Quantum Groups and Braid Group Statistics in Conformal Field Theory

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Abstract

A quantum universal enveloping algebra U_q and the braid group on n strands $\mathcal{B}_n$ mutually commute when acting on the n-fold tensor product of a U_q-module. Their combined action is applied to low dimensional systems – the only ones that admit a nontrivial monodromy and hence a braid group (rather than a permutation group) statistics.

The lectures introduce the notions of braid group and Hopf algebra and apply them to examples of 2-dimensional (rational) conformal field theory. The case of the $su(2)$ current algebra model, for which the deformation parameter q is an even root of unity, is considered in some detail. In particular, the solution to the Schwarz problem for the $su(2)$ Knizhnik-Zamolodchikov equation is reviewed.

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Preface

These lectures are meant as an introduction to quantum groups (with emphasis on quantum universal enveloping algebras (QUEA)), braid groups and their application to 2-dimensional conformal field theory. With a view of an audience of mixed background they purport to introduce the basic concepts encountered on the way: Hopf algebras, permutation and braid groups, the conformal group in two and higher dimensions, axiomatic quantum field theory – in various degree of detail. Thus the first four sections and Appendix A form a minicourse on braid groups and Hopf algebras viewing them in the context of (a deformed) Schur-Weyl duality. Section 4 and Appendix B also contain some less standard material: the general form of n-point $U_q(s\ell_2)$ invariants based on joint work with P. Furlan and Ya. S. Stanev of the 1990's. The Drinfeld double and the universal R-matrix are treated rather schematically (in Section 5); mastering this subject would require more work and further reading. This is even more true for the sketch of Wightman axioms (Appendix C) and of the axioms for a chiral vertex algebra (Appendix D) – subjects of monographs outlined here on a couple of pages.

The next three sections (6, 7, 8) provide another introductory course (for more advanced students) – on conformal field theory (CFT). We begin with the axioms of quantum field theory (supplemented with the requirement of conformal invariance) in D space-time dimension in order to stress the special features of the case $D = 2$ to which the rest of the lectures is devoted. Sections 7 and 8 deal with the $u(1)$ conformal current algebra – the simplest $2D$ CFT – and its local extensions, introducing on the way fractional charge fields with anyonic statistics. The survey of the $su(2)$ current algebra model corresponding to the Wess-Zumino-Novikov-Witten action (Sections 9, 10) is more schematic. A regular basis of solutions of the Knizhnik-Zamolodchikov equation (introduced by Ya. S. Stanev and the author) is displayed without derivation in Section 9. Section 10 includes a survey of the U_q oscillator algebra introduced by Pusz and Woronowicz which can be viewed as a deformation of the Schwinger model for $su(2)$.

The two topics, braid groups and QUEA, on one hand, and $2D$ CFT, on the other, are combined in Sections 11, 12 into the study of monodromy representations of the braid group $\mathcal{B}_4$. As an application we survey in Section 11 the solution of the Schwarz problem for the Knizhnik-Zamolodchikov equation (worked out by Stanev and myself). Section 12 introduces and studies the restricted and the Lusztig QUEA for q an even root of unity and reviews recent work of Furlan, Hadjivanov and the author on the subject. Section 13, the last one, contains an overview and provides references to adjacent topics (including Chern-Simons theory) that have been left out.

I thank my long term collaborators mentioned above who contributed to the understanding of the subject matter of these notes.

It is a common observation (see, *e.g.*, the Introduction to [Fr]) that public attention to fundamental physics is declining. In the light of this global phenomenon it was particularly rewarding to me to witness the keen interest of the young (and not so young) audience at the Universidade Federal do Espírito Santo in Vitória, Brazil, during the course of these lectures.

It is a pleasure to thank Clistenis Constantinidis, Olivier Piguet and Galen Sotkov for their hospitality in Vitoria where these lectures were presented. The hospitality and support of L'Institut des Hautes Études Scientifiques, Bures-sur-Yvette, where these notes were written is also gratefully acknowledged. I thank, in particular, Cécile Cheikhchoukh for her expert and expeditions typing which allowed to produce the present version without delay. This work is supported in part by the Research Training Network of the European Commission under contract MRTN-CT-2004-00514 and by the Bulgarian National Council for Scientific Research under contract Ph-1406.

1 Introduction

The concept of a group seems to be tailor made to match the notion of symmetry. It is economical and general: it just assumes that a composition of maps (or transformations) g_1 and g_2 is again a map, $g_1 g_2$ (of a set into itself), that the product is associative, $(g_1 g_2) g_3 = g_1 (g_2 g_3)$, and that for each transformation g there is an inverse, g^{-1} such that $gg^{-1} = g^{-1} g = 1$ (1 standing for the identity map which has the property $g \cdot 1 = 1 \cdot g = g$). For transformations depending on continuous parameters (like translations and rotations) one has the powerful concept of a Lie[1] group which allows to reduce in most cases the study of a symmetry to a local problem of Lie algebra.

Why then should we look for a more general concept like Hopf[2] algebra or "quantum group" (or even for some further extension thereof)?

A historical account answering this question from a mathematical point of view can be found in the lectures of Pierre Cartier [C]. I shall single out one aspect of his answer which also has a physical interpretation. Another view of the history of quantum groups is provided by Ludwig Faddeev [F] starting with integrable systems, in particular, spin chains.

A first principle of quantum theory is the *principle of superposition.* It tells us that *quantum states* form a vector space and symmetry groups act by (linear) representations on this space. If we assume, as usual, the standard probabilistic interpretation of state vectors, then we have to deal with *unitary representations* of the symmetry group. Furthermore, the state space of a pair of non-interacting systems is the *tensor product* of the spaces of individual systems. This leads us to considering the ring of representations closed under tensor products and direct sums.

Consider now a system of n identical non-relativistic particles of coordinates x_i and internal quantum number s_i $(i = 1, \ldots, n)$. Assume further, for the sake of definiteness that each s_i takes k values and that the internal symmetry group is $U(k)$. The state of such a system is described by a (fixed time, in general, multicomponent) wave function $\psi(x_1, s_1; \ldots; x_n, s_n)$ ($\in \mathcal{H}_1^{\otimes n}$ where $\mathcal{H}_1$ is the 1-particle space). It possesses two types of symmetry which commute with each other: (i) the internal symmetry, described by the n-fold tensor product of fundamental representations of $U(k)$ acting on the variables s_i; (ii) symmetry under permutation of the pairs of arguments (x_i, s_i), reflecting the indistinguishability of identical particles. For 1-component wave function we have the Fermi[3]-Bose[4] alternative: ψ is either invariant or changes sign under

[1] Marius Sophus Lie (1812-1899) Norvegian mathematician.

[2] Heinz Hopf (1894-1971) introduces the concept of Hopf algebra (in a topological context) in 1941 – see references to the original papers in [C].

[3] The Italian (later American) physicist Enrico Fermi (1901-1954) did his work on the Fermi-Dirac statistics while in Florence (1925-26). He received the Nobel Prize in Physics in 1938 for his work on induced radioactivity.

[4] Satyendra Nath Bose (1894-1974) is an Indian Bengali mathematical physicist. His work of 1922 on the Bose statistics was first rejected and then only accepted for publication after the author sent his manuscript to Einstein who presented it together with his own paper on

transposition of two (pairs of) arguments. In general (for a multicomponent ψ), it should transform under an irreducible representation (IR) of the permutation groups. (This relates to the Schur-Weyl duality reviewed in Appendix A.)

All this is fine if the *configuration space* $X_n = Y_n/\mathcal{S}_n$, where Y_n is the space of points $(x_1, \ldots, x_n)$ such that $x_i \neq x_j$ for $i \neq j$, is simply connected and hence carries single valued analytic functions. This is the case for space dimensions larger than two. If the x_i are points in a 2-dimensional plane however then the configuration space X_n is no longer simply connected. We shall see that the natural generalization of $\mathcal{S}_n$ in this case is the braid group $\mathcal{B}_n$ on n strands that will be introduced in Section 2. What is important for us here is the realization that when the permutation group acting on the tensor product of, say $U(k)$, representations is substituted by the braid group then the condition that "the symmetry commutes with the statistics" implies that tensor product of representations should be deformed to a *coproduct* (in general, not co-commutative) meaning that the concept of a symmetry group should be substituted by the more general notion of a Hopf algebra or quantum group.

A bibliographical note.

The relation between the possible particle statistics and the topology (in particular, the fundamental group) of configuration space was first pointed out by Leinaas and Myrheim [LM]. For a thought provoking recent review of spin and quantum statistics – see [Fr].

There are, by now, a number of texts on quantum groups – see, *e.g.*, [CP] [FK] [K] [L] [M] [Ma].

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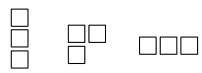
the same subject to Zeitschrift für Physik in 1924.

Appendix A. Young diagrams, Young tableaux and Schur-Weyl duality

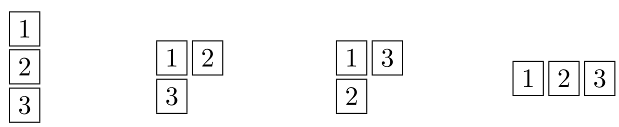
A *Young diagram*[5] of n boxes is a graphical expression of a *partition* of the natural number n into a sum of decreasing integers $n_1 \geq n_2 \geq \ldots n_k (\Sigma n_i = n)$. It consists of a finite number of boxes arranged in rows of decreasing length. All Young diagrams of three boxes are displayed on Figure A1



$\boldsymbol{Y}_a$ $\boldsymbol{Y}_b$ $\boldsymbol{Y}_c$

Figure A1: Young diagrams of three boxes

Young tableaux are Young diagrams in which each box carries a number. *Standard Young tableaux* of n boxes carry the numbers $1, \ldots, n$ of increasing order along rows and columns. There are four standard Young tableaux of three boxes displayed on Figure A2



Y_a Y_b^1 Y_b^2 Y_c

Figure A2: Standard Young tableaux of three boxes

Young diagrams Y of n boxes label the *irreducible representations* (IR) of the *symmetric group* $\mathcal{S}_n$ of permutations of n objects. The standard Young tableaux Y corresponding to a given diagram $\boldsymbol{Y}$ form a *basis* in the representation space of the IR $\boldsymbol{Y}$.

In general, the dimension of the representation corresponding to a Young diagram can be computed without writing down explicitely all Young tableaux corresponding to a given diagram $\boldsymbol{Y}$. To this end we shall introduce the *hook length* $h(x)$ of a box x of $\boldsymbol{Y}$. It is equal to the sum of the number of boxes to the right of x in the same row plus the number of boxes in the same column below x plus 1 (for x itself). In Figure A3 we give examples of hook lengths for two different diagrams

[5]The English mathematician Alfred Young (1873-1940) introduced these diagrams in 1900 while in Cambridge. For a systematic survey of Young tableaux and their applications – see [Fu].

$$\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 2 \\ \hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 & & \\ \cline{1-2}\end{array}$$

Figure A3: The number in each box gives its hook length

Then the dimension of $\boldsymbol{Y}$ is given by

$$\dim \boldsymbol{Y} = n!/\prod_{x\in Y} h(x)\,. \tag{A.1}$$

Exercise A.1. Find the dimension $d(\boldsymbol{Y})$ of the IRs of $\mathcal{S}_5$ and verify the formula $\sum\limits_{\boldsymbol{Y}} d^2(\boldsymbol{Y}) = 5!$.

To see how one reconstructs the action of the elements of $\mathcal{S}_n$ on a basis of Young tableaux, we will first say something more about the structure of the symmetric group.

$\mathcal{S}_n$ can be defined as a (finite) group of $(n-1)$ *generators* $s_1, \dots, s_{n-1}$ (where $s_i = P_{i\,i+1}$ plays the role of *transposition* (permutation) of the "objects" i and $i+1$), satisfying three sets of relations (the first of which tells us that the s_i are *reflections*):

$$s_i^2 = 1\,,\ i = 1, \dots, n-1\,;\ s_i\, s_j = s_j\, s_i \quad \text{for} \quad |i-j| > 1\,;$$
$$P_{i,i+2} = s_i\, s_{i+1}\, s_i = s_{i+1}\, s_i\, s_{i+1}\,,\ i = 1, \dots, m-2 \tag{A.2}$$

(P_{ij} playing the role of transposition of the objects (i,j) and satisfying $P_{ij}^2 = 1$; verify that indeed $P_{i\,i+2}^2 = 1$, as a consequence of (A.2)). Iterating the last relation (A.2) we can express any P_{ij}, $i \neq j$ as a word in the generators (of length $2\,|i-j| - 1$. If the indices i, j belong to a single column of the Young tableau Y then, by definition, $P_{ij}\, Y = -Y$. The permutation of two columns of equal lengths in a tableau Y leaves, by definition, Y invariant. The product of basic reflections determines the conjugacy class of the Coxeter element[6] of order n; in particular

$$c_{1n} := s_1 \dots s_{n-1} = c_{n1}^{-1} \quad (c_{n1} = s_{n-1} \dots s_1)\,, \quad c_{1n}^n = \mathbb{1}\,. \tag{A.3}$$

Exercise A.2. (i) Prove that the transposition P_{1n} has the form $P_{1n} = s_1 \dots s_{n-2}\, s_{n-1}\, s_{n-2} \dots s_1$; verify the relation $s_i\, P_{1n} = P_{1n}\, s_i$ for $i = 2, \dots, n-1$.
(ii) Prove (A.3). (*Hint*: use induction in n proving $(s_1 \dots s_n)^n = (s_1 \dots s_{n-1})^{n-1}\, P_{1n+1}$.)

[6] Harold Scott MacDonald Coxeter (1907-2003) born in London but worked for 60 years at the University of Toronto; he studied the product of generators in 1951 – see [C51].

Let us now describe, as a next exercise, the IRs of $\mathcal{S}_3$. The IRs $\boldsymbol{Y}_a$ and $\boldsymbol{Y}_c$ being 1-dimensional are easy to describe: $\boldsymbol{Y}_c$ is the trivial representation while $\boldsymbol{Y}_a$ is the *alternating* one: $s_1 Y_a = s_2 Y_a = -Y_A$. To construct the 2-dimensional representation $\boldsymbol{Y}_b$ we first note that the generators s_i are represented by 2×2 matrices of eigenvalues ± 1 (hence, $\det s_1 = \det s_2 = -1$, $\operatorname{tr} s_1 = \operatorname{tr} s_2 = 0$).

Exercise A.3. Using the relations $s_2 Y_b^1 = Y_b^2$, $s_2 Y_b^2 = Y_b^1$, $s_1 Y_b^2 = -Y_b^2$ find $s_1 Y_b^1$ and the matrix P_{13}.

($Answer:$ $s_1 Y_b^1 = Y_b^1 - Y_b^2$; $P_{13} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$.)

The IRs of the group $U(k)$ of complex unitary $k \times k$ matrices are again labeled by Young diagrams – of any number of boxes but of no more than k rows. The corresponding basis vectors can be represented by *semi-standard Young tableaux* in which the allowed numbers are $(1, \dots, k)$ that should increase monotoneously along rows and strictly along columns. Thus the representation corresponding to a single column of k boxes is 1-dimensional (given by the determinant). The IR associated with a single box is k-dimensional (with basis $\boxed{i}$, $i = 1, \dots, k$) and so is the representation of a single column of $(k-1)$-boxes.

Exercise A.4. Prove that the IR's of $U(k)$ corresponding to the 3-box diagrams of Figure A1 have dimensions $d_a = \binom{k}{3}$, $d_b = 2\binom{k+1}{3}$, $d_c = \binom{k+2}{3}$, respectively.

The Schur[7]-Weyl[8] theory concerns the decomposition of the n-fold tensor product of the defining (k-dimensional) representation $\Box$ of $U(k)$ into IRs of $U(k) \times \mathcal{S}_n$. (The permutations $s \in \mathcal{S}_n$ of different copies of the $U(k)$ module $\mathbb{C}^k$ commute with the $U(k)$ action.) We have

Proposition A.1. *Let $\boldsymbol{Y}$ run through the n-box Young diagrams with no more than k rows; then*

$$\Box_{U(k)}^{\otimes n} = \bigoplus_{\boldsymbol{Y}} \boldsymbol{Y}_{U(k)} \otimes \boldsymbol{Y}_{\mathcal{S}_n} \,. \tag{A.4}$$

In other words the representation $\Box_{U(k)}^{\otimes n}$ splits into a sum of tensor products of IRs of $U(k) \otimes \mathcal{S}_n$, the two IRs in each term corresponding to the *same Young diagram.*

[7] Issai Schur (January 10, 1875, Mogilov, Belarus, Russian empire – January 10, 1941, Tel Aviv, Palestine) studied and worked in Berlin; regarded himself as German and declined invitations to leave Germany for the US and Britain in 1934; dismissed from his chair in 1935 eventually emmigrated to Palestine in 1939. He is known for Schur's lemma and Schur's polynomials among many others.

[8] Hermann Weyl (1885, Elmshorn, near Hamburg – 1955, Zürich) worked in Göttingen, Zürich and Princeton. The duality in question appears in Weyl's 1928 book Gruppentheorie und Quantenmechanik. Concerning the Schur-Weyl duality – see, *e.g.*, [Z].

Exercise A.5. Using the result of Exercise A.4 verify that $k^3 = d_a + 2\,d_b + d_c$. Do the same exercise for $\square_{U(k)}^{\otimes 4}$ finding first the dimensions of all IRs of $\mathcal{S}_4$ and the dimensions of the IRs of $U(k)$ labeled by Young diagrams of four boxes.

2 Braid groups and Hecke algebras

In order to describe the fundamental group π_1 of the configuration space, we first introduce the Artin[9] *braid group* $\mathcal{B}_n$ *on n strands.* It is an infinite discrete group which can be defined in analogy with the symmetric group $\mathcal{S}_n$ (*cf.* Appendix A) as a group of $n-1$ generators $b_1, \ldots, b_{n-1}$ (and their inverses) obeying two sets of defining *braid relations* :

$$b_i\, b_{i+1}\, b_i = b_{i+1}\, b_i\, b_{i+1}\,,\ i = 1, \ldots, n-2\,; \quad b_i\, b_j = b_j\, b_i \quad \text{for} \quad |i-j| > 1\,. \tag{2.1}$$

Their intuitive meaning is illustrated on Figures. 2.1 and 2.2.

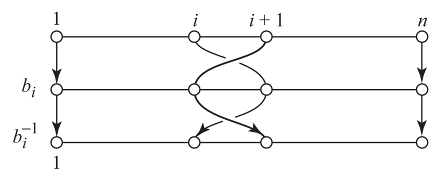


Figure 2.1: $b_i\, b_i^{-1} = 1\!\!1$

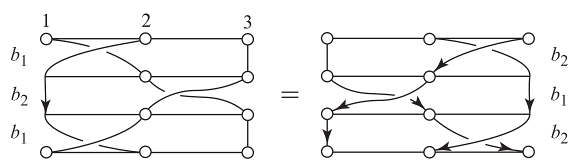


Figure 2.2: $b_1\, b_2\, b_1 = b_2\, b_1\, b_2$

If we ignore the path of a braid transformation and only follow its end point we obtain a permutation. This defines a homomorphism of $\mathcal{B}_n$ onto $\mathcal{S}_n$ whose kernel is, by definition, the *pure braid* or *monodromy* subgroup $\mathcal{M}_n$. This means that the following sequence of group homomorphisms is exact:

$$1 \to \mathcal{M}_n \to \mathcal{B}_n \to \mathcal{S}_n \to 1\,. \tag{2.2}$$

We are now ready to formulate a result that goes back to Hurwitz which has been then repeatedly rediscovered (see [M74] for a historical survey by an active participant in this work).

[9]Emil Artin (1898-1962), Theorie der Zöpfe (Hamburg, 1925). According to Wilhelm Magnus [M74] braid groups were implicit in Adolf Hurwitz's (1859-1919) work on monodromy (1891).

Let $(z_1, \dots, z_n)$ be n different points in the complex plane $\mathbb{C}$ and let z_0 be a fixed point in $\mathbb{C}$, different from $(z_1, \dots, z_n)$. Let $Y_n = \mathbb{C}\backslash(z_1, \dots, z_n)$ be the *n-punctured plane* and $X_n = Y_n/\mathcal{S}_n$ – the configuration space.

Theorem 2.1. [A] *The fundamental group of the configuration space $\pi_1(X_n, z_0)$ coincides with the braid group $\mathcal{B}_{n+1}$. The fundamental group of the n-punctured plane coincides with its monodromy subgroup, $\mathcal{M}_{n+1}$:*

$$\pi_1(X_n, z_0) \simeq \mathcal{B}_{n+1}\,, \quad \pi_1(Y_n, z_0) \simeq \mathcal{M}_{n+1}\,. \tag{2.3}$$

We just note that a path connecting two punctures, say z_i and z_{i+1}, is viewed as a closed path in X_n.

Introduce the analogues of the Coxeter elements (A.3):

$$B_{1n} = b_1 \dots b_{n-1}\,, \quad B_{n1} = b_{n-1} \dots b_1\,. \tag{2.4}$$

The powers of B_{1n} give rise to automorphisms that intertwine the $\mathcal{B}_n$ generators b_i among themselves:

$$B_{1n}^i\, b_1\, B_{1n}^{-i} = b_{i+1}\,, \quad i = 1, \dots, n-2\,. \tag{2.5}$$

We shall cite the following result on the structure of $\mathcal{B}_n$ (see [M74] for a concise review and references to the original papers).

Proposition 2.2. *The centre $Z_n = Z(\mathcal{B}_n)$ of $\mathcal{B}_n$ is generated by the element (of infinite order)*

$$\theta = B_{1n}^n \quad (= B_{n1}^n)\,; \tag{2.6}$$

θ and

$$\Omega = B_{1n}\, B_{n1} \text{ subject to the relation } \Omega^n = \theta^2 \tag{2.7}$$

give rise to a normal subgroup $\mathcal{N}_n$ of $\mathcal{B}_n$.

The fundamental group $\pi_1(\mathbb{S}^2, n)$ of the 2-sphere with n punctures can be presented as the quotient F_n^* of the free group F_n on n generators, $x_1, \dots, x_n$, by the single relation

$$x_1 \dots x_n = 1 \tag{2.8}$$

(expressing the fact that a loop encircling all n points on the sphere is contractible). The braid group $\mathcal{B}_n$ acts by automorphisms on F_n and on its quotient F_n^* ([M74]).

Proposition 2.3. *The automorphisms*

$$\beta_\nu(x_\nu) = x_{\nu+1}\,,\ \beta_\nu(x_{\nu+1}) = x_{\nu+1}^{-1}\, x_\nu\, x_{\nu+1}\,,\ \beta_\nu(x_\mu) = x_\mu \text{ for } \mu \neq \nu, \nu+1\,, \tag{2.9}$$

satisfy the defining relations (2.1) *for the generators of $\mathcal{B}_n$. Moreover, $\mathcal{B}_n$ is isomorphic to the automorphism group of F_n while its quotient with its centre gives* Aut F_n^**:*

$$\mathcal{B}_n \cong \mathrm{Aut}\, F_n\,,\ \mathcal{B}_n^* := \mathcal{B}_n/Z_n \cong \mathrm{Aut}\, F_n^*\,. \tag{2.10}$$

The mapping class group $M(\mathbb{S}^2, n)$ of $\mathbb{S}^2 \backslash \{z_1, \ldots, z_n\}$ – *i.e.*, the group of (isotopy classes of) orientation preserving self-homeomorphisms of the sphere with n-punctures – has been studied for nearly a century, starting with the work of Fricke-Klein[10] (1897 – following that of Hurwitz, mentioned above), followed by contributions by Artin, Magnus, Fadell, Van Buskirk, Arnold [A], Birman [B] among others. In the formulation of the main result below we follow the survey [M74] (containing over 60 references).

Theorem 2.4. *The braid group $\mathcal{B}_n(\mathbb{S}^2)$ of the 2-sphere arises from $\mathcal{B}_n$ by adjoining the single relation $\Omega = 1$ (where Ω is the generator of $\mathcal{N}_n$ defined in* (2.7)*). It has a single element θ* (2.6) *of order two (for $n > 2$) that generates its centre $\mathbb{Z}/2$. The mapping class group $M(\mathbb{S}^2, n)$ is obtained from $\mathcal{B}_n(\mathbb{S}^2)$ by setting $\theta = 1$.*

Remark 2.1. It follows from Theorem 2.4 that $\mathcal{B}_n(\mathbb{S}^2)$ is a non-splitting central extension of $M(\mathbb{S}^2, n)$ (just like $SU(2)$ is of $SO(3)$).

For quantum deformations of unitary (say, $SU(k)$) 1-particle symmetry (to be considered in Section 4, below) with deformation parameter q (such that $q = 1$ corresponds to the undeformed case) the group algebra of the fundamental representation of $\mathcal{B}_n$ is a *Hecke*[11] *algebra* characterized by the following relations

$$b_i^2 - (q - q^{-1})\, b_i - 1 = (b_i - q)(b_i + q^{-1}) = 0\,. \tag{2.11}$$

(The normalization of b_i has been chosen for convenience, so that the products of its eigenvalues is -1. Introducing in the next section a quasi-triangular R-matrix we shall naturally come to a different normalization which involves half-integer powers of the parameter q. In both cases, for $q \to 1$, Eq. (2.11) and its counterpart in Section 11 reduce to the involutivity condition for the reflections generating the symmetric group.)

It is convenient to express b_i in terms of the (non-normalized) projectors (antisymmetrizers) e_i:

$$e_i = q - b_i\,, \quad e_i^2 = (q + q^{-1})\, e_i\,, \tag{2.12}$$

which, in view of the braid relations (2.1) satisfy

$$e_i\, e_{i+1}\, e_i - e_i = e_{i+1}\, e_i\, e_{i+1} - e_{i+1} \tag{2.13}$$

$$e_i\, e_j = e_j\, e_i \quad \text{for} \quad |i - j| \geq 2\,. \tag{2.14}$$

[10]Karl Emmanuel Robert Fricke (1861-1930), professor of Higher Mathematics at the Technische Hochschule in Braunschweig, and Felix Christian Klein (1849-1925) (known for his influential Erlangen Program, 1872, and for his role in creating the model research centre at the University of Göttingen from 1886 on) wrote a four volume treatise on automorphic and elliptic modular functions over a period of about 20 years.

[11]Erich Hecke (1887-1947) studied in Göttingen with Hilbert, worked in Hamburg.

We shall introduce (also for later applications) the *q-numbers* $[n]$ $(\equiv [n]_q)$ setting

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad ([1] = 1\,,\ [2] = q + q^{-1}\,,\ [3] = q^2 + 1 + q^{-2}, \ldots)\,. \tag{2.15}$$

Exercise 2.1. Verify the relations (a) $[2][n] = [n-1] + [n+1]$, $[3][n] = [n-2] + [n] + [n+2]$; (b) if $q^N = -1$ then $[N] = 0$. For *generic* q, *i.e.* for q not a root of unity, $[N] \neq 0$ for any non-zero natural number N. Define for such q, following [GPS], the series of antisymmetrizers

$$P_-^1 = \mathbb{1}\,,\ P_-^{k+1} = \frac{1}{[k+1]}\left(q^k\,\mathbb{1} - q^{k-1}\,b_k + \ldots + (-1)^k\,b_1 \ldots b_k\right) P_-^k\,. \tag{2.16}$$

It is easy to check that

$$P_-^2 = \frac{e_1}{[2]}\,, \quad P_-^3 = \frac{1}{[3]!}\,(e_1\,e_2\,e_1 - e_1) = \frac{1}{[3]!}\,(e_2\,e_1\,e_2 - e_2) \tag{2.17}$$

where $[k]!$ is defined recursively by: $[0]! = 1$, $[k+1]! = [k]!\,[k+1]$ and $P_-^k = (P_-^k)_{12\ldots k}$.

Exercise 2.2. Prove that P_-^k are central projectors:

$$P_-^k\,b_i = b_i\,P_-^k = -q^{-1}\,P_-^k \quad \text{for} \quad 1 \leq i \leq k-1\,,$$

$$P_-^k\,P_{-j}^i = P_{-j}^i\,P_-^k = P_-^k \text{ for } P_{-j}^i = (P_-^i)_{j(j+1)\ldots(j+i-1)}\,,\ i + j - 1 \leq k\,; \tag{2.18}$$

furthermore,

$$P_-^k\,b_k\,P_-^k = \frac{q^k}{[k]}\,P_-^k - \frac{[k+1]}{[k]}\,P_-^{k+1} \tag{2.19}$$

(a relation that can be used as another recursive definition of P_-^k). It turns out that in applications to $2D$ conformal field theory q is precisely a root of 1. One has to deal with non-normalized projectors in that case.

We shall assume that the Hecke algebra of $\mathcal{B}_n$ is at most n-dimensional so that

$$\mathcal{P}_-^{n+1} = 0\,. \tag{2.20}$$

If k is the smallest positive integer for which $P_-^{k+1} = 0$ we say that we are dealing with an "even Hecke symmetry of rank k" in the terminology of [GPS]. Then P_-^k is an one-dimensional projector that can be written as a (tensor) product of two q-deformed Levi-Civita[12] tensors.

Assuming, on the other hand, that $P_-^3 = 0$ for $\mathcal{B}_\infty$ (*i.e.* that each of the expressions (2.13) vanish) we obtain the *Tempereley-Lieb algebra* which plays a prominent role in V.F.R. Jones theory of subfactors [J].

[12]Tullio Levi-Civita (1873-1941) published in 1900 "Méthodes de calcul différentiel absolu et leurs applications" together with his teacher Gregorio Ricci-Curbastro (1853-1925); Einstein used this book to master tensor calculus.

Here is an explicit realization of e_1 and e_2 (and hence of $\mathcal{B}_3$) satisfying

$$e_1\,e_2\,e_1 - e_1 = e_2\,e_1\,e_2 - e_2 = 0 \qquad (e_i^2 = [2]\,e_i) \tag{2.21}$$

in the triple tensor product of 2×2 matrices

$$e_{1\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} = \varepsilon^{\alpha_1\alpha_2}\,\varepsilon_{\beta_1\beta_2}\,\delta_{\beta_3}^{\alpha_3}\,,\ (e_2)_{\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} = \delta_{\beta_1}^{\alpha_1}\,\varepsilon^{\alpha_2\alpha_3}\,\varepsilon_{\beta_2\beta_3} \tag{2.22}$$

where $\varepsilon^{\alpha\beta}$ is the (rank 2) q-deformed Levi-Civita tensor

$$(\varepsilon^{\alpha\beta}) = \begin{pmatrix} 0 & -q^{1/2} \\ q^{-1/2} & 0 \end{pmatrix} = (\varepsilon_{\alpha\beta}) \tag{2.23}$$

satisfying

$$\varepsilon^{\alpha\sigma}\,\varepsilon_{\sigma\beta} = -\delta_\beta^\alpha\,,\ (\varepsilon^{\alpha\sigma}\,\varepsilon_{\beta\sigma}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\,. \tag{2.24}$$

Exercise 2.3. Verify (2.21) using (2.22) and the properties of $\varepsilon^{\alpha\beta}$.

