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Abstract

Recently, a new type of renormalizable $\phi^{\star 4}_4$ scalar model on the Moyal space was proved to be perturbatively renormalizable. It is translation-invariant and introduces in the action a $a/(\theta^2 p^2)$ term. We calculate here the β and γ functions at one-loop level for this model. The coupling constant β_λ function is proved to have the same behaviour as the one of the ϕ^4 model on the commutative $\mathbb{R}^4$. The β_a function of the new parameter a is also calculated. Some interpretation of these results are done.

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1 Introduction

Noncommutative (NC) quantum field theories (NCQFT) [1][2] is intensively investigated in the recent years [3]-[20]. A first renormalizable $\phi_4^{\star 4}$ model on the Moyal space, the Grosse-Wulkenhaar model (GWm), was proposed in [3]. Ever since several proofs of its renormalization were given and some of its properties were studied [4]-[7]. The β function of this model was proved to be vanishing at any order in perturbation theory [8, 9, 10]. These advances motivate to better scrutinize these NC models. Moreover, other renormalizable models have been highlighted. The $O(N)$ and $U(N)$ GWm were considered with respect to symmetry breaking issues [14]; their β functions were computed at one-loop in [15]. Finally, the GWm in a magnetic field was considered with respect to its parametric representation [16] and its β function computations at any order [17].

Nevertheless, the GWm mentioned above loses the usual translation invariance of a field theory. Moreover, the extension of this GW procedure for the construction of a renormalizable gauge theory seems unclear [18].

In [19], a different type of scalar model was proposed. This model preserves translation invariance and is also proved to be renormalizable at all order of perturbation theory [19]. These features come from a new term in the propagator, of the form $a/(\theta^2 p^2)$ and on which relies the "cure" of the UV/IR mixing. Finally, let us also mention that the extension of this mechanism for gauge theories was recently proposed [20].

In this paper, we consider this NC translation-invariant scalar model and compute its one-loop β functions for the coupling constant λ, the mass m and the new parameter a. We decompose the propagator of the theory as a sum of the usual commutative propagator and a NC correction. Different comparisons with the commutative ϕ^4 model are made. The sign of the β_λ function is proved to be the same as in the commutative theory.

The paper is organized as follows. The next section introduces the model and recalls some of the renormalization results of [19]. The third section proposes the decomposition mentioned above of the NC propagator. This decomposition allows us to calculate the γ and β functions of the model. Finally, some conclusions are drawn.

2 The model and its renormalization

The action [19] is

$$S_\theta[\phi] = \int d^4p \left\{ \frac{1}{2} p_\mu \phi\, p^\mu \phi + \frac{1}{2} m^2\, \phi\, \phi + \frac{1}{2}\, a\, \frac{1}{\theta^2 p^2}\, \phi\, \phi + \frac{\lambda}{4}\, V_\theta \right\}, \tag{2.1}$$

where a is some dimensionless parameter chosen such that $a \geq 0$ and V_θ is the Fourier transform of the potential $\frac{\lambda}{4}\phi(x)^{\star 4}$ in momentum space. Note that the 4 factor above (instead of the usual commutative 4! factor) comes from the fact that the Moyal vertex is invariant only under cyclic permutation of the incoming/outgoing fields. However, the comparison with the commutative results will become more difficult. The propagator writes

$$C(p) = \frac{1}{p^2 + m^2 + \frac{a}{\theta^2 p^2}}\,. \tag{2.2}$$

Note that the condition on a above ensures the positivity of $C(p)$. It is worthy to recall that the Moyal vertex can be written in terms of momenta as

$$V_\theta = -\frac{\lambda}{4!}\, \delta(p^1 + p^2 + p^3 + p^4)\, e^{\frac{i}{2}\sum_{1\le j<\ell\le 4} p^j_\mu \Theta^{\mu\nu} p^\ell_\nu} \tag{2.3}$$

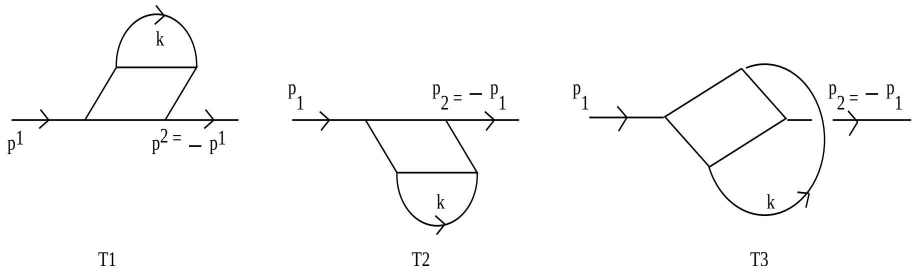

Figure 1: The tadpole graphs T_1, T_2 and T_3.

