Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence

Leonid POSITSELSKI

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Avril 2009

IHES/M/09/20

TWO KINDS OF DERIVED CATEGORIES, KOSZUL DUALITY, AND COMODULE-CONTRAMODULE CORRESPONDENCE

LEONID POSITSELSKI

Contents

Introduction

0.1. A common wisdom says that difficulties arise in Koszul duality because important spectral sequences diverge. What really happens here is that one considers the spectral sequence of a complex endowed with, typically, a decreasing filtration which is not complete. Indeed, the spectral sequence of a complete and cocomplete filtered complex always converges in the relevant sense [10]. The solution to the problem, therefore, is to either replace the complex with its completion, or choose a different filtration. In this paper, we mostly follow the second path. This involves elaboration of the distinction between two kinds of derived categories, as we will see below.

The first conclusion is that one has to pay attention to completions if one wants one's spectral sequences to converge. What this means in the case of the spectral sequence related to a bicomplex is that the familiar picture of two spectral sequences converging to the same limit splits in two halves when the bicomplex becomes infinite enough. The two spectral sequences essentially converge to the cohomology of two different total complexes. To obtains those, one takes infinite products in the "positive" direction along the diagonals and infinite direct sums in the "negative" direction (like in Laurent series). The two possible choices of the “positive” and “negative” directions give rise to the two completions. The word “essentially” here is to be understood as “ignoring the delicate, but often manageable issues related to nonexactness of the inverse limit”.

0.2. This alternative between taking infinite direct sums and infinite products when constructing the total complex leads to the classical distinction between differential derived functors of the first and the second kind [15]. Roughly speaking, one can consider a DG-module either as a deformation of its cohomology or as a deformation of itself considered with zero differential; the spectral sequences related to the former and the latter kind of deformation essentially converge to the cohomology of the differential derived functors of the first and the second kind, respectively.

Derived categories of the first and *the second kind* are intended to serve as the domains of the differential derived functors of the first and the second kind. This does not always work as smoothly as one wishes; one discovers that, for technical reasons, it is better to consider derived categories of the first kind for *algebras* and derived categories of the second kind for *coalgebras*. The distinction between the derived functors/categories of the first and the second kind is only relevant when certain finiteness conditions no longer hold; this happens when one considers either unbounded complexes, or differential graded modules.

Let us discuss the story of two derived categories in more detail. When the finiteness conditions do hold, the derived category can be represented in two simple ways. It is both the quotient category of the homotopy category by the thick subcategory of complexes with zero cohomology and the triangulated subcategory of the homotopy category formed by the complexes of projective or injective objects. In the general case, this simple picture splits in two halves. The derived category of the first kind is still defined as the quotient category of the homotopy category by the thick subcategory of complexes (DG-modules, …) with zero cohomology. It can be also obtained as a full subcategory of the homotopy category, but the description of this subcategory is more complicated [26, 16, 6]. On the other hand, the derived category of the second kind is defined as the quotient category of the homotopy category by a thick subcategory with a rather complicated description. At the same time, it is equivalent to the full subcategory of the homotopy category formed by complexes (DG-comodules, DG-contramodules, …) which become injective or projective when considered without the differential.

0.3. The time has come to mention that there exist two kinds of module categories for a coalgebra: besides the familiar *comodules*, there are also *contramodules* [11]. Comodules can be thought of as discrete modules which are unions of their finite-dimensional subcomodules, while contramodules are modules where certain infinite summation operations are defined. For example, the space of linear maps from a comodule to any vector space has a natural contramodule structure.

The derived category of the first kind is what is known as just *the derived category*: the unbounded derived category, the derived category of DG-modules, etc. The derived category of the second kind comes in two dual versions: the *coderived* and the *contraderived category*. The coderived category works well for comodules, while the contraderived category is useful for contramodules. The classical notion of a DG-(co)algebra itself can be generalized in two ways; the derived category of the first kind is well-defined for an A_∞-algebra, while the derived category of the second kind makes perfect sense for a *CDG-coalgebra*.

Other situations exist when derived categories of the second kind are well-behaved. One of them is that of a CDG-ring whose underlying graded ring has a finite homological dimension. In this case, the coderived and contraderived categories coincide. In particular, this includes the case of a CDG-algebra whose underlying graded algebra is free. Such CDG-algebras can be thought of as strictly unital *curved A_∞-coalgebras*; CDG-modules over the former with free and cofree underlying graded modules correspond to strictly unital curved A_∞-comodules and A_∞-contramodules over the latter.

The functors of forgetting the differentials, assigning graded (co/contra)modules to CDG-(co/contra)modules, play a crucial role in the whole theory of derived categories of the second kind. So it is helpful to have versions of these functors defined for arbitrary DG-categories. An attempt to obtain such forgetful functors leads to a nice construction of an *almost involution* on the category of DG-categories. The related constructions for CDG-rings and CDG-coalgebras are important for the nonhomogeneous quadratic duality theory, particularly in the relative case [25].

0.4. Now let us turn to (derived) Koszul duality. This subject originates from the classical Bernstein–Gelfand–Gelfand duality (equivalence) between the bounded derived categories of finitely generated graded modules over the symmetric and exterior algebras with dual vector spaces of generators [5]. Attempting to generalize this straightforwardly to arbitrary algebras, one discovers that many restricting conditions have to be imposed: it is important here that one works with algebras over a field, that the algebras and modules are graded, that the algebras are Koszul, that one of them is finite-dimensional, while the other is Noetherian (or at least coherent) and has a finite homological dimension.

The standard contemporary source is [4], where many of these restrictions are eliminated, but it is still assumed that everything happens over a semisimple base ring, that the algebras and modules are graded, and that the algebras are Koszul. In [3], Koszulity is not assumed, but positive grading and semisimplicity of the base ring still are. The main goal of this paper is to work out the Koszul duality for ungraded algebras and coalgebras over a field, and more generally, differential graded algebras and coalgebras. In this setting, the Koszulity condition is less important, although it allows to obtain a certain generalization of the duality result. As to the duality over a base more general than a field, we refer the reader to [25, Section 11], where a version of Koszul duality is obtained for a base coring over a base ring.

The thematic example of ungraded Koszul duality over a field is the relation between complexes of modules over a Lie algebra $\mathfrak{g}$ and DG-comodules over its standard homological complex. Here one discovers that, when $\mathfrak{g}$ is reductive, the standard homological complex with coefficients in a nontrivial irreducible $\mathfrak{g}$-module has zero cohomology—even though it is not contractible, and becomes an injective graded comodule when one forgets the differential. So one has to consider a version of derived category of DG-comodules where certain acyclic DG-comodules survive if one wishes this category to be equivalent to the derived category of $\mathfrak{g}$-modules. That is how derived categories of the second kind appear in Koszul duality [12, 19, 17].

0.5. Yet another very good reason for considering derived categories of the second kind is that in their terms a certain relation between comodules and contramodules can be established. Namely, the coderived category of CDG-comodules and the contraderived category of CDG-contramodules over a given CDG-coalgebra are naturally equivalent. We call this phenomenon the *comodule-contramodule correspondence*; it appears to be almost as important as the Koszul duality.

One can generalize the comodule-contramodule correspondence to the case of strictly unital curved A_∞-comodules and A_∞-contramodules over a curved A_∞-coalgebra by considering the derived category of the second kind for CDG-modules over a CDG-algebra whose underlying graded algebra is a free asssociative algebra.

0.6. This paper can be considered as an extended introduction to [25]; nevertheless, it contains many results not covered by [25]. The fact that exotic derived categories arise in Koszul duality was essentially discovered by Hinich [13], whose ideas were developed by Lefèvre-Hasegawa [19]; see also Fløystad [12], Huebschmann [14], and Nicolas [22]. The terminology of "coderived categories" was introduced in Keller's exposition [17]. However, the definition of coderived categories in [19, 17] was not entirely satisfactory, in our view, in that the right hand side of the purpoted duality is to a certain extent defined in terms of the left hand side. This defect is corrected in the present paper. In addition, we emphasize contramodules and CDG-coalgebras, whose role in the derived categories of the second kind and derived Koszul duality business does not seem to have been appreciated enough.

0.7. Now let us describe the content of this paper in more detail. In Section 1 we obtain two semiorthogonal decompositions of the homotopy category of DG-modules over a DG-ring, providing injective and projective resolutions for the derived category of DG-modules. We also consider flat resolutions and use them to define the derived functor Tor for a DG-ring. Besides, we construct a t-structure on the derived category of DG-modules over an arbitrary DG-ring.

The derived categories of DG-comodules and DG-contramodules and the differential derived functors $\mathrm{Cotor}^{C,I}$ and Coext_C^I of the first kind for a DG-coalgebra C are briefly discussed in Section 2. Partial results about injective and projective resolutions for the coderived and contraderived categories of a CDG-ring are obtained in Section 3. The Noetherian and Artinian cases and the finite homological dimension case are considered; in the latter situation, a natural definition of the differential derived functor $\mathrm{Tor}^{B,II}$ of the second kind for a CDG-ring B is given. In addition, we construct an "almost involution" on the category of DG-categories.

In Section 4 we construct semiorthogonal decompositions of the homotopy categories of CDG-comodules and CDG-contramodules over a CDG-coalgebra, providing injective and projective resolutions for the coderived category of CDG-comodules and the contraderived category of CDG-contramodules. We also define the differential derived functors Cotor, Coext, and Ctrtor for a CDG-coalgebra, and give a sufficient condition for a morphism of CDG-coalgebras to induce equivalences of the coderived and contraderived categories. The comodule-contramodule correspondence for a CDG-coalgebra is obtained in Section 5.

Koszul duality (or "triality", as there are actually two module categories on the coalgebra side) is studied in Section 6. Two versions of the duality theorem for (C)DG-modules, CDG-comodules, and CDG-contramodules are obtained, one valid for conilpotent CDG-coalgebras only and one applicable in the general case. We also construct an equivalence between natural localizations of the categories of DG-algebras (with nonzero units) and conilpotent CDG-coalgebras.

We discuss the derived categories of A_∞-modules and the co/contraderived categories of curved A_∞-co/contramodules in Section 7. We explain the relation between strictly unital A_∞-algebras and coaugmented CDG-coalgebra structures on graded tensor coalgebras, and use it to prove the standard results about strictly unital A_∞-modules. The similar approach to strictly unital curved A_∞-coalgebras yields the comodule-contramodule correspondence in the A_∞ case.

In a future version of this paper we plan to include the constructions of model category structures on the categories of (C)DG- (co,contra)modules and (co)algebras.

Acknowledgement. The author is grateful to Michael Finkelberg for posing the problem of constructing derived nonhomogeneous Koszul duality. I want to express my gratitude to Vladimir Voevodsky for very stimulating discussions and encouragement, without which this work would probably never have been done. I also benefited from discussions with Joseph Bernstein, Maxim Kontsevich, Henning Krause, Tony Pantev, and Alexander Efimov. I am grateful to Ivan Mirkovic, who always urged me to write down the material presented below. Most of the content of this paper was worked out when I was a Member of the Institute for Advanced Study, which I wish to thank for its hospitality. I also wish to thank the participants of an informal seminar at IAS, where I first presented these results in the Spring of 1999. The author was partially supported by P. Deligne 2004 Balzan prize and an INTAS grant while writing the paper up. Parts of this paper have been written when I was visiting the Institut des Hautes Études Scientifiques.

1. Derived Category of DG-Modules

1.1. **DG-rings and DG-modules.** A *DG-ring* $A = (A, d)$ is a pair consisting of an associative graded ring $A = \bigoplus_{i\in\mathbb{Z}} A^i$ and an odd derivation $d\colon A \longrightarrow A$ of degree 1 such that $d^2 = 0$. In other words, it is supposed that $d(A^i) \subset A^{i+1}$ and $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ for a, $b \in A$, where $|a|$ denotes the degree of a homogeneous element, i. e., $a \in A^{|a|}$.

A *left DG-module* (M, d_M) over a DG-ring A is a graded left A-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ endowed with a differential $d_M\colon M \longrightarrow M$ of degree 1 compatible with the derivation of A and such that $d_M^2 = 0$. The compatibility means that the equation $d_M(ax) = d(a)x + (-1)^{|a|}ad_M(x)$ holds for all $a \in A$ and $x \in M$.

A *right DG-module* (N, d_N) over A is a graded right A-module N endowed with a differential d_N of degree 1 satisfying the equations $d_N(xa) = d_N(x)a + (-1)^{|x|}xd(a)$ and $d_N^2 = 0$, where $x \in N^{|x|}$.

Let L and M be left DG-modules over A. The *complex of homomorphisms* $\mathrm{Hom}_A(L, M)$ from L to M over A is constructed as follows. The component $\mathrm{Hom}_A^i(L, M)$ consists of all homogeneous maps $f\colon L \longrightarrow M$ of degree i such that $f(ax) = (-)^{i|a|}af(x)$ for all $a \in A$ and $x \in L$. The differential in the complex $\mathrm{Hom}_A(L, M)$ is given by the formula $d(f)(x) = d_M(f(x)) - (-1)^{|f|}f(d_L(x))$. Clearly, one has $d^2(f) = 0$; for any composable morphisms of left DG-modules f and g one has $d(fg) = d(f)g + (-1)^{|f|}fd(g)$.

For any two right DG-modules R and N over A, the complex of homomorphisms $\mathrm{Hom}_A(R, N)$ is defined by the same formulas as above and satisfies the same properties, with the only difference that a homogeneous map $f\colon R \longrightarrow N$ belonging to $\mathrm{Hom}_A(R, N)$ must satisfy the equation $f(xa) = f(x)a$ for $a \in A$ and $x \in R$.

Let N be a right DG-module and M be a left DG-module over A. The *tensor product complex* $N \otimes_A M$ is defined as the graded quotient group of the graded abelian group $N \otimes_{\mathbb{Z}} M$ by the relations $xa \otimes y = x \otimes ay$ for $x \in N,\ a \in A,\ y \in M$, endowed with the differential given by the formula $d(x\otimes y) = d(x)\otimes y + (-1)^{|x|}x\otimes d(y)$. For any two right DG-modules R and N and any two left DG-modules L and M the natural map of complexes $\mathrm{Hom}_A(R, N) \otimes_{\mathbb{Z}} \mathrm{Hom}_A(L, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R \otimes_A L, N \otimes_A M)$

is defined by the formula $(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y)$. Here $\mathbb{Z}$ is considered as a DG-ring concentrated in degree 0.

For any DG-ring A, its cohomology $H(A) = H_d(A)$, defined as the quotient of the kernel of d by its image, has a natural structure of graded ring. For a left DG-module M over A, its cohomology $H(M)$ is a graded module over $H(A)$; for a right DG-module N, its cohomology $H(N)$ is a right graded module over $H(A)$.

A *DG-algebra* A over a commutative ring k is a DG-ring endowed with DG-ring homomorphism $k \longrightarrow A^0$ whose image is contained in the center of the algebra A, where k is considered as a DG-ring concentrated in degree 0; equivalently, a DG-algebra is a complex of k-modules with a k-linear DG-ring structure.

Remark. One can consider DG-algebras and DG-modules graded by an abelian group Γ different from $\mathbb{Z}$, provided that Γ is endowed with a parity homomorphism $\Gamma \longrightarrow \mathbb{Z}/2$ and an odd element $\mathbf{1} \in \Gamma$, so that the differentials would have degree $\mathbf{1}$. In particular, one can take $\Gamma = \mathbb{Z}/2$, that is have gradings reduced to parities, or consider fractional gradings by using some subgroup of $\mathbb{Q}$ consisting of rationals with odd denominators in the role of Γ. Even more generally, one can replace the parity function with a symmetric bilinear form $\sigma\colon \Gamma \times \Gamma \longrightarrow \mathbb{Z}/2$, to be used in the super sign rule in place of the product of parities; one just has to assume that $\sigma(\mathbf{1}, \mathbf{1}) = 1 \bmod 2$. All the most important results of this paper remain valid in such settings.

1.2. **DG-categories.** A *DG-category* is a category whose sets of morphisms are complexes and compositions are biadditive maps compatible with the gradings and the differentials. In other words, a DG-category DG consists of a class of objects, complexes of abelian groups $\mathrm{Hom}_{\mathsf{DG}}(X, Y)$, called the complexes of morphisms from X to Y, defined for any two objects X and Y, and morphisms of complexes $\mathrm{Hom}_{\mathsf{DG}}(Y, Z) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathsf{DG}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathsf{DG}}(X, Z)$, called the composition maps, defined for any three objects X, Y, and Z. The compositions must be associative and unit elements $\mathrm{id}_X \in \mathrm{Hom}_{\mathsf{DG}}(X, X)$ must exist; the equations $d(\mathrm{id}_X) = 0$ then hold automatically.

For example, left DG-modules over a DG-ring A form a DG-category, which we will denote by $\mathsf{DG}(A\text{–}\mathsf{mod})$. The DG-category of right DG-modules over A will be denoted by $\mathsf{DG}(\mathsf{mod}\text{–}A)$.

A *covariant DG-functor* $\mathsf{DG}' \longrightarrow \mathsf{DG}''$ consists of a map between the classes of objects and (closed) morphisms between the complexes of morphisms compatible with the compositions. A *contravariant DG-functor* is defined in the same way, except that one has to take into account the natural isomorphism of complexes $V \otimes W \simeq W \otimes V$ for complexes of abelian groups V and W that is given by the formula $v \otimes w \longmapsto (-1)^{|v||w|} w \otimes V$. (Covariant or contravariant) DG-functors between DG' and DG'' form a DG-category themselves. The complex of morphisms between DG-functors F and G is a subcomplex of the product of the complexes of morphisms from $F(X)$ to

$G(X)$ in DG'' taken over all objects $X \in \mathsf{DG}'$; the desired subcomplex is formed by all the systems of morphisms compatible with all morphisms $X \longrightarrow Y$ in DG'.

For example, a DG-ring A can be considered as a DG-category with a single object; covariant DG-functors from this DG-category to the DG-category of complexes of abelian groups are left DG-modules over A, while contravariant DG-functors between the same DG-categories can be identified with right DG-modules over A.

A *closed morphism* $f\colon X \longrightarrow Y$ in a DG-category DG is an element of $\mathrm{Hom}^0_{\mathsf{DG}}(X,Y)$ such that $d(f)=0$. The category whose objects are the objects of DG and whose morphisms are closed morphisms in DG is denoted by $Z^0(\mathsf{DG})$.

An object Y is called the *product* of a family of objects X_α (notation: $Y = \prod_\alpha X_\alpha$) if a closed isomorphism of contravariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(-,Y) \simeq \prod_\alpha \mathrm{Hom}_{\mathsf{DG}}(-,X_\alpha)$ is fixed. An object Y is called the *direct sum* of a family of objects X_α (notation: $Y = \bigoplus_\alpha X_\alpha$) if a closed isomorphism of covariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(Y,-) \simeq \prod_\alpha \mathrm{Hom}_{\mathsf{DG}}(X_\alpha,-)$ is fixed.

An object Y is called the *shift* of an object X by an integer i (notation: $Y = X[i]$) if a closed isomorphism of contravariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(-,Y) \simeq \mathrm{Hom}_{\mathsf{DG}}(-,X)[i]$ is fixed, or equivalently, a closed isomorphism of covariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(Y,-) \simeq \mathrm{Hom}_{\mathsf{DG}}(X,-)[-i]$ is fixed.

An object Z is called the *cone* of a closed morphism $f\colon X \longrightarrow Y$ (notation: $Z = \mathrm{cone}(f)$) if a closed isomorphism of contravariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(-,Z) \simeq \mathrm{cone}(f_*)$, where $f_*\colon \mathrm{Hom}_{\mathsf{DG}}(-,X) \longrightarrow \mathrm{Hom}_{\mathsf{DG}}(-,Y)$, is fixed, or equivalently, a closed isomorphism of covariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(Z,-) \simeq \mathrm{cone}(f^*)[-1]$, where $f^*\colon \mathrm{Hom}_{\mathsf{DG}}(Y,-) \longrightarrow \mathrm{Hom}_{\mathsf{DG}}(X,-)$, is fixed.

Let V be a complex of abelian groups and $p\colon V \longrightarrow V$ be an endomorphism of degree 1 satisfying the Maurer-Cartan equation $d(p)+p^2=0$. Then one can define a new differential on V by setting $d' = d+p$; let us denote the complex so obtained by $V(p)$. Let $q \in \mathrm{Hom}^1_{\mathsf{DG}}(X,X)$ be an endomorphism of degree 1 satisfying the equation $d(q)+q^2=0$. An object Y is called the *twist* of the object X with respect to q if a closed isomorphism of contravariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(-,X) \simeq \mathrm{Hom}_{\mathsf{DG}}(-,Y)(q_*)$ is fixed, where $q_*(g) = q \circ g$ for any morphism g whose target is X, or equivalently, a closed isomorphism of covariant DG-functors $\mathrm{Hom}_{\mathsf{DG}}(Y,-) \simeq \mathrm{Hom}_{\mathsf{DG}}(X,-)(-q^*)$ is fixed, where $q^*(g) = (-1)^{|g|} g \circ q$ for any morphism g whose source is Y.

As any representing objects of DG-functors, all direct sums, products, shifts, cones, and twists are defined uniquely up to a unique closed isomorphism. The direct sum of a finite set of objects is naturally also their product, and vice versa. Finite direct sums, products, shifts, cones, and twists are preserved by any DG-functors. One can express the cone of a closed morphism $f\colon X \longrightarrow Y$ as the twist of the direct sum $Y \oplus X[1]$ with respect to the endomorphism q induced by f.

Here is another way to think about cones of closed morphisms in DG-categories. Let $\mathsf{DG}^\#$ denote the category whose objects are the objects of DG and morphisms are the (not necessarily closed) morphisms in DG of degree 0. Let $X' \longrightarrow X \longrightarrow X''$ be a triple of objects in DG with closed morphisms between them that is split exact in $\mathsf{DG}^\#$. Then X is the cone of a closed morphism $X''[-1] \longrightarrow X'$. Conversely, for any closed morphism $X \longrightarrow Y$ in DG with the cone Z there is a natural triple of objects and closed morphisms $Y \longrightarrow Z \longrightarrow X[1]$, which is split exact in $\mathsf{DG}^\#$.

Let DG be a DG-category with shifts, twists, and infinite direct sums. Let $\cdots \longrightarrow X_i \longrightarrow X_{i-1} \longrightarrow \cdots$ be a complex of objects of DG with closed differentials ∂_i. Then the differentials ∂_i induce an endomorphism q of degree 1 on the direct sum $\bigoplus_i X_i[i]$ satisfying the equations $d(q) = 0 = q^2$. The twist of this direct sum with respect to this endomorphism is called the *total object* of the complex $X_\bullet$ *formed by taking infinite direct sums* and denoted by $\mathrm{Tot}^\oplus(X_\bullet)$. For a DG-category DG with shifts, twists, and infinite products, one can consider the analogous construction with the infinite direct sum replaced by the infinite product $\prod_i X_i[i]$. Thus one obtains the definition of the *total object formed by taking infinite products* $\mathrm{Tot}^\sqcap(X_\bullet)$.

For a finite complex $X^\bullet$, the two total objects coincide and are denoted simply by $\mathrm{Tot}(X_\bullet)$; this total object only requires existence of finite direct sums/products for its construction. Alternatively, the total objects Tot, $\mathrm{Tot}^\oplus$, and $\mathrm{Tot}^\sqcap$ can be defined as certain representing objects of DG-functors. The finite total object Tot can be also expressed in terms of iterated cones, so it is well-defined whenever cones exist in a DG-category DG, and it is preserved by any DG-functors.

A DG-functor $\mathsf{DG}' \longrightarrow \mathsf{DG}''$ is said to be *fully faithful* if it induces isomorphisms of the complexes of morphisms. A DG-functor is said to be an *equivalence of DG-categories* if it is fully faithful and every object of DG'' admits a closed isomorphism with an object coming from DG'. This is equivalent to existence of a DG-functor in the opposite direction for which both the compositions admit closed isomorphisms to the identity DG-functors. DG-functors $F\colon \mathsf{DG}' \longrightarrow \mathsf{DG}''$ and $G\colon \mathsf{DG}'' \longrightarrow \mathsf{DG}'$ are said to be *adjoint* if for every objects $X \in \mathsf{DG}'$ and $Y \in \mathsf{DG}''$ a closed isomorphism of complexes $\mathrm{Hom}_{\mathsf{DG}''}(F(X), Y) \simeq \mathrm{Hom}_{\mathsf{DG}'}(X, G(Y))$ is given such that these isomorphisms commute with the (not necessarily closed) morphisms induced by morphisms in DG' and DG''.

Let DG be a DG-category where (a zero object and) all shifts and cones exist. Then the *homotopy category* $H^0(\mathsf{DG})$ is the additive category with the same class of objects as DG and groups of morphisms given by $\mathrm{Hom}_{H^0(\mathsf{DG})}(X, Y) = H^0(\mathrm{Hom}_{\mathsf{DG}}(X, Y))$. The homotopy category is a triangulated category [8]. Shifts of objects and cones of closed morphisms in DG become shifts of objects and cones of morphisms in the triangulated category $H^0(\mathsf{DG})$. Any direct sums and products of objects of a DG-category are also their directs sums and products in the homotopy category. Adjoint functors between DG-categories induce adjoint functors between the corresponding categories of closed morphisms and homotopy categories.

Two closed morphisms $f,\ g\colon X \longrightarrow Y$ in a DG-category DG are called *homotopic* if their images coincide in $H^0(\mathsf{DG})$. A closed morphism in DG is called a *homotopy equivalence* if it becomes an isomorphism in $H^0(\mathsf{DG})$. An object of DG is called *contractible* if it vanishes in $H^0(\mathsf{DG})$.

All shifts, twists, infinite direct sums, and infinite direct products exist in the DG-categories of DG-modules. The homotopy category of (the DG-category of) left DG-modules over a DG-ring A is denoted by $\mathsf{Hot}(A\text{–}\mathsf{mod}) = H^0\mathsf{DG}(A\text{–}\mathsf{mod})$; the homotopy category of right DG-modules over A is denoted by $\mathsf{Hot}(\mathsf{mod}\text{–}A) = H^0\mathsf{DG}(\mathsf{mod}\text{–}A)$.

1.3. **Semiorthogonal decompositions.** Let H be a triangulated category and $\mathsf{A} \subset \mathsf{H}$ be a full triangulated subcategory. Then the quotient category H/A is defined as the localization of H with respect to the multiplicative system of morphisms whose cones belong to A. The subcategory A is called *thick* if it coincides with the full subcategory formed by all the objects of H whose images in H/A vanish. A triangulated subcategory $\mathsf{A} \subset \mathsf{H}$ is thick if and only if it is closed under direct summands in H [28, 20]. The following Lemma is essentially due to Verdier [27]; see also [2, 7].

Lemma. *Let H be a triangulated category and B, $\mathsf{C} \subset \mathsf{H}$ be its full triangulated subcategories such that $\operatorname{Hom}_{\mathsf{H}}(B, C) = 0$ for all $B \in \mathsf{B}$ and $C \in \mathsf{C}$. Then the natural maps $\operatorname{Hom}_{\mathsf{H}}(B, X) \longrightarrow \operatorname{Hom}_{\mathsf{H}/\mathsf{C}}(B, X)$ and $\operatorname{Hom}_{\mathsf{H}}(X, C) \longrightarrow \operatorname{Hom}_{\mathsf{H}/\mathsf{B}}(X, C)$ are isomorphisms for any objects $B \in \mathsf{B}$, $C \in \mathsf{C}$, and $X \in \mathsf{H}$. In particular, the functors $\mathsf{B} \longrightarrow \mathsf{H}/\mathsf{C}$ and $\mathsf{C} \longrightarrow \mathsf{H}/\mathsf{B}$ are fully faithful. Furthermore, the following conditions are equivalent:*

- (a) *B is a thick subcategory in H and the functor $\mathsf{C} \longrightarrow \mathsf{H}/\mathsf{B}$ is an equivalence of triangulated categories;*
- (b) *C is a thick subcategory in H and the functor $\mathsf{B} \longrightarrow \mathsf{H}/\mathsf{C}$ is an equivalence of triangulated categories;*
- (c) *B and C generate H as a triangulated category, i. e., any object of H can be obtained from objects of B and C by iterating the operations of shift and cone;*
- (d) *for any object $X \in \mathsf{H}$ there exists a distinguished triangle $B \longrightarrow X \longrightarrow C \longrightarrow B[1]$ with $B \in \mathsf{B}$ and $C \in \mathsf{C}$ (and in this case for any morphism $X' \longrightarrow X''$ in H there exists a unique morphism between any distinguished triangles of the above form for X' and X'', so this triangle is unique up to a unique isomorphism and depends functorially on X);*
- (e) *C is the full subcategory of H formed by all the objects $C \in \mathsf{H}$ such that $\operatorname{Hom}_{\mathsf{H}}(B, C) = 0$ for all $B \in \mathsf{B}$, and the embedding functor $\mathsf{B} \longrightarrow \mathsf{H}$ has a right adjoint functor (which can be then identified with the localization functor $\mathsf{H} \longrightarrow \mathsf{H}/\mathsf{C} \simeq \mathsf{B}$);*

(f) C *is the full subcategory of* H *formed by all the objects* $C \in \mathsf{H}$ *such that* $\mathrm{Hom}_{\mathsf{H}}(B, C) = 0$ *for all* $B \in \mathsf{B}$*,* B *is a thick subcategory in* H*, and the localization functor* $\mathsf{H} \longrightarrow \mathsf{H}/\mathsf{B}$ *has a right adjoint functor;*

(g) B *is the full subcategory of* H *formed by all the objects* $B \in \mathsf{H}$ *such that* $\mathrm{Hom}_{\mathsf{H}}(B, C) = 0$ *for all* $C \in \mathsf{C}$*, and the embedding functor* $\mathsf{C} \longrightarrow \mathsf{H}$ *has a left adjoint functor (which can be then identified with the localization functor* $\mathsf{H} \longrightarrow \mathsf{H}/\mathsf{B} \simeq \mathsf{C}$*);*

(h) B *is the full subcategory of* H *formed by all the objects* $B \in \mathsf{H}$ *such that* $\mathrm{Hom}_{\mathsf{H}}(B, C) = 0$ *for all* $C \in \mathsf{C}$*,* C *is a thick subcategory in* H*, and the localization functor* $\mathsf{H} \longrightarrow \mathsf{H}/\mathsf{C}$ *has a left adjoint functor.* □

1.4. **Projective resolutions.** A DG-module M is said to be acyclic if it is acyclic as a complex of abelian groups, i. e., $H(M) = 0$. The thick subcategory of the homotopy category $\mathsf{Hot}(A\text{–}\mathsf{mod})$ formed by the acyclic DG-modules is denoted by $\mathsf{Acycl}(A\text{–}\mathsf{mod})$. The *derived category* of left DG-modules over A is defined as the quotient category $\mathsf{D}(A\text{–}\mathsf{mod}) = \mathsf{Hot}(A\text{–}\mathsf{mod})/\mathsf{Acycl}(A\text{–}\mathsf{mod})$.