Another solution of the braid relations of $\mathcal{B}_3$, in which the expression (2.13) is a 1-dimensional projector,

$$e_1\,e_2\,e_1 - e_1 = e_2\,e_1\,e_2 - e_2 = \varepsilon^{\alpha_1\alpha_2\alpha_3}\,\varepsilon_{\beta_1\beta_2\beta_3} \tag{2.25}$$

where

$$\varepsilon^{123} = -q^{3/2} = -q\,\varepsilon^{132} = -q\,\varepsilon^{213} = q^2\varepsilon^{312} = q^2\varepsilon^{231} = -q^3\varepsilon^{321}\,,$$

$$\varepsilon^{\alpha\alpha\beta} = 0 = \varepsilon^{\alpha\beta\alpha} = \varepsilon^{\beta\alpha\alpha}\,, \quad (\varepsilon^{\alpha\beta\gamma}) = (\varepsilon_{\alpha\beta\gamma})\,, \tag{2.26}$$

is given by

$$e_{1\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} = \varepsilon^{\alpha_1\alpha_2\sigma}\,\varepsilon_{\sigma\beta_1\beta_2}\,\delta_{\beta_3}^{\alpha^3}\,, \quad e_{2\beta_1\beta_2\beta_3}^{\alpha_1\alpha_2\alpha_3} = \delta_{\beta_1}^{\alpha_1}\,\varepsilon^{\sigma\alpha_2\alpha_3}\,\varepsilon_{\beta_2\beta_3\sigma}\,. \tag{2.27}$$

Exercise 2.4. Use the identity

$$\varepsilon_{\beta\sigma_1\sigma_2}\,\varepsilon^{\sigma_1\sigma_2\alpha} = [2]\,\delta_\beta^\alpha \tag{2.28}$$

to verify the relations $e_i^2 = [2]\,e_i$, $i = 1, 2$. Verify (2.25) for e_i given by (2.27).

Note that the order of indices of the quantum Levi-Civita tensor in (2.27) is important. It is easy to check, for instance that substituting the first product by $\varepsilon^{\alpha_1\alpha_2\sigma}\,\varepsilon_{\beta_1\beta_2\sigma}$ would violate the condition $e_1^2 = [2]\,e_1$.

We end up our brief survey of the braid group with a simple application to physically interesting new statistics in two dimensions.

The permutation group $\mathcal{S}_n$ has exactly two 1-dimensional representations (corresponding to the Young diagrams $\boldsymbol{Y}_a$ and $\boldsymbol{Y}_c$ of Figure A1 for $n = 3$ – see Appendix A): the fully symmetric (trivial) representation, corresponding to bosons and the totally antisymmetric one, describing fermions. By contrast the braid group $\mathcal{B}_n$ has a 1-parameter family of 1-dimensional (unitary) representations given by

$$\pi_q(b_i) = q \;(\pi_q(b_i^{-1}) = \bar{q}\,,\; q\,\bar{q} = 1)\,. \tag{2.29}$$

(Note that this representation trivially satisfy the Hecke algebra condition (2.11).) This representation describes (according to presently accepted theoretical models – see [FKST] [FST]) the fractional quantum Hall effect. The story of how physicists got aware of the anyonic representations is told in [BLSW].

3 Bialgebras and Hopf algebras: classical examples and definition

A natural way to arrive at the Hopf algebra generalization of the notion of a group G is to study the duality between an algebra U that could be either the *group algebra*, say $\mathbb{C}G$, or the *universal enveloping algebra* (UEA) $U(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G, and the algebra $\mathcal{F}(G)$ of functions on the group. (The appropriate topology of $\mathcal{F}(G)$ depends on the class of groups one is considering – see the introduction to [C]. As these introductory remarks are just ment as motivation we will not burden them with topological considerations.) We shall thus first explain why ordinary groups and Lie algebras can be viewed as Hopf algebras and only then will give the formal definition.

Remark 3.1. Mathematicians would often replace the field $\mathbb{C}$ of complex numbers in the definition of a group algebra by an arbitrary field K (having in mind, *e.g.*, applications to algebraic groups – see [C]). Such generality may also be useful for some physical applications but again we refrain from complicating excessively this introductory note. The space of functions $\mathcal{F}(G)$ is sometimes denoted by $\mathbb{C}^G$ (or K^G – see [C]).

Usually $\mathbb{C}G$ and $U(\mathcal{G})$ are just viewed as associative algebras. It is important, however, that U is a *bialgebra*, *i.e.* that it is also equipped with a *coalgebra structure* consisting of two algebra homomorphisms: the *coproduct* $\Delta : U \to U \otimes U$ and the counit $\varepsilon : U \to \mathbb{C}$ such that

$$(\mathbb{1} \otimes \Delta)\, \Delta = (\Delta \otimes \mathbb{1})\, \Delta \tag{3.1}$$

$$(\mathbb{1} \otimes \varepsilon)\, \Delta\, (X) = (\varepsilon \otimes \mathbb{1})\, \Delta\, (X) = X\,,\ \forall\, X \in U\,. \tag{3.2}$$

The product $m : U \otimes U \to U$ $(m(X \otimes Y) \equiv X \cdot Y)$ and the coproduct Δ should also satisfy a compatibility condition which will be formulated later. It is the presence of the coproduct which allows to view the tensor product of any two representations of U again as a representation of U (rather than as a representation of $U \otimes U$ which is always possible for an associative algebra). The coproduct in U is related to the pointwise product of functions $f(g)$ in $\mathcal{F}(G)$ by

$$(A, f_1\, f_2) = (\Delta(A), f_1 \otimes f_2) = \sum_{(A)} (A_1, f_1)(A_2, f_2) \tag{3.3}$$

where we are using Sweedler's notation[13]

$$\Delta(A) = \sum_{(A)} A_1 \otimes A_2 \quad \text{for} \quad A \in U\ (\Rightarrow A_1, A_2 \in U)\,. \tag{3.4}$$

Any element of $\mathbb{C}G$ is, by definition a finite linear combination of elements of G with complex coefficients:

$$A = \sum_g a(g)\, g \Rightarrow (A, f) = \sum_g a(g)\, f(g) \quad (\in\ \mathbb{C})\,. \tag{3.5}$$

[13]It is a self-explaining notation introduced by Moss E. Sweedler in his book *Hopf Algebras* (W.A. Benjamin, N.Y. 1969) of the pre-quantum groups' era.

Applying this to the left hand side of (3.3) with $(f_1\, f_2)(g) := f_1(g)\, f_2(g)$, and comparing the result with the right hand side we deduce

$$\Delta\, g = g \otimes g \Rightarrow \Delta\, A = \sum_g a(g)\, g \otimes g\,. \tag{3.6}$$

Both $\mathbb{C}G$ and $\mathcal{F}(G)$ are *unital* associative algebras; in other words, they have unit elements: the group unit $\mathbb{1} \in G \subset \mathbb{C}G$ and the constant function $f_0(g) = 1 \in \mathcal{F}(G)$. This allows to define a *counit* in both $\mathbb{C}G$ and $\mathcal{F}$, setting

$$\varepsilon(A) = (A, 1) = \sum_g a(g)\,, \quad i.e.\ \ \varepsilon(g) = 1 \in \mathbb{C}\,, \quad \varepsilon_{\mathcal{F}}(f) = (\mathbb{1}, f) = f(\mathbb{1})\,. \tag{3.7}$$

If G is a *finite group* we can define in this simple algebraic manner a coproduct in $\mathcal{F}(G)$ as well, setting

$$\Delta_{\mathcal{F}}\, f(g_1, g_2) = f(g_1\, g_2)\,. \tag{3.8}$$

Remark 3.2. Note that for a finite group G the tensor product $\mathcal{F}(G) \otimes \mathcal{F}(G)$ is naturally isomorphic to the space $\mathcal{F}(G \times G)$ of functions of two group variables. For G infinite the tensor square of $\mathcal{F}(G)$ is a proper subset of $\mathcal{F}(G \times G)$.

Exercise 3.1. Verify that the coproducts Δ (3.6) and $\Delta_{\mathcal{F}}$ (3.8) and the counits ε and $\varepsilon_{\mathcal{F}}$ (3.7) satisfy the coalgebra conditions (3.1) and (3.2).

We may view $\mathbb{C}G$ and $\mathcal{F}(G)$ as *Hopf algebras* by introducing in each of these bialgebras the antipode S:

$$S : \mathbb{C}G \to \mathbb{C}G\,,\ Sg = g^{-1}\,; \quad S_{\mathcal{F}} : \mathcal{F}(G) \to \mathcal{F}(G)\,,\ (S_{\mathcal{F}}\, f)(g) = f(g^{-1})\,. \tag{3.9}$$

In both cases S is defined as a linear *antihomomorphism* of algebra: $S(A_1\, A_2) = S(A_2)\, S(A_1)$.

To end up with our classical examples of a Hopf algebra we display Δ, ε, and S for the UEA $U(\mathcal{G})$ of a Lie algebra $\mathcal{G}$ defining them for elements of $\mathcal{G}$:

$$\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X\,,\ \varepsilon(X) = 0\,,\ S(X) = -X\,,\ \forall\, X \in \mathcal{G}\,. \tag{3.10}$$

We observe that the (associative) algebras $\mathbb{C}G$ and $U(\mathcal{G})$ are, in general, non-commutative but the coproduct in both case equals the *permuted* (or *opposite*) one

$$\Delta'(X) := \sum_{(x)} x_2 \otimes x_1 = \sum_x x_1 \otimes x_2 = \Delta(X)\,. \tag{3.11}$$

We say in such a case that the algebra U is *co-commutative.* By contrast, the algebra $\mathcal{F}(G)$ dual to $\mathbb{C}G$ is commutative but not co-commutative. Here is, finally, the abstract definition of a Hopf algebra (over an arbitrary field K) in which one demands neither commutativity nor co-commutativity.

Definition 3.1. *An associative unital algebra* $\mathcal{B}$ *with multiplication* m *and unit* $1\!\!1$ *is called a* bialgebra *if there are* unital algebra homomorphisms

$$\Delta : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}\,, \quad \Delta(X \cdot Y) = \Delta(X)\,\Delta(Y)$$

$$\varepsilon : \mathcal{B} \to \mathbb{C}\,,\ \varepsilon(XY) = \varepsilon(X)\,\varepsilon(Y)\,; \quad \Delta(1\!\!1) = 1\!\!1 \otimes 1\!\!1\,,\ \varepsilon(1\!\!1) = 1 \in \mathbb{C} \tag{3.12}$$

such that Δ *and* ε *satisfy the* coalgebra conditions (3.1), (3.2) *and compatibility between multiplication and co-multiplication:*

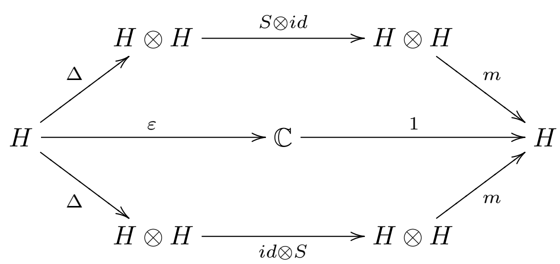
$$m \otimes m\, P_{23}\, \Delta(X) \otimes \Delta(Y) = \Delta(X \cdot Y)\,, \tag{3.13}$$

where P_{23} *stands for the permutation of the factors* 2 *and* 3 *in the* 4*-fold tensor product. In more detail, using* (3.4),

$$m \otimes m\, P_{23} \sum_{(X,Y)} (x_1 \otimes x_2 \otimes y_1 \otimes y_2) = \sum_{(X \cdot Y)} x_1 \cdot y_1 \otimes x_2 \cdot y_2 = \Delta(X \cdot Y)$$

(where we identify $x \cdot y$ *with* $m(x \otimes y)$*).*

Definition 3.2. *A* Hopf algebra H *is a bialgebra (over* $\mathbb{C}$*) equipped with a* $\mathbb{C}$*-linear antihomomorphism of algebras* $S : H \to H$*, the* antipode*, such that the following diagram commutes:*



(3.14)

Using the notation (3.4) *we can translate the content of* (3.14) *into the relation*

$$\sum_{(X)} S(x_1) \cdot x_2 = \sum_{(X)} x_1 \cdot S(x_2) = \varepsilon(X) 1\!\!1 \quad \forall\, X \in H\,. \tag{3.15}$$

4 Quantum universal enveloping algebras: the $U_q(A_r)$-case

An important example, in which neither commutativity nor co-commutativity holds, is given by the *quantum universal enveloping algebra* (QUEA) $U_q(\mathcal{G})$ of a (semi)simple Lie algebra $\mathcal{G}$. We shall spell out its definition for $\mathcal{G} = A_r = s\ell_{r+1}$ (the rank r Lie algebra of the special linear group of $(r+1)\times(r+1)$ matrices) and (complex) parameter $q \neq 0, \pm 1$. It combines the properties of $\mathbb{C}G$ and $U_q(\mathcal{G})$ being generated by a mixture of group like and Lie algebra like elements.

The QUEA $U_q(A_r)$ has r group-like generators K_i (and their inverses K_i^{-1}) which correspond to the *Cartan torus* and $2r$ Lie algebra-like ones (*raising and lowering operators*) E_i and F_i corresponding to simple roots. They obey the following commutation relations (CR):

$$K_i\, E_j\, K_i^{-1} = q^{(\alpha_i|\alpha_j)}\, E_j\ ,\ \ K_i\, F_j\, K_i^{-1} = q^{-(\alpha_i|\alpha_j)}\, F_j\ ,$$

$$[E_i, F_j] = \delta_{ij}\, \frac{K_i - K_i^{-1}}{q - q^{-1}}\ , \qquad i, j = 1, \ldots, r\, , \tag{4.1}$$

and the *Serre relations* (that are only non-trivial for $r > 1$):

$$E_i^{(2)}\, E_{i+1} + E_{i+1}\, E_i^{(2)} = E_i\, E_{i+1}\, E_i\ ,\ \ E_i\, E_{i+1}^{(2)} + E_{i+1}^{(2)}\, E_i = E_{i+1}\, E_i\, E_{i+1}\ ,$$

$$F_i^{(2)}\, F_{i+1} + F_{i+1}\, F_i^{(2)} = F_i\, F_{i+1}\, F_i\ ,\ \ F_i\, F_{i+1}^{(2)} + F_{i+1}^{(2)}\, F_i = F_{i+1}\, F_i\, F_{i+1}\ ,$$

for

$$X^{(n)} = \frac{1}{[n]!}\, X^n\ ;\ [E_i, E_j] = 0 = [F_i, F_j] \quad \text{for} \quad |i-j| > 1\, . \tag{4.2}$$

Here (in the first relation (4.1)) α_i are the *simple roots*, normalized to have square 2, so that $((\alpha_i \mid \alpha_j))$ is the A_r *Cartan*[14] *matrix* :

$$(\alpha_i \mid \alpha_i) = 2\ ,\ (\alpha_i \mid \alpha_{i+1}) = -1\ ,\ (\alpha_i \mid \alpha_j) = 0\ , \quad \text{for} \quad |i-j| > 1\, . \tag{4.3}$$

It is simple to display the "classical" ($q \to 1$) limit of these relations. Setting

$$K_i = q^{H_i} \qquad (K_i^{-1} = q^{-H_i})\ , \quad i = 1, \ldots, r \tag{4.4}$$

we find, at least formally, that the first two CR (4.1) are equivalent to the classical ones

$$[H_i, E_j] = (\alpha_i \mid \alpha_j)\, E_j\ , \quad [H_i, F_j] = -(\alpha_i \mid \alpha_j)\, F_j\ ,$$

while the third one has a classical limit:

$$[E_i, F_j] = [H_i]\, \delta_{ij} \quad (\to H_i\, \delta_{ij} \quad \text{for} \quad q \to 1) \tag{4.5}$$

[14]The French mathematician Élie Joseph Cartan (1869-1951) has introduced the general notion of antisymmetric differential forms (1894-1904) and the theory of spinors (1913) besides his major contribution to Lie algebras (his doctoral thesis of 1894) in which he completed Killing's work on the classification of semi-simple Lie algebras over the complex field.

where we have extended the notation $[n]$ (2.15) for a q-number to operator valued entries ($n \to H_i$). Note however, that Eq. (4.4) is not algebraic (it involves the exponential function). That's why purists only use $K_i^{(\pm 1)}$ in dealing with $U_q(A_r)$.

We define the coproduct, the counit, and the antipode on the generators of $U_q(A_r)$ as follows

$$\Delta(K_i) = K_i \otimes K_i \,, \quad \Delta(E_i) = E_i \otimes K_i + 1\!\!1 \otimes E_i$$

$$\Delta(F_i) = F_i \otimes 1\!\!1 + K_i^{-1} \otimes F_i \,, \quad i = 1, \ldots, r \,; \tag{4.6}$$

$$\varepsilon(K_i) = 1 \,, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0 \,; \tag{4.7}$$

$$S(E_i) = -E_i\, K_i^{-1} \,, \quad S(F_i) = -K_i\, F_i \,, \quad S(K_i) = K_i^{-1} \,. \tag{4.8}$$

Exercise 4.1. Verify (2.25) which can also be written in the form

$$m(1 \otimes S)\, \Delta(X) = m(S \otimes 1)\, \Delta(X) = \varepsilon(X)\, 1\!\!1 \tag{4.9}$$

for the generators of $U_q(A_r)$.

Exercise 4.2. Verify (on the generators) the relation $(\varepsilon \otimes 1)\, \Delta(X) = (1 \otimes \varepsilon)\, \Delta(X) = X$ (3.15).

Remark 4.1. It helps understanding both the origin and the meaning of quantum groups to observe that it is the coproduct that determines the deformation of the Lie algebra structure. To this end we note that the fundamental (undeformed) representation of A_r, given in terms of the Weyl matrices

$$(e_{ij})^k_\ell = \delta^k_i\, \delta_{j\ell} \quad (\Rightarrow e_{ij}\, e_{k\ell} = \delta_{jk}\, e_{i\ell}) \qquad i, j, k, \ell = 1, \ldots, r+1 \,, \tag{4.10}$$

by

$$E_i = e_{i\,i+1} \,, \quad F_i = e_{i+1\,i} \,, \quad H_i = e_{ii} - e_{i+1\,i+1} \,, \tag{4.11}$$

is also a representation of $U_q(A_r)$; in particular,

$$[E_i, F_j] = \delta_{ij} [H_i] = \delta_{ij}\, H_i \quad \text{for} \quad H_i = e_{ii} - e_{i+1\,i+1} \,. \tag{4.12}$$

More generally, if H is a hermitian matrix with eigenvalues $0, \pm 1$ then $[H] = H$. Furthermore, the Serre relations (4.2) (which involve the deformation parameter q) are also satisfied by the q-independent matrices of the defining $(r+1)$-dimensional representation of A_r since each term is separately equal to zero:

$$\begin{aligned} &E_i^2 = 0 = F_i^2 = E_i\, E_{i+1}\, E_i = F_i\, F_{i+1}\, F_i \\ &\text{for} \quad E_i = e_{i\,i+1} \,, \quad F_i = e_{i+1\,i}\ (= E_i^*) \,. \end{aligned} \tag{4.13}$$

It is the non-cocommutative coproduct which replaces the symmetric tensor product of representations, that yields modified higher dimensional representations and forces us to use the q-deformed CR (4.1) (4.2). (As we shall see below the coproduct is also directly related to the appearance of braid group representations in the q-deformed Schur-Weyl duality.)

It is instructive to see how 2- and higher point $U_q(A_r)$ invariants appear in tensor products of finite dimensional representations. We shall work out the solution to this problem for the simplest case of $U_q(A_1) \equiv U_q$.

For *generic* q ($q \neq 0, q$ not a root of unity) the theory of finite dimensional representations of U_q is essentially the same as that of the undeformed algebra $A_1 \simeq su(2)$. The irreducible representations (IRs) of U_q are again labeled by the isospin I, or by the dimension $p := 2I + 1$. An explicit realization of the U_q module $\mathcal{F}_p$ is given in terms of the *weight basis* $\{q^{\alpha_{pm}} \mid p, m\rangle\}$ for any choice of the (integer) exponents α_{pm}. Instead of the q-deformed Casimir operator $C_2^{(q)} := EF + FE + [2] \left[\frac{H}{2}\right]^2$ it is more convenient to use its rescaled version: $C = (q - q^{-1})^2 \, C_2^{(q)} + [2]$:

$$C := \lambda^2 EF + q^{H-1} + q^{1-H} = \lambda^2 FE + q^{H+1} + q^{-H-1} \,, \; \lambda := q - q^{-1} \quad (4.14)$$

we have, as part of the definition of the weight basis

$$(C - q^p - q^{-p}) \, q^{\alpha_{pm}} \mid p, m\rangle = 0 = (q^H - q^{2m-p+1}) \, q^{\alpha_{pm}} \mid p, m\rangle \,. \quad (4.15)$$

Exercise 4.3. Check that (4.14) (4.15) imply

$$(EF - [m][p-m]) \, q^{\alpha_{pm}} \mid p, m\rangle = 0 = (FE - [m+1][p-m-1]) \, q^{\alpha_{pm}} \mid p, m\rangle \quad (4.16)$$

(independent of the choice of α_{pm}).

We shall single out the *(real) canonical basis* $\{\mid p, m\rangle\}$ by the (α_{pm}-dependent) relations

$$E \mid p, m\rangle = [p - m - 1] \mid p, m + 1\rangle \,, \quad F \mid p, m\rangle = [m] \mid p, m - 1\rangle \,. \quad (4.17)$$

It follows, in particular, that $\mathcal{F}_p$ has both a lowest and a highest weight vector, $\mid p, 0\rangle$ and $\mid p, p - 1\rangle$, a property that is independent of the choice of α_{pm}:

$$E \, q^{\alpha_{p\,p-1}} \mid p, p - 1\rangle = 0 = F \, q^{\alpha_{p0}} \mid p, 0\rangle \,. \quad (4.18)$$

Another remarkable weight basis, that will be used shortly, is what we shall call an *E-basis*, $\{\mid p, m)\}$, for which

$$E \mid p, m) = (p - m - 1)_+ \mid p, m + 1) \,, \quad F \mid p, m) = q^{2-p} (m)_+ \mid p, m - 1) \,; \quad (4.19)$$

here the (complex for $q \, \bar{q} = 1$) q-numbers $(n)_+$ and $(n)_-$ (that will appear later) are defined by

$$(n)_+ := [n] \, q^{n-1} = 1 + q^2 + \cdots + q^{2n-2} = \frac{1 - q^{2n}}{1 - q^2} \,, \quad (4.20)$$

$$(n)_- = [n]\, q^{1-n} = 1 + q^{-2} + \cdots + q^{2-2n}\,.$$

Exercise 4.4. Verify that the vectors

$$\mid p, m) = q^{\alpha_{p,m}} \mid p, m\rangle\,, \quad \text{with} \quad \alpha_{pm} = \frac{m(m+3)}{2} - mp \tag{4.21}$$

satisfy (4.19) as a consequence of (4.17).

The E-basis allows to introduce U_q *coherent states* [FST] which are vector valued polynomials of degree $p - 1 = 2I$ in a formal variable u:

$$\Phi_I(u) := \sum_{m=0}^{2I} \binom{2I}{m}_+ u^m \mid 2I+1, m)\,, \quad \binom{n}{m}_+ = \frac{(n)_+!}{(m)_+!\,(n-m)_+!} \tag{4.22}$$

($(0)_+! = (1)_+! = 1$, $(n+1)_+! = (n)_+!\,(n+1)_+$).

Exercise 4.5. Verify the relation

$$(E - D_+)\, \Phi_I(u) = 0 \quad \text{for} \quad (D_\pm\, f)(u) = \frac{f(q^{\pm 2}\, u) - f(u)}{(q^{\pm 2} - 1)\, u}\,. \tag{4.23}$$

(*Hint* : use the relation $D_\pm\, u^m = (m)_\pm\, u^{m-1}$.)

A function $J^{(I)}(u_1, \dots, u_n)$ on the n-fold tensor product of $\Phi_I(u)$'s is U_q invariant if it is homogeneous – as a consequence of $K(= q^H)$ invariance,

$$K : q^{-2nI}\, J^{(I)}(q^2\, u_1, \dots, q^2\, u_n) = J^{(I)}(u_1, u_2, \dots, u_n)\,, \tag{4.24}$$

and E- and F-invariant:

$$E : \sum_{k=1}^{n} D_{k+}\, J^{(I)}(u_1, \dots, u_k, q^2\, u_{k+1}, \dots, q^2\, u_n)\, q^{2I(k-n)} = 0 \tag{4.25}$$

$$F : \sum_{k=1}^{n} u_k^{2I+2}\, q^{2I(k-1)}\, D_{k-}(u_k^{-2I}\, J^{(I)}(q^{-2}\, u_1, \dots, q^{-2}\, u_{k-1}, u_k, \dots, u_n)) = 0\,. \tag{4.26}$$

Exercise 4.6. Prove using just (4.24) and (4.25) that the general 2-point invariant is proportional to

$$J^{(I)}(u_1, u_2) = w_{2I}\left(u_1, u_2; \frac{1}{2}\right) = \prod_{n=0}^{2I-1} (q^{I-n}\, u_1 - q^{n-I}\, u_2) \tag{4.27}$$

$$= q^{-I} \sum_{m=0}^{2I} \begin{bmatrix} 2I \\ m \end{bmatrix} (q\, u_1)^m\, (-u_2)^{2I-m}\,,$$

where

$$w_k(u,v;\rho) = w_k(q^\rho\, u, q^{-\rho}\, v) = \prod_{\nu=0}^{k-1} \left(q^{\rho+\frac{k-1}{2}-\nu}\, u - q^{\nu-\rho-\frac{k-1}{2}}\, v \right) ,$$

$$w_k(x,y)(= w_k(x,y;0)) := \sum_{n=0}^{k} \begin{bmatrix} k \\ n \end{bmatrix} x^n (-y)^{k-n} = \prod_{\nu=0}^{k-1} (x - q^{k-1-2\nu}\, y)\,. \quad (4.28)$$

Proposition 4.1. *The space of* 4*-point* U_q*-invariants in the tensor product* $\mathcal{F}_p^{\otimes 4}$ *is* p*-dimensional and is spanned by*

$$\begin{aligned} J_\lambda^{(I)}(u_1,\dots,u_4) \quad = \quad & w_{2I-\lambda}\left(u_1,u_2;\frac{\lambda+1}{2}\right) w_{2I-\lambda}\left(u_3,u_4;\frac{\lambda+1}{2}\right) \\ & w_\lambda\left(u_2,u_3;I-\frac{\lambda-1}{2}\right) w_\lambda\left(u_1,u_4;\frac{\lambda+1}{2}-I\right) \\ & \lambda = 0,1,\dots,2I(=p-1)\,. \end{aligned} \quad (4.29)$$

(For a proof see Appendix B. A more general result is established in [FST].)

Clearly, the QUEA $U_q(A_r)$ is neither commutative nor co-commutative. The violation of co-commutativity however is not arbitrary: $U_q(A_r)$ is an *almost co-commutative quasi-triangular Hopf algebra*.

A Hopf algebra H is called *almost co-commutative* if there exists an invertible element R of $H\otimes H$ which intertwines the coproduct Δ (3.2) with its permuted one, Δ' (3.13)

$$R\Delta(X) = \Delta'(X)R\,, \qquad \forall\, X\in H\,. \quad (4.30)$$

It is called quasi-triangular if R satisfies, in addition

$$(\Delta\otimes 1\!\!1)(R) = R_{13}\, R_{23}\,, \quad (1\!\!1\otimes\Delta)(R) = R_{13}\, R_{12}\,. \quad (4.31)$$

Here we are using Faddeev's notation (see, *e.g.*, [FRT]) R_{ij} for the action of R on the triple tensor product $H\otimes H\otimes H$: define the algebra morphisms $\phi_{ij}: H\otimes H\to H\otimes H\otimes H$ ($i,j=1,2,3,\ i\neq j$) by

$$\phi_{12}(a\otimes b) = a\otimes b\otimes 1\,, \quad \phi_{23}(a\otimes b) = 1\otimes a\otimes b\,,$$

$$\phi_{13}(a\otimes b) = a\otimes 1\otimes b\,; \quad \text{then} \quad R_{ij} := \phi_{ij}\, R\,. \quad (4.32)$$

Applying $\varepsilon\otimes 1\!\!1$ to both sides of the first equation (4.31) and $1\!\!1\otimes\varepsilon$ to the second one and using Eq. (3.2) we find

$$(\varepsilon\otimes 1)\, R = (1\otimes\varepsilon)\, R = 1\,. \quad (4.33)$$

Exercise 4.7. Prove, using quasi-triangularity, the relations

$$R^{-1} = (S\otimes 1)\, R\,, \quad R = (1\otimes S)\, R^{-1} = (S\otimes S)\, R\,. \quad (4.34)$$

The quasi-triangularity further implies the *Yang-Baxter equation* (YBE) for the R matrix:

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12}\,. \tag{4.35}$$

A natural construction of the R-matrix, which allows to verify the above properties, requires yet another concept, the *quantum double*, and is given in Section 5 below. Here we shall reproduce instead a simple example of a space of non-commutative matrices which allows to understand the meaning of the YBE (4.35) and its connection to the basic relation among braid group generators.

Let $T = (T^\alpha_\beta\,,\ \alpha, \beta = 1, \ldots, n)$ be an $n \times n$ matrix whose entries do not commute but obey the *RTT relation* (see [FRT])

$$R_{12}\, T_1\, T_2 = T_2\, T_1\, R_{12} \quad \text{where} \quad T_1 = T \otimes \mathbb{1}\,,\ T_2 = \mathbb{1} \otimes T\,. \tag{4.36}$$

Natural examples of such T-matrices are provided by the Borel components of $U_q(A_r)$ (see Section 5). We then apply both sides of (4.35) to the triple product $T_1\, T_2\, T_3$:

$$\begin{aligned} & R_{12}\, R_{13}\, R_{23}\, T_1\, T_2\, T_3 = R_{12}\, R_{13}\, T_1\, T_3\, T_2\, R_{23} = \\ = \;& R_{12}\, T_3\, T_1\, T_2\, R_{13}\, R_{23} = T_3\, T_2\, T_1\, R_{12}\, R_{13}\, R_{23}\,; \end{aligned}$$

$$\begin{aligned} & R_{23}\, R_{13}\, R_{12}\, T_1\, T_2\, T_3 = R_{23}\, R_{13}\, T_2\, T_1\, T_3\, R_{12} = \\ = \;& R_{23}\, T_2\, T_3\, T_1\, R_{13}\, R_{12} = T_3\, T_2\, T_1\, R_{23}\, R_{13}\, R_{12}\,, \end{aligned} \tag{4.37}$$

where we have used the relations $R_{23}\, T_1 = T_1\, R_{23}$, $R_{13}\, T_2 = T_2\, R_{13}$, $R_{12}\, T_3 = T_3\, R_{12}$. Thus both sides of (4.35) when commuted with $T_1\, T_2\, T_3$ intertwine it with $T_3\, T_2\, T_1$, permuting the T_i in different order. Eq. (4.35) thus reflects the associativity of multiplication (of the elements) of T-matrices. (More on this interpretation of the YBE the reader will find in [Ma].)

