelliptic Haplhen elements*

$$\frac{\sqrt{X_{2g+2}} - \sqrt{Y_{2g+2}}}{x - y},$$

where X_{2g+2} is a polynomial of degree $2g + 2$ and $Y_{2g+2} = X_{2g+2}(y)$. It is obviously related to the theory of functions of the hyperelliptic curve

$$\Gamma : z^2 = X_{2g+2}$$

of genus g. We are also going to refer to this theory as *HH continued fractions.* We hope that this theory will find its way to concrete applications in modern technology. As a possibility we can mention development of *branched, multivalued algorithms* to be used in future cryptography. However, this requires strong interaction with experts of different fields.

Here, we give two geometric interpretations of the dynamics which lies in the basis of the HH continued fraction developments.

In the Section 10 we give geometric realization of the $2 \leftrightarrow g+1$ dynamics which deals with nets of polynomials and with polygons circumscribed about a conic K. It extends, not only in a flavor, the initial story of the Poncelet polygons. The dynamics is realized as a path of polygons of $g+1$ sides inscribed in a curve B of degree $2g$ and circumscribed about the conic K obtained by successive moves of certain type – so called *flips along edges.*

The second geometric realization, done in Section 11 starts with the notion of the generalized Cayley curve (see [6], [7]) and leads to a new interpretation of generalized Jacobians of a hyperelliptic curves following and continuing the program of [7].

2 Basic Algebraic Lemma

Given a polynomial X of degree $2g+2$ in x. We suppose that X is not a square of a polynomial. Assuming that the values of y and ε are finite and fixed, we are going to study HH elements in a neighborhood of ε. Then, X can be considered as a polynomial of degree $2g+2$ in s, where $s = x - \varepsilon$ is chosen as a variable in a neighborhood of ε.

Lemma 1 (Basic Algebraic Lemma) *Let X be a polynomial of degree $2g+2$ in x and $Y = X(y)$ its value at a given fixed point y. Then, there exists a unique triplet of polynomials A, B, C with $\deg A = g+1$, $\deg B = \deg C = g$ in x such that*

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y} - C = \frac{B(x-\varepsilon)^{g+1}}{\sqrt{X}+A}. \tag{1}$$

Proof. Put $s = x - \varepsilon$ and $t = y - \varepsilon$ and denote $X = X'(s) = \sum_{i=0}^{2g+2} p_i s^i$ and

$$A = \sum_{i=0}^{g+1} A_i s^i, \quad B = \sum_{i=0}^{g} B_i s^i, \quad C = \sum_{i=0}^{g} C_i s^i.$$

We are going to determine the coefficients A_i, B_i, C_i in the way that the equation (1) is satisfied. Taking into account that $\sqrt{X}$ is irrational, the last equation can be separated into two equations:

$$\begin{aligned} A - \sqrt{Y} - C(s-t) &= 0; \\ X - A\sqrt{Y} - AC(s-t) - Bs^{g+1}(s-t) &= 0. \end{aligned} \tag{2}$$

Add the following consequence of (2):

$$X - A^2 = Bs^{g+1}(s-t). \tag{3}$$

For $s = 0$, from (1) and (2), we get:

$$C_0 = \frac{\sqrt{Y} - \sqrt{p_0}}{t},$$
$$A_0 = -C_0 t + \sqrt{Y} = \sqrt{p_0}.$$

Then we calculate A_i, $i = 1, \dots, g$, from equation (3). From the first of equations (2), by putting $s = t$, we get

$$A(t) = \sqrt{Y}.$$

From the first equation (2), for $s = t$, we see that $\deg C = \deg A - 1$ and we compute all the coefficients C_i as functions of the coefficients of the polynomial A. For example, $C_g = A_{g+1}$.

The last step is to compute the polynomial B. Observe that the coefficients $A_0, \dots A_g$ of the polynomial A are obtained in a manner such that

$$s^{g+1} | X - A^2.$$

The leading coefficient A_{g+1} is such that $X - A^2 = 0$ for $s = t$. Thus, there is a unique polynomial B, $\deg B = g$ such that

$$X - A^2 = Bs^{g+1}(s-t).$$

□

3 Hyperelliptic Halphen-type continued fractions

Let us start with the factorization of the polynomial B:

$$B(s) = B_g \prod_{i=1}^{g} (s - t_1^i)$$

and denote $A(t_1^i) = -\sqrt{Y_1^i}$. Then we have

$$\frac{A + \sqrt{X}}{s - t_1^i} = P_A^g(t_1^i, s) + \frac{\sqrt{X} - \sqrt{Y_1^i}}{x - y_1^i},$$

with certain polynomial P_A^g of degree g in s and with coefficients depending on the coefficients of A and t_1^i.

Denote

$$Q_0 = \frac{\sqrt{X} - \sqrt{Y}}{x - y} - C.$$

Then we have

$$Q_0 = \frac{B_g \prod_{j=1, j\neq i}^{g}(s - t_1^j)s^{g+1}}{P_A^g(t_1^i, s) + \frac{\sqrt{X}-\sqrt{Y_1^i}}{x-y_1^i}}.$$

Now, by applying the Lemma 1 we obtain the polynomials $A^{(1,i)}, B^{(1,i)}, C^{(1,i)}$ of degree $g+1, g, g$ respectively, such that

$$\frac{\sqrt{X} - \sqrt{Y_1^i}}{x - y_1^i} - C^{(1,i)} = \frac{B^{(1,i)}(x-\varepsilon)^{g+1}}{\sqrt{X} + A^{(1,i)}}.$$

Denote

$$\alpha_1^{(i)} := P_A^g(t_1^i, s), \quad \beta_1^{(i)} := B_g \prod_{j=1, j\neq i}^{g}(s - t_1^j)s^{g+1},$$

and introduce $Q_1^{(i)}$ by the equation

$$Q_0 = \frac{\beta_1^{(i)}}{\alpha_1^{(i)} + Q_1^{(i)}}.$$

Observe that $\deg \alpha_1^{(i)} = g$ and $\deg \beta_1^{(i)} = 2g$.

Now, one can go further, step by step: to factorize $B^{(1,i)}$, to choose one of its zeroes t_2^j and to denote by $B^{i,j} := B^{(1,i)}/(s - t_2^j)$. Further, we denote

$$\alpha_2^{(i,j)} := P_{A^{1,i}}g(t_2^j, s), \quad \beta_2^{(i,j)} := B^{i,j}s^{g+1},$$

and calculate $Q_2^{(i,j)}$ from the equation

$$Q_1^{(i)} = \frac{\beta_2^{(i,j)}}{\alpha_2^{(i,j)} + Q_2^{(i,j)}}.$$

Thus we have

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = C + \frac{\beta_1^{(i)}}{\alpha_1^i + \frac{\beta_2^{(i,j)}}{\alpha_2^{(i,j)} + Q_2^{(i,j)}}}.$$

Following the same scheme, in the i-th step we introduce polynomials

$$A^{(i,j_1,\dots,j_i)}, \quad B^{(i,j_1,\dots,j_i)}, \quad C^{(i,j_1,\dots,j_i)}$$

of degrees $g+1$, g, g respectively. They satisfy the equations

$$\begin{aligned} A^{(i,j_1,\dots,j_i)} &= C^{(i,j_1,\dots,j_i)}(s - t_i^{j_1,\dots,j_i}) + \sqrt{Y_i^{j_1,\dots,j_i}}, \\ X - A^{(i,j_1,\dots,j_i)2} &= B^{(i,j_1,\dots,j_i)}s^{g+1}(s - t_i^{j_1,\dots,j_i}). \end{aligned} \tag{4}$$

We see that in the case $g > 1$ the formulae of the $(i+1)$–th step depend on the choice of one of the roots of the polynomial $B^{(i)}$ and of the choices from the previous steps. To avoid abuse of notations we are going to omit many times in future formulae the indexes $j_1, \dots, j_i$, which indicate the choices done in the first i steps, although we assume all the time that the choice has been done.