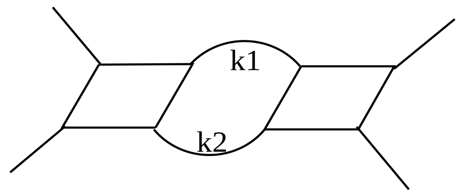

Figure 2: The bubble graph

Let us introduce the following terminology.

Definition 2.1 *Let g be the genus, L the number of lines, F the number of faces, B the number of faces broken by external legs of a graph.*

(i) A planar graph is a graph such that $g = 0$.

(ii) A nonplanar graph is a graph such that $g > 0$.

(iii) A planar regular graph is a planar graph such that $B = 1$.

(iv) A planar irregular graph is a planar graph such that $B \geq 2$.

As already stated in [19], the primitively divergent graphs of the model (2.1) are the two- and four-point ones. More precisely, one has the following:

- the planar regular two-point graphs are responsible for the wave function and mass renormalization,
- the planar regular four-point graphs are responsible for the coupling constant renormalization,
- the planar irregular two-point graphs are responsible for the renormalization of the parameter a.

The rest of the graphs are irrelevant to the renormalization process. Thus, the one-loop graphs to be considered are ones of Fig.1 and 2. Note that the tadpole graphs T_1 and T_2 are planar regular graphs while T_3 is a planar irregular graph.

3 The β functions of the model

3.1 Decomposition of the propagator: noncommutative correction

Before calculating the β and γ functions of this model, let us write some useful integral representation of the propagator (2.2). We propose to use the formula

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B}, \tag{3.1}$$

for

$$A = p^2 + m^2, \quad B = \frac{a}{\theta^2 p^2}. \tag{3.2}$$

Thus, the propagator (2.2) writes

$$\begin{aligned} C(p) &= \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\frac{a}{\theta^2 p^2(p^2+m^2)+a} \\ &= \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\frac{a}{\theta^2(p^2+m_1^2)(p^2+m_2^2)}, \end{aligned} \tag{3.3}$$

where $-m_1^2$ and $-m_2^2$ are the roots of the denominator of the second term in the lhs considered as a second order equation in p^2, namely

$$\frac{-\theta^2 m^2 \pm \sqrt{\theta^4 m^4 - 4\theta^2 a}}{2\theta^2} < 0, \tag{3.4}$$

with $a < \theta^2 m^4/4$. One can also use the following formula

$$\frac{1}{p^2+m_1^2}\frac{1}{p^2+m_2^2} = \frac{1}{m_2^2-m_1^2}\left(\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right). \tag{3.5}$$

This allows to write the propagator (3.3) as

$$C(p) = \frac{1}{p^2+m^2} - \frac{a}{\theta^2(m_2^2-m_1^2)}\frac{1}{p^2+m^2}\left(\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right). \tag{3.6}$$

Note that, in this paper, we will use the decomposition (3.3), the one given by (3.6) being equivalent. One can interpret the last term of (3.3) as some noncommutative correction to the propagator. Let us now prove that this correction leads only to irrelevant (*i.e.* finite) contribution when inserted into the one-loop diagrams of Fig.1 and 2.

Indeed, when inserted in the T_1 or T_2 tadpole graphs, we get an integral of the form

$$\lambda \int d^4p \frac{1}{(p^2+m^2+\frac{a}{\theta^2 p^2})}. \tag{3.7}$$

Thus the noncommutative correction obtained *via* the decomposition (3.3) is

$$\lambda \int d^4p \frac{a}{\theta^2}\frac{1}{(p^2+m^2)(p^2+m_1^2)(p^2+m_2^2)}, \tag{3.8}$$

which is convergent.