A left DG-module L over a DG-ring A is called *projective* if for any acyclic left DG-module M over A the complex $\mathrm{Hom}_A(L, M)$ is acyclic. The full triangulated subcategory of $\mathsf{Hot}(A\text{–}\mathsf{mod})$ formed by the projective DG-modules is denoted by $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}}$. The following Theorem says, in particular, that the homotopy category $\mathsf{H} = \mathsf{Hot}(A\text{–}\mathsf{mod})$ and its subcategories $\mathsf{B} = \mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}}$ and $\mathsf{C} = \mathsf{Acycl}(A\text{–}\mathsf{mod})$ satisfy the equivalent conditions of Lemma 1.3, and so describes the derived category $\mathsf{D}(A\text{–}\mathsf{mod})$.

Theorem. (a) *The category* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}}$ *is the minimal triangulated subcategory of* $\mathsf{Hot}(A\text{–}\mathsf{mod})$ *containing the DG-module* A *and closed under infinite direct sums.*

(b) *The composition of functors* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}} \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

Proof. First notice that the category $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}}$ is closed under infinite direct sums. It contains the DG-module A, since for any DG-module M over A there is a natural isomorphism of complexes of abelian groups $\mathrm{Hom}_A(A, M) \simeq M$. According to Lemma 1.3, it remains to construct for any DG-module M a morphism $f\colon F \longrightarrow M$ in the homotopy category of DG-modules over A such that the DG-module F belongs to the minimal triangulated subcategory containing the DG-module A and closed under infinite direct sums, while the cone of the morphism f is an acyclic DG-module. When A is a DG-algebra over a field k, it suffices to consider the bar-resolution of a DG-module M. It is a complex of DG-modules over A, and its total DG-module formed by taking infinite direct sums provides the desired DG-module F.

Let us give a detailed construction in the general case. Let M be a DG-module over A. Choose a complex of free abelian groups M' together with a surjective morphism of complexes $M' \longrightarrow M$ such that the cohomology $H(M')$ is also a free graded abelian group and the induced morphism of cohomology $H(M') \longrightarrow H(M)$ is also surjective. For example, one can take M' to be the graded abelian group with the components freely generated by nonzero elements of the components of M, endowed with the induced differential. Set $F_0 = A \otimes_{\mathbb{Z}} M'$; then there is a natural closed surjective morphism $F_0 \longrightarrow M$ of DG-modules over A and the induced morphism of cohomology $H(F_0) \longrightarrow H(M)$ is also surjective. Let K be the kernel of the morphism $F_0 \longrightarrow M$ (taken in the abelian category $Z^0\mathsf{DG}(A\text{–}\mathsf{mod})$ of DG-modules and closed morphisms between them). Applying the same construction to the DG-module K in place of M, we obtain the DG-module F_1, etc. Let F be the total DG-module of the complex of DG-modules $\cdots \longrightarrow F_1 \longrightarrow F_0$ formed by taking infinite direct sums. One can easily check that the cone of the morphism $F \longrightarrow M$ is acyclic, since the complex $\cdots \longrightarrow H(F_1) \longrightarrow H(F_0) \longrightarrow H(M) \longrightarrow 0$ is acyclic (it suffices to apply the result of [10] to the increasing filtration of the total complex of $\cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M$ coming from the silly filtration of this complex of complexes).

It remains to show that the DG-module F as an object of the homotopy category can be obtained from the DG-module A by iterating the operations of shift, cone, and infinite direct sum. Every DG-module F_i is a direct sum of shifts of the DG-module A and shifts of the cone of the identity endomorphism of the DG-module A. Denote by X_i the total DG-module of the finite complex of DG-modules $F_i \longrightarrow \cdots \longrightarrow F_0$. Then we have $F = \varinjlim X_i$ in the abelian category $Z^0\mathsf{DG}(A\text{–}\mathsf{mod})$. So there is an exact triple of DG-modules and closed morphisms $0 \longrightarrow \bigoplus X_i \longrightarrow \bigoplus X_i \longrightarrow F \longrightarrow 0$. Since the embeddings $X_i \longrightarrow X_{i+1}$ split in $\mathsf{DG}(A\text{–}\mathsf{mod})^\#$, the above exact triple also splits in this additive category. Thus F is a cone of the morphism $\bigoplus X_i \longrightarrow \bigoplus X_i$ in the triangulated category $\mathsf{Hot}(A\text{–}\mathsf{mod})$. □

1.5. **Injective resolutions.** A left DG-module M over a DG-ring A is said to be *injective* if for any acyclic DG-module L over A the complex $\mathrm{Hom}_A(L, M)$ is acyclic. The full triangulated subcategory of $\mathsf{Hot}(A\text{–}\mathsf{mod})$ formed by the injective DG-modules is denoted by $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}}$.

For any right DG-module N over A and any complex of abelian groups V the complex $\mathrm{Hom}_{\mathbb{Z}}(N, V)$ has a natural structure of left DG-module over A with the graded A-module structure given by the formula $(af)(n) = (-1)^{|a|(|f|+|n|)} f(na)$.

The following Theorem provides another semiorthogonal decomposition of the homotopy category $\mathsf{Hot}(A\text{–}\mathsf{mod})$ and another description of the derived category $\mathsf{D}(A\text{–}\mathsf{mod})$.

Theorem. (a) *The category* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}}$ *is the minimal triangulated subcategory of* $\mathsf{Hot}(A\text{–}\mathsf{mod})$ *containing the DG-module* $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ *and closed under infinite products.*

(b) *The composition of functors* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}} \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

Proof. The proof is analogous to that of Theorem 1.4. Clearly, the category $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}}$ is closed under infinite products. It contains the DG-module $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, since the complex $\mathrm{Hom}_A(L, \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})) \simeq \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Q}/\mathbb{Z})$ is acyclic whenever the DG-module L is. To construct an injective resolution of a DG-module M, one can embed in into a complex of injective abelian groups M' so that the cohomology $H(M')$ is also injective and $H(M)$ also embeds into $H(M')$. For example, one can take the components of M' to be the products of $\mathbb{Q}/\mathbb{Z}$ over all nonzero homomorphisms of abelian groups from the components of M to $\mathbb{Q}/\mathbb{Z}$. Take $J_0 = \mathrm{Hom}_{\mathbb{Z}}(A, M')$ and consider the induced injective morphism of DG-modules $M \longrightarrow J_0$. Set $K = J_0/M$, $J_{-1} = \mathrm{Hom}_{\mathbb{Z}}(A, K')$, etc., and $J = \mathrm{Tot}^{\sqcap}(J_\bullet)$. Then the morphism of DG-modules $M \longrightarrow J$ has an acyclic cone and the DG-module J is isomorphic in $\mathsf{Hot}(A\text{–}\mathsf{mod})$ to a DG-module obtained from $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ by iterating the operations of shift, cone, and infinite product. □

1.6. **Flat resolutions.** A right DG-module N over a DG-ring A is said to be *flat* if for any acyclic left DG-module M over A the complex $N \otimes_A M$ is acyclic. Flat left DG-modules over A are defined in the analogous way. The full triangulated subcategory of $\mathsf{Hot}(A\text{–}\mathsf{mod})$ formed by flat DG-modules is denoted by $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}}$.

We denote the thick subcategory of acyclic right A-modules by $\mathsf{Acycl}(\mathsf{mod}\text{–}A) \subset \mathsf{Hot}(\mathsf{mod}\text{–}A)$. The quotient category $\mathsf{Hot}(\mathsf{mod}\text{–}A)/\mathsf{Acycl}(\mathsf{mod}\text{–}A)$ is called the derived category of right DG-modules over A and denoted by $\mathsf{D}(\mathsf{mod}\text{–}A)$. The full triangulated subcategory of flat right DG-modules is denoted by $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}} \subset \mathsf{Hot}(\mathsf{mod}\text{–}A)$.

It follows from Theorems 1.4–1.5 and Lemma 1.3 that one can compute the right derived functor $\mathrm{Ext}_A(L, M) = \mathrm{Hom}_{\mathsf{D}(A\text{–}\mathsf{mod})}(L, M)$ for left DG-modules L and M over a DG-ring A in terms of projective or injective resolutions. Namely, one has $\mathrm{Ext}_A(L, M) \simeq H(\mathrm{Hom}_A(L, M))$ whenever L is a projective DG-module or M is an injective DG-module over A. The following Theorem allows to define a left derived functor $\mathrm{Tor}^A(N, M)$ for a right DG-module N and a left DG-module M over A so that it could be computed in terms of flat resolutions.

Theorem. (a) *The functor* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}}/(\mathsf{Acycl}(A\text{–}\mathsf{mod}) \cap \mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$ *induced by the embedding* $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}} \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

(b) *The functor* $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}}/(\mathsf{Acycl}(\mathsf{mod}\text{–}A) \cap \mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}}) \longrightarrow \mathsf{D}(\mathsf{mod}\text{–}A)$ *induced by the embedding* $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}} \longrightarrow \mathsf{Hot}(\mathsf{mod}\text{–}A)$ *is an equivalence of triangulated categories.*

The proof of Theorem is based on the following Lemma.

Lemma. *Let* H *be a triangulated category and* $\mathsf{A}, \mathsf{F} \subset \mathsf{H}$ *be full triangulated subcategories. Then the natural functor* $\mathsf{F}/\mathsf{A} \cap \mathsf{F} \longrightarrow \mathsf{H}/\mathsf{A}$ *is an equivalence of triangulated categories whenever one of the following two conditions holds:*

(a) *for any object* $X \in \mathsf{H}$ *there exists an object* $F \in \mathsf{F}$ *together with a morphism* $F \longrightarrow X$ *in* H *such that a cone of that morphism belongs to* A*, or*
(b) *for any object* $Y \in \mathsf{H}$ *there exists an object* $F \in \mathsf{F}$ *together with a morphism* $Y \longrightarrow F$ *in* H *such that a cone of that morphism belongs to* A*.*

Proof of Lemma. It is clear that the functor $\mathsf{F}/\mathsf{A}\cap\mathsf{F} \longrightarrow \mathsf{H}/\mathsf{A}$ is surjective on the isomorphism classes of objects under either of the assumptions (a) or (b). To prove that it is bijective on morphisms, represent morphisms in both quotient categories by fractions of the form $X \longleftarrow X' \longrightarrow Y$ in the case (a) and by fractions of the form $X \longrightarrow Y' \longleftarrow Y$ in the case (b). ☐

Proof of Theorem. Part (a): first notice that any projective left DG-module M over a DG-ring A is flat. Indeed, one has $\mathrm{Hom}_{\mathbb{Z}}(N\otimes_A M, \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}))$ for any right DG-module N over A, so whenever N is acyclic, and consequently $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z})$ is acyclic, the left hand side of this isomorphism is acyclic, too, and therefore $N \otimes_A M$ is acyclic. So it remains to use Theorem 1.4 together with Lemma 1.3 and the above Lemma. To prove part (b), switch the left and right sides by passing to the DG-ring A^{op} defined as follows. As a complex, A^{op} is identified with A, while the multiplication in A^{op} is given by the formula $a^{\mathrm{op}}b^{\mathrm{op}} = (-1)^{|a||b|}(ba)^{\mathrm{op}}$. Then right DG-modules over A are left DG-modules over A^{op} and vice versa. ☐

Now let us define the derived functor

$$\mathrm{Tor}^A\colon \mathsf{D}(\mathsf{mod}\text{–}A) \times \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod}^{\mathsf{gr}}$$

for a DG-algebra A over a commutative ring k, where $k\text{–}\mathsf{mod}^{\mathsf{gr}}$ denotes the category of graded k-modules. For this purpose, restrict the functor of tensor product $\otimes_A\colon \mathsf{Hot}(\mathsf{mod}\text{–}A) \times \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(k\text{–}\mathsf{mod})$ to either of the full subcategories $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}} \times \mathsf{Hot}(A\text{–}\mathsf{mod})$ or $\mathsf{Hot}(\mathsf{mod}\text{–}A) \times \mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}}$ and compose it with the cohomology functor $H\colon \mathsf{Hot}(k\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod}^{\mathsf{gr}}$. The functors so obtained factorize through the localizations $\mathsf{D}(\mathsf{mod}\text{–}A)\times\mathsf{D}(A\text{–}\mathsf{mod})$ and the two induced derived functors $\mathsf{D}(\mathsf{mod}\text{–}A)\times\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod}^{\mathsf{gr}}$ are naturally isomorphic to each other.

Indeed, the tensor product $N \otimes_A M$ by the definition is acyclic whenever one of the DG-modules N and M is acyclic, while the other one is flat. Let us check that the complex $N \otimes_A M$ is acyclic whenever either of the DG-modules N and M is simultaneously acyclic and flat. Assume that N is acyclic and flat; choose a flat left DG-module F over A together with a morphism of DG-modules $F \longrightarrow A$ with an acyclic cone. Then the complex $N \otimes_A F$ is acyclic, since N is acyclic; while the morphism $N \otimes_A F \longrightarrow N \otimes_A M$ is a quasi-isomorphism, since N is flat.

To construct an isomorphism of the two induced derived functors, it suffices to notice that both of them are isomorpic to the derived functor obtained by restricting the functor $\otimes_A$ to the full subcategory $\mathsf{Hot}(\mathsf{mod}\text{–}A)_{\mathsf{fl}}\times\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}}$. In other words, suppose that $G \longrightarrow N$ and $F \longrightarrow M$ are morphisms of DG-modules with acyclic cones, where the right DG-module G and the left DG-module F are flat. Then there are natural quasi-isomorphisms $G \otimes_A M \longleftarrow G \otimes_A F \longrightarrow N \otimes_A F$.

1.7. **Restriction and extention of scalars.** Let $f\colon A \longrightarrow B$ be a morphism of DG-algebras, i. e., a closed morphism of complexes preserving the multiplication. Then any DG-module over B can be also considered as a DG-module over A, which defines the restriction-of-scalars functor $R_f\colon \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$. This functor has a left adjoint functor E_f given by the formula $E_f(M) = B \otimes_A M$ and a right adjoint functor E^f given by the formula $E^f(M) = \mathrm{Hom}_A(B, M)$ (where the DG-module structure on $\mathrm{Hom}_A(B, M)$ is defined so that $\mathrm{Hom}_A(B, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(B, M)$ is a closed injective morphism of DG-modules).

The functor R_f obviously maps acyclic DG-modules to acyclic DG-modules, and so induces a functor $\mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$, which we will denote by $\mathbb{I}R_f$. The functor E_f has a left derived functor $\mathbb{L}E_f$ obtained by restricting E_f to either of the full subcategories $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{proj}}$ or $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{fl}} \subset \mathsf{Hot}(A\text{–}\mathsf{mod})$ and composing it with the localization functor $\mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$. The functor E^f has a right derived functor $\mathbb{R}E^f$ obtained by restricting E^f to the full subcategory $\mathsf{Hot}(A\text{–}\mathsf{mod})_{\mathsf{inj}} \subset \mathsf{Hot}(A\text{–}\mathsf{mod})$ and composing it with the localization functor $\mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$. The functor $\mathbb{L}E_f$ is left adjoint to the functor $\mathbb{I}R_f$ and the functor $\mathbb{R}E^f$ is right adjoint to the functor $\mathbb{I}R_f$.

Theorem. *The functors $\mathbb{I}R_f$, $\mathbb{L}E_f$, $\mathbb{R}E^f$ are equivalences of triangulated categories if and only if the morphism f induces an isomorphism $H(A) \simeq H(B)$.*

Proof. Morphisms in $\mathsf{D}(A\text{–}\mathsf{mod})$ between shifts of the DG-module A recover the cohomology $H(A)$ and analogously for the DG-algebra B, so the "only if" assertion follows from the isomorphism $\mathbb{L}E_f(A) \simeq B$. To prove the "if" part, we will show that the adjunction morphisms $\mathbb{L}E_f(\mathbb{I}R_f(N)) \longrightarrow N$ and $M \longrightarrow \mathbb{I}R_f(\mathbb{L}E_f(M))$ are isomorphisms for any left DG-modules M over A and N over B. The former morphism is represented by the composition $B \otimes_A G \longrightarrow B \otimes_A N \longrightarrow N$ for any flat DG-module G over A endowed with a quasi-isomorphism $G \longrightarrow N$ of DG-modules over A. This composition is a quasi-isomorphism, since the morphisms $B \otimes_A G \longleftarrow A \otimes_A G \longrightarrow A \otimes_A N \simeq N$ are quasi-isomorphisms. The latter morphism is represented by the fraction $M \longleftarrow F \longrightarrow B \otimes_A F$ for any flat DG-module F over A endowed with a quasi-isomorphism $F \longrightarrow M$ of DG-modules over A. The morphism $F \simeq A \otimes_A F \longrightarrow B \otimes_A F$ is a quasi-isomorphism. □

1.8. **DG-module t-structure.** An object Y of a triangulated category D is called an *extension* of objects Z and X if there is a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$. Let $\mathsf{D} = \mathsf{D}(A\text{–}\mathsf{mod})$ denote the derived category of left DG-modules over a DG-ring A. Let $\mathsf{D}^{\geqslant 0} \subset \mathsf{D}$ denote the full subcategory formed by all DG-modules M over A such that $H^i(M) = 0$ for $i < 0$ and $\mathsf{D}^{\leqslant 0} \subset \mathsf{D}$ denote the minimal full subcategory of $\mathsf{D}(A\text{–}\mathsf{mod})$ containing the DG-modules $A[i]$ for $i \geqslant 0$ and closed under extensions and infinite direct sums.

Theorem. (a) *The pair of subcategories* $(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0})$ *defines a t-structure* [2] *on the derived category* $\mathsf{D}(A\text{–}\mathsf{mod})$.

(b) *The subcategory* $\mathsf{D}^{\leqslant 0} \subset \mathsf{D}$ *coincides with the full subcategory formed by all DG-modules* M *over* A *such such that* $H^i(M) = 0$ *for* $i > 0$ *if and only if* $H^i(A) = 0$ *for all* $i > 0$.

Proof. Part (a): clearly, one has $\mathsf{D}^{\leqslant 0}[1] \subset \mathsf{D}^{\leqslant 0}$, $\mathsf{D}^{\geqslant 0}[-1] \subset \mathsf{D}^{\geqslant 0}$, and $\operatorname{Hom}_{\mathsf{D}}(\mathsf{D}^{\leqslant 0}, \mathsf{D}^{\geqslant 0}[-1]) = 0$. It remains to construct for any DG-module M over A a closed morphism of DG-modules $F \longrightarrow M$ inducing a monomorphism on H^1 and an isomorphism on H^i for all $i \leqslant 0$ such that F can be obtained from the DG-modules $A[i]$ with $i \geqslant 0$ by iterated extensions and infinite direct sums in the homotopy category of DG-modules. This construction is similar to that of the proof of Theorem 1.4, with the following changes. One chooses a surjective morphism $M' \longrightarrow M$ onto M from a complex of free abelian groups M' with free abelian groups of cohomology so that $H^i(M') = 0$ for $i > 0$ and the maps $H^i(M') \longrightarrow H^i(M)$ are surjective for all $i \leqslant 0$. Then for $F_0 = A \otimes_{\mathbb{Z}} M'$ and $K = \ker(F_0 \to M)$ one chooses a surjective morphism $K' \longrightarrow K$ onto K from a complex of free abelian groups K' with free abelian groups of cohomology so that $H^i(K') = 0$ for $i > 1$ and the maps $H^i(K') \longrightarrow H^i(K)$ are surjective for all $i \leqslant 1$ in order to put $F_1 = A \otimes_{\mathbb{Z}} K'$, etc. The DG-module F is constructed as the total DG-module of the complex $\cdots \longrightarrow F_1 \longrightarrow F_0$ formed by taking infinite direct sums. The "only if" assertion in part (b) is clear. To prove "if", replace A with its quasi-isomorphic DG-subring $\tau_{\leqslant 0}A$ with the components $(\tau_{\leqslant 0}A)^i = A^i$ for $i < 0$, $(\tau_{\leqslant 0}A)^0 = \ker(A^0 \to A^1)$, and $(\tau_{\leqslant 0}A)^i = 0$ for $i > 0$; then notice that the canonical filtrations on DG-modules over $\tau_{\leqslant 0}A$ considered as complexes of abelian groups are compatible with the action of the ring $\tau_{\leqslant 0}A$. ☐

Remark 1. The t-structure described in part (a) of Theorem can well be degenerate, though it is clearly nondegenerate under the assumptions of part (b). Namely, one can have $\bigcap \mathsf{D}^{\leqslant 0}[i] \neq 0$. For example, take $A = k[x]$ to be the graded algebra of polynomials with one generator x of degree 1 over a field k and endow it with the zero differential. Then the graded A-module $k[x, x^{-1}]$ considered as a DG-module with zero differential belongs to the above intersection, since it can be presented as the inductive limit of the DG-modules $x^{-n}k[x]$. Moreover, take $A = k[x, x^{-1}]$, where $\deg x = 1$ and $d(x) = 0$; then $\mathsf{D}^{\geqslant 0} = 0$ and $\mathsf{D}^{\leqslant 0} = \mathsf{D}$.

Remark 2. One might wish to define a dual version of the above t-structure on $\mathsf{D}(A\text{–}\mathsf{mod})$ where $\mathsf{D}^{\geqslant 0}$ would be the minimal full subcategory of D containing the DG-modules $\operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})[i]$ for $i \leqslant 0$ and closed under extensions and infinite products, while $\mathsf{D}^{\leqslant 0}$ would consist of all DG-modules M with $H^i(M) = 0$ for $i > 0$.

The dual version of the above proof does not seem to work in this case, however, because of a problem related to nonexactness of the countable inverse limit.

Remark 3. The above construction of the DG-module t-structure can be generalized in the following way. Let D be a triangulated category with infinite direct sums. An object $C\in\mathsf{D}$ is said to be compact if the functor $\mathrm{Hom}_{\mathsf{D}}(C,-)$ preserves infinite direct sums. Let $\mathsf{C}\subset\mathsf{D}$ be a subset of objects of D consisting of compact objects and such that $\mathsf{C}[1]\subset\mathsf{C}$. Let $\mathsf{D}^{\geqslant 0}$ be the full subcategory of D formed by all objects X such that $\mathrm{Hom}_{\mathsf{D}}(C,X[-1])=0$ for all $C\in\mathsf{C}$, and let $\mathsf{D}^{\leqslant 0}$ be the minimal full subcategory of D containing C and closed under extensions and infinite direct sums. Then $(\mathsf{D}^{\leqslant 0},\mathsf{D}^{\geqslant 0})$ is a t-structure on D. Indeed, let X be an object of D. Consider the natural map into X from the direct sum of objects from C indexed by morphisms from objects of C to X; let X_1 be the cone of this map. Applying the same construction to the object X_1 in place of X, we obtain the object X_2, etc. Let Y be the homotopy inductive limit of X_i, i. e., the cone of the natural map $\bigoplus_i X_i\longrightarrow\bigoplus_i X_i$. Then $Y\in\mathsf{D}^{\geqslant 0}[-1]$ and $\mathrm{cone}(X\to Y)[-1]\in\mathsf{D}^{\leqslant 0}$.

2. Derived Categories of DG-Comodules and DG-Contramodules

2.1. **Graded comodules.** Let k be a fixed ground field. A *graded coalgebra* C over k is a graded k-vector space $C=\bigoplus_{i\in\mathbb{Z}}C^i$ endowed with a comultiplication map $C\longrightarrow C\otimes_k C$ and a counut map $C\longrightarrow k$, which must be homogeneous linear maps of degree 0 satisfying the coassociativity and counity equations. Namely, the comultiplication map must have equal compositions with the two maps $C\otimes_k C\rightrightarrows C\otimes_k C\otimes_k C$ induced by the comultiplication map, while the compositions of the comultiplication map with the two maps $C\otimes_k C\longrightarrow C$ induced by the counit map must coincide with the identity endomorphism of C.

A *graded left comodule* M over C is a graded k-vector space $M=\bigoplus_{i\in\mathbb{Z}}M^i$ endowed with a left coaction map $M\longrightarrow C\otimes_k M$, which must be a homogeneous linear map of degree 0 satisfying the coassociativity and counity equations. Namely, the coaction map must have equal compositions with the two maps $C\otimes_k M\rightrightarrows C\otimes_k C\otimes_k M$ induced by the comultiplication map and the coaction map, while the composition of the coaction map with the map $C\otimes_k M\longrightarrow M$ induced by the counit map must coincide with the identity endomorphism of M. A *graded right comodule* N over C is a graded vector space endowed with a right coaction map $N\longrightarrow N\otimes_k C$ satisfying the analogous linearity, homegeneity, coassociativity, and counity equations.

The *cotensor product* of a graded right C-comodule N and a graded left C-comodule M is the graded vector space $N\mathbin{\square}_C M$ defined as the kernel of the pair of linear maps $N\otimes_k M\rightrightarrows N\otimes_k C\otimes_k M$, one of which is induced by the right coaction map and the other by the left coaction map. There are natural isomorphisms

$C \,\square_C\, M \simeq M$ and $N \,\square_C\, C \simeq N$ for any graded left C-comodule M and graded right C-comodule N.

Graded left C-comodules of the form $C \otimes_k V$, where V is a graded vector space, are called *cofree* graded left C-comodules; analogously for graded right C-comodules. The category of graded C-comodules is an abelian category with enough injectives; injective graded C-comodules are exactly the direct summands of cofree graded C-comodules.

For any graded left C-comodules L and M, the graded vector space $\mathrm{Hom}_C(L, M)$ consists of homogeneous linear maps $f\colon L \longrightarrow M$ satisfying the condition that the coaction maps of L and M form a commutative diagram together with the map f and the map $f_*\colon C \otimes_k L \longrightarrow C \otimes_k M$ given by the formula $f_*(c \otimes x) = (-1)^{|f||c|} c \otimes f(x)$. For any graded right C-comodules R and N, the graded vector space $\mathrm{Hom}_C(R, N)$ consists of homogeneous linear maps $f\colon R \longrightarrow N$ such that the coaction maps of R and N form a commutative diagram together with the map f and the map $f_*\colon R \otimes_k C \longrightarrow N \otimes_k C$ given by the formula $f_*(x \otimes c) = f_*(x) \otimes c$. For any left C-comodule L and any graded vector space V there is a natural isomorphism $\mathrm{Hom}_C(L,\, C \otimes_k V) \simeq \mathrm{Hom}_k(L, V)$; analogously in the right comodule case.

2.2. **Graded contramodules.** A *graded left contramodule* P over a graded coalgebra C is a graded k-vector space $P = \bigoplus_{i \in \mathbb{Z}} P^i$ endowed with the following structure. Let $\mathrm{Hom}_k(C, P)$ be the graded vector space of homogeneous linear maps $C \longrightarrow P$; then a homogeneous linear map $\mathrm{Hom}_k(C, P) \longrightarrow P$ of degree 0, called the left contraaction map, must be given and the following contraassociativity and counity equations must be satisfied. For any graded vector spaces V, W, and P, define the natural isomorphism $\mathrm{Hom}_k(V \otimes_k W,\, P) \simeq \mathrm{Hom}_k(W, \mathrm{Hom}_k(V, P))$ by the formula $f(w)(v) = (-1)^{|w||v|} f(v \otimes w)$. The comultiplication and the contraaction maps induce a pair of maps $\mathrm{Hom}_k(C \otimes_k C,\, P) \simeq \mathrm{Hom}_k(C, \mathrm{Hom}_k(C, P)) \rightrightarrows \mathrm{Hom}_k(C, P)$. These maps must have equal compositions with the contraaction map; besides, the composition of the map $P \longrightarrow \mathrm{Hom}_k(C, P)$ induced by the counit map with the contraaction map must coincide with the identity endomorphism of P.

The graded vector space of *cohomomorphisms* $\mathrm{Cohom}_C(M, P)$ from a graded left C-comodule M to a graded left C-contramodule P is defined as the cokernel of the pair of linear maps $\mathrm{Hom}_k(C \otimes_k M,\, P) \longrightarrow \mathrm{Hom}_k(M, P)$, one of which is induced by the left coaction map and the other by the left contraaction map. For any graded C-contramodule P there is a natural isomorphism $\mathrm{Cohom}_C(C, P) \simeq P$.

For any graded right C-comodule N and any graded vector space V there is a natural graded left C-contramodule structure on the graded vector space of homogeneous linear maps $\mathrm{Hom}_k(N, V)$ given by the left contraaction map $\mathrm{Hom}_k(C, \mathrm{Hom}_k(N, V)) \simeq \mathrm{Hom}_k(N \otimes_k C,\, V) \longrightarrow \mathrm{Hom}_k(N, V)$ induced by the right coaction map. For any graded left C-comodule M, graded right C-comodule N, and graded vector space V, there is a natural isomorphism $\operatorname{Hom}_k(N \mathbin{\square}_C M,\ V) \simeq \operatorname{Cohom}_C(M, \operatorname{Hom}_k(N, V))$.

Graded left C-contramodules of the form $\operatorname{Hom}_k(C, V)$ are called *free* graded left C-contramodules. The category of graded left C-contramodules is an abelian category with enough projectives; projective graded left C-contramodules are exactly the direct summands of free graded left C-contramodules.

The *contratensor product* of a graded right C-comodule N and a graded left C-contramodule P is the graded vector space $N \odot_C P$ defined as the kernel of the pair of linear maps $N \otimes_k \operatorname{Hom}_k(C, P) \longrightarrow N \otimes_k P$, one of which is induced by the left contraaction map, while the other one is obtained as the composition of the map induced by the right coaction map and the map induced by the evaluation map $C \otimes_k \operatorname{Hom}_k(C, P) \longrightarrow P$ given by the formula $c \otimes f \longmapsto (-1)^{|c||f|} f(c)$. For any graded right C-comodule N and any graded vector space V there is a natural isomorphism $N \odot_C \operatorname{Hom}_k(C, V) \simeq N \otimes_k V$.

For any graded left C-contramodules P and Q, the graded vector space $\operatorname{Hom}^C(P, Q)$ consists of all homogeneous linear maps $f\colon P \longrightarrow Q$ satisfying the condition that the contraaction maps of P and Q form a commutative diagram together with the map f and the map $f_*\colon \operatorname{Hom}_k(C, P) \longrightarrow \operatorname{Hom}_k(C, Q)$ given by the formula $f_*(g) = f \circ g$. For any graded left C-contramodule Q and any graded vector space V there is a natural isomorphism $\operatorname{Hom}^C(\operatorname{Hom}_k(C, V), Q) \simeq \operatorname{Hom}_k(V, Q)$. For any right C-comodule N, any graded left C-contramodule P, and any graded vector space V, there is a natural isomorphism $\operatorname{Hom}_k(N \odot_C P,\ V) \simeq \operatorname{Hom}^C(P, \operatorname{Hom}_k(N, V))$.

