Instanton Partition Functions and M-Theory

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1. Introduction

This is a short summary of the progress of the last few years in exact non-perturbative calculations in supersymmetric gauge and string theories, based on the formalism of the *instanton partition functions*. We discuss theories in various spacetime dimensions, and connect their partition functions to algebraic geometry, combinatorics, integrable systems, representation theory of infinite-dimensional algebras, and topology, on the mathematical side, and to the low-energy gauge theory dynamics, wall-crossing of the particle spectrum, crystal melting, random growth models, and the models of electrons in random fields, on the physical side. In these lectures we shall discuss instanton partition functions in two, four, and six dimensions. These partition functions capture some information about the spectrum of the supersymmetric gauge theories, more precisely their low-energy dynamics. Some of these theories are not defined as quantum field theories, and need string theory for their microscopic definition. Remarkably, as we shall discover, they know even about the M-theory. Our conjectures include the identities between the generalization of the MacMahon formula and the character of M-theory, compactified down to $0+1$ dimension. The organization of these notes is the following. We start, in the section **2**, with the quick review of M-theory. In the section **3** we discuss gauge theories in various spacetime dimensions, ranging from two up to eight. These theories, as we shall explain in section **3**, can be topologically twisted, or partially twisted, to give an integral representation of intersection theory on some moduli space $\mathcal{M}$, or its K-theoretic version. Mathematicians usually study the moduli spaces $\mathcal{M}_X$ of solutions of gauge theory equations defined over a compact manifold X (which can be two, four, or six dimensional in our problems). The twisted gauge theory correlation functions can be then used to define some invariants of X. When X has some symmetry group H, which preserves the gauge theory equations, the intersection theory and K-theory of $\mathcal{M}_X$ have the H-equivariant version. In the section **4** we introduce the main object of our study: the instanton partition functions. These are the partition functions defined using the H-equivariant theory on $\mathcal{M}_{\mathbf{R}^{2d}}$ for $H = SO(2d)$ or $U(d)$. The section **5** presents the instanton partition functions as the sums over N-tuples of various kinds of partitions, for $G = U(N)$. The section **6** collects various interesting facts about these partition functions. We relate the partition functions in 2 and $2+1$ dimensions to quantum integrable systems, the partition functions in 4 and $4+1$ dimensions to the representation theory of infinite-dimensional Lie algebras, $(2,0)$ tensor multiplet in six dimensions, statistical mechanical models and to algebraic integrable systems; finally, the partition functions in 6 dimensions are related to dimer models, free fermions, two dimensional lattice electrons in random fields. Our final partition function, that of $6+1$ dimensional theory is then related, for $G = U(1)$, to the partition function of the eleven dimensional linearized supergravity.

2. Combinatorial partition functions

Counting is in the human being's nature. We count sigarette buds, or steam locomotives, stars in the sky, or fish in a sea, or neighbour's possessions. Abstractly speaking, we count some objects, with, or without structure.

The first, trivial, generating function, just counts the objects without any structure, only paying attention to the total number of objects:

$$\varphi_1(q) = 1 + q + q^2 + \ldots = \frac{1}{1-q} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} q^n\right) \tag{2.1}$$

The next level of sophistication arises when we try to remember how a collection of n objects could have fallen on our hands. For example, it could have come as a union, or as a bound state, of more elementary collections, of which we only care about their total number, as in the first example. In this way we are led to the problem of counting partitions of natural numbers. We define $p_2(n)$ as the number of ways to represent n as a sum of natural numbers (up to permutation of summands). For example, $p_2(1) = 1$, $p_2(2) = 2$, as $2 = 2, 2 = 1 + 1$. Each partition λ, accounted for by $p_2(n)$ is a collection

$$\lambda = \left(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0 \right) \tag{2.2}$$

of integers, so that by definition:

$$n = |\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i \tag{2.3}$$

For the partition λ, $|\lambda| = n$ is called the size of the partition, and $\ell(\lambda)$, the number of non-zero entries, is called the length of the partition. The notation $p_2(n)$ comes from the two dimensional nature of the Young diagram of partition λ. The Young diagram is a collection of squares attached one to another, so that the first row has λ_1 squares, the second row has λ_2 squares, and so on. Given a partition λ, a dual partition λ^t is such that its Young diagram is the flipped Young diagram of λ. In other words $\lambda^t_i = \#\{ j \,|\, \lambda_j \geq i \}$. Of course, $|\lambda| = |\lambda^t|$.

The generating function of the numbers $p_2(n)$ is well-known:

$$\varphi_2(q) = \sum_{\lambda} q^{|\lambda|} = \sum_{n=1}^{\infty} p_2(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^n}\right) \tag{2.4}$$

The amazing property of this generating function is its modularity:

$$\varphi_2\left(e^{2\pi i\tau}\right) = \sqrt{\frac{\tau}{i}}e^{\frac{\pi i}{12}\left(\tau+\frac{1}{\tau}\right)}\varphi_2\left(e^{-\frac{2\pi i}{\tau}}\right) \tag{2.5}$$

which allows to estimate the large n behavior of $p_2(n)$ to a very good accuracy.

The simple counting of the two dimensional partitions can be generalized in a number of ways.

For example, the partitions λ of size $n = |\lambda|$ label the irreducible representations of the symmetric group $\mathcal{S}_n$. Indeed, the irreducible representations of a finite group are in one-to-one correspondence with the conjugacy classes. For the symmetric group the conjugacy classes are labeled by the multiplicities of the cycles of a given length. By ordering these lengths we obtain a partition. For any finite group Γ one can define a natural measure on the space $\Gamma^{\vee}$ of its irreducible representations:

$$\mu_\lambda = \left(\frac{\dim(\lambda)}{|\Gamma|}\right)^2 \tag{2.6}$$

which is normalized so that

$$\sum_{\lambda\in\Gamma^{\vee}} \mu_\lambda = \frac{1}{|\Gamma|} \tag{2.7}$$

This so-called Plancherel measure is the Fourier transform of the Haar measure on the group. For $\Gamma = \mathcal{S}_n$, the measure (2.6) can be viewed as a Boltzmann weight for some statistical mechanical model. This model describes boundary of the Young diagram, viewed as a discrete version of a sand pile. One rotates Young diagram by 135^{o} degrees counterclockwise. The rotated Young diagram can be obtained as follows. Start with the wedge, the plot of the function $f(x) = |x|$. Take n squares, rotated 45^{o} clockwise, and start dropping them, one by one, into the wedge. The squares will slide down, until they stop, at the wedge, or at the square below. The measure (2.6) is simply the quantum mechanical probablity of creating a pile of squares which corresponds to the Young diagram of a partition λ. In other words, the amplitude of getting λ is just the number of ways of arriving at it by dropping one square after another, assuming that the dropping position is chosen at random, dividing by the order of the permutation group, and the probability is the square of the amplitude. Thus, $\mu_{(1)} = 1$, $\mu_{(2)} = \mu_{(1,1)} = \frac{1}{2^2}$, $\mu_{(3)} = \frac{1}{6^2}, \mu_{(2,1)} = \frac{1}{3^2}, \mu_{(1,1,1)} = \frac{1}{6^2}$. The Boltzmann weight (2.6) can be also expressed as a product over the squares in the Young diagram, or as the product over the pairs of boundary squares. In other words, the boundary of Young diagram behaves as a chain of interacting beads. The energy of interaction is given by the logarithm of the so-called hook-length:

$$\mathcal{E}\left(x,y\right) = \log \left| \, h_{(i,j)} \, \right|^{2} \tag{2.8}$$

where $x = (i, \lambda_i)$, $y = (\lambda^t_j, j)$, $h_{(i,j)} = \lambda_i - j + \lambda^t_j - i + 1$. The Plancherel measure is thus the Boltzmann weight of the Coulomb-gas like interacting beads, at the temperature $\beta = 1$:

$$\mu_\lambda = \prod_{(i,j)\in\lambda} \frac{1}{h^2_{(i,j)}} = \exp\left[- \sum_{x,y\in\partial\lambda,\, x\neq y} \mathcal{E}(x,y) \right] \tag{2.9}$$

The Plancherel measure (2.9) has an asymmetric generalization, parametrized by two numbers (ϵ_1, ϵ_2):

$$\mu_\lambda(\epsilon_1,\epsilon_2) = \exp\left[- \sum_{x,y\in\partial\lambda,\, x\neq y} \mathcal{E}(x,y;\epsilon_1,\epsilon_2) \right] \tag{2.10}$$

where

$$\mathcal{E}(x,y;\epsilon_1,\epsilon_2) = \log\left(\epsilon_1(a_{i,j}+1) - \epsilon_2(l_{i,j})\right)\left(-\epsilon_1(a_{i,j}) + \epsilon_2(l_{i,j}+1)\right) \tag{2.11}$$

where the "arm-length" $a_{i,j} = \lambda_i - j$ and the "leg-length" $l_{i,j} = \lambda^t_j - i$ (of course, in order for (2.11) to define a meaningful energy, the parameters ϵ_1, ϵ_2 must obey some inequality, which we shall not discuss here). There are more generalizations of importance: the "massive",

$$\mu_\lambda(\epsilon_1,\epsilon_2,m) = \prod_{\square\in\lambda} \frac{\left(\epsilon_1(a_\square+1) - \epsilon_2(l_\square) + m\right)\left(-\epsilon_1(a_\square) + \epsilon_2(l_\square+1) + m\right)}{\left(\epsilon_1(a_\square+1) - \epsilon_2(l_\square)\right)\left(-\epsilon_1(a_\square) + \epsilon_2(l_\square+1)\right)} \tag{2.12}$$

the trigonomentric,

$$\mu_\lambda(q_1,q_2,\mathbf{m}) = \mathbf{m}^{-|\lambda|} \prod_{(i,j)\in\lambda} \frac{\left(1 - q_1^{a_{i,j}+1} q_2^{-l_{i,j}} \mathbf{m}\right)\left(1 - q_1^{-a_{i,j}} q_2^{l_{i,j}+1} \mathbf{m}\right)}{\left(1 - q_1^{a_{i,j}+1} q_2^{-l_{i,j}}\right)\left(1 - q_1^{-a_{i,j}} q_2^{l_{i,j}+1}\right)} \tag{2.13}$$

the elliptic, and so on. They interpolate between the Plancherel measure (2.9) and the uniform measure (2.4).