According to our notation we have

$$s - t_i | B^{(i-1)}$$

and

$$B^{(i-1)} = \frac{\beta_i}{s^{g+1}}(s - t_i)$$

or

$$B^{(i)} = \hat{\beta}_{i+1}(s - t_{i+1}),$$

where $\hat{\beta}_i = \beta_i / s^{g+1}$. From the equations (4) we have

$$\begin{aligned} X - A^{(i-1)2} &= \hat{\beta}_i(s - t_{i-1})s^{g+1}(s - t_i), \\ X - A^{(i)2} &= \hat{\beta}_{i+1}(s - t_{i+1})s^{g+1}(s - t_i) \end{aligned} \tag{5}$$

together with

$$\begin{aligned} A^{(i)}(t_i) &= \sqrt{Y_i}, \\ A^{(i-1)}(t_i) &= -\sqrt{Y_i}. \end{aligned}$$

We introduce λ_i by the relation

$$A^{(i)}_{g+1} = \sqrt{p_0}\lambda_i.$$

Theorem 1 *If λ_i is fixed, then t_i, $t^{(1)}_{i+1}$, $\dots$, $t^{(g)}_{i+1}\}$ are the roots of the following polynomial equation of degree $g+1$ in s:*

$$Q_X(\lambda_i, s) = 0.$$

The proof follows from the equations (5). In the same way, we get

Theorem 2 *If t_i is fixed, then λ_i and λ_{i-1} are the roots of the polynomial equation of degree 2 in λ:*

$$Q_X(\lambda_{i-1}, t_i) = 0, \qquad Q_X(\lambda_i, t_i) = 0.$$

One can easily calculate

$$B^{(i)}_g = p_{2g+2} - p_0\lambda_i^2,$$

thus

$$\beta_{i+1} = (p_{2g+2} - p_0\lambda_i^2)\prod_{j=2}^{g}(s - t^j_i)s^{g+1}.$$

We also have

$$\begin{aligned} A^{(i)} &= \sqrt{p_0}\left(1 + q_1 s + \dots + \lambda_i s^{g+1}\right), \\ C^{(i)} &= \sqrt{p_0}\left(q_1 + \dots + \lambda_i(s^g + s^{g-1}t_i + \dots + t^g_i)\right), \end{aligned}$$

and

$$\alpha_i = \sqrt{p_0}\left(2q_1 + \cdots + (\lambda_{i-1} + \lambda_i)(s^g + s^{g-1}t_i + \cdots + t_i^g)\right).$$

According to Theorem 2, the sum $\lambda_{i-1} + \lambda_i$ from the last equation, can be expressed through the coefficients of the polynomial $Q_X(\lambda, t_i)$ as a polynomial of the second degree in λ .

4 Basic examples: genus one case

The genus one case, or the elliptic case, has been studied by Halphen. Here we reproduce some of his formulae (see [9] for more details). The elliptic curve is given by a polynomial X of degree 4, in variable s in a neighborhood of ε:

$$X = S(s) = \sum_{i=0}^{4} p_i s^i$$

The development around the point ε of its square root has the form

$$\sqrt{X} = \sqrt{p_0}\left(1 + q_1 s + q_2 s^2 + q_3 s^3 + \dots\right),$$

with the following relations between q's and p's:

$$\begin{aligned} q_1 &= \frac{p_1}{2p_0}, \\ q_2 &= \frac{1}{8p_0^2}\left(4p_0p_2 - p_1^2\right), \\ q_3 &= \frac{1}{4p_0^3}\left(2p_0p_3 - p_0p_1p_2 + \frac{p_1^3}{4}\right), \\ q_4 &= \frac{1}{2p_0}\left(p_4 - 2q_1q_3p_0 - q_2^2p_0\right). \end{aligned}$$

Here we have

$$\frac{X}{p_0} = (1 + q_1 s + q_2 s^2)^2 + 2q_3 s^3 + 2(q_1q_3 + q_4)s^4.$$

From Basic Algebraic Lemma, applied to the case $g = 1$, we get the polynomials $A = A_0 + A_1 s + A_2 s^2$, $B = B_0 + B_1 s, C = C_0 + C_1 s$ which satisfy

$$\begin{aligned} A - \sqrt{Y} - C(s-t) &= 0; \\ X - A\sqrt{Y} - AC(s-t) - Bs^2(s-t) &= 0; \\ X - A^2 &= Bs^2(s-t). \end{aligned} \tag{6}$$

From the equations (6), one gets the formulae for the polynomials A, B, C:

$$\begin{aligned}
A_0 &= \sqrt{p_0}, \quad A_1 = q_1\sqrt{p_0}, \quad A_2 = \frac{\sqrt{Y} - (1+q_1t)\sqrt{p_0}}{t^2},\\
C_0 &= \frac{\sqrt{Y} - \sqrt{p_0}}{t}, \quad C_1 = A_2,\\
B_0 &= \frac{2\sqrt{p_0}}{t^3}\left(\sqrt{Y} - \sqrt{p_0}(1+q_1t+q_2t^2)\right),\\
B_1 &= \frac{2\sqrt{p_0}}{t^4}\left((1+q_1t)\sqrt{Y} - \sqrt{p_0}(1+2q_1t+(q_1^2+q_2)t^2+(q_1q_2+q_3)t^3)\right).
\end{aligned}$$

If we denote

$$P_A^{(1)}(t,s) := A_1 + A_2(s+t),$$

then we have

$$Q_0 = \frac{B_1s^2}{P_A^{(1)}(t_1,s) + \frac{\sqrt{X}-\sqrt{Y_1}}{x-y_1}},$$

and, step by step

$$\begin{aligned}
A^{(i)} &= \sqrt{Y_i} - C^{(i)}(s-t_i);\\
X - A^{(i)2} &= B^{(i)}s^2(s-t_i),
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
B^{(i-1)}(t_i) &= 0,\\
A^{(i)}(t_{i+1}) &= -\sqrt{Y_{i+1}}.
\end{aligned}$$

Finally, one gets

$$\begin{aligned}
\beta_i &= B_1^{(i-1)}s^2,\\
\alpha_i &= P^{(1)}_{A^{(i-1)}}(t_i,s) + C^{(i)}.
\end{aligned}$$

From equation (7) we get

$$\begin{aligned}
X - A^{(i-1)2} &= ms^2(s-t_{i-1})(s-t_i);\\
X - A^{(i)2} &= ns^2(s-t_i)(s-t_{i+1});\\
A^{(i)}(t_i) &= \sqrt{Y_i};\\
A^{(i-1)}(t_i) &= -\sqrt{Y_i};\\
A_2^{(i)} &= \sqrt{p_0}\lambda_i,
\end{aligned}$$

with some constants m, n, and then we have:

$$\begin{aligned}
\lambda_i &= \frac{1}{t_i^2}\left(\frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1+q_1t_i)\right),\\
\lambda_{i-1} &= \frac{1}{t_i^2}\left(-\frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1+q_1t_i)\right),
\end{aligned}$$

From the last equations one obtains:

Proposition 1 *If λ_i is fixed, then t_i, t_{i+1} are roots of the polynomial $Q_X(\lambda_i, s)$ quadratic in s:*

$$Q_X(\lambda_i, s) := (p_4 - p_0\lambda_i^2)s^2 + (p_3 - p_1\lambda_i)s + 2p_0(q_2 - \lambda_i) = 0.$$

Corollary 1 *The product of two consecutive t_i and t_{i+1} is:*

$$t_i t_{i+1} = \frac{2p_0(\lambda_i - q_2)}{p_0\lambda_i^2 - p_4},$$

and their sum is equal to:

$$t_i + t_{i+1} = \frac{p_1\lambda_i - p_3}{p_4 - p_0\lambda_i^2}.$$

Proposition 1 can be reformulated giving relation between two consecutive λ_{i-1} and λ_i:

Proposition 2 *If t_i is fixed, then λ_{i-1}, λ_i are solutions of quadratic equation:*

$$\lambda^2(p_0t_i^2) + \lambda(p_1t_i + 2p_0) - (p_4t_i^2 + p_3t_i + 2p_0q_2) = 0.$$

For the *normal form* of the elliptic HH c. f. we consider the case where

$$\alpha_i' = 1 + u_i s, \qquad \beta_i' = v_i s^2.$$

Then we have

$$t_i = -\frac{1}{q_1 + u_i}, \qquad \lambda_i = q_2 - 2v_{i+1}.$$

The recurrent relations are given with

$$\begin{aligned} u_i + u_{i-1} &= -q_1 + \frac{q_2}{2v_i}, \\ v_i + u_i u_{i-1} &= q_2 + \frac{q_4}{2v_i}. \end{aligned}$$

The second set of recurrent equations is done by

$$\begin{aligned} v_i + v_{i+1} &= q_2 + q_1 u_i + u_i^2, \\ 2v_i v_{i+1} &= -q_4 + q_3 u_i. \end{aligned}$$

5 Basic examples: genus two case

5.1 Notation

We start with a polynomial X of degree 6 in x and rewrite it as a polynomial in s in a neighborhood of ε

$$X = S(s) = \sum_{i=0}^{6} p_i s^i$$

and its square root developed around ε as

$$\sqrt{X} = \sqrt{p_0}(1 + q_1 s + q_2 s^2 + q_3 s^3 + q_4 s^4 + q_5 s^5 + q_6 s^6 + q_7 s^7 + \dots).$$

