On a class of hamiltonian fiber bundles

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ON A CLASS OF HAMILTONIAN FIBER BUNDLES

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Abstract. We study an interesting class of hamiltonian fiber bundles whose fibers are compact homogeneous symplectic manifolds. This construction establishes relation between the cohomology of the classifying space $B\,\mathrm{Ham}(K/V,\omega)$ and the image of the Matsushima's homomorphism in the cohomology of lattices in semisimple Lie groups of non-compact type.

1. Introduction

1.1. **Symplectic structures on locally homogeneous spaces.** This work is devoted to study of an interesting class of hamiltonian fiber bundles. A fiber bundle

$$(M,\omega) \to P \to B$$

is called symplectic, if its fiber is a symplectic manifold (M,ω), and its structural group is the group of symplectomorphisms $\mathrm{Symp}(M,\omega)$. A symplectic fiber bundle is *hamiltonian*, if the structure group reduces further to the group of hamiltonian symplectomorphisms $\mathrm{Ham}(M,\omega)$. Total spaces of hamiltonian fiber bundles admit coupling forms, that is, differential 2-forms Ω whose cohomology class restricts to the cohomology class $[\omega]$ of symplectic form on the fiber [LMcD]. Hamiltonian fiber bundles are a useful tool in studying topology of the group $\mathrm{Ham}(M,\omega)$. Examples of this approach can be found in [LMcD], [KMcD].

The aim of this paper is to prove the following results.

Theorem 1. *Let G be a noncompact real semisimple connected Lie group with no complex structure and let $K \subset G$ be a maximal compact subgroup. Assume that G is a real form of some complex semisimple Lie group G^c. Let $V \subset K$ be the centralizer of a torus $S \subset K$. Then*

(i) the manifold K/V admits a K-invariant symplectic structure,
(ii) the manifold G/V admits a G-invariant symplectic structure,
(iii) for any cocompact lattice $\Gamma \subset G$ the inclusion $K/V \to \Gamma\backslash G/V$ is a symplectic embedding and

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(iv) the bundle

$$K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K$$

is Hamiltonian with the structure group K.

The proof of this theorem follows from Theorem 4 and Theorem 8 in subsequent sections. The possibility of constructing symplectic forms on locally homogeneous complex manifolds was mentioned in [ABCKT].

1.2. **The Matsushima homomorphism and Hamiltonian characteristic classes.** We denote Lie algebras corresponding to Lie groups $G, H, ...,$ by the corresponding Gothic letters $\mathfrak{g}, \mathfrak{h}, ...$ Assume that G is a real form of a complex semisimple Lie group G^c and let $M \subset G^c$ be a maximal compact subgroup containing K. The homogeneous space M/K is called *the dual of the locally symmetric space* $\Gamma\backslash G/K$.

Let us define the *Matsushima homomorphism*

$$\nu : H^*(M/K;\mathbb{R}) \to H^*(B\Gamma).$$

If a group H acts on a smooth manifold Y, the symbol $\Omega^H(Y)$ denotes the complex of differential forms on Y which are invariant with respect the H-action on Y. It is well known that $\Omega^G(G/K)$ and $\Omega^M(M/K)$ consist of harmonic forms. Therefore

$$\Omega^G(G/K) = H^*(\mathfrak{g}, \mathfrak{k}),\ \Omega^M(M/K) = H^*(\mathfrak{m}_0, \mathfrak{k})$$

where the right-hand side terms denote the relative Lie algebra cohomology (here and in the sequel, as an exception, the Lie algebra of M is denoted by $\mathfrak{m}_0$). One can show that $H^*(\mathfrak{g}, \mathfrak{k}) = H^*(\mathfrak{m}_0, \mathfrak{k})$ (see [O]). Since M/K is compact, the Hodge theory yields $\Omega^M(M/K) = H^*(M, \mathbb{R})$. Hence, one can also identify the cohomology of M/K with the algebra of G-invariant forms on G/K:

$$\Omega^G(G/K) = H^*(M/K, \mathbb{R})$$

Consider the complex $\Omega^\Gamma(G/K)$ of Γ-invariant forms. The projection $G/K \to \Gamma \setminus G/K$ induces the isomorphism $\Omega^\Gamma(G/K) \cong \Omega(\Gamma \setminus G/K))$. Thus, $H^*(\Gamma \setminus G/K, \mathbb{R}) = H^*(\Omega^\Gamma(G/K))$. Since the elements of $\Omega^G(G/K)$ are closed forms, the inclusion $\Omega^G(G/K) \subset \Omega^\Gamma(G/K)$ induces a homomorphism $\nu : H^*(M/K, \mathbb{R}) \to H^*(\Gamma \setminus G/K, \mathbb{R})$. It was proved in [M] that ν is injective in all degrees, and is surjective in degrees $q \leq m(\mathfrak{g})$, where $m(\mathfrak{g})$ is some constant, explicitly determined in terms of $\mathfrak{g}$. It is proved by Okun in [O] that the Matsushima homomorphism is induced by a virtual map $B\Gamma \to M/K$. A virtual map is a map defined on a finite cover.

Theorem 2. *Let $c : B\Gamma := \Gamma\backslash G/K \to BK$ be the classifying map of the bundle*

$$K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K.$$

Then the image of the induced homomorphism $c^* : H^*(BK;\mathbb{R}) \to H^*(\Gamma;\mathbb{R})$ *is equal to the image of the Matsushima homomorphism.*

Proof. Since the groups $K \subset M$ are of equal rank, the inclusion $j : M/K \to BK$ induces the surjection on the real cohomology. Moreover we have that $c^* = \nu \circ j^*$. □

As we mentioned in Theorem 1, the bundle $K/V \to \Gamma\backslash G/V \to B\Gamma$ is Hamiltonian with the structure group K. Hence the classifying map $B\Gamma \to \mathrm{BHam}(K/V)$ factors through BK and it follows from the above result that the induced homomorphism factors through $H^*(M/K)$. That is, the Hamiltonian characteristic classes are contained in the image of the Matsushima homomorphism. In some cases the inclusion is an equality and then we have the following result.

Theorem 3. *Assume that the map* $BK \to B\,\mathrm{Ham}(K/V)$ *induces a surjection on (rational) cohomology. Let* $c : B\Gamma = \Gamma\backslash G/K \to B\,\mathrm{Ham}(K/V)$ *be the classifying map of the bundle*

$$K/V \to \Gamma\backslash G/V \to \Gamma\backslash G/K.$$

Then the image of the induced homomorphism

$$c^* : H^*(B\,\mathrm{Ham}(K/V);\mathbb{R}) \to H^*(\Gamma;\mathbb{R})$$

is equal to the image of the Matsusima homomorphism. In other words, all Matsushima classes are Hamiltonian characteristic classes.