The generating functions, summing over all partitions with the measures (2.13),(2.12), (2.10), (2.9) are quite beautiful, and exhibit unexpected symmetries, generalizing the modularity (2.5):

$$Z(\epsilon_1,\epsilon_2,m,q) = \sum_\lambda q^{|\lambda|} \mu_\lambda\left(\epsilon_1,\epsilon_2,m\right) = \varphi_2(q)^{\frac{(m+\epsilon_1)(m+\epsilon_2)}{\epsilon_1\epsilon_2}} \tag{2.14}$$

$$Z^{\text{inst}}(q_1,q_2;\mathbf{m},q) = \sum_\lambda q^{|\lambda|} \prod_{\square\in\lambda} \frac{(1-\mathbf{m}q_1^{a(\square)+1}q_2^{-l(\square)})(1-\mathbf{m}q_1^{-a(\square)}q_2^{l(\square)+1})}{\mathbf{m}\,(1-q_1^{a(\square)+1}q_2^{-l(\square)})(1-q_1^{-a(\square)}q_2^{l(\square)+1})}$$

$$Z^{\text{inst}}(q_1,q_2;\mathbf{m},q) = \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{n(1-q^n)} \frac{(1-\mathbf{m}^n q_1^n)(1-\mathbf{m}^n q_2^n)}{\mathbf{m}^n(1-q_1^n)(1-q_2^n)}\right) \tag{2.15}$$

Actually, the formula (2.15) is still a conjecture, and the formula (2.14) is proven in [1] in the special case $\epsilon_1+\epsilon_2=0$. An equivalent form of (2.15):

$$Z^{\text{inst}}(q_1,q_2;\mathbf{m},q) = \prod_{n=1}^{\infty}\prod_{a,b=1}^{\infty} \frac{(1-q_1^a q_2^{b-1}q^n)(1-q_1^{a-1}q_2^b q^n)}{(1-\mathbf{m}^{-1}q_1^{a-1}q_2^{b-1}q^n)(1-\mathbf{m}q_1^a q_2^b q^n)}$$

The next level of sophistication is counting the three dimensional partitions. The three dimensional partition π is a stack of two dimensional partitions, which are non-increasing in an obvious geometric sense:

$$\pi = \{\pi_{i,j}\,|\,\pi_{i,j}\in\mathbf{Z}_{\geq 0},\, h_{i,j}\geq \pi_{i+1,j}\,,\, \pi_{i,j}\geq\pi_{i,j+1}\,\} \tag{2.16}$$

For fixed a, $\lambda_i=\pi_{i,a}$ defines a partition, similarly $\lambda_j = h_{a,j}$ also defines a partition. More generally, for fixed a,b, $\lambda_i=\pi_{a+pi,b+qi}$ defines a partition, for $p,q\geq 0$. We can view the three dimensional partition as the set of points in $\mathbf{Z}_+^3$:

$$\begin{aligned}\pi &= \{\,(i,j,k)\,|\,i,j,k\in\mathbf{Z}_+\,,\,1\leq k\leq \pi_{i,j}\,\}\\ &= \{\,(i,j,k)\,|\,i,j,k\in\mathbf{Z}_+\,,\,1\leq i\leq \widetilde{\pi}_{j,k}\,\}\\ &= \{\,(i,j,k)\,|\,i,j,k\in\mathbf{Z}_+\,,\,1\leq j\leq \pi'_{i,k}\,\}\end{aligned} \tag{2.17}$$

Finally, the three dimensional partition π defines three two dimensional partitions π^x,π^y,π^z, its shadows on the coordinate planes yz, xz, xy, respectively:

$$\begin{aligned}(i,j)\in\pi^z &\Leftrightarrow \pi_{i,j}>0\\ (i,k)\in\pi^y &\Leftrightarrow \pi'_{i,k}>0\\ (j,k)\in\pi^x &\Leftrightarrow \widetilde{\pi}_{j,k}>0\end{aligned} \tag{2.18}$$

The size, or rather the volume, of the partition π, is the sum:

$$|\pi| = \sum_{(i,j)\in\pi^z}\pi_{i,j} = \sum_{(j,k)\in\pi^x}\widetilde{\pi}_{j,k} = \sum_{(i,k)\in\pi^y}\pi'_{i,k} \tag{2.19}$$

The generating function of the number of the three dimensional partitions of a given size is known as the MacMahon formula:

$$\varphi_3(q) = \sum_{\pi} q^{|\pi|} = \sum_{n=0}^{\infty} p_3(n) q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = \exp \sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{(1-q^n)^2} \tag{2.20}$$

The two dimensional random partitions and the three dimensional random partitions can be related in many ways. For example, one can define a measure on two dimensional partitions λ by counting all three dimensional partitions which project inside λ:

$$\mu_\lambda(q) = \sum_{\pi : \pi_{i,j} > 0 \Rightarrow (i,j) \in \lambda} q^{|\pi|} = \frac{1}{1 - q^{h(i,j)}} \tag{2.21}$$

The measures $\mu_\lambda(q_1, q_2, \mathbf{m})$ also have analogues for the three dimensional partitions:

$$\mu_\pi(q_1, q_2, q_3) = \prod_{x,y \in \partial\pi} e^{-\mathcal{E}_3(x,y)} \tag{2.22}$$

Here, for $x = (i, j, \pi_{i,j})$, $y = (\widetilde{\pi}_{j',k}, j', k)$:

$$e^{-\mathcal{E}_3(x,y)} = \frac{1 - q_1^{i - \widetilde{\pi}_{j',k}} q_2^{j - j' + 1} q_3^{\pi_{i,j} + 1 - k}}{1 - q_1^{i - \widetilde{\pi}_{j',k}} q_2^{j - j'} q_3^{\pi_{i,j} + 1 - k}} \times \text{similar factors} \tag{2.23}$$

We write the precise formula in the following sections. The generating function is conjectured to be (for $Q = q\,(q_1 q_2 q_3)^{-\frac{1}{2}}$):

$$\begin{aligned} Z(q_1, q_2, q_3, q) &= \sum_{\pi} q^{|\pi|} \mu_\pi(q_1, q_2, q_3) = \\ &\exp\left(- \sum_{n=1}^{\infty} \frac{Q^n}{n(1-Q^n)(1-Q^n q_1^n q_2^n q_3^n)} \frac{(1-q_1^n q_2^n)(1-q_1^n q_3^n)(1-q_2^n q_3^n)}{(1-q_1^n)(1-q_2^n)(1-q_3^n)} \right) \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{\sin\left(\frac{n\beta(\epsilon_1+\epsilon_2)}{2}\right) \sin\left(\frac{n\beta(\epsilon_1+\epsilon_3)}{2}\right) \sin\left(\frac{n\beta(\epsilon_2+\epsilon_3)}{2}\right)}{\sin\left(\frac{n\beta\epsilon_1}{2}\right) \sin\left(\frac{n\beta\epsilon_2}{2}\right) \sin\left(\frac{n\beta\epsilon_3}{2}\right) \sin\left(\frac{n\beta\epsilon_4}{2}\right) \sin\left(\frac{n\beta\epsilon_5}{2}\right)} \right) \end{aligned} \tag{2.24}$$

where

$$q_1 = e^{\beta\epsilon_1},\; q_2 = e^{\beta\epsilon_2},\; q_3 = e^{\beta\epsilon_3},\, q = e^{\beta\epsilon},$$

$$\begin{aligned} \epsilon_4 &= \epsilon - \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)\,, \\ \epsilon_5 &= -\epsilon - \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)\,, \\ &\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 = 0 \end{aligned} \tag{2.25}$$

In the rational limit $\beta \to 0$, q-finite, the measure (2.23) reduces to the so-called *equivariant vertex* measure [2],[3], and the partition function (2.24) reduces to:

$$Z(\epsilon_1, \epsilon_2, \epsilon_3, q) = \varphi_3(q)^{\frac{(\epsilon_1+\epsilon_2)(\epsilon_1+\epsilon_3)(\epsilon_2+\epsilon_3)}{\epsilon_1\epsilon_2\epsilon_3}} \tag{2.26}$$

This formula is actually rigorously proven, in [3].

3. Geometric partition functions

The partition functions we presented in the previous section arises in the following geometric problems. One studies K-theory or intersection theory of some moduli space $\mathcal{M}_n$, typically defined by some matrix polynomial equations. Among the matrices we shall have $n \times n$ matrices B_a, where $a = 1, \ldots, d$. In the first story $d = 1$, for the two dimensional partitions $d = 2$, and $d = 3$ for the three dimensional partitions. The space $\mathcal{M}_n$ is acted on by the group $\mathbf{T} = U(1)^d$. In addition it may be acted upon by some other group G. The partitions in one, two, and three dimensions are the $\mathbf{T}$-fixed points in $\mathcal{M}_n$. The geometric partition functions are the generating functions of the integrals

$$Z = \sum_{n=0}^{\infty} q^n \int_{\mathcal{M}_n} \mathcal{X}_n \tag{3.1}$$

where $\mathcal{X}_n$ is some $\mathbf{T}$-equivariant characteristic class of $\mathcal{M}_n$, which depends on a particular theory we wish to study. Let $\mathbf{t} = \mathrm{Lie}\mathbf{T}$. The localization with respect to the $\mathbf{T}$-action then expresses (3.1) as the sum over the fixed points:

$$Z = \sum_n q^n \sum_{f \in \mathcal{M}_n^{\mathbf{T}}} \frac{\mathcal{X}_n(f)}{\prod_i w_i(f)} \tag{3.2}$$

where $w_i(f) \in \mathbf{t}^*$ are the weights of the $\mathbf{T}$-action on $T_f\mathcal{M}_n$, and $\mathcal{X}_n(f)$ is the restriction of $\mathcal{X}_n$ at the fixed point f. In the equivariant K-theory the formula (3.2) is similar except that $\mathcal{X}_n$ stands for the equivariant K-theory class, and the denominator has $\left(1 - e^{-w_i(f)}\right)$ instead of $w_i(f)$.

$$Z^{\mathrm{K}} = \sum_n q^n \sum_{f \in \mathcal{M}_n^{\mathbf{T}}} \frac{\mathcal{X}_n(f)}{\prod_i \left(1 - e^{-w_i(f)}\right)} \tag{3.3}$$

3.1. One dimension

Let $d = 1$. Consider the space of pairs: (B, I), $B \in \mathrm{End}(\mathbf{C}^n), I \in \mathbf{C}^n$. Define $\mathcal{M}_n$ to be the symplectic quotient of that space by the action of $U(n)$:

$$\mathcal{M}_n = \{\, (B, I) \mid [B, B^\dagger] + I \otimes I^\dagger = r \cdot \mathbf{1}_n \}/(B, I) \sim (gBg^{-1}, gI) \tag{3.4}$$

for $g \in U(n)$. Incidentally, the space $\mathcal{M}_n$ is the phase space of Calogero-Moser integrable system, for $r > 0$. For $r < 0$ it is empty, and for $r = 0$ it is singular as a real manifold, but it is a smooth complex variety, isomorphic to $\mathbf{C}^n$.