Eq. (4.35) reminds us the Artin braid relation (2.1). To obtain the exact relation between the two we multiply both sides of (4.35) by the product of permutations $P_{12}\, P_{23}\, P_{12} = P_{13} = P_{23}\, P_{12}\, P_{23}$ and set

$$P_{i\, i+1}\, R_{i\, i+1} = \hat{R}_{i\, i+1}\ (= b_i)\,. \tag{4.38}$$

This gives for the left hand side of (4.35)

$$\begin{aligned} & P_{23}\, P_{12} (P_{23}\, R_{12}\, P_{23}) (P_{23}\, R_{13}\, P_{23})\, \hat{R}_{23} = P_{23}\, P_{12}\, R_{13}\, P^2_{12}\, R_{12}\, \hat{R}_{23} = \\ = \;& P_{23}\, R_{23}\, \hat{R}_{12}\, \hat{R}_{23} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23}\,, \end{aligned}$$

where we have used $P_{23}\, R_{13}\, P_{23} = R_{12}$ etc. Similarly, the right hand side of (4.35) is reduced to $\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12}$. Thus the YBE (4.15) is equivalent to the braid relation

$$\hat{R}_{12}\, \hat{R}_{23}\, \hat{R}_{12} = \hat{R}_{23}\, \hat{R}_{12}\, \hat{R}_{23} \tag{4.39}$$

for $\hat{R}_{i\, i+1}$ given by (4.38).

Appendix B. General form of n-point U_q-invariants

We shall construct in this Appendix a privileged basis of n-point (in particular, 4-point) U_q-invariants – something peculiar for the q-deformed case. We shall provide on the way the main ingredients of the proof of Proposition 4.1. We first comment on the meaning of the U_q-invariants (4.27)–(4.29) comparing them with the corresponding $SU(2)$ invariants. Then we discuss the role of the different U_q-invariance conditions (4.24)–(4.26) – proving on the way Proposition 4.1. Finally, we comment on the properties which distinguish the basis (4.29) of 4-point invariants. The Appendix may be viewed as a pedagogical introduction to the paper [FST] which gives a system of n-point U_q-invariants in the tensor product of irreducible U_q-modules $\mathcal{F}_{p_i}$ corresponding to different isospins I_i (and dimensions $p_i = 2I_i + 1$) (see Proposition B.1 below).

The basic 2-point invariant with respect to $SU(2)$ is the skew symmetric tensor $\epsilon^{AB} = -\epsilon^{BA}$ $(A, B = 1, 2)$. If we introduce the undeformed coherent states $\Phi_I(\zeta)$ (obtained from (4.22) in the limit $q \to 1$ – *cf.* the book [P]), it is given (for $I = \frac{1}{2}$) by the difference $\zeta_{12} = \zeta_1 - \zeta_2$ of formal variables. Its generalization to higher isospins I is nothing but the power ζ_{12}^{2I}. The product $J^{(I)}(u_1, u_2)$ (4.27) is a deformation of this simple monomial and can be obtained as follows.

K invariance (4.24) (for generic q) implies that $J^{(I)}(u_1, u_2)$ is a homogeneous polynomial in u_1, u_2 of degree $2I$:

$$J^{(I)}(u_1, u_2) = \sum_{m=0}^{2I} a_{Im}\, u_1^m (-u_2)^{2I-m}\,.$$

Applying to it the condition (4.25) of E invariance we find the recursive relation

$$(m)_+\, q^{2I-2m}\, a_{Im} = (2I - m + 1)_+\, a_{Im-1} \Leftrightarrow a_{Im} = \frac{[2I - m + 1]}{[m]}\, q\, a_{Im-1}\,.$$

Solving the recurrence for $a_0 = q^{-I}$ we obtain the right hand side of (4.27). In verifying F-invariance of the 2-point function so obtained one uses the identity $(-n)_- = -q^2 (n)_+$.

A 3-point invariant in the tensor product of three U_q-modules $\boldsymbol{I}_1, \boldsymbol{I}_2, \boldsymbol{I}_3$ of isospins I_1, I_2, I_3 only exists if $\boldsymbol{I}_3$ enters the tensor product expansion of $\boldsymbol{I}_1 \otimes \boldsymbol{I}_2$,

$$|I_1 - I_2| \le I_3 \le I_1 + I_2\,, \quad I_1 + I_2 - I_3 \in \mathbb{N}\,, \tag{B.1}$$

and then it is unique (up to normalization). Similar existence conditions hold for n-point invariants which may depend on at most $n-3$ (discrete) parameters. In particular, there are $p = 2I + 1$ 4-point invariants J_λ, $\lambda = 0, 1, \ldots, 2I$ in the 4-fold tensor product $\boldsymbol{I}^{\otimes 4} (= \mathcal{F}_p^{\otimes 4})$. The expressions (4.29) clearly obey the homogeneity condition (4.24). In order to verify that they are also E-invariant one uses the relations

$$D_{1+}\, w_k(q^\rho\, u_1, q^{-\rho}\, u_2) = q^{\rho + \frac{k-1}{2}}\, [k]\, w_{k-1}(q^{\rho+\frac{1}{2}}\, u_1, q^{-\rho-\frac{1}{2}}\, u_2)$$

$$D_{2+}\, w_k(q^\rho\, u_1, q^{-\rho}\, u_2) = -q^{-\rho-\frac{k-1}{2}}\, [k]\, w_{k-1}(q^{\rho-\frac{1}{2}}\, u_1, q^{\frac{1}{2}-\rho}\, u_2)\,. \qquad \text{(B.2)}$$

The following more general result is established in [FST].

Proposition B.1. *There exists a basis* $J^{(I_1,\dots,I_n)}_{\{\mu_{ij}\}}$ *of* U_q*-invariant monomials in the* n*-fold tensor product* $\boldsymbol{I}_1 \otimes \boldsymbol{I}_2 \otimes \ldots \otimes \boldsymbol{I}_n$*,*

$$J^{(I_1,\dots,I_n)}_{\{\mu_{ij}\}} = \prod_{1\le i<j\le n} w_{ij}\,, \quad w_{ij} = w_{k_{ij}}(u_i, u_j; \rho_{ij}) \qquad \text{(B.3)}$$

where $w_k(u,v;\rho)$ *is defined by* (4.28)*. The parameters* $k_{ij}(=k_{ji})$ *and* ρ_{ij} *have to satisfy (as a consequence of the invariance of* (B.3)*) the following conditions:*

$$\sum_j k_{ij} = 2\, I_i \qquad (k_{ii}=0)\,, \qquad \text{(B.4)}$$

$$k_{ij}\, k_{\ell m} = 0 \quad \textit{for} \quad i<\ell<j<m \quad (\textit{or} \;\; \ell<i<m<j)\,; \qquad \text{(B.5)}$$

if $k_{ij}>0$*, then*

$$\rho_{ij} + \frac{1}{2}\, k_{ij} = \sum_{s=i+1}^{j} I_s - \sum_{\substack{i\le\ell<m\le j \\ (\ell,m)\neq(i,j)}} k_{\ell m}\,, \qquad 1\le i<j\le n\,. \qquad \text{(B.6)}$$

There are $n-3$ *(integer valued) parameters among the* k_{ij} $(0\le k_{ij}\le \min(2I_i, 2I_j))$ *which label the general solution of* (B.3)–(B.6).

Exercise B.1. For $n=4$, $I_1=I_2=I_3=I_4=:I$ set $k_{14}=\lambda$ and determine the remaining k_{ij} that reproduce the solution (4.29).

We observe that the *selection rule* (B.5) has no counterpart in the undeformed case. There are, so to speak, fewer U_q- than $SU(2)$-invariants. The basis of invariant monomials expressed as products of elementary 2-point invariants is essentially unique. The invariants (4.29) (in contrast to other choices used in the literature) are well defined and linearly independent also for q a root of unity.

5 Quantum Gauss decomposition and the Drinfeld double

Every element g of a dense open neighbourhood of the group unit of the general linear group $\mathrm{GL}(n,\mathbb{C})$ admits a *Gauss decomposition* $g = b \cdot f$, where b is a lower triangular matrix while f is upper triangular with units on the diagonal. This corresponds to splitting of the Lie algebra into a Borel subalgebra generated (in the $s\ell_n$ case) by E_i and H_i and a nilpotent one, generated by F_i. In the q-deformed case, we see that such a splitting does not lead to Hopf subalgebras, since $\Delta(F_i)$ (4.4) also involves $K_i^{-1} (= q^{-H_i})$. A way out is to include the diagonal (Cartan) elements in both parts of the decomposition and then impose a relation among them. This allows to introduce the notion of quantum double and yields a streamlined construction of a universal quasi-triangular R-matrix. We shall outline this construction for the rank one case, $U_q(A_1)$ (the QUEA generated by K, E, F).

We introduce a pair of *quantum Borel*[15] *subalgebras* $U_q\, b_\pm$ of two generators each: (k, E) and $\tilde{k}, F$, such that $k\, \tilde{k} = K$ satisfying

$$k\, E = q\, E\, k\,, \qquad F\, \tilde{k} = q\, \tilde{k}\, F \tag{5.1}$$

and the mixed relations

$$[k, \tilde{k}] = 0\,, \quad F\, k = q\, k\, F\,, \quad [E, F] = \frac{k^2 - \tilde{k}^{-2}}{q - q^{-1}}\,. \tag{5.2}$$

(Ultimately, we shall set $k = \tilde{k}$.)

Introduce the triangular matrices

$$M_- = \begin{pmatrix} k & 0 \\ \lambda\, k^{-1} E & k^{-1} \end{pmatrix}\,, \quad M_+ = \begin{pmatrix} \tilde{k}^{-1} & -\lambda\, F\, \tilde{k} \\ 0 & \tilde{k} \end{pmatrix}\,, \quad \lambda = q - q^{-1}\,. \tag{5.3}$$

Exercise 5.1. Verify that the CR (5.1) are equivalent to the "RTT relations"

$$R_{12}\, (M_\pm)_1\, (M_\pm)_2 = (M_\pm)_2\, (M_{\pm 1})\, R_{12} \tag{5.4}$$

where R is expressed in terms of the Weyl matrices, $(e_{ij})^\alpha_\beta = \delta^\alpha_i\, \delta_{j\beta}$, as follows:

$$\begin{aligned} R &= q^{\frac{1}{2}} (e_{11} \otimes e_{11} + e_{22} \otimes e_{22}) + q^{-\frac{1}{2}} (e_{11} \otimes e_{22} + e_{22} \otimes e_{11} + \lambda\, e_{21} \otimes e_{12}) \\ &= q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\,. \end{aligned} \tag{5.5}$$

[15] The Swiss mathematician Armand Borel (1923-2003) is one of the creators of the theory of linear algebraic groups.

The coproduct in each $U_q\, b_\pm$ defined by

$$(\Delta\, M_\epsilon)^\alpha_\beta = M^\alpha_{\epsilon\,\sigma} \otimes M^\sigma_{\epsilon\,\beta}\,, \quad \epsilon = \pm\,, \tag{5.6}$$

is equivalent to

$$\Delta(k) = k \otimes k\,, \quad \Delta(E) = E \otimes k^2 + 1 \otimes E\,,$$
$$\Delta(\tilde{k}) = \tilde{k} \otimes \tilde{k}\,, \quad \Delta(F) = F \otimes 1 + \tilde{k}^{-2} \otimes F\,. \tag{5.7}$$

Similarly, the counit and the antipode acquire the form they have in the group algebra (Section 3) when expressed in terms of the matrices (5.3):

$$\varepsilon(M^\alpha_{\epsilon\,\beta}) = \delta^\alpha_\beta\,, \quad S(M^\alpha_{\epsilon\,\beta}) = (M_\epsilon^{-1})^\alpha_\beta\,. \tag{5.8}$$

The introduction of the pair of Borel Hopf algebras, $U_q\, b_\pm$, is justified by the following two facts.

(i) *There exists a unique bilinear pairing*

$$\langle Y, X\rangle \quad (\in \mathbb{C} \;\; \textit{for} \;\; X \in U_q\, b_-\,,\; Y \in U_q\, b_+)$$

such that

$$\langle Y Y', X\rangle = \langle Y \otimes Y', \Delta(X)\rangle = \sum_{(X)} \langle Y, X_1\rangle\, \langle Y', X_2\rangle$$

for

$$Y, Y' \in U_q\, b_+\,, \quad \Delta(X) = \sum_{(X)} X_1 \otimes X_2 \in U_q\, b_- \otimes U_q\, b_-\,, \tag{5.9}$$

$$\langle \Delta(Y), X \otimes X'\rangle = \sum_{(Y)} \langle Y_1, X\rangle\, \langle Y_2, X'\rangle = \langle Y, X' X\rangle \tag{5.10}$$

$$\langle 1\!\!1, X\rangle = \varepsilon(X)\,, \quad \langle S(Y), X\rangle = \langle Y, S^{-1}(X)\rangle\,, \quad \varepsilon(Y) = \langle Y, 1\!\!1\rangle\,. \tag{5.11}$$

It is given by

$$\langle \tilde{k}^m\, F^\mu, e_{n\nu}\rangle = \delta_{\mu\nu}\, \frac{[\mu]!}{\lambda^\mu}\; q^{\frac{mn - \mu(\mu-1)}{2}} \tag{5.12}$$

where $\{e_{n\nu}\}$ *is a Poincaré*[16]*-Birkhoff-Witt*[17] (PBW) *basis in* $U_q\, b_-$*:*

$$e_{n\nu} := k^n\, E^\nu\,. \tag{5.13}$$

[16] Jules-Henri Poincaré (1854-1912), more than anybody else may be called the prophet of 20$^{\text{th}}$ century mathematics. He is the founder of topology (called by him analysis situs), introducing, in particular, the concept of fundamental group, used in Section 2. He preceded Einstein in analyzing the relativity of time and simultaneity. Poincaré stated the PBW theorem in 1900.

[17] Garrett Birkhoff (1911-1996) son of the Harvard mathematician George David Birkhoff (1884-1944) is known for his contributions to abstract algebra. He and the German mathematician Ernst Witt (1911-1991) published independent proofs of Poincaré's statement in 1937.

The mixed relations (5.2) *are recovered provided the product* XY *is constrained by*

$$XY(\cdot) = \sum_{(X)} Y(S^{-1}(X_3)\cdot X_1)\, X_2 \tag{5.14}$$

for

$$\Delta^{(2)}(X) = (1\otimes\Delta)\,\Delta(X) = (\Delta\otimes 1)\,\Delta(X) = \sum_{(X)} X_1\otimes X_2\otimes X_3\,. \tag{5.15}$$

The dot $(\cdot)$ *in* (5.14) *stands for the argument (say* $Z \in U_q\, b_-$*) of the functional* $Y(Z) \equiv \langle Y, Z\rangle$*.*

(ii) *The universal* R*-matrix of the quantum double* $(U_q\, b_+, U_q\, b_-)$ *is given by*

$$R = \sum_{n,\nu} e_{n\nu}\otimes f_{n\nu} \quad \textit{for} \quad \langle f_{m\mu}, e_{n\nu}\rangle = \delta_{mn}\,\delta_{\mu\nu}\,. \tag{5.16}$$

(In the case of the restricted QUEA $\bar{U}_q$ for $q^k = -1$ – see Section 12 – the double construction has been worked out in [FGST] and [FHT].)

Remark 5.1. The matrices $M_\pm^{\pm 1}$ (5.3) provide the Gauss decomposition of the monodromy matrix M appearing in the context of the $su(2)$ current algebra model (Eq. (9.11))

$$M = q^{-\frac{3}{2}}\, M_+\, M_-^{-1} \quad \text{for} \quad q = e^{-\frac{i\pi}{h}} \tag{5.17}$$

(see Section 9 below; the choice of the phase factor is dictated by the result of Exercise 9.4).

The *double cover* D_q of $U_q(A_1)$ is obtained from the above quantum double by setting

$$k = \tilde{k} \qquad (k^2 = K)\,. \tag{5.18}$$

Its CR are then obtained from (5.1) and (5.2). Its significance stems from the fact that the universal R-matrix (5.16) of U_q belongs, in fact, to (a completion of) $D_q\otimes D_q$. It plays an important role in the physically interesting case of q a root of unity – see Section 11 below.

Writing the *Drinfeld-Jimbo universal R-matrix* requires introducing H (instead of K) and using transcendental functions, thus leaving the algebraic framework. We have (see [CP])

$$R = q^{\frac{1}{2}H\otimes H} \sum_{\nu=0}^{\infty} \frac{q^{\binom{\nu}{2}}\,\lambda^\nu}{[\nu]!}\, E^\nu\otimes F^\nu \qquad \left(\binom{\nu}{2} = \frac{\nu(\nu-1)}{2}\right)\,. \tag{5.19}$$

It turns out that an expression of the type (5.16)

$$R = \sum_\sigma e_\sigma\otimes f^\sigma\,, \quad e_\sigma \in U_q\, b_-\,, \quad f^\sigma \in U_q\, b_+\,, \quad \langle f^\rho, e_\sigma\rangle = \delta^\rho_\sigma \tag{5.20}$$

using a pair of dual bases of the quantum double is easier to handle (than (5.19)) for extracting properties of the R-matrix – even without knowing the explicit form of the basis $\{e_\sigma\}$ (a formula like (5.13)) of $U_q\, b_-$. To give an example, we shall establish the quasi-triangularity relation $(\Delta \otimes \mathbb{1})\, R = R_{13}\, R_{23}$ (4.31) for R given by (5.20).

As $\{e_\sigma\}$ form a basis in the Hopf algebra $U_q\, b_-$ one can expand the coproduct of e_σ into tensor products $e_\rho \otimes e_\tau$ and use the first equation (5.9) to determine the coefficients:

$$\Delta(e_\sigma) = \sum_{\rho,\tau} g_\sigma^{\rho\tau}\, e_\rho \otimes e_\tau\,, \quad g_\sigma^{\rho\tau} = \langle f^\rho \cdot f^\tau, e_\sigma \rangle\,. \tag{5.21}$$

Inserting in the left side of the above quasi-triangularity relation we find

$$(\Delta \otimes 1)\, R = \sum_{\sigma,\rho,\tau} g_\sigma^{\rho\tau}\, e_\rho \otimes e_\tau \otimes f^\sigma = \sum_{\rho,\tau} e_\rho \otimes e_\tau \otimes f^\rho \cdot f^\tau = R_{13}\, R_{23} \tag{5.22}$$

where we have used the relation

$$\sum_\sigma g_\sigma^{\rho\tau}\, f^\sigma = f^\rho\, f^\tau \tag{5.23}$$

which follows from (5.21) and the last equation (5.20). The relation $(1 \otimes \Delta)\, R = R_{13}\, R_{12}$ is established similarly.

Remark[18] 5.2. According to [CP] (p. 123) if (U_q, R) is a quasi-triangular Hopf algebra so is $(U_q, \tilde{R})$ for $\tilde{R} = R_{21}^{-1}$. The universal $\tilde{R}$-matrix can be obtained from the "transposed quantum double" $(U_q\, b_-, U_q\, b_+)$ by the same procedure which allowed us to construct R from $(U_q\, b_+, U_q\, b_-)$. The result is

$$\tilde{R} = \sum_{\nu=0}^{\infty} \frac{q^{-\binom{\nu}{2}} (-\lambda)^\nu}{[\nu]!}\, F^\nu \otimes E^\nu\, q^{-\frac{1}{2} H \otimes H}\,. \tag{5.24}$$

We shall see in Remark 12.1 below that the 4×4 matrix (5.5) is related to a finite dimensional counterpart of $\tilde{R}(q^{-1})$.

[18] I owe the (closely related) Remarks 5.2 and 12.1 to Ludmil Hadjiivanov.

6 Conformally invariant QFT in two and higher dimensions

Historically quantum field theory (QFT) arose (in the late 1920s) in an attempt to unify quantum mechanics with special relativity using the canonical Lagrangian (or Hamiltonian) approach and perturbation theory[19]. We shall base our treatment, instead, on the axiomatic framework developed in the second half of 20th century (see, *e.g.* [SW] [Jo] [BLOT] [H] [BH]) with the dual aim (i) to separate sense from nonsense in the formal manipulations with divergences and (ii) to clarify the basic principles of relativistic local quantum theory and their general implications. Adding the requirement of conformal invariance to the physically justified Wightman axioms [SW] (for a summary – see Appendix C) makes for the first time the axiomatic approach constructive (for surveys of axiomatic conformal field theory (CFT) in two and four dimensions – see [TMP] [FST] and [T07]).

Let us first recall the concept of conformal transformations and conformal invariance in any number D of space-time dimensions.

A transformation $g : x \to y$ of an open set O of space-time into another open set, gO is said to be conformal if the infinitesimal square interval dx^2 gets just multiplied by a (positive) factor: if $g : x\,(\in O) \to y(x,g)(\in gO)$ then

$$dy^2(x,g) = \frac{dx^2}{\omega^2(x,g)}\,, \quad \omega(x,g) \in \mathbb{R}\,, \quad \omega(x,g) \neq 0 \ \text{ for } \ x \in O\,. \tag{6.1}$$

Thus a conformal transformation is a generalization of an isometry (that would correspond to $\omega = 1$). To fix the ideas we shall consider conformal transformations of Minkowski[20] space M, setting

$$dx^2 = d\boldsymbol{x}^2 - (dx^0)^2\,, \quad d\boldsymbol{x}^2 = \sum_{i=1}^{D-1} (dx^i)^2\,. \tag{6.2}$$

One should, however, keep in mind that our discussion applies equally well to all conformally flat metrics (such that $ds^2 = \frac{dx^2}{\Omega^2(x)}$). In particular, all spaces of constant curvature – the positive curvature de Sitter[21] space and the negative

[19] I am unable to choose a single "best" textbook on QFT. An authoritative 3-volume treatise is Weinberg's [We]. For a selection of original papers on quantum electrodynamics reflecting the development up to the 1950s – see [Sc]; a clear and concise exposition of later developments in renormalization theory including the use of Becchi-Rouet-Stora cohomology is contained in [PS]. Different in style and purpose is the (often entertaining) book [BM] which surveys the inter-relations between gauge theory and modern mathematics.

[20] Hermann Minkowski (1864-1909) introduced the 4-dimensional space-time (in 1908 in Göttingen), thus completing the special theory of relativity of Hendrik Antoon Lorentz (1853-1928, Nobel Prize in Physics, 1902), Henri Poincaré and Albert Einstein (1879-1955, Nobel Prize in Physics, 1921).

[21] The Dutch mathematician, physicist and astronomer Willem de Sitter (1872-1934), who was interested in the concept of inertia in general relativity, introduced (as an alternative to Einstein's static universe) his constant curvature space (with a zero mass density) with a pos-

curvature anti de Sitter space (on top of the zero curvature Minkowski space) – are conformally flat. The assumption that the choice of metric within a given conformal class should not affect the physics in a CFT thus forces us to adopt a more general point of view on QFT.

Typically conformal transformations develop singularities: they cannot be defined on the whole of M. That is the reason we speak of open (sub)sets of M in the definition (6.1). By contrast, the *conformal Killing*[22] *vector* $K^\mu(x)(= K^\mu(x,g))$ corresponding to an infinitesimal conformal transformation

$$y^\mu(x) = x^\mu + \varepsilon\, K^\mu(x) + O(\varepsilon^2)\,, \quad \omega(x) = 1 - \varepsilon\, f(x) + O(\varepsilon^2) \qquad (6.3)$$

is well defined in M and satisfies the *conformal Killing equation*

$$\partial_\mu\, K_\nu + \partial_\nu\, K_\mu = 2 f \eta_{\mu\nu} \quad (\eta_{\mu\nu} = \mathrm{diag}\,(-, + + \ldots))\,. \qquad (6.4)$$

Exercise 6.1. Writing $dx^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu$, $dy^2 = \eta_{\mu\nu}\, dy^\mu\, dy^\nu$ and inserting (6.3) in (6.1) derive (6.4) by equating the terms of order ε.

Exercise 6.2. Demonstrate, using (6.4), that

$$\partial \cdot K (\equiv \partial_\mu\, K^\mu) = D f\,, \quad (D-2)\, \partial_\lambda\, \partial_\mu\, f = 0\,. \qquad (6.5)$$

Use the result to derive the following

Proposition 6.1. (Liouville[23] theorem) *The general form of the conformal Killing vector for $D > 2$ is given by*

$$K^\mu(x) = a^\mu + \alpha\, x^\mu + \lambda_{\mu\nu}\, x^\nu - 2(c \cdot x)\, x^\mu + x^2\, c^\mu\,, \quad \lambda_{\mu\nu} = -\lambda_{\nu\mu}\,. \qquad (6.6)$$

Exercise 6.3. Verify that the conformal group C of M is spanned by Poincaré transformations $y^\mu = \Lambda^\mu_\nu\, x^\nu + a^\mu$, uniform dilation $y^\mu = \rho\, x^\mu$, $\rho > 0$, and *special*

itive cosmological constant in 1917. Presently, it is believed that our universe is approaching a de Sitter space-time. Einstein and de Sitter wrote a joint paper in 1932 on what came to be called *dark matter* (whose presence is only detected by its gravitational field).

[22] Wilhelm Karl Joseph Killing (1847-1923), a student of Weierstrass and Kummer in Berlin, became a professor at the seminary college in Braumsberg. He invented Lie algebras, independently of Sophus Lie, around 1880. In 1888-1890 Killing classified (essentially) the complex simple Lie algebras, inventing the notions of a Cartan subalgebra and a Cartan matrix. He introduced the root systems and discovered the exceptional Lie algebra $\mathcal{G}_2$ (in 1887). For a popular article about Killing and his work on Lie algebras, see A. John Coleman, *The greatest mathematical paper of all time*, The Mathematical Intelligencer **11**:3 (1889) 29-38; see also, T. Hawkins, Wilhelm Killing and the structure of Lie algebras, Archive for History of Exact Science **26** (1982) 126-192.

[23] Joseph Liouville (1809-1882) published his theorem (for 3-dimensional Euclidean space) in a Note to the 5th edition of Gaspard Monge (1746-1818), *Application de l'analyse à la géométrie* (Paris, 1850) entitled "Extension au cas des trois dimensions de la question du tracé géographique" (pp. 609-616).

conformal transformations which can be defined as translations $T_c : x \to x + c$ sandwitched between two *conformal inversions* $R : x \to \frac{x}{x^2}$:

$$y(x,c) = R\,T_c\,R\,x = \Big(\frac{x}{x^2} + c\Big)\left[\Big(\frac{x}{x^2} + c\Big)^2\right]^{-1} = \frac{x + cx^2}{\omega(x,c)}\,,$$

$$\omega(x,c) = 1 + 2c \cdot x + c^2 x^2\,. \tag{6.7}$$

Clearly, the special conformal transformations (6.7) are singular (for $c \neq 0$) on the cone $\omega(x,c) = 0$ (that degenerates into a hyperplane for $c^2 = 0$). One can define, following Dirac[24] [D36], the conformal compactification of space-time $\bar{M}$ as a projective quadric in $D + 2$ dimensions:

$$\bar{M} = Q/\mathbb{R}^* \simeq \mathbb{S}^{D-1} \times \mathbb{S}^1/\pm 1\,, \quad Q = \left\{\vec{\xi} \in \mathbb{R}^{D,2}\,;\; \vec{\xi}^{\,2} = \sum_{\alpha=1}^{D} \xi_\alpha^2 - \xi_0^2 - \xi_{-1}^2 = 0\right\},$$

$$\mathbb{R}^* = \mathbb{R}\backslash\{0\}\,. \tag{6.8}$$

M is embedded in a dense open set of $\bar{M}$ in which $\kappa := \xi^D + \xi^{-1} (= \xi_D - \xi_{-1}) \neq 0$:

$$x^\mu = \frac{1}{\kappa}\,\xi^\mu \qquad \left(x^2 = \frac{\xi^{-1} - \xi^D}{\kappa}\right).$$

The quadric Q (6.8) is, clearly, invariant under the full orthogonal group $O(D,2)$. The reflection $(-\mathbb{1}) : \vec{\xi} \to -\vec{\xi}$ acts however as the identity transformation on $Q/\mathbb{R}^*$ so it is only the quotient group $O(D,2)/\pm 1$ which acts effectively on $\bar{M}$ and should be identified with the conformal group C (including reflections) of compactified Minkowski space. It is natural, following Segal[25] [S], to identify the *conformal energy* operator with the (hermitian) generator H of the centre of the Lie algebra $so(2) \times so(D)$ of the maximal compact subgroup of C, *i.e.*, with the infinitesimal rotation in the $(-1,0)$-plane. It can be expressed in terms of the Minkowski space energy operator P_0 (the zeroth component of the energy momentum vector) and its conjugate by the conformal inversion R as

$$H = \frac{1}{2}\,(P_0 + R\,P_0\,R)\,. \tag{6.9}$$

Here $R\,P_0\,R$ is a physical (hermitian) generator of the special conformal transformation (6.7) (in other words, the vector field corresponding to the Lie algebra element $i\,R\,P_0\,R$ is $\left[\frac{\partial}{\partial c^0}\,y^\nu(x,c)\right]_{c=0}\frac{\partial}{\partial x^\nu}$). In a unitary representation of (a covering of) the conformal group H (6.9) is positive whenever the Minkowski energy P_0 is positive.

The following exercise shows that for $D = 2$ the conformal group is infinite dimensional.

[24] Paul Adrien Maurice Dirac (1902-1984), Nobel Prize in Physics 1933, known for his equation and for the prediction of antiparticles, recollects (in his Varenna 1977 lecture) of his great appreciation of projective geometry since his student years at Bristol.

[25] Irving Ezra Segal (1918-1998) was a Professor in mathematics at the Massachusetts Institute of Technology.

Exercise 6.4. Let $f_\pm(z)$ be a pair of (non-constant) meromorphic functions (taking real values on the real line). Demonstrate that both changes of variables $x \to y$ such that

$$y^0 + y^1 = f_+(x^0 \pm x^1)\,, \quad y^0 - y^1 = f_-(x^0 \mp x^1)\,, \tag{6.10}$$

satisfy the condition (6.1) for a conformal mapping. Show that the upper sign in (6.10) corresponds to the connected component of the identity of the (infinite dimensional) group of meromorphic mappings, while the lower one belongs to the connected component of space reflections.