The proofs of the results of this subsection are not difficult; some details can be found in [25]. The assertions stated in the last paragraph can be used to deduce the assertions of the preceding two paragraphs.

Remark. Ungraded contramodules over ungraded coalgebras can be simply defined as graded contramodules concentrated in degree 0 over graded coalgebras concentrated in degree 0. One might wish to have a forgetful functor assigning ungraded contramodules over ungraded coalgebras to graded contramodules over graded coalgebras. The construction of such a functor is delicate in two ways. Firstly, to assign an ungraded contramodule to a graded contramodule P, one has to take the direct product of its grading components $\prod_{i \in \mathbb{Z}} P^i$ rather than the direct sum, while the undraded coalgebra corresponding to a graded coalgebra C is still constructed as the direct sum $\bigoplus_{i \in \mathbb{Z}} C^i$. Analogously, to assign an ungraded comodule to a graded comodule M one takes the direct sum $\bigoplus_{i \in \mathbb{Z}} M^i$, to assign an ungraded ring to a graded ring A one takes the direct sum $\bigoplus_{i \in \mathbb{Z}} A^i$, while to assign an ungraded module to a graded module M one can take *either* the direct sum $\bigoplus_{i \in \mathbb{Z}} M^i$, *or* the direct product $\prod_{i \in \mathbb{Z}} M^i$. Secondly, there is a problem of signs in the contraassociativity equation, which is unique to graded contramodules (no signs are present in the definitions of graded algebras, modules, coalgebras, or comodules); it is resolved as follows. A morphism between graded vector spaces $f\colon V \longrightarrow W$, when it is not necessarily even, can be thought of either as a *left* or as a *right* morphism. The left and the right morphisms correspond to each other according to the sign rule $f(x) = (-1)^{|f||x|}(x)f$, where $f(x)$ is the notation for the left morphisms and $(x)f$ for the right morphisms. The above definition of graded left contramodules P is given in terms of *left* morphisms $C \longrightarrow P$; to define the functor of forgetting the grading, one has to reinterpret it in terms of *right* morphisms. The exposition in [25] presumes *right* morphisms in this definition (even if the notation $f(x)$ is being used from time to time).

2.3. **DG-comodules and contramodules.** A *DG-coalgebra* C over a field k is a graded coalgebra endowed with a differential $d\colon C \longrightarrow C$ of degree 1 with $d^2 = 0$ such that the comultiplication map $C \longrightarrow C \otimes_k C$ and the counit map $C \longrightarrow k$ are morphisms of complexes. Here the differential on $C \otimes_k C$ is defined as on the tensor product of two copies of the complex C, while the differential on k is trivial.

A *left DG-comodule* M over a DG-coalgebra C is a graded comodule M over the graded coalgebra C together with a differential $d\colon M \longrightarrow M$ of degree 1 with $d^2 = 0$ such that the left coaction map $M \longrightarrow C \otimes_k M$ is a morphism of complexes. Here $C \otimes_k M$ is considered as the tensor product of the complexes C and M over k. *Right DG-comodules* are defined in the analogous way. A *left DG-contramodule* P over C is a graded contramodule P over C endowed with a differential $d\colon P \longrightarrow P$ of degree 1 with $d^2 = 0$ such that the left contraaction map $\mathrm{Hom}_k(C, P) \longrightarrow P$ is a morphism of complexes. Here $\mathrm{Hom}_k(C, P)$ is endowed with the differential of the complex of homomorphisms from the complex C to the complex P over k.

Whenever N is a right DG-comodule and M is a left DG-comodule over a DG-coalgebra C, the cotensor product $N \mathbin{\square}_C M$ of the graded comodules N and M over the graded coalgebra C is endowed with the differential of the subcomplex of the tensor product complex $N \otimes_k M$. Whenever M is a left DG-comodule and P is a left DG-contramodule over a DG-coalgebra C, the graded vector space of cohomomorphisms $\mathrm{Cohom}_C(M, P)$ is endowed with the differential of the quotient complex of the complex of homomorphisms $\mathrm{Hom}_k(M, P)$.

Whenever N is a right DG-comodule and P is a left DG-contramodule over a DG-coalgebra C, the contratensor product $N \odot_C P$ of the graded comodule N and the graded contramodule P over the graded coalgebra C is endowed with the differential of the quotient complex of the tensor product complex $N \otimes_k P$.

For any left DG-comodules L and M over a DG-coalgebra C, the graded vector space of homomorphisms $\mathrm{Hom}_C(L, M)$ between the graded comodules L and M over the graded coalgebra C is endowed with the differential of the subcomplex of the complex of homomorphisms $\mathrm{Hom}_k(L, M)$. Differentials on the graded vector spaces of homomorphisms $\mathrm{Hom}_C(R, N)$ and $\mathrm{Hom}^C(P, Q)$ for right DG-comodules R, N and left DG-contramodules P, Q over a DG-coalgebra C are constructed in the completely analogous way. These constructions define the DG-categories $\mathsf{DG}(C\text{–}\mathsf{comod})$, $\mathsf{DG}(\mathsf{comod}\text{–}C)$, and $\mathsf{DG}(C\text{–}\mathsf{contra})$ of left DG-comodules, right DG-comodules, and left DG-contramodules over C, respectively.

All shifts, twists, infinite direct sums, and infinite direct products exist in the DG-categories of DG-comodules and DG-contramodules. The homotopy category of (the DG-category of) left DG-comodules over C is denoted by $\mathsf{Hot}(C\text{–}\mathsf{comod})$, the homotopy category of right DG-comodules over C is denoted by $\mathsf{Hot}(\mathsf{comod}\text{–}C)$, and the homotopy category of left DG-contramodules over C is denoted by $\mathsf{Hot}(C\text{–}\mathsf{contra})$.

2.4. **Injective and projective resolutions.** A DG-comodule M or a DG-contramodule P is said to be acyclic if it is acyclic as a complex of vector spaces, i. e., $H(M) = 0$ or $H(P) = 0$, respectively. The classes of acyclic DG-comodules over a DG-coalgebra C are closed under shifts, cones, and infinite direct sums, while the class of acyclic DG-contramodules is closed under shifts, cones, and infinite products. The thick subcategories of the homotopy categories $\mathsf{Hot}(C\text{–}\mathsf{comod})$, $\mathsf{Hot}(\mathsf{comod}\text{–}C)$, and $\mathsf{Hot}(C\text{–}\mathsf{contra})$ formed by the acyclic DG-comodules and DG-contramodules over C are denoted by $\mathsf{Acycl}(C\text{–}\mathsf{comod})$, $\mathsf{Acycl}(\mathsf{comod}\text{–}C)$, and $\mathsf{Acycl}(C\text{–}\mathsf{contra})$, respectively. The *derived categories* of left DG-comodules, right DG-comodules, and left DG-contramodules over C are defined as the quotient categories $\mathsf{D}(C\text{–}\mathsf{comod}) = \mathsf{Hot}(C\text{–}\mathsf{comod})/\mathsf{Acycl}(C\text{–}\mathsf{comod})$, $\mathsf{D}(\mathsf{comod}\text{–}C) = \mathsf{Hot}(\mathsf{comod}\text{–}C)/\mathsf{Acycl}(\mathsf{comod}\text{–}C)$, and $\mathsf{D}(C\text{–}\mathsf{contra}) = \mathsf{Hot}(C\text{–}\mathsf{contra})/\mathsf{Acycl}(C\text{–}\mathsf{contra})$.

A left DG-comodule M over a DG-coalgebra C is called *injective* if for any acyclic left DG-comodule L over C the complex $\mathrm{Hom}_C(L, M)$ is acyclic. The full triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{comod})$ formed by the injective DG-comodules is denoted by $\mathsf{Hot}(C\text{–}\mathsf{comod})_{\mathsf{inj}}$. A left DG-contramodule P over a DG-coalgebra C is called *projective* if for any acyclic left DG-contramodule Q over C the complex $\mathrm{Hom}^C(P, Q)$ is acyclic. The full triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{contra})$ formed by the projective DG-contramodules is denoted by $\mathsf{Hot}(C\text{–}\mathsf{contra})_{\mathsf{proj}}$.

Theorem. (a) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{comod})_{\mathsf{inj}} \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{comod})$ *is an equivalence of triangulated categories.*

(b) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{contra})_{\mathsf{proj}} \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{contra})$ *is an equivalence of triangulated categories.*

Proof will be given in subsection 5.5.

2.5. **Cotor and Coext of the first kind.** Let N be a right DG-comodule and M be a left DG-comodule over a DG-coalgebra C. Consider the cobar bicomplex $N \otimes_k M \longrightarrow N \otimes_k C \otimes_k M \longrightarrow N \otimes_k C \otimes_k C \otimes_k M \longrightarrow \cdots$ and construct its total complex by taking infinite products. Let $\mathrm{Cotor}^{C,I}(N, M)$ denote the cohomology of this total complex.

Let M be a left DG-comodule and P be a left DG-contramodule over a DG-coalgebra C. Consider the bar bicomplex $\dotsb \longrightarrow \mathrm{Hom}_k(M\otimes_k C\otimes_k C,\ P)\longrightarrow \mathrm{Hom}_k(M\otimes_k C,\,P)\longrightarrow \mathrm{Hom}_k(C,P)$ and construct its total complex by taking infinite direct sums. Let $\mathrm{Coext}^I_C(M,P)$ denote the cohomology of this total complex.

Let $k\text{–}\mathsf{vect}$ denote the category of vector spaces over k and $k\text{–}\mathsf{vect}^{\mathsf{gr}}$ denote the category of graded vector spaces over k.

Proposition. (a) *The functor* $\mathrm{Cotor}^{C,I}$ *factorizes through the Carthesian product of the derived categories of right and left DG-comodules over* C*, so there is a well-defined functor* $\mathrm{Cotor}^{C,I}\colon \mathsf{D}(\mathsf{comod}\text{–}C)\times\mathsf{D}(C\text{–}\mathsf{comod})\longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$.

(b) *The functor* Coext^I_C *factorizes through the Carthesian product of the derived categories of left DG-comodules and left DG-contramodules over* C*, so there is a well-defined functor* $\mathrm{Coext}^I_C\colon \mathsf{D}(C\text{–}\mathsf{comod})\times\mathsf{D}(C\text{–}\mathsf{contra})\longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$.

Proof. This follows from the fact that a complete and cocomplete filtered complex is acyclic whenever the associated graded complex is acyclic [10]. □

Let N be a right DG-comodule and P be a left DG-contramodule over a DG-coalgebra C. Consider the bar bicomplex $\dotsb\longrightarrow N\otimes_k \mathrm{Hom}_k(C\otimes_k C,\ P)\longrightarrow N\otimes_k \mathrm{Hom}_k(C,P)\longrightarrow N\otimes_k P$ and construct its total complex by taking infinite direct sums. Let $\mathrm{Ctrtor}^{C,I}(N,P)$ denote the cohomology of this total complex.

Let L and M be left DG-comodules over a DG-coalgebra C. Consider the cobar bicomplex $\mathrm{Hom}_k(L,M)\longrightarrow \mathrm{Hom}_k(L,\,C\otimes_k M)\longrightarrow \mathrm{Hom}_k(L,\,C\otimes_k C\otimes_k M)\longrightarrow\dotsb$ and construct its total complex by taking infinite products. Let $\mathrm{Ext}^I_C(L,M)$ denote the cohomology of this total complex.

Let P and Q be left DG-contramodules over C. Consider the cobar bicomplex $\mathrm{Hom}_k(P,Q)\longrightarrow \mathrm{Hom}_k(\mathrm{Hom}_k(C,P),Q)\longrightarrow \mathrm{Hom}_k(\mathrm{Hom}_k(C\otimes_k C,\ P),\ Q)\longrightarrow\dotsb$ and construct its total complex by taking infinite products. Let $\mathrm{Ext}^{C,I}(P,Q)$ denote the cohomology of this total complex.

Just as in the above Proposition, the functors $\mathrm{Ctrtor}^{C,I}$, Ext^I_C, and $\mathrm{Ext}^{C,I}$ factorize through the derived categories of DG-comodules and DG-contramodules.

All of these constructions can be extended to A_∞-comodules and A_∞-contramodules over A_∞-coalgebras; see Remark 7.6.

Remark. Another approach to defining derived functors of cotensor product, cohomomorphisms, contratensor product, etc., whose domains would be Carthesian products of the derived categories of DG-comodules and DG-contramodules consists in restricting these functors to the full subcategories of injective DG-comodules and projective DG-contramodules in the homotopy categories. To obtain versions of derived functors Cotor^C and Coext_C in this way one would have to restrict the functors of cotensor product and cohomomorphisms to the homotopy categories of injective DG-comodules and projective DG-contramodules in *both* arguments; resolving only one of the arguments does not provide a functor factorizable through the derived category in the other argument [25, subsection 0.2.3]. To construct version of derived functors Ctrtor^C, Ext_C, and Ext^C, on the other hand, it suffices to resolve just *one* of the arguments (the second one, the second one, and the first one, respectively). The versions of Ext_C and Ext^C so obtained coincide with the functors Hom in the derived categories. It looks unlikely that the derived functors defined in the way of this Remark should agree with the derived functors defined above in this subsection.

3. Coderived and Contraderived Categories of CDG-Modules

3.1. **CDG-rings and CDG-modules.** A *CDG-ring* (curved differential graded ring) $B = (B, d, h)$ is a triple consisting of an associative graded ring $B = \bigoplus_{i\in\mathbb{Z}} B^i$, an odd derivation $d\colon B \longrightarrow B$ of degree 1, and an element $h \in B^2$ satisfying the equations $d^2(x) = [h, x]$ for all $x \in B$ and $d(h) = 0$. A *morphism* of CDG-rings $f\colon B \longrightarrow A$ is a pair $f = (f, a)$ consisting of a morphism of graded rings $f\colon B \longrightarrow A$ and an element $a \in A^1$ satisfying the equations $f(d_B(x)) = d_A(f(x)) + [a, x]$ and $f(h_B) = h_A + d_A(a) + a^2$ for all $x \in B$, where $B = (B, d_B, h_B)$ and $A = (A, d_A, h_A)$, while the bracket $[y, z]$ denotes the supercommutator of y and z. The composition of morphisms is defined by the rule $(f, a) \circ (g, b) = (f \circ g,\ a + f(b))$. Identity morphisms are the morphisms $(\mathrm{id}, 0)$.

The element $h \in B^2$ is called the *curvature element* of a CDG-ring B. The element $a \in A^1$ is called the *change-of-connection element* of a CDG-ring morphism f.

To any DG-ring structure on a graded ring A one can assign a CDG-ring structure on the same graded ring by setting $h = 0$. This defines a functor from the category of DG-rings to the category of CDG-rings. This functor is faithful, but not fully faithful, as non-isomorphic DG-rings may become isomorphic as CDG-rings.

A *left CDG-module* (M, d_M) over a CDG-ring B is a graded left B-module $M = \bigoplus_{i\in\mathbb{Z}} M^i$ endowed with a derivation $d_M\colon M \longrightarrow M$ compatible with the derivation d_B of B and such that $d_M^2(x) = hx$ for any $x \in M$. A *right CDG-module* (N, d_N) over a CDG-ring B is a graded right B-module $N = \bigoplus_{i\in\mathbb{Z}} N^i$ endowed with a derivation $d_N\colon N \longrightarrow N$ compatible with d_B and such that $d_N^2(x) = -xh$ for any $x \in N$.

Let $f = (f, a)\colon B \longrightarrow A$ be a morphism of CDG-rings and (M, d_M) be a left CDG-module over A. Then the left CDG-module $R_f M$ over B is defined as the graded abelian group M with the graded B-module structure obtained by the restriction of scalars via f, endowed with the differential $d'_M(x) = d_M(x) + ax$ for $x \in M$. Analogously, let (N, d_N) be a right CDG-module over A. Then the right CDG-module $R_f N$ over B is defined as the graded module N over B with the graded module structure induced by f endowed with the differental $d'_N(x) = d_N(x) - (-1)^{|x|} xa$.

For any left CDG-modules L and M over B, the *complex of homomorphisms* $\mathrm{Hom}_B(L, M)$ from L to M over B is constructed using exactly the same formulas as in 1.1. It turns out that these formulas still define a complex in the CDG-module case, as the h-related terms cancel each other. The same applies to the definitions in the subsequent two paragraphs in 1.1, which all remain applicable in the CDG-module case, including the definitions of the complex of homomorphisms between right CDG-modules, the *tensor product complex* of a left and a right CDG-module, etc.

The cohomology of CDG-rings and CDG-modules is not defined, though, as their differentials may have nonzero squares. A *CDG-algebra* over a commutative ring k is a graded k-module with a k-linear CDG-ring structure.

So left and right CDG-modules over a given CDG-ring B form DG-categories, which we denote, just as the DG-categories of DG-modules, by $\mathsf{DG}(B\text{–}\mathsf{mod})$ and $\mathsf{DG}(\mathsf{mod}\text{–}B)$, respectively. All shifts, twists, infinite direct sums, and infinite direct products exist in the DG-categories of CDG-modules. The corresponding homotopy categories are denoted by $\mathsf{Hot}(B\text{–}\mathsf{mod})$ and $\mathsf{Hot}(\mathsf{mod}\text{–}B)$.

Notice that there is no obvious way to define derived categories of CDG-modules, since it is not clear what should be meant by an acyclic CDG-module. Moreover, the functors of restriction of scalars R_f related to CDG-isomorphisms f between DG-rings may well transform acyclic DG-modules to non-acyclic ones.

For a CDG-ring B, we will sometimes denote by $B^{\#}$ the graded ring B considered without its differential and curvature element (or with the zero differential and curvature element). For a left CDG-module M and a right CDG-module N over B, we denote by $M^{\#}$ and $N^{\#}$ the corresponding graded modules (or CDG-modules with zero differential) over $B^{\#}$.

3.2. **Some constructions for DG-categories.** The reader will easily recover the details of the constructions sketched below.

Let DG be a DG-category. Define the DG-category $\mathsf{DG}^\natural$ by the following construction. An object of $\mathsf{DG}^\natural$ is a pair (Z, t), where Z in an object of DG and $t \in \mathrm{Hom}^{-1}_{\mathsf{DG}}(Z, Z)$ is a contracting homotopy with zero square, i. e., $d(t) = \mathrm{id}_X$ and $t^2 = 0$. Morphisms $(Z', t') \longrightarrow (Z'', t'')$ of degree n in $\mathsf{DG}^\natural$ are morphisms $f\colon Z' \longrightarrow Z''$ of degree $-n$ in DG such that $d(f) = 0$ in DG. The differential on the complex of morphisms in $\mathsf{DG}^\natural$ is given by the supercommutator with t, i. e., $d^\natural(f) = t''f - (-1)^{|f|}ft'$.

Obviously, all twists of objects by their Maurer–Cartan endomorphisms exist in the DG-category $\mathsf{DG}^\natural$. Shifts, (finite or infinite) direct sums, or direct products exist in $\mathsf{DG}^\natural$ whenever they exist in DG.

Let $B = (B, d, h)$ be a CDG-ring. Construct the DG-ring $B^\sim = (B^\sim, \partial)$ as follows. The graded ring $B^\sim$ is obtained by changing the sign of the grading in the ring $B[\delta]$, which is in turn constructed by adjoining to B an element δ of degree 1 with the relations $[\delta, x] = d(x)$ for $x \in B$ and $\delta^2 = h$. The differential $\partial = \partial/\partial\delta$ is defined by the rules $\partial(\delta) = 1$ and $\partial(x) = 0$ for all $x \in B$. This construction can be extended to an equivalence between the categories of CDG-rings and acyclic DG-rings [25] (here a DG-ring is called acyclic if its cohomology is the zero ring). There is a natural isomorphism of DG-categories $\mathsf{DG}(B\text{–}\mathsf{mod})^\natural \simeq \mathsf{DG}(B^\sim\text{–}\mathsf{mod})$.

Let DG be a DG-category with shifts and cones. Denote by $\natural\colon X \longmapsto X^\natural$ the functor $Z^0(\mathsf{DG}) \longrightarrow Z^0(\mathsf{DG}^\natural)$ assigning to an object X the object $\operatorname{cone}(\mathrm{id}_X)[-1]$ with its standard constructing homotopy t. This functor can be extended in a natural way to a fully faithful functor $\mathsf{DG}^\# \longrightarrow Z^0(\mathsf{DG}^\natural)$, since not necessarily closed morphisms of degree 0 also induce closed morphisms of the cones of identity endomorphisms commuting with the standard contracting homotopies.

The functor $\natural$ has left and right adjoint functors G^+, $G^-\colon Z^0(\mathsf{DG}^\natural) \longrightarrow Z^0(\mathsf{DG})$, which are given by the rules $G^+(Z,t) = Z$ and $G^-(Z,t) = Z[1]$, so G^+ and G^- only differ by a shift. Whenever all infinite direct sums (products) exist in the DG-category DG, the functors G^+, G^-, and $\natural$ preserve them.

In particular, when $\mathsf{DG} = \mathsf{DG}(B\text{–}\mathsf{mod})$, the category $Z^0(\mathsf{DG}^\natural)$ can be identified with the category of graded left $B^\#$-modules in such a way that the functor $\natural$ becomes the functor $M \longmapsto M^\#$ of forgetting the differential. The category $\mathsf{DG}(B\text{–}\mathsf{mod})^\#$ is then identified with the full subcategory consisting of all graded $B^\#$-modules that admit a structure of CDG-module over B.

For any DG-category DG, objects of the DG-category $\mathsf{DG}^{\natural\natural}$ are triples (W, t, s), where W is an object of DG and t, $s\colon W \longrightarrow W$ are endomorphisms of degree -1 and 1, respectively, satisfying the equations $t^2 = 0 = s^2$, $ts + st = \mathrm{id}_W$, $d(t) = \mathrm{id}_W$, and $d(s) = 0$. Assuming that shifts and cones exist in DG, there is a natural fully faithful functor $\mathsf{DG} \longrightarrow \mathsf{DG}^{\natural\natural}$ given by the formula $W = \operatorname{cone}(\mathrm{id}_X)[-1]$.

This functor is an equivalence of DG-categories whenever all twists of objects exist in DG and all images of idempotent endomorphisms exist in $Z^0(\mathsf{DG})$. Indeed, to recover the object X from the object W, it suffices to take the image of the closed idempotent endomorphism ts of the twisted object $W(-s)$.

In particular, there is a natural equivalence of DG-categories $\mathsf{DG}(B\text{–}\mathsf{mod}) \simeq \mathsf{DG}(B^{\sim\sim}\text{–}\mathsf{mod})$. So the DG-category of CDG-modules over an arbitrary CDG-ring is equivalent to the DG-category of DG-modules over a certain acyclic DG-ring.

The "almost involution" $\mathsf{DG} \longmapsto \mathsf{DG}^\natural$ is not defined on the level of homotopy categories. Indeed, if DG is the DG-category of complexes over an additive category A containing images of its idempotent endomorphisms, then all objects of the DG-category $\mathsf{DG}^\natural$ are contractible, while the DG-category $\mathsf{DG}^{\natural\natural}$ is again equivalent to the DG-category of complexes over A.

3.3. **Coderived and contraderived categories.** Let B be a CDG-ring. Then the category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$ of left CDG-modules and closed morphisms between them is an abelian category, so one can speak about exact triples of CDG-modules. We presume that morphisms constituting an exact triple are closed. An exact triple of CDG-modules can be also viewed as a finite complex of CDG-modules, so its total CDG-module can be assigned to it.

A left CDG-module L over B is called *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing the total CDG-modules of exact triples of left CDG-modules over B. The thick subcategory of absolutely acyclic CDG-modules is denoted by $\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod}) \subset \mathsf{Hot}(B\text{–}\mathsf{mod})$. The quotient category $\mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod}) = \mathsf{Hot}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ is called the *absolute derived category* of CDG-modules over B.

The thick subcategory $\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ is often too small and some ways of enlarging it turn out to be useful. A left CDG-module over B is called *coacyclic* if it belongs to the minimal triangulated subcategory of the homotopy category $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing the total CDG-modules of exact triples of left CDG-modules over B and closed under infinite direct sums. The coacyclic CDG-modules form a thick subcategory of the homotopy category, since a triangulated category with infinite direct sums contains images of its idempotent endomorphisms [21]. This thick subcategory is denoted by $\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod}) \subset \mathsf{Hot}(B\text{–}\mathsf{mod})$. It is the minimal thick subcategory of $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing $\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ and closed under infinite direct sums. The *coderived category* of left CDG-modules over B is defined as the quotient category $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod}) = \mathsf{Hot}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$.

Analogously, a left CDG-module over B is called *contraacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing the total CDG-modules of exact triples of left CDG-modules over B and closed under infinite products. The thick subcategory formed by all contraacyclic CDG-modules is denoted by $\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \subset \mathsf{Hot}(B\text{–}\mathsf{mod})$. It is the minimal thick subcategory of $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing $\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ and closed under infinite products. The *contraderived category* of left CDG-modules over B is defined as the quotient category $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) = \mathsf{Hot}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$.

All the above definitions can be repeated verbatim for right CDG-modules, so there are thick subcategories $\mathsf{Acycl}^{\mathsf{co}}(\mathsf{mod}\text{–}B)$, $\mathsf{Acycl}^{\mathsf{ctr}}(\mathsf{mod}\text{–}B)$, and $\mathsf{Acycl}^{\mathsf{abs}}(\mathsf{mod}\text{–}B)$ in $\mathsf{Hot}(\mathsf{mod}\text{–}B)$ with the corresponding quotient categories $\mathsf{D}^{\mathsf{co}}(\mathsf{mod}\text{–}B)$, $\mathsf{D}^{\mathsf{ctr}}(\mathsf{mod}\text{–}B)$, and $\mathsf{D}^{\mathsf{abs}}(\mathsf{mod}\text{–}B)$.

Remark 1. When B is a DG-ring, the coderived and contraderived categories of DG-modules over B still differ from the derived category of DG-modules and between each other, in general. Indeed, they can even all differ when B is simply a ring considered as a DG-ring concentrated in degree 0. For example, let $\Lambda = k[\varepsilon/\varepsilon^2]$ be the exterior algebra in one variable over a field k. Then there is an infinite in both directions, acyclic, noncontractible complex of free and cofree Λ-modules

$\cdots \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \cdots$, where the differentials are given by the action of ε. This complex of Λ-modules is neither coacyclic, nor contraacyclic. Furthermore, let $\cdots \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow k \longrightarrow 0$ and $0 \longrightarrow k \longrightarrow \Lambda \longrightarrow \Lambda \longrightarrow \cdots$ be the complexes of canonical truncation of the above doubly infinite complex. Then the former of these two complexes is contraacylic and the latter is coacyclic, but not the other way. There also exist finite-dimensional DG-modules (over finite-dimensional DG-algebras over fields) that are acyclic, but neither coacyclic, nor contraacyclic. The simplest example of this kind is that of the DG-algebra with zero differential $B = k[\varepsilon]/\varepsilon^2$, where $\deg \varepsilon = 1$, and the DG-module M over B constructed as the free graded B-module with one homogeneous generator m for which $d(m) = \varepsilon m$.

Remark 2. Much more generally, to define the coderived (contraderived) category, it suffices to have a DG-category DG with shifts, cones, and arbitrary infinite direct sums (products), for which the additive category $Z^0(\mathsf{DG})$ is endowed with an exact category structure. Examples of such a situation include not only the categories of CDG-modules, but also, e. g., the category of complexes over an exact category [25]. Then one considers the total objects of exact triples in $Z^0(\mathsf{DG})$ as objects of the homotopy category $H^0(\mathsf{DG})$ and takes the quotient category of $H^0(\mathsf{DG})$ by the minimal triangulated subcategory containing all such objects and closed under infinite direct sums (products). It may be advisable to require the class of exact triples in $Z^0(\mathsf{DG})$ to be closed with respect to infinite direct sums (products) when working with this construction. A deeper notion of an *exact DG-category* is discussed in Remark 3.5 below, where some results provable in this setting are formulated.

3.4. **Bounded cases.** Let B a DG-ring. Denote by $\mathsf{Hot}^+(B\text{–}\mathsf{mod})$ and $\mathsf{Hot}^-(B\text{–}\mathsf{mod})$ the homotopy categories of DG-modules over B bounded from below and from above, respectively. That is, $M \in \mathsf{Hot}^+(B\text{–}\mathsf{mod})$ iff $M^i = 0$ for all $i \ll 0$ and $M \in \mathsf{Hot}^-(B\text{–}\mathsf{mod})$ iff $M^i = 0$ for all $i \gg 0$. Set $\mathsf{Acycl}^\pm(B\text{–}\mathsf{mod}) = \mathsf{Acycl}(B\text{–}\mathsf{mod}) \cap \mathsf{Hot}^\pm(B\text{–}\mathsf{mod})$, and analogously for $\mathsf{Acycl}^{\mathsf{co},\pm}(B\text{–}\mathsf{mod})$ and $\mathsf{Acycl}^{\mathsf{ctr},\pm}(B\text{–}\mathsf{mod})$.

Clearly, the thick subcategories $\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ and $\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ are contained in the thick subcategory $\mathsf{Acycl}(B\text{–}\mathsf{mod})$ for any DG-ring B.

Theorem 1. *Assume that $B^i = 0$ for all $i > 0$. Then*

(a) $\mathsf{Acycl}^{\mathsf{co},+}(B\text{–}\mathsf{mod}) = \mathsf{Acycl}^+(B\text{–}\mathsf{mod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr},-}(B\text{–}\mathsf{mod}) = \mathsf{Acycl}^-(B\text{–}\mathsf{mod})$*;*

(b) *the natural functors* $\mathsf{Hot}^\pm(B\text{–}\mathsf{mod})/\mathsf{Acycl}^\pm(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ *are fully faithful; and*

(c) *the natural functors* $\mathsf{Hot}^\pm(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{co},\pm}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Hot}^\pm(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{ctr},\pm}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *are fully faithful;*

(d) *the triangulated subcategories* $\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *generate the triangulated subcategory* $\mathsf{Acycl}(B\text{–}\mathsf{mod})$.

