Quantum Integrability and Supersymmetric Vacua

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QUANTUM INTEGRABILITY AND SUPERSYMMETRIC VACUA

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Supersymmetric vacua of two dimensional $\mathcal{N}=4$ gauge theories with matter, softly broken by the twisted masses down to $\mathcal{N}=2$, are shown to be in one-to-one correspondence with the eigenstates of integrable spin chain Hamiltonians. Examples include: the Heisenberg $SU(2)$ XXX spin chain is mapped to the two dimensional $U(N)$ theory with fundamental hypermultiplets, the XXZ spin chain is mapped to the analogous three dimensional super-Yang-Mills theory compactified on a circle, the XYZ spin chain and eight-vertex model - are related to the four dimensional theory compactified on $\mathbf{T}^2$. A consequence of our duality is the isomorphism of the quantum cohomology ring of various quiver varieties, such as $T^*\mathrm{Gr}(N,L)$ and the ring of quantum integrals of motion of various spin chains. In all these cases the duality extends to any spin group, representations, boundary conditions, and inhomogeneity, including Sinh-Gordon and non-linear Schrödinger models as well as dynamical spin chains like Hubbard model. These more general spin chains correspond to quiver gauge theories with twisted masses, with classical gauge groups. We give the gauge-theoretic interpretation of Drinfeld polynomials and Baxter operators. In the classical weak coupling limit our results make contact with the construction of H. Nakajima of

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the quantum affine algebra action in the K-theory of quiver varieties. A lift to a four dimensional $\mathcal{N} = 2^*$ theory on $\mathbf{R}^2 \times \mathbf{S}^2$ leads to (instanton-) corrected Bethe equations, thereby possibly expanding the domain of integrable systems beyond the factorized two-body scattering S-matrix approach. We also relate the instanton corrections in the four dimensonal gauge theory to the finite-size effects in the integrable two dimensional field theories.

1. Gauge theories and integrable systems

The dynamics of gauge theory is a subject of long history and the ever growing importance.

In the last fifteen years or so it has become clear that the gauge theory dynamics in the vacuum sector is related to that of quantum many-body systems. A classic example is the equivalence of the pure Yang-Mills theory with gauge group $U(N)$ in two dimensons to the system of N free non-relativistic fermions on a circle. The same theory embeds as a supersymmetric vacuum sector of a (deformation of) $\mathcal{N} = 2$ super-Yang-Mills theory in two dimensions.

A bit less trivial example found in [1] is that the vacuum sector of a certain supersymmetric two dimensional $U(N)$ gauge theory with massive adjoint matter is described by the solutions of Bethe ansatz equations for the quantum Nonlinear Schrödinger equation (NLS) in the N-particle sector. The model of [1] describes the $U(1)$-equivariant intersection theory on the moduli space of solutions to Hitchin's equations [2], just as the pure Yang-Mills theory describes the intersection theory on moduli space of flat connections on a two dimensional Riemann surface. This subject was revived in [3], [4], by showing that the natural interpretation of the results of [1] is in terms of the equivalence of the vacua of the $U(N)$ Yang-Mills-Higgs theory in a sense of [3] and the energy eigenstates of the N-particle Yang system, i.e. a system of N non-relativistic particles on a circle with delta-function interaction. Furthermore, [3],[4] suggested that such a correspondence should be a general property of a larger class of supersymmetric gauge theories in various spacetime dimensions.

Prior to [1] a different connection to spin systems with long-range interaction appeared in two dimensional pure Yang-Mills theory with massive matter [5], [6]. Three dimensional lift of latter gauge theory describes relativistic interacting particles [7], while four dimensional theories lead to elliptic generalizations [8].

In this paper we formulate precisely the correspondence between the two dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and quantum integrable systems in a very general setup. The $\mathcal{N} = 2$ supersymmetric theories have rich algebraic structure surviving quantum corrections [9]. In particular, there is a distinquished class of operators $(\mathcal{O}_A)$, which do not have singularities in their operator product expansion and form a (super)commutative ring, called the chiral ring [9],[10]. These operators commute with some of the nilpotent supercharges of the supersymmetry algebra. The supersymmetric vacua of the theory form a representation of that ring. The space of supersymmetric vacua is thus naturally identified with the space of states of a quantum integrable system, whose Hamiltonians are the generators of the chiral ring. The duality states that the spectrum of the quantum Hamiltonians coincides with the spectrum of the chiral ring. The nontrivial result of this paper is that arguably all quantum integrable lattice models from the integrable systems textbooks correspond in this fashion to the $\mathcal{N} = 2$ supersymmetric *gauge* theories, essentially also from the (different) textbooks. More precisely, the gauge theories which correspond to the integrable spin chains and their limits (the non-linear Schrödinger equation and other systems encountered in [1], [3], [4] being particular large spin limits thereof) are the softly broken $\mathcal{N} = 4$ theories. It is quite important that we are dealing here with the gauge theories, rather then the general $(2, 2)$ models, since it is in the gauge theory context that the equations describing the supersymmetric vacua can be identified with Bethe equations of the integrable world.

