Dimensional Regularization of the Gravitational Interaction of Point Masses in the ADM Formalism

T. DAMOUR, P. JARANOWSKI and G. SCHAEFER

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Novembre 2008

IHES/P/08/55

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Thibault Damour
Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France
E-mail: damour@ihes.fr

Piotr Jaranowski
Institute of Theoretical Physics, University of Białystok, Białystok, Poland
E-mail: pio@alpha.uwb.edu.pl

Gerhard Schäfer
Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität, Jena, Germany
E-mail: Gerhard.Schaefer@uni-jena.de

The ADM formalism for two-point-mass systems in d space dimensions is sketched. It is pointed out that the regularization ambiguities of the 3rd post-Newtonian ADM Hamiltonian considered directly in $d = 3$ space dimensions can be cured by dimensional continuation (to complex d's), which leads to a finite and unique Hamiltonian as $d \to 3$. Some so far unpublished details of the dimensional-continuation computation of the 3rd post-Newtonian two-point-mass ADM Hamiltonian are presented.

Keywords: binary systems, equations of motion, point masses, dimensional regularization

1. Introduction

The problem of finding the equations of motion (EOM) of a two-body system within the post-Newtonian (PN) approximation of general relativity is solved up to the 3.5PN order of approximation for the case of compact and nonrotating bodies [by nPN approximation we mean corrections of order $(v/c)^{2n} \sim \big(Gm/(rc^2)\big)^n$ to Newtonian gravity]. The 3PN level of accuracy was achieved only recently. There exist two independent derivations of the 3PN EOM using distributional (Dirac delta's) sources: either ADM-Hamiltonian-based,[1,2] or harmonic-coordinate-based.[3,4] There also exists a third independent derivation of the 3PN EOM in harmonic coordinates using a surface-integral approach.[5]

To cure the self-field divergencies of point particles it is necessary to use some regularization method. It turned out that different such methods applied in $d = 3$ space dimensions were not able to give unique EOM at the 3PN order. Only by em-

*The research of P.J. has been partially supported by the KBN Grant no 1 P03B 029 27.

ploying dimensional continuation was it possible to obtain unambiguous results.[2,4] In this note we review the dimensional-continuation-based derivation of the 3PN two-point-mass ADM Hamiltonian.

2. ADM formalism for 2-point-mass systems in d space dimensions

We use units such that $c{=}16\pi G_{d+1}{=}1$. We work in an asymptotically flat $(d+1)$-dimensional spacetime with Minkowskian coordinates x^0, $\mathbf{x}{\equiv}(x^1,\dots,x^d)$. Particles are labeled by the index $a\in\{1,2\}$; masses, positions, and momenta of the particles are denoted by m_a, $\mathbf{x}_a{\equiv}(x_a^1,\dots,x_a^d)$, and $\mathbf{p}_a{\equiv}(p_{a1},\dots,p_{ad})$, respectively. We also define: $\mathbf{r}_a := \mathbf{x}-\mathbf{x}_a$, $r_a := |\mathbf{r}_a|$, $\mathbf{n}_a := \mathbf{r}_a/r_a$; $\mathbf{r}_{12} := \mathbf{x}_1-\mathbf{x}_2$, $r_{12} := |\mathbf{r}_{12}|$ ($|\mathbf{v}|$ means here the Euclidean length of the d-vector $\mathbf{v}$). The canonical variables of the theory consist of *matter* variables $(\mathbf{x}_a,\mathbf{p}_a)$ and *field* variables (γ_{ij},π^{ij}), where the space metric γ_{ij} is induced by the full space-time metric on the hypersurface $x^0{=}\text{const}$; its conjugate π^{ij} can be expressed in terms of the extrinsic curvature of that hypersurface.

Source terms in the *constraint equations* written down for two-point-mass systems are proportional to the d-dimensional Dirac delta functions $\delta(\mathbf{x}-\mathbf{x}_a)$. We use the ADM gauge defined by the conditions (TT $\equiv$ transverse-traceless):

$$\gamma_{ij} = \left(1+\frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)}\delta_{ij}+h_{ij}^{\mathrm{TT}},\quad \pi^{ii}=0. \tag{1}$$

The field momentum π^{ij} splits into a TT part π^{ij}_{TT} and a rest $\widetilde{\pi}^{ij}$ (traceless but expressible in terms of a vector), $\pi^{ij}=\widetilde{\pi}^{ij}+\pi^{ij}_{\mathrm{TT}}$. If both the constraint equations and the gauge conditions are satisfied, the ADM Hamiltonian can be put into its *reduced* form:

$$H\big(\mathbf{x}_a,\mathbf{p}_a,h_{ij}^{\mathrm{TT}},\pi^{ij}_{\mathrm{TT}}\big) = -\int \mathrm{d}^d x\,\Delta\phi\big(\mathbf{x}_a,\mathbf{p}_a,h_{ij}^{\mathrm{TT}},\pi^{ij}_{\mathrm{TT}}\big). \tag{2}$$