Proof. For a DG-module M over B, denote by $\tau_{\leqslant n} M$ the subcomplexes of canonical filtration of M considered as a complex of abelian groups. Due to the condition on B, these are DG-submodules. Notice that the quotient DG-modules $\tau_{\leqslant n+1}M/\tau_{\leqslant n}M$ are contractible for any acyclic DG-module M. Let $M \in \mathsf{Acycl}^{\mathsf{co},+}(B\text{–}\mathsf{mod})$. Then one has $\tau_{\leqslant n}M = 0$ for n small enough, hence $\tau_{\leqslant n}M$ is coacyclic for all n. It remains to use the exact triple $\bigoplus_n \tau_{\leqslant n}M \longrightarrow \bigoplus_n \tau_{\leqslant n}M \longrightarrow M$ in order to show that M is coacyclic. The proof of the second assertion of (a) is analogous. To check (b) and (c), it suffices to notice that whenever a DG-module M is acyclic, coacyclic, or contraacyclic, the DG-modules $\tau_{\leqslant n}M$ and $M/\tau_{\leqslant n}M$ belong to the same class. The same observation allows to deduce (d) from (a). □

Theorem 2. *Assume that $B^i = 0$ for all $i < 0$, the ring B^0 is semisimple, and $B^1 = 0$. Then*

(a) $\mathsf{Acycl}^{\mathsf{co},-}(B\text{–}\mathsf{mod}) = \mathsf{Acycl}^{-}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr},+}(B\text{–}\mathsf{mod}) = \mathsf{Acycl}^{+}(B\text{–}\mathsf{mod})$*;*

(b) *the natural functors* $\mathsf{Hot}^{\pm}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\pm}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ *are fully faithful; and*

(c) *the natural functors* $\mathsf{Hot}^{\pm}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{co},\pm}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Hot}^{\pm}(B\text{–}\mathsf{mod})/\mathsf{Acycl}^{\mathsf{ctr},\pm}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *are fully faithful;*

(d) *the triangulated subcategories* $\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *generate the triangulated subcategory* $\mathsf{Acycl}(B\text{–}\mathsf{mod})$*.*

Proof. Analogous to the proof of Theorem 1, with the only change that instead of the DG-submodules $\tau_{\leqslant n}M$ one uses the (nonfunctorial) DG-submodules $\sigma_{\geqslant n}M \subset M$, which are constructed as follows. For any DG-module M over B and an integer n, choose a complementary B^0-submodule $K \subset M^n$ to the submodule $\ker(d^n\colon M^n \to M^{n+1}) \subset M^n$. Set $(\sigma_{\geqslant n}M)^i = 0$ for $i < n$, $(\sigma^n_{\geqslant n}M)^n = K$, and $(\sigma_{\geqslant n}M)^i = M^i$ for $i > n$. Then $\sigma_{\geqslant n}M$ is a DG-submodule of M and the quotient DG-modules $\sigma_{\geqslant n-1}M/\sigma_{\geqslant n}M$ are contractible for any acyclic DG-module M over B. □

3.5. **Noetherian case.** Let B be a CDG-ring. Denote by $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{inj}})$ the full subcategory of the homotopy category of left CDG-modules over B formed by all the CDG-modules M for which the graded module $M^{\#}$ over the graded ring $B^{\#}$ is injective. Assume that the graded ring $B^{\#}$ is graded left Noetherian, i. e., satisfies the ascending chain condition for homogeneous left ideals. The next Theorem provides a semiorthogonal decomposition of the homotopy category $\mathsf{Hot}(B\text{–}\mathsf{mod})$ and describes the coderived category $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ in terms of injective resolutions.

Theorem. (a) *For any CDG-modules $L \in \mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ and $M \in \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{inj}})$, the complex $\mathrm{Hom}_B(L, M)$ is acyclic.*

(b) *The composition of functors* $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{inj}}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

The assertion (a) does not depend on the Noetherianity assumption on B. The assertion (b) apparently does.

Proof. Part (a): since the functor $\mathrm{Hom}_B(-,M)$ transforms shifts, cones, and infinite direct sums into shifts, shifted cones, and infinite products, it suffices to consider the case when L is the total CDG-module of an exact triple of CDG-modules. Since $M^\#$ is an injective graded $B^\#$-module, the complex $\mathrm{Hom}_B(L,M)$ is the total complex of an exact triple of complexes, and hence an acyclic complex, in this case.

Part (b): by Lemma 1.3, it suffices to construct for any CDG-module M a morphism $M \longrightarrow J$ in the homotopy category of CDG-modules over B such that the graded $B^\#$-module $M^\#$ is injective and the cone of the morphism $M \longrightarrow J$ is coacyclic. Choose an embedding $M^\# \longrightarrow I_0$ of the graded $B^\#$-module $M^\#$ into an injective graded $B^\#$-module I_0. For any graded left $B^\#$-module L, denote by $G^-(L)$ the CDG-module over B cofreely cogenerated by L. Explicitly, $G^-(L)$ as a graded abelian group consists of all formal expressions of the form $d^{-1}x + py$, where x, $y \in L$ and $\deg d^{-1}x = \deg x - 1$, $\deg py = \deg y$. The differential on $G^-(L)$ is given by the formulas $d(d^{-1}x) = px$ and $d(py) = d^{-1}(hy)$, where h is the curvature element of B. The action of B is given by the formulas $b(d^{-1}x) = (-1)^{|b|}d^{-1}(bx)$ and $b(px) = p(bx) + d^{-1}(d(b)x)$. There is a bijective correspondence between morphisms of graded $B^\#$-modules $f\colon M^\# \longrightarrow L$ and closed morphisms of CDG-modules $g\colon M \longrightarrow G^-(L)$ which is described by the formula $g(z) = d^{-1}(f(dz)) + pf(z)$. There is also an exact triple of graded $B^\#$-modules $L[-1] \longrightarrow G^-(L)^\# \longrightarrow L$. So, in particular, we have a closed embedding of CDG-modules $M \longrightarrow G^-(I_0)$, where the graded $B^\#$-module $G^-(I_0)$ is injective. Let K be the cokernel of the embedding $M \longrightarrow G^-(I_0)$ (taken in the abelian category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$ of CDG-modules and closed morphisms between them). Applying the same construction to the CDG-module K in place of M, we obtain the CDG-module $G^-(I_{-1})$, etc. Let J be the total CDG-module of the complex of CDG-modules $G^-(I_0) \longrightarrow G^-(I_{-1}) \longrightarrow \cdots$ formed by taking infinite direct sums. Since the graded ring $B^\#$ is Noetherian, the class of injective graded $B^\#$-modules is closed with respect to infinite direct sums, so the graded $B^\#$-module $J^\#$ is injective.

It remains to show that the cone of the closed morphism $M \longrightarrow J$ is coacyclic. Here one uses the general fact that the total CDG-module of an exact complex of CDG-modules $0 \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \cdots$ bounded from below is coacyclic. To prove this, notice that our total CDG-module E is the inductive limit of the total CDG-modules X_n of the finite exact complexes of canonical truncation $0 \longrightarrow E_0 \longrightarrow \cdots \longrightarrow E_n \longrightarrow K_n \longrightarrow 0$. So there is an exact triple of CDG-modules and closed morphisms $0 \longrightarrow \bigoplus_n X_n \longrightarrow \bigoplus_n X_n \longrightarrow E \longrightarrow 0$. Clearly, the CDG-modules X_n are coacyclic. Now it remains to notice either that the total CDG-module of this exact triple is coacyclic by definition, or that this exact sequence splits in $\mathsf{DG}(B\text{–}\mathsf{mod})^\#$, and consequently this total CDG-module is even contractible.

When B is a CDG-algebra over a field or a DG-ring, there is an alternative proof analogous to the proof of Theorem 4.4 below. □

Remark. The assertions of Theorem can be extended to the more general setting of *exact DG-categories* satisfing appropriate conditions. Namely, let DG be a DG-category with shifts and cones. The category DG is said to be an exact DG-category if an exact category structure is defined on $Z^0(\mathsf{DG}^\natural)$ such that the following conditions formulated in terms of the functor $\natural$ are satisfied. Firstly, morphisms in $Z^0(\mathsf{DG})$ whose images are admissible monomorphisms in $Z^0(\mathsf{DG}^\natural)$ must admit cokernel morphisms in $Z^0(\mathsf{DG}^\natural)$ coming from morphisms in $Z^0(\mathsf{DG})$. Analogously, morphisms in $Z^0(\mathsf{DG})$ whose images are admissible epimorphisms in $Z^0(\mathsf{DG}^\natural)$ must admit kernel morphisms in $Z^0(\mathsf{DG}^\natural)$ coming from morphisms in $Z^0(\mathsf{DG})$. Secondly, the natural triples $Z \longrightarrow G^+(Z)^\natural \longrightarrow Z[-1]$ in $Z^0(\mathsf{DG}^\natural)$ must be exact for all objects $Z \in \mathsf{DG}^\natural$. In this case the category $Z^0(\mathsf{DG})$ itself acquires an exact category structure in which a triple is exact if and only if its image is exact in $Z^0(\mathsf{DG}^\natural)$; moreover, the functors G^+ and G^- preserve and reflect exactness of triples. Assume further that all infinite direct sums exist in DG and the class of exact triples in $Z^0(\mathsf{DG}^\natural)$ is closed under infinite direct sums. Then the coderived category of DG is defined as the quotient category of the homotopy category $H^0(\mathsf{DG})$ by the minimal triangulated subcategory containing the total objects of exact triples in $Z^0(\mathsf{DG})$ and closed under infinite direct sums; the objects belonging to the latter subcategory are called coacyclic. The complex of homomorphisms from any coacyclic object to any object of DG whose image is injective with respect to the exact category $Z^0(\mathsf{DG}^\natural)$ is acyclic. Whenever there are enough injectives in the exact category $Z^0(\mathsf{DG}^\natural)$ and the class of injectives is closed under infinite direct sums, the full triangulated subcategory of the homotopy category $H^0(\mathsf{DG})$ consisting of all the objects whose images are injective in $Z^0(\mathsf{DG}^\natural)$ is equivalent to the coderived category of DG.

3.6. **Artinian case.** Let B be a CDG-ring. Denote by $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ the full subcategory of the homotopy category of left CDG-modules over B formed by all the CDG-modules L for which the graded $B^\#$-module $L^\#$ is projective. Assume that the graded ring $B^\#$ is graded right Artinian, i. e., satisfies the descending chain condition for homogeneous right ideals. The next Theorem provides another semiorthogonal decomposition of the homotopy category $\mathsf{Hot}(B\text{–}\mathsf{mod})$ and describes the contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ in terms of projective resolutions.

Theorem. (a) *For any CDG-modules* $L \in \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}})$ *and* $M \in \mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$, *the complex* $\mathrm{Hom}_B(L, M)$ *is acyclic.*

(b) *The composition of functors* $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ *is an equivalence of triangulated categories.*

The assertion (a) does not depend on the Artinianity assumption on B. In the assertion (b), the Artinianity assumption can be slightly weakened (see the proof).

Proof. Analogous to the proof of Theorem 3.5, with the following changes. Part (a) is simple. In part (b), one needs to use the construction of the CDG-module $G^+(L)$ over B freely generated by a graded $B^{\#}$-module L. Elements of $G^+(L)$ are formal expressions of the form $x + dy$ with x, $y \in L$; the action of B and the differential d on $G^+(L)$ are given by the equations in the definition of a CDG-module. There is a natural closed isomorphism of CDG-modules $G^+(L) \simeq G^-(L)[-1]$. Then one has to use the fact that the class of projective graded left modules over a right Artinian graded ring is closed under infinite products. Indeed, the class of flat graded left modules over a right coherent graded ring is closed under infinite products [9], and every flat graded left module over a left perfect graded ring is projective [1]. In the rest of the proof one can, e. g., use the Mittag-Leffler condition for vanishing of the derived functor of projective limit (of abelian groups). When B is a CDG-algebra over a field or a DG-ring, one can also argue as in the proof of Theorem 4.4 below. ☐

3.7. **Finite homological dimension case.** Let B be a CDG-ring. Assume that the graded ring $B^{\#}$ has a finite left homological dimension (i. e., the homological dimension of the category of graded left $B^{\#}$-modules is finite). The next Theorem identifies the coderived, contraderived, and absolute derived categories of left CDG-modules over B and describes them in terms of projective and injective resolutions.

Theorem. (a) *The three thick subcategories* $\mathsf{Acycl}^{\mathsf{co}}(B\text{–}\mathsf{mod})$, $\mathsf{Acycl}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$, *and* $\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *in the homotopy category* $\mathsf{Hot}(B\text{–}\mathsf{mod})$ *coincide.*

(b) *The compositions of functors* $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{inj}}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{proj}}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *are both equivalences of triangulated categories.*

Proof. We will show that the minimal triangulated subcategory of $\mathsf{Hot}(B\text{–}\mathsf{mod})$ containing the total CDG-modules of exact triples of CDG-modules and the triangulated subcategory $\mathsf{Hot}(B\text{–}\mathsf{mod})_{\mathsf{inj}}$ form a semiorthogonal decomposition of $\mathsf{Hot}(B\text{–}\mathsf{mod})$, as do the triangulated subcategory $\mathsf{Hot}(B\text{–}\mathsf{mod})_{\mathsf{proj}}$ and the same minimal triangulated subcategory. This implies both (b) and the assertion that this minimal triangulated subcategory is closed under infinite direct sums and products, which is even stronger than (a). It suffices to construct for any CDG-module M over B closed CDG-module morphisms $F \longrightarrow B \longrightarrow J$ whose cones belong to the mentioned minimal triangulated subcategory, while the graded $B^{\#}$-modules $F^{\#}$ and $J^{\#}$ are projective and injective, respectively. To do so, we start as in the proofs of Theorems 3.5 and 3.6, constructing an exact complex of CDG-modules $0 \longrightarrow M \longrightarrow G^-(I_0) \longrightarrow G^-(I_{-1}) \longrightarrow \cdots$ with injective graded $B^{\#}$-modules $G^-(I_{-n})^{\#}$, and an analogous left CDG-module resolution with projective graded $B^{\#}$-modules. Since the graded left homological dimension of $B^{\#}$ is finite, there exists a nonnegative integer d such that the image K of the morphism $G^-(I_{-d+1}) \longrightarrow G^-(I_{-d})$ taken in the category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$ is injective as a graded $B^{\#}$-module. It remains to take J to be the total CDG-module of the finite complex of CDG-modules $G^-(I_0) \longrightarrow \cdots \longrightarrow G^-(I_{-d-1}) \longrightarrow K$. ☐

Remark. Let DG be an exact DG-category in the sense of Remark 3.5. Assume that the exact category $Z^0(\mathsf{DG}^\natural)$ has a finite homological dimension and enough injectives. Then the minimal triangulated subcategory of $H^0(\mathsf{DG})$ containing the total objects of exact triples in $Z^0(\mathsf{DG})$ and the full triangulated subcategory of $H^0(\mathsf{DG})$ consisting of all the objects whose images are injective in $Z^0(\mathsf{DG}^\natural)$ form a semiorthogonal decomposition of $H^0(\mathsf{DG})$. In particular, this minimal triangulated subcategory is closed under infinite direct sums. It would be interesting to deduce the latter conclusion without assuming existence of injectives, but only finite homological dimension of the exact category $Z^0(\mathsf{DG}^\natural)$ together with existence of infinite direct sums in DG and their exactness in $Z^0(\mathsf{DG}^\natural)$.

So we have $\mathsf{D^{co}}(B\text{–}\mathsf{mod}) = \mathsf{D^{ctr}}(B\text{–}\mathsf{mod})$ for a CDG-ring B such that the graded ring $B^{\#}$ has a finite left homological dimension. There are also some other situations when the coderived and contraderived categories of CDG-modules over a given CDG-ring B are naturally equivalent.

In particular, suppose that the graded ring B is quasi-Frobenius, i. e., the classes of injective and projective graded left B-modules coincide. Then the class of injective graded left B-modules is closed under infinite direct sums and the class of projective graded left B-modules is closed under infinite products, so the conclusions of Theorems 3.5 and 3.6 hold. Thus we have $\mathsf{D^{co}}(B\text{–}\mathsf{mod}) \simeq \mathsf{Hot}(B\text{–}\mathsf{mod_{inj}}) = \mathsf{Hot}(B\text{–}\mathsf{mod_{proj}}) \simeq \mathsf{D^{ctr}}(B\text{–}\mathsf{mod})$. There is also a natural isomorphism $\mathsf{D^{co}}(B\text{–}\mathsf{mod}) \simeq \mathsf{D^{ctr}}(B\text{–}\mathsf{mod})$ when B is a CDG-algebra over a field k such that the underlying graded algebra $B^{\#}$ is finite-dimensional; see 5.2.

3.8. **Tor and Ext of the second kind.** Let B be a CDG-algebra over a commutative ring k. Our goal is to define differential derived functors of the second kind

$$\mathrm{Tor}^{B,II}\colon \mathsf{D^{abs}}(\mathsf{mod}\text{–}B) \times \mathsf{D^{abs}}(B\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod^{gr}}$$

$$\mathrm{Ext}_B^{II}\colon \mathsf{D^{abs}}(B\text{–}\mathsf{mod})^{\mathsf{op}} \times \mathsf{D^{abs}}(B\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod^{gr}}$$

and give a simple categorical interpretation of these definitions in the finite homological dimension case. First we notice that there are enough projective and injective objects in the category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$, and these objects remain projective and injective in the category of graded $B^{\#}$-modules. To construct these projectives and injectives, it suffices to apply the functors G^+ and G^- to projective and injective graded $B^{\#}$-modules (see the proofs of Theorems 3.5–3.6).

Let N and M be a right and a left CDG-module over B. We will consider CDG-module resolutions $\cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$ and $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, i. e., exact sequences of this form in the abelian categories $Z^0\mathsf{DG}(\mathsf{mod}\text{–}B)$

and $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$. To any such pair of resolutions we assign the total complex $T = \mathrm{Tot}^{\sqcap}(Q_\bullet \otimes_B P_\bullet)$ of the tricomplex $Q_n \otimes_B P_m$ formed by taking infinite products along the diagonal planes. Whenever all the graded right $B^\#$-modules $Q_n^\#$ are flat, the cohomology of the total complex T does not depend of the choice of the resolution $P_\bullet$ and vice versa. Indeed, whenever all graded modules $Q_n^\#$ are flat, the natural map $\mathrm{Tot}^{\sqcap}(Q_\bullet \otimes_B P_\bullet) \longrightarrow \mathrm{Tor}^{\sqcap}(Q_\bullet \otimes_B M)$ is a quasi-isomorphism, since it is an quasi-isomorphism on the quotient complexes by the components of the complete decreasing filtration induced by the canonical filtration of the complex $P_\bullet$.

Using existence of projective resolutions in the categories $Z^0\mathsf{DG}(\mathsf{mod}\text{–}B)$ and $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$, one can see that the assignment according to which the cohomology $H(T)$ of the total complex T corresponds to a pair (N, M) whenever either all the graded right $B^\#$-modules $Q_n^\#$, or all the graded left $B^\#$-modules $P_n^\#$ are flat defines a functor on the category $Z^0\mathsf{DG}(\mathsf{mod}\text{–}B) \times Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$. It factorizes through the Carthesian product of the homotopy categories, defining a triangulated functor of two variables $H^0\mathsf{DG}(\mathsf{mod}\text{–}B) \times H^0\mathsf{DG}(B\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod}^{\mathsf{gr}}$. The latter functor factorizes through the Carthesian product of the absolute derived categories, hence the functor which we denote by $\mathrm{Tor}^{B,II}$.

Analogously, let L and M be left CDG-modules over B. Consider CDG-module resolutions $\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow L \longrightarrow 0$ and $0 \longrightarrow M \longrightarrow R_0 \longrightarrow R_{-1} \longrightarrow \cdots$, i. e., exact sequences of this form in the category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$. To any such pair of resolutions we assign the total complex $T = \mathrm{Tot}^{\oplus}(\mathrm{Hom}_B(P_\bullet, R_\bullet))$ of the tricomplex $\mathrm{Hom}_B(P_n, R_m)$ formed by taking infinite direct sums along the diagonal planes. The rule according to which the cohomology $H(T)$ corresponds to a pair (L, M) whenever either all the graded left $B^\#$-modules $P_n^\#$ are projective, or all the graded left $B^\#$-modules $R_m^\#$ are injective defines a functor on the category $Z^0\mathsf{DG}(B\text{–}\mathsf{mod})^{\mathsf{op}} \times Z^0\mathsf{DG}(B\text{–}\mathsf{mod})$. This functor factorizes through the Cartesian product of the homotopy categories, defining a triangulated functor of two variables, which in turn factorizes through the Carthesian product of the absolute derived categories. Hence the functor which we denote by Ext_B^{II}.

Now assume that the graded ring $B^\#$ has a finite weak homological dimension, i. e., the homological dimension of the functor Tor between graded right and left $B^\#$-modules is finite. Let $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{fl}})$ and $\mathsf{Hot}(\mathsf{mod}_{\mathsf{fl}}\text{–}B)$ denote the homotopy categories of left and right CDG-modules over B that are flat as graded $B^\#$-modules. Using Lemma 1.6 and the construction from the proof of Theorem 3.6, one can show that the natural functor $\mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{fl}})/\mathsf{Acycl}^{\mathsf{abs}}(B\text{–}\mathsf{mod}) \cap \mathsf{Hot}(B\text{–}\mathsf{mod}_{\mathsf{fl}}) \longrightarrow \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ is an equivalence of triangulated categories. To prove the analogous assertion for right CDG-modules over B, it suffices to pass to the opposite CDG-ring $B^{\mathrm{op}} = (B^{\mathrm{op}}, d_{B^{\mathrm{op}}}, h_{B^{\mathrm{op}}})$, which coincides with B as a graded abelian group and has the multiplication, differential, and curvature element defined by the formulas $a^{\mathrm{op}}b^{\mathrm{op}} =$

$(-1)^{|a||b|}(ba)^{\mathrm{op}}$, $d_{B^{\mathrm{op}}}(b^{\mathrm{op}}) = d_B(b)^{\mathrm{op}}$, and $h_{B^{\mathrm{op}}} = -h_B^{\mathrm{op}}$, where b^{op} denotes the element of B^{op} corresponding to an element $b \in B$.

The tensor product of CDG-modules $N \otimes_B M$ over B is acyclic whenever one of the CDG-modules N and M is coacyclic and another one is flat as a graded $B^\#$-module. As in 1.6, it follows that $N \otimes_B M$ is also acyclic whenever one of the CDG-modules N and M is simultaneously coacyclic and flat as a graded $B^\#$-module. So restricting the functor of tensor product over B to either of the Carthesian products $\mathsf{Hot}(\mathsf{mod_{fl}}\text{–}B) \times \mathsf{Hot}(B\text{–}\mathsf{mod})$ or $\mathsf{Hot}(\mathsf{mod}\text{–}B) \times \mathsf{Hot}(B\text{–}\mathsf{mod_{fl}})$, one can construct the derived functor of tensor product of CDG-modules, which is defined on the Carthesian product of absolute derived categories and factorizes through the Carthesian product of coderived categories. Thus we get a functor

$$\mathsf{D^{co}}(\mathsf{mod}\text{–}B) \times \mathsf{D^{co}}(B\text{–}\mathsf{mod}) \longrightarrow k\text{–}\mathsf{mod^{gr}}.$$

This derived functor coincides with the above-defined functor $\mathrm{Tor}^{B,II}$, since one can use finite resolutions $P_\bullet$ and $Q_\bullet$ in the construction of the latter functor in the finite weak homological dimension case.

Analogously, whenever the graded ring $B^\#$ has a finite left homological dimension, the functor $\mathrm{Hom}_{\mathsf{D^{abs}}(B\text{–}\mathsf{mod})}(L, M)$ of homomorphisms in the absolute derived category coincides with the above-defined functor $\mathrm{Ext}_B^{II}(L, M)$.

4. Coderived Category of CDG-Comodules and Contraderived Category of CDG-Contramodules

4.1. **CDG-comodules and CDG-contramodules.** Let k be a fixed ground field. We will consider graded vector spaces V over k endowed with homogeneous endomorphisms d of degree 1 with *not* necessarily zero squares. The endomorphism d will be called "the differential". Given two graded vector spaces V and W with the differentials d, the differential on the graded tensor product $V \otimes_k W$ is defined by the usual formula $d(v \otimes w) = d(v) \otimes w + (-1)^{|v|} v \otimes d(w)$ and the differential on the graded vector space of homogeneous homomorphisms $\mathrm{Hom}_k(V, W)$ is defined by the usual formula $(df)(v) = d(f(v) - (-1)^{|f|} f(dv)$. The graded vector space k is endowed with the zero differential.

Using a version of Sweedler's notation, we will denote symbolically the comultiplication in a graded coalgebra C by $c \longmapsto c_{(1)} \otimes c_{(2)}$. The coaction in a graded left comodule M over C will be denoted by $x \longmapsto x_{(-1)} \otimes x_{(0)}$, while the coaction in a graded right comodule N over C will be denoted by $y \longmapsto y_{(0)} \otimes y_{(1)}$. Here $x_{(0)} \in M$, $y_{(0)} \in N$, and $x_{(-1)}$, $y_{(1)} \in C$. The contraaction map $\mathrm{Hom}_k(C, P) \longrightarrow P$ of a graded left contramodule P over C will be denoted by π_P.

The graded dual vector space $C^* = \mathrm{Hom}_k(C, k)$ to a graded coalgebra C is a graded algebra with the multiplication given by the formula $(\phi * \psi)(c) = \phi(c_{(2)})\psi(c_{(1)})$. Any graded left comodule M over C has a natural structure of a graded left C^*-module given by the rule $\phi * x = \phi(x_{(-1)})x_{(0)}$, while any graded right comodule N over C has a natural structure of a graded right C^*-module given by $y * \phi = (-1)^{|\phi|}\phi(y_{(1)})y_{(0)}$. In particular, the graded coalgebra C itself is a graded C^*-bimodule. Any graded left contramodule P over C has a natural structure of a graded left C^*-comodule given by $\phi * p = \pi_P(c \mapsto (-1)^{|\phi||p|}\phi(c)p)$.

A *CDG-coalgebra* over k is a graded coalgebra C endowed with a homogeneous endomorphism d of degree 1 (with a not necessarily zero square) and a homogeneous linear function $h\colon C \longrightarrow k$ of degree 2 (that is h vanishes on all the components of C except perhaps C^{-2}) satisfying the following equations. Firstly, the comultiplication map $C \longrightarrow C \otimes_k C$ and the counit map $C \longrightarrow k$ must commute with the differentials on C, k, and $C \otimes_k C$, where the latter two differentials are given by the above rules. Secondly, one must have $d^2(c) = h * c - c * h$ and $h(d(c)) = 0$ for all $c \in C$. A homogeneous endomorphism d of degree 1 acting on a graded coalgebra C is called an *odd coderivation* of degree 1 if it satisfies the first of these two conditions.

A morphism of CDG-coalgebras $C \longrightarrow D$ is a pair (f, a), where $f\colon C \longrightarrow D$ is a morphism of graded coalgebras and $a\colon C \longrightarrow k$ is a homogeneous linear function of degree 1 such that the equations $d_D(f(c)) = f(d_C(c)) + f(a * c) - (-1)^{|c|} f(c * a)$ and $h_D(f(c)) = h_C(c) + a(d_C(c)) + a^2(c)$ hold for all $c \in C$. The composition of morphisms is defined by the rule $(g, b) \circ (f, a) = (g \circ f,\ b \circ f + a)$. Identity morphisms are the morphisms $(\mathrm{id}, 0)$. So the *category of CDG-coalgebras* is defined.

A *left CDG-comodule* over C is a graded left C-comodule M endowed with a homogeneous linear endomorphism d of degree 1 (with a not necessarily zero square) satisfying the following equations. Firstly, the coaction map $M \longrightarrow C \otimes_k M$ must commute with the differentials on C and $C \otimes_k M$. Secondly, one must have $d^2(x) = h * x$ for all $x \in M$. A *right CDG-comodule* over C is a graded right C-comodule N endowed with a homogeneous linear endomorphism d of degree 1 such that the coaction map $N \longrightarrow N \otimes_k C$ commutes with the differentials and the equation $d^2(y) = -y * h$ holds. A *left CDG-contramodule* over C is a graded left C-contramodule P endowed with a homogeneous linear endomorphism d of degree 1 such that the contraaction map $\mathrm{Hom}_k(C, P) \longrightarrow P$ commutes with the differentials on $\mathrm{Hom}_k(C, P)$ and P, and the equation $d^2(p) = h * p$ holds. In each the the above three situations, a homogeneous k-linear endomorphism $d\colon M \longrightarrow M$ or $d\colon N \longrightarrow N$ of degree 1 is called an *odd coderivation* of degree 1 *compatible with* an odd coderivation $d\colon C \longrightarrow C$ of degree 1 or a homogeneous k-linear endomorphism $d\colon P \longrightarrow P$ of degree 1 is called an *odd contraderivation* of degree 1 *compatible with* an odd coderivation $d\colon C \longrightarrow C$ of degree 1 if the first of the two conditions is satisfied.

For any morphism of graded coalgebras $f\colon C \longrightarrow D$ there are restriction-of-scalars functors assigning to graded comodules and contramodules over C graded D-comodule and D-contramodule structures on the same graded vector spaces. Now let $f = (f,a)\colon C \longrightarrow D$ be a morphism of CDG-coalgebras and M be a left CDG-comodule over C. Then the left CDG-comodule $R_f M$ over D is defined by restricting scalars in the graded C-comodule M via the morphism of graded coalgebras $f\colon C \longrightarrow D$ and changing the differential d on M by the rule $d'(x) = d(x) + a * x$. Analogously, left N be a right CDG-module over C. Then the right CDG-comodule $R_f N$ over D is defined by restricting scalars in the graded comodule N and changing the differential d on N by the rule $d'(y) = d(y) - (-1)^{|y|} y * a$. Finally, let P be a graded left CDG-contramodule over C. Then the left CDG-contramodule $R_f P$ over D is defined by restricting scalars in the graded contramodule P via the morphism f and changing the differential d on P by the rule $d'(p) = d(p) + a * p$.

Whenever N is a right CDG-comodule and N is a left CDG-comodule over a CDG-coalgebra C, the tensor product $N \square_C M$ of the graded comodules N and M over the graded coalgebra C considered as a subspace of the tensor product $N \otimes_k M$ is preserved by the differential of $N \otimes_k M$. The restriction of this differential to $N \square_C M$ has a zero square, which makes $N \square_C M$ a complex. Whenever M is a left CDG-comodule and P is a left CDG-contramodule over a CDG-coalgebra C, the graded space of cohomomorphisms $\mathrm{Cohom}_C(M,P)$ is an invariant quotient space of the graded space $\mathrm{Hom}_k(M,P)$ with respect to the differential on $\mathrm{Hom}_k(M,P)$. The induced differential on $\mathrm{Cohom}_C(M,P)$ has a zero square, which makes $\mathrm{Cohom}_C(M,P)$ a complex.

Whenever N is a right CDG-comodule and P is a left CDG-contramodule over a CDG-coalgebra C, the contratensor product $N \odot_C P$ of the graded comodule N and the graded contramodule P over the graded coalgebra C is an invariant quotient space of the tensor product $N \otimes_k P$ with respect to the differential on $N \otimes_k P$. The induced differential on $N \odot_C P$ has a zero square, which makes $N \odot_C P$ a complex.