Conjecturally the surjectivity occurs if K is a simple compact group. For example, it is shown in [KMcD] that the action of $SU(n)$ on generalized flag manifolds

$$SU(n)/S(U(n_1) \times \cdots \times U(n_k)),\ n_1 + \cdots + n_k = n$$

induces a surjection of the rational cohomology on the classifying space level. The general case is the subject of our forthcoming paper.

1.3. **Hamiltonian characteristic classes from the coupling class.** Recall that the *coupling class* associated with a Hamiltonian fibration $p : P \to B$ with fibre (M,ω) is a cohomology class $\Omega \in H^2(P)$ that pulls back to the class of the symplectic form on the fibre and $p_!\Omega^{n+1} = 0$, where $\dim M = 2n$, and $p!$ denotes the fiber integration. If $p : P = M_{\mathrm{Ham}((M,\omega))} \to B\,\mathrm{Ham}((M,\omega))$ is the universal Hamiltonian fibration, then the fibre integrals

$$\alpha_k := p_!\Omega^{n+k} \in B\,\mathrm{Ham}(M,\omega)$$

form an infinite family of universal Hamiltonian characteristic classes.

Consider the Hamiltonian bundle as in Theorem 1

$$K/V \to \Gamma\backslash G/V \xrightarrow{\pi} B\Gamma.$$

If $H^2(B\Gamma;\mathbb{R}) = 0$ then the cohomology class of the symplectic form on $\Gamma\backslash G/V$ constructed in Theorem 1 is the coupling class. That is why its top power is nonzero and hence the fibre integral $\pi_!(\Omega^N)$ for $N = \dim G/V$ is nonzero as well. This proves the nontriviality of the class α_k, for $k = \dim B\Gamma$ (see Section 6 for details).

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2. Locally homogeneous complex manifolds

Here we describe a general construction of locally homogeneous compact complex manifolds from [GS]. Let G^c denote a connected complex semisimple Lie group, and $B \subset G^c$ a parabolic subgroup in it. It is known that $X = G^c/B$ is a compact homogeneous algebraic manifold. Let G be a real form of G^c such that $V = G \cap B$ is compact. Then, it can be shown that the G-orbit of $o = B$ is a connected open domain $D = G(o) = G/V \subset X$. Hence, $D = G/V$ is a complex open homogeneous manifold. Let $\Gamma \subset G$ be a discrete subgroup such that $N(\Gamma) \cap V = \{e\}$. Then Γ acts on D properly and discountinuosly, which yields a complex structure on the quotient $Y = \Gamma \setminus D$. Let $M \subset G^c$ denote the maximal compact subgroup in G^c and let $K \subset G$ be the maximal compact subgroup in G. One can choose M and K in a way that

$$M \supset K \supset V = G \cap B.$$

It is shown in [GS] that

$$G \cap B = K \cap B = M \cap B = V.$$

Also, M acts transitively on G^c/B, which shows that $G^{\mathbb{C}}/B = M/V$. Let $\mathfrak{k}$ denote the Lie algebra of K, and $\mathfrak{k}^c$ its complexification. Then $\mathfrak{k}$ is a reductive complex subalgebra of the Lie algebra $\mathfrak{g}^c$ of G^c. It is shown in [GS] that the Lie subgroup $K^c \subset G^c$ corresponding to $\mathfrak{k}^c$, has the property that

$$K^c \cap B \supset Z(K^c).$$

Therefore, $S = K^c/(K^c \cap B)$ is a homogeneous Kähler manifold. The compact subgroup K acts transitively on S, hence, S can be identified with K/V. Finally, we have obtained a fiber bundle

$$K/V \to G/V \to G/K$$

such that both G/V and K/V have complex structures, and all fibers K/V are complex submanifolds of G/V. Moreover, all such K/V are Kähler homogeneous. We can summarize the discussion as follows.

Theorem 4. *Let G^c be a complex semisimple Lie group, which has a real form G with no compact components and such that for some* *parabolic subgroup B, $G \cap B = V$ is compact. If there exists a cocompact lattice $\Gamma \subset G$ such that $N(\Gamma) \cap V = \{e\}$, then the fiber bundle*

$$K/V \to \Gamma \setminus G/V \to \Gamma \setminus G/K \tag{1}$$

has the following properties

(i) $\Gamma \setminus G/V$ *is a closed complex manifold,*
(ii) K/V *is a Kähler homogeneous manifold with respect to the complex structure on the total space* G/V*,*
(iii) the structure group of the bundle is K*.*

Proof. We have already proved (i) and (ii). It remains to show that the structure group of the bundle is K. This follows, since one can interpret the same bundle as associate with the principal bundle

$$K \to \Gamma \setminus G \to \Gamma \setminus G/K$$

with fiber K/V. To show that K acts freely, one uses the fact that Γ acts freely on G/K. Hence, if some element $k \in K$ had fixed point, say Γg, then $\gamma g k = \gamma_1 g$. Since Γ acts freely, $\gamma_1^{-1}\gamma = e$, and, hence, $k = e$. This also shows that the K-action has no isotropy. □

3. Preliminaries on Lie algebras and their root systems

Proofs in this article essentially use the Lie algebra techniques. All necessary information is contained in [OV1], [OV2]. Here we collect notation and results we use.

Let $\mathfrak{g}^c$ be a complex semisimple Lie algebra. We choose a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}^c$ and denote by $\Delta = \Delta(\mathfrak{g}^c, \mathfrak{t})$ the root system of $\mathfrak{g}^c$ with respect to $\mathfrak{t}$. Thus, we have a root space decomposition

$$\mathfrak{g}^c = \mathfrak{t} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha$ are one dimensional root subspaces, and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$, if $\alpha + \beta \in \Delta$, and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0$ otherwise. We say that a subalgebra $\mathfrak{u} \subset \mathfrak{g}^c$ is a Borel subalgebra, if it is a maximal solvable subalgebra in $\mathfrak{g}^c$. A subalgebra $\mathfrak{b} \subset \mathfrak{g}^c$ is called parabolic, if it contains a Borel subalgebra. In the sequel we use the following description of parabolic subalgebras in terms of root systems. We say that a subset $P \subset \Delta$ is closed, if

$$\alpha, \beta \in P, \alpha + \beta \in \Delta \implies \alpha + \beta \in P.$$

By definition, $\mathcal{F}(\mathfrak{t}, P)$ denotes the subalgebra of $\mathfrak{g}^c$ of the form

$$\mathcal{F}(\mathfrak{t}, P) = \mathfrak{t} \oplus \sum_{\alpha \in P} \mathfrak{g}^\alpha.$$

Choose a system $\Pi \subset \Delta$ of primitive roots in Δ. For any subset $M \subset \Pi$, denote by $[M]$ the minimal closed subsystem in Δ generated by M. Let Δ^+ denote a subset of positive roots in Δ. Here and in the sequel we choose and fix one of such subsets. We have the following characterization of parabolic subalgebras in $\mathfrak{g}^c$.