The case of the planar irregular tadpole graph T_3 induces the same integral when letting the external moment go to 0. Finally, the bubble graph of Fig.2 leads also to a finite integral which is irrelevant. This can be explicitly seen by writing the corresponding Feynman amplitude (at vanishing external momenta)

$$\lambda^2 \int d^4p \frac{1}{(p^2+m^2+\frac{a}{\theta^2 p^2})^2}. \tag{3.9}$$

Inserting now the decomposition (3.3) in the integral (3.9) implies the separation of the noncommutative correction

$$\lambda^2 \left[2\frac{a}{\theta^2} \int d^4p \frac{1}{(p^2+m^2)^2(p^2+m_1^2)(p^2+m_2^2)} + \frac{a^2}{\theta^4} \int d^4p \frac{1}{[(p^2+m^2)(p^2+m_1^2)(p^2+m_2^2)]^2} \right]. \tag{3.10}$$

Both these integrals are finite thus irrelevant.

3.2 One-loop β and γ functions

We briefly set the renormalization group (RG) flow framework used hereafter. Firstly, the dressed propagator $G^2(p)$ or connected two-point function is given by

$$G^2(p) = \frac{C(p)}{1-C(p)\Sigma(p)} = \frac{1}{C(p)^{-1}-\Sigma(p)}, \tag{3.11}$$

$$C(p)^{-1}-\Sigma(p) = p^2+m^2+\frac{a}{\theta^2 p^2} - \Sigma(p) \tag{3.12}$$

where $\Sigma(p)$ is the self-energy. One writes $\Sigma(p) = \langle \phi(p)\phi(-p)\rangle_{1PI}^t$, where by t we understand amputated. Furthermore, note that

$$\Sigma(p) = \Sigma_{\text{plr}}(p) + \Sigma_{\text{pli}}(p). \tag{3.13}$$

"plr" and "pli" refer to planar regular and irregular contributions, respectively.

We now want to compute at one-loop the renormalization equations

$$\lambda_r = -\frac{\Gamma^4(0,0,0,0)}{Z^2}, \qquad m_r^2 - m_b^2 = -\frac{\Sigma_{\text{plr}}}{Z}, \tag{3.14}$$

where by r, we mean "renormalized" and by b, we mean "bare". In addition, Z is the wave function renormalization and the amputated four-point function is

$$\Gamma^4(p^1,p^2,p^3,-p^1-p^2-p^3) = \langle \phi(p^1)\phi(p^2)\phi(p^3)\phi(-p^1-p^2-p^3)\rangle_{1PI}^t \tag{3.15}$$

The RG flow of the parameter a is now considered. In [19], it was already observed that this renormalization is *finite*, meaning that the coefficient of $1/p^2$ is finite. Indeed, an explicit computation of the Feynman amplitude of a planar irregular two-point function leads to this result (see again [19])

$$\mathcal{A} = \frac{1}{p^2} F(p), \tag{3.16}$$

where $\mathcal{A}$ is the corresponding amplitude and $F(p)$ is a function uniformly bounded by a constant for all p. This is related to the fact that the slice definition takes into consideration the mixing of high and low energies.

One has

Proposition 3.1 *At one-loop, the self-energy is given by*

$$\Sigma(p) = -\lambda \left(2S^{(1)}(0) + S^{(1)}(p)\right), \tag{3.17}$$

where

$$S^{(1)}(p) \quad = \quad \int d^4k \frac{e^{ik_\mu \Theta^{\mu\nu} p_\nu}}{k^2+m^2}. \tag{3.18}$$

$$\tag{3.19}$$

Proof. The self-energy can be obtained at first order in λ by

$$\Sigma(p) = \sum_{\mathcal{G}_i} K_{\mathcal{G}_i} S_{\mathcal{G}_i}(p) \tag{3.20}$$

where $\mathcal{G}_i$ runs over one-loop 1PI amputated two-point planar regular and irregular graphs with amplitude $S_{\mathcal{G}_i}(p)$, and $K_{\mathcal{G}_j}$ corresponds to some combinatorial factor. As discussed above, the graphs to be considered are the tadpole graphs T_1, T_2 and T_3 (see Fig.1), with the combinatorial factors

$$K_{T_1} = 4, \qquad K_{T_2} = 4, \qquad K_{T_3} = 4, \tag{3.21}$$

respectively. Since the noncommutative correction of the propagator produces an irrelevant contribution (see above), we obtain $S^{(1)}(0)$ for the amplitudes of the tadpole graphs T_1 and T_2 and $S^{(1)}(p)$ for the amplitude of the T_3 graph. □

Remark that the integral $S^{(1)}(0)$ is quadratically divergent while $S^{(1)}(p)$ is convergent; nevertheless it is this integral which leads to the UV/IR mixing (indeed, a $1/p^2$ contribution which, when inserting the corresponding planar irregular tadpole into a "bigger" graph will lead to a IR divergence).