For any left CDG-comodules L and M over a CDG-coalgebra C, the graded vector space of homomorphisms between the graded comodules L and M over the graded coalgebra C considered as a subspace of the graded space $\mathrm{Hom}_k(L,M)$ is preserved by the differential on $\mathrm{Hom}_k(L,M)$. The induced differential on $\mathrm{Hom}_C(L,M)$ has a zero square, which makes $\mathrm{Hom}_C(L,M)$ a complex. Differentials with zero squares on the graded vector spaces of homomorphisms $\mathrm{Hom}_C(R,N)$ and $\mathrm{Hom}^C(P,Q)$ for right CDG-comodules R, N and left CDG-contramodules P, Q over a CDG-coalgebra C are constructed in the analogous way. These constructions define the DG-categories $\mathsf{DG}(C\text{–}\mathsf{comod})$, $\mathsf{DG}(\mathsf{comod}\text{–}C)$, and $\mathsf{DG}(C\text{–}\mathsf{contra})$ of left CDG-comodules, right CDG-comodules, and left CDG-contramodules over C.

For a CDG-coalgebra C, we will sometimes denote by $C^{\#}$ the graded coalgebra C considered without its differential d and linear function h (or with the zero differential and linear function). For left or right CDG-comodules, or CDG-contramodules M, N, or P we will denote by $M^{\#}$, $N^{\#}$, and $P^{\#}$ the corresponding graded comodules and contramodules (or CDG-comodules and CDG-contramodules with zero differentials) over $C^{\#}$. Notice that for DG being the DG-category of DG-comodules or DG-contramodules over C, the corresponding additive category $Z^0(\mathsf{DG}^\natural)$ can be identified with the category of graded comodules or contramodules over $C^\#$; the functor $\natural$ is identified with the functor of forgetting the differential.

All shifts, twists, infinite direct sums, and infinite direct products exist in the DG-categories of CDG-modules and CDG-contramodules. The homotopy category of (the DG-category of) left CDG-comodules over C is denoted by $\mathsf{Hot}(C\text{–}\mathsf{comod})$, the homotopy category of right CDG-comodules over C is denoted by $\mathsf{Hot}(\mathsf{comod}\text{–}C)$, and the homotopy category of left CDG-contramodules over C is denoted by $\mathsf{Hot}(C\text{–}\mathsf{contra})$. Notice that there is no obvious way to define derived categories of CDG-comodules or CDG-contramodules, as there is no notion of cohomology of a CDG-comodule or a CDG-contramodule, and hence no class of acyclic CDG-comodules or CDG-contramodules.

4.2. **Coderived and contraderived categories.** The absolute derived categories of CDG-comodules and CDG-contramodules, the coderived categories of CDG-comodules, and the contraderived categories of CDG-contramodules are defined in the way analogous to that for CDG-modules. These are all particular cases of the general definition sketched in Remarks 3.3.2 and 3.5. Let us spell out these definitions in a little more detail.

Let C be a CDG-coalgebra. We will consider exact triples in the abelian categories $Z^0\mathsf{DG}(C\text{–}\mathsf{comod})$ and $Z^0\mathsf{DG}(C\text{–}\mathsf{contra})$, i. e., exact triples of left CDG-modules or lefr CDG-contramodules over C and closed morphisms between them. An exact triple of CDG-comodules or CDG-contramodules can be viewed as a finite complex of CDG-comodules or CDG-contramodules, so the total CDG-comodule or CDG-contramodule is defined for such an exact triple.

A left CDG-comodule or left CDG-contramodule over C is called *absolutely acyclic* if it belongs to the minimal thick subcategory of the homotopy category $\mathsf{Hot}(C\text{–}\mathsf{comod})$ or $\mathsf{Hot}(C\text{–}\mathsf{contra})$ containing the total CDG-comodules or CDG-contramodules of exact triples of left CDG-comodules or left CDG-contramodules over C. The thick subcategories of absolutely acyclic CDG-comodules and CDG-contramodules are denoted by $\mathsf{Acycl}^{\mathsf{abs}}(C\text{–}\mathsf{comod}) \subset \mathsf{Hot}(C\text{–}\mathsf{comod})$ and $\mathsf{Acycl}^{\mathsf{abs}}(C\text{–}\mathsf{contra}) \subset \mathsf{Hot}(C\text{–}\mathsf{contra})$. The quotient categories $\mathsf{D}^{\mathsf{abs}}(C\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{abs}}(C\text{–}\mathsf{contra})$ of the homotopy categories of left CDG-comodules and left CDG-contramodules by these thick subcategories are called the *absolute derived categories* of CDG-comodules and CDG-contramodules over C.

A left CDG-comodule over C is called *coacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{comod})$ containing the total CDG-comodules of exact triples of left CDG-comodules over C and closed under infinite direct sums. The thick subcategory formed by all coacyclic CDG-comodules is denoted by $\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \subset \mathsf{Hot}(C\text{–}\mathsf{comod})$. The *coderived category* of left

CDG-comodules over C is defined as the quotient category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) = \mathsf{Hot}(C\text{–}\mathsf{comod})/\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})$.

A left CDG-contramodule over C is called *contraacyclic* if it belongs to the minimal triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{contra})$ containing the total CDG-contramodules of exact triples of left CDG-contramodules over C and closed under infinite products. The thick subcategory formed by all contraacyclic CDG-contramodules is denoted by $\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \subset \mathsf{Hot}(C\text{–}\mathsf{contra})$. The *contraderived category* of left CDG-contramodules over C is defined as the quotient category $\mathsf{DG}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) = \mathsf{Hot}(C\text{–}\mathsf{contra})/\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.

All the above definitions for left CDG-comodules can be repeated verbatim for right CDG-comodules, so there are thick subcategories $\mathsf{Acycl}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$ and $\mathsf{Acycl}^{\mathsf{abs}}(\mathsf{comod}\text{–}C)$ in $\mathsf{Hot}(\mathsf{comod}\text{–}C)$ with the corresponding quotient categories $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$ and $\mathsf{D}^{\mathsf{abs}}(\mathsf{comod}\text{–}C)$.

4.3. **Bounded cases.** Let C be a DG-coalgebra. Denote by $\mathsf{Hot}^+(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}^+(C\text{–}\mathsf{contra})$ the homotopy categories of DG-comodules and DG-contramodules over C bounded from below, and denote by $\mathsf{Hot}^-(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}^-(C\text{–}\mathsf{contra})$ the homotopy categories of DG-comodules and DG-contramodules over C bounded from above. That is, $M \in \mathsf{Hot}^+(C\text{–}\mathsf{comod})$ iff $M^i = 0$ for all $i \ll 0$ and $P \in \mathsf{Hot}^-(C\text{–}\mathsf{contra})$ iff $P^i = 0$ for all $i \gg 0$; similarly for $\mathsf{Hot}^-(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}^+(C\text{–}\mathsf{contra})$. Set $\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{comod}) = \mathsf{Acycl}(C\text{–}\mathsf{comod}) \cap \mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})$ and analogously for $\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{contra})$; also set $\mathsf{Acycl}^{\mathsf{co},\pm}(C\text{–}\mathsf{comod}) = \mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \cap \mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})$ and analogously for $\mathsf{Acycl}^{\mathsf{ctr},\pm}(C\text{–}\mathsf{contra})$.

Clearly, one has $\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \subset \mathsf{Acycl}(C\text{–}\mathsf{comod})$ and $\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \subset \mathsf{Acycl}(C\text{–}\mathsf{contra})$ for any DG-coalgebra C.

Theorem 1. *Assume that $C^i = 0$ for $i < 0$. Then*

(a) $\mathsf{Acycl}^{\mathsf{co},+}(C\text{–}\mathsf{comod}) = \mathsf{Acycl}^+(C\text{–}\mathsf{comod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr},-}(C\text{–}\mathsf{contra}) = \mathsf{Acycl}^-(C\text{–}\mathsf{contra})$*;*

(b) *the natural functors* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})/\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{contra})/\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{contra})$ *are fully faithful;*

(c) *the natural functors* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})/\mathsf{Acycl}^{\mathsf{co},\pm}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{contra})/\mathsf{Acycl}^{\mathsf{ctr},\pm}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *are fully faithful.*

Proof. Analogous to the proof of Theorem 3.4.1. ☐

For an ungraded CDG-coalgebra E, the following conditions are equivalent: (i) the category of left comodules over E is semisimple; (ii) the category of right comodules over E is semisimple; (iii) the category of left contramodules over E is semisimple; (iv) E is the sum of its simple subcoalgebras, where a coalgebra is called *simple* if it contains no nonzero proper subcoalgebras. A coalgebra E satisfying these equivalent conditions is called *cosemisimple* [25, Appendix A].

Theorem 2. *Assume that $C^i = 0$ for $i > 0$, the coalgebra C^0 is cosemisimple, and $C^{-1} = 0$. Then*

(a) $\mathsf{Acycl}^{\mathsf{co},-}(C\text{–}\mathsf{comod}) = \mathsf{Acycl}^{-}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Acycl}^{\mathsf{ctr},+}(C\text{–}\mathsf{contra}) = \mathsf{Acycl}^{+}(C\text{–}\mathsf{contra})$*;*

(b) *the natural functors* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})/\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{contra})/\mathsf{Acycl}^{\pm}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{contra})$ *are fully faithful;*

(c) *the natural functors* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{comod})/\mathsf{Acycl}^{\mathsf{co},\pm}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Hot}^{\pm}(C\text{–}\mathsf{contra})/\mathsf{Acycl}^{\mathsf{ctr},\pm}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *are fully faithful.*

Proof. Analogous to the proof of Theorem 3.4.2. □

4.4. **Injective and projective resolutions.** Let C be a CDG-coalgebra. Denote by $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$ the full triangulated subcategory of the homotopy category of left CDG-comodules over C consisting of all the CDG-comodules M for which the graded comodule $M^{\#}$ over he graded coalgebra $C^{\#}$ is injective. Analogously, denote by $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}})$ the full triangulated subcategory of the homotopy category of left CDG-contramodules over C consisting of all the CDG-contramodules P for which the graded contramodule $P^{\#}$ over the graded coalgebra $C^{\#}$ is projective. The next Theorem provides semiorthogonal decompositions of the homotopy categories $\mathsf{Hot}(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}(C\text{–}\mathsf{contra})$, and describes the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ in terms of injective and projective resolutions, respectively.

Theorem. (a) *For any CDG-comodules $L \in \mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $M \in \mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$, the complex $\mathrm{Hom}_C(L, M)$ is acyclic.*

(b) *For any CDG-contramodules $P \in \mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}})$ and $Q \in \mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$, the complex $\mathrm{Hom}^C(P, Q)$ is acyclic.*

(c) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *is an equivalence of triangulated categories.*

(d) *The composition of functors* $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *is an equivalence of triangulated categories.*

Proof. Parts (a) and (b) are easy; see the proof of Theorem 3.5(a). Parts (c) and (d) can be proven in the way analogous to that of Theorems 3.5(b) and 3.6(b), or alternatively in the following way. Let us first consider the case of a DG-coalgebra C. For any DG-comodule M over C, consider the cobar resolution $C \otimes_k M \longrightarrow C \otimes_k C \otimes_k M \longrightarrow \cdots$. This is a complex of DG-comodules over C and closed morphisms between them; denote by J the total DG-comodule of this complex formed by taking infinite direct sums. Then the graded $C^{\#}$-comodule $J^{\#}$ is injective and the cone of the closed morphism $M \longrightarrow J$ is coacyclic. Analogously, for a DG-contramodule P over C one considers the bar resolution $\cdots \longrightarrow \mathrm{Hom}_k(C \otimes_k C,\, P) \longrightarrow \mathrm{Hom}_k(C, P)$ and forms its total DG-contramodule by taking infinite products. In the case of a CDG-coalgebra

C, the construction has to be modified as follows. Let M be a left CDG-comodule over C; consider the graded left $C^\#$-comodule $J=\bigoplus_{n=0}^\infty(C^{\otimes n+1}\otimes_k M)[-n]$, where the comodule structure on $C^{\otimes n+1}\otimes_k M$ comes from the comodule structure on the leftmost factor C and the shift of the grading introduces the appropriate sign in the graded comodule structure. The differential on J is described as the sum of three components ∂, d, and δ given by the formulas $\partial(c_0\otimes c_1\otimes\cdots\otimes c_n\otimes x)=\mu(c_0)\otimes c_1\otimes\cdots\otimes x-c_0\otimes\mu(c_1)\otimes c_2\otimes\cdots\otimes x+\cdots+(-1)^{n+1}c_0\otimes c_1\otimes\cdots\otimes c_n\otimes\nu(x)$, where $\mu\colon C\longrightarrow C\otimes C$ and $\nu\colon M\longrightarrow C\otimes M$ are the comultiplication and coaction maps, $(-1)^n d(c_0\otimes c_1\otimes\cdots\otimes c_n\otimes x)=d(c_0)\otimes c_1\otimes\cdots\otimes x+(-1)^{|c_0|}c_0\otimes d(c_1)\otimes c_2\otimes\cdots\otimes x+\cdots+(-1)^{|c_0|+\cdots+|c_n|}c_0\otimes\cdots\otimes c_n\otimes d(x)$, and $\delta(c_0\otimes c_1\otimes\cdots\otimes c_n\otimes x)=h(c_1)c_0\otimes c_2\otimes c_3\otimes\cdots\otimes x-h(c_2)c_0\otimes c_1\otimes c_3\otimes c_4\otimes\cdots\otimes x+\cdots+(-1)^{n-1}h(c_n)c_0\otimes\cdots c_{n-1}\otimes x$. The graded $C^\#$-comodule J with the differential $\partial+d+\delta$ is a CDG-comodule over C; it is endowed with a closed morphism of CDG-comodules $M\longrightarrow J$. Denote by $\tau_{\leqslant n}J$ the subspaces of canonical filtration of the vector space J considered as a complex with the grading n and the differential ∂; then $\tau_{\leqslant n}J$ are CDG-subcomodules of J. The quotient CDG-comodules $\tau_{\leqslant n}J/\tau_{\leqslant n-1}J$ (taken in the abelian category of CDG-comodules and closed morphisms) are contractible CDG-comodules, the contracting homotopy being given by the map inverse to the differential induced by the differential ∂. The only exception is the CDG-comodule $\tau_{\leqslant 0}J$, which is isomorphic to M. It follows that the cone of the closed morphism $M\longrightarrow J$ is coacyclic. ☐

Remark. The following results, which are particular cases of Remark 3.5, generalize simultaneously the above Theorem and Theorems 3.5–3.6. A topological graded abelian group (with additive topology) is a graded abelian group endowed with a system of graded subgroups closed under finite intersections and containing with any subgroup all the larger graded subgroups; graded subgroups belonging to the system are called open. To any topological graded abelian group one can assign an (ungraded) topological abelian group by taking the projective limit of the direct sums of all grading components of the graded quotient groups by open graded subgroups. A topological graded abelian group with a graded ring structure is called a topological graded ring if its multiplication can be extended to a topological ring structure on the associated ungraded topological abelian group. Let us restrict ourselves to separated and complete topological graded rings B where open two-sided graded ideals form a base of the topology; these are exactly the graded projective limits of discrete graded rings. Let (B,d,h) be a CDG-ring structure on B such that the differential d is continuous; one can easily check that open two-sided graded differential ideals form a base of the topology of B in this case, so B is a projective limit of discrete CDG-rings. First assume that all discrete graded quotient rings of B are left Noetherian. A graded left B-module is called discrete if the annihilator of every its homogeneous element is an open left ideal in B. Consider the DG-category $\mathsf{DG}(B\text{–}\mathsf{mod})$ of discrete graded left B-modules with CDG-module structures. The corresponding coderived category $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ is defined in the obvious way. The graded left B-module of continuous homogeneous abelian group homomorphisms from B into any (discrete) injective graded abelian group is an injective object in the category of discrete graded left B-modules. A discrete graded left B-module M is injective if and only if for any open two-sided graded ideal $J \subset B$ the annihilator of J in M is an injective graded left B/J-module. It follows that there are enough injectives in the abelian category of discrete graded left B-modules and the class of injectives is closed under infinite direct sums. For any discrete graded left B-module M the graded left B-module $G^-(M)$ is also discrete. So the category $Z^0(\mathsf{DG}(B\text{–}\mathsf{mod})^\natural)$ can be identified with the category of discrete graded left B-modules and the result of Remark 3.5 applies. Thus the coderived category $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$ is equivalent to the homotopy category of discrete left CDG-modules over B that are injective as discrete graded modules. Analogously, assume that all discrete graded quotient rings of B are right Artinian. It is not difficult to define the DG-category $\mathsf{DG}(B\text{–}\mathsf{contra})$ of graded left CDG-contramodules over B (cf. [25, Appendix A]). The corresponding contraderived category $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{contra})$ is equivalent to the homotopy category of graded left CDG-contramodules that are projective as graded contramodules. The key step is to show that a graded left CDG-contramodule P over B is projective if and only if for any open two-sided graded ideal $J \subset B$ the maximal quotient contramodule of P whose B-contramodule structure comes from a B/J-(contra)module structure is a projective B/J-module.

4.5. **Finite homological dimension case.** Let E be a graded coalgebra. Then the homological dimensions of the categories of graded right E-comodules, graded left E-comodules, and graded left E-contramodules coincide, as they coincide with the homological dimensions of the derived functors of cotensor product and cohomomorphisms on the abelian categories of comodules and contramodules [25]. The common value of these three homological dimensions we will call the homological dimension of the graded coalgebra E.

Let C be a CDG-coalgebra. Assume that the graded coalgebra $C^\#$ has a finite homological dimension. The next Theorem identifies the coderived category of C-comodules and the contraderived category of C-contramodules with the corresponding absolute derived categories.

Theorem. (a) *The two thick subcategories* $\mathsf{Acycl}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{Acycl}^{\mathsf{abs}}(C\text{–}\mathsf{comod})$ *in the homotopy category* $\mathsf{Hot}(C\text{–}\mathsf{comod})$ *coincide.*

(b) *The two thick subcategories* $\mathsf{Acycl}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *and* $\mathsf{Acycl}^{\mathsf{abs}}(C\text{–}\mathsf{contra})$ *in the homotopy category* $\mathsf{Hot}(C\text{–}\mathsf{contra})$ *coincide.*

Proof. The proof is analogous to that of Theorem 3.7 and can be based on either the constructions from the proof of Theorem 4.4 or appropriate versions of the constructions from the proofs of Theorems 3.5–3.6. □

4.6. **Cotor, Coext, and Ctrtor.** Let us define the derived functors

$$\operatorname{Cotor}^C\colon \mathsf{D^{co}}(\mathsf{comod}\text{–}C)\times\mathsf{D^{co}}(C\text{–}\mathsf{comod})\longrightarrow k\text{–}\mathsf{vect^{gr}}$$
$$\operatorname{Coext}_C\colon \mathsf{D^{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{D^{ctr}}(C\text{–}\mathsf{contra})\longrightarrow k\text{–}\mathsf{vect^{gr}}$$
$$\operatorname{Ctrtor}^C\colon \mathsf{D^{co}}(\mathsf{comod}\text{–}C)\times\mathsf{D^{ctr}}(C\text{–}\mathsf{contra})\longrightarrow k\text{–}\mathsf{vect^{gr}}$$

for a CDG-coalgebra C. We denote by $\mathsf{Hot}(\mathsf{mod_{inj}}\text{–}C)$ the full subcategory of $\mathsf{Hot}(\mathsf{comod}\text{–}C)$ formed by all the right CDG-comodules N over C for which the graded $C^\#$-comodule $N^\#$ is injective. To check that the composition of functors $\mathsf{Hot}(\mathsf{mod_{inj}}\text{–}C)\longrightarrow\mathsf{Hot}(\mathsf{comod}\text{–}C)\longrightarrow\mathsf{D^{co}}(\mathsf{comod}\text{–}C)$, one can pass to the opposite CDG-coalgebra $C^{\mathrm{op}}=(C^{\mathrm{op}},d^{\mathrm{op}},h^{\mathrm{op}})$, which coincides with C as a graded vector space and has the comultiplication, differential, and curvature defined by the formulas $(c^{\mathrm{op}})_{(1)}\otimes(c^{\mathrm{op}})_{(2)}=(-1)^{|c_{(1)}||c_{(2)}|}c_{(2)}^{\mathrm{op}}\otimes c_{(1)}^{\mathrm{op}}$, $d^{\mathrm{op}}(c^{\mathrm{op}})=d(c)^{\mathrm{op}}$, and $h^{\mathrm{op}}(c^{\mathrm{op}})=-h(c)$, where c^{op} denotes the element of C^{op} corresponding to an element $c\in C$.

To define the functor Cotor^C, restrict the functor of cotensor product $\square_C\colon$ $\mathsf{Hot}(\mathsf{comod}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{Hot}(k\text{–}\mathsf{vect})$ to either of the full subcategories $\mathsf{Hot}(\mathsf{mod_{inj}}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod})$ or $\mathsf{Hot}(\mathsf{comod}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})$. The functors so obtained factorize through the localization $\mathsf{D^{co}}(\mathsf{comod}\text{–}C)\times\mathsf{D^{co}}(C\text{–}\mathsf{comod})$ and the two induced derived functors $\mathsf{D^{co}}(\mathsf{comod}\text{–}C)\times\mathsf{D^{co}}(C\text{–}\mathsf{comod})\longrightarrow k\text{–}\mathsf{vect^{gr}}$ are naturally isomorphic to each other. Indeed, the cotensor product $N\,\square_C\,M$ is acyclic whenever one of the CDG-comodules N and M is coacyclic and the other is injective as a graded comodule. This follows from the fact that the functor of cotensor product with an injective graded comodule sends exact triples of graded comodules to exact triples of graded vector spaces. To construct an isomorphism between the two induced derived functors, it suffices to notice that both of them are isomorphic to the derived functor obtained by restricting the functor $\square_C$ to the full subcategory $\mathsf{Hot}(\mathsf{mod_{inj}}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})$.

To define the functor Coext_C, restrict the functor of cohomomorphisms $\operatorname{Cohom}_C\colon$ $\mathsf{Hot}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{Hot}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{Hot}(k\text{–}\mathsf{vect})$ to either of the full subcategories $\mathsf{Hot}(C\text{–}\mathsf{comod_{inj}})^{\mathsf{op}}\times\mathsf{Hot}(C\text{–}\mathsf{contra})$ or $\mathsf{Hot}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{Hot}(C\text{–}\mathsf{contra_{proj}})$. The functors so obtained factorize through the localization $\mathsf{D^{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{D^{ctr}}(C\text{–}\mathsf{contra})$ and the two induced derived functors $\mathsf{D^{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}}\times\mathsf{D^{ctr}}(C\text{–}\mathsf{contra})\longrightarrow k\text{–}\mathsf{vect^{gr}}$ are naturally isomorphic. Indeed, the complex of cohomomorphisms $\operatorname{Cohom}_C(M,P)$ is acyclic whenever either the CDG-comodule M is coacyclic and the CDG-contramodule P is projective as a graded contramodule, or the CDG-comodule M is injective as a graded comodule and the CDG-contramodule P is contraacyclic.

To define the functor Ctrtor^C, restrict the functor of contratensor product $\odot_C\colon\mathsf{Hot}(\mathsf{comod}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{Hot}(k\text{–}\mathsf{vect})$ to the full subcategory $\mathsf{Hot}(\mathsf{comod}\text{–}C)\times\mathsf{Hot}(C\text{–}\mathsf{contra_{proj}})$. The functor so obtained factorizes through the localization $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \times \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$, so one obtains the desired derived functor. Indeed, the contratensor product $N \odot_C P$ is acyclic whenever the CDG-comodule N is coacyclic and the CDG-contramodule P is projective as a graded contramodule. Notice that one can only obtain the functor Ctrtor as the derived functor in its second argument, but apparently not in its first argument, as comodules adjusted to contratensor product most often do not exist.

By Lemma 1.3, one can also compute the functor $\operatorname{Ext}_C = \operatorname{Hom}_{\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})}$ of homomorphisms in the coderived category of left CDG-comodules in terms of injective resolutions of the second argument and the functor $\operatorname{Ext}^C = \operatorname{Hom}_{\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})}$ of homomorphisms in the contraderived category of left CDG-contramodules in terms of projective resolutions of the first argument. Namely, one has $\operatorname{Ext}_C(L, M) = H(\operatorname{Hom}_C(L, M))$ whenever the CDG-comodule M is injective as a graded $C^{\#}$-comodule and $\operatorname{Ext}^C(P, Q) = H(\operatorname{Hom}^C(P, Q))$ whenever the CDG-contramodule P is projective as a graded $C^{\#}$-contramodule.

For any right CDG-comodule N over C and any complex of k-vector spaces V the differential on $\operatorname{Hom}_k(N, V)$ defined by the usual formula provides a structure of CDG-contramodule over C on $\operatorname{Hom}_k(N, V)$. The functor $\operatorname{Hom}_k(-, V)$ assigns contraacyclic CDG-contramodules to coacyclic CDG-comodules N and so induces a functor $\mathbb{I}\operatorname{Hom}_k(-, V)\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ on the level of coderived and contraderived categories. There are natural isomorphisms of functors of two arguments $\operatorname{Hom}_k(\operatorname{Cotor}^C(N, M), H(V)) \simeq \operatorname{Coext}_C(M, \mathbb{I}\operatorname{Hom}_k(N, V))$ and $\operatorname{Hom}_k(\operatorname{Ctrtor}^C(N, P), H(V)) \simeq \operatorname{Ext}^C(P, \mathbb{I}\operatorname{Hom}_k(N, V))$, where where $H(V)$ denotes the cohomology of the complex V.

4.7. **Restriction and extension of scalars.** Let $f\colon C \longrightarrow D$ be a morphism of CDG-coalgebras. Then any CDG-comodule or CDG-contramodule over C can be also considered as a CDG-comodule or CDG-contramodule over D, as explained in 4.1. This defines the restriction-of-scalars functors $R_f\colon \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(D\text{–}\mathsf{comod})$ and $R^f\colon \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(D\text{–}\mathsf{contra})$. The functor R_f has a right adjoint functor E_f given by the formula $E_f(N) = C \mathbin{\square}_D N$, while the functor R^f has a left adjoint functor given by the formula $E^f(Q) = \operatorname{Cohom}_D(C, Q)$; to define the differentials on $E_f(N)$ and $E^f(Q)$, it is simplest to decompose f into an isomorphism of CDG-coalgebras followed by a morphism of CDG-coalgebras with a vanishing linear function a.

The functors R_f and R^f obviously map coacyclic CDG-comodules and contraacyclic CDG-contramodules to CDG-comodules and CDG-contramodules of the same kind, and so induce functors $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{contra})$, which we denote by $\mathbb{I}R_f$ and $\mathbb{I}R^f$. The functor E_f has a right derived functor $\mathbb{R}E_f$ obtained by restricting E_f to the full subcategory $\mathsf{Hot}(D\text{–}\mathsf{comod}_{\mathsf{inj}}) \subset \mathsf{Hot}(D\text{–}\mathsf{comod})$ and composing it with the localization functor

$\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$. The functor E^f has a left derived functor $\mathbb{L}E^f$ obtained by restricting E^f to the full subcategory $\mathsf{Hot}(D\text{–}\mathsf{contra}_{\mathsf{proj}}) \subset \mathsf{Hot}(D\text{–}\mathsf{contra})$ and composing it with the localization functor $\mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$. The functor $\mathbb{R}E_f$ is right adjoint to the functor $\mathbb{I}R_f$ and the functor $\mathbb{L}E^f$ is left adjoint to the functor $\mathbb{I}R^f$.

For any two CDG-coalgebras E and F, a *CDG-bicomodule* K over E and F is a graded vector space endowed with commuting structures of a graded left E-comodule and a graded right F-comodule and a differential d compatible with both the differentials in E and F and satisfying the equation $d^2(x) = h_E * x - x * h_F$ for all $x \in K$. Notice that a CDG-bicomodule over E and F has no natural structures of a left CDG-comodule over E or right CDG-comodule over F, as the equations for d^2 are different for CDG-comodules and CDG-bicomodules.

CDG-bicomodules over E and F form a DG-category with morphisms of CDG-bicomodules being homogeneous linear maps satisfying the compatibility equations for both the graded left E-comodule and graded right F-comodule structures; the differential on morphisms of CDG-bicomodules is defined by the usual formula. The class of *coacyclic* CDG-bicomodules over E and F is constructed in the same way as the class of coacyclic CDG-comodules, i. e., one considers exact triples of CDG-bicomodules and closed morphisms between them, and generates the minimal triangulated subcategory of the homotopy category of CDG-bicomodules containing the total CDG-bicomodules of exact triples of CDG-bicomodules and closed under infinite direct sums. A CDG-bicomodule over E and E is called simply a CDG-bicomodule over E.

Assume that a graded coalgebra C is endowed with an increasing filtration by graded vector subspaces $F_0C \subset F_1C \subset \cdots$, $C = \bigcup_n F_nC$ that is compatible with the comultiplication and the differential, that is $\mu(F_nC) \subset \sum_{i+j=n} F_iC \otimes F_jC$ and $d(F_nC) \subset F_nC$, where μ denotes the comultiplication map. Then the associated quotient object $\mathrm{gr}_F C = \bigoplus_n F_nC/F_{n-1}C$ becomes a CDG-coalgebra with the comultiplication and differential induced by those in C, and the counit and the curvature linear function h obtained by restricting the corresponding linear functions on C to F_0C. In particular, F_0C is also a CDG-coalgebra; it is simultaneously a CDG-subcoalgebra of both C and $\mathrm{gr}_F C$ and a quotient CDG-coalgebra of $\mathrm{gr}_F C$. The associated quotient space $\mathrm{gr}_F C$, in addition to a CDG-coalgebra structure, has a structure of CDG-bicomodule over F_0C.

Now suppose that both CDG-coalgebras C and D are endowed with increasing filtrations F as above and the morphism of CDG-coalgebras $f\colon C \longrightarrow D$ preserves the filtrations. Moreover, let us assume that the morphism of CDG-coalgebras $F_0C \longrightarrow F_0D$ induced by f is an isomorphism and the cone of the morphism $\mathrm{gr}_F C \longrightarrow \mathrm{gr}_F D$ of CDG-bicomodules over F_0D is a coacyclic CDG-bicomodule.