At this point we should clarify a possible confusion about the rôle of integrable systems in the description of the dynamics of supersymmetric gauge theories.

It is known that the low energy dynamics of the four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories is governed by the classical algebraic integrable systems [11]. Moreover, the natural gauge theories lead to integrable systems of Hitchin type, which are equivalent to many-body systems [12] and conjecturally to spin chains [13].

We emphasize, however, that the correspondence between the gauge theories and integrable models we discuss in the present paper and in [1],[3],[4] is of a different nature. The low energy effective theory in four dimensions is described by the classical algebraic integrable systems of type [11], while the vacuum states we discuss presently are mapped to the quantum eigenstates of a different, quantum integrable system.

Another possible source of confusion is the emergence of the Bethe ansatz and the spin chains in the $\mathcal{N} = 4$ supersymmetric gauge theory in four dimensions. In the work [14] and its further developments [15] the anomalous dimensions of local operators of the $\mathcal{N} = 4$

supersymmetric Yang-Mills theory are shown (to a certain loop order in perturbation theory) to be the eigenvalues of some spin chain Hamiltonian. The gauge theory is studied in the 't Hooft large N limit. In our story the gauge theory has less supersymmetry, N is finite, and the operators we consider are from the chiral ring, i.e. their conformal dimensions are not corrected quantum mechanically. Our goal is to determine their vacuum expectation values.

The gauge theories we shall study in two dimensions, as well as their string theory realizations, have a natural lift to three and four dimensions, while keeping the same number of supersymmetries, modulo certain anomalies. Indeed, the $\mathcal{N} = 2$ super-Yang-Mills theory in two dimensions is a dimensional reduction of the $\mathcal{N} = 1$ four dimensional Yang-Mills theory (this fact is useful in the superspace formulation of the theory [16]). Instead of the dimensional reduction one can take the compactification on a two dimensional torus. That way the theory will look macroscopically two dimensional, but the effective dynamics will be different due to the contributions of the Kaluza-Klein modes (the early examples of these corrections in the analogous compactifications from five to four dimensions can be found in [17]). This is seen, for example, in the geometry of the (classical) moduli space of vacua, which is compact for the theory obtained by compactification from four to two dimensions (it is isomorphic to the moduli space Bun_G of holomorphic $G_{\mathbf{C}}$-bundles on elliptic curve), and is non-compact in the dimensionally reduced theory. Quantum mechanically, though, the geometry of the moduli space of vacua is more complicated, in particular it will acquire many components. The twisted superpotential is meromorphic function on the moduli space. We shall show that the critical points of this function determine the Bethe roots of the anisotropic spin chain, the XYZ magnet. Its XXZ limit will be mapped to the three dimensional gauge theory compactified on a circle. We thus see a satisfying picture of the elliptic, trigonometric, and rational theories corresponding to the four dimensional, three dimensional and the two dimensional theories respectively.

The gauge theories we consider also have a natural lift to string theory constructions, involving the stacks of D–branes and $NS5$–branes. These constructions make natural the appearence of various ingredients of the algebraic Bethe ansatz used to solve the dual spin chains. In particular, we interpret the $B(\lambda)$ and $C(\lambda)$ operators defined as the matrix elements of the monodromy matrix as the brane creating and annihilating operators in string theory.

We hope that the duality between the gauge theories and the quantum integrable systems we established in this paper can be used to enrich both subjects. In particular, the notions of *special coordinates, topological anti-topological fusion* and so on have not been appreciated so far in the world of quantum integrable systems.

In this note we shall mostly discuss one example, that one relating the XXX spin chain for the $SU(2)$ group, and the $\mathcal{N}=4$ two dimensional theory with the gauge group $U(N)$ and L fundamental hypermultiplet, whose supersymmetry is broken down to $\mathcal{N}=2$ by the choice of the twisted masses. In some limit the theory reduces to the supersymmetric sigma model on the noncompact hyperkähler manifold, that of the cotangent bundle to the Grassmanian $\mathrm{Gr}(N,L)$ of the N-dimensional complex planes in $\mathbf{C}^L$. Our main statement then maps the equivariant quantum cohomology algebra of $T^*\mathrm{Gr}(N,L)$ to the algebra of quantum integrals of motion of the $XXX_{1/2}$ spin chain.