Let us assume that $r > 0$, and take $\mathcal{X}_n = 1$. Take $\mathbf{T} = U(1)$ acting on $\mathcal{M}_n$ by sending the class of (B, I) to $(q_1 B, I)$, $q_1 \in U(1)$. The fixed points of the $\mathbf{T}$-action are easy to classify: these are the operators B and the vector I such that the $U(1)$-transformation can be undone by the $U(n)$ transformation:

$$q_1 B = g(q_1) B g(q_1)^{-1} \, , \; g(q_1) I = I \tag{3.5}$$

One can find a basis $e_1, e_2, \ldots, e_n$ in $\mathbf{C}^n$, where $g(q_1)$ is a diagonal matrix, and

$$g(q_1) e_i = q_1^{i-1} e_i \, , \; I = \sqrt{n\, r} e_1 \, , \; B = \sum_{i=1}^{n-1} \sqrt{r\,(n-i)}\, e_{i+1} \otimes e_i^{\dagger} \tag{3.6}$$

Thus, there is only one $U(1)$-fixed point f on $\mathcal{M}_n$. The tangent space to $\mathcal{M}_n$ at f is a representation of $U(1)$. Its character:

$$\sum_i e^{w_i(f)} = V^* - (1 - q_1) V V^* = \sum_{i=1}^{n} q_1^{n+1-i} \tag{3.7}$$

where $V = \operatorname{tr}_{\mathbf{C}^n} g(q_1) \, , V^* = \operatorname{tr}_{\mathbf{C}^n} g(q_1)^{-1}$, and the K-theoretic partition function is equal to:

$$\begin{aligned} Z(q_1; q) &= \sum_{n=0}^{\infty} \frac{q^n}{(1-q_1^{-1})(1-q_1^{-2})\ldots(1-q_1^{-n})} \\ &= \exp\left(- \sum_{n=1}^{\infty} \frac{q^n}{n} \frac{q_1^n}{1-q_1^n} \right) \\ &= \prod_{n=1}^{\infty} (1 - q_1^n \, q) \end{aligned} \tag{3.8}$$

If we choose as $\mathcal{X}_n$ a class of the virtual bundle:

$$\mathcal{X}_n = \sum_{j=0}^{n} (-1)^j \mathbf{m}^j \Lambda^j \mathcal{T}^*_{\mathcal{M}_n} \, , \tag{3.9}$$

then the contribution of the fixed point f is modified to:

$$\frac{(1 - \mathbf{m}\, q_1^{-1}) \ldots (1 - \mathbf{m}\, q_1^{-n})}{(1-q_1^{-1})\ldots(1-q_1^{-n})}$$

while the partition function of the "massive theory" is equal to:

$$\begin{aligned} Z(q_1; \mathbf{m}, q) &= \sum_{n=0}^{\infty} q^n \, \frac{(1 - \mathbf{m}\, q_1^{-1}) \ldots (1 - \mathbf{m}\, q_1^{-n})}{(1-q_1^{-1})\ldots(1-q_1^{-n})} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{n} \frac{\mathbf{m}^n - q_1^n}{1-q_1^n} \right) \\ &= \prod_{n=1}^{\infty} \frac{1 - q_1^n \, q}{1 - \mathbf{m}\, q_1^{n-1}\, q} \end{aligned} \tag{3.10}$$

which for $\mathbf{m} = 1$, the Euler characteristic, gives our first generating function $\varphi_1(q)$ (2.1).

3.2. Two dimensions

In this problem the space $\mathcal{M}_n$ is the Hilbert scheme of n points on $\mathbf{C}^2$, which can be given a description similar to (3.4):

$$\begin{aligned}\mathcal{M}_n^{d=2} = \{ \, (B_1, B_2, I, J) \, | \, B_{1,2} \in \mathrm{End}(\mathbf{C}^n) \, , I \in \mathbf{C}^n \, , J \in \mathbf{C}^{n*} \, , \\ [B_1, B_2] + IJ = 0 \, , \, [B_1, B_1^{\dagger}] + [B_2, B_2^{\dagger}] + I \otimes I^{\dagger} - J^{\dagger} \otimes J = r \, \mathbf{1}_n \} / U(n) \\ (B_{1,2}, I, J) \sim (g B_{1,2} g^{-1}, gI, Jg^{-1}) \, , \text{for } g \in U(n)\end{aligned} \tag{3.11}$$

The space $\mathcal{M}_n$ is acted upon by the group $U(2)$ under which (B_1, B_2) transform as a doublet, and $(I, J^{\dagger})$ as a doublet of $SU(2) \subset U(2)$. With respect to the torus $\mathbf{T} = U(1) \times U(1) \subset U(2)$, the data $(B_{1,2}, I, J)$ transforms as follows:

$$(B_1, B_2, I, J) \mapsto (q_1 B_1, q_2 B_2, (q_1 q_2)^{1/2} \, I, (q_1 q_2)^{1/2} \, J) \tag{3.12}$$

Let us assume that $r > 0$ for definiteness. Then one can show that $J = 0$, and that the set of polynomials $P(x_1, x_2)$ such that $P(B_1, B_2)I = 0$ form an ideal in the ring of polynomials in two variables, of codimension n. The fixed points of the $\mathbf{T}$-action on $\mathcal{M}_n$ are the monomial ideals, which are in one-to-one correspondence with the partitions λ of size $|\lambda| = n$. The ideal $\mathcal{I}_\lambda$ corresponding to the partition λ is generated by

$$x_1^{i-1} x_2^{\lambda_i} \, , \; i = 1, \ldots, \ell(\lambda) \; .$$

Equivalently, it is generated by

$$x_1^{\lambda_j^t} x_2^{j-1} \, , \; j = 1, \ldots, \lambda_1 \; .$$

The character of the tangent space to $\mathcal{M}_n$ at λ is

$$\begin{aligned}\mathrm{tr}_{\, T_\lambda \mathcal{M}_n} (q_1, q_2) = q_1 q_2 V + V^* - (1 - q_1)(1 - q_2) V V^* = \\ \sum_{\square \in \lambda} q_1^{a_\square + 1} q_2^{-l_\square} + q_1^{-a_\square} q_2^{l_\square + 1}\end{aligned} \tag{3.13}$$

which explains our "arm-leg" measures on the space of two dimensional partitions, if we use (3.9) as the equivariant K-theory class.

3.2.1. *Non-abelian version*

The moduli space of $U(N)$ instantons on (noncommutative) $\mathbf{R}^4$ have a similar description:

$$
\begin{aligned}
&\mathcal{M}_n^{U(N)} = \{\, (B_1, B_2, I, J) \,| \\
&B_{1,2} \in \mathrm{End}(\mathbf{C}^n)\,, I \in \mathrm{Hom}(\mathbf{C}^N, \mathbf{C}^n)\,, J \in \mathrm{Hom}(\mathbf{C}^n, \mathbf{C}^N)\,, \\
&\quad [B_1, B_2] + IJ = 0\,,\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = r\,\mathbf{1}_n \}/U(n) \\
&\quad (B_{1,2}, I, J) \sim (gB_{1,2}g^{-1}, gI, Jg^{-1})\,, \text{for } g \in U(n)
\end{aligned}
\tag{3.14}
$$

3.3. Three dimensions

Here the story is more involved. Define $\mathcal{M}_n$ as the space of quadruples (B_1, B_2, B_3, Y) of $n \times n$ matrices and a vector $I \in \mathbf{C}^n$ subject to the equations:

$$
\begin{aligned}
[B_1, B_2] + [B_3^\dagger, Y] &= 0 \\
[B_3, B_1] + [B_2^\dagger, Y] &= 0 \\
[B_2, B_3] + [B_1^\dagger, Y] &= 0 \\
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + [B_3, B_3^\dagger] + [Y, Y^\dagger] + I \otimes I^\dagger &= r\,\mathbf{1}_n \\
YI &= 0
\end{aligned}
\tag{3.15}
$$

viewed up to the action of $U(n)$ via:

$$
(B_1, B_2, B_3, Y, I) \mapsto \left(gB_1g^{-1}, gB_2g^{-1}, gB_3g^{-1}, gYg^{-1}, gI\right)
\tag{3.16}
$$

On the solutions (3.15) we have: $Y = 0$, $[B_i, B_j] = 0$, and one gets an ideal $\mathcal{I}$ in $\mathbf{C}[x_1, x_2, x_3]$ similarly to the two dimensional construction. The space $\mathcal{M}_n$ being a quotient of the space of eight Hermitian $n \times n$ matrices and a complex n-vector, by the action of $U(n)$, subject to 7 matrix and one vector equations, should have dimension zero. Instead, it has the generic complex dimension $3n$. What it means is that each point in $\mathcal{M}_n$ one has an obstruction vector space, of the same dimension as the tangent space. The integral over $\mathcal{M}_n$ should be viewed as the integral in the perfect obstruction theory [4] which allows localization with respect to the torus action.

The three dimensional torus $\mathbf{T} = U(1) \times U(1) \times U(1) \subset SU(4)$ acts on $\mathcal{M}_n$. The fixed points of the torus action are the monomial ideals, which are in one-to-one correspondence with the three dimensional partitions.