Theorem. (a) *Assume that* $\mathrm{gr}_F C^\#$ *and* $\mathrm{gr}_F D^\#$ *are injective graded right* $F_0D^\#$*-comodules. Then the adjoint functors* $\mathbb{I}R_f$ *and* $\mathbb{R}E_f$ *are equivalences of triangulated categories.*

(b) *Assume that* $\mathrm{gr}_F C^\#$ *and* $\mathrm{gr}_F D^\#$ *are injective graded left* $F_0D^\#$*-comodules. Then the adjoint functors* $\mathbb{I}R^f$ *and* $\mathbb{L}E^f$ *are equivalences of triangulated categories.*

Proof. We will prove (a); the proof of (b) is analogous. Let N be a CDG-comodule over D such that the graded comodule $N^\#$ over $D^\#$ is injective. We have to check that the cone of the morphism of CDG-comodules $R_f(E_f(N)) = C \mathbin{\square}_D N \longrightarrow N$ is a coacyclic CDG-comodule over D. Introduce an increasing filtration F on N by the rule $F_nN = \nu^{-1}(F_nD\otimes_k N)$, where $\nu\colon N\longrightarrow D\otimes_k N$ denotes the coaction map. The filtration F is compatible with the graded comodule structure and the differential on N. The induced filtration F on the cotensor product $C\mathbin{\square}_D N$ can be obtained by the same construction applied to the CDG-comodule $C\mathbin{\square}_D N$ over C. It suffices to check that the associated quotient object $\mathrm{gr}_F\,\mathrm{cone}(C\mathbin{\square}_D N\to N)$ is a coacyclic CDG-comodule over F_0D. But this associated quotient object is isomorphic to the cotensor product $\mathrm{cone}(\mathrm{gr}_F C\to \mathrm{gr}_F D)\mathbin{\square}_{F_0D} F_0N$, so it remains to notice that the cotensor product of the CDG-bicomodule $\mathrm{cone}(\mathrm{gr}_F C\to\mathrm{gr}_F D)$ over F_0D with any left CDG-comodule L over F_0D is a coacyclic left CDG-comodule over F_0D in our assumptions. To check the latter, one can choose a morphism from L into a CDG-comodule that is injective as a graded comodule such that the cone of that morphism of CDG-comodules is coacyclic. Now let M be a CDG-comodule over C. We have to check that the cone of the morphism $M\longrightarrow \mathbb{R}E_f(\mathbb{I}R_f(M))$ in the coderived category of CDG-comodules over C is trivial. To do so, we will need an injective resolution of $R_f(M)$ that is natural enough, so that filtrations of C, D, and M would induce a filtration of the resolution in a way compatible with the passage to the associated quotient objects. One can use either the construction from the proof of Theorem 4.4, or a version of the construction from the proof of Theorem 3.5 with the coaction map in the role of a natural embedding of a graded comodule into an injective graded comodule. Computing the object $\mathbb{R}E_f(\mathbb{I}R_f(M))$ in terms of such a natural resolution J of the CDG-comodule $R_f(M)$ over D, we find out that it suffices to check that the cone of the morphism $\mathrm{gr}_F M\longrightarrow \mathrm{gr}_F(C\mathbin{\square}_D J)$ is a coacyclic CDG-comodule over F_0D. But the cones of the morphisms $\mathrm{gr}_F M\longrightarrow\mathrm{gr}_F J$ and $\mathrm{gr}_F(C\mathbin{\square}_D J)\longrightarrow \mathrm{gr}_F J$ are coacyclic CDG-comodules over F_0D, the latter one in view of the above argument applied to the CDG-comodule $N=\mathrm{gr}_F J$ over the CDG-coalgebra $\mathrm{gr}_F D$ endowed with a morphism of CDG-coalgebras $\mathrm{gr}_F C\longrightarrow \mathrm{gr}_F D$. ☐

5. Comodule-Contramodule Correspondence

5.1. **Functors Φ_C and Ψ_C.** Let C be a CDG-coalgebra over a field k. For any left CDG-contramodule P over C, let $\Phi_C(P)$ denote the left CDG-comodule over C constructed in the following way. The underlying graded vector space of $\Phi_C(P)$ is, by the definition, the contratensor product $C \odot_C P$, which is a graded quotient space of the tensor product $C \otimes_k P$. The graded left C-comodule structure and the differential on $\Phi_C(P)$ are induced by the graded left C-comodule structure and the differential on $C \otimes_k P$. The graded left C-comodule structure on $C \otimes_k P$ comes from the graded left C-comodule structure on C, while the differential on $C \otimes_k P$ is given by the standard rule as the differential on the tensor product of graded vector spaces with differentials. It is straightforward to check that $\Phi_C(P)$ is a left CDG-comodule.

For any left CDG-comodule M over C, let $\Psi_C(M)$ denote the left CDG-contramodule over C constructed as follows. The underlying graded vector space of $\Psi_C(M)$ is, by the definition, the space of comodule homomorphisms $\mathrm{Hom}_C(C, M)$. The graded left contramodule structure and the differential on $\Psi_C(P)$ are obtained by restricting the graded left contramodule structure and the differential on $\mathrm{Hom}_k(C, M)$ to this graded vector subspace. The graded left C-contramodule structure on $\mathrm{Hom}_k(C, M)$ is induced by the graded right C-comodule structure on C, while the differential on $\mathrm{Hom}_k(C, M)$ is given by the standard rule as the differential on the space of homogeneous linear maps between graded vector spaces with differentials.

To a morphism $f\colon L \longrightarrow M$ in the DG-category of left CDG-comodules over C, one assigns the morphism $\Phi_C(f)$ given by the formula $c\otimes x \longmapsto (-)^{|f||c|}c\otimes f(x)$. To a morphism $f\colon P \longrightarrow Q$ in the DG-category of left CDG-contramodules over C, one assigns the morphism $\Psi_C(f)$ given by the formula $g \longmapsto f\circ g$. These rules define DG-functors $\Phi_C\colon \mathsf{DG}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{DG}(C\text{–}\mathsf{comod})$ and $\Psi_C\colon \mathsf{DG}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{DG}(C\text{–}\mathsf{contra})$. The isomorphism between the complexes of morphisms induced by our standard isomorphism $\mathrm{Hom}_k(C \otimes_k P,\ M) \simeq \mathrm{Hom}_k(P, \mathrm{Hom}_k(C, M))$ makes the DG-functor Φ_C left adjoint to the DG-functor Ψ_C. So there are induced adjoint functors $\mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod})$ and $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra})$, which we also denote by Φ_C and Ψ_C.

5.2. **Correspondence Theorem.** Restricting the functor Φ_C to the full triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}}) \subset \mathsf{Hot}(C\text{–}\mathsf{contra})$ and composing it with the localization functor $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, we obtain the left derived functor $\mathbb{L}\Phi_C\colon \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{comod})$. Restricting the functor Ψ_C to the full triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}}) \subset \mathsf{Hot}(C\text{–}\mathsf{comod})$ and composing it with the localization functor $\mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$, we obtain the right derived functor $\mathbb{R}\Psi_C\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.

Theorem. *The functors $\mathbb{L}\Phi_C$ and $\mathbb{R}\Psi_C$ are mutually inverse equivalences between the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and the contraderived category $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$.*

Proof. One can easily see that the functors Φ_C and Ψ_C between the homotopy categories of CDG-contramodules and CDG-comodules over C map the full triangulated subcategories $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}})$ and $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}})$ into each other and their restrictions to these subcategories are mutually inverse equivalences between them. □

In particular, let $B = (B, d, h)$ be a CDG-algebra over a field k such that the graded algebra B is (totally) finite-dimensional. Then there is a natural equivalence of triangulated categories $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$. Indeed, let $C = B^*$ be the graded dual vector space to B with the CDG-coalgebra structure defined by the formulas $c(ab) = c_{(2)}(a)c_{(1)}(b)$, $d_C(c)(b) = (-1)^{|c|}c(d(b))$, and $h_C(c) = c(h)$ for a, $b \in B$ and $c \in C$. Then the DG-categories of left CDG-modules over B, left CDG-comodules over C, and left CDG-contramodules over C are all isomorphic, so $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod}) = \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) = \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$.

5.3. **Coext and Ext, Cotor and Ctrtor.** For any left CDG-comodules L and M over a CDG-coalgebra C, there is a natural closed morphism of complexes of vector spaces $\mathrm{Cohom}_C(L, \Psi_C(M)) \longrightarrow \mathrm{Hom}_C(L, M)$, which is an isomorphism whenever either of the graded left $C^{\#}$-comodules $L^{\#}$ and $M^{\#}$ is injective. For any left CDG-contramodules P and Q over C, there is a natural morphism of complexes of vector spaces $\mathrm{Cohom}_C(\Phi_C(P), Q) \longrightarrow \mathrm{Hom}^C(P, Q)$, which is an isomorphism whenever either of the graded left $C^{\#}$-contramodules $P^{\#}$ and $Q^{\#}$ is projective.

For any right CDG-comodule N and left CDG-contramodule P over C, there is a natural closed morphism of complexes of vector spaces $N \odot_C P \longrightarrow N \,\square_C\, \Phi_C(P)$, which is an isomorphism whenever either the graded right $C^{\#}$-comodule N is injective, or the graded left $C^{\#}$-contramodule P is projective.

It follows that there are natural isomorphisms of derived functors of two arguments $\mathrm{Ext}_C(M, \mathbb{L}\Phi_C(P)) \simeq \mathrm{Coext}_C(M, P) \simeq \mathrm{Ext}^C(\mathbb{R}\Psi_C(M), P)$ and $\mathrm{Cotor}_C(N, M) \simeq \mathrm{Ctrtor}_C(N, \Psi_C(M))$ for $M \in \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, $N \in \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$, and $P \in \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$. In other words, the comodule-contramodule correspondence transforms the functor Coext_C into the functors Ext_C and Ext^C, and also it transforms the functor Cotor^C into the functor Ctrtor^C.

5.4. **Relation with extension of scalars.** Let $f\colon C \longrightarrow D$ be a CDG-coalgebra morphism. For any left CDG-comodule N over D such that the graded $D^{\#}$-comodule $N^{\#}$ is injective, there is a natural closed isomorphism $\Psi_C(E_f(N)) \simeq E^f(\Psi_D(N))$ of CDG-contramodules over C provided by the isomorphisms $\mathrm{Hom}_C(C,\, C \,\square_D\, N) \simeq \mathrm{Hom}_D(C, N) \simeq \mathrm{Cohom}_D(C, \mathrm{Hom}_D(D, N))$ of graded vector spaces. Analogously, for any left CDG-contramodule Q over D such that the graded $D^{\#}$-contramodule $Q^{\#}$ is projective, there is a natural closed isomorphism $\Phi_C(E^f(Q)) \simeq E_f(\Phi_D(Q))$ of CDG-comodules over C provided by the isomorphisms $C \odot_C \mathrm{Cohom}_D(C, Q) \simeq C \odot_D Q \simeq C \,\square_D\, (D \odot_D Q)$.

Notice also that the functors E_f and E^f preserve the classes of CDG-comodules and CDG-contramodules that are injective or projective as graded comodules and contramodules, while the functors Φ and Ψ map these classes into each other. Thus we have natural isomorphisms of compositions of derived functors $\mathbb{R}\Psi_C \circ \mathbb{R}E_f \simeq \mathbb{L}E^f \circ \mathbb{R}\Psi_D$ and $\mathbb{L}\Phi_C \circ \mathbb{L}E^f \simeq \mathbb{R}E_f \circ \mathbb{L}\Phi_D$; in other words, the equivalences between the coderived and contraderived categories of comodules and contramodules transform the derived functor $\mathbb{R}E_f$ into the derived functor $\mathbb{L}E^f$. Identifying $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ with $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ and $\mathsf{D}^{\mathsf{co}}(D\text{–}\mathsf{comod})$ with $\mathsf{D}^{\mathsf{ctr}}(D\text{–}\mathsf{contra})$, one can say that the functor $\mathbb{I}R_f$ is left adjoint to the functor $\mathbb{R}E_f = \mathbb{L}E^f$, while the functor $\mathbb{I}R^f$ is right adjoint to the same functor $\mathbb{R}E_f = \mathbb{L}E^f$.

5.5. **Proof of Theorem 2.4.** Our first aim is to show that the triangulated category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ is compactly generated for any CDG-coalgebra C. A triangulated category D where all infinite direct sums exist is said to be compactly generated if it contains a set of compact objects C (see Remark 1.8.3 for the definition) such that D coincides with the minimal triangulated subcategory of D containing C and closed under infinite direct sums.

We will work with the coderived category $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$. It follows from Theorem 4.4 that any finite-dimensional CDG-comodule over C represents a compact object in $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$, since the full triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod}_{\mathsf{inj}}) \subset \mathsf{Hot}(C\text{–}\mathsf{comod})$ is closed under infinite direct sums. Let us check that any CDG-comodule over C up to an isomorphism in $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ can be obtained from finite-dimensional CDG-comodules by iterated operations of cone and infinite direct sum.

A graded coalgebra E is called *cosemisimple* if its homological dimension is equal to zero, or equivalently E is cosemisimple as an ungraded coalgebra. For any graded coalgebra E there exists a unique maximal cosemisimple graded subcoalgebra $E^{\mathrm{ss}} \subset E$, which coincides with the maximal cosemisimple subcoalgebra of the ungraded coalgebra E. The quotient coalgebra (without counit) E/E^{ss} is *conilpotent*, i. e., for any element $e \in E/E^{\mathrm{ss}}$ the image of e under the iterated comultiplication map $E/E^{\mathrm{ss}} \longrightarrow (E/E^{\mathrm{ss}})^{\otimes n}$ vanishes for n large enough. One can easily prove these results, e. g., using the fact that any graded coalgebra is the union of its finite-dimensional graded subcoalgebras together with the graded version of the structure theory of finite-dimensional associative algebras.

Let $E = C^{\sim}$ be the graded coalgebra for which the category of CDG-comodules over C and closed morphisms between them is equivalent to the category of graded comodules over E (see [25] for an explicit construction). Let $F_nE \subset E$ be the graded subspace formed by all elements $e \in E$ whose images vanish in $(E/E^{\mathrm{ss}})^{\otimes n+1}$. Then $F_0E = E^{\mathrm{ss}}$, $E = \bigcup_n F_nE$, and the filtration F_nE is compatible with the coalgebra structure on E. For any graded left comodule M over E, set F_nM to be the full preimage of $F_nE \otimes_k M$ under the comultiplication map $M \longrightarrow E \otimes_k M$. Then the filtrations F on E and M are compatible with the coaction map; in paricular, F_nM are E-subcomodules of M and the quotient comodules $F_nM/F_{n-1}M$ are comodules over F_0E. Since F_0E is a semisimple graded coalgebra, the comodules $F_nM/F_{n-1}M$ are direct sums of irreducible comodules, which are finite-dimensional.

Any left CDG-comodule M over C can be viewed as a left graded comodule over E; the above construction provides a filtration F on M such that F_nM are CDG-comodules over C, the embeddings $F_nM \longrightarrow M$ are closed morphisms, and the quotient CDG-comodules $F_nM/F_{n-1}M$ taken in the abelian category $Z^0\mathsf{DG}(C\text{–}\mathsf{comod})$ are direct sums of finite-dimensional CDG-comodules over C. It follows that M belongs to the minimal triangulated subcategory of $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ containing finite-dimensional CDG-comodules and closed under infinite direct sums. So we have proven that $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ is compactly generated.

Let us point out that in the similar way one can prove that $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ is the minimal triangulated subcategory of itself containing finite-dimensional CDG-contramodules and closed under infinite products. As for comodules, the category of CDG-contramodules over C and closed morphisms between them is equivalent to the category of graded contramodules over E. Even though the natural decreasing filtration $F^nP = \operatorname{im}\operatorname{Hom}_k(E/F_{n-1}E, P)$ on a graded contramodule P over E associated with the filtration F of E is not always separated, it is always separated and complete for projective graded contramodules and their graded subcontramodules, which is sufficient for the argument to work [25].

Now let C be a DG-coalgebra. To prove Theorem 2.4(a), it suffices to notice that $\mathsf{D}(C\text{–}\mathsf{comod})$ is the quotient category of $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ by the thick subcategory which can be represented as the kernel of the forgeful functor $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(k\text{–}\mathsf{vect})$ or the kernel of the homological functor $H\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$. Both this forgetful functor and this homological functor preserve infinite direct sums. It follows that this thick subcategory is well-generated [18] and therefore the localization functor $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{comod})$ has a right adjoint. The localization functor $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ has a right adjoint by Theorem 4.4, thus the localization functor $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{comod})$ also has a right adjoint.

To prove Theorem 2.4(b), consider the object $P = \operatorname{Hom}_k(C, k) \in \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$. Notice that $\mathsf{D}(C\text{–}\mathsf{contra})$ is the quotient category of $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ by the thick subcategory of all objects Q such that $\operatorname{Hom}_{\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})}(P, Q) = 0$. Consider the minimal triangulated subcategory of $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ containing P and closed under infinite direct sums. This triangulated category is well-generated and therefore the functor of its embedding into $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{contra})$ has a right adjoint functor. It follows that the localization functor $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{contra})$ has a left adjoint functor whose image coincides with the minimal triangulated subcategory of $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ containing P and closed under infinite direct sums. The localization functor $\mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ has a left adjoint functor by Theorem 4.4, thus the localization functor $\mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(C\text{–}\mathsf{contra})$ also has a left adjoint.

In addition to the assertions of Theorem, we have proven that the triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra})_{\mathsf{proj}}$ coincides with the minimal triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{contra})$ containing the DG-contramodule $\mathrm{Hom}_k(C,k)$ and closed under infinite direct sums. Indeed, this is so in the triangulated category $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ and the triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{contra}_{\mathsf{proj}}) \subset \mathsf{Hot}(C\text{–}\mathsf{contra})$, which is equivalent to $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$, contains $\mathrm{Hom}_k(C,k)$, and is closed under infinite direct sums. We do not know whether the triangulated subcategory $\mathsf{Hot}(C\text{–}\mathsf{comod})_{\mathsf{inj}}$ coincides with the minimal triangulated subcategory of $\mathsf{Hot}(C\text{–}\mathsf{comod})$ containing the DG-comodule C and closed under infinite products; the former subcategory certainly contains the latter one. Notice that the DG-comodule C and the DG-contramodule $\mathrm{Hom}_k(C,k)$ correspond to each other under the equivalence of categories $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$. ☐

6. Koszul Duality: Conilpotent and Nonconilpotent Cases

6.1. **Bar and cobar constructions.** Let $B = (B, d, h)$ be a CDG-algebra over a field k. We assume that B is nonzero, i. e., the unit element $1 \in B$ is not equal to 0, and consider $k = k \cdot 1$ as a graded vector subspace in B. Let $v\colon B \longrightarrow k$ be a homogeneous k-linear retraction of the graded vector space B to its subspace k; set $V = \ker v \subset B$. The direct sum decomposition $B = V \oplus k$ allows one to split the multiplication map $m\colon V \otimes_k V \longrightarrow B$, the differential $d\colon V \longrightarrow B$, and the curvature element $h \in B$ into the components $m = (m_V, m_k)$, $d = (d_V, d_k)$, and $h = (h_V, h_k)$, where $m_V\colon V \otimes_k V \longrightarrow V$, $m_k\colon V \otimes_k V \longrightarrow k$, $d_V\colon V \longrightarrow V$, $d_k\colon V \longrightarrow k$, $h_V \in V$, and $h_k \in k$. Notice that the restrictions of the multiplication map and the differential to $k \otimes_k V$, $V \otimes_k k$, $k \otimes_k k$, and k are uniquely determined by the axioms of a graded algebra and its derivation. One has $h_k = 0$ for the dimension reasons when B is $\mathbb{Z}$-graded, but h_k may be nonzero when B is $\mathbb{Z}/2$-graded (see Remark 1.1).

Set $B_+ = B/k$. Let $\mathrm{Bar}(B) = \bigoplus_{i=0}^{\infty} B_+^{\otimes i}$ be the tensor coalgebra generated by the graded vector space B_+. The comultiplication in $\mathrm{Bar}(B)$ is given by the rule $b_1 \otimes \dots \otimes b_i \longmapsto \sum_{j=0}^{i} (b_1 \otimes \dots \otimes b_j) \otimes (b_{j+1} \otimes \dots \otimes b_i)$ and the counit is the projection to the component $B_+^{\otimes 0} \simeq k$. The coalgebra $\mathrm{Bar}(B)$ is a graded coalgebra with the grading given by the rule $\deg(b_1 \otimes \dots \otimes b_i) = \deg(b_1) + \dots + \deg(b_i) - i$.

Odd coderivations of degree 1 on $\mathrm{Bar}(B)$ are determined by their compositions with the projection of $\mathrm{Bar}(B)$ to the component $B_+^{\otimes 1} \simeq B_+$; conversely, any linear map $\mathrm{Bar}(B) \to B_+$ of degree 2 gives rise to an odd coderivation of degree 1 on $\mathrm{Bar}(B)$. Let d_{Bar} be odd coderivation of degree 1 on $\mathrm{Bar}(B)$ whose compositions with the projection $\mathrm{Bar}(B) \longrightarrow B_+$ are given by the rules $b_1 \otimes \dots \otimes b_i \longmapsto 0$ for

$i \geqslant 3$, $b_1 \otimes b_2 \longmapsto (-1)^{|b_1|+1} m_V(b_1 \otimes b_2)$, $b \longmapsto -d_V(b)$, and $1 \longmapsto h_V$, where B_+ is identified with V and $1 \in B_+^{\otimes 0}$. Let $h_{\mathrm{Bar}} \colon \mathrm{Bar}(B) \longrightarrow k$ be the linear function given by the formulas $h_{\mathrm{Bar}}(b_1 \otimes \dots \otimes b_i) = 0$ for $i \geqslant 3$, $h_{\mathrm{Bar}}(b_1 \otimes b_2) = (-1)^{|b_1|+1} h_k(b_1 \otimes b_2)$, $h_{\mathrm{Bar}}(b) = -d_k(b)$, and $h_{\mathrm{Bar}}(1) = h_k$. Then $\mathrm{Bar}_v(B) = (\mathrm{Bar}(B), d_{\mathrm{Bar}}, h_{\mathrm{Bar}})$ is a CDG-coalgebra over k. The CDG-coalgebra $\mathrm{Bar}_v(B)$ is called the *bar-construction* of a CDG-algebra B endowed with a homogeneous k-linear retraction $v \colon B \longrightarrow k$.

A retraction $v \colon B \longrightarrow k$ is called an *augmentation* of a CDG-algebra B if $(v, 0) \colon (B, d, h) \longrightarrow (k, 0, 0)$ is a morphism of CDG-algebras; equivalently, v is an augmentation if it is a morphism of graded algebras satisfying the equations $v(d(b)) = 0$ and $v(h) = 0$. A k-linear retraction v is an augmentation if and only if the CDG-coalgebra $\mathrm{Bar}_v(B)$ is actually a DG-coalgebra, i. e., $h_{\mathrm{Bar}} = 0$.

Let $C = (C, d, h)$ be a CDG-coalgebra over k. We assume that C is nonzero, i. e., the counit map $\varepsilon \colon C \longrightarrow k$ is a nonzero linear function. Let $w \colon k \longrightarrow C$ be a homogeneous k-linear section of the surjective map of graded vector spaces ε; set $W = \operatorname{coker} w$. The direct sum decomposition $C = W \oplus k$ allows one to split the comultiplication map $\mu \colon C \longrightarrow W \otimes_k W$, the differential $d \colon C \longrightarrow W$, and the curvature linear function $h \colon C \longrightarrow k$ into the components $\mu = (\mu_W, \mu_k)$, $d = (d_W, d_k)$, and $h = (h_W, h_k)$, where $\mu_W \colon W \longrightarrow W \otimes_k W$, $\mu_k \in W \otimes_k W$, $d_W \colon W \longrightarrow W$, $d_k \in W$, $h_W \colon W \longrightarrow k$, and $h_k \in k$. Notice that the compositions of the comultiplication map with the projections $C \otimes_k C \longrightarrow k \otimes_k W$, $W \otimes_k k$, $k \otimes_k k$ and the composition of the differential with the projection (counit) $C \longrightarrow k$ are uniquely detemined by the axioms of a graded coalgebra and a differential compatible with the coalgebra structure. One has $h_k = 0$ for dimension reasons when B is $\mathbb{Z}$-graded, but h_k may be nonzero when B is $\mathbb{Z}/2$-graded.

Set $C_+ = \ker \varepsilon$. Let $\mathrm{Cob}(C) = \bigoplus_{i=0}^{\infty} C_+^{\otimes i}$ be the tensor (free associative) algebra, generated by the graded vector space C_+. The multiplication in $\mathrm{Cob}(C)$ is given by the rule $(c_1 \otimes \dots \otimes c_j)(c_{j+1} \otimes \dots \otimes c_i) = c_1 \otimes \dots \otimes c_i$ and the unit element is $1 \in k \simeq C_+^{\otimes 0}$. The algebra $\mathrm{Cob}(C)$ is a graded algebra with the grading given by the rule $\deg(c_1 \otimes \dots \otimes c_i) = \deg c_1 + \dots + \deg c_i + i$.

Odd derivations of degree 1 on $\mathrm{Cob}(C)$ are determined by their restrictions to the component $C_+ \simeq C_+^{\otimes 1} \subset \mathrm{Cob}(C)$; conversely, any linear map $C_+ \longrightarrow \mathrm{Cob}(C)$ of degree 2 gives rise to an odd derivation of degree 1 on $\mathrm{Cob}(C)$. Let d_{Cob} be the odd derivation on $\mathrm{Cob}(C)$ whose restriction to C_+ is given by the formula $d(c) = (-1)^{|c_{(1,W)}|+1} c_{(1,W)} \otimes c_{(2,W)} - d_W(c) + h_W(c)$, where C_+ is identified with W and $\mu_W(c) = c_{(1,W)} \otimes c_{(2,W)}$. Let $h_{\mathrm{Cob}} \in \mathrm{Cob}(C)$ be the element given by the formula $h_{\mathrm{Cob}} = (-1)^{|\mu_{(1,k)}|+1} \mu_{(1,k)} \otimes \mu_{(2,k)} - d_k + h_k$, where $\mu_k = \mu_{(1,k)} \otimes \mu_{(2,k)}$. Then $\mathrm{Cob}_w(C) = (\mathrm{Cob}(C), d_{\mathrm{Cob}}, h_{\mathrm{Cob}})$ is a CDG-algebra over k. The CDG-algebra $\mathrm{Cob}_w(C)$ is called the *cobar-construction* of a CDG-coalgebra C endowed with a homogeneous k-linear section $w \colon k \longrightarrow C$ of the counit map $\varepsilon \colon C \longrightarrow k$.

A section $w\colon k \longrightarrow C$ is called a *coaugmentation* of a CDG-coalgebra C if $(w,0)\colon (k,0,0) \longrightarrow (C,d,h)$ is a morphism of CDG-coalgebras; equivalently, w is a coaugmentation if it is a morphism of graded coalgebras satisfying the equations $d\circ w = 0$ and $h\circ w = 0$. A k-linear section w is a coaugmentation if and only if the CDG-algebra $\mathrm{Cob}_w(C)$ is actually a DG-algebra, i. e., $h_{\mathrm{Cob}} = 0$.

For any CDG-algebra B with a k-linear retraction v, the k-linear section $w\colon k \longrightarrow \mathrm{Bar}_v(B)$ given by the embedding of $k \simeq B_+^{\otimes 0}$ into $\mathrm{Bar}(B)$ is a coaugmentation of the CDG-coalgebra $\mathrm{Bar}_v(B)$ if and only if $h = 0$ in B, i. e., B is a DG-algebra. For any CDG-coalgebra C with a k-linear section w, the k-linear retraction $v\colon \mathrm{Cob}_w(C) \longrightarrow k$ given by the projection of $\mathrm{Cob}(C)$ onto $C_+^{\otimes 0} \simeq k$ is an augmentation of the CDG-algebra $\mathrm{Cob}_w(C)$ if and only if $h = 0$ on C, i. e., C is a DG-coalgebra. So a (co)augmentation on one side of the (co)bar-construction corresponds to the vanishing of the curvature element on the other side.

Given a CDG-algebra B, changing a retraction $v\colon B \longrightarrow k$ to another retraction $v'\colon B \longrightarrow k$ given by the formula $v'(b) = v(b) + a(b)$ leads to an isomorphism of CDG-coalgebras $(\mathrm{id}, a)\colon \mathrm{Bar}_v(B) \longrightarrow \mathrm{Bar}_{v'}(B)$, where $a\colon B_+ \longrightarrow k$ is a linear function of degree 0 identified with the corresponding linear function $\mathrm{Bar}(B) \longrightarrow B_+ \longrightarrow k$ of degree 1. Given a CDG-coalgebra C, changing a section $w\colon k \longrightarrow C$ to another section $w'\colon k \longrightarrow C$ given by the rule $w'(1) = w(1) + a$ leads to an isomorphism of CDG-algebras $(\mathrm{id}, a)\colon \mathrm{Cob}_{w'}(C) \longrightarrow \mathrm{Cob}_w(C)$, where $a \in C_+$ is an element of degree 0 identified with the corresponding element of $\mathrm{Cob}(C) \supset C_+$ of degree 1. To an isomorphism of CDG-coalgebras of the form $(\mathrm{id}, a)\colon (C,d,h) \longrightarrow (C,d',h')$ one can assign an isomorphism of the corresponding cobar-constructions of the form $(f_a, 0)\colon \mathrm{Cob}_w(C,d,h) \longrightarrow \mathrm{Cob}_w(C,d',h')$ with the automorphism f_a of the graded algebra $\mathrm{Cob}(C)$ given by the rule $c \longmapsto c + a(c)$ for $c \in C_+$. Here $a\colon C \longrightarrow k$ is a linear function of degree 1.

Consequently, there is a functor from the category of CDG-coalgebras to the category of CDG-algebras assigning to a CDG-coalgebra C its cobar-construction $\mathrm{Cob}_w(C)$. The cobar-construction is also a functor from the category of coaugmented CDG-coalgebras to the category of DG-algebras, from the category of DG-coalgebras to the category of augmented CDG-algebras, and from the category of coaugmented DG-coalgebras to the category of augmented DG-algebras.

Furthermore, let us call a morphism of CDG-algebras $(f,a)\colon B \longrightarrow A$ *strict* if one has $a = 0$. Then there is a functor from the category of CDG-algebras and strict morphisms between them to the category of CDG-coalgebras assigning to a CDG-algebra B its bar-construction $\mathrm{Bar}_v(B)$. The bar-construction is also a functor from the category of DG-algebras to the category of coaugmented CDG-coalgebras, from the category of augmented CDG-algebras and strict morphisms between them to the category of DG-coalgebras, and from the category of augmented DG-algebras to the category of coaugmented DG-coalgebras.