Then, the relations between coefficients p_i and q_j are

$$\begin{aligned}
p_1 &= 2p_0 q_1, \\
p_2 &= p_0(2q_2 + q_1^2), \\
p_3 &= 2p_0(q_3 + q_1 q_2), \\
p_4 &= p_0(2q_4 + q_2^2 + 2q_1 q_3), \\
p_5 &= 2p_0(q_5 + q_2 q_3 + q_1 q_4), \\
p_6 &= p_0(2q_6 + 2q_1 q_5 + 2q_2 q_4 + q_3^2),
\end{aligned}$$

with relations between q_i such as:

$$0 = q_7 + 2q_1 q_6 + 2q_2 q_5 + 2q_3 q_4.$$

Conversely, q_i's can be expressed through p's:

$$\begin{aligned}
q_1 &= \frac{p_1}{2p_0}, \\
q_2 &= \frac{1}{2p_0^2}\left(p_2 p_0 - \frac{p_1^2}{4}\right), \\
q_3 &= \frac{1}{2p_0^3}\left(p_3 p_0^2 - \frac{p_1 p_2 p_0}{2} + \frac{p_1^3}{8}\right), \\
q_4 &= \frac{1}{2p_0^4}\left\{p_4 p_0^3 - \frac{4p_2 p_0 - p_1^2}{8} - \frac{p_1}{16}(8p_3 p_0^2 - 4p_1 p_2 p_0 + p_1^3)\right\}, \\
q_5 &= \frac{p_5}{2p_0} - q_2 q_3 - q_1 q_4, \\
q_6 &= \frac{1}{2p_0}(p_6 - 2q_1 q_5 p_0 - 2q_2 q_4 p - 0 - q_3^2 p_0).
\end{aligned}$$

The initial polynomial X can be expressed through q_i's:

$$\frac{X}{p_0} = (1 + q_1 s + q_2 s^2 + q_3 s^3)^2 + 2q_4 s^4 + 2(q_1 q_4 + q_5)s^5 + 2(q_1 q_5 + q_2 q_4 + q_6)s^6.$$

5.2 The case of Basic Algebraic Lemma

We are going to determine polynomials A, B, C of degrees $\deg A = 3$, $\deg B = \deg C = 2$. Denote $A = A(s) = A_0 + A_1 s + A_2 s^2 + A_3 s^3$, $B = B(s) = B_0 + B_1 s + B_2 s^2$, $C = C(s) = C_0 + C_1 s + C_2 s^2$. Then the equations (2) and (3) become

$$\begin{aligned}
A - \sqrt{Y} - C(s-t) &= 0; \\
X - A\sqrt{Y} - AC(s-t) - Bs^3(s-t) &= 0; \\
X - A^2 &= Bs^3(s-t).
\end{aligned} \tag{8}$$

For $s=0$, we obtain:

$$C_0 = \frac{\sqrt{Y}-\sqrt{p_0}}{t}, \qquad A_0 = \sqrt{p_0}.$$

Then we calculate $A_i,\ i=1,2$ from the last equation of (8) by comparing polynomials X and A term by term up to the second degree:

$$A_1 = \frac{p_1}{2\sqrt{p_0}}, \qquad A_2 = \frac{1}{2\sqrt{p_0}}\frac{4p_2p_0-p_1^2}{4p_0},$$

thus

$$A = \sqrt{p_0}(1+q_1s+q_2s^2+\lambda_1 s^3).$$

From the relation $A(t)=\sqrt{Y}$ we get

$$A_3 = \frac{1}{t^3}\left[\sqrt{Y}-(\sqrt{p_0}+\frac{p_1}{2\sqrt{p_0}}t+\frac{4p_0p_2-p_1^2}{8p_0^{3/2}}t^2)\right].$$

The coefficients of C are $C_1=A_2$ and $C_2=A_3$. The coefficients of the polynomial B are

$$\begin{aligned} B_2 &= p_6 - A_3^2,\\ B_1 &= B_2t + p_5 - 2A_2A_3,\\ B_0 &= B_1t + p_4 - (2A_1A_3 + A_2^2). \end{aligned}$$

We factorize it

$$B = B_2(s-t_1^0)(s-t_1^1),$$

and denote

$$A(t_1^0) = -\sqrt{Y_1^0}, \qquad A(t_1^1) = -\sqrt{Y_1^1}.$$

Now, we have

$$\begin{aligned} \frac{A+\sqrt{X}}{s-t_1^0} &= \frac{A+\sqrt{Y_1^0}}{s-t_1^0} + \frac{\sqrt{X}-\sqrt{Y_1^0}}{x-y_1^0}\\ &= \frac{A(s)-A(t_1^0)}{s-t_1^0} + \frac{\sqrt{X}-\sqrt{Y_1^0}}{x-y_1^0}\\ &= A_1 + A_2(s+t_1^0) + A_3(s^2+st_1^0+t_1^{02}) + \frac{\sqrt{X}-\sqrt{Y_1^0}}{x-y_1^0}. \end{aligned}$$

Denote

$$P_A^{(2)}(t,s) := A_1 + A_2(s+t) + A_3(s^2+st+t^2).$$

Then we have finally

$$Q_0 = \frac{B_2(s-t_1^1)s^3}{P_A^{(2)}(t_1^0,s) + \frac{\sqrt{X}-\sqrt{Y_1^0}}{x-y_1^0}}.$$

Step by step we get

$$\begin{aligned} A^{(i)} &= \sqrt{Y_i} - C^{(i)}(s - t_i); \\ X - A^{(i)2} &= B^{(i)} s^3 (s - t_i), \end{aligned} \tag{9}$$

where

$$\begin{aligned} &B^{(i-1)}(t_i) = 0, \quad t_i := t_i^0, \\ &A^{(i)}(t_{i+1}) = -\sqrt{Y_{i+1}}. \end{aligned}$$

Now, we have

$$\begin{aligned} \beta_i &= B_2^{(i-1)}(s - t_i^1) s^3, \\ \alpha_i &= P^{(2)}_{A^{(i-1)}}(t_i, s) + C^{(i)}. \end{aligned} \tag{10}$$

We can represent the HH continued fraction in the following manner

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = C + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \cdots + \frac{\beta_i|}{|\alpha_i + Q_i},$$

where

$$Q_i = \frac{\sqrt{X} - \sqrt{Y_i}}{x - y_i} - C^{(i)} = \frac{B^{(i)} s^3}{\sqrt{X} + A^{(i)}}$$

and

$$Q_i = \frac{\beta_{i+1}}{\alpha_{i+1} + Q_{i+1}}.$$

5.3 Relations between λ_i and t_i

From equation (9), we get:

$$\begin{aligned} X - A^{(i-1)2} &= B_2^{(i-1)}(s - t_i^1) s^3 (s - t_{i-1})(s - t_i); \\ X - A^{(i)2} &= B_2^{(i)}(s - t_{i+1}^1) s^3 (s - t_i)(s - t_{i+1}); \\ A^{(i)}(t_i) &= \sqrt{Y_i}; \\ A^{(i-1)}(t_i) &= -\sqrt{Y_i}; \\ A_3^{(i)} &= \sqrt{p_0}\lambda_i. \end{aligned} \tag{11}$$

From (11), we have:

$$\begin{aligned} \lambda_i &= \frac{1}{t_i^3}\left(\frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_1 t_i + q_2 t_i^2)\right), \\ \lambda_{i-1} &= \frac{1}{t_i^3}\left(-\frac{\sqrt{Y_i}}{\sqrt{p_0}} - (1 + q_1 t_i + q_2 t_i^2)\right), \end{aligned}$$

and thus

$$t_i^3\sqrt{Y_{i+1}} + t_{i+1}^3\sqrt{Y_i} = \sqrt{p_0}(t_{i+1} - t_i)[t_{1+1}^2 + t_{i+1}t_i + q_1 t_i t_{i+1}(t_{i+1} + t_i) + q_2 t_i^2 t_{i+1}^2].$$

From equations (11) we also get

$$\begin{aligned} \lambda_{i-1} + \lambda_i &= -\frac{2}{t_i^3}(1 + q_1 t_i + q_2 t_i^2), \\ \lambda_{i-1}\lambda_i &= -\frac{1}{t_i^6}\left[(1 + q_1 t_i + q_2 t_i^2)^2 - \frac{Y_i}{p_0}\right]. \end{aligned} \tag{12}$$

Finally, we have

Proposition 3 *If λ_i is fixed, then t_i, t_{i+1}, t^1_{i+1} are roots of the polynomial $Q_X(\lambda_i, s)$ of the degree 3 in s:*

$$\begin{aligned} Q_X(\lambda_i, s) := (p_6 - p_0\lambda_i^2)s^3 + (p_5 - 2p_0 q_2 \lambda_i)s^2 + (p_4 - 2p_0 q_1 \lambda_i - q_2^2 p_0)s + \\ + (p_3 - 2p_0\lambda_i - 2p_0 q_1 q_2) = 0. \end{aligned}$$