Exercise 6.5. (a) Show that the only complex conformal transformations which transform circles into circles or straight lines are the non-singular fractional linear (also called *Möbius*[26]) transformations

$$z \to z' = gz \equiv \frac{az+b}{cz+d} \qquad ad - bc \neq 0\,. \tag{6.11}$$

(b) They preserve the real line if the matrix entries of the 2×2 matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are real. They preserve the upper half plane if the determinant of g is positive (then one can set $\det g = ad - bc = 1$, thus identifying the Möbius group with $SL(2,\mathbb{R})$).

(c) The transformation (6.11) preserves the unit circle iff $d = \bar{a}$, $c = \bar{b}$. For $g \in SU(1,1)$ (*i.e.* for $g = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, $\det g = |a|^2 - |b|^2 = 1$) the Möbius transformation (6.11) preserves the interior (as well as the exterior) of the unit circle. For $|a|^2 - |b|^2 < 0$ it exchanges $|z| < 1$ with $|z| > 1$.

There is a complex Möbius map g_c of the upper half plane $\tau\,(\mathrm{Im}\,\tau > 0)$ onto the unit disk ($|z| < 1$) intertwining the $SL(2,\mathbb{R})$ and the $SU(1,1)$ actions. Choosing $g_c\, i = 0$, $g_c\, 0 = 1$ we find

$$g_c : \tau \to z = \frac{1+i\tau}{1-i\tau} \qquad \left(\tau = i\,\frac{1-z}{1+z}\right)\,. \tag{6.12}$$

It maps the real light ray $\tau = t\,(= x^0 + x^1)$ onto the unit circle, sending the point at infinity to -1. Thus g_c plays the role of a *compactification map* for the light ray.

Exercise 6.6. Demonstrate that the non-singular conformal transformation $z \to f(z)$ is a Möbius transformation iff the Schwarz[27] derivative

$$\{f, z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{6.13}$$

[26] August Ferdinand Möbius (1790-1868). The Möbius group $SL(2,\mathbb{C})$ is a double cover of the (connected component of the) Lorentz group $SO^\uparrow(3,1)$.

[27] Karl Hermann Amadeus Schwarz (1843-1921) a student of Karl Weierstrass (1815-1897); introduced his derivative in 1872.

vanishes.

We shall now exhibit a 4-dimensional quaternionic analogue of (6.12). Consider the Lie algebra $u(2)$ of 2×2 anti-hermitian matrices

$$i\,\tilde{x} = i\,x^0\,\sigma_0 - i\,x^j\,\sigma_j \qquad (\in u(2))$$

where σ_j are the Pauli[28] matrices, $\sigma_0 = \mathbb{1}$ is the 2×2 unit matrix. The Cayley[29] map from the Lie algebra $u(2)$ to the group $U(2)$ of 2×2 unitary matrices,

$$i\,\tilde{x} \to u = \frac{1 + i\,\tilde{x}}{1 - i\,\tilde{x}} \in U(2) \quad \text{for} \quad x^\mu \in \mathbb{R}\,, \tag{6.14}$$

can be viewed as an alternative of the conformal compactification (6.8).

Exercise 6.7. Writing u (6.14) in the form $u = u^4\,\mathbb{1} - i\,u^j\,\sigma_j$ prove that u^α are related to $\vec{\xi}$ in (6.8) and (6.9) by

$$u^\alpha = \frac{\xi^\alpha}{\xi^{-1} + i\,\xi^0}\,, \quad \alpha = 1,2,3,4\ \left(\sum_{\alpha=1}^{4} u^\alpha\,\bar{u}^\alpha = 1\right). \tag{6.15}$$

Exercise 6.8. Prove that the Lie algebra $su(2,2)$ of the pseudo-unitary group $SU(2,2)$ coincides with the conformal Lie algebra $so(4,2)$.

(*Hint* : use the realization of Appendix C to [NT] for $D = 4$.)

Exercise 6.9. Prove that $SU(2,2)$ is a 4-fold cover of the conformal group $C_0 \simeq SO_0(4,2)/\pm 1$ of 4-dimensional Minkowski space or, equivalently, a 2-fold cover of $SO_0(4,2)$:

$$SO_0(4,2) \simeq SU(2,2)/\pm 1\,. \tag{6.16}$$

Discrete masses of atoms and elementary particles violate "the great principle of similitude"[30] (*i.e.* scale and, *a fortiori*, conformal invariance). The situation in QFT is still more involved – and more interesting: dimensional parameters arise in the process of renormalization even if they are absent in the classical theory. Dilation and conformal invariance can only be preserved for a

[28]Wolfgang Ernest Pauli (born in Vienna 1900, died in 1958 in room 137 of a hospital in Zürich). During his stay in Hamburg (1923-1928) he discovered the exclusion principle (1925), for which he was awarded the Nobel Prize in Physics in 1945, and introduced the Pauli matrices (in 1927).

[29]Arthur Cayley (1821-1895) after studying at Trinity became (at 25) a lawyer for 14 years in London writing during that period over 200 mathematical papers. He was first to define the modern way the concept of a group. The Cayley transform originally appeared (1846) as a mapping between skew symmetric and special orthogonal matrices.

[30]See Lord Rayleigh, *The principle of similitude*, Nature **95**:2368 (March 1915) 66-68 and 644. John William Strutt – Lord Rayleigh (1842-1919) was awarded the 1904 Nobel Prize for his discovery of the inert gas argon.

renormalization group fixed point, *i.e.*, for a *critical theory*, the QFT counterpart of a point of phase transition. One may hope that the study of an idealized critical theory with no dimensional parameters will prove to be an essential step in understanding QFT – just as Galilei's[31] law of inertia, that neglects friction, has been crucial in formulating and understanding classical mechanics. (For a more comprehensive discussion of the relevance of conformal invariance see the Introduction to [T07].)

The case of 2-*dimensional conformal field theory* ($2D$ CFT), to which are devoted the next two sections, is attractive from several points of view. It not only provides soluble QFT models satisfying the axioms, but the euclidean version of such models applies to $2D$ critical phenomena. String vacua are also described by a class of $2D$ CFT. (For a survey of QFT and strings addressed to mathematicians – see [QFS].)

Before going to the discussion of a class of $2D$ CFT models we shall make a general remark pertinent to a CFT in any even number of space-time dimensions.

It is important to distinguish in axiomatic QFT between *local observables*, such as the stress-energy tensor and conserved local currents on one hand, and gauge dependent *charged fields* which intertwine among different representations (or *superselection sectors*) of the algebra of observables, on the other. (This is stressed, in particular, in Haag's approach to local quantum physics, [H], in which a compact gauge group of the first kind is derived from intrinsic properties of the observable algebra.) In the framework of axiomatic CFT we postulate that local observables are *globally conformal invariant* (GCI) – *i.e.*, invariant under finite conformal transformations in Minkowski spaces, [NT01]. This is a highly non-trivial requirement since a finite interval $(x_1 - x_2)^2$ goes under special conformal transformations (6.7) into

$$[y_1(x_1, c) - y_2(x_2, c)]^2 = \frac{(x_1 - x_2)^2}{\omega(x_1, c)\, \omega(x_2, c)}\,. \tag{6.17}$$

The product of ω-factors (unlike the square in the infinitesimal law (6.1)) may change sign. The local commutativity for space-like separations implies Huygens[32] locality: the commutator of local fields has support on light-like separations (it vanishes for both space-like and time-like $x_1 - x_2$). Moreover, one can express the strong (Huygens) locality between two observable Bose fields by the algebraic relation

$$[(x_1 - x_2)^2]^N\, [\phi(x_1), \psi(x_2)] = 0 \quad \text{for} \quad N \gg 0 \tag{6.18}$$

($N \gg 0$ meaning "for sufficiently large N"). This allows a formulation of GCI QFT in terms of formal power series (instead of distributions), [N], [BN]. Combined with the remaining Wightman axioms it implies rationality of correlation

[31]Galileo Galilei (1564-1642) amplified his views on mechanics in his last dialogue (1638) written when exiled to his villa at Arcetri.

[32]The Dutch physicist, mathematician and astronomer Christian Huygens (1629-1695) is the originator of the wave theory of light.

functions of observable fields [NT01], long believed to be a peculiarity of chiral observable fields in 1 + 1 dimension (for a review – see [NT]). It should be noted, however, that canonical free fields and the stress energy tensor in odd space-time dimensions violate Huygens locality (and hence, GCI).

Exercise 6.10. Use the Schwinger[33] α-representation $\left(\frac{1}{p^2} = \int_0^\infty e^{-\alpha p^2}\, d\alpha\right)$ to derive for euclidean p and x in D-dimensional space-time the relation

$$\int \frac{e^{ipx}}{p^2} \frac{d^D p}{(2\pi)^{D/2}} = \frac{\Gamma\left(\frac{D}{2} - 1\right)}{4\pi^{D/2}} (x^2)^{1-\frac{D}{2}} \,. \tag{6.19}$$

Deduce from here, using energy positivity that the Minkowski space 2-point function $w(x_{12}) = \langle 0 \mid \varphi(x_1)\, \varphi(x_2) \mid 0 \rangle$ for a free massless field in $D = 3$ space time dimensions (with euclidean propagator $\frac{1}{p^2}$) is

$$w(x) = \frac{1}{4\pi(x^2 + i0x^0)^{1/2}} \,. \tag{6.20}$$

Thus, the GCI postulate is only appropriate for even D. A survey of both standard (infinitesimal) and global conformal invariance in QFT in four dimensions is contained in [T07] (see also the introduction to [NT05]).

[33] Julian Seymour Schwinger (1918-1994) shared the 1965 Nobel Prize in Physics with Richard Feynman (1918-1988) and Sin-Itiro Tomonaga (1906-1979) for his work in quantum electrodynamics.

Appendix C. Informal summary of Wightman axioms

Pure quantum states are described by *unit rays* in a *complex* (positive metric) *Hilbert space* $\mathcal{H}$ which carries a *unitary positive energy ray representation* of the proper orthochronous Poincaré ($\equiv$ inhomogeneous Lorentz) group $\mathcal{P}_+^\uparrow$. A *ray* (or *projective*) representation of $\mathcal{P}_+^\uparrow$ is equivalent to a *single valued representation* $U(A,a)$ of its *universal covering group* (which is, by definition, simply connected). For $D > 2$ the covering of $\mathcal{P}_+^\uparrow$ is obtained by substituting the Lorentz group $SO^\uparrow(D-1,1)$ by its double cover, the spin group $\mathrm{Spin}\,(D-1,1)$. For $D = 4$ this double cover is isomorphic to the group $SL(2,\mathbb{C})$ of complex 2×2 matrices of determinant 1. We have, denoting the 2×2 unit matrix by σ_0, $A\,\sigma_\mu\, x^\mu A^* = \sigma_\mu\,\Lambda_\nu^M x^\nu$ for $A \in SL(2,\mathbb{C}) \simeq \mathrm{Spin}\,(3,1)$

$$\Lambda = \Lambda(A) \in SO^\uparrow(3,1) \simeq SL(2,\mathbb{C})/\pm 1\,. \tag{C.1}$$

Positive energy means that the hermitian generator of translation, the energy momentum vector P_μ has joint spectrum in the forward light cone; moreover the unique translation invariant state is the *vacuum* $|0\rangle$:

$$P_0 \geq |\boldsymbol{P}|\,,\ |\boldsymbol{P}|^2 = \sum_{i=1}^{D-1} P_i^2\,, \quad P_\mu \mid 0\rangle = 0\,, \quad \mu = 0,1,\ldots,D-1\,. \tag{C.2}$$

Quantum fields $\phi(x)$ are *operator (spin-tensor) valued distributions*[34] which transform covariantly under $U(A,a)$:

$$U(A,a)\,\phi(x)\,U(A,a)^* = V(A^{-1})\,\phi(\Lambda(A)\,x+a) \qquad (U^* = U^{-1})\,, \tag{C.3}$$

V being a finite dimensional representation of the "quantum mechanical Lorentz group" $\mathrm{Spin}\,(D-1,1)$.

It is a consequence of energy positivity that the vector valued function $\phi(x)\mid 0\rangle$ admits analytic continuation to complex $z^\mu = x^\mu + iy^\mu$ in the *forward tube* (noting that in our conventions $U(a,\mathbb{1}) = e^{iP\cdot a}$):

$$\frac{\partial}{\partial \overline{z}}\,\phi(z)\mid 0\rangle = 0 \quad \text{for} \quad z \in \mathcal{T}_+ = \{z = x+iy \in \mathbb{C}^D\,; \quad y^0 > |\boldsymbol{y}|\}\,. \tag{C.4}$$

Exercise C.1. Prove that $\mathcal{T}_+$ is invariant under the action of the connected component C_0 of the (real) conformal group. (*Hint* : verify that $\mathcal{T}_+$ is invariant under the Weyl inversion

$$z \to wz = \frac{I_s\,z}{z^2} \qquad I_s(z^0,\boldsymbol{z}) = (z^0,-\boldsymbol{z}) \tag{C.5}$$

[34] More precisely, we are dealing with tempered distributions introduced by Laurent Schwartz (1915-2002).

and notice that C_0 is generated by w and by real translations. For a stronger result, known to V. Glaser (1924-1984) – see [U].)

Exercise C.2. Extend the projective quadric construction to the conformal compactification of complexified Minkowski space $M_{\mathbb{C}}$ and verify that the stabilizer of $z = (i, \mathbf{0})$ ($\in \mathcal{T}_+$) is the maximal compact subgroup of C_0.

Observable (Bose) fields commute for space-like separations:

$$[\phi(x_1), \psi(x_2)] = 0 \quad \text{for} \quad (x_1 - x_2)^2 > 0 \; (\textit{local commutativity}). \tag{C.6}$$

The vacuum is assumed to be a *cyclic vector* with respect to the set of (relativistic) local fields, so that every vector in $\mathcal{H}$ can be written as a strong limit of linear combinations of vectors of the form $\phi_1(x_1) \ldots \phi_n(x_n) \mid 0\rangle$ (smeared with test functions). It follows that the full content of the theory can be expressed in terms of *(Wightman) correlation functions* – vacuum expectation values of fields products.

Exercise C.3. Prove that the Cayley map (6.14) extends to points z of the tube domain $\mathcal{T}_+$. The image T_+ of $\mathcal{T}_+$ under this map is given by

$$T_+ = \left\{ z \in \mathbb{C}^4 \, ; \; |z^2| < 1 \, , \; |z|^2 = \sum_{\alpha=1}^{4} |z_\alpha|^2 < \frac{1}{2} \left(1 + |z^2|^2\right) \right\} . \tag{C.7}$$

Extend the map (6.14) (and T_+) to any number D of space-time dimensions.

Remark C.1. The tube domain (C.7) is biholomorphically equivalent to the *classical Cartan domain of type* IV (see *e.g.* [SV] pp. 182-192).

7 Two-dimensional conformal current algebras

Basic objects in QFT are the *correlation functions* – vacuum expectation values of products of local fields which satisfy certain symmetry properties and can be viewed as boundary values of analytic functions as a consequence of the spectral conditions (energy positivity). Conformal invariance plays the role of a dynamical principle: it allows to determine 2-point functions (uniquely, up to normalization) and 3-point functions (up to a few constants). (Four point functions and higher can only be determined in a GCI QFT.) The 2-point function of two currents $j_\mu(x)$ of scale dimension $D-1$ in D space-time dimensions has the form ([TMP]):

$$W^\mu_\nu(x_{12}) := \langle 0 \mid j^\mu(x_1)\, j_\nu(x_2) \mid 0\rangle = N_J\, r^\mu_\nu(x_{12})\, \rho_{12}^{1-D}\ , \ \rho_{12} = x^2_{12} + i\, 0\, x^0_{12} \quad (7.1)$$

where the $i\,0\,x^0_{12}$ defines the right hand side of (7.1) as a distribution,

$$x_{12} = x_1 - x_2\,, \quad x^2 = \boldsymbol{x}^2 - (x^0)^2\,, \quad \boldsymbol{x}^2 = \sum_{i=1}^{D-1} x_i^2\,, \quad r^\mu_\nu(x) = \delta^\mu_\nu - 2\,\frac{x^\mu x_\nu}{\rho} \quad (7.2)$$

($r^2 = \mathbb{I}$, $r^\mu_\nu\, x^\nu = -x^\mu$). $W^\mu_\nu(x)$ satisfies the conservation law

$$\partial_\mu\, W^\mu_\nu(x) = 0 \qquad (\text{for } \partial_\mu = \frac{\partial}{\partial\, x^\mu}) \quad (7.3)$$

implying (in view of Wightman positivity and the Reeh-Schlieder theorem [SW], [BLOT]) that the current itself is conserved (as an operator valued distribution):

$$\partial_\mu\, j^\mu(x) = 0\,. \quad (7.4)$$

For $D = 2$ we see, in addition, that W^μ_ν is a gradient:

$$W^\mu_\nu(x) = \partial_\nu\, N_J\, \frac{x^\mu}{\rho} \qquad (\rho = x^2 + i\,0\,x^0)\,, \quad (7.5)$$

hence the curl of j is also zero:

$$\partial_\mu\, j_\nu(x) - \partial_\nu\, j_\mu(x) = 0\,, \quad \mu, \nu = 0, 1\,. \quad (7.6)$$

Exercise 7.1. Prove that Eqs. (7.4) and (7.6) imply that the current splits into two chiral components, depending on a single light cone variable $x^0 \pm x^1$ each:

$$\frac{1}{\sqrt{2}}\,(j^0 - j^1(x)) =: j(x^0 + x^1)\,, \quad \frac{1}{\sqrt{2}}\,(j^0 + j^1) =: \bar{j}(x^0 - x^1)\,. \quad (7.7)$$

As a consequence of energy positivity both vector valued function $j(t) \mid 0\rangle$ and $\bar{j}(\bar{t}) \mid 0\rangle$ are boundary values of functions analytic in the upper half plane. It is now convenient to use the compactification map g_c (6.12) from the upper half-plane onto the unit disk D_1. It gives rise to the *z-picture fields* $\phi(z)$ that are naturally identified with their formal Laurent[35] expansions yielding convergent in D_1 Taylor[36] series for the vector valued function $\phi(z) \mid 0\rangle$. The compact picture current, is identified by equating the corresponding 1-forms:

$$J(z)\,\frac{dz}{2\pi i} = j(t)\,dt\,, \tag{7.8}$$

i.e.

$$J(z) = 2\pi i\,\frac{dt}{dz}\,j(t(z)) = \frac{4\pi}{(1+z)^2}\,j\left(i\,\frac{1-z}{1+z}\right) \tag{7.9}$$

where we have devided by the length of the unit circle (*i.e.* of the compactified light ray) with respect to the (complex) measure $\frac{dz}{z}$. $J(z)$ is more convenient to work with (than $j(t)$), since its *mode expansion* is given by the (formal) Laurent series

$$J(z) = \sum_{n\in\mathbb{Z}} J_n\,z^{-n-1} \quad (J_n = \mathrm{Res}_z(z^n\,J(z))) \tag{7.10}$$

(that repalces the integral Fourier[37] transform of $j(t)$).

Similarly, the conserved traceless stress energy tensor θ for $D = 2$ also splits into two chiral components,

$$\begin{aligned} \Theta(x^0+x^1) &= \frac{1}{\sqrt{2}}\,(\Theta^0_0 - \Theta^1_0) \quad (= \frac{1}{2\sqrt{2}}\,(\Theta^0_0 - \Theta^1_0 + \Theta^0_1 - \Theta^1_1)) \\ \overline{\Theta}(x^0-x^1) &= \frac{1}{\sqrt{2}}\,(\Theta^0_0 + \Theta^1_0)\,. \end{aligned} \tag{7.11}$$

Exercise 7.2. Use the conservation law, $\partial_\mu\,\theta^\mu_\nu = 0$, and the tracelessness, $\theta^\mu_\mu = 0$ of θ^μ_ν to prove that $\frac{\partial\theta}{\partial\bar{t}} = 0 = \frac{\partial\bar{\theta}}{\partial t}$ for $t = x^0 + x^1$, $\bar{t} = x^0 - x^1$ and θ and $\bar{\theta}$ defined by the right hand side of the first and second equation (7.11).

Exercise 7.3. Equating the *quadratic differentials*

$$\Theta(t)\,dt^2 = T(z)\,\frac{dz^2}{2\pi} \tag{7.12}$$

express $T(z)$ in terms of $\Theta(t)$ for $t = i\,\frac{1-z}{1+z}$.

[35] Pierre Alphonse Laurent (Paris 1813-1854) introduced in 1843 the Laurent series in a memoir submitted for the "Grand Prix de l'Académie des Sciences", but the submission was after the due date, and the paper was not published until after his death (at the age of 41).

[36] The English mathematician Brook Taylor (1685-1731) proved a theorem about power series expansions (following ideas of Isaac Newton, 1642-1727) in a paper of 1715 which remained unrecognized until 1772 when Joseph-Louis Lagrange (1736-1813) proclaimed it the basic principle of differential calculus.

[37] The French mathematician and physicist Jean Baptiste Joseph Fourier (1768-1830) went with Napoleon Bonaparte on his Egyptian expedition in 1798; was governor of Lower Egypt (until 1801). In his "Théorie analytique de la chaleur" (1822) he introduced the Fourier series (exhibiting discontinuous functions with convergent Fourier series). His claims were made precise and proven by Johann Peter Gustav Lejeune Dirichlet (1805-1859).

Remark 7.1. The (conserved) current $j^\mu(x)$ in D dimensions should have conformal dimension $D-1$ (in mass units) in order to allow interpreting $j^0(x)$ as the charge density of a dimensionless charge. Similarly, Θ^μ_ν has dimension D in D-dimensional space-time so that one may interpret Θ^0_0 as an energy density. This accounts for the difference between (7.9) and (7.12). The factor $(2\pi)^{-1}$ in (7.12) is chosen to simplify the 2-point function of $T(z)$ in the theory of a free Weyl fermion.

Exercise 7.4. (a) Given (7.1) for $D=2$, compute the 2-point function for $j(t)$. (*Answer* :

$$\langle 0 \mid j(t_1)\, j(t_2) \mid 0 \rangle = \frac{-N_J}{(t_1 - t_2 - i0)^2}\,; \tag{7.13}$$

hint : use the fact that $\rho_{12} = i0(t_{12}+\bar{t}_{12}) - t_{12}\,\bar{t}_{12}$, for $t_{12} = t_1 - t_2$, $t_i = x_i^0 + x_i^1$.)

(b) Viewing the right hand side of (7.13) as a rational function of t_{12} (*i.e.* neglecting the $i\,0$ prescription) and setting $N_J = (2\pi)^{-2}$ prove that the 2-point function of $J(z)$ (7.9) is

$$\langle 0 \mid J(z_1)\, J(z_2) \mid 0 \rangle = \frac{1}{z_{12}^2}\,, \quad z_{12} = z_1 - z_2\,. \tag{7.14}$$

Remark 7.2. The solution of the $2D$ massless Dirac equation $(\gamma^0\,\partial_0 + \gamma^1\,\partial_1)\,\Psi = 0$ for

$$\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \left(\gamma_0\,\gamma_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\gamma^0\,\gamma^1\right)$$

assumes the form $\Psi = \begin{pmatrix} \psi(t) \\ \bar{\psi}(\bar{t}) \end{pmatrix}$. If we define $j(t)$ in the theory of a free Weyl field $\psi(t)$ from the *operator product expansion*

$$\frac{1}{2}\left(\psi^*(t_1)\,\psi(t_2) - \psi(t_1)\,\psi^*(t_2)\right) = j\left(\frac{t_1 + t_2}{2}\right) + O(t_{12}^2) \tag{7.15}$$

then the 2-point function (7.13) of j will indeed involve the normalization constant $N_J = (2\pi)^{-2}$ (see [FST] Appendix C).

If we write the mode expansion of the z-picture stress-energy tensor as

$$T(z) = \sum_{n\in\mathbb{Z}} L_n\, z^{-n-2} \qquad \left(\bar{T}(\bar{z}) = \sum_n \bar{L}_n\, \bar{z}^{-n-2}\right) \tag{7.16}$$

then the conformal energy H (6.9) is the sum of the left and right mover's zero modes:

$$H = L_0 + \bar{L}_0\,. \tag{7.17}$$

We shall identify accordingly the chiral energy operator with L_0. We have

$$[L_0, J(z)] = \left(z \frac{d}{dz} + 1 \right) J(z) = \frac{d}{dz} \left(z \, J(z) \right) \Rightarrow [L_0, J_n] = -n \, J_n \, ; \tag{7.18}$$

$$[L_0, T(z)] = \left(z \frac{d}{dz} + 2 \right) T(z) \Rightarrow [L_0, L_n] = -n \, L_n \, . \tag{7.19}$$

More generally, if $W(z)$ is a chiral Bose field of integer dimension d,

$$W(z) = \sum_n W_n \, z^{-n-d} \, , \tag{7.20}$$

then

$$[L_0, W(z)] = \left(z \frac{d}{dz} + d \right) W(z) \Rightarrow [L_0, W_n] = -n \, W_n \, . \tag{7.21}$$

It follows from the analysis of Section 6 (see, in particular, Exercises 6.2 and 6.4) that there is an infinite parameter set of invertible local conformal transformations $z \to f(z)$ of a neighbourhood of the origin (in which $f'(z) \neq 0$). If the theory is assumed to be invariant under such an "infinite conformal group" then the correlation functions would have been independent of z which would mean that all chiral fields (including the stress-energy tensor) would vanish. What actually happens is that the vacuum state is not invariant under the infinite dimensional conformal group. Correlation functions of chiral fields, like (7.14) or

$$\langle 0 \mid T(z_1) \, T(z_2) \mid 0 \rangle = \frac{c}{2 \, z_{12}^4} \qquad (c > 0) \tag{7.22}$$

are only invariant under the Möbius group of fractional linear transformations (see Exercise 6.6). Noting the Lie algebras $s\ell(2, \mathbb{R})$, $su(1,1)$ and $so(2,1)$ are isomorphic we can say that the correlation functions of a D-dimensional CFT are $so(D,2)$ invariant for all $D \geq 1$. As we shall see shortly the chiral Möbius Lie algebra is spanned by L_0, $L_{\pm 1}$; the CR (7.19) should be completed by

$$[L_1, L_{-1}] = 2 \, L_0 \quad \text{so that} \quad [L_m, L_n] = (m-n) \, L_{m+n} \, , \quad m, n = 0, \pm 1 \, . \tag{7.23}$$

The z-picture correlation functions (like (7.14) (7.22)) having the same form as the x-space ones are, in particular, translation invariant, the (non hermitian) generator of translations of the complex variable z being L_{-1} which should also annihilate the vacuum:

$$L_{-1} \mid 0 \rangle = 0 \, , \quad [L_{-1}, W(z)] = \frac{dW(z)}{dz} \Rightarrow [W_n, L_{-1}] = (n+d-1) \, W_{n-1} \, . \tag{7.24}$$

The upper half plane, the analyticity domain of $\phi(\tau) \mid 0 \rangle$ for any chiral field ϕ, is mapped by the complex Möbius transformation (7.8) onto the unit disk. Thus we expect that z-picture fields applied to the vacuum give rise to Taylor expansions convergent for $|z| < 1$. To formulate the precise statement we need the notion of z-picture conjugate of a hermitian chiral field $W(z)$ of dimension d and expansion (7.20):

$$(W(z))^* = \frac{1}{\bar{z}^{2d}} W\left(\frac{1}{\bar{z}}\right) . \tag{7.25}$$

Proposition 7.1. (a) *The vector valued function $W(z) \mid 0\rangle$ for a hermitian scalar field W (7.20) of a positive integral dimension d has the form*

$$W(z) \mid 0\rangle = \sum_{n=0}^{\infty} W_{-n-d}\, z^n \mid 0\rangle\,, \quad \textit{i.e.} \quad W_n \mid 0\rangle = 0 \quad \textit{for} \quad n+d>0\,. \tag{7.26}$$

(b) *The norm square of this vector is given by a power series convergent for $\bar{z}\, z < 1$.*

Proof. (a) $W_n \mid 0\rangle = 0$ for $n > 0$ because $(L_0 + n)\, W_n \mid 0\rangle = 0$ and we have assumed energy positivity. Hence $W(z) \mid 0\rangle$ may at most have a finite number (no more than d) negative powers of z in its Laurent expansion. Hence the formal power series

$$F(z,w) := e^{wL_{-1}}\, W(z) \mid 0\rangle$$

can be written in the form $F(z,w) = \frac{v_0(w)}{z^N} + \frac{v_1(z,w)}{z^{N-1}}$ where v_0 and v_1 only involve non-negative powers of z and w in their (formal) Laurent expansions. On the other hand, Eq. (7.24) implies that $\frac{\partial F}{\partial z} = \frac{\partial F}{\partial w}$. This is only possible if $N = 0$, implying (7.26). Thus the lowest energy state generated by the W modes is $W_{-d} \mid 0\rangle$ of energy d.

Remark 7.3. We have thus proved that, under the assumption of energy positivity, any translation covariant formal power series $W(z) \mid 0\rangle$ involves no negative powers of z. Thus the vector $W(0) \mid 0\rangle$ is well defined (and determines $W(z)$ – see Appendix C). A more general result of this type, applicable to higher dimensional GCI theories, is contained in Proposition 3.2 (a) of [NT05].

(b) the 2-point function of W is determined from translation and dilation invariance to have the form

$$\langle 0 \mid W(z_1)\, W(z_2) \mid 0\rangle = \frac{N_W}{z_{12}^{2d}}\,. \tag{7.27}$$

(Hilbert space positivity demands $N_W > 0$.) It follows from here and from the conjugation rule (7.25) that the norm square of the vector (7.26),

$$\| W(z) \mid 0\rangle \|^2 = \frac{N_W}{(1 - z\,\bar{z})^{2d}} = N_W \sum_{n=0}^{\infty} \binom{2d+n-1}{n} (z\,\bar{z})^n\,, \tag{7.28}$$

indeed converges for $|z|^2 < 1$. □

Proposition 7.1 implies that the 2-point correlator (7.27) should be viewed as a boundary value of a function analytic in the domain $|z_2| < |z_1|$ where it is defined as a (convergent) power serie in $\frac{z_2}{z_1}$. The same rational function in the domain $|z_1| < |z_2|$ will be written as $(z_2 - z_1)^{-2d}$.