Theorem 5. [OV2], (Theorem 1.4, Chapter 6). *By associating with a subset $M \subset \Pi$ the subalgebra $\mathcal{P}(M)$ of the form*

$$\mathcal{P}(M) = \mathfrak{t} \oplus \sum_{\alpha \in [M] \cup \Delta^+} \mathfrak{g}^\alpha,$$

one obtains a one-to-one correspondence between subsets of Π and a class of conjugate parabolic subalgebras of $\mathfrak{g}^c$. The subalgebra

$$\mathcal{F}(\mathfrak{t}, [M])$$

is a reductive Levi subalgebra of $\mathcal{P}(M)$.

A real Lie subalgebra algebra $\mathfrak{g} \subset \mathfrak{g}^c$ is called a real form of $\mathfrak{g}^c$, if the natural embedding of (real) Lie algebras $\mathfrak{g} \subset \mathfrak{g}^c$ can be extended to an isomorphism of complex Lie algebras $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{g}^c$. There is a one-to-one correspondence between real forms and $\mathbb{C}$-*anti-linear* involutive automorphisms $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$. Thus, if $\mathfrak{g}$ is a real form of $\mathfrak{g}^c$,

$$\mathfrak{g} = (\mathfrak{g}^c)^\sigma,\ \sigma^2 = \mathrm{id},$$

that is, $\mathfrak{g}$ is a set of fixed points of some anti-linear involutive automorphism σ. This correspondence will be used in the sequel. A root space decomposition yields a convenient set of vector space generators of $\mathfrak{g}^c$, which we will all the Chevalley generators. Any semisimple Lie algebra has non-degenerate Killing form, which will be denoted in the sequel by $\mathcal{K}$ in the complex case, and by κ in the real case. For any $\alpha \in \Delta$, one can choose vectors $X_\alpha \in \mathfrak{g}^\alpha$ and $H_\alpha \in \mathfrak{t}$ such that

(i) $\mathcal{K}(X_\alpha, X_\beta) = \delta_{\alpha,-\beta}$, $[X_\alpha, X_{-\alpha}] = H_\alpha$,
(ii) $[X_\alpha, X_\beta] = 0$, if $\alpha \neq -\beta$, and $\alpha + \beta \neq 0$,
(iii) $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$, if $\alpha, \beta, \alpha + \beta \in \Delta$, $N_{\alpha,\beta} \in \mathbb{R}$.

4. Symplectic forms on locally homogeneous complex manifolds

Here we want to show that locally homogeneous closed manifold

$$\Gamma \setminus G/V$$

admits a pseudo-Kähler structure. By definition, this means that $\Gamma \setminus G/V$ admits a pseudo-Riemannian metric g such that $g(Jv, Jw) = g(v, w)$ for all tangent vectors v, w and such that the fundamental form ω determined by $\omega(v, w) = g(v, Jv)$ is closed. Note that this implies the symplecticness of ω. The following conditions are known to be equivalent (see [KN], Theorem 4.3):

(i) $d\omega = 0$, and J is integrable,

(ii) $\nabla_v J = 0$ for any tangent vector v, and the Levi-Civitta connection of the metric g.

Consider the principal bundle

$$K \to G \to G/K$$

and the associated bundle (1). We claim that for the given *G-invariant integrable almost complex structure* J there is a G-invariant pseudo-Riemannian metric g on G/V such that it is preserved by J and $\nabla_v J = 0$ for any $v \in T(G/V)$. By invariance, these structures descend onto $\Gamma \setminus G/V$, as required.

Recall first general facts about invariant metrics, almost complex structures and connections on homogeneous spaces which can be found in [KN] (Prop. 6.2, 6.5, Cor. 3.2 and Theorem 3.3, Chapter 10). Assume that we are given a reductive homogeneous space G/V with the reductive decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m},\ [\mathfrak{v}, \mathfrak{m}] \subset \mathfrak{m}.$$

In the sequel we will also assume that V is connected, which is the case for $V = B \cap G$ [GS].

(i) There is one-to-one correspondence between G-invariant almost complex structures and linear operators

$$J : \mathfrak{m} \to \mathfrak{m},\ J^2 = -\operatorname{Id}_{\mathfrak{m}}$$

such that

$$J[Y, X]_{\mathfrak{m}} = [Y, JX]_{\mathfrak{m}}.$$

(ii) an almost complex structure J from (i) is integrable if and only if

$$[JX, JY]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - J[X, JY]_{\mathfrak{m}} - J[X, Y]_{\mathfrak{m}} = 0.$$

(iii) There is a one-to-one correspondence between pseudo-Riemannian metrics on G/V and a non-degenerate bilinear forms μ on $\mathfrak{m} \times \mathfrak{m}$ such that

$$\mu([Z, X], Y) + \mu(X, [Z, Y]) = 0$$

for any $X, Y \in \mathfrak{m}$ and any $Z \in \mathfrak{v}$.

(iv) There is a one-to-one correspondence between Riemannian connections corresponding to (iii) and bilinear forms

$$\Lambda(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y)$$

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is defined by the equation

$$2\mu(U(X, Y), Z) = \mu(X, [Z, Y]_{\mathfrak{m}}) + \mu([Z, X]_{\mathfrak{m}}, Y).$$

We will derive $\nabla_v J = 0$ from the above data, and our assumptions on the homogeneous space G/V. Note that since V is compact, we can always assume that there is a reductive decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m},\ [\mathfrak{v}, \mathfrak{m}] \subset \mathfrak{m}.$$

This is known from [KN] (Example 2.1, Chapter 10). Use the formula for the covariant derivative $\nabla_X J$ and the Lie derivative $\mathcal{L}_X$ from [McDS]:

$$(\nabla_X J)Y = (\mathcal{L}_X J)Y + \nabla_{JY} X - J\nabla_Y X.$$

Since in case of the homogeneous space G/V the corresponding formulas involve only left-invariant vector fields generated by vectors $X \in \mathfrak{m}$, one concludes that

$$\mathcal{L}_X J = 0.$$

Indeed, X has the flow generated by $\exp tX \subset G$, and J is G-invariant. Hence the cited formula together with the previous considerations yields

$$(\nabla_X J)Y = \frac{1}{2}[JY, X]_{\mathfrak{m}} - \frac{1}{2}J[Y, X]_{\mathfrak{m}} + U(JY, X) - JU(Y, X).$$

This gives the following summary.

