Two Dimensional Topological Strings and Gauge Theory

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We study topological A-type string on an arbitrary two dimensional target space. Using the Virasoro constraints, proven by A. Okounkov and R. Pandharipande, we find an explicit formula for the partition function. The target space field theory reproducing this partition function is proposed. This field theory has infinite set of deformations which are overlooked by the standard definition of the topological string. We also discuss the relations to the multi-trace deformations of gauge theories, and make contact with quantum integrable systems. In addition, the target space theory can be in turn coupled to gravity, thereby realizing the topological string version of M. Green's "worldsheets for worldsheets" idea.

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1.

Topological strings are a continuous source of inspiration for gauge and string theorists. They can be studied on their own, for the purely mathematical reasons. Sometimes the amplitudes of the topological string can be viewed as the subset of the "physical" superstrings. The topological strings produce exact all-loop results [1], from which one hopes to gain some intuition about the quantum theory of gravity, perhaps even at the non-perturbative level. For example, the topological strings give a realization of the quantum space foam picture [2]. The topological strings of A and B type play a crucial rôle in describing the compactifications of II string theories on Calabi-Yau manifolds, which gives rise to the $\mathcal{N} = 2$ theories in four dimension. The partition function $\mathcal{Z}(t)$ of a topological string, of A or B type, depends on a some set of couplings t, which correspond to the cohomology of the target space of string theory, valued in some sheaf. For example, for the B model on a Calabi-Yau manifold X of complex dimension d, the coupling constants t belong to

$$H_B(X) = \bigoplus_{p,q=0}^{d} H^p(X, \Lambda^q \mathcal{T}_X) \approx H^{d-*,*}(X) \ ,$$

while for A model the couplings are valued in

$$H_A(X) = \bigoplus_{p,q=0}^{d} H^p(X, \Lambda^q \mathcal{T}^*_X) \approx H^{*,*}(X) \ ,$$

In addition, every operator $\mathcal{O}$, describing these couplings, comes with the so-called gravitational descendents $\sigma_k(\mathcal{O})$, $k = 0, 1, 2, \ldots$. Thus the full set of couplings of the topological string is an infinite dimensional space

$$H_{A,B}(X) \otimes \mathbf{C}[[z]]$$

where we using a formal variable z to label the gravitational descendents:

$$\sigma_k(\mathcal{O}) \leftrightarrow \mathcal{O} \otimes z^k$$

In the case $d = 3$ the gravitational descendents decouple for $k > 0$, except for the dilaton $\sigma_1(1)$, which corresponds to the string coupling constant $\hbar$. The (disconnected) partition function of the topological string

$$\mathcal{Z}_X(t; \hbar) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g(t) \ ,$$

where $t \in H_{A,B}(X)$, is a generating function of genus g topological string diagrams. For the B model these diagrams can be identified with Feynman diagrams of a certain quantum field theory on X, the so-called Kodaira-Spencer theory [1]. For the A model the analogous theory, the so-called theory of Kähler gravity [3] is expected to be non-local and is constructed only in the large volume limit where the non-local effects are exponentially suppressed.

In this note we shall construct the Kähler gravity theory for the two dimensional X and will find that it is a local theory of an infinite number of fields. The proofs and derivations will appear in a companion paper [4].

Duality CY vs. $\mathbf{R}^4$*: topological string −− supersymmetric gauge theory.*