Corollary 2 *The product of two consecutive t_i and t_{i+1} is*

$$t_i t_{i+1} = \frac{p_3 - 2p_0\lambda_i - 2p_0 q_1 q_2}{t^1_{i+1}(p_6 - p_0\lambda_i^2)}.$$

Proposition 3 can be reformulated to give a relation between two consecutive λ_{i-1} and λ_i:

Proposition 4 *If t_i is fixed, then λ_{i-1}, λ_i are solutions of quadratic equation*

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0,$$

with

$$\begin{aligned} \alpha &= -p_0 t_i^3, \\ \beta &= -2p_0 q_2 t_i^2 - 2p_0 q_1 t_i - 2p_0, \\ \gamma &= p_6 t_i^3 + p_5 t_i^2 + (p_4 - q_2^2 p_0)t_i + p_3 - 2p_0 q_1 q_2. \end{aligned}$$

5.4 Normal form of genus 2 HH c. f. Recurrent relations

Using equations (12) and (10), we get formulae for α_i:

$$\alpha_i = \sqrt{p_0}\left(-\frac{2}{t_i} + (2q_2 + \lambda_{i-1}t_i)s - \frac{2}{t_i^3}(1 + q_1 t_i + q_2 t_i^2)s^2\right).$$

Given HH c. f. with α_i, β_i, it can be transformed to the equivalent one with

$$\alpha_i' = c_i\alpha_i, \qquad \beta_i' = c_{i-1}c_i\beta_i.$$

Here, we chose coefficients

$$c_i = -\frac{t_i}{2\sqrt{p_0}}$$

and get

$$\begin{aligned}\alpha_i' &= 1 + w_i s + u_i s^2,\\ \beta_i' &= v_i \frac{s - t_i^1}{t_i^1} s^3,\end{aligned} \tag{13}$$

where

$$\begin{aligned}u_i &= \frac{1 + q_1 t_i + q_2 t_i^2}{t_i^2},\\ w_i &= -(q_2 t_i + \frac{\lambda_{i-1}}{2} t_i^2),\\ v_i &= -\frac{\lambda_{i-1}}{2} + q_3.\end{aligned} \tag{14}$$

We will refer to the form (13) as to *normal form* of the given HH c. f.

From the equations (13), we get

$$\lambda_{i-1} = -2v_i + q_3 \tag{15}$$

and

$$\begin{aligned}(q_2 - u_i)t_i^2 + q_1 t_i + 1 = 0,\\ \frac{\lambda_{i-1}}{2} t_i^2 + q_2 t_i + w_i = 0.\end{aligned} \tag{16}$$

From the equations (16), we have

$$t_i = \frac{(u_i - q_2)w_i + (v_i - \frac{q_3}{2})}{q_2(q_2 - u_i) - q_1(\frac{q_3}{2} - v_i)}.$$

From Proposition 4 and equation (16), one obtains:

$$\lambda^2 \left(-\frac{t_i}{2}\right) - u_i\lambda + [q_6 t_i + q_5(q_1 t_i + 1) + q_3 u_i + q_4 u_i t_i] = 0,$$

having two zeroes λ_{i-1} and λ_i. From the last equation, we get

$$t_i = \frac{u_i(\lambda - q_3) - q_5}{-\frac{\lambda^2}{2} + q_6 + q_5 q_1 + q_4 u_i}.$$

By using the second of equations (16) and equating the right sides of the last equation for λ_{i-1} and λ, we get

Lemma 2 *The following relation between u_i and v_i, v_{i+1} holds:*

$$-\frac{1}{2}(2u_i v_{i+1} + q_5)(q_3 - 2v_i)^2 + 2u_i(q_6 + q_5 q_1 + q_4 u_i)(v_{i+1} - v_i) + \\ + \frac{1}{2}(2u_i v_i + q_5)(q_3 - 2v_{i+1})^2 = 0.$$

This Lemma implies the following

Corollary 3 *If* $v_i \neq v_{i+1}$ *then*

$$0 = q_3^2 u_i - 4u_i v_i v_{i+1} - 2q_5(v_i + v_{i+1}) + 2q_5 q_3 - 2u_i(q_6 + q_5 q_1 + q_4 u_i).$$

From the equations (12), (15), (14) we get

Proposition 5 *The recurrence equations connecting* v_i *and* v_{i+1} *for fixed* u_i *and* t_i *are:*

$$v_i + v_{i+1} = \frac{u_i}{t_i} + q_3,$$
$$4v_i v_{i+1} = (-2q_6 - 2q_1 q_5 - 2q_4 u_i) + q_3^2 - 2\frac{q_5}{t_i}.$$

Rewrite polynomial $Q_X(\lambda_i, s)$ in the form

$$Q_X(\lambda_i, s) = Q_3 s^3 + Q_2 s^2 + Q_1 s + Q_0,$$

where

$$Q_3 = q_6 + q_1 q_5 + q_4 q_2 + \frac{q_3^2}{2} - \frac{\lambda_i}{2},$$
$$Q_2 = q_5 + q_1 q_4 + q_2(q_3 - \lambda_i),$$
$$Q_1 = q_4 + q_1(q_3 - \lambda_i),$$
$$Q_0 = q_3 - \lambda_i.$$

Summing the relations $Q_X(\lambda_i, t_i) = 0$ and $Q_X(\lambda_i, t_{i+1}) = 0$, we get

$$(u_i + u_{i+1})(q_3 - \lambda_i) = (t_i + t_{i+1})\left(\frac{\lambda_i^2}{2} - q_6 - q_5 q_1 - q_4 q_2\right) - q_4 \frac{t_i + t_{i+1}}{t_i t_{i+1}} - 2q_5 - 2q_1 q_4.$$

From the last equation, using the Viète formulae for polynomial $Q_X(s)$ and equation (15), we get

Proposition 6

$$u_i + u_{i+1} = \frac{1}{2} v_{i+1} [\left(\frac{Q_2}{Q_3} - t_{i+1}^1\right)\left(\frac{(-2v_{i+1} + q_3)^2}{2} - q_6 - q_5 q_1 - q_4 q_2\right) -$$
$$- q_4 \left(\frac{Q_2}{Q_3} - t_{i+1}^1\right) \frac{Q_3}{Q_0} t_{i+1}^1 - 2q_5 - 2q_1 q_4].$$

6 Periodicity and symmetry

6.1 Definition and the first properties

According to Theorem 2, in the case

$$t_h = t_k$$

for some h, k, there are two possibilities:

$$(I) \quad \lambda_{h-1} = \lambda_{k-1}, \qquad \lambda_h = \lambda_k;$$
$$(II) \quad \lambda_{h-1} = \lambda_k, \qquad \lambda_h = \lambda_{k-1}.$$

The first possibility leads to *periodicity*:

$$t_{h+s} = t_{k+s}, \qquad \lambda_{h+s} = \lambda_{k+s}$$

for any s and with appropriate choice of roots. If $p = h - k$ and $r \equiv s \pmod p$ then

$$\alpha_r = \alpha_s, \qquad \beta_r = \beta_s.$$

The second possibility leads to *symmetry*:

$$t_{h+s} = t_{k-s}, \qquad \lambda_{h+s} = \lambda_{k-s-1}$$

for any s. More precisely, we introduce

Definition 1

(i) If $h + k = 2n$ we say that HH c. f. is **even symmetric** *with*

$$\alpha_{n-i} = \alpha_{n+i}, \qquad \beta_{n-i} = \beta_{n+i-1}.$$

for any i and with α_n as the **centre of symmetry**.

(ii) If $h + k = 2n + 1$ we say that HH c. f. is **odd symmetric** *with*

$$\alpha_{n-i} = \alpha_{n+i-1}, \qquad \beta_{n-i} = \beta_{n+i}.$$

for any i and with β_n as the **centre of symmetry**.

Now we can formulate some initial properties connecting periodicity and symmetry.

Proposition 7

(A) If a HH c. f. is periodic with the period of $2r$ and even symmetric with α_n as the centre, then it is also even symmetric with respect α_{n+r}.

(B) If a HH c. f. is periodic with the period of $2r$ and odd symmetric with respect β_n, then it is also odd symmetric with respect β_{n+r}.

(C) If a HH c. f. is periodic with the period of $2r - 1$ and even symmetric with respect α_n, then it is also odd symmetric with respect β_{n+r}. The converse is also true.

Proposition 8 *If a HH c. f. is double symmetric, then it is periodic. Moreover:*

(A) If a HH c. f. is even symmetric with respect α_m and α_n, $n < m$ then the period is $2(n - m)$.

(B) If a HH c. f. is odd symmetric with respect β_m and β_n, $n < m$ then the period is $2(m - n)$.

(C) *If a HH c. f. is even symmetric with respect* α_n *and* β_m*, then the period is* $2(n-m)+1$ *in the case* $m \leq n$ *and the period is* $2(m-n)-1$ *when* $m > n$.