Furthermore, we point out that the decomposition (3.13) of the self-energy into a planar regular and a planar irregular part corresponds in (3.17) to $2S^{(1)}(0)$ for the planar regular part (the wave function and mass renormalization) and to $S^{(1)}(p)$ (the renormalization of the parameter a). Hence, one has a splitting of this self-energy into two distinct parts, responsible for the renormalization of two distinct parameters, m and a. This is a major difference with respect to the commutative ϕ^4 model.

Let us calculate the wave function renormalization $Z = 1 - \partial_{p^2}\Sigma_{\text{plr}}(p)|_{p=0}$. Since $S^{(1)}(0)$ has no dependence on the external momenta p, the following one-loop result is reached

$$Z = 1. \tag{3.22}$$

Then, the γ function of the model is

$$\gamma = 0 + \mathcal{O}(\lambda^2). \tag{3.23}$$

Note that we have proved that the results (3.22) and (3.23) are at one-loop, for the reasons explained above, nothing but the ones of the ϕ^4 theory on commutative space.

In the following, we investigate the RG flows of the parameters m. As a straightforward consequence of Proposition 3.1, the tadpole graphs T_1 and T_2 represent 2/3 of $\Sigma(0)$. The total self-energy at vanishing external momenta $\Sigma(0)$ is nothing but the one of the commutative ϕ^4 model (for a proper rescaling of λ). We have

$$\beta_m \propto \beta_m^{\text{commutative}}. \tag{3.24}$$

As a consequence of the above discussion of the *finite* renormalization of the parameter a we have

$$\beta_a = 0. \tag{3.25}$$

We want to emphasize that the splitting (3.13) of the self-energy can also be associated to some mechanism for taking the commutative limit, as already indicated in [19].

In the following, the RG flow of the coupling constant λ is calculated. The following statement holds.

Theorem 3.2 *At one-loop, the RG flow of the coupling λ satisfies*

$$\lambda_r = \lambda \left(1 - 2\lambda\, \mathcal{S}^{(2)}\right), \tag{3.26}$$

with

$$\mathcal{S}^{(2)} = \int d^4k \frac{1}{(k^2+m^2)^2}. \tag{3.27}$$

Proof. The noncommutative correction of the propagator corresponds to an irrelevant contribution in Γ^4. Only the bubble graph of Fig.2 has to be considered. Its combinatorial factor is

$$4 \cdot 4 \cdot 4. \tag{3.28}$$

The Feynman amplitude of the bubble graph includes the integral $\mathcal{S}^2$. Thus, one gets

$$\Gamma^4(0,0,0,0) \quad = \quad -\lambda + \lambda^2 \frac{1}{2!4^2} 4^3 \mathcal{S}^2 \tag{3.29}$$

which completes the proof. □

Note that the divergence of the integral (3.27) is logarithmic when removing the UV cutoff. The β_λ function of the model (2.1) is thus a simple fraction of the β_λ function of the commutative ϕ^4 model. The difference is due to the fact that one has to take into considerations only the planar regular bubble graph of Fig.2. In other words, the symmetry factor of the noncommutative graph of Fig.2 is only a part of the symmetry factor of the corresponding commutative graph (for a commutative theory, the planar irregular or regular four-point graphs are indistinguishable).

With our conventions, after performing the solid angle integration of d^4k (introducing a $2\pi^2$ factor), one obtains

$$\beta_\lambda = 4\pi^2\lambda^2 + \mathcal{O}(\lambda^3). \tag{3.30}$$

We have thus computed here the one-loop β and γ functions of the NC translation-invariant renormalizable scalar model (2.1). The β_λ function is proved to have the same behaviour as in the commutative ϕ^4 case. This result is a direct consequence of the fact that β_λ is given only by the planar regular sector of the theory and this sector is not affected by noncommutativity (note that this is the same as for the "naive" NC $\phi^{\star 4}$ model, *i.e.* the NC scalar model without an harmonic GW x^2 or a $1/p^2$ term which presents the UV/IR mixing). Indeed, the "new" planar irregular sector which, in the case of a NC theory, is qualitatively different of the planar regular one, is responsible only for the renormalization of the constant a (as already observed in [19]). Finally, we have also calculated the running β_a of this new constant a.

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