Proposition 1. *The complex homogeneous space G/V admits an invariant pseudo-Kähler structure whenever there exists a non-degenerate bilinear form $\mu : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ such that*

(i) $\mu([Z, X], Y) + \mu(X, [Z, Y]) = 0$ *for all* $X, Y \in \mathfrak{m}, Z \in \mathfrak{v}$;

(ii) $\mu(JX, JY) = \mu(X, Y)$;

(iii) a bilinear form $U : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ determined by the equation

$$2\mu(U(X, Y), Z) = \mu(X, [Z, Y]_{\mathfrak{m}}) + \mu([Z, X]_{\mathfrak{m}}, Y)$$

satisfies the equality

$$\frac{1}{2}[JY, X]_{\mathfrak{m}} - \frac{1}{2}J[Y, X]_{\mathfrak{m}} + U(JY, X) - JU(Y, X) = 0$$

for any $X, Y \in \mathfrak{m}$.

Proposition 2. *Under the assumptions on G/V, there exists a bilinear form $g : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ satisfying the assumptions of Proposition 1. Hence, there is a G-invariant pseudo-Kähler metric on G/V.*

Proof. The proof is divided into three steps. We will use the following formulas and notation.

(i) Recall that $\mathcal{K}$ denotes the Killing form of $\mathfrak{g}^c$, and κ the Killing form of $\mathfrak{g}$.

(ii) Let $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$ denote the antilinear involutive automorphism of $\mathfrak{g}^c$.

(iii) Let h_σ denote the hermitian form on $\mathfrak{g}^c$ defined by

$$h_\sigma(A, B) = -\mathcal{K}(A, \sigma(B)).$$

(iv) the following relations hold [OV1, OV2]:

$$h_\sigma|_{\mathfrak{g}} = \kappa,\ \mathcal{K}|_{\mathfrak{g}} = \kappa.$$

Step 1. We begin with the proof of the following fact.
Consider the orthogonal decomposition

$$\mathfrak{g}^c = \mathfrak{b} \oplus \mathfrak{m}^c$$

with respect to h_σ. Let

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$$

be the orthogonal decomposition with respect to κ. Then $\mathfrak{m}$, as a real vector space, is spanned by vectors

$$X + \sigma(X),\ \ X \in \mathfrak{m}^c.$$

The proof of the latter statement goes as follows. For $A \in \mathfrak{v}, Z = X + \sigma(X) \in \mathfrak{m}$ we have

$$\kappa(A, Z) = \kappa(A, X + \sigma(X)) = h_\sigma(A, X + \sigma(X)) =$$

$$h_\sigma(A, X) + h_\sigma(A, \sigma(X)) = 0$$

The latter follows, since $h_\sigma(A, X) = 0$ by the choice of $\mathfrak{m}$, while $h_\sigma(A, \sigma(X)) = 0$, since h_σ is hermitian. Indeed,

$$h_\sigma(A, \sigma(X)) = \overline{h_\sigma(\sigma(X), \sigma(A))} =$$

$$-\overline{\mathcal{K}(\sigma(X), A))} = -\overline{\mathcal{K}(A, \sigma(X))} = \overline{h_\sigma(A, X)}.$$

Here we used also $\sigma(A) = A$. Hence

$$\langle X + \sigma(X), iX + \sigma(iX\rangle_{\mathbb{R}} \subset \mathfrak{m}.$$

Summarizing, we write ($\mathfrak{m}_1$ considered as a real vector space)

$$\mathfrak{m}_1 = \langle X + \sigma(X)\,, X \in \mathfrak{m}^c\rangle.$$

Hence, one can define a surjection

$$\varphi : \mathfrak{m}^c \to \mathfrak{m}_1,\ \mathfrak{m}_1 \subset \mathfrak{m},$$

by the rule

$$\varphi(X) = X + \sigma(X),\ X \in \mathfrak{m}^c.$$

Now, if φ were injective, by dimensional reasons, the only possibility would be

$$\varphi(\mathfrak{m}^c) = \mathfrak{m}.$$

If this were not the case, $\varphi(X) = 0$, for $X \in \mathfrak{m}^c$. We will show that this is impossible. Recall the following facts from [GS] (p.263-265):

(i) Let $\mathfrak{k}$ be the maximal compact subalgebra in $\mathfrak{g}$, $\mathfrak{k}^c$ its complexification, $\mathfrak{v} = \mathfrak{g} \cap \mathfrak{b}$, and
$$\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$$
an $\operatorname{ad}\mathfrak{k}^c$-invariant complement. Let $\mathfrak{p} = \mathfrak{p}^c \cap \mathfrak{g}$. Then
$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$
is a Cartan decomposition,
$$\mathfrak{m}_0 = \mathfrak{k} \oplus i\mathfrak{p}$$
is a compact real form of $\mathfrak{g}^c$ which contains $\mathfrak{v}$.

(ii) If σ and τ denote, the conjugations with respect to $\mathfrak{g}$ and $\mathfrak{m}_0$, then
$$\sigma\tau = \tau\sigma,$$
and $\theta = \sigma\tau$ is an involutive true automorphism of $\mathfrak{g}^c$.

(iii) $\mathfrak{k}^c$ and $\mathfrak{p}^c$ are $(+1)$ and (-1)-eigenspaces of θ,

(iv) Root spaces of $\mathfrak{g}^c$ are contained either in $\mathfrak{k}^c$ or in $\mathfrak{p}^c$, which implies
$$\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}},$$
$$\mathfrak{k}^c = \mathfrak{t}^c \oplus \sum_{\alpha\in\Delta_{\mathfrak{k}}} \mathfrak{g}^\alpha$$
$$\mathfrak{p}^c = \sum_{\alpha\in\Delta_{\mathfrak{p}}} \mathfrak{g}^\alpha.$$

(v) Under the choices made,
$$\tau(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}.$$

(vi) $\mathfrak{m}^c = \sum_{\alpha\in\Delta_1} \mathfrak{g}^\alpha$, where $\Delta_1 = \Delta \setminus [M]$. The latter follows from the fact that root space decomposition is orthogonal with respect to h_σ form.