Remark. There is *no* isomorphism of bar-constructions corresponding to an isomorphism of CDG-algebras that is not strict. The reason is, essentially, that there exist no morphisms of tensor coalgebras $\mathrm{Bar}(B)$ that do not preserve their coaumentations $k \simeq B_+^{\otimes 0} \longrightarrow \mathrm{Bar}(B)$, while there do exist coderivations of $\mathrm{Bar}(B)$ not compatible with the coaugmentation. Moreover, for any augmented CDG-algebra $B = (B, d, h)$ with $h \neq 0$ the DG-coalgebra $\mathrm{Bar}_v(B)$ is acyclic, i. e., its cohomology is the zero coalgebra. Indeed, consider the dual DG-algebra $\mathrm{Bar}_v(B)^*$. Its subalgebra of cocycles of degree zero $Z^0(\mathrm{Bar}_v(B)^*)$ is complete in the adic topology of its augmentation ideal $\ker(Z^0(\mathrm{Bar}_v(B)^*) \to k)$, while the ideal of coboundaries $\operatorname{im} d^{-1} \subset Z^0(\mathrm{Bar}_v(B)^*)$ contains elements not belonging to the augmentation ideal. Thus $\operatorname{im} d^{-1} = Z^0(\mathrm{Bar}_v(B)^*)$ and the unit element $1 \in \mathrm{Bar}_v(B)^*$ is a coboundary. One can show that the DG-coalgebra $\mathrm{Bar}_v(B)$ considered up to DG-coalgebra isomorphisms carries *no* information about a coaugmented CDG-algebra B except for the dimensions of its raded components. Furthermore, for any CDG-algebra (B, d, h) with $h \neq 0$ and any left CDG-module M over B, the CDG-comodule $\mathrm{Bar}_v(B) \otimes^{\tau_{B,v}} M$ and the CDG-contramodule $\mathrm{Hom}^{\tau_{B,v}}(\mathrm{Bar}_v(B), M)$ over the CDG-coalgebra $\mathrm{Bar}_v(B)$ are contractible (see Remark 7.3).

6.2. **Twisting cochains.** Let $C = (C, d_C, h_C)$ be a CDG-coalgebra and $B = (B, d_B, h_B)$ be a CDG-algebra over the same field k. We introduce a CDG-algebra structure on the graded vector space of homogeneous homomorphisms $\mathrm{Hom}_k(C, B)$ in the following way. The multiplication in $\mathrm{Hom}_k(C, B)$ is given by the formula $(fg)(c) = (-1)^{|g||c_{(1)}|} f(c_{(1)}) g(c_{(2)})$. The differential is given by the standard rule $d(f)(c) = d_B(f(c)) - (-1)^{|f|} f(d_C(c))$. The curvature element is defined by the formula $h(c) = \varepsilon(c) h_B - h_C(c) \cdot 1$, where 1 is the unit element of B and ε is the counit map of C. A homogeneous linear map $\tau \colon C \longrightarrow B$ of degree 1 is called a *twisting cochain* [19, 14, 22] if it satisfies the equation $\tau^2 + d\tau + h = 0$ with respect to the above-defined CDG-algebra structure on $\mathrm{Hom}_k(C, B)$.

Let C be a CDG-coalgebra and $w \colon k \longrightarrow C$ be a homogeneous k-linear section of the counit map ε. Then the composition $\tau = \tau_{C,w} \colon C \longrightarrow \mathrm{Cob}(C)$ of the homogeneous linear maps $C \longrightarrow W \simeq C_+ \simeq C_+^{\otimes 1} \longrightarrow \mathrm{Cob}(C)$ is a twisting cochain for C and $\mathrm{Cob}_w(C)$. Let B be a CDG-algebra and $v \colon C \longrightarrow k$ be a homogeneous k-linear retraction. Then minus the composition $\mathrm{Bar}_v(B) \longrightarrow B_+^{\otimes 1} \simeq B_+ \simeq V \longrightarrow B$ is a twisting cochain $\tau = \tau_{B,v} \colon \mathrm{Bar}_v(B) \longrightarrow B$ for $\mathrm{Bar}_v(B)$ and B.

Let $\tau \colon C \longrightarrow B$ be a twisting cochain for a CDG-coalgebra C and a CDG-algebra B. Then for any left CDG-module M over B there is a natural structure of left CDG-comodule over C on the tensor product $C \otimes_k M$. Namely, the coaction of C in $C \otimes_k M$ is induced by the left coaction of C in itself, while the differential on $C \otimes_k M$ is given by the formula $d(c \otimes x) = d(c) \otimes x + (-1)^{|c|} c \otimes d(x) + (-1)^{|c_{(1)}|} c_{(1)} \otimes \tau(c_{(2)}) x$. We will denote the tensor product $C \otimes_k M$ with this CDG-comodule structure by $C \otimes^\tau M$.

Furthermore, for any left CDG-comodule N over C there is a natural structure of left CDG-module over B on the tensor product $B \otimes_k N$. Namely, the action of B in $B \otimes_k N$ is induced by the left action of B in itself, while the differential on $B \otimes_k N$ is given by the formula $d(b\otimes y) = d(b)\otimes y + (-1)^{|b|} b\otimes d(y) + (-1)^{|b|+1} b\tau(n_{(-1)})\otimes n_{(0)}$. We will denote the tensor product $B \otimes_k N$ with this CDG-module structure by $B \otimes^\tau N$.

The correspondences assigning to a CDG-module M over B the CDG-comodule $C \otimes^\tau M$ over C and to a CDG-comodule N over C the CDG-module $B \otimes^\tau N$ over B can be extended to DG-functors whose action on morphisms is given by the standard formulas $f_*(c \otimes x) = (-1)^{|f||c|} c \otimes f_*(x)$ and $g_*(b \otimes y) = (-1)^{|g||b|} b \otimes g_*(y)$. The DG-functor $C \otimes^\tau -\colon \mathsf{DG}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{DG}(C\text{–}\mathsf{comod})$ is right adjoint to the DG-functor $B \otimes^\tau -\colon \mathsf{DG}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{DG}(B\text{–}\mathsf{mod})$.

Analogously, for any right CDG-module M over B there is a natural structure of right CDG-comodule over C on the tensor product $M \otimes_k C$. The coaction of C in $M \otimes_k C$ is induced by the right coaction of C in itself and the differential on $M \otimes_k C$ is given by the formula $d(x \otimes c) = d(x) \otimes c + (-1)^{|x|} x \otimes d(c) + (-1)^{|x|+1} x\tau(c_{(1)}) \otimes c_{(2)}$. We will denote the tensor product $M \otimes_k C$ with this CDG-comodule structure by $M \otimes^\tau C$. For any right CDG-comodule N over C there is a natural structure of right CDG-module over B on the tensor product $N \otimes_k B$. Namely, the action of B in $N \otimes_k B$ is induced by the right action of B in itself and the differential on $N \otimes_k B$ is given by the formula $d(y\otimes b) = d(y)\otimes b + (-1)^{|y|} y\otimes d(b) + (-1)^{|y_{(0)}|} y_{(0)} \otimes \tau(y_{(1)})b$. We will denote the tensor product $N \otimes_k B$ with this CDG-module structure by $N \otimes^\tau B$.

For any left CDG-module P over B there is a natural structure of left CDG-contramodule over C on the graded vector space of homogeneous linear maps $\mathrm{Hom}_k(C, P)$. The contraaction of C in $\mathrm{Hom}_k(C, P)$ is induced by the right coaction of C in itself as explained in 2.2. The differential on $\mathrm{Hom}_k(C, P)$ is given by the formula $d(f)(c) = d(f(c)) - (-1)^{|f|} f(d(c)) + (-1)^{|f||c_{(1)}|}\tau(c_{(1)})f(c_{(2)})$ for $f \in \mathrm{Hom}_k(C, P)$. We will denote the graded vector space $\mathrm{Hom}_k(C, P)$ with this CDG-contramodule structure by $\mathrm{Hom}^\tau(C, P)$. For any left CDG-contramodule Q over C there is a natural structure of left CDG-module over B on the graded vector space of homogeneous linear maps $\mathrm{Hom}_k(B, Q)$. The action of B in $\mathrm{Hom}_k(B, Q)$ is induced by the right action of B in itself as explained in 1.5 and 1.7. The differential on $\mathrm{Hom}_k(B, Q)$ is given by the formula $d(f)(b) = d(f(b)) - (-1)^{|f|} f(d(b)) + \pi(c \mapsto (-1)^{|f|+1+|c||b|} f(\tau(c)b))$, where π denotes the contraaction map $\mathrm{Hom}_k(C, Q) \longrightarrow Q$. We will denote the graded vector space $\mathrm{Hom}_k(B, Q)$ with this CDG-module structure by $\mathrm{Hom}^\tau(B, Q)$.

The correspondences assigning to a CDG-module P over B the CDG-contramodule $\mathrm{Hom}^\tau(C, P)$ over C and to a CDG-contramodule Q over C the CDG-module $\mathrm{Hom}^\tau(B, Q)$ over B can be extended to DG-functors whose action on morphisms is given by the standard formula $g_*(f) = g \circ f$ for $f\colon C \longrightarrow P$ or $f\colon B \longrightarrow Q$. The DG-functor $\mathrm{Hom}^\tau(C, -)\colon \mathsf{DG}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{DG}(C\text{–}\mathsf{contra})$ is left adjoint to the DG-functor $\mathrm{Hom}^\tau(B, -)\colon \mathsf{DG}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{DG}(B\text{–}\mathsf{mod})$.

6.3. **Duality for bar-construction.** Let $A = (A, d)$ be a DG-algebra over a field k. Choose a homogeneous k-linear retraction $v\colon A \longrightarrow k$ and consider the bar-construction $C = \operatorname{Bar}_v(A)$; then C is a coaugmented CDG-coalgebra. Let $\tau = \tau_{A,v}\colon C \longrightarrow A$ be the natural twisting cochain.

Theorem. (a) *The functors* $C \otimes^\tau -\colon \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod})$ *and* $A \otimes^\tau -\colon \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$, *which are mutually inverse equivalences of triangulated categories.*

(b) *The functors* $\operatorname{Hom}^\tau(C, -)\colon \mathsf{Hot}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra})$ *and* $\operatorname{Hom}^\tau(A, -)\colon \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(A\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$, *which are mutually inverse equivalences of triangulated categories.*

(c) *The above equivalences of triangulated categories* $\mathsf{D}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}(A\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *form a commutative diagram with the equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *provided by the derived functors* $\mathbb{L}\Phi_C$ *and* $\mathbb{R}\Psi_C$ *of Theorem 5.2.*

Proof. Part (a): first notice that for any coacyclic CDG-comodule N over C the DG-module $A \otimes^\tau N$ over A is contractible. Indeed, whenever N is the total CDG-module of an exact triple of CDG-modules $A \otimes^\tau N$ is the total DG-module of an exact triple of DG-modules that splits as an exact triple of graded A-modules. Secondly, let us check that for any acyclic DG-module M over A the CDG-comodule $C \otimes^\tau M$ over C is coacyclic. Introduce an increasing filtration F on the coalgebra $C = \operatorname{Cob}(A)$ by the rule $F_i \operatorname{Cob}(A) = \bigoplus_{j\leqslant i} A_+^{\otimes i}$. There is an induced filtration on $C \otimes^\tau M$ given by the formula $F_i(C \otimes^\tau M) = F_i C \otimes_k M$. This is a filtration by CDG-subcomodules and the quotient CDG-comodules $F_i(C \otimes^\tau M)/F_{i-1}(C \otimes^\tau M)$ have trivial C-comodule structures. So these quotient CDG-comodules can be considered simply as complexes of vector spaces, and as such they are isomorphic to the complexes $A_+^{\otimes i} \otimes_k M$. These complexes are acyclic, and hence coacyclic, whenever M is acyclic. So the CDG-comodule $C \otimes^\tau M$ is coacyclic. Since it is cofree as a graded C-comodule, it is even contractible. We have shown that there are induced functors $\mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ and $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$; it remains to check that they are mutually inverse equivalences. For any DG-module M over A, the DG-module $A \otimes^\tau C \otimes^\tau M$ is isomorphic to the total DG-module of the reduced bar-resolution $\cdots \longrightarrow A \otimes A_+ \otimes A_+ \otimes M \longrightarrow A \otimes A_+ \otimes M \longrightarrow A \otimes M$. So the cone of the adjunction morphism $A \otimes^\tau C \otimes^\tau M \longrightarrow M$ is acyclic, since the reduced bar-resolution remains exact after passing to the cohomology $\cdots \longrightarrow H(A) \otimes H(A_+) \otimes H(M) \longrightarrow H(A) \otimes H(M) \longrightarrow H(M) \longrightarrow 0$, as explained in the proof of Theorem 1.4. For a CDG-comodule N over C, let K denote the cone of the adjunction morphism $N \longrightarrow C \otimes^\tau A \otimes^\tau N$. Let us show that the CDG-comodule K is absolutely acyclic.

Introduce a finite increasing filtration on the graded C-comodule K by the rules $F_{-2}K = 0$, $F_{-1}K = N[1]$, $F_0K = N[1] \oplus C \otimes_k k \otimes_k N \subset N[1] \oplus C \otimes_k A \otimes_k N$, and $F_1K = K$, where $C \otimes_k k \otimes_k N$ is embedded into $C \otimes_k A \otimes_k N$ by the map induced by the unit element of A. The differential d on K does not preserve this filtration; still one has $d(F_iK) \subset F_{i+1}K$. Let ∂ denote the differential induced by d on the associated quotient C-comodule $\mathrm{gr}_F K$. Then $(\mathrm{gr}_F K, \partial)$ is an exact complex of graded C-comodules; indeed, it is isomorphic to the standard resolution of the graded comodule N over the graded tensor coalgebra C. Set $L = F_{-1}K + d(F_{-1}K) \subset K$; it follows that both L and K/L are contractible CDG-comodules over C. Part (a) is proven; the proof of part (b) is completely analogous (up to duality). To prove (c), it suffices to notice the natural isomorphisms $\Psi_C(C \otimes^\tau M) \simeq \mathrm{Hom}^\tau(C, M)$ and $\Phi_C(\mathrm{Hom}^\tau(C, M)) \simeq C \otimes^\tau M$ for a DG-module M over A. □

6.4. **Conilpotent duality for cobar-construction.** A graded coalgebra E without counit is called *conilpotent* (cf. 5.5) if it is the union of the kernels of iterated comultiplication maps $E \longrightarrow E^{\otimes n}$. A graded coalgebra D endowed with a coaugmentation (morphism of coalgebras) $w\colon k \longrightarrow D$ is called *conilpotent* if the graded coalgebra without counit $D/w(k)$ is conilpotent. One can easily see that a conilpotent graded coalgebra has a unique coaugmentation.

For a conilpotent graded coalgebra D set F_nD to be the kernel of the composition $D \longrightarrow D^{\otimes n+1} \longrightarrow (D/w(k))^{\otimes n+1}$; then the increasing filtration F on D is compatible with the coalgebra structure. We will call a CDG-coalgebra $C = (C, d, h)$ *conilpotent* if it is coaugmented as a CDG-coalgebra and conilpotent as a graded coalgebra. A DG-coalgebra is *conilpotent* if it is conilpotent as a CDG-coalgebra. For a conilpotent CDG-coalgebra C, the filtration F defined above is compatible with the CDG-coalgebra structure, i. e., one has $d(F_nC) \subset F_nC$, and in addition, $h(F_0C) = 0$.

Let C be a conilpotent CDG-coalgebra and $w\colon k \longrightarrow C$ be its coaugmentation map. Consider the cobar-construction $A = \mathrm{Cob}_w(C)$; then A is a DG-algebra. Let $\tau = \tau_{C,w}\colon C \longrightarrow A$ be the natural twisting cochain.

Theorem. *All the assertions of Theorem 6.3 hold for the DG-algebra A, CDG-coalgebra C, and twisting cochain τ as above in place of A, C, and τ from 6.3.*

Proof. Just as in the proof of Theorem 6.3 one shows that for any coacyclic CDG-comodule N over C the DG-module $A \otimes^\tau N$ over A is contractible. To check that the CDG-comodule $C \otimes^\tau M$ is coacyclic (and even contractible) for any acyclic DG-module M over A, one uses the filtration F on the coalgebra C that was constructed above and the induced filtration of the CDG-comodule $C \otimes^\tau M$ by its CDG-subcomodules $F_i(C \otimes^\tau M) = F_iC \otimes_k M$. The quotient CDG-comodules $F_i(C \otimes^\tau M)/F_{i-1}(C \otimes^\tau M)$ are simply the complexes $F_iC/F_{i-1}C \otimes_k M$ with the trivial C-comodule structures, so they are coacyclic whenever M is acyclic. For any

CDG-comodule N over C, the CDG-comodule $C\otimes^\tau A\otimes^\tau N$ is isomorphic to the reduced version of the curved cobar-resolution introduced in the proof of Theorem 4.4. So the same argument with the canonical filtration with respect to the cobar differential ∂ proves that the cone of the adjunction morphism $N \longrightarrow C\otimes^\tau A\otimes^\tau N$ is coacyclic. For a DG-module M over A, denote by K the cocone of the adjunction morphism $A\otimes^\tau C\otimes^\tau M\longrightarrow M$. We will show that the DG-module K is absolutely acyclic. Introduce a finite decreasing filtration F on the graded A-module K by the rules $K/F^{-1}K=0$, $K/F^0K=M[-1]$, $K/F^1K=A\otimes_k k\otimes_k M\oplus M[-1]$, and $K/F^2K=K$, where $A\otimes_k C\otimes_k M$ maps onto $A\otimes_k k\otimes_k M$ by the map induced by the counit of C. The differential d on K does not preserve this filtration; still one has $d(F^iK)\subset F^{i-1}K$. Let ∂ denote the differential induced by d on the associated quotient A-module $\mathrm{gr}_F K$. Then $(\mathrm{gr}_F K,\partial)$ is an exact complex of graded A-modules; indeed, it is isomorphic to the standard resolution of the graded module M over the graded tensor algebra A. Set $L=F^1K+d(F^1K)\subset L$; it follows that both L and K/L are contractible DG-modules over A. The proof of part (b) is similar (up to duality), and the proof of (c) is analogous to the proof of Theorem 6.3(c). ☐

6.5. **Acyclic twisting cochains.** Let C be a coaugmented CDG-coalgebra with a coaugmentation w and A be a DG-algebra. Then there is a natural bijective correspondence between morphisms of DG-algebras $\mathrm{Cob}_w(C)\longrightarrow A$ and twisting cochains $\tau\colon C\longrightarrow A$ such that $\tau\circ w=0$. Whenever C is a DG-coalgebra, so that $\mathrm{Cob}_w(C)$ is an augmented DG-algebra, and A is also an augmented DG-algebra with an augmentation v, a morphism of DG-algebras $\mathrm{Cob}_w(C)\longrightarrow A$ preserves the augmentations if and only if one has $v\circ\tau=0$ for the corresponding twisting cochain τ.

Let us assume from now on that C is a conilpotent CDG-coalgebra. Then a twisting cochain $\tau\colon C\longrightarrow A$ with $\tau\circ w=0$ is said to be *acyclic* if the corresponding morphism of DG-algebras $\mathrm{Cob}(C)\longrightarrow A$ is a quasi-isomorphism.

Theorem. (a) *The functors* $C\otimes^\tau-\colon \mathsf{Hot}(A\text{–}\mathsf{mod})\longrightarrow\mathsf{Hot}(C\text{–}\mathsf{comod})$ *and* $A\otimes^\tau-\colon \mathsf{Hot}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{Hot}(A\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}(A\text{–}\mathsf{mod})\longrightarrow\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})\longrightarrow\mathsf{D}(A\text{–}\mathsf{mod})$*, the former of which is right adjoint to the latter. These functors are mutually inverse equivalences of triangulated categories if and only if the twisting cochain* τ *is acyclic.*

(b) *The functors* $\mathrm{Hom}^\tau(C,-)\colon \mathsf{Hot}(A\text{–}\mathsf{mod})\longrightarrow\mathsf{Hot}(C\text{–}\mathsf{contra})$ *and* $\mathrm{Hom}^\tau(A,-)\colon \mathsf{Hot}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{Hot}(A\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}(A\text{–}\mathsf{mod})\longrightarrow\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})\longrightarrow\mathsf{D}(A\text{–}\mathsf{mod})$*, the former of which is left adjoint to the latter. These functors are mutually inverse equivalences of triangulated categories if and only if the twisting cochain* τ *is acyclic.*

(c) *Whenever the twisting cochain* τ *is acyclic, the above equivalences of triangulated categories* $\mathsf{D}(A\text{–}\mathsf{mod})\simeq\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}(A\text{–}\mathsf{mod})\simeq\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *form a* *commutative diagram with the equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *provided by the derived functors* $\mathbb{L}\Phi_C$ *and* $\mathbb{R}\Psi_C$ *of Theorem 5.2.*

So in particular the twisting cochain $\tau = \tau_{A.v}$ of 6.3 is acyclic; the twisting cochain $\tau = \tau_{C,w}$ of 6.4 is acyclic by the definition.

Notice that for any acyclic twisting cochain τ the above equivalences of derived categories (of the first and the second kind) transform the trivial CDG-comodule k over C into the free DG-module A over A and the trivial CDG-contramodule k over C into the cofree DG-module $\mathrm{Hom}_k(A,k)$ over A. When C is a DG-coalgebra, A is an augmented DG-algebra with an augmentation v, and one has $v \circ \tau = 0$, these equivalences of exotic derived categories also transform the trivial DG-module k over A into the cofree DG-comodule C over C and into the free DG-contramodule $\mathrm{Hom}_k(C,k)$ over C. Here the trivial comodule, contramodule, and module structures on k are defined in terms of the coaugmentation w and augmentation v.

Proof. Part (a): Just as in the proofs of Theorems 6.3 and 6.4 one shows that the functor $N \longmapsto A \otimes^\tau M$ sends coacyclic CDG-comodules to contractible DG-modules and the functor $M \longmapsto C \otimes^\tau M$ sends acyclic DG-modules to contractible CDG-comodules. In order to see that the induced functors are adjoint it suffices to recall that adjointness of functors can be expressed in terms of adjunction morphisms and equations they satisfy; these morphisms obviously continue to exist and the equations continue to hold after passing to the induced functors between the quotient categories. To prove that these functors are equivalences of triangulated categories if and only if τ is an acyclic twisting cochain, it suffices to apply Theorem 6.4 and Theorem 1.7 for the morphism of DG-algebras $f\colon \mathrm{Cob}_w(C) \longrightarrow A$. Indeed, there are obvious isomorphisms of functors $C \otimes^\tau M \simeq C \otimes^{\tau_{C,w}} R_f(M)$ for a DG-module M over A and $A \otimes^\tau N \simeq E_f(\mathrm{Cob}_w(C) \otimes^{\tau_{C,w}} N)$ for a CDG-comodule N over C. The proof of part (b) is completely analogous and uses the functor E^f instead of E_f. Notice that for any twisting cochain τ, for any CDG-comodule N over C the DG-module $A \otimes^\tau N$ over A is projective and for any CDG-contramodule Q over C the DG-module $\mathrm{Hom}^\tau(A,Q)$ over A is injective, as one can prove using either the adjointness of the τ-related functors between the homotopy categories, or the facts that $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ is generated by the trivial CDG-comodule k as a triangulated category with infinite direct sums and $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ is generated by the trivial CDG-contramodule k as a triangulated category with infinite products (see 5.5 for some details). The proof of part (c) is identical to the proof of Theorem 6.3(c). ☐

Notice that for any acyclic twisting cochain $\tau\colon C \longrightarrow A$ and any left CDG-comodule N over C the complex $A \otimes^\tau N$ computes $\mathrm{Cotor}^C(k,N)$. Indeed, let $A \otimes^\tau C$ be the image of the right DG-module A over A under the functor $M \longmapsto M \otimes^\tau C$. Then by the right version of Theorem the right CDG-comodule $A \otimes^\tau C$ over C is isomorphic to the trivial right CDG-comodule k in the coderived category. This CDG-comodule is also cofree as a graded comodule and one has $A\otimes^\tau N \simeq (A\otimes^\tau C)\,\square_C N$. Analogously, for any acyclic twisting cochain τ and any left CDG-contramodule Q over C the complex $\mathrm{Hom}^\tau(A,Q)\simeq \mathrm{Cohom}_C(C\otimes^\tau A,\,Q)$ computes $\mathrm{Coext}_C(k,Q)$.

Now let C be a conilpotent DG-algebra, A be an augmented DG-algebra with an augmentation v, and $\tau\colon C\longrightarrow A$ be an acyclic twisting cochain for which $v\circ\tau=0$. Then for any left DG-module M over A the complex $C\otimes^\tau M$ computes $\mathrm{Tor}^A(k,M)$. Indeed, let $C\otimes^\tau A$ be the image of the right DG-comodule C over C under the functor $N\longmapsto N\otimes^\tau A$. Then by the right version of Theorem the right DG-module $C\otimes^\tau A$ over A is isomorphic to the trivial right DG-module k in the derived category. This DG-module is also projective, as mentioned in the above proof, and one has $C\otimes^\tau M \simeq (C\otimes^\tau A)\otimes_A M$. Analogously, for any left DG-module P over A the complex $\mathrm{Hom}^\tau(C,P)\simeq \mathrm{Hom}_A(A\otimes^\tau C,\,P)$ computes $\mathrm{Ext}_A(k,P)$.

It follows from the above Theorem that our definion of the coderived category of CDG-comodules is equivalent to the definition of Lefèvre-Hasegawa and Keller [19, 17] for a conilpotent CDG-coalgebra C.

6.6. **Koszul generators.** Let A be a DG-algebra over a field k. Suppose that A is endowed with an increasing filtration by graded subspaces $k=F_0A\subset F_1A\subset F_2A\subset \dots\subset A$ which is compatible with the multiplication, preserved by the differential, and cocomplete, i. e., $A=\bigcup_i F_iA$. Let $\mathrm{gr}_F A=\bigoplus_i F_iA/F_{i-1}A$ be the associated quotient algebra; it is a bigraded algebra with a grading n induced by the grading of A and a nonnegative grading i coming from the filtration F. Assume that the algebra gr_FA is Koszul [23, 24, 25] in its nonnegative grading i.

Choose a graded subspace $\overline{V}\subset F_1A$ complementary to $k=F_0A$ in F_1A. Notice that the filtration F on A is determined by the subspace $\overline{V}\subset A$, as a Koszul algebra is generated by its component of degree 1 We will call F a *Koszul filtration* and $\overline{V}$ a *Koszul generating subspace* of A. Extend $\overline{V}$ to a subspace $V\subset A$ complementary to k in A and denote by $v\colon A\longrightarrow k$ the projection of A to k along V.

Let $C\subset \bigoplus_i (F_1A/k)^{\otimes i}$ be the Koszul coalgebra quadratic dual to gr_FA. Recall that C is constructed as the direct sum of intersections of the form $C=\bigoplus_{i=0}^\infty \bigcup_{s=1}^{i-1}(F_1A/k)^{\otimes s-1}\otimes_k R\otimes_k (F_1A/k)^{\otimes i-s-1}$, where $R\subset (F_1A/k)\otimes_k(F_1A/k)$ is the kernel of the multiplication map $(F_1A/k)^{\otimes 2}\longrightarrow F_2A/F_1A$. In particular, $C_0=k$, $C_1=F_1A/k$, and $C_2=R$ are the low-degree components of C in the grading i. We will consider C as a subcoalgebra of the tensor coalgebra $\mathrm{Bar}(A)=\bigoplus_i (A/k)^{\otimes i}$ and endow C with the total grading inherited from the grading of $\mathrm{Bar}(A)$.

One can easily see that the graded subcoalgebra $C\subset \mathrm{Bar}_v(A)$ is preserved by the differential of $\mathrm{Bar}_v(A)$, which makes it a CDG-algebra and a CDG-subcoalgebra of $\mathrm{Bar}_v(A)$. The CDG-algebra structure on C does not depend on the choice of a subspace $V\subset A$, but only on the subspace $\overline{V}\subset F_1A$. Define the homogeneneous linear map $\tau\colon C\longrightarrow A$ of degree 1 as minus the composition $C\longrightarrow C_1=F_1A/k\simeq$

$\overline{V} \longrightarrow F_1A \subset A$. Clearly, C is a conilpotent CDG-coalgebra with the coaugmentation $w\colon k \simeq C_0 \longrightarrow C$ and $\tau \circ w = 0$.

Theorem. *The map τ is an acyclic twisting cochain.*

Proof. The element $\tau \in \operatorname{Hom}_k(C, A)$ satisfies the equation $\tau^2 + d\tau + h = 0$, since it is the image of the twisting cochain $\tau_{A,v} \in \operatorname{Hom}_k(\operatorname{Bar}_v(A), A)$ under the natural strict surjective morphism of CDG-algebras $\operatorname{Hom}_k(\operatorname{Bar}_v(A), A) \longrightarrow \operatorname{Hom}_k(C, A)$ induced by the embedding $C \longrightarrow \operatorname{Bar}_v(A)$. To check that the morphism of DG-algebras $\operatorname{Cob}_w(C) \longrightarrow A$ is a quasi-isomorphism, it suffices to pass to the associated quotient objects with respect to the increasing filtration F on A and the increasing filtration F on $\operatorname{Cob}_w(C)$ induced by the increasing filtration F on C associated with the grading i. Then it remains to use the fact that the coalgebra C is Koszul [24]. □

Let A be a DG-algebra with a Koszul generating subspace $\overline{V}$ and the corresponding Koszul filtration F. Consider the CDG-coalgebra C and the twisting cochain $\tau\colon C \longrightarrow A$ constructed above. Let M be a left DG-module over A; suppose that M is endowed with an increasing filtration by graded subspaces $F_0M \subset F_1M \subset \cdots \subset M$ that is compatible with the filtration on A and the action of A on M, preserved by the differential on M, and cocomplete, i. e., $M = \bigcup_i F_iM$. Assume that the associated quotient module $\operatorname{gr}_F M$ over the associated quotient algebra $\operatorname{gr}_F A$ is Koszul in its nonnegative grading i. Define the graded subcomodule $N \subset C \otimes^\tau M$ as the intersection $C \otimes_k F_0M \cap C \otimes_k S \subset C \otimes_k M$, where $S \subset (F_1A/k) \otimes F_0M$ is the kernel of the action map $(F_1A/k) \otimes_k F_0M \longrightarrow F_1M/F_0M$. This is the Koszul comodule quadratic dual to the Koszul module $\operatorname{gr}_F M$ over $\operatorname{gr}_F A$. The subcomodule N is preserved by the differential on $C \otimes^\tau M$, so it is a CDG-comodule over C. The natural morphism of DG-modules $A \otimes^\tau N \longrightarrow M$ over A is a quasi-isomorphism, as one can show in the way analogous to the proof of the above Theorem. A dual result holds for DG-modules P over A endowed with a complete decreasing filtration satisfying the Koszulity condition and the CDG-contramodules P quadratic dual to them.