4. Gauge theories

We shall study theories in $2d$ or $2d+1$ dimensions, for $d = 1, 2, 3$. Our theories will have a gauge field A, fermionic one-form ψ, a scalar fermion η and a complex boson σ in $2d$ dimensions, a real scalar φ in $2d+1$ dimensions. In addition, there are some fermions χ, and their bosonic superpartners H, which are $2d-2$ forms with some constraints. All these fields are in the adjoint representation. In addition, one may add some matter fields. We shall be discussing the topologically twisted theories.

4.1. Supersymmetry

Supersymmetry, if discovered, is both a remarkable manifestation of the possible presence of extra dimensions of the physical space-time and a great theoretical tool in the almost century-long attempt in unification of all fundamental interactions. Mathematically, supersymmetry is a close cousin of such well-studied and deep notions as de Rham or Dolbeault complexes, equivariant cohomology, and Dirac operators.

For example, we shall study four dimensional gauge theory, with $\mathcal{N} = 2$ supersymmetry. Mathematically the $\mathcal{N} = 2$ supersymmetry (more precisely, what we describe here is the so-called twisted supersymmetry algebra) is the algebra of odd derivations of the differential graded algebra $\mathbf{A}_{\mathbf{R}^4} = \Omega^{\bullet}\left(\mathcal{A}_{\mathbf{R}^4}\right)$, of differential forms on the space of connections on a principal G-bundle $\mathcal{P}$ over $\mathbf{R}^4$. It is generated by eight supercharges. Four of them are the de Rham operator and the three $\overline{\partial}$-operators, $\overline{\partial}_I, \overline{\partial}_J, \overline{\partial}_K$, which correspond to the three complex structures I, J, K on $\mathbf{R}^4$. The other four supercharges are the operators $G_m = \iota_{\frac{\partial}{\partial x^m}}$ of contractions with the translation vector fields on $\mathbf{R}^4$. These operators anti-commute to the Lie derivatives along the translational vector fields, or their $(1,0)$ and $(0,1)$ components.

All our theories have a fermionic scalar symmetry Q which is a twisted version of the supersymmetry of the physical theory.

The path integral computing the correlation functions of Q-invariant observables localizes onto the field configurations, preserved by Q. These configurations form a moduli space $\mathcal{M}$ which depends on the spacetime manifold X. In $2d$ dimensions this moduli space $\mathcal{M}_X$ is finite dimensional, and can be compactified, for compact X. In $2d+1$ dimensions, for the spacetime of the form $X \times \mathbf{S}^1$, these configurations should be viewed as constant loops in $\mathcal{M}_X$, the moduli space of the $2d$-dimensional theory on X.

The equations of gauge theory which define $\mathcal{M}_X$ may depend on the metric on X, or on its conformal class, or on the complex structure of X and some choice of polarization. The group H in all these cases should preserve these structures. We shall study our theories on $X = \mathbf{R}^{2d}$, where $H = SO(2d)$, for $d = 2, 4$ or $H = U(3)$ for $d = 3$.

4.2. Group theory notations

Let G denote a compact Lie group, $\mathbf{g} = \mathrm{Lie}G$ its Lie algebra, $T \subset G$ its maximal torus, and $\mathbf{t} \subset \mathbf{g}$ its Lie algebra, Cartan subalgebra of $\mathbf{g}$. Let $r = \mathrm{dim}\mathbf{t}$ denotes the rank of G. We have Cartan decomposition: $\mathbf{g} = \mathbf{t} \oplus \mathbf{n}_+ \oplus \mathbf{n}_-$. Let $\mathcal{W}$ denote the Weyl group of G, Δ_+ the set of positive roots, $\Delta = \Delta_+ \cup \Delta_-$ the set of all roots. Each root $\alpha \in \Delta_+$ corresponds to an element $e_\alpha \in \mathbf{n}_+$, also sometimes called a positive root and to an element $e_{-\alpha} \in \mathbf{n}_-$, called the negative root. The root e_α being an eigenvector for the adjoint action of $\mathbf{t}$ on $\mathbf{g}$ also defines an element of $\mathbf{t}^*$. Let ρ denote half the sum of the positive roots:

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Delta_+} \alpha \tag{4.1}$$

which we view as an element of $\mathbf{t}^*$. Finally, $\Lambda_{\mathrm{w}} \subset \mathbf{t}^*$ denotes the weight lattice. It contains the root lattice Λ_{r}, which is integrally generated by $e_\alpha \in \mathbf{t}^*$, $\alpha \in \Delta$. The quotient $\Lambda_{\mathrm{w}}/\Lambda_{\mathrm{r}}$ is isomorphic to the center $Z(G)$ of G.

We shall be using the notation ϕ for vectors in $\mathbf{g}$, φ for vectors in $\mathbf{t}$. An $Ad(G)$-invariant function $F(\phi)$ on $\mathbf{g}$ is uniquely determined by its restriction $f(\varphi)$ on $\mathbf{t}$, where it defines a $\mathcal{W}$-invariant function. As such, it can be also expressed in terms of the Chevalley generators (elementary symmetric polynomials in the case of $G = SU(r+1)$), $\sigma_1, \ldots, \sigma_r$:

$$\mathbf{C}[\mathbf{t}]^{\mathcal{W}} = \mathbf{C}[\sigma_1, \ldots, \sigma_r] \tag{4.2}$$

We shall sometimes use the same notation for the $\mathcal{W}$-invariant function f on $\mathbf{t}$ and for the function on $\mathbf{t}/\mathcal{W}$:

$$f(\varphi) \sim f(\sigma_1, \ldots, \sigma_r) \tag{4.3}$$

An important rôle in what follows will be played by the identity:

$$d\sigma_1 \wedge \ldots \wedge d\sigma_r = \prod_{\alpha \in \Delta_+} \langle \alpha, \varphi \rangle \; d\varphi_1 \wedge \ldots \wedge d\varphi_r \tag{4.4}$$

4.3. Two dimensions

4.3.1. *Pure super-Yang-Mills theory*

The fields of the $\mathcal{N}=2$ theory in two dimensions are: the gauge field A_m, the fermion ψ_m, two scalar fermions χ, η, and a complex scalar σ, all in the adjoint representation. The Lagrangian is given by:

$$L = \left\{ Q, \, \mathrm{tr}\, \left(\chi \wedge (F - \star H) + \psi \star D_A \overline{\sigma} + \eta \star [\sigma, \overline{\sigma}] \right) \right\} \tag{4.5}$$

where the Q-operator acts as follows:

$$\begin{aligned} QA = \psi \, , \, Q\psi = D_A \phi \\ Q\overline{\sigma} = \eta \, , \, Q\eta = [\sigma, \overline{\sigma}] \\ Q\chi = H \, , \, QH = [\sigma, \chi] \\ Q\phi = 0 \end{aligned} \tag{4.6}$$

4.3.2. *Coupling to matter*

Let Y be a Kähler manifold with G-isometry. For $\sigma \in \mathbf{g}$ let $V_\sigma \in Vect(X)$ denote the corresponding vector field. Let z^m denote the coordinates on Y, y^i the holomorphic coordinates, $y^{\bar{i}}$ the antiholomorphic coordinates. Let $\mu : Y \to \mathbf{g}^*$ denote the moment map, corresponding to the G-action.

Then the gauge theory can be coupled to the sigma model (type A topological sigma model [5], which computes the number of pseudoholomorphic curves [6]) with the target space Y. The Q-symmetry acts as follows:

$$\begin{aligned} Qz^m = \chi^m \, , \, Q\chi^m = V_\sigma^m(z) \\ Q\pi_i = p_i \, , \, Qp_i = \mathcal{L}_{V_\sigma} \pi_i \\ Q\pi_{\bar{i}} = p_{\bar{i}} \, , \, Qp_{\bar{i}} = \mathcal{L}_{V_\sigma} \pi_{\bar{i}} \end{aligned} \tag{4.7}$$

The Lagrangian (4.5) generalizes to:

$$\begin{aligned} L = \Big\{ Q, \, \mathrm{tr}\, (\chi \wedge (F + \star (\mu(z) - H)) + \\ \pi_i \overline{\partial}_A y^i + \pi_{\bar{i}} \partial_A y^{\bar{i}} + g^{i\bar{i}} \left(p_i \pi_{\bar{i}} - p_{\bar{i}} \pi_i \right) + \ldots \\ \psi \star D_A \overline{\sigma} + \eta \star [\sigma, \overline{\sigma}]) \Big\} \end{aligned} \tag{4.8}$$

where ... stands for the terms with three fermions, which are irrelevant to our discussion.

The Q-fixed points are the solutions to the following equations:

$$\begin{aligned} & \overline{\partial}_A y^i \equiv \overline{\partial} y^i + V^i_{\overline{A}}(y) = 0 \\ & F_A + \mu(z)\text{vol} = 0 \\ & D_A\sigma = 0\,,\; V_\sigma(z) = 0 \end{aligned} \tag{4.9}$$

where vol is a volume form on the Riemann surface X constructed using the metric. The moduli space $\mathcal{M}_X$ of solutions to (4.9) is a fibration over Bun_G, the moduli stack of holomorphic $G_{\mathbf{C}}$-bundles on X. The fiber over $\mathcal{P}$ without automorphisms is the space of holomorphic sections $H^0\left(X, \mathcal{P} \times_{G_{\mathbf{C}}} Y\right)$. For X a Riemann surface of genus $g > 1$, $Bun_{G_{\mathbf{C}}}$ is a positive dimension space. For X a Riemann sphere, the case of interest for our further investigation, the stack $G_{\mathbf{C}}$-bundles have no continuous moduli, only automorphisms, so the stack nature of $Bun_{G_{\mathbf{C}}}$ is quite important. However, for sufficiently ample Y the moduli space $\mathcal{M}_X$ is a positive dimension space and one can define an integration theory over it. If $\mathcal{P}$ has automorphisms, these translate to the non-trivial solutions of the equation $D_A\sigma = 0$. Such a solution $\sigma \neq 0$ defines a solution of (4.9) only if $V_\sigma(z) = 0$, i.e. we land in one of the Aut$\mathcal{P} \subset G_{\mathbf{C}}$-fixed points in Y. The "stability" condition $F_A + \star\mu = 0$ then implies that A is a constant curvature $\mathbf{T}$-connection on X which, in turn, implies some quantization condition on $\langle \mu, \sigma \rangle$ which can only be met for special values of r. As r crosses such a value, the correlation functions may jump. This is an example of the wall-crossing behavior of the topological correlation functions.