A topic which keeps attracting attention of many researchers in the field, is the duality between the topological strings on local Calabi-Yau manifolds and the chiral sector in the four dimensional $\mathcal{N} = 2$ and $\mathcal{N} = 1$ supersymmetric gauge theories. The simplest example of that duality is the geometrical engineering of [5]. One starts with an ADE singularity, i.e. a quotient $\mathbf{C}^2/\Gamma_G$, fibered over a $\mathbf{CP}^1$ so that the total space is a (singular) Calabi-Yau manifold. By resolving the singularities one obtains a smooth non-compact Calabi-Yau manifold X_G. If one views the IIA string on $X_G \times \mathbf{R}^{1,3}$ as a large volume limit of a compactification on a Calabi-Yau manifold with the locus of ADE singularities over an isolated rational curve, then the effective four dimensional theory will decouple from gravity. Moreover one can be model the effective theory on the four dimensional $\mathcal{N} = 2$ theory with the MacKay dual ADE gauge group G, where the resolution of singularities of X_G corresponds to fixing a particular vacuum expectation value of the adjoint scalar. Then the prepotential of the low-energy effective theory is given by the genus zero prepotential of the type A topological string on X_G (more precisely, it is the prepotential of the five-dimensional gauge theory compactified on a circle which arises in this way [6][7], in order to see the four dimensional prepotential one has to go to a certain scaling limit in the CY moduli space [5]).

Duality Σ *vs.* $\mathbf{R}^4$*: topological string −− supersymmetric gauge theory.*

Another remarkable duality between the chiral sector of the four dimensional $\mathcal{N} = 2$ theories and the topological strings on the two dimensional manifolds was discovered in [8] and further studied in [9]. It is based on the comparison of the instanton calculus in the four dimensional gauge theory [10] and the Gromov-Witten/Hurwitz correspondence of [11]. The physics of that correspondence involves the theory on a fivebrane wrapped on a Riemann surface. One can actually stretch the duality beyond the realm of the physical superstrings and conjecture a powerful S-duality at the level of the topological strings only [12], leading to the concept of the topological string version of M-theory, or Z-theory [13] [14].

The duality of [8](see also a paper on the mathematically related subject [15] and recent works on the duality with $\mathcal{N} = 1$ four dimensional theories [16]) identified the disconnected partition function of the topological string on $\mathbf{CP}^1$ in the background with the arbitrary topological descendents of the Kähler form $\sigma_k(\omega)$ turned on. The couplings t_k^ω (up to a k-dependent factor) are identified with the couplings of the operators

$$\int \mathrm{d}^4\vartheta \ \ \mathrm{tr} \ \ \Phi^{k+2}$$

in the $\mathcal{N} = 2$ gauge theory:

$$\sum_{k=0}^{\infty} \frac{t_k^\omega}{k!} \int_C \sigma_k(\omega) \leftrightarrow \sum_{k=0}^{\infty} \frac{t_k^\omega}{(k+2)!} \int_{\mathbf{R}^{4|4}} \mathrm{d}^4x \mathrm{d}^4\vartheta \ \ \mathrm{tr} \ \ \Phi^{k+2} \tag{1.1}$$

where in the left hand side we write the worldsheet couplings. In this paper we shall deepen the duality discovered in the original paper [8].

Duality Σ vs.Σ: topological string −− two dimensional gauge theory.

About fifteen years ago D. Gross has proposed to attack the problem of finding the large N gauge theory description in terms of some kind of string theory via the analysis of the two dimensional gauge theories. By carefully analyzing the 't Hooft limit of the two dimensional Yang-Mills theory on a Riemann surface Σ D. Gross and W. Taylor have identified many features of the corresponding string theory, while [17] have proposed a new kind of topological string theory. An important aspect of the construction of [17] was the realization of the fact that the topological Yang-Mills theory (which is the perturbative limit of the physical Yang-Mills theory) can be described by the Hurwitz theory. The latter counts ramified coverings of a Riemann surface Σ. In this paper we shall find a different version of the string field theory, the one corresponding to the A type topological strings on a Riemann surface Σ. It will turn out to be a kind of an infinite N gauge theory, but most likely not the ordinary 't Hooft large N limit of the gauge group with the finite dimensional gauge group like $SU(N)$ or $SO(N)$.

Worldsheets for worldsheets.