<u>A longer version.</u> This note is a shortened version of [18]. In [18] we give the precise microscopic description of the matter sector, superpotential and twisted superpotential of the theories under consideration. We also explain how one lifts these theories to three and four dimensions. We then compute the twisted effective superpotential $\widetilde{W}^{\mathrm{eff}}(\sigma)$ on the Coulomb branch for the all our models. We then derive the exact equations describing quantum-mechanical supersymmetric ground states:

$$\exp\left(\frac{\partial\widetilde{W}^{\mathrm{eff}}(\sigma)}{\partial\sigma^i}\right)=1 \tag{1.1}$$

We present several examples, and remind the connection to quantum cohomology of various homogeneous spaces, like Grassmanians and flag varieties. We then discuss the theories with $\mathcal{N}=4$ supersymmetric matter content softly broken down to $\mathcal{N}=2$ by the twisted masses. In this case the equations (1.1) are identified with the Bethe equations of the dual quantum integrable systems. Also [18] reviews the methods used to solve exactly spin chains and related quantum integrable systems, like the eight vertex model, Hubbard model, Gaudin model, non-linear Schrödinger system and so on. Finally, after all these preparations we formulate the duality dictionary between the gauge theories and spin chains. We show that the Bethe eigenvectors in quantum integrable systems correspond to the supersymmetric ground states in gauge theory. We identify the so-called Yang-Yang (YY) function of quantum integrable system with the effective twisted superpotential $\widetilde{W}$ of the gauge theory. The vacuum equation (1.1) then coincides with the Bethe equation, while the Hamiltonians correspond to the chiral ring observables. We discuss the naturalness of the gauge theories which we map to quantum integrable systems. We show that the pattern of the twisted masses which at first appears highly fine tuned is in fact the generic pattern of twisted masses compatible with the superpotential of the microscopic theory. We show that the gauge theories with $U(N)$ group map to periodic spin chains, the gauge theories with $SO(N), Sp(N)$ gauge groups are mapped to open spin chains with particular boundary conditions. We show that the A, D, E-type, as well as the supergroup spin chains, with various representations at the spin sites correspond to quiver gauge theories with the $\times_i U(N_i)$ gauge groups. Furthermore, [18] provides the string theory construction of some of these theories. The string theory point of view makes some of the tools of the algebraic Bethe ansatz more transparent. In particular, the raising and the lowering generators of the Yangian algebra are identified with the brane creation and annihilation operators. In [18] we develop the gauge theory/quantum integrable system correspondence further, by looking at the more exotic gauge theories, coming from higher dimensions via a compactification on a sphere, with a partial twist, or via a localization on a fixed locus of some rotational symmetry. We also discuss the relation of our duality to the familiar story of the classical integrability describing the geometry of the space of vacua of the four dimensional $\mathcal{N} = 2$ theories [11].

In [18] the Hamiltonians of the quantum integrable system are identified with the operators of quantum multiplication in the equivariant cohomology of the hyperkähler quotients, corresponding to the Higgs branches of our gauge theories. In particular, the length L inhomogeneous $XXX_{\frac{1}{2}}$ chain (with all local spins equal to $\frac{1}{2}$) corresponds to the equivariant quantum cohomology of the cotangent bundle $T^*Gr(N, L)$ to the Grassmanian $Gr(N, L)$. This result complements nicely the construction of H. Nakajima and others of the action of the Yangians [19] and quantum affine algebras [20] on the classical cohomology and K-theory respectively of certain quiver varieties. Next, [18] applies our results to the two dimensional topological field theories. We discuss various twists of our supersymmetric gauge theories. The correlation functions of the chiral ring operators map to the equivariant intersection indices on the moduli spaces of solutions to various versions of the two dimensional vortex equations, with what is mathematically called the Higgs fields taking values in various line bundles (in the case of Hitchin equations the Higgs field is valued in the canonical line bundle). The main body of [18] has essentially shown that all known Bethe ansatz-soluble integrable systems are covered by our correspondence. However, there are more supersymmetric gauge theories which lead to the equations (1.1) which can be viewed as the deformations of Bethe equations. For example, a four dimensional $\mathcal{N} = 2^*$ theory compactified on $\mathbf{S}^2$ with a partial twist leads to a deformation of the non-linear Schrödinger system with interesting modular properties. Another interesting model comes from the quantum cohomology of instanton moduli spaces and the Hilbert scheme of points.

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2. Grassmanian and the XXX spin chain.