Observations:

(i) A HH c. f. can be at the same time even symmetric and odd symmetric.

(ii) If $\lambda_i = \lambda_{i-1}$ then the symmetry is even; if $t_i = t_{i+1}$ then the symmetry is odd.

6.2 Further results

Theorem 3 *An HH c. f. is even-symmetric with the central parameter* y *if* $X(y) = 0$.

The proof follows from the fact that even-symmetry is equivalent to the condition $\lambda_p = \lambda_{p-1}$, which is equivalent to equality $Y_p = 0$.

For odd-symmetry, let us start with the example of genus two case. From relations:

$$Q_X(\lambda, s) = 0, \qquad \frac{d}{ds} Q_X(\lambda, s) = 0$$

we get the system:

$$\begin{aligned} 3Q_3s^2 + 2Q_2s + Q_1 &= 0 \\ Q_2s^2 + 2Q_1s + 3Q_0 &= 0. \end{aligned} \tag{17}$$

From the last system we get

$$v_{i+1} = -\frac{s[q_5 + q_1q_4)s + q_4]}{2q_2s^2 + 4q_1s + 6}$$

or, equivalently

$$\lambda_i = \frac{p_5s^2 + 2(p_4 - q_2^2p_0)s + 3(p_3 - 2p_0q_1q_2)}{2p_0q_2s^2 + 4p_0q_2s + 6p_0}.$$

By replacing any of the last two relations in the first equation of (17), we get the equation of the sixth degree in s. On the other hand, from (17) we get:

$$s = \frac{9Q_0Q_3 - Q_1Q_2}{2Q_2^2 - 6Q_1Q_3}.$$

Now, by replacing the last formula in the first equation of (17) we get the equation of the eight degree in λ_i.

7 General case

7.1 Invariant approach

Now we pass to the general case, with polynomial X of degree $2g+2$. Relation

$$Q_X(\lambda, s) = 0 \tag{18}$$

defines *a basic curve* Γ_X. Denote its genus by G and consider its projections p_1 to the λ–plane, and p_2 to the s–plane.

Denote by R_e the ramification points of the second projection and call them *even-symmetric points* of the basic curve.

The set R_{o+r} of the ramification points of the first projection is the union of sets of *the odd-symmetric points* and of *the gluing points.*

The gluing points represent situation where some of the roots of the polynomial $B^{(i)}$ coincide. For example in genus 2 case the gluing points correspond to the condition $t_{i+1} = t'_{i+1}$.

From Theorem 3, we get

$$\deg R_e = 2g + 2.$$

Applying the Riemann-Hurvitz formula, we have

$$\begin{aligned} 2 - 2G &= 4 - \deg R_e; \\ 2 - 2g &= 2(g+1) - \deg R_{o+r}. \end{aligned}$$

Thus

$$\text{genus}\,(\Gamma_X) = G = g$$

and

$$\deg R_{o+r} = 4g.$$

We get a birational morphism

$$f : \Gamma \to \Gamma_X$$

by the formulae

$$f : (x, s) \mapsto (t, \lambda),$$

where

$$\begin{aligned} t &= x, \\ \lambda &= \frac{1}{t^{g+1}} \left(\frac{s}{\sqrt{p_0}} - Q_g(t) \right), \\ Q_g(t) &= 1 + q_1 t + \cdots + q_g t^g. \end{aligned}$$

Function f satisfies the following commuting relation:

$$f \circ \tau_\Gamma = \tau_{\Gamma_X} \circ f,$$

where τ_Γ and τ_{Γ_X} are natural involutions on hyperelliptic curves Γ and Γ_X respectively.

7.2 Multi-valued divisor dynamics

The inverse image of a value z of the function λ is a divisor of degree $g+1$:

$$\lambda^{-1}(z) =: D(z), \quad \deg D(z) = g+1.$$

Now, the HH-continued fractions development can be described as a multi-valued discrete dynamics of divisors $D_k^j = D(z_k^j)$. Here, the lower index k denotes the k–th step of the dynamics and the upper index j goes in the range from 1 to $(g+1)k$ denoting branches of multivaluedness. More precisely, the discrete divisor dynamics which governs HH-continued fraction development can be described as follows.

Suppose the development has started with a point $P_0 = P_0^1$. It leads to the divisor

$$D_0 := D(\lambda(P_0)) = P_0^1 + P_0^2 + \cdots + P_0^{g+1},$$

with $\lambda(P_0^i) = \lambda(P_0^j)$. In the next step, we get $g+1$ divisors of degree $g+1$:

$$D_1^j := D\left(\lambda(\tau_\Gamma(P_0^j))\right).$$

And we continue like this. In each step, the divisor:

$$D_{k-1}^j = P_{k-1}^{(j,1)} + \cdots + P_{k-1}^{(j,(g+1))}$$

from the previous step, gives $g+1$ new divisors

$$D_k^{(j-1)(g+1)+l} := D\left(\lambda(\tau_\Gamma(P_{k-1}^{(j,l)}))\right), \qquad l = 1, \dots, g+1.$$

In the case of genus one, this dynamics can be traced out from the $2-2$–correspondence $Q_\Gamma(\lambda, t) = 0$. According to [9], for example, there exist constants a, b, c, d, T such that for every i we have

$$\lambda_i = \frac{ax(u_i+T)+b}{cx(u_i+T)+d},$$

where u is an uniformizing parameter on the elliptic curve. The involution is the symmetry at the origin and since the function x is even, the two parameters corresponding to the fixed value λ_i are u_i and $\bar{u}_i = -u_i - 2T$. Thus

$$\begin{aligned} u_{i+1} &= u_i + 2T, \\ \lambda_{i+1} &= \frac{ax(u_i+3T)+b}{cx(u_i+3T)+d}. \end{aligned}$$

In the cases of higher genera the dynamics is much more complicated. Thus we pass to the consideration of generalized Jacobians.

7.3 Generalized Jacobians

A natural environment for consideration divisors of degree $g+1$ on the curve Γ of genus g is generalized Jacobian $\mathrm{Jac}(\Gamma, \{Q_1, Q_2\})$ of Γ, obtained by gluing a pair of points Q_1, Q_2 of Γ (see [8]).

This Jacobian can be introduced as a set of classes of relative equivalence among the divisors on Γ of certain degree. Two divisors of the same degree D_1 and D_2 are called *equivalent relative to points* Q_1, Q_2, if there exists a meromorphic function f on Γm such that $(f) = D_1 - D_2$ and $f(Q_1) = f(Q_2)$.

The generalized Abel map is defined with

$$\widetilde{\mathcal{A}}(P) = (\mathcal{A}(P), \mu_1(P), \mu_2(P)), \quad \mu_i(P) = exp \int_{P_0}^{P} \Omega_{Q_iQ_0}, i = 1, 2,$$

where $\mathcal{A}(P)$ is the standard Abel map. Here $\Omega_{Q_iQ_0}$ denotes the normalized differential of the third kind, with poles at the point Q_i and at arbitrary fixed point Q_0.

Here we consider the case where $Q_1 = +\infty$ and $Q_2 = -\infty$ on the curve Γ of genus g. The divisors we are going to consider are those of degree $g+1$ of the form $D_i = D(z_i)$ where usually $z_i = \lambda(P_i)$. The divisors of degree $g+1$ up to the equivalence relative to the points Q_1 and Q_2 are uniquely determined by their generalized Abel image on the generalized Jacobian.

Thus, in order to measure the distance between relative classes of $D_1 = D(z_1) = D(\lambda(P_1))$ and of $D_2 = D(z_2) = D(\lambda(P_2))$ we introduce the following index

$$I(D_1, D_2) = I(z_1, z_2) = I(P_1, P_2) := \frac{\lim_{P\to+\infty} \frac{\lambda(P)-z_1}{\lambda(P)-z_2}}{\lim_{P\to-\infty} \frac{\lambda(P)-z_1}{\lambda(P)-z_2}}.$$

We are interested in the case $P_2 = \tau_\Gamma(P_1)$ and we have

$$I(P_1) := I(P_1, \tau_\Gamma(P_1)) = \lim_{P\to+\infty} \frac{\frac{\lambda(P)-\lambda(P_1)}{\lambda(P)-\lambda(\tau(P_1))}}{\frac{\lambda(\tau(P))-\lambda(P_1)}{\lambda(\tau(P))-\lambda(\tau(P_1))}}$$

After some calculations we get

Lemma 3 *The index of the point is given by the formula*

$$I(P_1) = 1 + \frac{2\sqrt{p_{2g+2}}\big(\lambda(\tau(P_1)) - \lambda(P_1)\big)}{p_{2g+2} - \sqrt{p_{2g+2}}\big(\lambda(\tau(P_1)) - \lambda(P_1)\big) - \lambda(P_1)\lambda(\tau(P_1))}.$$

8 Irregular terms

The parameters t that appear to be infinite or zero, we call *irregular*.