We see that the above properties contradict the possibility
$$\sigma(X_\alpha) + X_\alpha = 0.$$
Indeed, if $\alpha \in \Delta_{\mathfrak{k}}$, then $\theta(X_\alpha) = X_\alpha$. But
$$X_\alpha = \theta(X_\alpha) = \sigma\tau(X_\alpha) = \sigma(X_{-\alpha}).$$
Applying σ to both sides of the above equality, and using involutivity, one obtains
$$\sigma(X_\alpha) = X_{-\alpha}.$$
In the same way, if $\alpha \in \Delta_{\mathfrak{p}}$, equality $\theta(X_\alpha) = -X_\alpha$ implies
$$\sigma(X_\alpha) = -X_{-\alpha}.$$

Let $X = X_{\alpha_1} + \cdots + X_{\alpha_s}$ be in the kernel of φ, that is $\sigma(X) + X = 0$. Then

$$(\sigma(X_{\alpha_1}) + X_{\alpha_1}) + \cdots + (\sigma(X_{\alpha_s}) + X_{\alpha_s}) = 0.$$

Applying $\theta = \tau\sigma$ to the above equation, we get

$$\tau(X_{\alpha_1}) + \theta(X_{\alpha_1}) + \cdots + \tau(X_{\alpha_s}) + \theta(X_{\alpha_s}) = 0.$$

This is the same as

$$\pm X_{-\alpha_1} \pm X_{\alpha_1} \pm \cdots \pm X_{-\alpha_s} \pm X_{\alpha_s} = 0$$

which implies all X_{α_j} being zero.

Step 2 (constructing a pseudo-riemannian metric). Now we construct a bilinear form $g : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ which satisfies the required properties to determine a pseudo-Kähler structure on G/V. This follows from the following statement.

Let there be given decompositions

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m},\ \mathfrak{g}^c = \mathfrak{b} \oplus \mathfrak{m}^c$$

as in Step 1. Suppose there is an $\mathbb{R}$-linear map $\psi : \mathfrak{m} \to \mathfrak{m}^c$ such that

$$\psi \circ J = i \circ \psi,\ \operatorname{Ad} s \circ \psi = \psi \circ \operatorname{Ad} s,\ \text{for any } s \in V.$$

Consider the real bilinear form

$$g : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R},\ g(X, Y) = \operatorname{Re} h_\sigma(\psi(X), \psi(Y)).$$

Then

(i) g is a real part of hermitian form $\tilde{h}_\sigma = h_\sigma \circ \psi$;
(ii) g is $\operatorname{Ad} V$-invariant;
(iii) the real bilinear operator $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ determined by the equation

$$2g(U(X, Y), Z) = g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y)$$

has the property

$$U(JX, Y) = JU(X, Y).$$

Here is the proof of the statement. Property (i) follows from definition. To prove (ii), write

$$g(\operatorname{Ad} s(X), \operatorname{Ad} s(Y)) = \operatorname{Re} h_\sigma(\psi(\operatorname{Ad} s(X)), \psi(\operatorname{Ad} s(Y)))$$

$$= \operatorname{Re} h_\sigma(\operatorname{Ad} s(\psi(X)), \operatorname{Ad} s(\psi(Y)))$$

$$= \operatorname{Re} K(\operatorname{Ad} s(\psi(X)), \sigma(\operatorname{Ad} s(\psi(Y)))$$

$$= \operatorname{Re} K(\operatorname{Ad} s(\psi(X)), \operatorname{Ad} s(\sigma(\psi(Y)))$$

$$= \operatorname{Re} K(\psi(X), \sigma(\psi(Y)) = \operatorname{Re} h_\sigma(\psi(X), \psi(Y)) = g(X, Y).$$

In due course, we used the equality $\sigma \circ \operatorname{Ad} s = \operatorname{Ad} s \circ \sigma$, which follows from the fact that $\mathfrak{g}$ is the set of fixed points of σ, and also the $\operatorname{Ad} V$-invariance of the Killing form.

To prove (iii), consider the hermitian form $\tilde{h}_\sigma$ on the complex vector space $(\mathfrak{m}, J)$. Define a *real* bilinear operator $\tilde{U} : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ by the formula

$$2\tilde{h}_\sigma(\tilde{U}(X, Y), Z) = \tilde{h}_\sigma(X, [Z, Y]_\mathfrak{m}) + \tilde{h}_\sigma([Z, X]_\mathfrak{m}, Y).$$

Note that the non-degeneracy of $\tilde{h}_\sigma$ ensures the existence of $\tilde{U}$, but the right-hand side is only $\mathbb{R}$-linear, since $\mathfrak{g}$ is not complex as a Lie algebra. However, the latter definition yields

$$2 \operatorname{Re} \tilde{h}_\sigma(\tilde{U}(X, Y), Z) = \operatorname{Re} \tilde{h}_\sigma(X, [Z, Y]_\mathfrak{m}) + \operatorname{Re} \tilde{h}_\sigma([Z, X]_\mathfrak{m}, Y)$$

$$2 \operatorname{Im} \tilde{h}_\sigma(\tilde{U}(X, Y), Z) = \operatorname{Im} \tilde{h}_\sigma(X, [Z, Y]_\mathfrak{m}) + \operatorname{Im} \tilde{h}_\sigma([Z, X]_\mathfrak{m}, Y).$$

It follows that $\tilde{U} = U$, and for U both equalities for real and imaginary parts, hold. Hence

$$2g(JU(X, Y), Z) = 2 \operatorname{Re} h_\sigma(\psi(JU(X, Y), \psi(Z)) =$$

$$2 \operatorname{Re} h_\sigma(i\psi(U(X, Y), \psi(Z)) = 2 \operatorname{Re} i h_\sigma(\psi(U(X, Y)), \psi(Z))) =$$

$$-2 \operatorname{Im} h_\sigma(\psi(U(X, Y)), \psi(Z)) = -2 \operatorname{Im} \tilde{h}_\sigma(U(X, Y), Z).$$

On the other hand,

$$2g(U(JX, Y), Z) = g(JX, [Z, Y]_\mathfrak{m}) + g([Z, JX]_\mathfrak{m}, Y) =$$

$$2 \operatorname{Re} h_\sigma(\psi(JX), \psi[Z, Y]_\mathfrak{m}) + 2 \operatorname{Re} h_\sigma(\psi([Z, JX]_\mathfrak{m}), \psi(Y)) =$$

$$2 \operatorname{Re} h_\sigma(i\psi(X), \psi([Z, Y]_\mathfrak{m})) + 2 \operatorname{Re} h_\sigma(\psi(J[Z, X]_\mathfrak{m}, \psi(Y)) =$$

$$-2 \operatorname{Im} h_\sigma(\psi(X), \psi([Z, Y]_\mathfrak{m})) - 2 \operatorname{Im} h_\sigma(\psi([Z, X]_\mathfrak{m}), \psi(Y)) =$$

$$-2 \operatorname{Im} \tilde{h}_\sigma(X, [Z, Y]_\mathfrak{m}) - 2 \operatorname{Im} \tilde{h}_\sigma([Z, X]_\mathfrak{m}, Y).$$

Here we used the equality

$$[Z, JX]_{\mathfrak{m}} = J[Z, X]_{\mathfrak{m}}$$

which follows from the invariance of J. Finally,

$$g(JU(X,Y), Z) = g(U(JX,Y), Z).$$

This proves the statement.