Note that the first line in (4.9), the equation describing the holomorphic section of the associated bundle $\mathcal{P} \times_{G_{\mathbf{C}}} Y$, is conformally invariant, i.e. it only depends on the complex structure of X. The second line, the equation fixing the Hermitian structure on $\mathcal{P}$, depends on the conformal factor of the metric on X. In the limit where the metric on X is scaled to infinity, i.e. when X is very large, almost everywhere on X the contribution of $\mu(z)$ in (4.9) dominates. In other words, the holomorphic section of $\mathcal{P} \times_{G_{\mathbf{C}}} Y$ lands in the zero locus of the moment map in Y, thereby defining a holomorphic map to the Kähler quotient $Y//G = \mu^{-1}(0)/G = Y^s/G_{\mathbf{C}}$. However, at some points on X the section of $\mathcal{P} \times_{G_{\mathbf{C}}} Y$ passes through the unstable points $Y \backslash Y^s$ in Y, i.e. the points which cannot be translated by the action of $G_{\mathbf{C}}$ to the zero locus of μ. Thus, roughly speaking, the moduli space $\mathcal{M}_X$ of solutions to (4.9) is a completion of the moduli space of holomorphic maps $X \to Y//G$ by the "ideal instantons", or "freckles", which are points on X colored by components of the set of critial points of μ on Y. This compactification is related to Drinfeld's "quasimaps", or to Uhlenbeck compactification.

4.3.3. *Examples*

Take $G = U(1)$, $Y = \mathbf{C}^L$, with the standard action of $U(1)$ by multiplying all coordinates by the same phase. The moment map is just a function:

$$\mu(z) = \sum_{i=1}^{L} |y^i|^2 - r \tag{4.10}$$

where r is some constant. The moduli space $\mathcal{M}_X$, for $X = \mathbf{CP}^1$ is a projective space $\mathbf{P}^{Ld+L-1}$, where $d = c_1(\mathcal{P})$ is the first Chern class:

$$d = \frac{1}{2\pi i} \int_X F \tag{4.11}$$

The instanton partition function for this example:

$$Z(\epsilon; \sigma, Q) = Q^{\frac{\sigma}{\epsilon}} \sum_{d=0}^{\infty} Q^d \prod_{n=1}^{d} \frac{1}{(\sigma + n\epsilon)^L} \tag{4.12}$$

4.3.4. *Generalized Gromov-Witten invariants*

The topological sigma model coupled to the topological gauge theory is a simplified version of the topological string. One can view the correlation functions of the Q-invariant observables (which are nothing but the G-equivariant cohomology classes of Y) as the cohomology classes of $Bun_{G_{\mathbf{C}}}$. One can also couple the theory to the topological gravity (the subtle point is the metric dependence of (4.9)) to get cohomology classes of the moduli space of curves with holomorphic bundles on them, thus .

4.3.5. *Quantum integrable systems*

Every topological sigma model defines an abstract quantum integrable system. The three-point functions C_{ijk} of the Q-invariant observables $\mathcal{O}_i$ on the sphere define a commutative associative algebra. This algebra acts in the space $\mathcal{H}$ of the cohomology of the target space of the sigma model. One can view $\mathcal{O}_i' s$ as the quantum Hamiltonians. The spectrum of the algebra of $\mathcal{O}_i' s$ can be quite interesting. For example, for $G = U(N)$, $Y \subset \mathrm{Hom}\left(\mathbf{C}^N, \mathbf{C}^L\right) \times \mathrm{Hom}\left(\mathbf{C}^L, \mathbf{C}^N\right)$ being the incidence subvariety $\{(A, B) \mid AB = 0\}$, the corresponding quantum integrable system is the Heisenberg spin chain [7].

4.4. Three dimensions

We shall view the three dimensional theory as a lift of a two dimensional one. The 2+1 dimensional theory corresponding to our two dimensional models is quite simple. It has the same fields, which now depend on an extra coordinate, t, except that the field σ which used to be a complex adjoint scalar now mutates to the component A_t of the gauge field and a real scalar φ. All formulae remain the same except for the change: $\sigma \to \partial_t + A_t + i\varphi$.

The perturbative $2+1$ dimensional partition function is given by:

$$Z^{3d}_{\rm pert}(q;u) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} F_2\left(q^n; u^n\right)\right) \tag{4.13}$$

where $u \in \mathbf{T}$, and where the single particle partition function is given by the supertrace:

$$F_2(q;u) = \frac{1}{(1-q)(1-q^{-1})} \left[\sum_{\alpha\in\Delta} u^{\alpha}\right] \times \left(q + q^{-1} + 1 - 1 - 1 - q^{-1}\right) \tag{4.14}$$

where the expression in the brackets comes from the contribution of fields: A $(q+q^{-1})$, φ $(+1)$, ψ, χ, η $(-1-q^{-1})$, gauge invariance (-1), so that $Z_{\rm pert}$ is given by an interesting infinite product:

$$Z^{3d}_{\rm pert}(q;u) = \prod_{\alpha\in\Delta} \prod_{n=1}^{\infty} \frac{1}{(1-q^n u^{\alpha})} \tag{4.15}$$

The two dimensional partition function is given by the limit $\beta \to 0$ where:

$$q = e^{\beta\epsilon}\,,\ u = e^{\beta a}\,,\ a \in \mathbf{t} \tag{4.16}$$

4.4.1. *Instanton partition function in 2+1 dimensions*

When the Higgs field ϕ has a vacuum expectation value, the gauge group G is broken down to the maximal torus $\mathbf{T}$, and the gauge bundle (which is topologically trivial principal G-bundle $\mathcal{P}$ for the simple G) reduces to a possibly non-trivial $\mathbf{T}$-bundle. It is classified by $\lambda \in H_2(\Sigma, \pi_1(\mathbf{T}))$. Take $G = U(N)$ for simplicity. Then the possible topologies of the $\mathbf{T}$-bundles are labeled by the vectors $\vec{d} = (d_1, \ldots, d_N) \in \mathbf{Z}^N$, and the partition function will be a kind of a theta function obtained by summing over $\vec{d}$'s. We shall not write an explicit formula here.

5. Higher dimensions

5.1. Gauge theory in four dimensions

The fields of the pure $\mathcal{N} = 2$ twisted super-Yang-Mills theory in four dimensions are the gauge field A, the adjoint fermion one-form ψ, the fermion self-dual two form χ^{+}, the fermion scalar η, and the complex adjoint scalar σ. The supersymmetric field configurations are the solutions to the instanton equations

$$F_A^{+} = 0$$

This theory [8] gives rise to the Donaldson invariants of the four dimensional manifolds, and their K-theoretic analogues when lifted to $4+1$ dimensions [9].

5.1.1. *The instanton partition function*

The moduli space of framed instantons $\mathcal{M}_n$ of charge n,

$$n = -\frac{1}{8\pi^2}\int_X \mathrm{tr}\, F_A \wedge F_A \tag{5.1}$$

is a Riemannian manifold, with the metric induced from that on X:

$$g\left(\delta_1 A, \delta_2 A\right) = \int_X v_g \ \mathrm{tr}\ \left(\delta_1 A \wedge \star\delta_2 A\right) \tag{5.2}$$

for

$$\delta_1 A, \delta_2 A \in \Omega^1(X) \otimes \mathbf{g}\ .$$

The moduli space $\mathcal{M}_n$ is acted upon by the group

$$H = G \times U_X\ ,$$

where G is the gauge group, and U_X is the group of isometries, preserving the framing locus. Let (a, ϵ) be an element of the Cartan subalgebra of H, $a \in \mathrm{Lie}G$, $\epsilon \in \mathrm{Lie}U_X$. Let

$$V(a, \epsilon) \in \mathrm{Vect}(\mathcal{M}_n) \tag{5.3}$$

be the corresponding vector field. Let

$$\lambda(\overline{a}, \overline{\epsilon}) = \iota_{V(\overline{a},\overline{\epsilon})} g \tag{5.4}$$

denote the corresponding one-form, where

$$\overline{a} \in \mathrm{Lie}G\, ,\ \overline{\epsilon} \in \mathrm{Lie}U_X$$

$$[a, \overline{a}] = 0\, ,\ [\overline{\epsilon}, \epsilon] = 0$$

Then

$$Z(a, \epsilon; q) = \sum_n q^n \int_{\mathcal{M}_n} \exp\left[\mathrm{d}\lambda(\overline{a}, \overline{\epsilon})\right] e^{-g(V(\overline{a},\overline{\epsilon}), V(a,\epsilon))} \tag{5.5}$$

5.1.2. *Perturbative calculation in $4+1$ dimensions*

The perturbative partition function in $4+1$ dimensions, for pure super-Yang-Mills is given by:

$$\begin{aligned} Z^{5d}_{\text{pert}}(q_1,q_2;u) = \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}F_4\left(q_1^n,q_2^n;u^n\right)\right) \\ F_4(q_1,q_2;u) = \prod_{i=1}^{2}\frac{1}{(1-q_i)\left(1-q_i^{-1}\right)}\left[\sum_{\alpha\in\Delta}u^{\alpha}\right]\times \\ \times\left(\sum_{i=1}^{2}\left(q_i+q_i^{-1}\right)+1-1-1-\left(\sum_{i=1}^{2}q_i^{-1}\right)-q_1q_2\right) \\ = -\left[\sum_{\alpha\in\Delta}u^{\alpha}\right]\times\prod_{i=1}^{2}\frac{1}{1-q_i^{-1}} \end{aligned} \tag{5.6}$$

where the following fields contribute to the sum

$$\sum_{i=1}^{2}\left(q_i+q_i^{-1}\right)+1-1-1-\left(\sum_{i=1}^{2}q_i^{-1}\right)-q_1q_2$$

$$\begin{aligned} A &: \sum_{i=1}^{2}\left(q_i+q_i^{-1}\right) \\ \varphi &: +1 \\ \eta,\chi &: -1-q_1q_2 \\ \text{gauge invariance} &: -1 \\ \psi &: -\sum_{i=1}^{2}q_i^{-1} \end{aligned} \tag{5.7}$$