8.1 t_h − infinite

Suppose $t_0 = \infty$. We start from the following relation:

$$X - A^2 = Bs^{g+1}.$$

Then, HH continued fraction is based on the relation

$$\sqrt{X} - \sqrt{p_{2g+2}}s^{g+1} = C + \frac{Bs^{g+1}}{\sqrt{X} + A}.$$

Proposition 9 *Irregular HH c. f. with $t_h = \infty$ is even symmetric if and only if $p_{2g+2} = 0$.*

8.2 $t_h = 0$

Let $t_0 = 0$. In that case the basic relation of HH continued fraction is

$$\frac{\sqrt{X} - \sqrt{p_0}}{x - \varepsilon} - C = \frac{B(x-\varepsilon)^{g+1}}{\sqrt{X} + A}.$$

Then we have also

$$A - \sqrt{p_o} = Cs,$$
$$X - A^2 = Bs^{g+2}.$$

An HH continued fraction is developed through the following relations

$$\sqrt{X} = A + \frac{Bs^{g+2}}{\sqrt{X} + A},$$
$$\sqrt{X} = A + \frac{Bs^{g+2}}{P_A^{(g)}} + \frac{\sqrt{X} - \sqrt{Y_1}}{x - y_1}$$

Proposition 10 *The condition $t_h = 0$ is equivalent to $v_{h+1} = \infty$. Such an HH c. f. is odd symmetric with respect to β_{h+1}.*

8.3 ε − infinite

The starting relation in the case $\varepsilon = \infty$ is

$$X - A^2 = B(x - y).$$

Changing the variables: $x = 1/s$, $y = 1/t$, we come to:

$$X' - A'^2 = -\frac{1}{t}B' s^{g+1}(s - t).$$

The HH c. f. takes the form

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = C + \frac{B_0|}{|A_1} + \frac{B_0^{(1)}|}{|A_2} + \cdots + \frac{B_0^{(i-1)}|}{|A_i + Q_i},$$

where $\deg B_0^{(i)} = g - 1$, $B_0^{(i)} = B^{(i)}/(x - t_i^0)$, $\deg C = g$, $\deg A_i = g$. Appropriate HH c. f. is obtained from the last one after the change of variables.

Lemma 4 *The following identity holds:*

$$y^{g+1}\frac{\sqrt{X}-\sqrt{Y}}{x-y} = (x^g + x^{g-1}y + \cdots + xy^{g-1} + y^g)\sqrt{Y} + \frac{y^{g+1}\sqrt{X} - x^{g+1}\sqrt{Y}}{x-y}.$$

Proposition 11 *The HH element $(\sqrt{X}-\sqrt{Y})/(x-y)$ around $x=\infty$ has the same coefficient as $(\sqrt{X'}-\sqrt{Y'})/(s-t)$ around $s=0$.*

9 Remainders, continuants and approximation

We consider an HH c. f. of an element f

$$f = C + \frac{\beta_1|}{|\alpha_1} + \frac{\beta_2|}{|\alpha_2} + \ldots.$$

Together with *the remainder of rank i* Q_i, where

$$Q_i = \frac{B^{(i)s^{g+1}}}{\sqrt{X}+A^{(i)}},$$

we consider *the continuants* (G_i) and (H_i) and *the convergents* G_i/H_i such that

$$\begin{bmatrix} G_m & G_{m-1} \\ H_m & H_{m-1} \end{bmatrix} = T_C T_1 \cdots T_m. \tag{19}$$

Here

$$T_i = \begin{bmatrix} \alpha_i & 1 \\ \beta_i & 0 \end{bmatrix}, \quad T_C = \begin{bmatrix} C & 1 \\ 1 & 0 \end{bmatrix}.$$

By taking the determinant of (19) we get

$$\begin{aligned} G_m H_{m-1} - G_{m-1}H_m &= (-1)^{m-1}\beta_1\beta_2\ldots\beta_m \\ &= \delta_m s^{(g+1)m} \\ \deg\delta_m &= (g-1)m. \end{aligned}$$

We also have the following relations:

$$f = \frac{(\alpha_m+Q_m)G_{m-1}+\beta_m G_{m-2}}{(\alpha_m+Q_m)H_{m-1}+\beta_m H_{m-2}} = \frac{G_m+Q_mG_{m-1}}{H_m+Q_mH_{m-1}},$$

and

$$Q_m = -\frac{G_m - H_m f}{G_{m-1}-H_{m-1}}.$$

Proposition 12 *The degree of the continuants is* $\deg G_m = g(m+1)$, $\deg H_m = gm$.

Let us introduce

$$\widehat{G}_m = G_m + \frac{H_m}{s-t}\sqrt{Y}$$
$$\widehat{H}_m = \frac{H_m}{s-t}.$$

Then we have

$$Q_m = -\frac{\widehat{G}_m - \widehat{H}_m\sqrt{X}}{\widehat{G}_{m-1} - \widehat{H}_{m-1}\sqrt{X}}$$

and also

$$\widehat{G}_m A^{(m)} + \widehat{G}_{m-1} B^{(m)} s^{g+1} = \widehat{H}_m X$$
$$\widehat{H}_m A^{(m)} + \widehat{H}_{m-1} B^{(m)} s^{g+1} = \widehat{G}_m X.$$

From the last equations we get

$$\delta_m s^{(g+1)m} A^{(m)} = P_1(s)$$
$$\delta_m s^{(g+1)(m+1)} B^{(m)} = P_2(s),$$

with

$$P_1(s) := H_m H_{m-1}\frac{X_Y}{x-y} - (G_m H_{m-1} - G_{m-1}H_m)\sqrt{Y} - G_m G_{m-1}(s-t)$$
$$P_2(s) := G_m^2(s-t) + 2G_m H_m\sqrt{Y} - H_m^2\frac{X-Y}{x-y}.$$

Theorem 4

(A) The polynomial $G_m H_{m-1} - H_m G_{m-1}$ *is of degree* $2gm$*. The first* $(g+1)m$ *coefficients are zero.*

(B) The polynomial P_1 *is of degree* $2mg+g+1$*. Its first* $(g+1)m$ *coefficients are zero.*

(C) The polynomial P_2 *is of degree* $2mg+2g+1$ *and its* $(g+1)(m+1)$ *first coefficients are zero.*

Lemma 5 *The following relations hold*

$$\frac{G_{m-1}(t_m)}{H_{m-1}(t_m)} = -A^{(m)}(t_m) = A^{(m-1)}(t_m)$$
$$\widehat{G}_m - \widehat{H}_m\sqrt{X} = (-1)^{m+1}Q_0Q_1Q_2\ldots Q_m.$$

Theorem 5 *If* $X(\varepsilon) \neq 0$ *and* $\varepsilon \neq y$*, then the element*

$$\widehat{G}_m - \widehat{H}_m\sqrt{X} = G_m - H_m\frac{\sqrt{X}-\sqrt{Y}}{x-y}$$

has a zero of order $(g+1)(m+1)$ *at* $s=0$*. If* $H(0) \neq 0$ *then the differences*

$$\frac{\sqrt{X}-\sqrt{Y}}{x-y} - \frac{G_m}{H_m}, \qquad \sqrt{X} - \frac{\widehat{G}_m}{\widehat{H}_m}$$

have developments starting with the order of $s^{(g+1)(m+1)}$*.*

Now, we consider $\sqrt{X}$ and its development as HH c. f. In that case, starting from

$$\frac{\sqrt{X}-\sqrt{p_0}}{x-\varepsilon},$$

we have

$$\deg G_0 = g+1, \quad H_0 = 1, \quad H_1 = \alpha_1, \quad G_1 = \alpha_1 G_0 + \beta_1 s^{g+2}$$

and

$$G_m = \alpha_m G_{m-1} + \beta_m G_{m-2},$$
$$H_m = \alpha_m H_{m-1} + \beta_m H_{m-2}.$$

From the last relation, we have

Theorem 6

(A) The degree of the continuants in this case is $\deg G_m = g(m+1)+1$, $\deg H_m = gm$.

(B) If $y=\varepsilon$ *then the development of the difference*

$$\sqrt{X} - \frac{\hat{G}_m}{\hat{H}_m}$$

starts with the order $s^{(g+1)(m+1)+1}$.

Theorem 7

(A) The polynomial $G_m H_{m-1} - H_m G_{m-1}$ *is of degree* $2gm+1$ *in* s*. The first* $(g+1)m+1$ *coefficients are* 0.