Step 3.*Under the conditions of Proposition 2, the complex structure $J : \mathfrak{m} \to \mathfrak{m}$ induced by the complex structure on G^c/B has the properties:*

(i) it is Ad V*-invariant;*

(ii) the map $\psi : \mathfrak{m} \to \mathfrak{m}^c$, $\psi(X) = X + \sigma(X)$ is $((J, i)$-linear;

(iii) $\psi \circ \operatorname{Ad} s = \operatorname{Ad} \circ \psi$ *for any $s \in V$.*

Now we prove the statement of Step 3. Define $J : \mathfrak{m} \to \mathfrak{m}$ by the formula

$$J(X + \sigma(X)) = iX + \sigma(iX).$$

Clearly, $J^2 = -\operatorname{id}$. Also,

$$\psi(J(X + \sigma(X)) = \psi(iX + \sigma(iX) = iX = i\psi(X + \sigma(X)).$$

Now we want to show that this definition yields the same complex structure on $\mathfrak{m}$ which is induced by the complex structure on G/V inherited from the complex structure on G^c/B. Note that the latter is obtained as follows. Identifying $T_o(G/V)$ with $\mathfrak{g}/\mathfrak{v}$, and $T_o(G^c/B)$ with $\mathfrak{g}^c/\mathfrak{b}$, we see that the latter can be identified with the complements in the direct sum decompositions

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}, \ \ \mathfrak{g}^c = \mathfrak{b} \oplus \mathfrak{m}.$$

Then, if $p : \mathfrak{g}^c \to \mathfrak{m}$ denote the projection, one has

$$J(A) = p(iA), \ A \in \mathfrak{m}.$$

Choose the Chevalley generators $\{X_\alpha, \ \alpha \in \Delta\}$ of $\mathfrak{g}^c$. Recall that

$$\sigma(X_\alpha) = X_{-\alpha}, \ \ \alpha \in \Delta, \text{if } \alpha \ \text{ is a compact root}$$

and

$$\sigma(X_\alpha) = -X_{-\alpha}, \ \text{ if } \alpha \text{ is a noncompact root}$$

Let $X_\alpha + \sigma(X_\alpha) \in \mathfrak{m}$, where $X_\alpha \in \mathfrak{m}^c$. Note that we have shown that $\mathfrak{m} = \langle X_\alpha + \sigma(X_\alpha)\rangle$, and that $\mathfrak{g}^c$ has the compatible root space decomposition into root spaces belonging either to $\mathfrak{b}$, or to $\mathfrak{m}^c$. Hence, if we choose $X_\alpha + \sigma(X_\alpha) \in \mathfrak{m}$, then

$$J(X_\alpha + \sigma(X_\alpha)) = p(iX_\alpha + i\sigma(X_\alpha)).$$

Hence, writing

$$iX_\alpha + i\sigma(X_\alpha) = \hat{B} + \sigma(Z) + Z, \hat{B} \in \mathfrak{b}, Z \in \mathfrak{m}^c$$

and

$$\hat{B} = X_{\alpha_1} + \cdots + X_{\alpha_k},\ Z = X_{\beta_1} + \cdots + X_{\beta_s},\ \alpha_l \in [M],\ \beta_j \in \Delta_1,$$

we see that there is only one way to represent $iX_\alpha + \sigma(X_\alpha)$ as a sum of $\hat{B}$ and $\sigma(Z) + Z$, namely

$$iX_\alpha + i\sigma(X_\alpha) = \hat{X}_{-\alpha} + (\sigma(\tilde{X}_\alpha) + \tilde{X}_\alpha),$$

and, moreover, $\hat{X}_{-\alpha}$ necessarily belongs to $\mathfrak{b}$. Assume that $\alpha \in \Delta_1$ is noncompact. Then $\sigma(X_\alpha) = X_{-\alpha}$. Then

$$iX_\alpha + i\sigma(X_\alpha) = iX_\alpha + iX_{-\alpha} = iX_\alpha + \sigma(iX_\alpha) + 2iX_{-\alpha}.$$

In the same way, in the compact case,

$$iX_\alpha + \sigma(X_\alpha) = iX_\alpha - iX_{-\alpha} = iX_\alpha + \sigma(iX_\alpha) - 2iX_{-\alpha}.$$

The latter equalities show that J defined at the beginning of proof, coincides with the one induced by the complex structure on G^c/B. Let us check that it is $\operatorname{Ad} V$-invariant.

$$\operatorname{Ad} sJ(X + \sigma(X)) = \operatorname{Ad} s(iX + \sigma(iX)) =$$

$$\operatorname{Ad} s(iX) + \operatorname{Ad} s\sigma(iX)) = i\operatorname{Ad} s(X) - i\sigma(\operatorname{Ad} s(X)).$$

On the other hand

$$J\operatorname{Ad} s(X + \sigma(X)) = J(\operatorname{Ad} s(X) + \operatorname{Ad}(\sigma(X))) =$$

$$J(\operatorname{Ad} s(X) + \sigma(\operatorname{Ad} s(X))) = \operatorname{Ad} s(iX) + \sigma(i\operatorname{Ad} s(X))) =$$

$$i\operatorname{Ad} s(X) - i\sigma(\operatorname{Ad} s(X)).$$

We have used here the fact that σ in an involutive antilinear automorphism of the Lie algebra $\mathfrak{g}^c$, which yielded the possibility of changing the order in the formulas. Now, we will check (ii). Since V is connected, it is sufficient to check (ii) on the Lie algebra level. Let $S \in \mathfrak{v}$.

$$\operatorname{ad} S \circ \psi(X + \sigma(X)) = [S, X].$$

$$\psi \circ \operatorname{ad} S(X + \sigma(X)) = \psi([S, X] + [S, \sigma(X)] =$$

$$\psi([S, X] + \sigma([S, X]) = [S, X].$$

In calculation, we have used $\sigma(S) = S$ for any $S \in \mathfrak{v}$, as well as the inclusion

$$\mathfrak{v} \subset \mathfrak{b}$$

which implies $[\mathfrak{v}, \mathfrak{m}^c] \subset \mathfrak{m}^c$. The proof of the statement is complete.

The completion of proof of Proposition 2. By Proposition 1, we need to construct a bilinear form $\mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$ satisfying conditions (i)-(iii). By Steps 1, and 3, the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$ with $\mathfrak{m}$ generated by vectors $X + \sigma(X)$ admits a complex structure $J : \mathfrak{m} \to \mathfrak{m}$, which is $\mathrm{Ad}\, V$-invariant and which yields the complex structure on G/V inherited from G^c/B. By Step 2, there is a non-degenerate bilinear form $g : \mathfrak{m} \times \mathfrak{m} \to \mathbb{R}$, which is $\mathrm{Ad}\, V$-invariant, and satisfies property (iii) of Proposition 1. It has property (ii) of Proposition 1, since by Step 2, it is a real part of a hermitian form $\tilde{h}_\sigma$. Property (i) of Proposition 1 is a consequence of $\mathrm{Ad}\, V$-invariance of g ([KN], Corollary 3.2, Chapter 10).