Proposition 7.2. *In a chiral theory satisfying both Hilbert space and energy positivity the modes J_n of a local current $J(z)$ with 2-point function (7.14) satisfy the Heisenberg* CR

$$[J_n, J_m] = n\,\delta_{n,-m}\,. \tag{7.29}$$

Proof. Local commutativity implies

$$[J(z_1), J(z_2)] = \sum_{n=0}^{n_J} A_n(z_2)\,\partial_2^n\,\delta(z_1 - z_2)\,. \tag{7.30}$$

Here the z picture δ-function is given by a formal Laurent series and obeys the defining property of a δ-function when applied to an analytic function f of z:

$$\delta(z_1 - z_2) = \sum_{n\in\mathbb{Z}} \frac{z_2^n}{z_1^{n+1}} \left(= \frac{1}{z_{12}} + \frac{1}{z_{21}} \right) \qquad \mathrm{Res}_{z_2}\,\delta(z_1 - z_2)\,f(z_2) = f(z_1)\,. \tag{7.31}$$

(Here $\frac{1}{z_{12}} = \frac{1}{z_1}\sum\limits_{n=0}^{\infty}\left(\frac{z_2}{z_1}\right)^n$, $\frac{1}{z_{21}} = \frac{1}{z_2}\sum\limits_{n=0}^{\infty}\left(\frac{z_1}{z_2}\right)^n$ have disjoint convergence domains. For a distribution F given by a formal Laurent series $F(z) = \sum\limits_n F_n\,z^n$, we set $\mathrm{Res}_z\,F = F_{-1}$.) Using conservation of scale dimension and the fact that $\partial_2^n\,\delta(z_1 - z_2)$ has dimension $n+1$ we conclude that the field A_n in (7.30) has dimension $1-n$.

Lemma 7.1. *If the dimension d of the chiral field W with 2-point function* (7.27) *is a negative integer, $d = -N$, then W violates both energy and Hilbert space positivity.*

Proof (of Lemma). The 2-point function $(z_{12})^{2N}$ corresponds to a minimal energy state $W_N \mid 0\rangle \neq 0$ of energy $-N$. The norm square (7.28) then goes into

$$(1 - z\,\bar{z})^{2N} = \sum_{n=0}^{2N} (-1)^n \binom{2N}{n} (z\,\bar{z})^n = \sum_{n=0}^{2N} \|W_{N-n} \mid 0\rangle\|^2\,(z\,\bar{z})^n$$

giving, in particular, $\|W_{N-1} \mid 0\rangle\|^2 = -2N\|W_N \mid 0\rangle\|^2$. □

Thus, our assumptions imply that $n_J = 1$: only two terms – with $n = 0$ and $n = 1$ – contribute to the sum (7.30). The uniqueness of the vacuum implies that $A_1(z)$ is a constant multiple of the identity. Comparison with (7.14) tells us that this constant is 1. The antisymmetry of the commutator under the exchange $z_1 \rightleftarrows z_2$ implies, on the other hand, $A_0 = 0$. Thus,

$$[J(z_1), J(z_2)] = \partial_2\, \delta(z_1 - z_2) = \sum_n n\, \frac{z_2^{n-1}}{z_1^{n+1}} = \sum_{m,n} [J_m, J_{-n}]\, \frac{z_2^{n-1}}{z_1^{m+1}} \tag{7.32}$$

which yields (7.29).

Exercise 7.5. Let $T(z)$ (7.16) with 2-point function (7.22) satisfy locality, energy and Hilbert space positivity. Derive the Virasoro CR:

$$[L_n, L_m] = (n - m)\, L_{n+m} + \frac{c}{12}\, (n^3 - n)\, \delta_{n,-m}\,. \tag{7.33}$$

(This is the *Lüscher-Mack theorem* – see [M88]; for a complete proof – see [FST]; for related work done in Brazil – see [Sc] [SS].)

It is the *central charge* c in (7.33), the *conformal anomaly*, which expresses the violation of the infinite dimensional conformal symmetry by the vacuum state. Note that its coefficient vanishes for $n = 0, \pm 1$, so that (7.33) reproduces the Möbius CR (7.23) as a special case.

A field ϕ is said to be *primary* if it transforms homogeneously (without anomaly) with respect to commutations with the *chiral algebra* $\mathcal{A}$. For instance *the current* $J(z)$ *is a Virasoro primary field.* It is covariant under infinitesimal reparametrizations:

$$[L_n, J(z)] = \frac{d}{dz}\, (z^{n+1}\, J(z)) \tag{7.34}$$

(J is however not primary with respect to the current algebra since (7.29) is inhomogeneous). More generally, a $2D$ field $\phi(z, \bar{z})$ is said to be primary of weight $(\Delta, \bar{\Delta})$ with respect to the *Virasoro algebra* Vir if

$$[L_n, \phi] = z^n \left(z\, \frac{\partial}{\partial z} + (n+1)\, \Delta \right) \phi\,, \quad [\bar{L}_n, \phi] = \bar{z}^n \left(\bar{z}\, \frac{\partial}{\partial \bar{z}} + (n+1)\, \bar{\Delta} \right) \phi\,. \tag{7.35}$$

The difference $s = \Delta - \bar{\Delta}$ is called the *spin* (or the *helicity*) of ϕ. Usually only fields with $2s \in \mathbb{Z}$ are encountered. Such fields live on a cylinder – *i.e.* their x-space counterparts satisfy

$$\varphi(x^0, x^1 + 2\pi) = (-1)^{2s}\, \varphi(x^0, x^1)\,. \tag{7.36}$$

Primary fields are relatively local to the observables. To check the locality of ϕ with respect to T we note that (7.35) is essentially equivalent to the *operator product expansion* (OPE)

$$T(z_1)\, \phi(z_2) = \Delta\, \frac{\phi(z_2)}{z_{12}^2} + \frac{1}{z_{12}}\, \phi'(z_2) + 0(1) \tag{7.37}$$

($O(1)$ standing for non-singular terms in z_{12}). This indeed amounts to local CR since

$$\frac{1}{z_{12}} + \frac{1}{z_{21}} = \delta(z_{12})\,, \quad \frac{1}{z_{12}^2} - \frac{1}{z_{21}^2} = \frac{\partial}{\partial z_2}\,\delta(z_{12})\,. \tag{7.38}$$

A $2D$ CFT is called *rational* if the chiral algebra $\mathcal{A}$ has a finite number of *unitary positive energy irreducible representations* (UPEIR) related to the (defining) *vacuum representation* of $\mathcal{A}$ by primary fields, relatively local to the observables. An example of a *rational conformal field theory* (RCFT) is provided by the Virasoro *minimal models* [BPZ] corresponding to central charge $c = c(m) = 1 - \frac{6}{(m+2)(m+3)}$, $m = 1, 2, \ldots$. The first chiral theory of this series, $c(1) = \frac{1}{2}$, can be viewed as generated by a free real fermion field, the Majorana[38]-Weyl field

$$\psi(z) = \sum_n \psi_{n-\frac{1}{2}}\, z^{-n}\,,\ [\psi_\rho, \psi_\sigma]_+ = \delta_{\rho,-\sigma}\,,\ \psi_\rho^* = \psi_{-\rho}\,, \quad \rho, \sigma \in \mathbb{Z} + \frac{1}{2}\,. \tag{7.39}$$

Exercise 7.6. Prove that

$$T(z) = -\frac{1}{4} \lim_{z_{1,2}\to z} \frac{\partial^2}{\partial z_1\, \partial z_2}\,\{z_{12}\,\psi(z_1)\,\psi(z_2)\} = \frac{1}{2} : \psi'(z)\,\psi(z) : \tag{7.40}$$

has all properties of the stress energy tensor with central charge $c = \frac{1}{2}$; in particular,

$$T(z_1)\,\psi(z_2) = \frac{\psi'(z_1)}{2\,z_{12}} + \frac{\psi(z_1)}{2\,z_{12}^2} + O(1) = \frac{\psi'(z_2)}{z_{12}} + \frac{1}{2}\frac{\psi(z_2)}{z_{12}^2} + O(1)$$

$$\langle 0 \mid T(z_1)\, T(z_2) \mid 0 \rangle = \frac{1}{4\, z_{12}^4}\,. \tag{7.41}$$

It can be demonstrated that the Virasoro algebra has three sectors in this case of weights $\Delta = 0, \frac{1}{16}, \frac{1}{2}$. For a general study of RCFT – see [MS] (see also the book on CFT [DMS]).

[38]Ettore Majorana (1906-1938?), one of the "ragazzi di via Panisperna"; their leader, Enrico Fermi, compares his genius with that of Galileo and Newton, adding: "Ettore had what nobody else in the world has. Unfortunately, he lacked what is instead easy to find in other men: simple common sense." Majorana disappeared on March 25, 1938 (listed among the passengers in a boat trip from Palermo to Napoli). The (real) $4D$ Majorana spinors, introduced in his last paper (of 1937), are now used to describe a massive neutrino.

Appendix D. Axioms for a chiral vertex algebra

Chiral CFT has become, starting with the work of Borcherds [B86], a domain in pure mathematics under the name of *vertex algebras*, that is already a subject of several books – see, *e.g.* [FLM] [Ka] [FB-Z] [Ga]. Our brief survey, following [DGM] and [Ka], should be viewed as a formalization and extension of the discussion of Section 7. Accordingly, we shall formulate the axioms for bosonic graded vertex algebras only, mentioning the fermionic (and superalgebra) case in a subsequent remark.

A *graded vertex algebra* consists of a $\mathbb{Z}_+$ graded pre-Hilbert vector space,

$$\mathcal{V} = \bigoplus_{n=0}^{\infty} \mathcal{V}_n \,, \quad \dim \mathcal{V}_0 = 1 \,, \quad \dim \mathcal{V}_n < \infty \tag{D.1}$$

equipped with a *translation operator* T $(= L_{-1})$ and a *state field correspondence* $Y : \mathcal{V} \to (\mathrm{End}\, V)[[z, z^{-1}]]$ (read: Y is a map from $\mathcal{V}$ to the space of formal Laurent series $Y(v,z)$, $v \in \mathcal{V}$ whose coefficients are endomorphisms – *i.e.* linear operators from $\mathcal{V}$ to $\mathcal{V}$) satisfying the following axioms.

(i) *Vacuum* : the 1-dimensional space $\mathcal{V}_0$ is spanned by the *vacuum vector* $\mid 0\rangle$ such that

$$T \mid 0\rangle = 0 \,, \quad \langle 0 \mid 0\rangle = 1 \,. \tag{D.2}$$

(ii) *Translation covariant fields* : to each vector $v \in \mathcal{V}$ there corresponds a formal Laurent series $Y(v,z)$ with operator valued coefficients such that (a) *the vector valued function*

$$Y(v,z) \mid 0\rangle = e^{zT}\, v \,, \tag{D.3}$$

is analytic (in the norm topology) for $|z| < 1$; furthermore

$$[T, Y(v,z)] = \frac{d}{dz}\, Y(v,z) \,. \tag{D.4}$$

(b) Assuming *linearity* in the vector argument v, *i.e.* requiring

$$Y(c_1\, v_1 + c_2\, v_2, z) = c_1\, Y(v_1, z) + c_2\, Y(v_2, z) \quad \text{for} \quad v_1, v_2 \in \mathcal{V} \,,\ c_1, c_2 \in \mathbb{C} \,, \tag{D.5}$$

we can define Y by first displaying its properties for *homogeneous elements*, $v_\ell \in \mathcal{V}_\ell$; then

$$Y(v_\ell, z) = \sum_n Y_n(v_\ell)\, z^{-n-\ell} \tag{D.6}$$

where (c) $Y_n(v_\ell)$ *changes the grading* by $-n$:

$$Y_n(v_\ell) : \mathcal{V}_k \to \mathcal{V}_{k-n} \qquad (\mathcal{V}_{k-n} = \{0\} \ \text{ for } \ k < n) \,. \tag{D.7}$$

Eq. (D.7), together with (D.1), is our *energy positivity* requirement. We identify the chiral vertex algebra Hamiltonian with the *Virasoro energy* L_0 satisfying

$$(L_0 - n)\, \mathcal{V}_n = 0 \,, \quad [L_0, T] = T \,. \tag{D.8}$$

Formal Laurent series of different arguments, $Y(v_1, z_1)$, $Y(v_2, z_2)$, can be multiplied giving a formal Laurent series $Y(v_1, z_1) \cdot Y(v_2, z_2)$ of two variables.

(iii) *Local commutativity* :

$$(z_{12})^N [Y(v_1, z_1), Y(v_2, z_2)] = 0 \quad \text{for} \quad N \gg 0 \,. \tag{D.9}$$

We denote by $\mathcal{A}(\mathcal{V})$ the set of formal power series Y satisfying the axioms (i–iii). The following Proposition, singled out by Goddard (see [DGM]), justifies the notation $Y(v, z)$.

Proposition D.1. *If two formal Laurent series $Y_1(v, z)$ and $Y_2(v, z)$ belong to $\mathcal{A}(\mathcal{V})$ (and hence satisfy* (D.3) *with the same v) then they coincide.*

Sketch of proof. Using locality one finds that the difference $Y_1(v, z) - Y_2(v, z)$ vanishes not just on the vacuum but on any other vector $v_1 \in \mathcal{V}$. □

This uniqueness result has a number of applications. We single out the following

Corollary D.1. *It follows from Proposition* D.1 *that*

(a) $Y(\mid 0\rangle, z) = \mathbb{I}$;

(b) $Y(Tv, z) = \frac{d}{dz}\, Y(v, z)$.

Exercise D.1. Prove, using energy positivity, that the Laurent series $Y(v_1, z)\, v_2$ has a finite number of negative powers of z. Demonstrate that for energy eigenstates, $(L_0 - d_i)\, v_i = 0$ for $i = 1, 2$, the leading negative power does not exceed $d_1 + d_2$.

Studying OPE of products of elements of $\mathcal{A}(\mathcal{V})$, it is useful to extend the definition of $Y(v, z)$ to v of the form $Y(v_1, w)\, v_2$ (which is not a finite energy state).

Exercise D.2. Demonstrate that both sides of the equality

$$Y(v_1, z_1)\, Y(v_2, z_2) \mid 0\rangle = Y(Y(v_1, z_{12})\, v_2, z_2) \mid 0\rangle \tag{D.10}$$

define analytic (in the Hilbert norm topology) vector valued functions for $|z_2| < |z_1| < 1$ and sufficiently small $|z_{12}|$ and that the equality (D.10) holds.

The stress-energy tensor $T(z)$ can be identified with $Y(L_{-2} \mid 0\rangle, z)$.

A field $Y(v, z)$ is primary of Vir (*cf.* (7.35)) if v is a *ground state*:

$$L_n\, v = 0 \,, \quad \text{for} \quad n > 0 \,, \;\; (L_0 - d)\, v = 0 \,. \tag{D.11}$$

8 Extensions of the $u(1)$ current algebra and their representations

Given a finite dimensional Lie algebra $\mathcal{G}$ with (real) *structure constants* $f_c^{ab} = -f_c^{ba}$ (satisfying the *Jacobi*[39] *identity* $f_s^{ab}\, f_d^{sc} + f_s^{ca}\, f_d^{sb} + f_s^{bc}\, f_d^{sa} = 0$), we can define a *Kac-Moody algebra* generated by (hermitian) currents $J^a(z)$ given by formal power series

$$J^a(z) = \sum_{n\in\mathbb{Z}} J_n^a\, z^{-n-1}\ (J_n^{a*} = J_{-n}^a)\,, \quad a = 1,\dots,d_{\mathcal{G}} := \dim\mathcal{G}\,, \tag{8.1}$$

where the modes J_n^a satisfy the CR

$$[J_m^a, J_n^b] = i\, f_c^{ab}\, J_{m+n}^c + k\, m\, \delta_{m,-n}\, g^{ab} \tag{8.2}$$

(g^{ab} standing for a $\mathcal{G}$ invariant positive metric).

The representation theory of the $u(1)$ current algebra (7.29) (which appears as a special case of (8.2) for $d_{\mathcal{G}} = 1$ and $f_c^{ab} = 0$) is relatively simple – this is, in fact, an infinite Heisenberg[40] algebra whose positive energy representations are labeled by the eigenvalues of the charge operator J_0.

We define the normal product: $J(z_1)\, J(z_2)$: of two $u(1)$ currents through their OPE

$$J(z_1)\, J(z_2) = \frac{1}{z_{12}^2} + : J(z_1)\, J(z_2) : \ . \tag{8.3}$$

Normal products $: J^n(z) :$ belong to the chiral algebra $\mathcal{A}(\mathcal{V}_J)$ where $\mathcal{V}_J$ is the space generated by polynomial of the current's negative modes J_{-n} acting on the vacuum.

Exercise 8.1. The *Sugawara stress tensor* of $\mathcal{A}(\mathcal{V}_J)$,

$$T(z) = \frac{1}{2} : J^2(z) :\,, \tag{8.4}$$

satisfies the defining OPEs for a J, T system (*cf.* (7.37)):

$$T(z_1)\, J(z_2) = \frac{1}{z_{12}^2}\, J(z_1) + O(1) = \frac{1}{z_{12}^2}\, J(z_2) + \frac{1}{z_{12}}\, J'(z_2) + O(1)\,, \tag{8.5}$$

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[39] The German mathematician Carl Gustav Jacob Jacobi (1804-1851) was considered to be the most inspiring teacher of his time. Bourbaki, in particular, Jean Dieudonné (1906-1992), have taken as a motto his words (from a letter to Legendre of 1830, deploring the fact that Fourier introduces his series just as an application to the heat equation): "le but unique de la science c'est l'honneur de l'esprit humain". The phrase "Invert, always invert" is associated with Jacobi who believed that many hard problems are solved when addressed backwards. Most of his papers were published post humously.

[40] Werner Heisenberg (1901-1976) has been awarded in 1932 the Nobel Prize in Physics for the creation of quantum mechanics (1925). The CR $[q,p] = i\hbar$ first appeared in work of Born-Jordan and of Dirac.

$$T(z_1)\,T(z_2) = \frac{1}{2}\,\frac{1}{z_{12}^4} + \frac{:J(z_1)\,J(z_2):}{z_{12}^2} + O(1)$$
$$= \frac{1}{2}\,\frac{1}{z_{12}^4} + 2\,\frac{T(z_2)}{z_{12}^2} + \frac{T'(z_2)}{z_{12}} + O(1)\,. \qquad (8.6)$$

Deduce that for a chiral CFT generated by $J(z)$ the Virasoro central charge is $c = 1$.

Exercise 8.2. Prove that Eq. (8.4) allows to write the Virasoro modes in terms of J_ℓ:

$$L_0 = \frac{1}{2}\,J_0^2 + \sum_{\ell=1}^{\infty} J_{-\ell}\,J_\ell\,, \quad L_n = \frac{1}{2}\sum_{\ell\in\mathbb{Z}} J_{n-\ell}\,J_\ell \quad \text{for} \quad n \neq 0\,. \qquad (8.7)$$

Verify the CR (7.33) (7.34) for these expressions.

Proposition 8.1. *The unitary irreducible positive energy representations (UIPERs) of $\mathcal{A}(\mathcal{V}_J)$ correspond to ground states $\mid g\rangle$ labeled by a real number g such that*

$$(J_0 - g) \mid g\rangle = 0 = J_n \mid g\rangle \quad \text{for} \quad n > 0 \qquad (8.8)$$

($g = 0$ corresponding to the defining vacuum UIPER of $\mathcal{A}(\mathcal{V}_J)$). To each $g \neq 0$ corresponds a pair $\psi(z, \pm g)$ of hermitian conjugate primary fields of dimension g^2 such that for each of them

$$\psi(z, g) \mid 0\rangle = e^{zL_{-1}} \mid g\rangle\,, \quad [J(z_1), \psi(z_2, g)] = g\,\delta(z_{12})\,\psi(z_2, g)\,. \qquad (8.9)$$

(Thus all $\psi(z, g)$ are relatively local to the current.) $\psi(z, \pm g)$ *locally commute among themselves iff g^2 is an even integer; then*

$$(z_{12})^{g^2}[\psi(z_1, g), \psi(z_2, -g)] = 0 (= [\psi(z_1, \pm g), \psi(z_2, \pm g)])\,. \qquad (8.10)$$

Sketch of proof (for a comprehensive discussion – see [BMT]). Introduce the *abelian*[41] (*i.e.* commutative) group $\{E^{ng}, n \in \mathbb{Z}\}$ of unitary operators such that

$$E^g \mid 0\rangle = \mid g\rangle \quad (E^g)^* = E^{-g} = (E^g)^{-1}\,, \quad [J(z), E^g] = \frac{g}{z}\,. \qquad (8.11)$$

Introduce further the indefinite integrals of the frequency parts of the current:

$$\varphi_+(z) = \sum_{n=1}^{\infty} \frac{1}{n}\,J_{-n}\,z^n\,, \quad \varphi_-(z) = -\sum_{n=1}^{\infty} \frac{1}{n}\,J_n\,z^{-n}\,; \qquad (8.12)$$

then the *chiral vertex operator* (CVO) $\psi(z, g)$ can be written in the form

$$\psi(z, g) = E^g\,e^{g\varphi_+(z)}\,z^{gJ_0}\,e^{g\varphi_-(z)}\,. \qquad (8.13)$$

[41] Named after the Norwegian mathematician Niels Henrik Abel (1802-1829) who proved in 1824 the impossibility to solve the general fifth degree equation in radicals and created (in 1825, in Freiburg) the theory of elliptic, hyperelliptic (and, more generally, *abelian*) functions.

To verify the current-field CR (8.9) we use (8.11) and

$$[J(z_1), \varphi_+(z_2)] = \frac{1}{z_{12}} - \frac{1}{z_1}\,, \quad [J(z_1), \varphi_-(z_2)] = \frac{1}{z_{21}}\,. \tag{8.14}$$

We have $\left(L_0 - \frac{g^2}{2}\right) \mid g\rangle = 0$ as a consequence of (8.7) and (8.8); hence, $\psi(z, \pm g)$ have scale dimension $\frac{g^2}{2}$, so that the (non-vanishing) 2-point function is

$$\langle 0 \mid \psi(z_1, g)\, \psi(z_2, -g) \mid 0\rangle = (z_{12})^{-g^2}\,. \tag{8.15}$$

As the sign of g is not fixed and g^2 is even the 2-point function is symmetric (viewed as a rational function) with respect to the exchange of factors.

Exercise 8.3. Verify (8.15) using (8.13).

Eq. (8.15) and the remark that the singularity of the 2-point function dominates those of higher point correlation functions as a consequence of Wightman positivity imply the strong locality condition (8.10). □

Exercise 8.4. Use the CR $[J_0, E^g] = g\, E^g$ to prove

$$\langle 0 \mid E^g \mid 0\rangle = \delta_{g0} \qquad (i.e.\ \langle 0 \mid E^g \mid 0\rangle = 0 \quad \text{for} \quad g \neq 0\,, \quad E^0 = 1\!\!1)\,. \tag{8.16}$$

Exercise 8.5. Use (8.13) to compute the n-point function

$$\langle 0 \mid \psi(z_1, g_1) \dots \psi(z_n, g_n) \mid 0\rangle = \prod_{1 \le i < j \le n} (z_{ij})^{g_i g_j}\,. \tag{8.17}$$

It follows from Proposition 8.1 that the CFT of the chiral algebra $\mathcal{A}(\mathcal{V}_J)$ has a continuum of inequivalent UIPERs and hence, is *not* rational (recall the definition at the end of Section 7). On the other hand, if g^2 is a (positive) even integer then the algebra $\mathcal{A}(g^2)$ generated by the pair of oppositely charged Bose fields $\psi(z, \pm g)$ provides a *local extension* of $\mathcal{A}(\mathcal{V}_J)$. Indeed, the current J is contained in the OPE of the product $\psi(z_1, g)\, \psi(z_2, -g)$ which defines a bilocal field ([FST]):

$$\begin{aligned} & z_{12}^{g^2}\, \psi(z_1, g)\, \psi(z_2, -g) =: e^{g \int_{z_2}^{z_1} J(z) dz} := 1\!\!1 + g \int_{z_2}^{z_1} J(z)\, dz \\ + \quad & 6g^2 \int_{z_2}^{z_1} \frac{(z_1 - z)(z - z_2)}{z_{12}}\, T(z)\, dz + g^3\, z_{12}^3\, R_3(z_1, z_2; g)\,, \end{aligned}$$

$$\begin{aligned} \langle 0 \mid J(z_1)\, T(z_2) \mid 0\rangle = 0 \quad &= \quad \langle 0 \mid J(z_1)\, R(z_1, z_2; g) \mid 0\rangle \\ &= \quad \langle 0 \mid T(z_1)\, R(z_1, z_2; g) \mid 0\rangle\,. \end{aligned} \tag{8.18}$$

The following set of exercises is designed to establish (8.18) (explaining on the way its meaning).

Exercise 8.5. Prove the equivalence of the following two definitions of the normal exponent in the first equation (8.18):

$$: e^{g\int_{z_2}^{z_1} J(z)\,dz} := e^{g(\varphi_+(z_1)-\varphi_+(z_2))}\, z_{12}^{gJ_0}\, e^{g(\varphi_-(z_1)-\varphi_-(z_2))} \tag{8.19}$$

$$: e^{g\int_{z_2}^{z_1} J(z)dz} := \frac{e^{g\int_{z_2}^{z_1} J(z)dz}}{\langle 0 \mid e^{g\int_{z_2}^{z_1} J(z)dz} \mid 0\rangle}\,. \tag{8.20}$$

(Eq. (8.20) should be understood as an expansion in powers of g defining the corresponding normal products iteratively.) Use (8.19) to verify the first equation (8.18).

Exercise 8.6. Prove that the CR (8.9) is equivalent to the following CR between the frequency parts of the current and the charged field ψ:

$$[\psi(z_1,g), J_{(+)}(z_2)] = -\frac{g}{z_{12}}\,\psi(z_1,g)\,,\ [J^{(-)}(z_1),\psi(z_2,g)] = \frac{g}{z_{12}}\,\psi(z_2,g) \tag{8.21}$$

for

$$J_{(+)}(z) = \sum_{n=1}^{\infty} J_{-n}\, z^{n-1} = \varphi_+'(z)\,,\ J^{(-)}(z) = \sum_{n=0}^{\infty} \frac{J_n}{z^{n+1}} = \varphi_-'(z)\,. \tag{8.22}$$

Exercise 8.7. Use (8.21) and the vacuum conditions

$$J^{(-)}(z) \mid 0\rangle = 0 = \langle 0 \mid J_{(+)}(z) \tag{8.23}$$

to prove the *Ward*[42] *identity* for current-field correlation functions:

$$\begin{aligned} &\langle 0 \mid \psi(z_1,g_1)\ldots\psi(z_k,g_k)\, J(z)\,\psi(z_{k+1},g_{k+1})\ldots\psi(z_n,g_n) \mid 0\rangle \\ =\ &\left(\sum_{j=k+1}^{n} \frac{g_j}{z-z_j} - \sum_{i=1}^{k} \frac{g_i}{z_i - z}\right) \langle 0 \mid \psi(z_1,g_1)\ldots\psi(z_n,g_n) \mid 0\rangle\,. \end{aligned} \tag{8.24}$$

Thus the Ward identities allow to express current-charge fields correlation functions in terms of charged fields correlations. We find, in particular, the 3-point function

$$\langle 0 \mid \psi(z_1,g)\,\psi(z_2,-g)\,J(z_3) \mid 0\rangle = \frac{g}{z_{13}\,z_{23}}\, z_{12}^{1-g^2}\,. \tag{8.25}$$

[42] John Clive Ward (1924-2000), British physicist; the Ward identity in quantum electrodynamics relates the renormalization of the wave function of the electron to its vertex function renormalization.

Exercise 8.8. Derive in a similar manner the Ward-Takahashi identities for correlation functions of ψ's with the stress energy tensor; deduce from this the expression for the 3-point function

$$\langle 0 \mid \psi(z_1, g)\, \psi(z_2, -g)\, T(z_3) \mid 0 \rangle = \frac{g^2}{2}\, \frac{z_{12}^{2-g^2}}{z_{13}^2\, z_{23}^2}\,. \tag{8.26}$$

Exercise 8.9. Use (8.25) (8.26) and the orthogonality relations in Eq. (8.18) to verify the third equation (8.18).

Remark 8.1. The algebra $\mathcal{A}(g^2)$ contains charged fields $\psi(z, ng)$ of all charges multiple of g ($n \in \mathbb{Z}$). They appear as "composite fields" in OPEs of $\psi(z, \pm g)$. We have the following iterative rule:

$$\psi(z_1, g)\, \psi(z_2, ng) = z_{12}^{ng^2} \{\psi(z_2, (n+1)g) + O(z_{12})\}\,. \tag{8.27}$$

Thus the (isomorphic to $\mathbb{Z}$) group of all powers of U_g, introduced in the "Sketch of proof" of Proposition 8.1, is realized in the vacuum representation of $\mathcal{A}(g^2)$. It follows that $\mathcal{A}(m^2\, g^2)$ (for $m > 1$ integer) is a proper subalgebra if $\mathcal{A}(g^2)$.

One can, sure, also consider the CVO $\psi(z, g)$ for any (positive) integer g^2; the odd g^2 then correspond to Fermi fields. The local commutativity property (8.10), extends in this case to a *graded local commutativity*:

$$z_{12}^{g^2}\, \psi(z_1, g)\, \psi(z_2, -g) = z_{21}^{g^2}\, \psi(z_2, -g)\, \psi(z_1, g) \quad \text{for} \quad g^2 \in \mathbb{N}\,. \tag{8.28}$$

The chiral algebra $\mathcal{A}(4(2\nu + 1))$ appears as the bosonic part of the ($\mathbb{Z}_2$ graded) chiral superalgebra $\mathcal{A}(2\nu + 1)$, $\nu = 0, 1, \ldots$.

Exercise 8.10. Let $G^{\pm}(z) = \sqrt{\frac{2}{3}}\, \psi(z, \pm\sqrt{3})$ and normalize the associated $u(1)$ current $J(z)$ so that to exclude irrationalities in the OPE (8.18), setting $3\, z_{12}^2 \langle 0 \mid J(z_1)\, J(z_2) \mid 0 \rangle = 1$. Prove the anticommutation relations

$$[G^+(z_1), G^-(z_2)]_+ = 2\, T(z_2)\, \delta(z_{12}) + (J(z_1) + J(z_2))\, \partial_2\, \delta(z_{12}) + \frac{1}{3}\, \partial_2^2\, \delta(z_{12})$$

$$\left(\langle 0 \mid G^+(z_1)\, G^-(z_2) \mid 0 \rangle = \frac{2}{3\, z_{12}^3} \right)\,. \tag{8.29}$$

Setting further

$$G^{\pm}(z) = \sum_n G^{\pm}_{n-\frac{1}{2}}\, z^{-n-1}\,, \tag{8.30}$$

deduce the modes' (anti)commutation relations

$$[G^{\pm}_{n-\frac{1}{2}}, G^{\mp}_{\frac{1}{2}-m}]_+ = 2\, L_{n-m} \pm (n + m - 1)\, J_{n-m} + \frac{n(n-1)}{3}\, \delta_{nm}$$

$$[J_n, J_m] = \frac{n}{3}\,\delta_{n,-m}\,, \quad [J_n, G^{\pm}_{\rho}] = \pm\, G^{\pm}_{n+\rho}\,. \tag{8.31}$$

This is the (vacuum) *Neveu-Schwarz sector* of the $N = 2$ (extended) superconformal model ([BFK] [G88]).