(B) The polynomial $H_m H_{m-1} X - G_m G_{m-1}$ *is of degree* $2mg+g+2$*. Its first* $(g+1)m+1$ *coefficients are zero.*

(C) The polynomial $G_m^2 - H_m^2 X$ *is of degree* $2mg+2g+2$ *and its* $(g+1)(m+1)+1$ *coefficients are zero.*

There are infinite ways to calculate $\sqrt{X}$ in the neighborhood of ε, depending on choice of the parameter y. The best approximation one obtains for the choice

$$y = \varepsilon.$$

To conclude the last observation we need to check the case $y=\infty$. In this case we have:

$$G_0 = \sqrt{p_0}(1+q_1 s + \cdots + q_g s^g), \quad H_0 = 1, \quad G_1 = \alpha_1 G_0 + \beta_1, \quad H_1 = \alpha_1,$$

and we denote

$$\widehat{G}_m = G_m + \sqrt{a_0} H_m s^{g+1}, \quad \widehat{H}_m = H_m.$$

Then we have

Proposition 13

(A) The degree of continuants is $\deg G_m = g(m+1)$*,* $\deg H_m = gm$*.*

(B) If $X(\varepsilon) \neq 0$ *and suppose the parameters* $t_1, \dots, t_{m+1}$ *are finite and different from zero, then*

$$\sqrt{X} - \frac{\widehat{G}_m}{\widehat{H}_m}$$

has the development starting with order $2m+2$ *in* s*.*

10 First geometric realization of the $2 \leftrightarrow g+1$ dynamics

Let us start with equation (18) which defines basic curve Γ_X. Consider polynomial $Q_X(\lambda, s)$ as *a quadratic pencil* or *a net* of polynomials a, b, c of degree $g+1$ in s:

$$Q_X(\lambda, s) = a(s)\lambda^2 + b(s)\lambda + c(s).$$

Linear pencils of polynomials were considered, for example, in [4, 15, 5].

Following Darboux, let us start with a fixed conic K given by the equation

$$z_1^2 = 4z_0z_2,$$

in the plane with standard coordinates (z_0, z_1, z_2). This conic can be rationally parameterized by $(s^2, 2s, 1)$. The tangent line to K through the point with the parameter s_0 is given by the equation

$$t_K(s_0) \ : \ z_2s_0^2 + z_1s_0 + z_0 = 0.$$

On the other hand, to a given point P in the plane, with coordinates $P = (\hat{z}_0, \hat{z}_1, \hat{z}_2)$, we may join two solutions ρ, ρ_1 of the following equation, which is quadratic in s:

$$\hat{z}_2s^2 + \hat{z}_1s + \hat{z}_0 = 0.$$

Each solution corresponds to a tangent to conic K from point P. We will call the pair (ρ, ρ_1) *the Darboux coordinates* of the point P. One finds immediately

$$\frac{\hat{z}_0}{\rho\rho_1} = -\frac{\hat{z}_1}{\rho + \rho_1} = \hat{z}_2.$$

Now, we interpret the net condition

$$a(s)\lambda^2 + b(s)\lambda + c(s) = 0 \tag{20}$$

as a correspondence between values of λ and sets of $g+1$ tangents to the conic K: Denote by $s_1, \dots, s_{g+1}$ the set of solutions of equation (20) for fixed λ and consider the tangents $t_K(s_1), \dots, t_K(s_{g+1})$.

Moreover, we associate to the polynomial X a plane curve B_X such that the Darboux coordinates ρ, ρ_1 of a point of the curve B_X satisfy the equation (20) with a fixed λ.

Definition 2 *We will call the curve B_X the* boundary curve *associated with the polynomial X and the conic K.*

Collecting together the results of classics: Jacobi, Steiner, Liuoville, Hesse, Cremona, Darboux, we may formulate the following statements.

Theorem 8

(a) The curve B_X is of degree $2q$ and in general it has $g(g-1)/2$ double points.

(b) For a fixed value of λ there correspond $g+1$ solutions $\rho_1, \dots, \rho_{g+1}$ of the equation (20) determining a $g+1$-polygon inscribed in B_X and circumscribed about the conic K and satisfying the following system of differential equations:

$$\frac{\rho_1^i d\rho_1}{\sqrt{b^2(\rho_1) - 4a(\rho_1)c(\rho_1)}} + \cdots + \frac{\rho_{g+1}^i d\rho_{g+1}}{\sqrt{b^2(\rho_{g+1}) - 4a(\rho_{g+1})c(\rho_{g+1})}} = 0, \quad (21)$$

where $i = 0, \dots, g-1$.

(c) There exist $2g+2$ lines tangent to the conic K which are tangent to every integral curve of the system of equations (21). Each of these tangents is tangent to each integral curve in g points.

We give more detailed presentation of the cases of genus one and two.

Example 1 *For $g = 1$, the system (21) consists of one equation. This is the Euler equation. The integral curves of the Euler equation are conics, which are, together with the conic K, inscribed in a quadrilateral.*

For a given value $s = \rho_1$ there are two solutions of the equation (20), denote them by λ_1 and λ_2. Let ρ_1, ρ be the solutions of equation (20) for λ_1, and ρ_1, ρ_2 the solutions for λ_2. The pairs of lines (ρ, ρ_1) and (ρ_1, ρ_2) form two angles inscribed in a conic B and circumscribed about the conic K. The involution which corresponds to the shift from λ_1 to λ_2 is realized as passage from the first angle to the second one.

Example 2 *For $g = 2$, the system (21) consists of two equations:*

$$\begin{gathered}\frac{d\rho_1}{\sqrt{b^2(\rho_1) - 4a(\rho_1)c(\rho_1)}} + \frac{d\rho_2}{\sqrt{b^2(\rho_2) - 4a(\rho_2)c(\rho_2)}} + \\ + \frac{d\rho_3}{\sqrt{b^2(\rho_3) - 4a(\rho_{g+1})c(\rho_3)}} = 0, \\ \frac{\rho_1 d\rho_1}{\sqrt{b^2(\rho_1) - 4a(\rho_1)c(\rho_1)}} + \frac{\rho_2 d\rho_2}{\sqrt{b^2(\rho_2) - 4a(\rho_2)c(\rho_2)}} + \\ + \frac{\rho_3 d\rho_3}{\sqrt{b^2(\rho_3) - 4a(\rho_{g+1})c(\rho_3)}} = 0\end{gathered}$$

giving the first generalization of the Euler equation. The integral curves are of degree four with one double point. Together with the conic K, they are inscribed in a hexagon.

Given one of the integral curves B and a tangent $t_K(\rho)$ to the conic K with the Darboux coordinate ρ.

For the value $s = \rho$ there are two solutions λ_1, λ_2 of equation (20). Let ρ_1, ρ_2 be the solutions of (20) for λ_1 that are different than ρ, and similarly, let ρ_3, ρ_4 be the solutions for λ_2. The triplets of lines (ρ, ρ_1, ρ_2) and (ρ, ρ_3, ρ_4) form two triangles inscribed in the degree four curve B and circumscribed about the conic K. The involution which corresponds to the shift from λ_1 to λ_2 is realized this time as the passage from the first triangle to the second one.

The line $t_K(\rho)$ intersects the degree four curve B in four points T_1, T_2, T_3, T_4, $T_i \in \rho_i$. The involution defined by

$$(\rho, \lambda_1) \mapsto (\rho, \lambda_2)$$

corresponds to the decomposition of the set $\{T_1, T_2, T_3, T_4\}$ on two subsets of the same number of elements: $\{T_1, T_2\}$ and $\{T_3, T_4\}$.

The last observation in the previous example gives an insight how to understand $2 \leftrightarrow g+1$ dynamics in general situation.

[Geometric realization of the dynamics 1] *Given a boundary curve B of degree $2g$ and a tangent to the conic K with the Darboux coordinate ρ. The line $t_K(\rho)$ intersects the degree $2g$ curve B in $2g$ points $T_1, \dots T_g, T_{g+1}, \dots, T_{2g}$. By condition $T_i \in \rho_i$, $2g$ new tangents to the conic K are determined. The involution defined by:*

$$(\rho, \lambda_1) \mapsto (\rho, \lambda_2)$$

corresponds to the decomposition of set $\{T_1, \dots T_g, T_{g+1}, \dots, T_{2g}\}$ to two subsets of the same number of elements, say: $\{T_1, \dots, T_g\}$ and $\{T_{g+1}, \dots, T_{2g}\}$. This means that ρ, together with ρ_i, $i = 1, \dots, g$, form the set of solutions of equation (20) with $\lambda = \lambda_1$, while ρ with ρ_i, $i = g+1, \dots, 2g$ form the set of solutions of (20) with $\lambda = \lambda_2$.