2.1. The gauge theory

Consider the $\mathcal{N} = (2, 2)$ supersymmetric two dimensional theory with the gauge group $U(N)$, which is a compactification of the $N_f = L$, $N_c = N$, four dimensional $\mathcal{N} = 2$ theory

[a] The IHES seminars (Bures-sur-Yvette, June 2007, April 2008), at the IAS Workshop on “Gauge Theory and Representation Theory” (Princeton, November 2007), at the YITP/RIMS conference “30 Years of Mathematical Methods in High Energy Physics ” in honour of Prof. Eguchi's 60th birthday (Kyoto, March 2008), at the London Mathematical Society lectures at Imperial College (London, April 2008), at the conferences on theoretical physics dedicated to the 50th anniversary of IHES (Bures-sur-Yvette, June 2008) and to L. Landau's 100th anniversary (Chernogolovka, June 2008), at the Cargese Summer Institute (Cargese, June 2008), at the Sixth Simons Workshop “Strings, Geometry and the LHC” (Stony Brook, July 2008), at the ENS summer institute (Paris, August 2008)

on a two-torus. In four dimensions this theory has an $SU(N_f)$ global symmetry group, in addition to the $SU(2)$ non-anomalous and $U(1)$ anomalous (for $N_f \neq 2N_c$) R-symmetry groups. We turn on a Wilson loop for these symmetry groups (ignoring the anomaly issue for a moment). The condition of unbroken supersymmetry requires these Wilson loops be flat. In the limit of the vanishing two-torus, if these Wilson loops are scaled appropriately, the resulting two dimensional theory has the so-called twisted mass couplings. The flatness condition means that the twisted masses belong to the complexification of the Lie algebra of the maximal torus of the global symmetry group. Note that there are certain mass couplings, the so-called complex mass terms, which one can turn on already in four dimensions. We first discuss the theory with both complex and twisted masses vanishing.

2.2. The T^*Grassmanian sigma model

Let us analyze the low energy field configurations of the theory whose matter content we just presented. The four dimensional theory and its two dimensional reduction have a superpotential

$$W = \operatorname{tr}_{\mathbf{C}^L} \widetilde{Q}\Phi Q \tag{2.1}$$

where Q is in the representation $(\mathbf{N}, \overline{\mathbf{L}})$ of $U(N) \times SU(L)$, $\widetilde{Q}$ is in $(\overline{\mathbf{N}}, \mathbf{L})$ of $U(N) \times SU(L)$, and Φ is in the adjoint of $U(N)$.

The low energy configurations have vanishing F- and D-terms. The vanishing of the F-terms means:

$$Q\widetilde{Q} = 0\,,\ \Phi Q = 0\,,\ \widetilde{Q}\Phi = 0 \tag{2.2}$$

while the vanishing of the D-terms means, in the presence of the Fayet-Illiopoulos term r:

$$QQ^\dagger - \widetilde{Q}^\dagger\widetilde{Q} + [\Phi, \Phi^\dagger] = r \cdot \mathbf{1}_N \tag{2.3}$$

Finally we identify the solutions to (2.2) and (2.3) which differ by the $U(N)$ gauge transformations. The low energy limit of the gauge theory is a sigma model on the cotangent bundle to the Grassmanian[b]. This is a hyperkähler manifold, as required by the $\mathcal{N} = 4$ supersymmetry in two dimensions.

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[b] Let us assume $r > 0$ (the case $r < 0$ is similar, the case $r = 0$ is complicated and will not be discussed). By taking the absolute squares of the norms of (2.2) and (2.3) we can deduce that $\Phi = 0$, while Q has a maximal rank. Then, $(Q, \widetilde{Q})$ obeying (2.3) with $\Phi = 0$, define a point in the Grassmanian $\mathrm{Gr}(N, L)$, as follows: define a positive definite Hermitian matrix $H = H^\dagger$ as the

The superpotential (2.1) is $SU(L)$ invariant (it is actually $U(L)$ invariant, but $U(1)$ is a part of the gauge group). The maximal torus of $SU(L)$ acts as follows:

$$\left(\widetilde{Q}, Q\right) \mapsto \left(e^{-im}\,\widetilde{Q}\,,\, Q\,e^{im}\right) \tag{2.7}$$

where

$$m = \mathrm{diag}\left(m_1, \ldots, m_L\right)\ , \tag{2.8}$$

with

$$\sum_{a=1}^{L} m_a = 0$$

In addition it is invariant under the $U(1)$ symmetry acting as:

$$e^{+iv} : \left(\widetilde{Q}, \Phi, Q\right) \mapsto \left(e^{-iv}\widetilde{Q}, e^{+2iv}\Phi, e^{-iv}Q\right) \tag{2.9}$$

These symmetries allow us to turn on the twisted masses. We have $L-1$ twisted masses corresponding to the $SU(L)$ symmetry, which we shall parametrize as the mass u which corresponds to the $U(1)$ symmetry (2.9), and L masses $\mu_a u$, which sum up to zero:

$$\sum_{a=1}^{L} \mu_a = 0\ . \tag{2.10}$$

In the sigma model description these symmetries correspond to the isometries of $T^*\mathrm{Gr}(N, L)$.