Thus,

$$Z^{5d}_{\text{pert}}(q_1,q_2;u) = \prod_{k,l=1}^{\infty}\prod_{\alpha\in\Delta}\left(1-q_1^kq_2^lu^{\alpha}\right) \tag{5.8}$$

5.1.3. *Four dimensional limit*

Again, we set $q_1=e^{iR\epsilon_1}, q_2=e^{iR\epsilon_2}$, $u=e^{iRa}$, and take the limit $R\to 0$ while keeping a,ϵ_1,ϵ_2 finite. Again, the plethystic sum becomes the integral:

$$\sum_{n=1}^{\infty}\frac{1}{n}f(e^{inRx}) \longrightarrow \left.\frac{d}{ds}\right|_{s=0}\left[\frac{\Lambda^s}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{d}t}{t}t^sf(e^{itx})\right] \tag{5.9}$$

where the right hand side is understood in a sense of analytic continuation in s. The whole point of introducing the $\frac{(t\Lambda)^s}{\Gamma(s)}$ factor is to eliminate a possible singularity near $t = 0$, where the argument of $f(\cdot)$ approaches 1. In the original $4+1$ expression the sum over n started with $n = 1$, and the singularity was absent. The price we pay for the regularization is the (dynamical) generation of the scale, Λ. It is related to the β-function of the supersymmetric Yang-Mills theory.

Thus:

$$Z^{4d}_{\mathrm{pert}}(\epsilon_1, \epsilon_2; a) = \exp \sum_{\alpha\in\Delta} \gamma_{\epsilon_1,\epsilon_2}\left(\langle \alpha, a \rangle \right) \tag{5.10}$$

where $\gamma_{\epsilon_1,\epsilon_2}(x)$ is an analytic function solving the difference equation:

$$\gamma_{\epsilon_1,\epsilon_2}(x) - \gamma_{\epsilon_1,\epsilon_2}(x+\epsilon_1) - \gamma_{\epsilon_1,\epsilon_2}(x+\epsilon_2) + \gamma_{\epsilon_1,\epsilon_2}(x+\epsilon_1+\epsilon_2) = -\log\left(\frac{(x+\epsilon_1+\epsilon_2)}{\Lambda}\right) \tag{5.11}$$

which is an $R \to 0$ limit of the five dimensional function

$$\xi_{q_1,q_2}(y) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{(yq_1q_2)^n}{(1-q_1^n)(1-q_2^n)} \tag{5.12}$$

which solves:

$$\xi_{q_1,q_2}(y) - \xi_{q_1,q_2}(yq_1) - \xi_{q_1,q_2}(yq_2) + \xi_{q_1,q_2}(yq_1q_2) = -\log\left(1 - yq_1q_2\right) \tag{5.13}$$

via:

$$\xi_{e^{iR\epsilon_1},e^{iR\epsilon_2}}(e^{iRx}) = \gamma_{\epsilon_1,\epsilon_2}(x) + \frac{x^2}{2\epsilon_1\epsilon_2}\log\left(\Lambda R\right) \tag{5.14}$$

Explicitly, for $\mathrm{Re}(x) < 0, \epsilon_1, \epsilon_2 \in \mathbf{C}\backslash\mathbf{R}$,

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}\bigg|_{s=0} \left[\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{\mathrm{d}t}{t} t^s \frac{e^{tx}}{(1-e^{-t\epsilon_1})(1-e^{-t\epsilon_2})} \right] \tag{5.15}$$

5.1.4. *Instanton corrections in $4+1$ dimensions*

The instanton corrections can be represented as a sum over N-tuples of ordinary (two dimensional) partitions $\vec{\lambda} = \left(\lambda^1, \ldots, \lambda^N\right)$. A partition λ can be idenified with the set $\{(i,j) \in \mathbf{Z}^2_+ \,|\, 1 \leq j \leq \lambda_i\}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell(\lambda)}$, $|\lambda| = \sum_i \lambda_i$. The character of the partition

$$\mathrm{ch}_\lambda(q_1, q_2) = \sum_{(i,j)\in\lambda} q_1^{i-1} q_2^{j-1} = \frac{1}{1-q_2} \sum_i q_1^{i-1}(1-q_2^{\lambda_i}) = \frac{1}{1-q_1} \sum_j q_2^{j-1}(1-q_1^{\lambda'_j})$$

where λ' is the partition, dual to λ. For the N-tuple $\vec{\lambda}$ we define $|\vec{\lambda}| = \sum_l |\lambda^l|$.

The instanton corresponding to $\vec{\lambda}$ contributes

$$Z_{\vec{\lambda}}(q_1, q_2; u, Q, k) = Q^{|\vec{\lambda}|} \chi^k_{\vec{\lambda}} \prod_{\Upsilon} \frac{1}{1 - e^{-x_\Upsilon}} \tag{5.16}$$

where:

$$\begin{aligned}
\mathcal{T}_{\vec{\lambda}} = \sum_{\Upsilon} e^{x_\Upsilon} &= \frac{WW^* - EE^*}{(1 - q_1^{-1})(1 - q_2^{-1})} \\
&= WV^* + VW^* q_1 q_2 - (1 - q_1)(1 - q_2) VV^* \\
= \sum_{m,l} u_m u_l^{-1} &\left[\sum_{(i,j)\in\lambda^l} q_1^{1-i} q_2^{1-j} + \sum_{(i,j)\in\lambda^m} q_1^i q_2^j + \sum_{i=1}^{\ell(\lambda^l)} \sum_{j=1}^{\lambda_1^m} q_1^i q_2^{1-j} (1 - q_1^{-\lambda_j^{m\prime}})(1 - q_2^{\lambda_i^l}) \right] \\
E &= W - (1 - q_1)(1 - q_2) V \\
W &= \sum_l u_l \,,\; V = \sum_l u_l \, \mathrm{ch}_{\lambda_l}(q_1, q_2)
\end{aligned} \tag{5.17}$$

For $N = 1$ one can further simplify:

$$\begin{aligned}
\mathcal{T}_\lambda = \sum_{(i,j)\in\lambda} & q_1^{\lambda'_j - i + 1} q_2^{j - \lambda_i} + q_1^{i - \lambda_j} q_2^{-j + 1 + \lambda_i} = \\
& \sum_{\square\in\lambda} q_1^{\mathrm{arm}(\square)+1} q_2^{-\mathrm{leg}(\square)} + q_1^{-\mathrm{arm}(\square)} q_2^{\mathrm{leg}(\square)+1}
\end{aligned} \tag{5.18}$$

5.1.5. *$SU(2)$ specification*

A certain simplifcation occurs for the special value of the rotation parameters: $q_1 = q_2^{-1} = q$. In addition, if one assumes that $u_l = q^{M_l}$, $M_l \in \mathbf{Z}$, then one can map the non-abelian problem to the abelian one [1] and express the partition function in terms of the representations of the chiral algebra of the system of N chiral fermions.

5.2. Six dimensions

The perturbative partition function in $6 + 1$ dimensions, for pure super-Yang-Mills (this is a unique theory up to a choice of the gauge group, it is the dimensional reduction of the $\mathcal{N}=1$ super-Yang-Mills theory in ten dimensions), is given by:

$$\begin{aligned} Z^{7d}_{\text{pert}}(q_1,q_2,q_3;u) &= \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}F_6\left(q_1^n,q_2^n,q_3^n;u^n\right)\right) \\ F_6(q_1,q_2,q_3;u) &= \prod_{i=1}^{3}\frac{1}{\left(1-q_i\right)\left(1-q_i^{-1}\right)}\left[\sum_{\alpha\in\Delta}u^{\alpha}\right]\times \\ &\times\left(\chi_A+\chi_{\vec{\varphi}}-\chi_{\Psi}-1\right) \\ &= -\left[\sum_{\alpha\in\Delta}u^{\alpha}\right]\times\prod_{i=1}^{3}\frac{1}{1-q_i^{-1}} \end{aligned} \tag{5.19}$$

where the following fields contribute to the sum:

$$\begin{aligned} &A\,:\,\sum_{i=1}^{3}\left(q_i+q_i^{-1}\right) \\ &\vec{\varphi}\,:\,+1+q_1q_2q_3+q_1^{-1}q_2^{-1}q_3^{-1} \\ &\eta,\chi\,:\,-1-(q_1q_2q_3)^{-1}-\sum_{i=1}^{3}q_i^{-1}-\sum_{i<j}^{3}q_iq_j \\ &\text{gauge invariance}\,:\,-1 \\ &\psi:-\sum_{i=1}^{3}q_i^{-1} \end{aligned} \tag{5.20}$$

Thus,

$$Z^{7d}_{\text{pert}}(q_1,q_2,q_3;u) = \prod_{k_1,k_2,k_3=1}^{\infty}\prod_{\alpha\in\Delta}\left(1-q_1^{k_1}q_2^{k_2}q_3^{k_3}u^{\alpha}\right) \tag{5.21}$$