The $(g+1)$–tuples of lines $(\rho, \rho_1, \dots \rho_g)$ and $(\rho, \rho_{g+1}, \dots, \rho_2 g)$ form two $(g+1)$–polygons inscribed in the degree $2g$ curve B and circumscribed about the conic K. These two polygons have a pair of sides belonging to the same line — ρ. The involution which corresponds to the shift from λ_1 to λ_2 is realized as the passage from the first polygon to the second one. We can call this move **the flip along the edge**.

The dynamics is a path of polygons of $g+1$ sides inscribed in the curve B of degree $2g$ and circumscribed about the conic K obtained by successive flips along edges.

11 The second geometric realization of the $2 \leftrightarrow g+1$ dynamics

In order to give another geometric realization of the $2 \leftrightarrow g+1$ dynamics which is governed by the HH continued fractions, first we are going to realize given hyperelliptic curve

$$\Gamma : z^2 = X_{2g+2}$$

of genus g, as a generalized Cayley's curve of Dragović and Radnović (see [6], [7]). By a birational isomorphism which maps one of the zeros of polynomial X_{2g+2} to the infinity, one can realize the curve Γ in the form:

$$y^2 = \mathcal{P}_{2g+1}(x),$$

where the polynomial $\mathcal{P}_{2g+1}$ is of odd degree equal to $2g+1$. Assuming that zeros of polynomial $\mathcal{P}_{2g+1}$ are real and different, one can order them as:

$$b_1 < b_2 < \cdots < b_{2d-1}.$$

Now, decompose the set of zeros of the polynomial $\mathcal{P}_{2g+1}$ in one of the ways that satisfies:

$$\{b_1, \dots, b_{2d-1}\} = \{a_1, \dots, a_{g+1}, \alpha_1, \dots, \alpha_g\},$$

where

$$\alpha_j \in \{b_{2j-1}, b_{2j}\}, \quad \text{for} \quad 1 \le j \le g. \tag{22}$$

Introduce the following family of confocal quadrics in the $g+1$–dimensional Euclidean space $\mathbf{E}^{g+1}$:

$$\mathcal{Q}_\lambda \ : \ \frac{x_1^2}{a_1 - \lambda} + \cdots + \frac{x_{g+1}^2}{a_{g+1} - \lambda} = 1 \qquad (\lambda \in \mathbf{R}), \tag{23}$$

where $a_1, \dots, a_d$ are different real constants chosen above.

By Chasles theorem, we know that a given line in $\mathbf{E}^{g+1}$ is tangent to g quadrics from given confocal family.

The g constants $\alpha_1, \dots, \alpha_g$, determine g quadrics from family (23). Since the constants satisfy conditions (22), there exist lines in $\mathbf{E}^{g+1}$ that are tangent to g distinct non-degenerate quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_g}$ from the confocal family.

Let ℓ be a line not contained in any quadric of the given confocal family and tangent to the given set of g quadrics $\mathcal{Q}_{\alpha_1}, \dots, \mathcal{Q}_{\alpha_g}$. The *generalized Cayley curve* $\mathcal{C}_\ell$ is the variety of hyperplanes tangent to quadrics of the confocal family at the points of ℓ.

The Figure 5, represents three planes which correspond to one point of the line ℓ in the 3-dimensional space.

The generalized Cayley curve is a hyperelliptic curve of genus g, for $g \ge 2$. Its natural realization in $\mathbf{P}^{g+1\,*}$ is of degree $2g+1$.

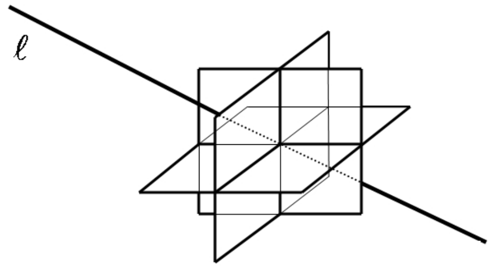

Figure 5: Three points of the generalized Cayley curve in dimension 3

There is a natural involution τ_ℓ on the generalized Cayley's curve $\mathcal{C}_\ell$ which maps to each other the two hyperplanes tangent to the same quadric of the confocal family. It is easy to see that the fixed points of this involution are hyperplanes corresponding to the g quadrics that are touching ℓ and to $g+2$ degenerate quadrics of the confocal family.

Now we come to the essential observation. The generalized Cayley's curve $\mathcal{C}_\ell$ is automatically equipped with a meromorphic function of degree $g+1$, namely with the projection

$$p_\ell : \mathcal{C}_\ell \mapsto \mathbb{P}^1(\ell).$$

The projection p_ℓ maps to a point t from ℓ the $g+1$ hyperplanes from $\mathcal{C}_\ell$ that contain t.

Now, we can give the second geometric realization of the dynamics governed by the hyperelliptic Halphen continued fractions.

[Geometric realization of the dynamics 2] *For a suitably chosen line ℓ, choose a point $t_1 \in \ell$ and a tangent hyperplane $T_{1,1}$ to a quadric $\mathcal{Q}_{1,1}$ at t_1. Find the other intersection of quadric $Q_{1,1}$ and line ℓ, and denote it as t_2. Let $T_{2,1}$ be the tangent hyperplane to $\mathcal{Q}_{1,1}$ at t_2. Denote by $T_{2,j}$ the tangent hyperplanes to quadrics $\mathcal{Q}_{2,j}$ at t_2, $j \in \{2, \dots, g+1\}$. Choose one of them, and denote the chosen tangent hyperplane with $T_{2,2}$. Find the other intersection of the quadric $\mathcal{Q}_{2,2}$ with the line ℓ and denote it with t_3. Denote the tangent hyperplane to the quadric $\mathcal{Q}_{2,2}$ at the point t_3 as $T_{3,1}$. Denote all other tangent hyperplanes to quadrics $\mathcal{Q}_{3,j}$ at t_3, by $T_{3,j}$ where $j \in \{2, \dots, g+1\}$. Choose one of the tangent hyperplanes, say $T_{3,3}$ and find the other intersection point of the quadric $\mathcal{Q}_{3,3}$ with the line ℓ. Denote the intersection point as t_4 and so on.*

By using notation of functions, we may say that

$$\tau_\ell(T_{i,1}) = T_{i+1,1}, \quad p_\ell(T_{i+1,1}) = t_{i+1}.$$

Also we have

$$p_\ell^{-1}(t_i) = \{T_{i,1}, T_{i,2}, \dots, T_{i,g+1}\}.$$

Even the case $g = 1$ gives a new geometric representation of the famous Euler–Chasles correspondence. What we give here is one its asymmetric realization. For $g = 1$, the projection p_ℓ is two-to-one, and it induces another involution

$$\mu_\ell \ : \ \mathcal{C}_\ell \to \mathcal{C}_\ell$$

which exchanges the elements of the inverse image of p_ℓ:

$$p_\ell(x) = p_\ell(\mu_\ell(x)).$$

The dynamics of Halphen continued fractions is executed by a shift L done by composition of the two involutions:

$$L = \mu_\ell \circ \tau_\ell.$$

Even in this basic case, $g = 1$, the construction we made leads to new geometric properties of lines in plane and dynamics of points of intersection with given family of confocal conics. These properties are reflections of the Poncelet porism for confocal conics and the Euler–Chasles correspondences. Thus, if a sequence of the dynamics described above forms a cycle starting from a point of a line, then the cycle of the same length will appear in this dynamics starting from any other point of the line.

As a trivial example, one may consider a horizontal or a vertical line and a standard confocal system of conics in a plane. The confocal system decomposes a horizontal or vertical line on cycles of length 2.

Now, we are going back to a general case. We follow the line of [7] where the set $\mathcal{A}_\ell$ of lines in $g+1$ dimensional space tangent to the fixed set of g quadrics of a given confocal family is equipped with a structure of Abelian variety. In the same spirit, we may consider tautological line bundle $\mathcal{L}\mathcal{A}_\ell$ as a generalized Abelian variety. Tautological bundle consists of pairs (line, point) with incidence relation that point belongs to a line.

12 Conclusion: Polynomial growth and integrability

Due to the well-known facts, the Padé approximants of hyperelliptic functions are unique up to the scalar factors. The approximants discussed in the previous section in the case of genus higher than 1 are neither unique nor of the Padé type. At the first glance, it seems that, by the construction, they have an exponential growth. However, a more careful analysis of their degrees compared to the degrees of approximation done in the previous section indicates their polynomial growth. After Veselov, one can consider a discrete multi-valued dynamics to be integrable if it has polynomial growth instead of an exponential one. In that sense, we can say that the multi-valued discrete dynamics associated with HH-continued fractions is an integrable dynamics.

In the case of genus one, it can be seen as multi-valued discrete dynamics associated with the Euler-Chasles 2–2 correspondence, which has been studied by Veselov (see [16]) and Veselov and Buchstaber (see [3]). It would be quite interesting to consider higher genus dynamics from the point of view of n-valued groups and their actions, following Buchstaber (see [2]).

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