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unique square root $H = \left(r\mathbf{1}_N + \widetilde{Q}^{\dagger}\widetilde{Q}\right)^{1/2}$. Define:

$$E = H^{-1}Q\,,\ E^{\dagger} = Q^{\dagger}H^{-1} \tag{2.4}$$

Then E defines an orthonormal set of N vectors in $\mathbf{C}^L$:

$$EE^{\dagger} = \mathbf{1}_N \tag{2.5}$$

This is our point in the Grassmanian. Now, given E, the rest of our data is $F = \widetilde{Q}H^{-1}$ obeying $EF = 0$. Indeed, given F, such that $\|F\| < 1$, we can reconstruct H:

$$F^{\dagger}F = \mathbf{1}_N - rH^{-2} \tag{2.6}$$

The matrix $\widetilde{Q}$ defines a point in the cotangent space to the Grassmanian at the point E.

2.3. Supersymmetric ground states

The main subject of our stortу is the space of supersymmetric ground states of the gauge theory. In the supersymmetric sigma model description, which is a kind of a Born-Oppenheimer approximation, the ground states correspond to the cohomology of the target space[d].

For the cotangent bundle of the Grassmanian the cohomology space is isomorphic to the N-th exterior power of the L-dimensional vector space:

$$H^{*}\left(T^{*}\mathrm{Gr}(N,L),\mathbf{C}\right)=\wedge^{N}\mathbf{C}^{L} \tag{2.11}$$

The isomorphism (2.11) is a little bit mysterious since the grading in the cohomology group is not obvious on the right hand side of (2.11). The space $\mathbf{C}^L$ in the right hand side of (2.11) has nothing to do with the space $\mathbf{C}^L$ whose N-dimensional subspaces are parametrized by the Grassmanian $\mathrm{Gr}(N,L)$. Perhaps a bit more geometric description of the cohomology of $T^{*}\mathrm{Gr}(N,L)$ is via the cohomology of the Grassmanian $\mathrm{Gr}(N,L)$ itself. The latter is generated by the Chern classes of the rank N tautological vector bundle E. Let

$$\mathbf{Q}(x)=x^{N}-c_{1}(E)x^{N-1}+c_{2}(E)x^{N-2}-\ldots+(-1)^{N}c_{N}(E) \tag{2.12}$$

be the Chern polynomial of E. The cohomology ring of the Grassmanian $\mathrm{Gr}(N,L)$ is generated by $c_1(E),c_2(E),\ldots,c_N(E)$. Let us now describe the relations. Let $W\approx\mathbf{C}^L$ be the topologically trivial vector bundle over $\mathrm{Gr}(N,L)$. Let $E^{\perp}=W/E$ be the dual tautological bundle. Let $\mathbf{Q}^{\perp}(x)=x^{L-N}-c_1(E^{\perp})x^{L-N-1}+c_2(E^{\perp})x^{L-N-2}+\ldots+(-1)^{L-N}c_{L-N}(E^{\perp})$ be the Chern polynomial of $E^{\perp}$. Then

$$\mathbf{Q}(x)\mathbf{Q}^{\perp}(x)=x^{L} \tag{2.13}$$

defines the relations among the generators of the cohomology ring[c].

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[d] For the non-compact target spaces one should use some kind of L^2-cohomology theory. Most of our discussion will be about the theory with twisted masses, where there are no flat directions.

[c] For example, if $N=1$, then the generator is $\sigma=c_1(E)$, and the relation (2.13) reads: $(x-\sigma)\mathbf{Q}^{\perp}(x)=x^{L}$, which implies $\sigma^{L}=0$, and $c_k(E^{\perp})=(-1)^k\sigma^k$. The cohomology ring is, in this case, the L-dimensional vector space $\mathbf{C}[\sigma]/\sigma^{L}$. In general the cohomology ring is the space of symmetric polynomials in N variables $\sigma_1,\ldots,\sigma_N$, subject to the relations $\sigma_i^L=0$, $i=1,\ldots,N$. A basis in this quotient can be chosen to be the monomial sums: $\mathbf{m}_{\psi}=\sum_{\pi\in\mathcal{S}_N}\prod_{i=1}^{N}\sigma_{\pi(i)}^{\psi_i+i-N-1}$, where $L\geq\psi_1>\psi_2>\ldots>\psi_N\geq 1$. In the quantum cohomology of the Grassmanian itself these relations are modified to $\sigma_i^L=Q$, see [21].

2.4. Chiral ring

Every two dimensional $\mathcal{N} = (2,2)$ supersymmetric field theory comes with an interesting algebra: the so-called chiral ring. It is the cohomology of one of the nilpotent supercharges of the theory in the space of operators.

This algebra, thanks to the state-operator correspondence and the standard supersymmetry arguments, is isomorphic, as a vector space, to the space of the supersymmetric ground states, i.e. to the cohomology of the Grassmanian, in our example. The ring structure, however, need not be isomorphic to the ring structure of the classical cohomology. It is parametrized, in fact, by the Kähler moduli of the target space. The deformed cohomology ring is called the *quantum cohomology*, [10], [22], and is by now well-studied mathematically [23]. In our case this is a one-parametric deformation of the classical cohomology ring of $T^*\mathrm{Gr}$.