In [18] M. Green has proposed to study the two dimensional string backgrounds as the theories of worldsheets for yet another string theories. With the advent of the string dualities a few interesting examples of this construction were invented. For example, M-theory fivebrane wrapped on **K3** becomes a heterotic string on $\mathbf{T}^3$. This is not exactly a realization of the [18] idea as we are using the localized soliton to generate the string. One could try to study the **CY4** or **Spin(7)** compactifications of the Type II string [19]. but this is difficult due to the lack of the detailed knowledge of the moduli spaces of these manifolds. In this paper we shall approach this problem in the context of the topological string.

The conventional formulation of the A model assigns to every cohomology class $\mathbf{e}_\alpha \in H^*(X)$ of the target space X an infinite sequence of observables $\sigma_k(\mathbf{e}_\alpha)$, $k = 0, 1, 2, \ldots$. The corresponding couplings t_k^α parameterize the so-called *large phase space*. For $k = 0$ one gets the *small phase space*. Viewed from the worldsheet, the observable $\sigma_k(\mathbf{e}_\alpha)$ is the k-th descendent of $\mathbf{e}_\alpha$. However, if we think of these observables in terms of the target space we should say that $\sigma_k(\mathbf{e}_\alpha)$ is the $\dim(X) - \deg(\mathbf{e}_\alpha)$-descendent of some local BRST invariant observable $\mathcal{O}_k$:

$$\sigma_k(\mathbf{e}_\alpha) \sim \int_X \mathbf{e}_\alpha \wedge \mathcal{O}_k^{(\dim(X)-\deg(\mathbf{e}_\alpha))} \tag{1.2}$$

The gravitational descendents of the top cohomology class of X therefore correspond to the zero-observables $\mathcal{O}_k^{(0)}$ of the target space theory, and as such they are the simplest to study. This is why we shall use as the starting point the so-called *stationary sector* of the theory [11], where only the couplings of these observables are turned on. The observables which are the hardest ones to study are the descendents of the puncture, i.e. unit operator. These correspond to the $\dim(X)$-observables constructed out of $\mathcal{O}_k$ and in the standard paradigm of the topological field theory correspond to the deformations of the space-time Lagrangian.

When the topological theory is a twisted version of the supersymmetric field theory, these deformations correspond to the F-terms of the supersymmetric theory. In two dimensions they are the superpotential deformations, in four dimensions they are the prepotential deformations. Whatever is their interpretation, the target space theory has more observables. Indeed, the product of two local observables $\mathcal{O}_k$ and $\mathcal{O}_l$ and higher order products cannot be expressed, in general, as linear combinations of $\mathcal{O}_k$. In analogy with the gauge theory which we shall make much more precise, the observables $\mathcal{O}_k$ correspond to the *single trace operators*, while the products $\mathcal{O}_{k_1}\mathcal{O}_{k_2}\ldots\mathcal{O}_{k_p}$, for $p > 1$, correspond to the *multi-trace operators*. Thus the full space of deformations of the target space theory will involve couplings $\mathbf{T}_{\vec{k}}^{\alpha,\nu}$, where α label the cohomology of X, ν label the gravitational descendents in the sense of the topological gravity on X (in the problem studied in this paper, X is a two dimensional manifold and ν is a non-negative integer), and $\vec{k} = (k_1 \geq k_2 \geq k_3 \geq \ldots \geq k_p)$ is a partition labelling the multi-trace operators. We call the space of all these couplings the *Very large phase space*. We shall write an expression for the partition function of the topological string on the *Very large phase space* in genus zero (target space). The problem of finding the special coordinates on the *Very large phase space*, which is in a sense equivalent to the problem of constructing the full quantum gravity dressed string theory partition function, is beyond the scope of the present paper. Nevertheless the formulation of the problem for the general target space X is more important then the possible solution of the problem we can anticipate from the gauge theory analogy for $X = \Sigma$, a Riemann surface.