□

5. Kähler fibers K/V

The aim of this section is a description of all possible fibers K/V in fiber bundles determined by locally homogeneous complex manifolds. For convenience of reference, we recall several known facts on homogeneous symplectic manifolds and root systems of complex Lie algebras.

Theorem 6. [B] *Let either G/U be homogeneous Kähler, or G be compact, and G/U be symplectic. Then U is compact, connected, and equal to the centralizer of a torus of G. Conversely, let G be compact semisimple and U be the centralizer of a torus. Then G/U is homogeneous Kähler and algebraic.*

In the sequel we need the following characterization of centralizers of abelian Lie subalgebras.

Theorem 7. [OV2] (Theorem 1.3, Chapter 6). *Let $M \subset \Pi$ be any subset of Π. Then the subalgebra $\mathcal{F}(\mathfrak{t}, [M])$ is of the form*

$$\mathcal{F}(\mathfrak{t}, [M]) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$$

for some subspace $\mathfrak{c} \subset \mathfrak{g}^c$. Conversely, any subalgebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$, where $\mathfrak{c} \subset \mathfrak{t}$, is of that form.

Our main result now reads as follows.

Theorem 8. *Let K be a compact Lie group which can be embedded as a maximal compact subgroup in a semisimple real Lie group G without compact components, and without complex structure. Let V be any closed subgroup which is a centralizer of some torus in K. Then*

(i) K/V is Kähler;

(ii) there exists an associate fiber bundle

$$K/V \to \Gamma \setminus G/V \to \Gamma \setminus G/K$$

such that the total space $\Gamma \setminus G/V$ is symplectic, and the inclusion of the Kähler fiber K/V is symplectic as well.

Proof. The fiber K/V is Kähler by Theorem 6. In view of Proposition 2, to prove symplecticness of $\Gamma \setminus G/K$, it is sufficient to show that if $V = Z_K(S)$ for some torus $S \subset K$, there exists a parabolic subgroup B in G^c such that $G \cap B = V$. Note that it is also sufficient to proceed on the Lie algebra level, and we will do it in the sequel. Let $\mathfrak{k}$ be the Lie algebra of K. Consider its complexification $\mathfrak{k}^c$. Note that

$$\mathfrak{v} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \Rightarrow \mathfrak{v}^c = \mathfrak{z}_{\mathfrak{k}^c}(\mathfrak{a}^c), \mathfrak{a}^c \subset \mathfrak{t}.$$

Therefore, by Theorem 7

$$\mathfrak{v}^c = \mathcal{F}(\mathfrak{t}, [M]),$$

for some Cartan subalgebra chosen as a common subalgebra in $\mathfrak{k}^c$ and $\mathfrak{g}^c$ and being the complexification of some maximal abelian subalgebra in $\mathfrak{v}$. Recall the decomposition $\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$ with respect to the sets $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$ of compact and non-compact roots. Since the algebra $\mathfrak{v}$ is compact, we know that

$$\mathfrak{v}^c = \mathfrak{t} \oplus \sum_{\alpha \in [M_{\mathfrak{k}}]} \mathfrak{g}^{\alpha},$$

where $M_{\mathfrak{k}} \subset \Pi_{\mathfrak{k}}$ is a subset of the set of compact primitive roots. Define

$$\mathcal{P}(M) = \mathfrak{t} \oplus \sum_{\alpha \in [M]_{\mathfrak{k}}} \mathfrak{g}^{\alpha} + \sum_{\beta \in \Delta^+} \mathfrak{g}^{\beta}.$$

By definition, the latter is a parabolic subalgebra. Since

$$\mathfrak{k}^c = \mathfrak{t} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}}} \mathfrak{g}^{\alpha},$$

we see that

$$\mathfrak{k}^c \cap \mathcal{P}(M) = \mathfrak{v}^c.$$

It follows that

$$\mathfrak{v} = \mathfrak{k} \cap \mathcal{P}(M).$$

The latter follows from the obvious inclusion $\mathfrak{v} \subset \mathfrak{k} \cap \mathcal{P}(M)$, the equality for complexifications, and dimensional reasons. The latter equality implies

$$\mathfrak{g} \cap \mathcal{P}(M) = \mathfrak{k} \cap \mathcal{P}(M).$$

The latter can be shown as follows. Recall that $\mathfrak{g}$ is a subalgebra of fixed points of the involutive anti-linear automorphism of $\mathfrak{g}^c$. We have already used the fact that $\sigma(X_\alpha) = X_{-\alpha}$ for compact roots α, and $\sigma(X_\alpha) = -X_{-\alpha}$. Hence, if $X \in \mathfrak{g} \cap \mathcal{P}(M)$, it is represented as a sum

$$X = X_{\alpha_1} + X_{-\alpha_1} + \cdots + X_{\alpha_k} + X_{-\alpha_k}$$

where α_i must be compact. Thus, necessarily, $\alpha_i \in [M_{\mathfrak{k}}]$, and the proof follows.

Finally, we want to prove symplecticness of the fiber inclusion. Let K^c be a complex Lie subgroup in G^c corresponding to $\mathfrak{k}^c$. Note that K acts transitively on $K^c/K^c \cap B$. Hence, there is a diffeomorphism $K/V \cong K^c/K^c \cap B$ determined by inclusion of K into K^c. Moreover, the equality $\mathfrak{v}^c = \mathfrak{k}^c \cap \mathcal{P}(M)$ enables one to get the decomposition

$$\mathfrak{k} = \mathfrak{v} \oplus \mathfrak{n}$$

compatible with the decomposition $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{m}$, that is

$$\mathfrak{n} \subset \mathfrak{m}, \;\; J(\mathfrak{n}) \subset \mathfrak{n}.$$

Also, h_σ, restricted to $\mathfrak{k}^c$ will produce Kähler metric on K/V. To do this, one simply repeats the proof of Proposition 2, taking $\mathfrak{k}^c$ and $\mathfrak{k}$ instead of $\mathfrak{g}^c$ and $\mathfrak{g}$, and $\mathcal{P}(M)$ instead of $\mathfrak{b}$. The necessary change is only to work with compact roots instead of all roots. Also, there is one more small change caused by the fact that $\mathfrak{k}^c$ is not always semisimple. However, it is reductive, and $\mathfrak{k}^c$ and $\mathfrak{v}^c$ have common center, which means that the argument for $\mathfrak{n}$ goes through.