In addition, if the target space has isometries, as is the case of the Grassmanian or its cotangent bundle, the theory, and the corresponding chiral algebra can be deformed by the twisted masses. The mathematical counterpart of this theory is called the *equivariant Gromov-Witten theory*, it was introduced in [24].

The chiral algebra of our theory is, therefore, the $U(1) \times SU(L)$ equivariant quantum cohomology of the cotangent bundle to the Grassmanian $T^*\mathrm{Gr}(N, L)$. It can be described using the twisted effective superpotential [18] on the "Coulomb branch" of the theory. It is a function of the scalars σ in the vector multiplet $\mathbf{\Sigma}$ of the gauge group $U(N)$, obtained by integrating out the massive matter fields:

$$\begin{aligned}\widetilde{W}(\sigma) = \sum_{a=1}^{L}\sum_{i=1}^{N} [(\sigma_i - \mu_a u + u/2)\,(\log\,(\sigma_i - \mu_a u + u/2) - 1) + \\ (-\sigma_i + \mu_a u + u/2)\,(\log\,(-\sigma_i + \mu_a u + u/2) - 1)] + \\ + \sum_{i,j=1}^{N} (\sigma_i - \sigma_j - u)\,(\log(\sigma_i - \sigma_j - u) - 1) \\ + 2\pi i t \sum_{i=1}^{N} \sigma_i\end{aligned} \tag{2.14}$$

where $t = r + i\vartheta$ is the linear combination of the Fayet-Illiopoulos term and the theta angle. The quantum cohomology algebra is generated by the symmetric polynomials c_k, s.t.

$$\mathbf{Q}(x) = \prod_{i=1}^{N}(x - \sigma_i) = x^N - c_1 x^{N-1} + c_2 x^{N-2} - \ldots + (-1)^N c_N \tag{2.15}$$

subject to the relations (1.1), which become[e]:

$$\prod_{a=1}^{L} \frac{\sigma_i - \mu_a u + u/2}{\sigma_i - \mu_a u - u/2} = e^{2\pi i t} \prod_{j \neq i} \frac{\sigma_i - \sigma_j + u}{\sigma_i - \sigma_j - u} \tag{2.16}$$

The solutions to (2.16) are the supersymmetric vacua of the gauge theory.

2.5. The XXX spin chain

Fig.1 The XXX spin chain

We shall not present here the XXX spin chain in the way it was originally defined. To save time, we shall proceed directly with its algebraic formulation. The spin chain is a quantum integrable system, whose commuting Hamiltonians are the generators of an abelian subalgebra of a larger, noncommutative associative algebra, called the Yangian. In our story we shall be dealing with the Yangian $Y(\mathbf{gl}_2)$ of the $\mathbf{gl}_2$ Lie algebra. The generators of $Y(\mathbf{gl}_n)$ look like the (negative) loops in $\mathbf{gl}_n$:

$$T_{ij}(x) = \delta_{ij} + \sum_{p=1}^{\infty} t_{ij}^{(p-1)} x^{-p} \tag{2.17}$$

where $i, j = 1, \ldots, n$. The defining relations of the Yangian are:

$$[t_{ij}^{(p+1)}, t_{kl}^{(q)}] - [t_{ij}^{(p)}, t_{kl}^{(q+1)}] = -\left(t_{kj}^{(p)} t_{il}^{(q)} - t_{kj}^{(q)} t_{il}^{(p)} \right) \tag{2.18}$$

[e] In the limit $u \to \infty$ with $m_a = \mu_a u$ finite, upon the redefinition $\sigma_i \to \sigma_i + u/2$, and the renormalization $t \to t + \frac{L}{2\pi i} \log(-u)$ these equations go over to the equivariant quantum cohomology of $\mathrm{Gr}(N, L)$

or, in terms of $T(x)$:

$$[T_{ij}(x), T_{kl}(y)] = -\frac{T_{kj}(x)T_{il}(y) - T_{kj}(y)T_{il}(x)}{x-y} \tag{2.19}$$

It follows from (2.19) that for any diagonal matrix $q = \mathrm{diag}\,(q_1, \ldots, q_n)$ the following operators

$$\tau(x,q) = \sum_{i=1}^{n} q_i T_{ii}(x) \tag{2.20}$$

commute:

$$[\tau(x,q), \tau(y,q)] = 0 \tag{2.21}$$

Given a representation $\mathcal{H}$ of the Yangian $Y(\mathbf{gl}_n)$ one can define a set of commuting quantum Hamiltonians acting in $\mathcal{H}$ as the coefficients of expansion of $\tau(x,q)$ at $x = \infty$:

$$\tau(x,q) = H_0 + H_1 x^{-1} + H_2 x^{-2} + \ldots \tag{2.22}$$

where $H_0 = \sum_{i=1}^{N} q_i$, and

$$[H_l, H_m] = 0\,, \;\; l,m = 1, \ldots \tag{2.23}$$

Whether the resulting quantum integrable system is interesting or not, depends on the representation $\mathcal{H}$.