The partition function $\mathcal{Z}_X$

In this paper we study the case where X is a Riemann surface of genus h. The partition function $\mathcal{Z}_X$ of the A-model on a Riemann surface X is a function of an infinite set of couplings, $\mathbf{t} = (t_n^\alpha)$ where $\alpha = 1, \ldots, \mathrm{dim} H^*(X) = 2h+2$ and $n \in \mathbf{Z}_{\geq 0}$. We introduce some additive basis $\mathbf{e}_\alpha$ of the cohomology of X, $\mathbf{e}_\alpha \in H^*(X, \mathbf{C})$. We have:

$$\mathcal{Z}_X(\mathbf{t}; \hbar, q) = \exp \sum_{g,n;\beta=0}^{\infty} \sum_{\vec{k},\vec{\alpha}} \frac{\hbar^{2g-2} q^\beta}{n!} \prod_{i=1}^{n} t_{k_i}^{\alpha_i} \int_{\overline{\mathcal{M}_{g,n}(X,\beta)}} \bigwedge_{i=1}^{n} \mathrm{ev}_i^*(\mathbf{e}_{\alpha_i}) \wedge \psi_i^{k_i} \tag{1.3}$$

where we used the standard notations [20] for the moduli space $\overline{\mathcal{M}_{g,n}(X,\beta)}$ of degree β genus g stable maps to X with n punctures, the evaluation maps:

$$\mathrm{ev}_i : \overline{\mathcal{M}_{g,n}(X,\beta)} \longrightarrow X \tag{1.4}$$

defined as:

$$\mathrm{ev}_i\,(C, x_1, \ldots, x_n; \phi) = \phi(x_i) \tag{1.5}$$

where $(C, x_1, \ldots, x_n; \phi)$ is the stable map with the n punctures $x_1, \ldots, x_n$. Finally, in (1.3) we have the first Chern classes of the tangent lines $\psi_i = c_1(T_{x_i}C)$ at the i'th marked point. Following [21] it is convenient to think of the partition function $\mathcal{Z}_X$ as of the functional on the space of positive loops valued in $H^*(X)$. Thus, let us introduce the $H^*(X)$-valued function:

$$\mathbf{t}(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{t}_n z^n \ , \ \mathbf{t}_n = \sum_{\alpha=1}^{2h+2} t_n^\alpha \, \mathbf{e}_\alpha \in H^*(X) \tag{1.6}$$

of a formal variable z. In addition we introduce another function, related to $\mathbf{t}(z)$, the Legendre transform of the antiderivative $\partial^{-1}(z-\mathbf{t}(z))$,

$$\mathbf{F}_{\mathbf{t}}(x) = x\,\mathbf{z}(x) - \tfrac{1}{2}\mathbf{z}^2(x) + \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\mathbf{t}_k\,\mathbf{z}^{k+1}(x), \tag{1.7}$$

where $\mathbf{z}(x) \in H^*(X)$ solves:

$$x = \mathbf{z}(x) - \mathbf{t}\left(\mathbf{z}\left(x\right)\right) \tag{1.8}$$

and is given by the following formal power series in $\mathbf{t}_k$'s:

$$\mathbf{z}(x) \equiv x + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\mathbf{t}^n(x)\right]^{(n-1)} = \, x + \mathbf{t}(x) + \mathbf{t}(x)\cdot\mathbf{t}'(x) + \ldots \tag{1.9}$$

Note that even though $x \in \mathbf{C} = H^0(X)$, $\mathbf{z}(x) \in H^*(X)$ is an inhomogeneous cohomology class.

We shall present our results in two forms: the mathematical and the physical. The mathematical formula is explicit but is not fully transparent. The physical formula is conceptually more appealing but it requires some preparations so we shall present it afterwards.

represents $\mathcal{Z}_X(\mathbf{t})$ as a sum over partitions λ (the basic notions of the theory of partitions are recalled in the main body of the paper). Given a partition $\lambda = (\lambda_i)_{i\geq 1}$, let $f_\lambda(x)$ denote its profile:

$$\begin{aligned} f_\lambda(x) \;&=\; |\,x\,| + \\ &\sum_{i=1}^{\infty} |\,x - \hbar\,(\lambda_i - i + 1)\,| \,-\, |\,x - \hbar\,(\lambda_i - i)\,| \,+\, |\,x - \hbar\,(-i)\,| \,-\, |\,x - \hbar\,(-i+1)\,| \end{aligned} \tag{1.10}$$