5.2.1. *Instanton corrections*

The supersymmetry preserving equations in six dimensions are:

$$\begin{aligned} &F_A^{2,0}=\partial_A^{\dagger}u \\ &F_A^{0,2}=\overline{\partial}_A^{\dagger}\overline{u} \\ &F_A^{1,1}\wedge\omega_X\wedge\omega_X=[u,\overline{u}] \end{aligned} \tag{5.22}$$

where

$$u\in\Omega^{3,0}(X)\otimes\mathbf{g} \tag{5.23}$$

We now take $G = U(N)$ and follow the familiar, by now, route. The instanton corrections organize themselves into the sum over the coloured three dimensional partitions $\vec{\pi} = (\pi_1, \ldots, \pi_N)$. Each partition π defines a polynomial in (q_1, q_2, q_3), the character ch_π:

$$\mathrm{ch}_\pi(q_1, q_2, q_3) = \sum_{(i,j,k)\in\pi} q_1^{i-1} q_2^{j-1} q_3^{k-1} \tag{5.24}$$

where the three dimensional partition is identified with the set $\pi = \{(i, j, k) \in \mathbf{Z}_+^3 \mid 1 \leq k \leq h_{i,j}\}$, and $h_{i,j}$ is the so-called height function, $h_{i,j} \geq h_{i+1,j}$, $h_{i,j} \geq h_{i,j+1}$, $h_{i,j} \geq 0$. The size of the partition $|\pi|$ is the total number of elements in the corresponding set:

$$|\pi| = \sum_{i,j=1}^{\infty} h_{i,j} \tag{5.25}$$

For the N-tuple $\vec{\pi}$ we define:

$$|\vec{\pi}| = \sum_{l=1}^{N} |\pi_l|$$

We shall only consider the finite size partitions. Now, define:

$$\begin{aligned}
E &= W - (1-q_1)(1-q_2)(1-q_3)V \\
E^* &= W^* - (1-q_1^{-1})(1-q_2^{-1})(1-q_3^{-1})V^* \\
W &= \sum_{l=1}^{N} u_l \,,\; W^* = \sum_{l=1}^{N} u_l^{-1} \\
V &= \sum_{l=1}^{N} u_l \, \mathrm{ch}_{\pi_l}(q_1, q_2, q_3) \\
V^* &= \sum_{l=1}^{N} u_l^{-1} \, \mathrm{ch}_{\pi_l}(q_1^{-1}, q_2^{-1}, q_3^{-1})
\end{aligned} \tag{5.26}$$

The instanton partition function is defined as:

$$Z^{7d}(q_1, q_2, q_3; u, Q) = \sum_{\vec{\pi}} (-Q)^{|\vec{\pi}|} \prod_{\Upsilon} \frac{e^{\frac{x_\Upsilon}{2}} - e^{-\frac{x_\Upsilon}{2}}}{e^{\frac{y_\Upsilon}{2}} - e^{-\frac{y_\Upsilon}{2}}} \tag{5.27}$$

where the exponents e^{x_Υ}, e^{y_Υ} are all products of the form $(u_l, u_m^{-1}, u_l u_m^{-1}) \times q_1^{k_1} q_2^{k_2} q_3^{k_3}$, for $k_1, k_2, k_3 \in \mathbf{Z}$, and are defined as follows:

$$\begin{aligned}
\mathcal{T}_\pi &\equiv \sum_{\Upsilon} e^{y_\Upsilon} - e^{x_\Upsilon} \\
&= \frac{WW^* - EE^*}{(1-q_1^{-1})(1-q_2^{-1})(1-q_3^{-1})} \\
&= V^*W - VW^* q_1 q_2 q_3 - VV^*(1-q_1)(1-q_2)(1-q_3)
\end{aligned} \tag{5.28}$$

5.2.2. *U(1) theory*

The case $N = 1$ is already quite interesting.

Let us list the first few contributions to the partition function. For the single box partition $|\pi| = 1$, $V = 1$,

$$\mathcal{T}_\pi = \sum_{i=1}^{3} \left(q_i - \frac{q_1 q_2 q_3}{q_i} \right)$$

and the contribution to the partition function:

$$Z_{\text{one-box}} = -\frac{q}{(q_1 q_2 q_3)^{1/2}} \frac{(1-q_1 q_2)(1-q_1 q_3)(1-q_2 q_3)}{(1-q_1)(1-q_2)(1-q_3)} \tag{5.29}$$

there are three two-box partitions, aligned in the $1, 2, 3$d directions respectively. The sum of their contributions is:

$$Z_{\text{two-boxes}} = -\frac{q}{(q_1 q_2 q_3)^{1/2}} Z_{\text{one-box}} \times \left[\sum_{i=1}^{3} \frac{(1 - q_1 q_2 q_3 / q_i^2)}{q_i (1 - q_i^2)} \prod_{j \neq i} \frac{(1 - q_j q_i^2)}{(1 - q_j / q_i)} \right] \tag{5.30}$$

Amasingly, after extracting the single particle contribution, i.e. by rewriting the partition function in the form:

$$Z_{\text{inst}}(q_1, q_2, q_3; q) = \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} F(q_1^n, q_2^n, q_3^n, q^n) \right) \tag{5.31}$$

the simplification emerges:

$$F(q_1, q_2, q_3, q) = \frac{Q}{(1-Q)(1-Q(q_1 q_2 q_3))} \frac{(1-q_1 q_2)(1-q_2 q_3)(1-q_1 q_3)}{(1-q_1)(1-q_2)(1-q_3)} \tag{5.32}$$

where

$$Q = \frac{q}{(q_1 q_2 q_3)^{\frac{1}{2}}}$$

which brings us to the combinatorial formula (2.24). The formula (5.32) also has an infinite product form:

$$Z_{\text{inst}}(q_1, q_2, q_3; q) =$$
$$\prod_{n=1}^{\infty} \left[\frac{(1-Q^n)}{(1-\widetilde{Q}^n)^2} \times \prod_{m=1}^{\infty} \frac{1}{1 - Q^n \widetilde{Q}^m} \frac{(1-q_1^n Q^m)(1-q_2^n Q^m)(1-q_3^n Q^m)}{(1-q_1^n \widetilde{Q}^m)(1-q_2^n \widetilde{Q}^m)(1-q_3^n \widetilde{Q}^m)} \right] \tag{5.33}$$

where

$$\widetilde{Q} = Q q_1 q_2 q_3$$
$$q_4 = Q \, , \; q_5 = \widetilde{Q}^{-1}$$

6. M-theory in a flash

M-theory [10] is the elusive quantum eleven dimensional theory whose low energy $E \ll M_{\rm Planck}$ limit is the unique eleven dimensional supergravity and its compactification on a circle of a finite radius R is equivalent to the IIA superstring theory with finite coupling $g_s = (M_{\rm Planck}R)^{\frac{3}{2}}$, and string tension $M_s^2 = M_{\rm Planck}^3 R$. The microscopic definition of the theory is still lacking. It is known to contain some stable extended objects, the membrane $M2$ and the fivebrane $M5$. The tensions of these extended objects are $M_{\rm Planck}^3$ and $M_{\rm Planck}^6$, respectively. The very IIA string should come from the membrane. The graviton Kaluza-Klein modes, which have the mass scale $1/R$, become the solitonic particles/black holes in the IIA theory, described in the weak coupling limit as $D0$-branes, whose mass is $M_s/g_s = 1/R$. The $M2$-brane descends to the $D2$-brane, whose tension is $M_s^3/g_s = M_{\rm Planck}^3$. In addition, the IIA theory has other extended objects which we shall encounter later. The $NS5$-brane is the descendent of the "elementary" $M5$-brane, the magnetic dual to $M2$ in eleven dimensions. Its tension is $M_{\rm Planck}^6 = M_s^6/g_s^2$. In the IIA string theory it is described as a closed string theory soliton, hence the $1/g_s^2$ dependence of its tension. The $M5$-brane wrapped on a circle of M-theory becomes $D4$-brane, whose tension is $M_{\rm Planck}^6 R = M_s^5/g_s$. The $D6$-brane is the M-theory on $TN_R \times \mathbf{R}^7$, where TN_R is the Taub-Nut space, a hyperkähler four dimensional manifold which asymptotically looks locally like $\mathbf{R}^3 \times \mathbf{S}^1$, where metrically $\mathbf{S}^1$ has a finite radius R, however globally at infinity the $\mathbf{S}^1$ is non-trivially fibered over $\mathbf{S}^2_\infty$ (so that the total space is $\mathbf{S}^3$, topologically), and in TN_R the circle $\mathbf{S}^1$ can be contracted to a point. The space TN_R has a $U(1)$-isometry which preserves the hyperkähler structure. The hyperkähler moment map [11] sends $m : TN_R \to \mathbf{R}^3$, the fiber being $\mathbf{S}^1$ everywhere except the origin in $\mathbf{R}^3$, where the fiber is just a point. The origin in $\mathbf{R}^3$ is interpreted by the IIA string as the location of the $D6$-brane. The open strings ending on the $D6$-brane are nothing but disks in TN_R which bound some particular fiber of the projection m. The Taub-Nut space has a multi-center version $TN_{N,R}$, which looks asymptotically as a quotient of $\mathbf{R}^3 \times \mathbf{S}^1/\mathbf{Z}_N$, so that the three-sphere $\mathbf{S}^3_\infty$ of the Taub-Nut space is replaced by the Lens space $\mathbf{S}^3/\mathbf{Z}_N$. When the moduli of the $TN_{N,R}$ space are generic, the space has $N-1$-noncontractible 2-cycles, on which the $M2$-branes can wrap, giving rise to the massive particles in $6+1$ dimensions, whose mass goes to zero as the moduli are adjusted so that $TN_{N,R}$ space develops an orbifold singularity. This is translated to the IIA string statement that when N parallel $D6$ branes coincide, their worldvolume supports a maximal supersymmetric Yang-Mills theory, with the gauge group $U(N)$.

If this theory is further compactified on a circle $\mathbf{S}^{1'}$, one can get another IIA string description of the resulting six dimensional theory, by taking $\mathbf{S}^{1'}$ to be the M-theory circle, rather then the $U(1)$-fibration of the Taub-Nut space. In this way one concludes that when the IIA string theory is compactified on a K3 manifold with the singularity of A, D, E type then the effective six dimensional theory has, in addition to the obvious spectrum of the Kaluza-Klein fields coming from the ten dimensional IIA supergravity, a gauge multiplet, with the gauge group of the same A, D, E type. If, instead of the K3 manifold, one takes a fibration over $\mathbf{P}^1$ with the K3 fibers, such that the total space has a Calabi-Yau metric, and the fiber has the A, D, E singularity fibered over the base $\mathbf{P}^1$ then the four dimensional gauge theory is the $\mathcal{N} = 2$ supersymmetric Yang-Mills theory with the gauge group G of the same A, D, E type. This is the way string theory realizes McKay duality between the discrete $SU(2)$ subgroups Γ, the corresponding singularities $\mathbf{C}^2/\Gamma$ and the simple Lie groups G_Γ. In order to get pure gauge theory, so that the massive string modes are decoupled and the gravity is non-dynamical, one takes a limit where the K3 manifold becomes infinitely large, while the singularity remains. In other words one studies IIA string in a non-compact background $Y \times \mathbf{R}^4$ where Y is a $\mathbf{C}^2/\Gamma$-bundle over $\mathbf{P}^1$ such that the total space is a non-compact Calabi-Yau threefold [12].