□

6. Applications to hamiltonian characteristic classes

Here we present an example of the use of general results obtained in this article. For any fiber bundle

$$M \longrightarrow P \xrightarrow{p} B$$

we denote by $p_!$ the fiber integration $p_! : H^*(P) \to H^*(M)$ [GuS]. We begin with the following easy observation.

Proposition 3. *Let $(M, \omega) \to E \to B$ be a Hamiltonian fibration. Suppose that*

(i) the fibration admits a compatible symplectic form. That is there is a symplectic form Ω on E such that it pulls back to the symplectic form on each fibre.

(ii) the second Betti number of the base is zero and $\dim B = 2k$

Then the Hamiltonian characteristic class $\alpha_k \in \mathrm{BHam}(M, \omega)$ is non-trivial.

Proof. Let $c : B \to \mathrm{BHam}(M, \omega)$ be the classifying map. Since the second Betti number of the base is zero the cohomology class of the compatible symplectic form is actually the coupling class. Since it is symplectic we have that

$$\langle c^*(\alpha_k), [B] \rangle = \int_B p_!(\Omega^{n+k}) = \int_E \Omega^{n+k} \neq 0$$

which completes the proof. □

This observation leads to the following results on the non-vanishing of certain hamiltonian characteristic classes.

Theorem 9. *Let $G \subset G^c$ be a semisimple Lie group which is a real form of a complex Lie group G^c. Let $\Gamma \subset G$ be a cocompact lattice such that $H^2(\Gamma; \mathbb{R}) = 0$. Let $(M, \omega) = K/V$ be a closed homogeneous symplectic manifold, where $K \subset G$ is a maximal compact subgroup with $H^1(K; \mathbb{R}) = 0$ and $V = B \cap G$, where $B \subset G^c$ is a parabolic subgroup. Then the characteristic class $\alpha_k \in H^{2k}(\mathrm{BHam}(M, \omega))$ is nontrivial for $2k = \dim G - \dim K$.*

Proof. Consider the Hamiltonian fibration (1). It follows from Theorem 1 that the fibration admits a compatible symplectic form. Since $\Gamma \backslash G/K = B\Gamma$ we have that the second Betti number of the base is zero and hence the compatible symplectic form represents the coupling class. Thus we are in the situation of Proposition 3 and the statement follows. □

Example 1. To give explicit examples, we refer to the following result proved in [KaNa]. We use standard notation for types of classical and exceptional simple Lie groups ([OV1]).

Theorem 10. *Let G be a non-compact real simple Lie group and Γ be a discrete subgroup of G with compact quotient space $\Gamma \setminus G$. Then the second Betti number $b_2(\Gamma \setminus G)$ of $\Gamma \setminus G$ equals zero if the type of G is E_6^i, $(i = 1, 2)$, E_7^i, $i = 1, 2, 3$, or F_4^1, or if G is classical and satisfies the following conditions*

(i) $SL(l + 1, \mathbb{R}), l \geq 6$,
(ii) $SU^*(2l), l \geq 6$,
(iii) $SU(p, q), p + q = l + 1, \frac{l+1}{2} \geq p \geq 5$,
(iv) $SO(p, q), p + q = 2l + 1, \min(\frac{p}{2}, \frac{2l+1-p}{2}) > 2$,
(v) $SO(p, q), p + q = 2l, \frac{l}{2} \geq \frac{p}{2} > 2$
(vi) $SO^*(2l), l \geq 7$
(vii) $Sp(l, \mathbb{R}), l \geq 7$
(viii) $Sp(p, q), p + q = l, \frac{l}{2} \geq p \geq 3$.

Now, assume that G/K is not Hermitian symmetric. Then K is semisimple, and we have $b_i(K) = 0$ for $i < 3$. The long cohomology exact sequence for the fibration $K \to \Gamma \backslash G \to \Gamma \backslash G/K$ yields vanishing of the second cohomology $H^2(\Gamma \setminus G/K) = 0$. Comparing the list of G for which $H^2(\Gamma \setminus G) = 0$ with the classification of all non-compact real forms of complex simple Lie algebras $\mathfrak{g}^c$ [OV1] (Table 9), we find that all types of simple real Lie groups G from Kanuyuki-Nagano theorem except (ii) and (vii) can be realized by a fibration (1). Moreover, looking through this table, one can list all semisimple K which may occur. These are

(i) $G = SL(l+1, \mathbb{R}), K = SO(l+1)$,
(ii) $SO(p,q), K = SO(p) \times SO(q)$
(iii) $SP(p,q), K = Sp(p) \times Sp(q)$
(iv) $G = E_6^1, K = Sp(4)$
(v) $G = E_6^2, K = SU(2) \times SU(6)$
(vi) $G = E_7^1, K = SU(8)$,
(vii) $G = E_7^2, K = SU(2) \times SO(12)$,
(viii) $G = F_4^1, K = SU(2) \times Sp(3)$.

Example 2. Let

$$(M, \omega) \to M_{\mathrm{Ham}} \to \mathrm{BHam}(M, \omega)$$

be the universal Hamiltonian fibration. Define the characteristic classes $\alpha_k \in H^{2k}(\mathrm{BHam}(M, \omega))$ by

$$\alpha_k := p!(\Omega^{n+k}).$$

Let $(M, \omega) = SO(2l)/U(l)$. Applying theorem 9 to the fibration

$$(M, \omega) \to \Gamma \backslash SO(k, 2l)/SO(k) \times U(l) \to \Gamma \backslash SO(k, 2l)/SO(k) \times SO(2l)$$

we get that $\alpha_{kl} \in H^{2kl}(\mathrm{BHam}(M, \omega))$ are nontrivial for all integers $k > 1$.

Example 3. Using Theorem 2 one can look for non-zero cohomology classes in $H^*(B\,\mathrm{Ham}(K/V), \mathbb{R})$. Indeed, they are non-zero in any degree p , for which the Matsushima map μ is onto, $H^p(M/K)$ is non-zero, and $BK \to B\,\mathrm{Ham}(K/V)$ induces a surjection in cohomology. For example, μ is an isomorphism in any degree p in case of real forms of classical Lie groups:

(1) $SL(l, \mathbb{R})$, $p < \frac{l+2}{4}$,
(2) $SU^*(2l)$, $p < \frac{l-1}{2}$,
(3) $SO(i, 2l+1-i)$, $p < \min(\frac{i}{2}, \frac{2l-i+1}{2})$,
(4) $SO(i, 2l-i)$, $p < \frac{i}{2} \leq \frac{l}{2}$,
(5) $Sp(i, l-i)$, $p < i \leq \frac{l}{2}$.

The whole list can be found in [KaNa]. Note that manifolds M/K are compact Riemannian symmetric spaces, and their Poincaré polynomials are well known [GHV]

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