Define $\mathcal{S}_\lambda(\mathbf{t})$ as:

$$\begin{aligned} \mathcal{S}_\lambda(\mathbf{t}) \;&=\; \frac{1}{2\hbar}\int_{\mathbf{R}} \mathrm{d}x\, f''_\lambda(x) \int_X \mathbf{F}_{\mathbf{t}}(x) + \\ &\frac{1}{2\hbar}\int_{\mathbf{R}} \mathrm{d}x\, f''_\lambda(x) \int_X e(X)\cdot \Delta_{\mathbf{t}}(x) + \\ &\frac{1}{8}\int\!\!\int_{\mathbf{R}^2} \mathrm{d}x_1 \mathrm{d}x_2\, f''_\lambda(x_1) f''_\lambda(x_2) \int_X e(X)\cdot \mathbf{G}_{\mathbf{t}}(x_1,x_2) \end{aligned} \tag{1.11}$$

where $e(X) = c_1(TX)$ is the Euler class of X,

$$\chi(X) = \int_X e(X) = 2 - 2h\ ,$$

and the functions $\Delta_{\mathbf{t}}$ and $\mathbf{G}_{\mathbf{t}}$ are the particular solutions to the finite difference equations:

$$\begin{aligned} &\Delta_{\mathbf{t}}(x + \tfrac{1}{2}\hbar) - \Delta_{\mathbf{t}}(x - \tfrac{1}{2}\hbar) = \\ &\qquad (x + \mathbf{t}_0)\log\left(\frac{x + \mathbf{t}_0}{\mathbf{z}(x)}\right) - \sum_{l=2}^{\infty} \frac{1}{l!}\left\{\sum_{m=2}^{l} \frac{1}{m}\right\} \mathbf{t}_l\, \mathbf{z}^l(x)\ , \end{aligned} \tag{1.12}$$

(the right hand side is a formal power series in x)

$$\begin{aligned} &\mathbf{G}_{\mathbf{t}}(x_1 + \tfrac{1}{2}\hbar, x_2 + \tfrac{1}{2}\hbar) - \mathbf{G}_{\mathbf{t}}(x_1 - \tfrac{1}{2}\hbar, x_2 + \tfrac{1}{2}\hbar) \\ &\qquad - \mathbf{G}_{\mathbf{t}}(x_1 - \tfrac{1}{2}\hbar, x_2 + \tfrac{1}{2}\hbar) + \mathbf{G}_{\mathbf{t}}(x_1 - \tfrac{1}{2}\hbar, x_2 - \tfrac{1}{2}\hbar) \\ &\qquad\qquad = \log\left(\frac{\mathbf{z}(x_1) - \mathbf{z}(x_2)}{\hbar}\right)\ , \end{aligned} \tag{1.13}$$

which we specify in [4]. The mathematical formula is:

$$\boxed{\mathcal{Z}_X\left(\mathbf{t};\hbar,q\right) = \sum_\lambda (-q)^{|\lambda|} \exp\left(\frac{\mathcal{S}_\lambda(\mathbf{t})}{\hbar}\right)} \tag{1.14}$$

We derive it in [4] using the Virasoro constraints proven in [22].

The physical formula: A model version

identifies $\mathcal{Z}_X(\mathbf{t})$ with the partition function of a two dimensional gauge theory on X. The gauge theory in question is a twisted $\mathcal{N}=2$ super-Yang-Mills theory with the gauge group $\mathbf{G}$, to be specified momentarily, perturbed by all single-trace operators, commuting with the scalar supercharge Q. More precisely, $\mathcal{Z}_X$ is equal to the generating function of the correlators of all 2, 1, and 0-observables (we remind the relevant notions in the main body of the paper), constructed out of the single-trace operators