The chiral algebras $\mathcal{A}(g^2)$ for integer $g^2 > 0$ provide te simplest examples of *rational CFT*.

Proposition 8.2. *The algebra $\mathcal{A}(g^2)$ for $g^2 = 2, 4, 6, \dots$ has g^2 UIPERs generated by primary CVO $\psi(z, e_k)$, relatively local to $\psi(z, g)$. They correspond to $g\,e_k = k$, $1 - \frac{g^2}{2} \le k \le \frac{g^2}{2}$. The fusion rules for the primary fields $\psi(z, e_k)$ are given by the multiplication rules of the finite cyclic group of g^2 elements*

$$\frac{\mathbb{Z}\,e_1}{\mathbb{Z}\,g} \simeq \frac{\mathbb{Z}}{g^2\,\mathbb{Z}}\,. \tag{8.32}$$

Sketch of proof (see [BMT]). Any UIPER of $\mathcal{A}(g^2)$ gives rise to a fully reducible unitary positive energy representation of the $u(1)$ current algebra $\mathcal{A}(\mathcal{V}_J)$ whose spectrum of J_0 is contained in the set $e + \mathbb{Z}\,g$ for some (real) e. The OPE

$$\psi(z, g)\,\psi(0, e) \mid 0\rangle = \psi(z, g) \mid e\rangle = z^{ge}(\mathbb{I} + O(z)) \mid g + e\rangle \tag{8.33}$$

only corresponds to a relatively local ground state $\mid e\rangle$ if it is single valued, *i.e.* if the power ge of z is an integer. Noting that e is determined $\mod ng$ ($n \in \mathbb{Z}$) we can choose $|e| \le \frac{g}{2}$. The rest is straightforward. □

The *field algebra* $\mathcal{F}\left(\frac{1}{g^2}\right)$ ($\supset \mathcal{A}(g^2)$) generated by the pair of charged primary fields $\psi\left(z, \pm\frac{1}{g}\right)$ admits a finite cyclic *group of global gauge transformations* acting on the state space by powers of the operator

$$U (= U_{1/g}) = e^{2\pi i \frac{J_0}{g}}\,, \quad U^{g^2}\mathcal{F}\left(\frac{1}{g^2}\right) = \mathbb{I}\,\mathcal{F}\left(\frac{1}{g^2}\right)\,. \tag{8.34}$$

It generates an automorphism of the field algebra such that

$$U\,\psi(z, e)\,U^{-1} = e^{2\pi i \frac{e}{g}}\,\psi(z, e) \Rightarrow U A\,U^{-1} = A \quad \text{for} \quad A \in \mathcal{A}(g^2)\,. \tag{8.35}$$

Remark 8.2. The primary chiral vertex operator $\psi(z, e)$ is a multivalued function of z. In fact, the extension of the relation (8.33) to two primary charges e_1, e_2 ($e_i\,g \in \mathbb{Z}$),

$$\psi(z, e_1) \mid e_2\rangle = z^{e_1 e_2}(1 + O(z)) \mid e_1 + e_2\rangle$$

is a multivalued function of z unless $e_1 e_2$ is also an integer. Setting $z = e^{it}$ we find a charge dependent *twisted periodicity* condition for ψ as a function of t:

$$\psi(e^{i(t+2\pi)}, e_1) \mid e_2\rangle = e^{2\pi i e_1 e_2}\;\psi(e^{it}, e_1) \mid e_2\rangle\,. \tag{8.36}$$

The exchange relations of $\psi(z, e)$ with itself give rise to a nontrivial one-dimensional representation of the braid group which defines for non-integer e^2 an *anyonic statistics*. (The idea for such statistics appears already in [LM]. More on the ancestry of the "anyon" can be found in [BLSW] – *cf.* Section 2.) For $eg \in \mathbb{Z}$ such an anyonic representation of $\mathcal{B}_2$ is isomorphic to a finite cyclic group. (A bound state of g^2 anyons obeys the Bose-Fermi alternative.) More general lattice vertex algebras yielding anyonic statistics are applied to the description of the fractional quantum Hall effect, [FKST] [FST] (see also [CGT] where the intriguing plateau with Hall conductivity $\sigma_H = \frac{5}{2}\,\frac{e^2}{h}$ is considered).

9 The $su(2)$ current algebra model. Knizhnik-Zamolodchikov equation

The simplest models associated with a (non-abelian) braid group statistics are the *affine Kac-Moody current algebra models* with a chiral algebra $\mathcal{A}_k(\mathcal{G})$ determined by a simple Lie algebra $\mathcal{G}$ and an *integer level*[43] k $(= 1, 2, \ldots)$. The simplest among the simple Lie algebras (corresponding to a compact Lie group) is $\mathcal{G} = su(2)$ spanned by three generators J_0^a, $a = 1, 2, 3$ satisfying

$$[J_0^a, J_0^b] = i\,\varepsilon^{abc}\,J_0^c \tag{9.1}$$

where ε^{abc} is the totally antisymmetric Levi-Civita tensor ($\varepsilon^{123} = 1 = \varepsilon^{312} = -\varepsilon^{321} = \ldots$). The corresponding local currents are given by (8.1) (with $d_{su(2)} = 3$), the current modes satisfying the CR (8.2) with

$$f_c^{ab} = \varepsilon^{abc}\,, \qquad g^{ab} = \frac{1}{2}\,\delta^{ab}\,. \tag{9.2}$$

The resulting infinite dimensional Lie algebra is denoted by $\widehat{su}(2)_k$.

Exercise 9.1. Verify that the $\widehat{su}(2)_k$ Sugawara chiral stress-energy tensor

$$T(z) = \frac{1}{h} : \vec{J}^2(z) :\equiv \frac{1}{h}\sum_{a=1}^{3} : [J^a(z)]^2 :\,, \quad h = k+2 \tag{9.3}$$

gives rise to the OPE (8.5) with $J(z)$ substituted by $J^a(z)$ while the second equation (8.6) is replaced by

$$T(z_1)\,T(z_2) = \frac{c_k}{2\,z_{12}^4} + \frac{T(z_1)+T(z_2)}{z_{12}^2} + O(1)\,, \quad c_k = \frac{3k}{k+2}\,. \tag{9.4}$$

The *renormalized level* $h = k+2$ is also called the *height* of $\mathcal{A}_k(A_1) = \widehat{su}(2)_k$.

Exercise 9.2. Prove that for $k = 1$ $(= c_1)$ and $J(z) = \sqrt{2}\,J^3(z)$, the stress tensor (9.3) coincides with (8.4) while the "charged components" of the current are reproduced by the vertex operator construction (8.13):

$$J^{\pm}(z) := J^1(z) \pm i\,J^2(z) = E^{\pm\sqrt{2}}\,e^{\pm\sqrt{2}\,\varphi_+(z)}\,z^{\pm\sqrt{2}\,J_0}\,e^{\pm\sqrt{2}\,\varphi_-(z)} \tag{9.5}$$

[43] Speaking of an associative chiral algebra $\mathcal{A}_k$ characterized by a natural number k (or of the corresponding infinite dimensional Lie algebra) we depart from the terminology of the theory of affine Kac-Moody algebras $\hat{\mathcal{G}}$ ([Kac]) in which the central charge, say K, is an operator commuting with the current modes. The algebra $\mathcal{A}_k$ would then correspond to a representation of $\hat{\mathcal{G}}$ in which we have chosen a particular eigenvalue k of the central charge K (thus specifying the vacuum of the theory).

with $\varphi_\pm$ given by (8.12). (*Hint* : verify that $T(z)$ (8.4) (for $J(z) = \sqrt{2}\, J^3(z)$) and $J^\pm(z)$ satisfy the correct OPE (or CR); deduce furthermore the OPE

$$J^+(z_1)\, J^-(z_2) = \frac{1}{z_{12}^2} \left\{ 1 + 2 \int_{z_2}^{z_1} J^3(z)\, dz + z_{12}(T(z_1) + T(z_2)) + O(z_{12}^2) \right\} \tag{9.6}$$

for $[J^1(z_1), J^2(z_2)] = J^3(z_2)\, \delta(z_{12})$, $\langle 0 \mid J^a(z_1)\, J^b(z_2) \mid 0 \rangle\, z_{12}^2 = \frac{1}{2}\, \delta^{ab}$.)

Exercise 9.2 demonstrates that we may only expect to encounter non-abelian braid group statistics for $k \geq 2$.

The representation theory of affine Kac-Moody algebras [Kac] tells us that $\widehat{su}(2)_k$ admits $k+1$ UIPERs labeled by the values I of the isospin of the ground states of *integrable representations*, such that $2I \leq k$. We use here the physicist terminology: a *ground state* is a lowest energy state with respect to the *conformal energy operator* $H_c = L_0 + \bar{L}_0$. As L_0 and $\bar{L}_0$ commute, it factorizes into a product a ground states for the left and right movers' current algebras. We shall only mention *diagonal representations* (with $I = \bar{I}$) in this brief synopsis of the $\widehat{su}(2)_k$ CFT model and will spell out the properties of the chiral (say, left movers') representations.

Exercise 9.3. (a) Prove, using (7.21), that the chiral energy operator L_0 commutes with the currents' zero modes: $[L_0, J_0^a] = 0$, $a = 1, 2, 3$. Deduce, as a corollary that the subspace of ground states of isospin I has dimension $p \equiv 2I+1$. A basis $|k, II_3\rangle$ of ground states in which J_0^3 is diagonal is characterized by

$$J_n^a \mid k, II_3\rangle = 0 \quad \text{for} \quad n > 0\,, \quad (J_0^3 - I_3) \mid k, II_3\rangle = 0\,,$$
$$I_3 = -I, 1-I, \ldots, I \quad (2I = 0, 1, \ldots, k)\,. \tag{9.7}$$

(b) Prove as a consequence of (9.7) and the Sugawara formula (9.3) that

$$L_0 = \frac{1}{h} \left(\vec{J}_0^{\,2} + 2 \sum_{n=1}^{\infty} \vec{J}_{-n}\, \vec{J}_n \right), \quad (L_0 - \Delta_I) \mid k, II_3\rangle = 0\,,$$
$$\Delta_I = \frac{I(I+1)}{h} \quad (h = k+2)\,. \tag{9.8}$$

The 2D primary field $\phi_I(z, \bar{z})$ which intertwines the vacuum representation of $\widehat{su}(2)_k \oplus \widehat{su}(2)_k$ with the one of weight (I, I) is thus a $(2I+1) \times (2I+1)$ component isospin tensor. In particular, the *step operator* $\phi_{\frac{1}{2}}$ can be viewed as an $SU(2)$ "group valued" field $g(z, \bar{z}) = \{g(z, \bar{z})_B^A\,,\ A, B = 1, 2\}$. The quotation marks should remind us that the quantum field $g(z, \bar{z})$ is actually a 2×2 matrix of operator valued distributions; only its classical counterpart can be assumed to belong to $SU(2)$. The 2-point function of g factorizes:

$$\langle 0 \mid g(z_1, \bar{z}_1)_{B_1}^{A_1}\, g(z_2, \bar{z}_2)_{B_2}^{A_2} \mid 0 \rangle = \frac{\epsilon^{A_1 A_2}}{(z_{12})^{2\Delta}}\ \frac{\epsilon_{B_1 B_2}}{(\bar{z}_{12})^{2\Delta}}$$

$$(\epsilon^{12} = \epsilon_{12} = 1 = -\epsilon_{21})\,, \quad \Delta = \Delta_{\frac{1}{2}} = \frac{3}{4h}\,.$$

Its 4-point function, however, does not (for $k > 1$). This suggests writing $g(z, \bar{z})$ as a matrix product of chiral fields:

$$g(z, \bar{z})^A_B = g(z)^A_\alpha\, \bar{g}^{-1}(\bar{z})^\alpha_B \left(\equiv \sum_{\alpha=1}^{2} g(z)^A_\alpha\, \bar{g}^{-1}(\bar{z})^\alpha_B\right). \tag{9.9}$$

The 2D field $g(z, \bar{z})$ provides an example of a conformal but not GCI field, as its correlation functions are not rational. It is also locally commuting but violates the stronger Huygens locality (6.18). Note that for Euclidean compactified space-time $t \to it$, z and $\bar{z}$ are complex conjugate ($z = e^{-t+ix}$, $\bar{z} = e^{-t-ix}$); locality then implies that $g(z, \bar{z})$ should be periodic in x (*i.e.* single valued in z):

$$\begin{aligned} e^{2\pi i(L_0 - \bar{L}_0)}\, g(z, \bar{z})\, e^{2\pi i(\bar{L}_0 - L_0)} &= g(e^{2\pi i}\, z, e^{-2\pi i}\, \bar{z}) = g(z, \bar{z})\,, \\ e^{2\pi i L_0}\, g(z, \bar{z})\, e^{-2\pi i L_0} &= e^{2\pi i \Delta}\, g(e^{2\pi i}\, z, \bar{z})\,. \end{aligned} \tag{9.10}$$

This only implies that the chiral components of $g(z, \bar{z})$ appearing in the right hand side of (9.9), should have the same *monodromy* M:

$$\begin{aligned} e^{2\pi i L_0}\, g(z)\, e^{-2\pi i L_0} &= e^{2\pi i \Delta}\, g(z\, e^{2\pi i}) = g(z)\, M\ (= g(z)^A_\sigma\, M^\sigma_\alpha)\,, \\ e^{-2\pi i \Delta}\, \bar{g}(\bar{z}\, e^{-2\pi i}) &= \bar{g}(\bar{z})\, M \end{aligned} \tag{9.11}$$

which will then cancel in the product (9.9).

Exercise 9.4. Use (9.11) to prove $(M - q^{-3/2}) \mid 0\rangle = 0$ for $q = e^{-\frac{i\pi}{h}}$.

The chiral fields $g(z)$ and $\bar{g}(\bar{z})$ satisfy a differential equation involving the $\widehat{su}(2)$ currents which follow from the Ward(-Takahashi) identities and from the Sugawara formula. In order to write it down it is convenient to combine the three components $J^a(z)$ of the current into a second degree polynomial in a formal variable ζ:

$$J(z, \zeta) = J^-(z) + 2\,\zeta\, J^3(z) - \zeta^2\, J^+(z) \quad (J^\pm = J^1 \pm i\, J^2)\,. \tag{9.12}$$

We leave it to the reader to verify that then the 2- and 3-point functions of the current assume the form:

$$\begin{aligned} \langle 0 \mid J(z_1, \zeta_1)\, J(z_2, \zeta_2) \mid 0\rangle &= -k\, \frac{\zeta_{12}^2}{z_{12}^2} \quad (\zeta_{12} = \zeta_1 - \zeta_2) \\ \langle 0 \mid J(z_1, \zeta_1)\, J(z_2, \zeta_2)\, J(z_3, \zeta_3) \mid 0\rangle &= k\, \frac{\zeta_{12}\, \zeta_{23}\, \zeta_{13}}{z_{12}\, z_{23}\, z_{13}}\,. \end{aligned} \tag{9.13}$$

A chiral $\widehat{su}(2)_k$ primary field ϕ_I (of isospin I) has both $SU(2)$ and U_q indices (like $g(z)^A_\alpha$ for $I = \frac{1}{2}$) and can be viewed, alternatively, as a polynomial (of degree $2I$) in two formal variables ζ and u, respectively. We shall expand its 4-point function in the U_q-invariant amplitudes $\mathcal{J}_\lambda^{(I)}(u_1, \dots, u_4)$ (4.29):

$$\begin{aligned} & \langle 0 \mid \phi_I(z_1,\zeta_1;u_1)\,\phi_I(z_2,\zeta_2;u_2)\,\phi_I(z_3,\zeta_3;u_3)\,\phi_I(z_4,\zeta_4;u_4) \mid 0\rangle \\ = & \sum_{\lambda=0}^{2I} w_\lambda(z_1,\zeta_1;z_2,\zeta_2;z_3,\zeta_3;z_4,\zeta_4)\,\mathcal{J}_\lambda^{(I)}(u_1,u_2,u_3,u_4)\,. \end{aligned} \tag{9.14}$$

The properties of the primary field ϕ_I are determined by its commutation relations with the modes $J_n(\zeta)$ of the current encoded in:

$$\begin{aligned} [J^{(-)}(z_1,\zeta_1),\phi_I(z_2,\zeta_2;u)] &= -\frac{1}{z_{12}}\,(\zeta_{12}^2\,\partial_{\zeta_2} + 2I\,\zeta_{12})\,\phi_I(z_2,\zeta_2;u) \\ [\phi_I(z_1,\zeta_1;u),J_{(+)}(z_2,\zeta_2)] &= \frac{1}{z_{12}}\left(\zeta_{12}^2\,\frac{\partial}{\partial\zeta_1} - 2I\,\zeta_{12}\right)\phi_I(z_1,\zeta_1;u)\,, \end{aligned} \tag{9.15}$$

where the frequency parts of the current, $J^{(-)}$ and $J_{(+)}$ are defined as in (8.22); setting similarly $T^{(-)}(z) = \sum\limits_{n=0}^{\infty} \frac{L_{n-1}}{z^{n+1}}$, $T_{(+)}(z) = \sum\limits_{n=0}^{\infty} L_{-n-2}\, z^n$ we find

$$\begin{aligned} [T^{(-)}(z_1),\phi_I(z_2,\zeta;u)] &= \frac{\Delta_I}{z_{12}^2}\,\phi_I(z_2,\zeta;u) + \frac{1}{z_{12}}\,\partial_{z_2}\,\phi_I(z_2,\zeta;u) \\ [\phi_I(z_1,\zeta;u),T_{(+)}(z_2)] &= \frac{\Delta_I}{z_{12}^2}\,\phi_I(z_1,\zeta;u) - \frac{1}{z_{12}}\,\partial_{z_1}\,\phi_I(z_1,\zeta;u)\,. \end{aligned} \tag{9.16}$$

Proposition 9.1. *Let $\phi_I(z,\zeta;u)$ be an $\widehat{su}(2)_k$ primary fields, that is a field satisfying* (9.15) *and* (9.16)*. Then Δ_I is given by* (9.8) *and ϕ_I satisfies the Knizhnik*[44]*-Zamolodchikov (KZ) equation*

$$h\,\partial_z\,\phi_I(z,\zeta;u) = I : \partial_\zeta\, J(z,\zeta)\,\phi_I(z,\zeta;u) : - : J(z,\zeta)\,\partial_\zeta\,\phi_I(z,\zeta;u) : \tag{9.17}$$

where the normal product is defined by the non-singular term in the current-field OPE and is expressed simply in terms of the frequency parts of J:

$$: J(z,\zeta_1)\,\phi_I(z,\zeta_2) := J_{(+)}(z,\zeta_1)\,\phi_I(z,\zeta_2) + \phi_I(z,\zeta_2)\,J^{(-)}(z,\zeta_1)\,. \tag{9.18}$$

Sketch of proof. Eq. (9.17) folows from the known CR $[L_n,\phi_I(z)]$, $[J_m,\phi_I(z)]$ derived from (9.15) (9.16) and from the Sugawara expression (9.3) for T. (See [FST] Chapter 5 for details.) □

It is instructive to display the KZ equation for the basic group valued field $g(z,\bar z)$ in a matrix form, spelling out in this case the meaning of the right hand

[44]Vadim Genrikhovich Knizhnik (Kiev 1962-Moscow 1987) was a student of A.M. Polyakov.

side of (9.17). To this end we introduce the matrix valued current $\underset{\sim}{J}(z) = J^i(z)\,\sigma_i$ related to $J(z,\zeta)$ (9.12) by

$$J(z,\zeta) = (\zeta,1)\,\underset{\sim}{J}(z)\begin{pmatrix}1\\-\zeta\end{pmatrix} = (\zeta,1)\begin{pmatrix}J^3(z) & J^+(z)\\ J^-(z) & -J^3(z)\end{pmatrix}\begin{pmatrix}1\\-\zeta\end{pmatrix}. \tag{9.19}$$

Exercise 9.5. Prove that Eq. (9.17) (for $I = \frac{1}{2}$) is equivalent to

$$h\,\frac{\partial}{\partial z}\,g(z,\bar z)^A_B = -:\underset{\sim}{J}(z)^A_S\,g(z,\bar z)^S_B:. \tag{9.20}$$

(*Solution* : setting $g(z,\bar z;\zeta)_B = \zeta\,g(z,\bar z)^1_B + g(z,\bar z)^2_B$ we find

$$\begin{aligned}
&\frac{1}{2}:\frac{\partial J(z,\zeta)}{\partial\zeta}\,g(z,\bar z;\zeta)_B:-:J(z,\zeta)\,\frac{\partial}{\partial\zeta}\,g(z,\bar z;\zeta)_B:\\
=\ &:(J^3(z)\,g(z,\bar z)^2_B - J^-(z)\,g(z,\bar z)^1_B - (J^3(z)\,g(z,\bar z)^1_B + J^+(z)\,g(z,\bar z)^2_B)\,\zeta):\\
=\ &-:(\zeta\,\underset{\sim}{J}(z)^1_S + \underset{\sim}{J}(z)^2_S)\,g(z,\bar z)^S_B:.)
\end{aligned}$$

Proposition 9.2. *The operator KZ equation* (9.17) *and the (operator) Ward identities* (9.15) (9.16) *allow to write down the KZ equation for any correlation function of* ϕ_I. *In particular,* $SU(2)$ *and conformal invariant amplitudes* $f_\lambda(\xi,\eta)$ *of the 4-point functions* w_λ (9.14)*, defined by*

$$w_\lambda(z_1,\zeta_1;\ldots;z_4,\zeta_4) = P_I(z_{ij},\zeta_{ij})\,f_\lambda(\xi,\eta)\,,\ \xi = \frac{\zeta_{12}\,\zeta_{34}}{\zeta_{13}\,\zeta_{24}}\,,\ \eta = \frac{z_{12}\,z_{34}}{z_{13}\,z_{24}}\,, \tag{9.21}$$

$$p_I(z_{ij},\zeta_{ij}) = \left(\frac{z_{13}\,z_{24}}{z_{12}\,z_{23}\,z_{34}\,z_{14}}\right)^{2\Delta_I}(\zeta_{13}\,\zeta_{24})^{2I}\,, \tag{9.22}$$

satisfy the KZ equation

$$\left(h\,\frac{\partial}{\partial\eta} - \frac{C_{12}}{\eta} + \frac{C_{23}}{1-\eta}\right) f_\lambda = 0\,, \tag{9.23}$$

where $C_{ij} = (\vec I_i + \vec I_j)^2$ *are the Casimir invariants which can be expressed as second order differential operators in* ξ*:*

$$C_{12} = 2I(2I+1-2I\xi) - [4I(1-\xi)-\xi]\,\xi\,\frac{\partial}{\partial\xi} + \xi^2(1-\xi)\,\frac{\partial^2}{\partial\xi^2}\,,$$

$$C_{23} = 2I(2I+1-2I(1-\xi)) + (4I\xi+1-\xi)(1-\xi)\,\frac{\partial}{\partial\xi} + \xi(1-\xi)^2\,\frac{\partial^2}{\partial\xi^2}\,. \tag{9.24}$$

Sketch of proof. Applying Eq. (9.17) to $\phi_I(z_2,\zeta_2;u_2)$ and moving $J^{(-)}(z_2,\zeta_2)$ to the right and $J_{(+)}(z_2,\zeta_2)$ to the left, using in both cases Eq. (9.15) as well as

$J^{(-)}(z,\zeta) \mid 0\rangle = 0 = \langle 0 \mid J_{(+)}(z,\zeta)$, we find for the full 4-point function (9.14) and hence for each w_λ the equation

$$\left(h\, \partial_{z_2} + \frac{2\vec{I}_1 \cdot \vec{I}_2}{z_{12}} - \frac{2\vec{I}_2 \cdot \vec{I}_3}{z_{23}} - \frac{2\vec{I}_2 \cdot \vec{I}_4}{z_{24}} \right) w_\lambda(z_1,\zeta_1;\ldots;z_4,\zeta_4) = 0\,, \tag{9.25}$$

where $2\vec{I}_i \cdot \vec{I}_j = 2 \sum_{a=1}^{3} I_i^a\, I_j^a$ is the *polarized* $su(2)$ *Casimir operator* that can be expressed as a differential operator in ζ_i and ζ_j. Inserting (9.19) into (9.23) and using the identity

$$(\vec{I}_1 + \vec{I}_2 + \vec{I}_3 + \vec{I}_4)\, w_\lambda = 0 = (2\vec{I}_1 \cdot \vec{I}_2 + 2\vec{I}_2 \cdot \vec{I}_3 + 2\vec{I}_2 \cdot \vec{I}_4 + 2I(I+1))\, w_\lambda \tag{9.26}$$

we obtain (9.21). Using further the relations

$$2\vec{I}_i \cdot \vec{I}_j = 2I\,[I + \zeta_{ij}(\partial_j - \partial_i)] - \zeta_{ij}^2\, \partial_i\, \partial_j \tag{9.27}$$

for $\vec{I}_i^{\,2} = I(I+1)$, $\partial_i = \frac{\partial}{\partial \zeta_i}$, we find (9.24). □

The basis $\{w_\lambda$ (or $f_\lambda)\}$ of solutions to (9.25) (or (9.23)) is fixed by the requirement that the full 4-point function (9.14) is invariant under the diagonal action of the braid group $\mathcal{B}_4$ on w_λ and $\mathcal{J}_\lambda^{(I)}$. We shall write down this solution expanding w_λ in a set $\{J_\ell^{(I)}\}$ of $SU(2)$ invariants obtained from $\mathcal{J}_\lambda^{(I)}$ in the limit $q \to 1$:

$$J_\ell^{(I)}(\zeta_1,\ldots,\zeta_4) = (\zeta_{12}\,\zeta_{34})^{2I-\ell}(\zeta_{14}\,\zeta_{23})^{\ell} = (\zeta_{13}\,\zeta_{24})^{2I}\, \xi^{2I-\ell}\,(1-\xi)^{\ell}\,. \tag{9.28}$$

The result is ([STH])

$$f_\lambda(\xi,\eta) = \sum_{\ell=0}^{2I} \xi^{2I-\ell}\,(1-\xi)^{\ell}\,\eta^{\ell}(1-\eta)^{2I-\ell}\, g_\lambda^{\ell}(\eta) \tag{9.29}$$

where g_λ^ℓ is given by the $2I$-fold integral

$$g_\lambda^\ell(\eta) = \int_0^\eta dt_1 \int_0^{t_1} dt_2 \quad \ldots \quad \int_0^{t_{\lambda-1}} dt_\lambda \int_\eta^1 dt_{\lambda+1} \int_{t_{\lambda+1}}^1 dt_{\lambda+2} \\ \ldots \quad \int_{t_{2I-1}}^1 dt_{2I}\, P_\lambda^\ell(\eta,t_i)\,, \tag{9.30}$$

$$P_\lambda^\ell(\eta,t_i) \;=\; \prod_{i=1}^{2I} t_i^{\frac{1}{h}}(1-t_i)^{\frac{1}{h}} \prod_{i=1}^{\lambda}(\eta-t_i)^{\frac{1}{h}-1} \prod_{j=\lambda+1}^{2I}(t_j-\eta)^{\frac{1}{h}-1} \prod_{1\le i<j\le 2I}(\epsilon_{\lambda j}\, t_{ij})^{\frac{2}{h}} \\ \times \quad \sum_{\sigma\in\mathcal{S}_{2I}} \prod_{i=1}^{\ell} t_{\sigma i}^{-1} \prod_{j=\ell+1}^{2I}(1-t\sigma_j)^{-1}\,, \tag{9.31}$$

the sum being spread over all permutations $\sigma : (1, \ldots, 2I) \to (\sigma 1, \ldots, \sigma I)$. In order to verify the braid invariance of the resulting 4-point function (9.14) one computes separately the $\mathcal{B}_4$ action on the U_q invariants $\mathcal{J}_\lambda^{(I)}(u_1, \ldots, u_4)$ (using the braid operator $\hat{R}$ (4.38)) and of the (analytically continued) functions (9.21)–(9.23), (9.29)–(9.31), taking into account the transformation properties of the $SU(2)$ invariants $J_\ell^{(I)}(\zeta_1, \ldots, \zeta_4)$ under permutation:

$$1 \rightleftarrows 2 : J_\ell^{(I)}(\zeta_2, \zeta_1, \zeta_3, \zeta_4) = (-1)^{2I-\ell} \sum_{s=0}^{\ell} \binom{\ell}{s} J_s^{(I)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$$

$$2 \rightleftarrows 3 : J_\ell^{(I)}(\zeta_1, \zeta_3, \zeta_2, \zeta_4) = (-1)^{\ell} \sum_{s=\ell}^{2I} \binom{2I-\ell}{2I-s} J_s^{(I)}(\zeta_1, \zeta_2, \zeta_3, \zeta_4) \,. \qquad (9.32)$$

We shall not work out here the details (see [STH] and [ST96] where the case of different isospins is also outlined) but will write down the resulting lower and upper triangular braid matrices in Section 11 below.

10 Canonical approach; WZNW action. Quantum matrix algebra

Although we are using throughout the axiomatic approach to conformal current algebras (combined with the representation theory of affine Kac-Moody algebras) we shall also give here a sketchy overview of the canonical action principle set forth by Witten [W84] (developing ideas of [WZ] and [N82] – a few months before the KZ equation was published).

We begin with an outlook of the *first order Lagrangian* (also called *covariant Hamiltonian*) formalism following [G].

In general, a field theory lives on a fibre bundle locally equivalent to $\mathcal{M} \times \mathcal{F}$ with a D-dimensional base space-time manifold $\mathcal{M}$ and a fiber $\mathcal{F}$ of field configurations. We shall use, correspondingly, two exterior differentials, a *horizontal* one, d, acting on (the tangent space to) $\mathcal{M}$, and a *vertical* one, δ, acting on $\mathcal{F}$, so that the total exterior differential $\boldsymbol{d}$ on $\mathcal{M} \times \mathcal{F}$ appears as their sum:

$$\boldsymbol{d} = d + \delta \, , d^2 = \delta^2 = 0 = [d, \delta]_+ \; (\equiv d\delta + \delta d) \Rightarrow \boldsymbol{d}^2 = 0 \, . \tag{10.1}$$

Whenever an action density (Lagrangian) exists it gives rise to a D-form $\boldsymbol{L}$ on $\mathcal{M} \times \mathcal{F}$ that will be assumed linear in the field differentials. The $(D+1)$-form

$$\omega := \boldsymbol{d} \boldsymbol{L} \quad (\Rightarrow \boldsymbol{d} \omega = 0) \tag{10.2}$$

provides an invariant characterization of the system: the pull-back of its contraction with verticle vector fields $\frac{\delta}{\delta\phi}$ reproduces the equations of motion, while the integral of ω over a $(D-1)$-dimensional space-like (say, equal time) surface in $\mathcal{M}$ defines a symplectic form on the fields. Such a closed $(D+1)$-form may exist also when there is no single valued local action. This is precisely the case with the (classical) Wess[45]-Zumino-Novikov-Witten (WZNW) *model* which we proceed to describe for the $su(2)$ current algebra.