The PN expansion of the reduced Hamiltonian is worked out up to the 3.5PN order:

$$H=\sum_{a=1}^{2} m_a + H_{\mathrm{N}} + H_{1\mathrm{PN}} + H_{2\mathrm{PN}} + H_{2.5\mathrm{PN}} + H_{3\mathrm{PN}} + H_{3.5\mathrm{PN}} + \mathcal{O}\big((v/c)^8\big). \tag{3}$$

3. Dimensional regularization of the 3PN Hamiltonian

In Refs. 1 it was shown that the Riesz-implemented Hadamard regularization of the 3PN two-point-mass Hamiltonian performed in $d=3$ space dimensions gives ambiguous results. The ambiguities were parametrized by two numerical coefficients called ambiguity parameters and denoted by $\omega_{\mathrm{kinetic}}$ and ω_{static}.

Dimensional continuation consists in obtaining the 3-dimensional Hamiltonian as $\lim_{d\to3} H_{3\mathrm{PN}}(d)$, where $H_{3\mathrm{PN}}(d)$ is the Hamiltonian computed in d space dimensions. This can be done straightforwardly if no poles proportional to $1/(d-3)$ arise when $d\to3$ (or if one shows that these poles can be renormalized away, as happens in harmonic coordinates[4]). Reference 2 has shown that out of all terms building up the Hamiltonian density there are ten terms $T_A(d)$, $A = 1, \ldots, 10$, giving rise to poles when $d \to 3$. It was checked that the poles produced by these terms cancel each other, thus $\lim_{d\to 3} H_{3\mathrm{PN}}(d)$ exists. Moreover, it was shown that for all other terms the 3-dimensional regularization give the same results as dimensional continuation.

Let $H^{\mathrm{Had}}_{3\mathrm{PN}}$ be the 3PN Hamiltonian obtained in Refs. 1 by using an Hadamard "partie finie" (Pf) regularization defined in $d = 3$ space dimensions. To correct this Hamiltonian one needs to compute the difference $\Delta H_{3\mathrm{PN}} := \lim_{d\to 3} H_{3\mathrm{PN}}(d) - H^{\mathrm{Had}}_{3\mathrm{PN}}$. Only ten terms T_A contribute to $\Delta H_{3\mathrm{PN}}$, therefore

$$\Delta H_{3\mathrm{PN}} = \lim_{d\to 3} \int \mathrm{d}^d x \sum_{A=1}^{10} T_A(d) - \mathrm{Pf} \int \mathrm{d}^3 x \sum_{A=1}^{10} T_A(3). \tag{4}$$

Below we present three different methods which we used to compute $\Delta H_{3\mathrm{PN}}$. The details of the 2nd and 3rd method were not published so far. Knowing $\Delta H_{3\mathrm{PN}}$ one determines the values of both ambiguity parameters: $\omega_{\mathrm{kinetic}} = 41/24$, $\omega_{\mathrm{static}} = 0$.

1st method. In Ref. 2 $\Delta H_{3\mathrm{PN}}$ was computed by means of the analysis of the local behaviour of the terms T_A around the particle positions $\mathbf{x} = \mathbf{x}_a$.

2nd method. It is possible to compute all d-dimensional integrals in Eq. (4) explicitly. To do this one uses the Riesz formula

$$\int \mathrm{d}^d x \, r_1^{\alpha} \, r_2^{\beta} = \pi^{d/2} \frac{\Gamma\big((\alpha+d)/2\big)\Gamma\big((\beta+d)/2\big)\Gamma\big(-(\alpha+\beta+d)/2\big)}{\Gamma(-\alpha/2)\Gamma(-\beta/2)\Gamma\big((\alpha+\beta+2d)/2\big)} r_{12}^{\alpha+\beta+d}, \tag{5}$$

and the distributional differentiation of homogeneous functions, e.g.,

$$\frac{\partial^2}{\partial x^i x^j} \frac{1}{r_a^{d-2}} = \mathrm{Pf}\Big((d-2)\frac{d\, n_a^i n_a^j - \delta_{ij}}{r_a^d}\Big) - \frac{4\pi^{d/2}}{d\,\Gamma(d/2-1)} \delta_{ij} \delta(\mathbf{x}-\mathbf{x}_a). \tag{6}$$

3rd method. Instead of d-dimensional Dirac distributions δ one uses d-dimensional Riesz kernels δ_{ε_a} to model point particles:

$$\delta(\mathbf{x}-\mathbf{x}_a) = \lim_{\varepsilon_a\to 0} \delta_{\varepsilon_a}(\mathbf{x}-\mathbf{x}_a), \quad \delta_{\varepsilon_a}(\mathbf{x}-\mathbf{x}_a) := \frac{\Gamma\big((d-\varepsilon_a)/2\big)}{\pi^{d/2}\, 2^{\varepsilon_a}\, \Gamma(\varepsilon_a/2)} r_a^{\varepsilon_a - d}. \tag{7}$$

Then one uses the formula (5) to calculate the integrals in Eq. (4) and, at the end of the calculation, one takes the limit $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$. No distributional differentiation is needed.

We have shown that these three methods yield the same final results.

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