$$\mathcal{O}_k = \mathrm{Coeff}_{u^k}\, \mathrm{Tr}_{\mathcal{H}}\, e^{u\phi} \ , \tag{1.15}$$

that is:

$$\boxed{\begin{aligned} \mathcal{Z}_X(\mathbf{t};\hbar,q) \ &= \\ &\left\langle \exp - \int_X \left[\sum_{k=0}^{\infty} \sum_{\alpha=1}^{2h+2} \hbar^{k-1+\frac{\mathrm{deg} e_\alpha}{2}}\, \widehat{t}_k^{\alpha}\, e_\alpha \wedge \mathcal{O}_{k+1}^{(2-\mathrm{deg} e_\alpha)} \right] \right\rangle \\ \widehat{t}_0^{2h+2} &= t_0^{2h+2} - \log(q) + \chi(X) \log(\hbar) \\ \widehat{t}_1^{1} &= t_1^1 - 1 \end{aligned}} \tag{1.16}$$

and the other times $\widehat{t}_k^{\alpha} = t_k^{\alpha}$. The gauge group $\mathbf{G}$ consists of certain unitary transformations of a Hilbert space $\mathcal{H}$. Its definition will be given in the main paper [4].

The physical formula: B model version

represents $\mathcal{Z}_X(\mathbf{t})$ as a partition function of a Landau-Ginzburg theory with the worldsheet X. The $\mathcal{N}=2$ supersymmetric Landau-Ginzburg theory without topological gravity is determined by the following data: a target space, which is a complex manifold $\mathcal{U}$, a holomorphic function $\mathcal{W}$, and a top degree holomorphic form Ω on $\mathcal{U}$. The target space $\mathcal{U}$ is an infinite-dimensional disconnected space. Its connected components $\mathcal{U}_\lambda$ are labelled by partitions λ. Each component is isomorphic to $\mathbf{C}^\infty$, the space of finite sequences of complex numbers. The superpotential is given by the regularized infinite sum

$$\mathcal{W} = \sum_{i=1}^{\infty} \left[\left(\lambda_i - i + \tfrac{1}{2}\right) z_i - \tfrac{1}{2} z_i^2 + \sum_{k=0}^{\infty} \frac{t_k}{(k+1)!} z_i^{k+1} \right]$$

The top degree form is given by the formal product:

$$\Omega = \bigwedge_{i=1}^{\infty} \mathrm{d}\varepsilon_i \tag{1.17}$$

where

$$1 + \sum_{i=1}^{\infty} \varepsilon_i t^i = \prod_{i=1}^{\infty} (1 + t z_i)$$

Of course the infinite-dimensionality of various ingredients involved means that this is not the conventional B model. However the theory provides a regularization of the infinite products and sums above.

Very large phase space extension

In the worldsheet formulation the Very large phase space observables are non-local. In the language of topological string, the insertion of the observable $\mathcal{O}_{0,0,\ldots,0}(x_1,\ldots,x_k)[\alpha]$, where $x_i \in C, [\alpha] \in H_*(X)$ corresponds to the condition that the points $x_1,\ldots,x_k$ of the worldsheet are mapped to the same point $f \in X$ sitting in a cycle representing $[\alpha]$.

Note that the non-local string theories describing multi-trace deformations of gauge theories were recently studied in the context of the AdS/CFT correspondence [23].

On the very large phase space the function $\mathcal{W}$ becomes a generic symmetric function of z_i's which is formally close to the function $\sum_{i=1}^{\infty}(\lambda_i - i + \frac{1}{2})z_i - \frac{1}{2}z_i^2$:

$$\begin{aligned} \mathcal{W}_\lambda &= \sum_{i=1}^{\infty}(\lambda_i - i + \tfrac{1}{2})z_i - \tfrac{1}{2}p_2 + w(p_1,p_2,\ldots) \\ w(p) &= \sum_{\vec{k}} \frac{\widetilde{T}_{k_1 k_2 \ldots k_l}}{k_1! k_2! \ldots k_l!} p_{k_1} p_{k_2} \ldots p_{k_l} \ , \end{aligned} \tag{1.18}$$