The spin chains correspond to H which is obtained by the tensor product of the so-called evaluation representations. Let $n = 2$, and let $V_a \approx \mathbf{C}^2$, $a = 1, \ldots, L$ be L copies of the two dimensional representation of the $\mathbf{gl}_2$ Lie algebra. Let

$$\mathcal{H} = \bigotimes_{a=1}^{L} V_a \tag{2.24}$$

Let e_{ij} be the standard matrix with zeroes everywhere except for the i-th column and the j-th row, where it has 1. Each V_a can be viewed as a representation of $Y(\mathbf{gl}_2)$, via:

$$T_{ij}(x)[a] = \delta_{ij} + \frac{e_{ij}}{x - \mu_a} \tag{2.25}$$

where $\mu_a \in \mathbf{C}$ is an arbitrary complex parameter. The representations of the Yangian have these extra parameters, since the commutation relations (2.19) are translation invariant. Consider an operator $L_a(x)$ which acts in the tensor product of V_a and the auxiliary $\mathbf{C}^2$:

$$\mathbf{L}_a(x) = \begin{pmatrix} T_{11}(x)[a] & T_{12}(x)[a] \\ T_{21}(x)[a] & T_{22}(x)[a] \end{pmatrix} = \mathbf{1} + \frac{1}{x-\mu_a} \begin{pmatrix} e_{11}[a] & e_{12}[a] \\ e_{21}[a] & e_{22}[a] \end{pmatrix} \in \mathrm{End}\left(V_a \otimes \mathbf{C}^2\right) \tag{2.26}$$

(the index $[a]$ should not be confused with the superscript (p) in the definition of the expansion modes of the Yangian generators). Note that:

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = P_{12} \tag{2.27}$$

is the permutation matrix acting in $\mathbf{C}^2 \otimes \mathbf{C}^2$. Now let us define the operators $T_{ij}(x) \in \mathrm{End}(\mathcal{H})$ via:

$$\begin{pmatrix} T_{11}(x) & T_{12}(x) \\ T_{21}(x) & T_{22}(x) \end{pmatrix} = \mathbf{L}_1(x)\mathbf{L}_2(x)\ldots\mathbf{L}_L(x) \tag{2.28}$$

where the product is the matrix product in the auxiliary space $\mathbf{C}^2$. More precisely, we should view the operators e_{ij} in (2.26) as acting in $\mathcal{H}$, as:

$$e_{ij}[a] = \mathbf{1} \otimes \ldots \otimes \underset{\mathrm{a'th}\downarrow\ \mathrm{factor}}{e_{ij}} \otimes \ldots \otimes \mathbf{1} \tag{2.29}$$

The commutation relations (2.19) can be written in the so-called R-matrix form, which makes it obvious that (2.28) defines a representation of $Y(\mathbf{gl}_2)$ in $\mathcal{H}$. Since the scaling of q by an overall factor simply multiplies $\tau(x, q)$ by the same factor, one can set $q_1 = 1$ without much loss of the generality.

The parameter $Q = q_2/q_1$ is called the twist parameter. The parameters μ_a are called the impurities. The translation invariance of (2.19) again implies that the shift of all μ_a's by the same amount leads to physically equivalent system.

2.6. Special cases and limits

The Heisenberg spin chain corresponds to $\mu_a = 0$, $Q = 1$.

The higher spins can be obtained by arranging the impurities in a special way and then restricting onto an irreducible submodule. In this way one can get the local spins to be in arbitrary representations of $\mathbf{sl}_2$. This is done by choosing, e.g. $\{\mu_a\} = \{\overline{\mu}_A + k_A\}$, where $k_A = -s_A + 1/2, -s_A + 3/2, \ldots, s_A - 3/2, s_A - 1/2$, where $2s_A \in \mathbf{Z}_+$, $A = 1, \ldots, \ell$. Thus, one can view the spin chain of length ℓ with the local spins $s_1, \ldots, s_\ell$ as a subsystem in the spin $\frac{1}{2}$ chain of the length $L = \sum_A 2s_A$.

Let us now assume that all local spins are taken to infinity, and at the same time the number of spin sites L is also sent to infinity, so that

$$\sum_{a=1}^{L} \frac{1}{s_a} = \frac{1}{c} \text{ finite} \tag{2.30}$$

In this limit the spin chain goes over to the non-linear Schrödinger system, NLS.