Space-time is taken to be the 2-dimensional ($2D$) cyclinder

$$\mathcal{M} = \mathbb{R} \times \mathbb{S}^1 = \{ x := (x^0, x^1) \equiv (t, \boldsymbol{x}) \, , \; t \in \mathbb{R} \, , \; \boldsymbol{x} \in \mathbb{R}/2\pi \, \mathbb{Z} \} \, . \tag{10.3}$$

In the first order formalism $\mathcal{F}$ is taken to consist of a pair of (periodic in $\boldsymbol{x}$) maps (g, J) such that

$$g(x) \in SU(2) \, , \quad g(t, \boldsymbol{x} + 2\pi) = g(t, \boldsymbol{x}) \, , \tag{10.4}$$

$$J(x) = j_\mu(x) \, dx^\mu \, , \quad j_\mu(x) \in su(2) \, , \quad j_\mu(t, \boldsymbol{x} + 2\pi) = j_\mu(t, \boldsymbol{x}) \, . \tag{10.5}$$

(Note that the $su(2)$-valued 1-form $J(x)$ is horizontal.) The basic 3-form ω is defined by

$$4\pi \, \omega = \boldsymbol{d} \, \mathrm{tr} \left(i \, g^{-1} \, \boldsymbol{d} \, g + \frac{1}{2k} \, J \right) \; {}^*J + k \, \theta(g) \, , \tag{10.6}$$

[45] Julius Wess (1934-2007), Austrian physicist, a student of Hans Thirring (1888-1976).

where ${}^*J$ is the Hodge[46] dual to J,

$$ {}^*J(x) = j^0(x)\, dx^1 - j^1(x)\, dx^0 \equiv \epsilon_{\mu\nu}\, j^\mu(x)\, dx^\nu \,, \qquad (10.7) $$

while the *Wess-Zumino term* $\theta(g)$ is the canonical 3-form on the group:

$$ \theta(g) = \frac{1}{3}\, \mathrm{tr}(g^{-1}\, \boldsymbol{d}g)^3 \quad ((g^{-1}\boldsymbol{d}g)^3 \equiv g^{-1}\boldsymbol{d}g \wedge g^{-1}\boldsymbol{d}g \wedge g^{-1}\boldsymbol{d}g)\,. \qquad (10.8) $$

(We omit throughout this section the wedge product of 1-forms.)

Exercise 10.1. The trace of the product of 1-forms $a_1 \dots a_n$ obeys the graded cyclic property $\mathrm{tr}(a_1 a_2 \dots a_n) = (-1)^{n-1}\, \mathrm{tr}(a_n a_1 \dots a_{n-1})$. Deduce from here that the 3-form θ is closed,

$$ d\theta(g) = 0\,, \qquad (10.9) $$

but not exact: there is no globally defined single valued 2-form $\alpha(g)$ on $SU(2)$ such that $\theta = d\alpha$.

Exercise 10.2. Using the relation $dx^\mu\, dx^\nu = -\epsilon^{\mu\nu}\, dx^0\, dx^1$ ($\epsilon^{\mu\nu} = -\epsilon_{\mu\nu}$) derive the following expressions for $J\,{}^*J$ and its exterior differential:

$$ J\,{}^*J = j_\mu\, j^\mu\, dx^0\, dx^1 (= -\,{}^*J\, J)\,, \quad \boldsymbol{d}\, J\,{}^*J = 2 j_\mu\, \delta j^\mu\, dx^0\, dx^1\,. \qquad (10.10) $$

Varying ω with respect to ${}^*J$ and "pulling back" (*i.e.* projecting on horizontal differentials) we find the *classical KZ equation* :

$$ i\, g^{-1}\, dg + \frac{1}{k}\, J = 0\,. \qquad (10.11) $$

To see the precise relation between (9.20) and (10.11) we set $J = \underset{\sim}{J}(z)\, dz + \underset{\sim}{\bar{J}}(\bar{z})\, d\bar{z}$ and multiply both sides of (10.11) by $k\, g(z, \bar{z})$. The effect of quantization then consists in replacing operator products with normal products and the level k with (its renormalized value) the height h.

Varying further the 3-form (10.6) with respect to g we find the second equation of motion

$$ d\,{}^*J = -ik(g^{-1}\, dg)^2 = \frac{i}{k}\, J^2 \Leftrightarrow \partial_\mu\, j^\mu \equiv \partial_1\, j_1 - \partial_0\, j_0 = \frac{i}{k}\, [j_0, j_1]\,. \qquad (10.12) $$

Taking the exterior derivative of (10.11) and comparing with (10.12) we deduce

$$ d(J + {}^*J) = 0 \Leftrightarrow \partial_+\, j_R = 0\,, \quad \partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_0)\,; \qquad (10.13) $$

[46] The Scottish mathematician William V.D. Hodge (1903-1975) discovered topological relations between algebraic and differential geometry. (See M. Atiyah, *William Valance Douglas Hodge*, Bull. London Math. Soc. **9** (1) (1977) 99-118.)

the left- and right-movers' currents are

$$j_R = \frac{1}{2}(j^0 + j^1)\,,\; j_L = \frac{1}{2}\, g(j^1 - j^0)\, g^{-1} (= -ik(\partial_+ g)\, g^{-1})\,,\; \partial_- j_L = 0\,. \quad (10.14)$$

The symplectic form of the model can be written in three equivalent forms:

$$\begin{aligned}\Omega^{(2)} &= \frac{1}{4\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\,(k\, g^{-1}\, g' - i\, j^0)(g^{-1}\,\delta g)^2 + i\,\delta j^0\, g^{-1}\,\delta g)\\ &= -\frac{1}{2\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\left(i\,\delta(j_L\,\delta g\, g^{-1}) + \frac{k}{2}\,\delta g\, g^{-1}(g^{-1}\,\delta g)'\right)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} dx^1\, \mathrm{tr}\left(i\,\delta(j_R\, g^{-1}\,\delta g) + \frac{k}{2}\, g^{-1}\,\delta g(g^{-1}\,\delta g)'\right) \qquad (10.15)\end{aligned}$$

where $f(x^0, x^1)'$ stands for partial derivative of f in x^1.

Exercise 10.3. Use the relations

$$j^0 = 2j^R + ik\, g^{-1}\, g' = 2\, g^{-1}\, j_L\, g - ik\, g^{-1}\, g' = j_R - g^{-1}\, j_L\, g\,,$$

and

$$ik\, \mathrm{tr}(\delta g\, g^{-1}\, (\delta g\, g^{-1})') = \mathrm{tr}(\delta j_1\, g^{-1}\, \delta g)$$

to verify the equivalence of the three expressions (10.15).

The general solution of the equations of motion (10.11)–(10.14),

$$\partial_+(g^{-1}\,\partial_-\, g) = 0 \Leftrightarrow \partial_-((\partial_+\, g)\, g^{-1}) = 0 \qquad (10.16)$$

can be written in a factorized form

$$g(x^+, x^-) = g_L(x^+)\, g_R^{-1}(x^-)\,, \quad x^\pm = x^1 \pm x^0 \qquad (10.17)$$

(*cf.* (9.9)). The following result of Gawedzki [G] allows to split the symplectic form (10.15) into a left- and right-movers' part as well.

Proposition 10.1. *One can split $\Omega^{(2)}$ as a sum of two chiral forms which only differ in sign,*

$$\Omega^{(2)} = \Omega_c(g_L) - \Omega_c(g_R) \qquad (10.18)$$

$$\Omega_c(g) = \frac{k}{4\pi}\, \mathrm{tr}\left\{\int_{-\pi}^{\pi} dx\, g^{-1}\,\delta g(g^{-1}\,\delta g)' + b^{-1}\,\delta b\,\delta M M^{-1}\right\}\,, \qquad (10.19)$$

where

$$b := g(-\pi) \quad (g = g_L \;\; or \;\; g_R)\,, \quad M = b^{-1}\, g(\pi) \qquad (10.20)$$

with M independent of the chirality: $M = b_L^{-1}\, g_L(\pi) = b_R^{-1}\, g_R(\pi)$.

As we have seen in our survey of the axiomatic approach to the $su(2)$ current algebra model (Section 9) only the chiral components of g ($g(z)$ and $\bar g(\bar z)$ in the z-picture, $g_\pm(x_\pm)$ in the present context) give room to – indeed display – a quantum group symmetry. In the canonical approach such a splitting is suggested by the fact that chiral (left and right) currents j_C are periodic in their respective arguments,

$$j_C(x+2\pi) = j_C(x) \quad \text{for} \quad C = L, R \tag{10.21}$$

(thus appearing as *chiral observables*) and *Poisson commute*

$$\{j_L(x^+), j_R(y^-)\} = 0\,. \tag{10.22}$$

(Computing Poisson brackets from a given symplectic form $\Omega = \frac{1}{2}\,\omega_{ij}\, d\xi^i \wedge d\xi^j$ amounts to inverting the skew symmetric matrix (ω_{ij}):

$$\{f(\xi), g(\xi)\} = (\omega^{-1})^{ij}\, \frac{\partial f}{\partial \xi^i}\, \frac{\partial g}{\partial \xi^j}\,.$$

In the infinite dimensional case at hand this requires, in general, some work – see [G]. The trivial Poisson bracket relations (10.22) follow however simply from the splitting (10.18)–(10.20) of the form $\Omega^{(2)}$ into chiral parts and from the fact that j_L and j_R are periodic and hence commute with the monodromy M. They are also a consequence of the observation that j_L and j_R appear as Noether[47] currents for two commuting, left and right, symmetries.) Eq. (10.22) is the classical counterpart of the *local commutativity* of observable Bose fields.

The chiral group valued fields $g_L(x^+)$ and $g_R(x^-)$ are determined by the corresponding currents and the *classical chiral KZ equations* (the chiral counterparts of (10.11)):

$$k\,\partial_+\, g_L(x^+) = i\, j_L(x^+)\, g_L(x^+)\,, \quad k\,\partial_-\, g_R(x^-) = -i\, j_R(x^-)\, g(x^-)\,. \tag{10.23}$$

The solution of (the quantum counterpart of) these equations involves the introduction of the chiral zero modes a^i_α of g_C which diagonalize the monodromy:

$$g_L(x)^A_\alpha = u^A_i(x)\, a^i_\alpha (= u^A_1(x)\, a^1_\alpha + u^A_2(x)\, a^2_\alpha)\,, \quad a^i_\sigma\, M^\sigma_\alpha = (M_p)^i_j\, a^j_\alpha \tag{10.24}$$

where M_p is a diagonal unitary matrix depending on the operator p whose eigenvalues are the dimensionalities, $2I+1$, of the IRs of U_q. The (quantized) zero modes a^i_α behave as q-deformed creation (for $i=1$) and annihilation (for $i=2$) operators whose Fock space will be displayed in Section 12 below.

In summary, the canonical approach allows to reproduce the results of the axiomatic treatment of Section 9. This is a long story with no complete pedagogical treatment in the literature. (Its systematic study starts with [G]; further developments can be traced back from [FHT].) Here we shall only elaborate on the U_q properties of the above group valued zero modes a^i_α (the "qantum oscillators") as they are related to the main topic of these lectures. Such "twisted

[47](Amalie) Emmy Noether (1882-1935) became in 1919 the first woman professor at the University of Göttingen.

oscillators" were first considered for their own sake (with no relation to the WZNWmodel) in [PW].

We assume that a^i_α $(i, \alpha = 1,2)$ belong to an associative algebra $\mathcal{A}_q$ that contains U_q as a subalgebra. For any $a \in \mathcal{A}_q$ and $X \in U_q$ we define the *adjoint action* Ad_X of X on a by

$$\mathrm{Ad}_X(a) = \sum_{(X)} x_1\, a\, S(x_2) \quad \text{for} \quad \Delta X = \sum_{(X)} x_1 \otimes x_2\,. \tag{10.25}$$

In particular, for $X = E$ and $X = K$ we find (using (4.6) and (4.8))

$$\mathrm{Ad}_E(a) = E\, a\, K^{-1} - a\, E\, K^{-1}\,, \quad \mathrm{Ad}_K(a) = K\, a\, K^{-1}\,.$$

Let, for any $X \in U_q$, X^f be the fundamental 2×2 matric representation (given by (4.11) for $i = 1$). We define the U_q transformation law of a^i_α by

$$\mathrm{Ad}_X(a^i_\alpha) = a^i_\beta\, (X^f)^\beta_\alpha\,. \tag{10.26}$$

Exercise 10.4. Derive the CR of a^i_α with E, F and K.
(*Answer* : $[E, a^i_1] = 0$, $[E, a^i_2] = a^i_1\, K$, $F\, a^i_1 - q^{-1}\, a^i_1\, F = a^i_2$, $F\, a^i_2 - q\, a^i_2\, F = 0$, $K\, a^i_1 = q\, a^i_1\, K$, $K\, a^i_2 = q^{-1}\, a^i_2\, K$, $i = 1,2$.)

The U_q *quantum matrix algebra* $\mathcal{A}_q$ is generated by a^i_α and $q^{\pm p}$ satisfying

$$q^p\, a^1_\alpha = a^1_\alpha\, q^{p+1}\,, \quad q^p\, a^2_\alpha = a^2_\alpha\, q^{p-1}\,, \quad q^p\, q^{-p} = \mathbb{1} \tag{10.27}$$

$$\begin{gathered} a^2_\alpha\, a^1_\beta = a^1_\alpha\, a^2_\beta + [p]\, \varepsilon_{\alpha\beta}\,, \quad a^i_\alpha\, a^i_\beta\, \varepsilon^{\alpha\beta} = 0\,, \quad i = 1,2\,, \\ a^2_\alpha\, a^1_\beta\, \varepsilon^{\alpha\beta} = [p+1]\,, \quad a^1_\alpha\, a^2_\beta\, \varepsilon^{\alpha\beta} = -[p-1]\,, \end{gathered} \tag{10.28}$$

where $\varepsilon^{\alpha\beta}$ $(= \varepsilon_{\alpha\beta})$ is the q-deformed Levi-Civita tensor (2.23). The relations (10.28) can be recast into a (quantum) determinant condition,

$$\det_q a := \frac{1}{[2]}\, \epsilon_{ij}\, a^i_\alpha\, a^j_\beta\, \varepsilon^{\alpha\beta} = [p]\,, \quad (\epsilon_{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,, \tag{10.29}$$

and the homogeneous R-matrix relation which allow a straightforward generalization to the $U_q(A_r)$ case [HIOPT]

$$R(p)^{ij}_{\ell m}\, a^m_\beta\, a^\ell_\alpha = a^i_\rho\, a^j_\sigma\, R^{\rho\sigma}_{\alpha\beta} \tag{10.30}$$

where $R = (R^{\rho\sigma}_{\alpha\beta})$ is the 4×4 matrix (5.5) while $R(p)$ is the *dynamical R-matrix* :

$$R(p) = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \frac{[p-1]}{[p]} & \frac{q^p}{[p]} & 0 \\ 0 & -\frac{q^{-p}}{[p]} & \frac{[p+1]}{[p]} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}\,. \tag{10.31}$$

Exercise 10.5. Demonstrate that (10.27)–(10.28) imply (10.29)–(10.30) while the determinant condition (10.29) alone yields (10.28).

Exercise 10.6. Assuming that $q^{\pm p}$ commute with U_q and using (10.26) and Exercise 10.4 prove the U_q-covariance of (10.27)–(10.28).

11 Monodromy representations of the braid group

The following braid relations have been derived in [STH] for the regular basis (9.21) (9.29)–(9.30). Let b_j stand for the exchange of the variables (z_j, ζ_j) with (z_{j+1}, ζ_{j+1}) along a path (in z-space) for which $z_{j\,j+1} \to e^{-i\pi}\, z_{j\,j+1}$; then

$$b_1\, f_\mu^{(I)}(\xi,\eta) = (1-\xi)^{2I}\,(1-\eta)^{4\Delta_I}\, f_\mu^{(I)}\left(\frac{\xi}{\xi-1}, \frac{\eta\, e^{-i\pi}}{1-\eta}\right) = f_\lambda^{(I)}(\xi,\eta)\, B_{1\mu}^{\lambda}\,,$$

$$b_2\, f_\mu^{(I)}(\xi,\eta) = \xi^{2I}\,\eta^{4\Delta_I}\, f_\mu^{(I)}\left(\frac{1}{\xi}, \frac{1}{\eta}\right) = f_\lambda^{(I)}(\xi,\eta)\, B_{2\mu}^{\lambda}\,, \qquad (11.1)$$

where B_1 is a lower triangular, B_2 is an upper triangular matrix:

$$B_{1\,\mu}^{\lambda} = (-1)^{2I-\lambda}\, q^{\lambda(\mu+1)-2I(I+1)} \begin{bmatrix} \lambda \\ \mu \end{bmatrix} = B_{2\,2I-\mu}^{2I-\lambda}\,, \qquad (11.2)$$

and we have $B_3 = B_1$. Here q may be any *primitive h-th root* of -1:

$$q^h = -1 \qquad (q^n \neq -1 \quad \text{if} \quad 0 < n < h)\,. \qquad (11.3)$$

It follows, in particular, that q is a phase factor ($q\,\bar{q} = 1$).

Exercise 11.1. Verify that the inverse matrices to $B_i(q)$ for q satisfying (11.3) are obtained by complex conjugation:

$$B_i(q)\, B_i(\bar{q}) = \mathbb{1} \quad \text{for} \quad q\,\bar{q} = 1 \qquad (11.4)$$

($\mathbb{1}$ standing for the $(2I+1)\times(2I+1)$ unit matrix).

Exercise 11.2. Verify the braid relation

$$B_1\, B_2\, B_1 = B_2\, B_1\, B_2 = (-1)^{2I}\, \bar{q}^{2I(I+1)}\, F\,, \quad F_\mu^\lambda = \delta_{2I-\mu}^{\lambda} \qquad (11.5)$$

(the *fusion matrix* F is, thus, a permutation matrix satisfying $F^2 = \mathbb{1}$). Verify that

$$B_2 = F\, B_1\, F \qquad (B_1 = F\, B_2\, F\,, \quad F = F^{-1})\,. \qquad (11.6)$$

Remark 11.1. It can be demonstrated that, for q satisfying (11.3), $B_{i,i=1,2}^{2h}$ is a multiple of the unit matrix for $2I+1 < h$ but is not diagonalizable for $2I+1 \geq h$.

Exercise 11.3. Verify the statement of Remark 11.1 for small values of h and $2I$.

It follows from Remark 11.1 that the braid matrices B_1 and B_2 are not diagonalizable – and hence not unitarizable for non-integrable representations of the $su(2)$ current algebra (*i.e.* for representations violating the upper bound $2I \leq k$ (9.7)). Note that the eigenvalues of B_i have absolute value 1, hence the matrices B_i are unitarizable exactly when they are diagonalizable. This explains why the B_1-diagonal basis, used in most of the literature, is ill defined beyond the unitarity limit, and justifies the attribute "regular" for the above triangular basis which always makes sense.

In order to give the reader a better feeling of this monodromy representation of the braid group $\mathcal{B}_4$ we shall consider in more detail the simplest representation corresponding to $2I = 1$ (*i.e.* to the braiding properties of the 4-point function of the chiral group-valued field $g(z)$).

Exercise 11.4. Verify that the normalized 2×2 braid matrices (of determinant -1)

$$b_1 = q^{\frac{1}{2}}\, B_1^{(2I=1)} = \begin{pmatrix} -\bar{q} & 0 \\ 1 & q \end{pmatrix}, \quad b_2 = q^{\frac{1}{2}}\, B_2^{(2I=1)} = \begin{pmatrix} q & 1 \\ 0 & -\bar{q} \end{pmatrix} \tag{11.7}$$

satisfy the Hecke algebra relations (2.18).

Remark 11.2. The general Hecke algebra representation of $\mathcal{B}_4$ realized on the 4-fold tensor product $(\mathbb{C}^2)^{\otimes 4}$ of the space $\mathbb{C}^2$ of 2-component isospinors is 16 dimensional. It can be constructed in terms of the Tempereley-Lieb projectors (2.19) as follows:

$$b_i = q\,\mathbb{1} - e_i\,, \quad i = 1,2,3\,; \quad e_1 = (\varepsilon^{\alpha_1\alpha_2}\,\varepsilon_{\beta_1\beta_2}\,\delta^{\alpha_3}_{\beta_3}\,\delta^{\alpha_4}_{\beta_4})\,,$$
$$e_2 = (\delta^{\alpha_1}_{\beta_1}\,\varepsilon^{\alpha_2\alpha_3}\,\varepsilon_{\beta_2\beta_3}\,\delta^{\alpha_4}_{\beta_4})\,, \quad e_3 = (\delta^{\alpha_1}_{\beta_1}\,\delta^{\alpha_2}_{\beta_2}\,\varepsilon^{\alpha_3\alpha_4}\,\varepsilon_{\beta_3\beta_4}) \tag{11.8}$$

where $\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta}$ is the q-deformed Levi-Civita tensor (2.23). The above 2-dimensional representation of $\mathcal{B}_4$ is a subrepresentation of this 16-dimensional one, spanned by the $SU(2)$ invariant tensors in $(\mathbb{C}^2)^{\otimes 4}$. We leave it to the reader to work out the details of this projection.

We shall end up our study of the 2-dimensional representation of $\mathcal{B}_4$ by answering the following question.

The Schwarz problem: for which values of h $(= 3, 4, \ldots)$ and q satisfying (11.3) is the matrix group generated by the 2×2 matrices b_i (11.7), a finite group?

The answer to this question determines when the KZ equation (for $2I = 1$) admits elementary (algebraic) solutions.

As $b_1^{2h} = b_2^{2h} = \mathbb{1}$ for $h = 3, 4, \ldots$, it is enough to study the *commutator subgroup*, generated by the pair

$$b = b_1^{-1}\, b_2 = b_2\, b_1\, b_2^{-1}\, b_1^{-1} = \begin{pmatrix} -q^2 & -q \\ q & 1 - \bar{q}^2 \end{pmatrix}, \quad \bar{b} = b_1, b_2^{-1}\,. \tag{11.9}$$

The argument we shall present in solving the problem (a special case of [ST]) is interesting in that it applies some elementary number theoretic methods.

Proposition 10.1. *The real symmetric matrix*

$$A = \begin{pmatrix} [2]^2 & [2] \\ [2] & [2]^2 \end{pmatrix} = \ {}^tS \begin{pmatrix} [3] & 0 \\ 0 & [2]^2 \end{pmatrix} S\,, \quad S = \begin{pmatrix} 1 & 0 \\ \frac{1}{[2]} & 1 \end{pmatrix} \tag{11.10}$$

(S being the matrix diagonalizing b_1,

$$S\, b_1\, S^{-1} = \begin{pmatrix} -\bar{q} & 0 \\ 0 & q \end{pmatrix}\,, \tag{11.11}$$

which is well defined for $h > 2$) is $\mathcal{B}_4$ invariant:

$$b^* A\, b = A\,, \quad \forall\, b \in \mathcal{B}_4 \Leftrightarrow {}^tb_i\, A = A\, b_i\,, \quad i = 1, 2\,, \quad \textit{for} \quad \bar{b}_i = b_i^{-1}\,, \tag{11.12}$$

where ${}^tS$ (and ${}^tb$) denotes the transposed of S (and b).

Proof. For b_1 Eq. (11.12) is a consequence of (11.11):

$${}^tb_1\, A = {}^tb_1\, {}^tS \begin{pmatrix} [3] & 0 \\ 0 & [2]^2 \end{pmatrix} S = {}^tS \begin{pmatrix} -[3]\, \bar{q} & 0 \\ 0 & [2]^2\, q \end{pmatrix} S = A\, b_1\,;$$

for b_2 both sides of (11.12) give $[2] \begin{pmatrix} 1 + q^2 & q \\ q & -\bar{q}^2 \end{pmatrix}$. The equivalence of the two invariance conditions for the realization (11.7) of b_i follows from (11.4). □

Remark 11.3. The eigenvalues $[2]^2 \pm [2]$ of A differ from those of the diagonal matrix diag$([3], [2]^2)$ of (11.10). However the positivity conditions for both are equivalent because of the *inertia law for non-degenerate quadratic forms.*

Corollary 11.1. *The above* 2*-dimensional representation of $\mathcal{B}_4$ is unitarizable provided*

$$([2] =)\, q + \bar{q} = 2 \cos \frac{\pi}{h}\,, \quad \textit{i.e.} \quad q = e^{\pm i \frac{\pi}{h}} \quad (\textit{for } h > 3)\,. \tag{11.13}$$

(For $h = 3$ the form A (11.10) is degenerate since then $[3] = 0$.) Indeed, for $h \geq 4$ the matrix A is positive definite since then $[2] > 1$ ($[3] \geq 1$).

Eq. (11.13) that guarantees the positivity of A is the only one which depends on the choice of a primitive root of (11.3). To stress this point we introduce the notion of a *Galois*[48] *automorphism* for the *cyclotomic field* defined by (11.3). The map $q \to q^n$ is a Galois automorphism of the field $\mathbb{Q}\,[q]$ of polynomials in q

[48]The legendary Evariste Galois (1811-1832) was only appreciated posthumously. His major work on algebraic equations was finally published in 1846 (following a positive review by Liouville 3 years earlier) – some 14 years after his fatal duel. In the night before the duel Galois, 20, composed a letter to his friend Auguste Chevalier outlining his mathematical ideas. Here is what Hermann Weyl had to say about this "testament": "This letter, if judged by the novelty and profundity of ideas it contains, is perhaps the most substantial piece of writing in the whole literature of mankind".

(obeying (11.3)) with rational coefficients, iff $(n, 2h) = 1$ – *i.e.* iff n is coprime with $2h$.

Exercise 11.5. (a) Prove that the Galois group for $h = 10$ is isomorphic to the product of cyclic groups of two and four elements, $\mathbb{Z}/(2) \times \mathbb{Z}/(4)$. (*Hint* : it is spanned by the exponents ± 1, ± 3, ± 7, ± 9 with multiplication mod 20.)

(b) Prove, similarly, that the Galois group for $h = 30$ is a 16 element group isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(2) \times \mathbb{Z}/(4)$.

Remark 11.4. The solution of Exercise 11.5(b) is related to the Coxeter exponents $(1, 7, 11, 13, 17, 19, 23, 29)$ of the exceptional group E_8. (For an application of the Coxeter exponents to the classification of the $\widehat{su}(2)_k$ conformal invariant theories – see [CIZ].)

A form A with coefficients in $\mathbb{Q}[q]$ is said to be *totally positive* if it is positive for all Galois transforms $q \to q^n$, $(n, 2h) = 1$ of q. The relevance of this concept to our problem is revealed by the following crucial lemma.

Proposition 11.2. *If the form* (11.10) *is totally positive,* i.e. *if* $[3] = q^2 + 1 + \bar{q}^2 > 0$ *for all primitive roots of* (11.3)*, then the* 2*-dimensional representation of* $\mathcal{B}_4$*, which leaves the non-degenerate form A invariant, is a finite matrix group. Conversely, if the invariant hermitian form is unique (or, equivalently, if the representation of* $\mathcal{B}_4$ *under consideration is irreducible), then the total positivity of A is necessary for its finiteness.*

The *proof* is based on the fact that the invariance group of a totally positive form A over a cyclotomic field is compact. Since $\mathcal{B}_4$ is discrete it would follow that the matrix group generated by b_1, b_2 (11.7) is finite.

As any finite dimensional representation of a compact group is unitarizable the unique invariant form A should be positive together with all its Galois transforms. □

Proposition 11.3. *The commutator subgroup of the* 2×2 *matrix* $(2I = 1)$ *realization of* $\mathcal{B}_4$ *generated by the matrices b and $\bar{b}$* (10.9) *is only finite for* $h = 4, 6, 10$*. It is isomorphic to:* (i) *the* 24 *element double cover* $\tilde{\mathcal{A}}_4$ *of the tetrahedral group for* $h = 4$*;* (ii) *the* 8 *element group of quaternion units for* $h = 6$*; and* (iii) *the* 120 *element double cover* $\tilde{\mathcal{A}}_5$ *of the icosahedral group for* $h = 10$*. (Here* $\mathcal{A}_n$ *stands for the alternating subgroup of even permutations of* $\mathcal{S}_n$ *represented by* 3×3 *orthogonal matrices,* $\tilde{\mathcal{A}}_n$ *is its double cover belonging to* $SU(2)$*.)*

Proof. For both $h = 4$ $([3] = 1)$ and $h = 6$ $([3] = 2)$ the q-number $[3]$ is independent of the choice of a primitive h-th root of -1 – and is positive. In general, we have to verify for which h

$$[3]_{q^n} = 1 + 2\cos\frac{2n\pi}{h} \geq 0 \text{ for all } n \text{ such that } (2h, n) = 1\,. \tag{11.14}$$

For $h = 4m - 1$, $m = 2, 3, \ldots$, we can set $n = 2m - 1$, for $h = 4m + 1$, $m = 1, 2, \ldots$, we may choose $n = 2m + 1$, violating in both cases the inequality (11.14) (the maximal value of $[3]_{q^{2m+1}}$ occuring for $m = 1$: $[3]_3 = 1 + 2\cos\frac{6\pi}{5} = 1 - 2\cos\frac{\pi}{5} = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} < 0$). For $h = 4m$, $m \geq 2$ we have $[3]_{q^{2m+1}} = 1 - 2\cos\frac{\pi}{2m} \leq 1 - 2\cos\frac{\pi}{4} = 1 - \sqrt{2} < 0$. Finally, for $h = 4m + 2$, $m \geq 2$ we have $[3]_{q^{2m-1}} = 1 + 2\cos\left(\frac{2m-1}{2m+1}\,\pi\right) = 1 - 2\cos\frac{2\pi}{2m+1}$ which implies

$$[3]_{q^3} \stackrel{(h=10)}{=\!=} 1 - 2\cos\frac{2\pi}{5} = \frac{3-\sqrt{5}}{2} > 0\,, \quad [3]_{q^{2m-1}} \leq 1 - 2\cos\frac{2\pi}{7} < 0 \text{ for } m \geq 3\,. \tag{11.15}$$

We conclude that the exceptional properties of the *golden ratio* (*i.e.* of $x = 2\cos\frac{\pi}{5}$ $(= \frac{1+\sqrt{5}}{2})$ satisfying $x^2 = x + 1$) ensure the positivity of $[3]_{q^3}$ thus verifying total positivity for ($h = 6$ and) $h = 10$ only (among $h = 4m + 2$).