while the holomorphic top form is given by (1.17). The three point function on a sphere is given by the regularized version of Grothendieck residue:

$$C_{\alpha\beta\gamma} = \sum_{\lambda} \sum_{p_\lambda : d\mathcal{W}_\lambda(p_\lambda)=0} \frac{\Phi_\alpha(p_\lambda)\Phi_\beta(p_\lambda)\Phi_\gamma(p_\lambda)}{\mathrm{Hess}_\Omega(\mathcal{W}_\lambda)} \tag{1.19}$$

where α, β, γ are partitions, $\Phi_{\alpha,\beta,\gamma}$ are some formal power seria in p_k's, generalizing Schur functions,

$$p_k = \sum_i z_i^k := k!\,\mathrm{Coeff}_{u^k} \sum_i e^{u z_i}$$

and $\mathrm{Hess}_\Omega(\mathcal{W}_\lambda)$ is the regularized determinant, defined as a ratio of the determinant of the derivative map $\mathrm{Det}\,(d\mathcal{W}_\lambda) : \det\mathcal{T} \longrightarrow \det\mathcal{T}^*$ and Ω^2 viewed as an element of $\det^{\otimes 2}\mathcal{T}^*$, where $\mathcal{T}$ is the tangent space $T_{p_\lambda}\mathcal{U}$ at the critical point p_λ of $\mathcal{W}_\lambda$.

The couplings $\widetilde{T}_{\vec{k}}$ in (1.18) are most likely not the flat (or special) coordinates on the Very large phase space. The flat coordinates $T_{\vec{k}}$ are obtained from $\widetilde{T}_{\vec{k}}$ by a formal diffeomorphism. To find them is the first step in understanding the target space quantum gravity.

So far we were discussing the standard topological string on X. The target space turns out to be a topological field theory. In two dimensions such a theory is a relatively simple construction.

All one needs to determine is a commutative associative algebra $\mathbf{A}$, and a functional $\langle\cdot\rangle : \mathbf{A} \to \mathbf{C}$.

In our case the algebra is just the algebra of symmetric functions. Indeed, we discussed so far the generators of this algebra, $\mathcal{O}_k$, $k = 0, 1, 2, \ldots$. We should be able to multiply $\mathcal{O}_k$'s. In this way we shall get arbitrary polynomials of $\mathcal{O}_k$'s. We may allow a formal power series in the generators $\mathcal{O}_k$'s with the assumption that such a series is well-defined once we substitute

$$\mathcal{O}_k(\lambda) = \mathrm{Coeff}_{u^{k+1}} \sum_{i=1}^{\infty} e^{u\,\mathbf{z}\left(\lambda_i - i + \frac{1}{2}\right)} \tag{1.20}$$

for an arbitrary partition λ.

We computed the functional for the large phase space:

$$\langle\mathcal{O}\rangle = \sum_{\lambda} e^{-\mathbf{r}(\lambda,\mathbf{t^1})}\mathcal{O}(\lambda) \tag{1.21}$$

We should consider the algebra $\mathbf{A}$ together with its space of deformations. The latter is the space of the couplings $T_{\vec{k}}$.

The two dimensional topological field theory can be coupled to the topological gravity. This is the target space gravity.

We can describe its observables directly in target space. We can also discuss its worldsheet definition. The latter is potentially interesting for more realistic quantum gravity theories.

The target space definition is the following. Consider the moduli space $\mathcal{M}_X$ of complex structures on X. The topological string amplitudes are independent of the choice of the complex structure on X. However, one can generalize them, so that they would define closed differential forms on $\mathcal{M}_X$. Moreover, we can consider non-compact Riemann surfaces X, i.e. curves with punctures.

Michael B. Green has proposed in [18] to study the two dimensional string backgrounds as the theories of worldsheets for yet another string theories. Our approach gives a concrete realization of that proposal.

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