6.1. Eleven dimensional supergravity

The fields of the eleven dimensional supergravity are the metric $g_{IJ} = \eta_{mn} e^m_I e^n_J$, with the elfbein e^m_I, and η_{mn} the Minkowski metric of the signature $-1^1 + 1^{10}$, the three form $C = \frac{1}{3!} C_{IJK} dx^I \wedge dx^J \wedge dx^K$, and the Rarita-Schwinger fermion field $\psi_I dx^I$, where the components $\psi_I \in \mathbf{S} \approx \mathbf{R}^{32}$ are the Majorana spinors. One represents the Clifford algebra $\{\Gamma_I, \Gamma_J\} = 2 g_{IJ}$ in $\mathbf{S}$. There is a real symplectic form $\mathcal{C} \in \Lambda^2 \mathbf{S}^*$. The Γ-matrices are $\mathcal{C}$-symmetric:

$$\mathcal{C}(a, \Gamma_I b) = \mathcal{C}(b, \Gamma_I a)\,,\ a, b \in \mathbf{S} \tag{6.1}$$

For the spinor $\psi^\alpha \in \mathbf{S}$ define $\overline{\psi}_\alpha = \mathcal{C}_{\alpha\beta} \psi^\beta$. We also define

$$\Gamma^{I_1 I_2 \ldots I_n} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} (-1)^\sigma \Gamma^{I_{\sigma(1)}} \Gamma^{I_{\sigma(2)}} \ldots \Gamma^{I_{\sigma(n)}} \tag{6.2}$$

– the representation of $\Lambda^n \mathbf{R}^{11}$ in $\mathbf{S}$.

The Lagrangian of the eleven dimensional supergravity reads, up to the terms, quartic in fermions:

$$
\begin{aligned}
L_S = \frac{1}{\kappa^2}\int_{M^{11}} \mathbf{v}_g \Big(-\tfrac{1}{2}R - \tfrac{1}{2}\overline{\psi}_I \Gamma^{IJK} D_J \psi_K - \frac{1}{48} G_{IJKL} G^{IJKL} \\
-\frac{\sqrt{2}}{192}\left(\overline{\psi}_I \Gamma^{IJKLMN}\psi_N + 12\overline{\psi}^J \Gamma^{KL}\psi^M\right) G_{JKLM} \Big) \\
-\sqrt{2} C \wedge G \wedge G
\end{aligned}
\qquad (6.3)
$$

where

$$
G = dC, \ C\wedge G \wedge G = \frac{1}{3!4!4!} C_{I_1 I_2 I_3} G_{I_4 \ldots I_7} G_{I_8 \ldots I_{11}} \epsilon^{I_1 I_2 \ldots I_{11}} \qquad (6.4)
$$

In writing (6.3) one views the spin connection Ω as an independent variable. The Riemann tensor is the curvature of Ω.

The local supersymmetry acts as follows:

$$
\begin{aligned}
\delta e_I^m &= \tfrac{1}{2} e_J^m \overline{\eta} \Gamma^J \psi_I \\
\delta C_{IJK} &= -\frac{\sqrt{2}}{8} \overline{\eta} \Gamma_{[IJ}\psi_{K]} \\
\delta \psi_I &= D_I \eta + \frac{\sqrt{2}}{288}\left(\Gamma_I^{JKLM} - 8\delta_I^J \Gamma^{KLM}\right)\eta G_{JKLM} + \ldots
\end{aligned}
\qquad (6.5)
$$

6.2. M-theory calculations

Consider M-theory compactified on a manifold X_{11} which is a $\mathbf{R}^{10}$-bundle over $\mathbf{S}^1$, with the locally flat metric, such that the fiber is rotated by an element $g \in SO(10)$ as one is looping around the base circle. The metric depends only on the conjugacy class of g, i.e. on the five angles ϑ_α, $\alpha = 1, \ldots, 5$:

$$
g = \exp\Omega = \begin{pmatrix} R_1 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 \\ 0 & 0 & R_3 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & 0 & R_5 \end{pmatrix}, \ R_\alpha = \begin{pmatrix} \cos(\vartheta_\alpha) & \sin(\vartheta_\alpha) \\ -\sin(\vartheta_\alpha) & \cos(\vartheta_\alpha) \end{pmatrix} \qquad (6.6)
$$

This background (which is the direct analogue of the Ω-background of [13]) has a generalization in the 11d sugra. The rotation (6.6) does not change the flat metric. However, the global eleven dimensional metric is non-trivial:

$$
\begin{aligned}
\mathrm{d}s_{11}^2 &= r^2 (\mathrm{d}t)^2 + \eta_{mn}\left(\mathrm{d}x^m + v^m \mathrm{d}t\right)\left(\mathrm{d}x^n + v^n \mathrm{d}t\right), \ v = \Omega \cdot x \\
\mathrm{d}s_{11}^2 &= r^2 (\mathrm{d}t)^2 + \eta_{mn} \mathrm{d}y^m \mathrm{d}y^n, \ y = e^{t\Omega}\cdot x
\end{aligned}
\qquad (6.7)
$$

6.3. Equivariant K-vertex

The bound states of the single D6 brane and several D0 branes can be studied with the help of an index, which is an equivariant version of Witten's index for the supersymmetric quantum mechanics, describing the low energy dynamics of the open strings stretched between the $D0$'s and between $D0$'s and the $D6$ brane (this problem was analyzed in [14] for the system of $D0$ branes only).

This quantum mechanics [15] is a dimensional reduction of the supersymmetric Yang-Mills theory down to $0+1$ dimensions, augmented with some extra fields. Whether this quantum mechanics is supersymmetric or not depends on the B-field, which can be turned on along the $D6$ brane.

The index, however, does not depend on the actual value of the B-field. More precisely, it does not change as the B-field is continuously varied. The index may jump, however, when the B-field crosses certain critical values, i.e. the walls of marginal stability of the $D0-D6$ system.

The quantum mechanical variables describing the motion of k $D0$-branes, are nine Hermitian matrices X^A, $A=1,\ldots,9$. Of these nine matrices six describe the motion along the $D6$ brane, and three describe the transverse motion. It is convenient to combine the six longitudinal matrices into three complex matrices B_α, $\alpha=1,2,3$, via, e.g.

$$X^\alpha = \frac{1}{\sqrt{2}}\left(B_\alpha + B_\alpha^\dagger\right)\ ,\ X^{3+\alpha} = \frac{i}{\sqrt{2}}\left(B_\alpha - B_\alpha^\dagger\right)\ .$$

The quantum mechanics of the $D0$-branes has the $U(k)$ gauge symmetry. The matrices X^A are all in the adjoint representation of $U(k)$. In addition, there is a matrix I, which is in the fundamental representation of $U(k)$. This is a lowest energy mode of the $0-6$ string.

Mathematically we are dealing with the space of four complex matrices B_α and Y, all in $\mathrm{End}(\mathbf{C}^k)$, and $I \in \mathbf{C}^k$, considered up to the $GL(k,\mathbf{C})$ action:

$$(B_1,B_2,B_3,Y;I) \sim (gB_1g^{-1}, gB_2g^{-1}, gB_3g^{-1}, gYg^{-1}; gI) \tag{6.8}$$

On this space we consider the holomorphic function (the superpotential) W,

$$W = \mathrm{tr}\ (B_1[B_2,B_3]) \tag{6.9}$$

We would like to study the space of the critical points of W. Ideally, it should have dimension zero. However, in reality it has a positive dimension ($3k$). This discussion brings us precisely to the setup of the three dimensional geometric partition function. The $D0-D6$ partition function therefore is identical to our formula (2.24). This result can also be interpreted in terms of summing over geometries, a la [16].

6.4. M-theory flesh

We now come to the conclusion of our analysis. The M-theory lift of the $D0-D6$ system in the Ω-background is the $TN_R \times \mathbf{R}^6$ fiber bundle over the circle $\mathbf{S}^1$. By deforming it to $\mathbf{R}^{10} = \mathbf{R}^4 \times \mathbf{R}^6$ bundle, one gets the M-theory Ω-background. The chemical potential for the $D0$-brane charge travelling around the circle $\mathbf{S}^1$ translates to the extra twist parameter of the TN space. The $SU(4)$ rotation of the $D6$ theory together with this twist becomes the $SU(5)$ rotation of $\mathbf{R}^{10}$. The eleven dimensional supergravity partition function in this background can be calculated, similarly to the calculation we performed for gauge theories so far. The result is:

$$Z^{\mathrm{11d\,sugra}}(q_1, q_2, q_3, q_4, q_5) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} F^{\mathrm{11d\,sugra}}\left(q_1^n, q_2^n, q_3^n, q_4^n, q_5^n\right)\right) \tag{6.10}$$

where

$$F^{\mathrm{11d\,sugra}}\left(q_1, q_2, q_3, q_4, q_5\right) = \frac{\sum_i q_i}{\prod_i (1-q_i)} + \frac{\sum_i q_i^{-1}}{\prod_i (1-q_i^{-1})} \tag{6.11}$$

which coincides with (2.24), for $q_4 = Q(q_1 q_2 q_3)^{-1/2}, q_5 = Q^{-1}(q_1 q_2 q_3)^{-1/2}$, after subtracting the perturbative part:

$$F^{\mathrm{11d\,sugra}}(q_1, q_2, q_3, q_4, q_5) = \widetilde{F}^{\mathrm{7d\,pert}}(q_1, q_2, q_3) + \mathcal{F}(q_1, q_2, q_3, q)\ . \tag{6.12}$$

where

$$\widetilde{F}^{\mathrm{7d\,pert}}(q_1, q_2, q_3) = F_6(q_1, q_2, q_3) + F_6(q_1^{-1}, q_2^{-1}, q_3^{-1}) = F^{\mathrm{11d\,sugra}}(q_1, q_2, q_3, q_4, q_5)|_{q=0}$$

(we take the $U(1)$ version of (5.19)).

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