In order to identify the various finite groups we use

$$b^3 = \bar{b}^3 = -1 = (b^{-1}\,\bar{b})^2 \quad \text{for} \quad h = 4\,, \tag{11.16}$$

$$b^2 = \bar{b}^2 = (b^{-1}\,\bar{b})^2 = -1 \quad \text{for} \quad h = 6\,, \tag{11.17}$$

$$(b^{-1}\,\bar{b}^2)^2 = (b^{-1}\,\bar{b})^3 = \bar{b}^5 = -1 \quad \text{for} \quad h = 10\,. \tag{11.18}$$

□

Remark 11.5. Propositions 11.2 and 11.3 are special cases of Lemma 3.2 and Theorem 3.3 of [ST] where *all* monodromy representations of $\mathcal{B}_4$ (for the $su(2)$ current algebra) realized by finite matrix groups are classified. The results for the 2-dimensional representations displayed here have been derived earlier (by quite different methods) by V. Jones (see the first reference [J]). Note that the exceptional values $4, 6$ and 10 of the height h correspond to levels $k = h - 2$ equal to the (real) dimensions $2, 4, 8$ of the field of complex numbers and of the division algebras of quaternions and octonions.

The following corollary of Proposition 11.3 (much as the end of the proof of that Proposition) require deeper familiarity with finite groups defined in terms of generators and relations than we have given here.

Exercise 11.6. Prove as a corollary of Proposition 11.3 that the groups generated by the matrices b_i are central extensions of (i) the 48 element binary octahedral group (the double cover of the permutation group $\mathcal{S}_4$, isomorphic to the symmetry group of the octahedron) – for $h = 4$; (ii) the 24 element binary tetrahedral group $\tilde{\mathcal{A}}_4$ – for $h = 6$; (iii) the binary icosahedral group $\tilde{\mathcal{A}}_5$ – the only one coinciding with its commutator subgroup – for $h = 10$. (For background on discrete groups defined by generators and relations – see [CM].)

The knowledge of the $(2I + 1)$-dimensional realization of $\mathcal{B}_4$ in the space of $su(2)$ current algebra 4-point blocks allows to establish another type of duality relation between quantum group and braid group representations. In order to display its full content we need to say something more about the representation theory of $U_q(A_1)$ for q an even root of unity. This will be the starting point of the next section.

12 Restricted and Lusztig QUEA for $q^h = -1$ and their representations

$A_1 \simeq s\ell(2)$ (as well as its compact real form $su(2)$) is a *simple Lie algebra*: it admits no non-trivial ideals. The same is true for the deformation $U_q(A_1)$ of its universal enveloping algebra for *generic* q (*i.e.* $q \neq 0$ and q *not* a root of unity). By contrast, if q satisfies (11.3) then $U_q(A_1)$ admits a huge proper ideal. Technically, this comes out because the q-numbers $[nh]$ vanish for $q^h = -1$.

Exercise 12.1. Prove the CR

$$[E, F^n] = [n]\, F^{n-1}\, [H + 1 - n]\,, \quad [E^n, F] = [n]\, E^{n-1}\, [H + n - 1]\,. \tag{12.1}$$

Deduce that these commutators vanish iff n is a multiple of h.

The result of Exercise 12.1 allows to prove that E^h and F^h generate an ideal of $U_q(A_1)$ for $q^h = -1$. In order to find a *maximal ideal* which contains these two elements we shall first construct a *model space* of $U_q(A_1)$ for generic q. (We recall that a vector space $\mathcal{F}$ is a model space for a Lie algebra $\mathcal{G}$ or for its UEA $U(\mathcal{G})$ if $\mathcal{F}$ is the direct sum of its finite dimensional irreducible modules, each encountered with multiplicity one.) To this end, we introduce the direct sum $\mathcal{F}$ ($= \mathcal{F}(q)$) of p-dimensional $U_q(A_1)$ modules $\mathcal{F}_p$ defined in Section 4:

$$\mathcal{F} = \bigoplus_{p=1}^{\infty} \mathcal{F}_p\,, \quad \mathcal{F}_p = \text{Span}\,\{|p, m\rangle\,,\ 0 \le m \le p - 1\} \tag{12.2}$$

where the canonical basis $\{|p, m\rangle\}$ is defined by the relations (4.15) (4.17).

Exercise 12.2. (a) Derive the relations

$$E^n \mid p, m\rangle = \frac{[p - m - 1]!}{[p - m - n - 1]!} \mid p, m + n\rangle\,,\ F^n \mid p, m\rangle = \frac{[m]!}{[m - n]!} \mid p, m - n\rangle\,. \tag{12.3}$$

(b) Verify, using (10.26)–(10.28), that $\mathcal{F}$ appears as a Fock space for a^i_α:

$$a^2_\alpha \mid 1, 0\rangle = 0\,, \quad \mid p, m\rangle = (a^1_1)^m\, (a^1_2)^{p-1-m} \mid 1, 0\rangle\,. \tag{12.4}$$

(c) Deduce that for $q^h = -1$ the following identities hold on $\mathcal{F}$:

$$E^h\, \mathcal{F} = 0 = F^h\, \mathcal{F} = (K^{2h} - 1)\, \mathcal{F}\,. \tag{12.5}$$

Exercise 12.3. Assuming the knowledge of the PBW basis of $U_q(A_1)$ (viewed as a vector space – *cf.* Section 5),

$$\{E^\mu\, K^n\, F^\nu\,,\ \mu, \nu = 0, 1, \dots, h - 1\,, \quad n \in \mathbb{Z}\} \tag{12.6}$$

prove that the quotient space with respect to the two-sided ideal defined by the kernel (12.5) of the representation of $U_q(A_1)$ in $\mathcal{F}$,

$$\bar{U}_q := U_q(A_1)/J_h\,, \quad J_h = \{E^h, F^h, K^h - K^{-h}\}\,, \tag{12.7}$$

is $2\,h^3$-dimensional.

The quotient $\bar{U}_q$ is called the *restricted QUEA* in [FGST] and [FHT].

We can similarly define the $4\,h^3$-dimensional quotient $\bar{D}_q$ of the double cover D_q of U_q (introduced in Section 5) by the same ideal J_h expressed in terms of k instead of $K = k^2$:

$$J_h = \{E^h, F^h, k^{2h} - k^{-2h}\}\,, \quad \bar{D}_q = D_q/J_h\,. \tag{12.8}$$

It allows to give meaning to the *universal R-matrix* of type (5.16) as a polynomial in the $\bar{D}_q$ generators, without invoking topology and completion. The reader will find the proof of the following result in [FGST] (see also Sections 2.2 and 3.1 of [FHT], whose conventions we have adopted here).

Proposition 12.1. (a) *The PBW bases in $\bar{U}_q(b_-)$ and $\bar{U}_q(b_+)$,*

$$e_{n\nu} = k^n\, E^\nu \in \bar{U}_q(b_-)\,;\ f_{m\mu} = \frac{\lambda^\mu\, q^{\frac{\mu(\mu-1)}{2}}}{4h\,[\mu]!} \sum_{s=0}^{4h-1} q^{-\frac{ms}{2}}\, \tilde{k}^s\, F^\mu\,,$$

$$m,n = 0,\ldots,4h-1\,,\ \mu,\nu = 0,\ldots,h-1\,, \tag{12.9}$$

are dual to each other with respect to the bilinear form defined in Section 5 (see Eq. (5.12)):

$$\langle f_{m\mu}, e_{n\nu}\rangle = \delta_{mn}\,\delta_{\mu\nu}\,,\ m,n = 0,1,\ldots,4h-1\,,\ \mu,\nu = 0,1,\ldots,h-1\,. \tag{12.10}$$

(b) *The R-matrix of the ($16h^4$-element) quantum double is given by*

$$\mathcal{R}^{\rm double} = \sum_{\nu=0}^{h-1} \sum_{n=0}^{4h-1} e_{n\nu} \otimes f_{n\nu}\,. \tag{12.11}$$

It reduces for $\tilde{k} = k$ (5.18) to the R-matrix of the ($4h^3$-element) double cover $\bar{D}_q$ of $\bar{U}_q$:

$$\mathcal{R} = \frac{1}{4h} \sum_{\nu=0}^{h-1} \sum_{m,n=0}^{4h-1} \frac{\lambda^\nu}{[\nu]!}\, q^{\frac{\nu(\nu-1)-mn}{2}}\, k^m\, E^\nu \otimes k^n\, F^\nu \in \bar{D}_q \otimes \bar{D}_q\,, \tag{12.12}$$

which satisfies the quasi-triangularity condition (4.31).

Exercise 12.4. Derive from (12.12) the expression transposed to (5.5) for the 2-dimensional representation of D_q for which

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & \bar{q}^{\frac{1}{2}} \end{pmatrix} \quad (E^2 = 0 = F^2)\,. \tag{12.13}$$

Remark 12.1. The second universal R-matrix (5.24) also has a finite dimensional counterpart $\tilde{R}$, such that

$$\tilde{R}(q^{-1}) = \frac{1}{4h} \sum_{\nu=0}^{h-1} \frac{\lambda^\nu}{[\nu]!}\, q^{\binom{\nu}{2}}\, F^\nu \otimes E^\nu \sum_{m,n=0}^{4h-1} q^{-\frac{mn}{2}}\, k^m \otimes k^n \,. \tag{12.14}$$

It is easy to verify that substituting E, F and k in (12.14) by their 2-dimensional representation (12.13) we reproduce the 4×4 R-matrix (5.5).

In order to display a new duality relation between $\mathcal{B}_4$ and U_q representations for the non-unitary extended chiral $su(2)$ WZNW model we need the *Lusztig extension* of the restricted QUEA $\bar{U}_q$ (see [L]). We first introduce, following [L], the *devided powers*

$$E^{(n)} = \frac{1}{[n]!}\, E^n\,, \quad F^{(n)} = \frac{1}{[n]!}\, F^n \tag{12.15}$$

satisfying $X^{(m)}\, X^{(n)} = \left[{n+m \atop n} \right] X^{(n+m)}$ ($\left[{n+m \atop n} \right] = \frac{[n+m]!}{[n]!\,[m]!}$; $X = E, F$),

$$[E^{(m)}, F^{(n)}] = \sum_{s=1}^{\min(m,n)} F^{(n-s)} \begin{bmatrix} H + 2s - m - n \\ s \end{bmatrix} E^{(m-s)} \,. \tag{12.16}$$

The right hand side of (12.15) only has a clear meaning for $n < h$ (since $[h] = 0$). The subsequent relations, however, make sense for all positive integers m, n and can serve as an implicit definition for higher devided powers. It is sufficient to add two new elements $E^{(h)}$ and $F^{(h)}$ in order to obtain an infinite extension U_h of $\bar{U}_q$. Indeed, their powers generate a sequence of new elements.

Exercise 12.5. (a) Defining the ratio $\frac{[nh]}{[h]}$ as a polynomial in $q^{\pm 1}$, deduce

$$\frac{[nh]}{[h]} = \sum_{\nu=0}^{n-1} q^{(n-1-2\nu)h} = (-1)^{n-1}\, n \,, \quad \begin{bmatrix} nh \\ n \end{bmatrix} = (-1)^{(n-1)h}\, n \,. \tag{12.17}$$

(*Hint* : use the identity $[nh + m] = (-1)^n\, [m]$.)
(b) Derive the general formula

$$\begin{bmatrix} Mh + m \\ Nh + n \end{bmatrix} = (-1)^{(M-1)Nh/mN - nM} \begin{bmatrix} m \\ n \end{bmatrix} \begin{pmatrix} M \\ N \end{pmatrix} \tag{12.18}$$

for $M \in \mathbb{Z}$, $N \in \mathbb{Z}_+$, $0 \leq m, n \leq h-1$, $\begin{pmatrix} M \\ N \end{pmatrix} = \frac{M(M-1)\ldots(M-N+1)}{N!}$.

For $n < h$ it is easy, using exercise 12.2(a), to verify the formulae

$$E^{(n)} \mid p, m\rangle = \begin{bmatrix} p - m - 1 \\ n \end{bmatrix} \mid p, m+n\rangle \,, \; F^{(n)} \mid p, m\rangle = \begin{bmatrix} m \\ n \end{bmatrix} \mid p, m-n\rangle \tag{12.19}$$

which allow to extend the action of $E^{(n)}$ and $F^{(n)}$ on the canonical basis to all positive n.

We shall now describe the *irreducible representations* (IRs) of $\bar{U}_q$ and will then single out the IRs of U_h in $\mathcal{F}$.

It is convenient to introduce an operator $q^{\hat{p}}$ (and its inverse, $\bar{q}^{\hat{p}}$) which is diagonal on the canonical basis and has $2h$ different eigenvalues (that fix, in particular, the Casimir invariant (10.4)):

$$(q^{\hat{p}} - q^p) \mid p, m\rangle = 0\,, \quad C = q^{\hat{p}} + \bar{q}^{\hat{p}}\,, \quad q^{h\hat{p}} = q^{-h\hat{p}}\,. \tag{12.20}$$

The IRs of $\bar{U}_q$ are classified in [FGST] (these authors do not use, however, the operator $q^{\hat{p}}$ and introduce bases inequivalent to ours).

Proposition 12.2. *The finite dimensional QUEA* $\bar{U}_q$ *has exactly* $2h$ *IRs* $V_p^{\pm}$, *labeled by their* dimension p *and* parity ϵ *such that*

$$(q^{\hat{p}} - \epsilon\, q^p)\, V_p^\epsilon = 0\,, \quad \dim V_p^\epsilon = p\,, \quad p = 1, \dots, h\,, \quad \epsilon = \pm\,. \tag{12.21}$$

The $\bar{U}_q$ *module* V_p^ϵ *can be equipped with a canonical basis* $|p, m\rangle^\epsilon$, $0 \le m \le p-1$ $(1 \le p \le h)$ *such that*

$$(q^H - \epsilon\, q^{2m-p+1}) \mid p, m\rangle^\epsilon = 0\,, \quad E \mid p, p-1\rangle^\epsilon = 0 = F \mid p, 0\rangle^\epsilon\,. \tag{12.22}$$

Corollary. *Eqs.* (12.21), (12.22) *and* (12.4) *imply the relations*

$$(EF - \epsilon\,[m][p-m]) \mid p, m\rangle^\epsilon = 0 = (FE - \epsilon\,[m+1][p-m-1]) \mid p, m\rangle^\epsilon\,. \tag{12.23}$$

We shall identify in what follows the irreducible $\bar{U}_q$ modules V_p^ϵ in the ($U_q(A_1)$-model) space $\mathcal{F}$. We will not reproduce the proof of [FGST] that these representations exhausts the IRs of $\bar{U}_q$.

The identification $V_p^+ = \mathcal{F}_p$ for $1 \le p \le h$ is immediate.

Exercise 12.6. Prove that the spaces $\mathcal{F}_{h+p}$, $1 \le p \le h$ admit two p-dimensional $\bar{U}_q$-invariant subspaces isomorphic to V_p^-. Verify that $\mathcal{F}_{h+p}$ is indecomposable for $0 < p < h$ and that the quotient $\mathcal{F}_{h+p}/V_p^- \oplus V_p^-$ is isomorphic to V_{h-p}^+. (*Hint* : identify $V_p^- \oplus V_p^-$ with the invariant subspace of $\mathcal{F}_{h+p}$ spanned by $\{\mid h+p, m\rangle\} \oplus \{\mid h+p, h+m\rangle\}$, $0 \le m \le p-1$.)

Remark 12.2. The actions of E and F on the two copies of V_p^- are equivalent albeit not identical:

$$\begin{aligned} E \mid h+p, m\rangle &= -[p-m-1] \mid h+p, m+1\rangle\,, \\ F \mid h+p, m\rangle &= [m] \mid h+p, m-1\rangle \\ E \mid h+p, h+m\rangle &= [p-m-1] \mid h+p, h+m+1\rangle\,, \end{aligned} \tag{12.24}$$

$$F \mid h+p, h+m\rangle = -[m] \mid h+p, h+m-1\rangle\,, \tag{12.25}$$

both yielding (12.23). (We may identify $|p,m\rangle^-$ with either $\mid h+p, h+m\rangle$ or $(-1)^m \mid h+p, m\rangle$.)

Exercise 12.7. Prove that the $\bar{U}_q$-modules $\mathcal{F}_{2h+p}$ $(1 \le p \le h)$ admit three p-dimensional invariant subspaces, each isomorphic to V_p^+, while the quotient space $\mathcal{F}_{2h+p}/V_p^+ \oplus V_p^+ \oplus V_p^+$ (for $p < h$) is isomorphic to $V_{h-p}^- \oplus V_{h-p}^-$. Describe the structure of $\mathcal{F}_{Nh+p}$, $N \in \mathbb{N}$, $1 \le p < h$.

Using the term *subquotient* for either on $\bar{U}_q$ submodule or a quotient we have the following easily verifiable result.

Proposition 12.3. *The direct sum of irreducible $\bar{U}_q$-modules that appear as subquotient of $\mathcal{F}_{Nh+p}$ of a given parity ϵ spans a single IR $\mathcal{V}_p^\epsilon$ of U_h; we have the following exact sequence (for $0 < p < h$) of U_h modules:*

$$0 \to \mathcal{V}_p^{\epsilon_N} \to \mathcal{F}_{Nh+p} \to \mathcal{V}_{h-p}^{-\epsilon_N} \to 0\,, \quad \epsilon_N = (-1)^N\,,$$

$$\mathcal{V}_p^{\epsilon_N} = \bigoplus_0^N V_p^{\epsilon_N}\,, \quad \mathcal{V}_{h-p}^{-\epsilon_N} = \bigoplus_1^N V_{h-p}^{-\epsilon_N}\,. \tag{12.26}$$

Sketch of proof. A straightforward application of (12.18) and (12.19) gives

$$E^{(h)} \mid Nh+p, nh+m\rangle = \begin{bmatrix} (N-n)\,h+p-m-1 \\ h \end{bmatrix} \mid Nh+p, (n+1)\,h+m\rangle$$

$$= (-1)^{(N-n-1)h+p-m-1}(N-n) \mid Nh+p, (n+1)\,h+m\rangle$$

$$0 \le n < N\,, \quad 0 \le m < p \le h \tag{12.27}$$

$$E^{(h)} \mid Nh+p, nh+p+m\rangle = \begin{bmatrix} (N-n)\,h-m-1 \\ h \end{bmatrix} \mid Nh+p, (n+1)\,h+p+m\rangle$$

$$0 \le n < N-1\,, \quad 0 \le m \le h-p-1\,, \tag{12.28}$$

and similar relations for $F^{(h)}$. These relations imply the irreducibility with respect to U_h of the direct sums $\mathcal{V}_p^{\epsilon_N}$ and $\mathcal{V}_{h-p}^{\epsilon_N}$ (12.26) of IRs of $\bar{U}_q \subset U_h$. □

On the other hand, the relations

$$E^{(h)}\,\mathcal{F}_p = 0 = F^{(h)}\,\mathcal{F}_p \quad \text{for} \quad 1 \le p \le h \tag{12.29}$$

tell us that the "Lusztig quantum group" U_h only plays a role in $\mathcal{F}_p$ for $p > h$. Our aim will be to establish a duality relation between the indecomposable representations of U_h in $\mathcal{F}_{Nh+p}$ displayed in Proposition 12.3 and the representations (11.2) of $\mathcal{B}_4$ for $2I+1 = Nh+p$. We denote the corresponding p-dimensional $\mathcal{B}_4$ module of 4-point blocks by $S_4(p)$.

Proposition 12.4. (see Theorem 4.1 of [FHT]) (a) *The $\mathcal{B}_4$ modules $\mathcal{S}_4(p)$ are irreducible for $0 < p < h$ and for $p = Nh$.*

(b) *For $N > 0$ and $0 < p < h$, $S_4(Nh+p)$ is indecomposable with structure dual to that of $\mathcal{F}_{Nh+p}$ displayed in Proposition* 12.3*. It has a $N(h-p)$-dimensional invariant subspace*

$$S(N, h-p) = \text{Span}\,\{f_\mu^{(Nh+p)}\,,\ \mu = nh+p, \ldots, (n+1)\,h-1\}_{n=0}^{N-1} \qquad (12.30)$$

which carries an IR of $\mathcal{B}_4$. The $(N+1)p$-dimensional quotient space $\tilde{S}(N+1, p)$ also carries an IR of the braid group.

Proof. The $\mathcal{B}_4$-invariance of $S(N, h-p)$ (12.30) follows from the proportionality of the $(Nh+p)$-dimensional matrices (11.2) to the q-binomial coefficients:

$$\begin{aligned} B_{1\,nh+\beta}^{mh+\alpha} &\sim \begin{bmatrix} nh+\alpha \\ nh+\beta \end{bmatrix} \sim \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \binom{m}{n} = 0 \\ B_{2\,nh+\beta}^{mh+\alpha} &\sim \begin{bmatrix} (N-m)\,h+p-\alpha-1 \\ (N-n-1)\,h+h+p-\beta-1 \end{bmatrix} \\ &\sim \begin{bmatrix} p-\alpha-1 \\ h+p-\beta-1 \end{bmatrix} \binom{N-m}{N-n-1} = 0 \end{aligned}$$

for $m = 0, \ldots, N, 0 \le \alpha \le p-1$, $n = 0, \ldots, N-1$, $p \le \beta \le h-1$; they vanish since $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$ for $0 \le \alpha < \beta$ (α, β integers). An inspection of the same expression (11.2) allows to conclude that the space $S(N, h-p)$ has no $\mathcal{B}_4$-invariant complement in $\mathcal{S}_4(Nh+p)$ which is, thus, indeed indecomposable. It is also readily verified that the quotient space

$$\tilde{S}(N+1, p) = \mathcal{S}_4(Nh+p)/S(N, h-p)$$

carries an IR of $\mathcal{B}_4$. □

We thus see that the indecomposable representations $\mathcal{F}_{Nh+p}$ (of U_h) and $\mathcal{S}_4(Nh+p)$ (of $\mathcal{B}_4$) contain the same number (two) of irreducible components (of the same dimensions) but the arrows of the exact sequences are reversed. This sums up the meaning of duality for indecomposable representations.

Remark 12.2. Note that the difference of conformal dimensions

$$\Delta_{Nh+I} - \Delta_I = N(Nh+p) \quad (0 < p = 2I+1 < h)$$

is a (positive) integer; this explains the similarity of the corresponding braid group representations $\mathcal{S}_4(p)$ and $\mathcal{S}_4(2Nh+p)$. There is a unique 1-dimensional subspace $S(1,1) \subset S_4(2h-1)$ among the $\mathcal{B}_4$-invariant subspaces displayed in Proposition 11.4 corresponding to a non-unitary local field of isospin and conformal dimension $h-1$:

$$\Delta_{h-1} = \frac{(h-1)\,h}{h} = h-1\,. \tag{12.31}$$

It has rational correlation functions; in particular, the 4-point amplitude $f_{h-1}^{(h-1)}$ (ξ,η) (9.29)–(9.31) is a polynomial [HP]. It therefore gives rise to a non-unitary local extension of the $\widehat{su}(2)_k$ current algebra that deserves a further study.

13 Outlook

In conclusion we shall sum up the philosophy underlying these notes – and the ensuing choice of material – and will then list some related topics which appear to be interesting and important but remain outside the scope of the present lectures.

It is natural from both physical and mathematical point of view to associate with any "symmetry" (meaning symmetry group, Lie algebra or a generalization thereof) a family (or "category") of representations equipped with a *tensor product*. The fact that the tensor product of representations is again a representation (of the same symmetry) leads us to the concept of a *coproduct*. The *commutant* of a tensor product representation yields the notion of a *braid group* which reduces to a *permutation group* when the symmetry is described by an ordinary group. If we think of *irreducible representations* as describing *elementary objects* (particles, excitation) then the behaviour under braiding (that exchanges elementary objects) would determine the particle *statistics*. We are thus led to consider the pair *symmetry and statistics* as a whole. The generalization or *deformation* of one requires a similar deformation of the other.

Quantum groups are coupled to *braid group statistics* (as already the title of these lectures suggests). Existing attempts at phenomenological applications of "q-symmetries" (viewing q as one more parameter to fit data), that ignore the (necessarily!) accompanying it braid group statistics, are, in my opinion, ill conceived.

The appearance of *monodromy* (a *normal subgroup* of the braid group) is a sign of the presence of multivalued correlation functions which naturally arise in a non-simply connected *configuration space* – that is the case of dimension two. In higher dimensions the *fundamental group* π_1 of configuration space is trivial. Indeed, the deep analysis of Doplicher-Haag-Roberts of the structure of *superselection sectors* in a local relativistic quantum theory (a work spanned over more than 20 years, culminating in [DR], and recounted in [H]) demonstrates that the gauge symmetry (of the first kind) of local observables is implemented by a compact group and is thus accompanied by a permutation group (para) statistics (reduced, essentially, to the familiar Bose and Fermi statistics). We, hence, only consider applications of quantum symmetry and braid group statistics to 2-dimensional ($2D$) conformal field theory. (Our analysis of such "applications" is restricted to the formalism. The relevance, say, of anyonic statistics to the theory of fractional quantum Hall effect is only alluded to.) We have given more room to the (mathematically) intriguing non-abelian QUEA which appear as gauge symmetries of chiral conformal fields. (A gauge symmetry, by its definition, does *not* affect observables. Accordingly, it is only manifest after one splits the observable $2D$ fields into *chiral vertex operators*, corresponding to the splitting of $2D$ correlation functions (single valued in the Euclidean domain) into multivalued *conformal blocks*).

Among the big omissions from the present survey the closest in spirit – and thereby particularly regrettable – is the Chern[49]-Simons theory about which we shall just say a few words and give a few references.

The *Chern-Simons theory* is a *topological gauge theory* on a three (space-time) dimensional manifold M. Here by topological we mean that its action does not depend on the metric on M. Let $\boldsymbol{A}$ be a connection one-form with values in a Lie algebra $\mathcal{G}$. (For $\mathcal{G} = u(n)$ – a commonly encountered example – this means that $\boldsymbol{A}$ is an antihermitian $n \times n$ matrix of 1-forms.) The curvature 2-form $\boldsymbol{F}$ is defined, as usual, by

$$\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A}\,. \tag{13.1}$$

An example of a topological action density in four space time dimensions is given by the so called "θ-term"[50] the 4-form $\mathrm{tr}(\boldsymbol{F} \wedge \boldsymbol{F})$ $(= \boldsymbol{F}^{\alpha}_{\beta} \wedge F^{\beta}_{\alpha})$, which is a total derivative (when expressed in terms of $\boldsymbol{A}$). The *Chern-Simons form*

$$\omega_3 = \mathrm{tr}\left(\boldsymbol{A} \wedge d\boldsymbol{A} + \frac{2}{3}\,\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A}\right) \tag{13.2}$$

is defined to satisfy

$$d\,\omega_3 = \mathrm{tr}(\boldsymbol{F} \wedge \boldsymbol{F})\,. \tag{13.3}$$

In order to verify (13.3) (for ω_3 given by (13.2) and F given by (13.1)) one has to use the cyclicity of the trace and the anticommutativity of 1-forms to deduce

$$\mathrm{tr}\,\boldsymbol{A}^{12k}(= \boldsymbol{A}^{\alpha_1}_{\alpha_2} \wedge \boldsymbol{A}^{\alpha_2}_{\alpha_3} \wedge \ldots \wedge A^{\alpha_{2k}}_{\alpha_1}) = 0 \tag{13.4}$$

(*cf.* Exercise 10.1). (More generally, the Chern-Simons $(2k-1)$-form ω_{2k-1} is defined to satisfy $d\,\omega_{2k-1} = \mathrm{tr}(\boldsymbol{F}^{\wedge k})$. Verify that

$$\omega_5 = \mathrm{tr}\left(\boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{A} - \frac{1}{2}\,\boldsymbol{F} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} + \frac{1}{10}\,\boldsymbol{A}^{\wedge 5}\right) \tag{13.5}$$

satisfies $d\,\omega_5 = \mathrm{tr}\,\boldsymbol{F}^{\wedge 3}$. Note that ω_3 may be also written as $\omega_3 = \mathrm{tr}\,(\boldsymbol{F} \wedge \boldsymbol{A} - \frac{1}{3}\,\boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A})$.

Varying the (conformally invariant!) Chern-Simons action

$$S = \frac{k}{4\pi} \int_M \mathrm{tr}\left(\boldsymbol{A} \wedge d\boldsymbol{A} + \frac{2}{3}\,\boldsymbol{A}^{\wedge 3}\right) \tag{13.6}$$

we find the equation of motion

$$0 = \frac{\delta S}{\delta \boldsymbol{A}} = \frac{k}{2\pi}\,\boldsymbol{F} \tag{13.7}$$

[49] The Chinese American mathematician Shiing-Shen Chern (1911-2004), a leading differential geometer of 20$^{\mathrm{th}}$ century, wrote the paper on Chern-Simons forms in 1974 with his student Jim Simons.

[50] The term $\theta\,\mathrm{tr}(F \wedge F)$ is much discussed in connection with the problem of strong CP violation, [SVZ]; for an instance of a subsequent theoretical study – see [W98].

which says that the curvature is zero, or, in other words, the *connection $\boldsymbol{A}$ is flat.* Flat connections are determined entirely by *holonomies* around noncontractible cycles. If $\mathcal{K}$ is an oriented knot[51] then one considers the trace of the holonomy of the gauge connection around $\mathcal{K}$ in a given IR of $U(n)$, which gives the *Wilson loop operator*, the trace of the path-ordered exponent

$$W_R^{\mathcal{K}}(\boldsymbol{A}) = \mathrm{tr}_R \left(P \exp \oint_{\mathcal{K}} \boldsymbol{A} \right) . \tag{13.8}$$

Witten [W] discovered that the vacuum expectation value of this operator for $n = 2$ reproduces the *Jones polynomial* invariant [J]. (The appearance of topological invariants in QFT has been suggested earlier by Albert Schwarz.) For M a 3-manifold with boundary Σ Witten demonstrates that the Chern-Simons theory on M with a compact Lie group G and action (13.6) gives rise to a WZNW theory on Σ corresponding to the current algebra $\hat{\mathcal{G}}$ of level k.

For a review on Chern-Simons theory with applications to topological strings – see [M05]. For the quantization of the Hamiltonian Chern-Simons theory and for the representation theory of Chern-Simons observables (not covered in [M05]) – see [AGS] and [AS].

A second important topic outside the scope of these lectures is the application of Hopf algebra techniques to QFT renormalization initiated by Dirk Kreimer and further developed by Connes and Kreimer – see for recent reviews [CoMa] [Kr] (*cf.* also [C]).

We have not touched upon the study of *quantum homogeneous spaces* and their possible application as candidates for non-commutative space-time manifolds. Here we feel that a more general point of view, not necessarily related to quantum groups is preferable – see [Co, CoMa]. For interesting purely mathematical results in this direction – see [CD-V] and [DLSSV].

[51] For a survey of modern knot theory – see [Li] and [PS].

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Only names of past scientists are listed for whom some biographical information is given in the text.