Now keep L and s_a finite, but take $\mu_a \to \infty$, instead. In this case the spin chain goes over to the so-called Gaudin system.

2.7. Bethe ansatz

Solution of the quantum integrable system consists of finding the common eigenvectors of the commuting Hamiltonians. There is the following ansatz for the spin chains eigenfunctions:

$$\mathbf{\Psi}_\lambda = T_{12}(\lambda_1)T_{12}(\lambda_2)\dots T_{12}(\lambda_N)\,\Omega \tag{2.31}$$

where $\Omega \in \mathcal{H}$ is the highest weight vector:

$$\begin{aligned} &T_{21}(x)\,\Omega = 0 \\ &T_{11}(x)\,\Omega = r_+(x)\,\Omega \\ &T_{22}(x)\,\Omega = r_-(x)\,\Omega \end{aligned} \tag{2.32}$$

where $r_\pm(x)$ are the easily computed rational functions:

$$r_\pm(x) = \frac{P(x \pm \frac{1}{2})}{P(x)}\,,\ P(x) = \prod_{a=1}^{L}\,(x - \mu_a) \tag{2.33}$$

In fact, the vector Ω is just the tensor product of the spin down states over all L factors V_a. The vector (2.31) is the eigenvector of $\tau(x,q)$ for all x iff λ's solve the following system of equations, the *Bethe equations*:

$$\prod_{a=1}^{L} \frac{\lambda_i - \mu_a + \frac{1}{2}}{\lambda_i - \mu_a - \frac{1}{2}} = Q \prod_{j\neq i} \frac{\lambda_i - \lambda_j + 1}{\lambda_i - \lambda_j - 1} \tag{2.34}$$

Equivalently, the polynomial, the so-called Baxter operator, [25]:

$$\mathbf{Q}(x) = \prod_{i=1}^{N}(x - \lambda_i) \tag{2.35}$$

has to obey the following difference equation:

$$r_+(x)\mathbf{Q}(x-1) + Q\,r_-(x)\mathbf{Q}(x+1) = \epsilon(x)\mathbf{Q}(x) \tag{2.36}$$

where $\epsilon(x)P(x)$ is a degree L polynomial in x, to be determined from (2.36) at the same time as $\mathbf{Q}(x)$. The equations (2.34) imply that the left hand side of (2.36) vanishes at

$x = \lambda_i$ and therefore is divisible by $\mathbf{Q}(x)$. The rational function $\epsilon(x)$ is the eigenvalue of the transfer matrix $\tau(x, q)$.

Finally, the equations (2.34) can be interpreted as the critical point equations for the YY function:

$$
\begin{aligned}
Y(\lambda) = \sum_{a=1}^{L}\sum_{i=1}^{N} \Big[\left(\lambda_i - \mu_a + \tfrac{1}{2}\right)\left(\log\left(\lambda_i - \mu_a + \tfrac{1}{2}\right) - 1\right) + \\
\left(-\lambda_i + \mu_a + \tfrac{1}{2}\right)\left(\log\left(-\lambda_i + \mu_a + \tfrac{1}{2}\right) - 1\right)\Big] + \\
\sum_{i,j=1}^{N} (\lambda_i - \lambda_j - 1)\left(\log(\lambda_i - \lambda_j - 1) - 1\right)
\end{aligned}
\tag{2.37}
$$

2.8. Dictionary

We can now formulate the precise dictionary. The operators (2.15) of the gauge theory map to Baxter operator (2.35), upon the rescaling, $\sigma_i = \lambda_i u$, the twisted superpotential is identified with the YY function, $\widetilde{W}(\sigma) = u\,Y(\sigma/u)$, and the supersymmetric vacua are the Bethe eigenstates. The twisted masses corresponding to the $SU(L)$ symmetry are the impurities, while the twisted mass corresponding to the $U(1)$ symmetry (2.9) set the scale of the twisted masses. The roots of the Drinfeld polynomial $P(x)$ are the masses of the fundamental hypermultiplets. Baxter's equations (2.36) become the Ward identities of the chiral ring:

$$
\left\langle \left[P(x+u/2)\frac{\mathbf{Q}(x+u)}{\mathbf{Q}(x)} + Q\,P(x-u/2)\frac{\mathbf{Q}(x-u)}{\mathbf{Q}(x)} \right]_{-} \right\rangle = 0 \tag{2.38}
$$

where $[\ldots]_{-}$ denotes a negative in x part in the expansion near $x = \infty$.

In the string theory realization u is mapped to the topological string coupling constant. The identification of the Kähler parameter t and the element of the Cartan subalgebra of $\mathbf{sl}_2$ (and the analogous identification in the higher rank case) is a particular case of the famous string realization of McKay duality between the ADE singularities and the Lie groups.

* * *

All these questions are discussed in greater detail in [18].

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