Example. Let $\mathfrak{g}$ be a Lie algebra and $A = U\mathfrak{g}$ be its universal enveloping algebra considered as a DG-algebra concentrated in degree 0. Let F be the standard filtration on $U\mathfrak{g}$ and $\overline{V} = \mathfrak{g} \subset U\mathfrak{g}$ be the standard generating subspace; they are well-known to be Koszul. Then A is augmented, so C is a DG-coalgebra; it can be identified with the standard homological complex $C_*(\mathfrak{g})$. The functors $M \longmapsto C \otimes^\tau M$ and $P \longmapsto \operatorname{Hom}^\tau(C, P)$ are isomorphic to the functors of standard homological and cohomological complexes $M \longmapsto C_*(\mathfrak{g}, M)$ and $P \longmapsto C^*(\mathfrak{g}, P)$ with coefficients in complexes of $\mathfrak{g}$-modules M and P. Hence we see that these functors induce equivalences between the derived category of $\mathfrak{g}$-modules, the coderived category of DG-comodules over $C_*(\mathfrak{g})$, and the contraderived category of DG-contramodules over $C_*(\mathfrak{g})$. When $\mathfrak{g}$ and consequently $C_*(\mathfrak{g})$ are finite-dimensional, DG-comodules and

DG-contramodules over $C_*(\mathfrak{g})$ can be identified with DG-modules over the standard cohomological complex $C^*(\mathfrak{g})$, so the functors $M \longmapsto C_*(\mathfrak{g}, M)$ and $P \longmapsto C^*(\mathfrak{g}, P)$ induce equivalences between the derived category of $\mathfrak{g}$-modules, the coderived category of DG-modules over $C^*(\mathfrak{g})$, and the contraderived category of DG-modules over $C^*(\mathfrak{g})$. These results can be extended to the case of a central extension of Lie algebras $0 \longrightarrow k \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow 0$ with the kernel k and the enveloping algebra $U'\mathfrak{g} = U\mathfrak{g}'/(1_{U\mathfrak{g}'} - 1_{\mathfrak{g}'})$ governing representations of $\mathfrak{g}'$ where the central element $1 \in k \subset \mathfrak{g}'$ acts by the identity. Choose a section $\mathfrak{g} \longrightarrow \mathfrak{g}'$ of our central extension and define the generating subspace $\overline{V} \subset \mathfrak{g}' \subset U'\mathfrak{g}$ accordingly; then the corresponding CDG-coalgebra C coincides with the DG-coalgebra $C_*(\mathfrak{g})$ as a graded coalgebra with a coderivation; the 2-cochain $C_2(\mathfrak{g}) \longrightarrow k$ of the central extension $\mathfrak{g}' \longrightarrow \mathfrak{g}$ is the curvature linear function of C. The derived category of $U'\mathfrak{g}$-modules is equivalent to the coderived category of CDG-comodules and the coderived category of CDG-contramodules over this CDG-coalgebra C.

6.7. **Nonconilpotent duality for cobar-construction.** Let C be a CDG-coalgebra endowed with a homogeneous k-linear section $w\colon k \longrightarrow C$ of the counit map ε, and let B be a CDG-algebra. Then there is a natural bijective correspondence between morphisms of CDG-algebras $\mathrm{Cob}_w(C) \longrightarrow B$ and twisting cochains $\tau\colon C \longrightarrow B$. A morphism of CDG-algebras $\mathrm{Cob}_w(C) \longrightarrow B$ is strict if and only if one has $\tau \circ w = 0$ for the corresponding twisting cochain τ. Whenever C is a DG-coalgebra, so that $\mathrm{Cob}_w(C)$ is an augmented CDG-algebra, and B is also an augmented CDG-algebra with an augmentation v, a morphism of CDG-algebras $\mathrm{Cob}_w(C) \longrightarrow B$ preserves the augmentations if and only if one has $v \circ \tau = 0$ for the corresponding twisting cochain τ.

Given a CDG-coalgebra C with a k-linear section $w\colon k \longrightarrow C$, set $B = \mathrm{Cob}_w(C)$ and $\tau = \tau_{C,w}\colon C \longrightarrow B$.

Theorem. (a) *The functors* $C \otimes^{\tau} -\colon \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{comod})$ *and* $B \otimes^{\tau} -\colon \mathsf{Hot}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow \mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$*, which are mutually inverse equivalences of triangulated categories.*

(b) *The functors* $\mathrm{Hom}^{\tau}(C, -)\colon \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(C\text{–}\mathsf{contra})$ *and* $\mathrm{Hom}^{\tau}(B, -)\colon \mathsf{Hot}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{Hot}(B\text{–}\mathsf{mod})$ *induce functors* $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow \mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$*, which are mutually inverse equivalences of triangulated categories.*

(c) *The above equivalences of triangulated categories* $\mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *and* $\mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *form a commutative diagram with the equivalence of triangulated categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *provided by the derived functors* $\mathbb{L}\Phi_C$ *and* $\mathbb{R}\Psi_C$ *of Theorem 5.2.*

Whenever C is an coaugmented CDG-coalgebra and accordingly B is a DG-algebra, the above equivalences of triangulated categories transform the trivial CDG-comodule k over C into the free DG-module B over B and the trivial CDG-contramodule k over C into the cofree DG-module $\mathrm{Hom}_k(B,k)$ over B. Whenever C is a DG-coalgebra and accordingly B is an augmented CDG-algebra, the same equivalences of triangulated categories transform the trivial CDG-module k over B into the cofree DG-comodule C over C and into the free DG-contramodule $\mathrm{Hom}_k(C,k)$ over C.

Proof. The assertions about existence of induced functors in (a) and (b) hold for any CDG-coalgebra C, CDG-algebra B, and twisting cochain $\tau\colon C\longrightarrow B$. Indeed, the functors $M\longmapsto C\otimes^\tau M$ and $N\longmapsto B\otimes^\tau N$ send coacyclic objects to contractible ones, while the functors $P\longmapsto \mathrm{Hom}^\tau(C,P)$ and $Q\longmapsto \mathrm{Hom}^\tau(B,Q)$ send contraacyclic objects to contractible ones, for the reasons explained in the proof of Theorem 6.3. One also finds that the induced functors are adjoint to each other, as explained in the proof of Theorem 6.5. Now when $B=\mathrm{Cob}_w(C)$ and $\tau=\tau_{C,w}$, the adjunction morphisms are isomorphisms, as it was shown in the proof of Theorem 6.4. Notice that $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})=\mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})=\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$ in this case by Theorem 3.7. The proof of part (c) is identical to that of Theorem 6.3(c). ☐

Let us continue to assume that $B=\mathrm{Cob}_w(C)$ and $\tau=\tau_{C,w}$. Whenever C is a coaugmented CDG-coalgebra, for any left CDG-comodule N over C the complex $B\otimes^\tau N$ computes $\mathrm{Cotor}^C(k,N)$ and for any left CDG-contramodule Q over C the complex $\mathrm{Hom}^\tau(B,Q)$ computes $\mathrm{Coext}_C(k,Q)$, as explained in 6.5. Whenever C is a DG-coalgebra, for any left CDG-modules M and P over B the complex $C\otimes^\tau M$ computes $\mathrm{Tor}^{B,II}(k,M)$ and the complex $\mathrm{Hom}^\tau(C,P)$ computes $\mathrm{Ext}_B^{II}(k,P)$.

Corollary. *Let C be a conilpotent CDG-coalgebra, $w\colon k\longrightarrow C$ be its coaugmentation, and $A=\mathrm{Cob}_w(C)$ be its cobar-construction. Then the derived category $\mathsf{D}(A\text{–}\mathsf{mod})$ and the absolute derived category $\mathsf{D}^{\mathsf{abs}}(A\text{–}\mathsf{mod})$ coincide; in other words, any acyclic DG-module over A is absolutely acyclic.*

Proof. Compare Theorem 6.4 and Theorem 6.7. ☐

For a counterexample showing that the conilpotency condition is necessary in this Corollary, see Remark 6.9.

Now let $\tau\colon C\longrightarrow B$ be any twisting cochain between a CDG-coalgebra C and a CDG-algebra B. Let us discuss the adjunction properties of our functors between the homotopy categories in some more detail. Notice that the functor $M\longmapsto C\otimes^\tau M$ is the composition of left adjoint functors $M\longmapsto \mathrm{Hom}^\tau(C,M)$ and $Q\longmapsto \Phi_C(Q)$. So the corresponding composition of right adjoint functors $N\longmapsto \mathrm{Hom}^\tau(B,\Psi_C(N))$ is right adjoint to the functor $M\longmapsto C\otimes^\tau M$. At the same time, the functor $N\longmapsto B\otimes^\tau N$ is left adjoint to the functor $M\longmapsto C\otimes^\tau M$. Analogously, the functor $M\longmapsto \mathrm{Hom}^\tau(C,M)$ is the composition of right adjoint functors $M\longmapsto C\otimes^\tau M$

and $N \longmapsto \Psi_C(N)$. So the corresponding composition of left adjoint functors $Q \longmapsto B \otimes^\tau \Phi_C(Q)$ is left adjoint to the functor $M \longmapsto \mathrm{Hom}^\tau(C, M)$. At the same time, the functor $Q \longmapsto \mathrm{Hom}^\tau(B, Q)$ is right adjoint to the functor $M \longmapsto \mathrm{Hom}^\tau(C, M)$.

6.8. **Cotor and Tor, Coext and Ext, Ctrtor and Tor.** Let C be a conilpotent CDG-coalgebra, A be a DG-algebra, and $\tau\colon C \longrightarrow A$ be an acyclic twisting cochain. Notice that by the right version of Theorem 6.5 the functors $M \longmapsto M \otimes^\tau C$ and $N \longmapsto N \otimes^\tau A$ induce an equivalence of triangulated categories $\mathsf{D}(\mathsf{mod}\text{–}A) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$.

Theorem 1. (a) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \simeq \mathsf{D}^(\mathsf{mod}\text{–}A)$ *and* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ *transform the functor* Cotor^C *into the functor* Tor^A.

(b) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ *transform the functor* Coext_C *into the functor* Ext_A.

(c) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \simeq \mathsf{D}(\mathsf{mod}\text{–}A)$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \simeq \mathsf{D}(A\text{–}\mathsf{mod})$ *transform the functor* Ctrtor^C *into the functor* Tor^A.

Proof. To prove part (a), it suffices to use either of the natural isomorphisms of complexes $N' \square_C (C \otimes^\tau M'') \simeq (N' \otimes^\tau A) \otimes_A M''$ or $(M' \otimes^\tau C) \square_C N'' \simeq M' \otimes_A (A \otimes^\tau N'')$. To check that one obrains the same isomorphism of functors in these two ways, notice that the two compositions $N' \square_C N'' \longrightarrow N' \square_C (C \otimes^\tau A \otimes^\tau N'') \simeq (N' \otimes^\tau A) \otimes_A (A \otimes^\tau N'')$ and $N' \square_C N'' \longrightarrow (N' \otimes^\tau A \otimes^\tau C) \square_C N'' \longrightarrow (N' \otimes^\tau A) \otimes_A (A \otimes^\tau N'')$ coincide. To prove (b), use either of the isomorphisms $\mathrm{Cohom}_C(C \otimes^\tau M,\, Q) \simeq \mathrm{Hom}_A(M, \mathrm{Hom}^\tau(A, Q))$ or $\mathrm{Cohom}_C(N, \mathrm{Hom}^\tau(C, P)) \simeq \mathrm{Hom}_A(A \otimes^\tau N, P)$. Alternatively, use the result of 5.3 and Theorem 6.5. To check (c), use the natural isomorphism $(N \otimes^\tau A) \otimes_A P \simeq N \odot_C \mathrm{Hom}^\tau(C, P)$. □

Now let C be a CDG-algebra endowed with a k-linear section $w\colon k \longrightarrow C$ and $B = \mathrm{Cob}_w(C)$ be its cobar-construction. By the right version of Theorem 6.7, the functors $M \longmapsto M \otimes^\tau C$ and $N \longmapsto N \otimes^\tau B$ induce an equivalence of triangulated categories $\mathsf{D}^{\mathsf{abs}}(\mathsf{mod}\text{–}B) \simeq \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$.

Theorem 2. (a) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \simeq \mathsf{D}^{\mathsf{abs}}(\mathsf{mod}\text{–}B)$ *and* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *transform the functor* Cotor^C *into the functor* $\mathrm{Tor}^{B,II}$.

(b) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \simeq \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \simeq \mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$ *transform the functor* Coext_C *into the functor* Ext_B^{II}.

(c) *The equivalences of triangulated categories* $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \simeq \mathsf{D}(\mathsf{mod}\text{–}B)$ *and* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \simeq \mathsf{D}(B\text{–}\mathsf{mod})$ *transform the functor* Ctrtor^C *into the functor* $\mathrm{Tor}^{B,II}$.

Proof. See the proof of Theorem 1. □

6.9. **Bar duality between algebras and coalgebras.** Graded tensor coalgebras are cofree objects in the category of conilpotent graded coalgebras. More precisely, for any conilpotent graded coalgebra C with the coaugmentation $w\colon k \longrightarrow C$ and any graded vector space U there is a bijective correspondence between graded coalgebra morphisms $C \longrightarrow \bigoplus_{i=0}^{\infty} U^{\otimes i}$ and homogeneous k-linear maps $C/w(k) \longrightarrow U$ of degree zero. Notice that the graded coalgebra $\bigoplus_{i=0}^{\infty} U^{\otimes i}$ is conilpotent and any morphism of conilpotent graded coalgebras preserves the coaugmentations. Let us emphasize that the above assertion in *not* true when the graded coalgebra C is not conilpotent.

Let B be a CDG-algebra and $v\colon B \longrightarrow k$ be a homogeneous k-linear retraction. Let C be a CDG-coalgebra that is conilpotent as a graded coalgebra; denote by $w\colon k \longrightarrow C$ the coaugmentation map (which does not have to be a coaugmentation of C as a CDG-algebra). Then there is a natural bijective correspondence between morphisms of CDG-coalgebras $C \longrightarrow \mathrm{Bar}_v(B)$ and twisting cochains $\tau\colon C \longrightarrow B$ such that $\tau \circ w = 0$. Whenever C is a DG-coalgebra and v is an augmentation of B, a CDG-coalgebra morphism $C \longrightarrow \mathrm{Bar}_v(B)$ is actually a morphism of DG-coalgebras if and only if one has $v \circ \tau = 0$ for the corresponding twisting cochain τ.

Let $k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}$ denote the category of conilpotent CDG-coalgebras, $k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}$ denote the category of conilpotent DG-coalgebras, $k\text{–}\mathsf{alg}_{\mathsf{dg}}$ denote the category of DG-algebras with nonzero units, and $k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}$ denote the category of augmented DG-algebras (over the ground field k). It follows from the above that the functor of conilpotent cobar-construction $\mathrm{Cob}_w\colon k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}} \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}$ is left adjoint to the functor of DG-algebra bar-construction $\mathrm{Bar}_v\colon k\text{–}\mathsf{alg}_{\mathsf{dg}} \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}$. Analogously, the functor of conilpotent DG-coalgebra cobar-construction $\mathrm{Cob}_w\colon k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}} \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}$ is right adjoint to the functor of augmented DG-algebra bar-construction $\mathrm{Bar}_v\colon k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}} \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}$.

A morphism of conilpotent CDG-coalgebras $(f,a)\colon C \longrightarrow D$ is called a *filtered quasi-isomorphism* if there exist increasing filtrations F on C and D satisfying the following conditions. The filtrations F must be compatible with the comultiplications and differentials on C and D; one must have $F_0C = w_C(k)$ and $F_0D = w_D(k)$, so that, in particular, the associated quotient objects $\mathrm{gr}_F C$ and $\mathrm{gr}_F D$ are DG-coalgebras; and the induced morphism $\mathrm{gr}_F f\colon \mathrm{gr}_F C \longrightarrow \mathrm{gr}_F D$ must be a quasi-isomorphism of graded complexes of vector spaces. A morphism of conilpotent DG-coalgebras is a filtered quasi-isomorphism if it is a filtered quasi-isomorphism as a morphism of conilpotent CDG-coalgebras. The classes of filtered quasi-isomorphisms will be denoted by $\mathsf{FQuis} \subset k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}$ and $\mathsf{FQuis} \subset k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}$. Let us emphasize that there is *no* claim that the classes of filtered quasi-isomorphisms are closed under composition of morphisms. The classes of quasi-isomorphisms of DG-algebras and augmented DG-algebras will be denoted by $\mathsf{Quis} \subset k\text{–}\mathsf{alg}_{\mathsf{dg}}$ and $\mathsf{Quis} \subset k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}$.

Theorem. (a) *The functors* $\mathrm{Cob}_w\colon k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}} \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}$ *and* $\mathrm{Bar}_v\colon k\text{–}\mathsf{alg}_{\mathsf{dg}} \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}$ *induce functors between the localized categories* $k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}[\mathsf{FQuis}^{-1}] \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}[\mathsf{Quis}^{-1}]$ *and* $k\text{–}\mathsf{alg}_{\mathsf{dg}}[\mathsf{Quis}^{-1}] \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{cdg}}^{\mathsf{conilp}}[\mathsf{FQuis}^{-1}]$, *which are mutually inverse equivalences of categories.*

(b) *The functors* $\mathrm{Cob}_w\colon k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}} \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}$ *and* $\mathrm{Bar}_v\colon k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}} \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}$ *induce functors between the localized categories* $k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}[\mathsf{FQuis}^{-1}] \longrightarrow k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}[\mathsf{Quis}^{-1}]$ *and* $k\text{–}\mathsf{alg}_{\mathsf{dg}}^{\mathsf{aug}}[\mathsf{Quis}^{-1}] \longrightarrow k\text{–}\mathsf{coalg}_{\mathsf{dg}}^{\mathsf{conilp}}[\mathsf{FQuis}^{-1}]$, *which are mutually inverse equivalences of categories.*

Proof. We will prove part (a); the proof of part (b) is similar. For any filtered quasi-isomorphism of conilpotent CDG-coalgebras $(f,a)\colon C \longrightarrow D$ the induced morphism of cobar-constructions $\mathrm{Cob}_w(f,a)\colon \mathrm{Cob}_w(C) \longrightarrow \mathrm{Cob}_w(D)$ is a quasi-isomorphism of DG-algebras. Indeed, let F denote the increasing filtrations on the cobar-constructions induced by the filtrations F of C and D; then the morphism of associated graded DG-algebras $\mathrm{gr}_F \mathrm{Cob}_w(f,a)$ is a quasi-isomorphism, since the tensor products and the cones of morphisms of complexes preserve quasi-isomorphisms. Conversely, for any quasi-isomorphism of DG-algebras $g\colon A \longrightarrow B$ the induced morphism of bar-constructions $\mathrm{Bar}_v(g)\colon \mathrm{Bar}_v(A) \longrightarrow \mathrm{Bar}_v(B)$ is a filtererd quasi-isomorphism. Indeed, it suffices to consider the increasing filtrations of bar-constructions associated with their nonnegative gradings i by the number of factors in tensor powers. So the induced functors exist; it remains to check that they are mutually inverse equivalences. For any DG-algebra A, the adjunction morphism $\mathrm{Cob}_w(\mathrm{Bar}_v(A)) \longrightarrow A$ is a quasi-isomorphism. One can prove this by passing to the associated quotients with respect to the increasing filtration F on A defined by the rules $F_0A = k$ and $F_1A = A$, and the induced filtration on $\mathrm{Cob}_w(\mathrm{Bar}_v(A))$. Finally, for any conilpotent CDG-coalgebra C, the adjunction morphism $C \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(C))$ is a filtered quasi-isomorphism. Indeed, consider the natural increasing filtration F on C defined in 6.4 and the induced filtration F on $\mathrm{Bar}_v(\mathrm{Cob}_w(C))$. We have to prove that our adjunction morphism becomes a quasi-isomorphism after passing to the associated quotient objects, i. e., the morphism of graded DG-coalgebras $\mathrm{gr}_F C \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(\mathrm{gr}_F C))$ is a quasi-isomorphism. Here it suffices to consider the decreasing filtration G on $\mathrm{gr}_F C$ defined by the rules $G^0\mathrm{gr}_F C = \mathrm{gr}_F C$, $G^1\mathrm{gr}_F C = \ker(\varepsilon\colon \mathrm{gr}_F C \to k)$, and $G^2\mathrm{gr}_F C = 0$. The induced filtration on $\mathrm{Bar}_v(\mathrm{Cob}_w(\mathrm{gr}_F C))$ stabilizes at every degree of the nonnegative grading coming from the filtration F and the morphism $\mathrm{gr}_F C \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(\mathrm{gr}_F C))$ can be easily seen to become a quasi-isomorphism after passing to the associated quotient objects with respect to the filtration G. ☐

Remark. Notice that the notion of a filtered quasi-isomorphism makes sense for conilpotent CDG-coalgebras only, as any CDG-coalgebra admitting an increasing filtration F satisfying the conditions in the definition of a filtered quasi-isomorphism is conilpotent. And one cannot even speak about conventional (nonfiltered) quasi-isomorphisms of CDG-coalgebras, as the latter are not complexes. Furthermore, the assertions of Theorem do not hold with the filtered quasi-isomorphisms replaced with conventional quasi-isomorphisms of DG-coalgebras, or with the conilpotency condition dropped. Indeed, let A be any DG-algebra; consider it as a DG-algebra without unit and add a unit formally to it, obtaining an augmented DG-algebra $k \oplus A$ with the augmentation v. Then the morphisms of augmented DG-algebras $k \longrightarrow k \oplus A \longrightarrow k$ induce quasi-isomorphisms of bar-constructions $\mathrm{Bar}_v(k) \longrightarrow \mathrm{Bar}_v(k \oplus A) \longrightarrow \mathrm{Bar}_v(k)$; applying the cobar-construction, we find that the morphisms $\mathrm{Cob}_w(\mathrm{Bar}_v(k)) \longrightarrow \mathrm{Cob}_w(\mathrm{Bar}_v(k \oplus A)) \longrightarrow \mathrm{Cob}_w(\mathrm{Bar}_v(k))$ are *not* quasi-isomorphisms, since the middle term is quasi-isomorphic to $k \oplus A$. Analogously, let D be any DG-coalgebra; consider it as a DG-coalgebra without counit and add a counit formally to it, obtaining a coaugmented DG-coalgebra $k \oplus D$ with the coaugmentation w. Then the morphisms of augmented DG-algebras $k \longrightarrow \mathrm{Cob}_w(k \oplus D) \longrightarrow k$ are quasi-isomorphisms and it follows that the induced morphisms of bar-constructions $\mathrm{Bar}_v(k) \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(k \oplus D)) \longrightarrow \mathrm{Bar}_v(k)$ are also quasi-isomorphisms, hence the cohomology of the DG-coalgebra $\mathrm{Bar}_v(\mathrm{Cob}_w(k \oplus D))$ is different from that of $k \oplus D$. And there is even no natural morphism between C and $\mathrm{Bar}_v(\mathrm{Cob}_w(C))$ for a non-conilpotent DG-coalgebra C. Finally, let D be a DG-coalgebra and N be a left DG-comodule over D. Consider N as a DG-comodule over the above DG-coalgebra $k \oplus D$; then the DG-module $\mathrm{Cob}_w(k \oplus D) \otimes^{\tau_{k \oplus D, w}} N$ is acyclic. It follows that the assertions of Theorem 6.4 and Corollary 6.7 do not hold without the conilpotency assumption on the coaugmented CDG-coalgebra C.

7. A_∞-Algebras and Curved A_∞-Coalgebras

7.1. **Nonunital A_∞-algebras.** Let A be a graded vector space over a field k. Consider the graded tensor coalgebra (cofree conilpotent coassociative graded coalgebra) $\bigoplus_{i=0}^\infty A[1]^{\otimes i}$ with its coaugmentation $w\colon k \simeq A[1]^{\otimes 0} \longrightarrow \bigoplus_i A[1]^{\otimes i}$. A *nonunital A_∞-algebra* structure on A is, by the definition, a coaugmented DG-coalgebra structure on $\bigoplus_i A[1]^{\otimes i}$, i. e., an odd coderivation d of degree 1 on $\bigoplus_i A[1]^{\otimes i}$ such that $d^2 = 0$ and $d \circ w = 0$. Since a coderivation of $\bigoplus_i A[1]^{\otimes i}$ is uniquely determined by its composition with the projection $\bigoplus_i A[1]^{\otimes i} \longrightarrow A[1]^{\otimes 1} \simeq A[1]$, a nonunital A_∞-algebra structure on A can be considered as a sequence of linear maps $m_i\colon A^{\otimes i} \longrightarrow A$, $i = 1, 2, \ldots$ of degree $2 - i$. More precisely, define the maps m_i by the rule that the image of the element $d(a_1 \otimes \cdots \otimes a_i)$ under the projection to A

equals $(-1)^{i+\sum_{s=1}^i (i-s)(|a_s|+1)} m_i(a_1\otimes\dots\otimes a_i)$ for $a_s\in A$. The sequence of maps m_i must satisfy a sequence of quadratic equations corresponding to the equation $d^2=0$ on the coderivation d. We will not write down these equations explicitly.

A morphism of nonunital A_∞-algebras $f\colon A\longrightarrow B$ over k is, by the definition, a morphism of (coaugmented) DG-coalgebras $\bigoplus_i A[1]^{\otimes i}\longrightarrow\bigoplus_i B[1]^{\otimes i}$. Since a graded coalgebra morphism into a graded tensor coalgebra $\bigoplus_i B[1]^{\otimes i}$ is determined by its composition with the projection $\bigoplus_i B[1]^{\otimes i}\longrightarrow B[1]^{\otimes 1}\simeq B[1]$ and any morphism of conilpotent graded coalgebras preserves coaugmentations, a morphism of nonunital A_∞-algebras $f\colon A\longrightarrow B$ can be considered as a sequence of linear maps $f_i\colon A^{\otimes i}\longrightarrow B$, $i=1$, 2, … of degree $1-i$. More precisely, define the maps f_i by the rule that the image of the element $f(a_i\otimes\dots\otimes a_i)$ under the projection to B equals $(-1)^{i-1+\sum_{s=1}^i (i-s)(|a_s|+1)} f_i(a_1\otimes\dots\otimes a_i)$ for $a_s\in A$. The sequence of maps f_i must satisfy a sequence of polynomial equations corresponding to the equation $d\circ f=f\circ d$ on the morphism f.

Let A be a nonunital A_∞-algebra over a field k and M be a graded vector space over k. A structure of *nonunital left A_∞-module* over A on M is, by the definition, a structure of DG-comodule over the DG-coalgebra $\bigoplus_i A[1]^{\otimes i}$ on the cofree graded left comodule $\bigoplus_i A[1]^{\otimes i}\otimes_k M$ over the graded coalgebra $\bigoplus_i A[1]^{\otimes i}$. Analogously, a structure of *nonunital right A_∞-module* over A on a graded vector space N is defined as a structure of DG-comodule over $\bigoplus_i A[1]^{\otimes i}$ on the cofree graded right comodule $N\otimes_k\bigoplus_i A[1]^{\otimes i}$. Since a coderivation of a cofree graded comodule $\bigoplus_i A[1]^{\otimes i}\otimes_k M$ compatible with a given coderivation of the graded coalgebra $\bigoplus_i A[1]^{\otimes i}$ is determined by its composition with the projection $\bigoplus_i A[1]^{\otimes i}\otimes_k M\longrightarrow M$ induced by the counit map $\bigoplus_i A[1]^{\otimes i}\longrightarrow k$, a nonunital left A_∞-module structure on M can be considered as a sequence of linear maps $n_i\colon A^{\otimes i}\otimes M\longrightarrow M$, $i=0$, 1, … of degree $1-i$. More precisely, define the maps n_i by the rule that the image of the element $d(a_1\otimes\dots\otimes a_i\otimes x)$ under the projection to M equals $(-1)^{i+\sum_{s=1}^i (i-s)(|a_s|+1)} n_i(a_1\otimes\dots\otimes a_i\otimes x)$ for $a_s\in A$ and $x\in M$. The sequence of maps n_i must satisfy a system of nonhomogeneous quadratic equations corresponding to the equation $d^2=0$ on the coderivation d on $\bigoplus_i A[1]^{\otimes i}\otimes_k M$. Analogously, a nonunital right A_∞-module structure on N can be considered as a sequence of linear maps $n_i\colon N\otimes A^{\otimes i}\longrightarrow N$ defined by the rule that the image of the element $d(y\otimes a_1\otimes\dots\otimes a_i)$ under the projection to N equals $(-1)^{i|y|+\sum_{s=1}^i (i-s)(|a_s|+1)} n_i(y\otimes a_1\otimes\dots\otimes a_i)$ for $a_s\in A$ and $y\in N$.

The complex of morphisms between nonunital left A_∞-modules L and M over a nonunital A_∞-algebra A is, by the definition, the complex of morphisms between left DG-comodules $\bigoplus_i A[1]^{\otimes i}\otimes_k L$ and $\bigoplus_i A[1]^{\otimes i}\otimes_k M$ over the DG-coalgebra $\bigoplus_i A[1]^{\otimes i}$. Analogously, the complex of morphisms between nonunital right A_∞-modules R and N over A is, by the definition, the complex of morphisms between right DG-comodules $R\otimes_k\bigoplus_i A[1]^{\otimes i}$ and $N\otimes_k\bigoplus_i A[1]^{\otimes i}$ over the DG-coalgebra $\bigoplus_i A[1]^{\otimes i}$. A morphism of nonunital left A_∞-modules $f\colon L \longrightarrow M$ of degree j is the same that a sequence of linear maps $f_i\colon A^{\otimes i}\otimes_k L \longrightarrow M$, $i = 0, 1, \dots$ of degree $j - i$. More precisely, define the maps f_i by the rule that the image of the element $f(a_1\otimes\dots\otimes a_i\otimes x)$ under the projection to M equals $(-1)^{i+\sum_{s=1}^i(i-s)(|a_s|+1)}f_i(a_1\otimes\dots\otimes a_i\otimes x)$ for $a_s\in A$ and $x\in L$. Any sequence of linear maps f_i corresponds to a (not necessarily closed) morphism of nonunital A_∞-modules f. Analogously, a morphism of nonunital right A_∞-modules $g\colon R \longrightarrow N$ of degree j is the same that a sequence of linear maps $g_i\colon A^{\otimes i}\otimes_k R\longrightarrow N$ of degree $j - i$. More precisely, define the maps g_i by the rule that the image of the element $g(y\otimes a_1\otimes\dots\otimes a_i)$ under the projection to N equals $(-1)^{i|y|+\sum_{s=1}^i(i-s)(|a_s|+1)}g_i(y\otimes a_1\otimes\dots\otimes a_i)$ for $a_s\in A$ and $y\in R$.

For any CDG-coalgebra C, the functors Φ_C and Ψ_C of 5.1–5.2 provide an equivalence between the DG-category of left CDG-comodules over C that are cofree as graded C-comodules and the DG-category of left CDG-contramodules over C that are free as graded C-contramodules. So one can alternatively define a nonunital left A_∞-module M over a nonunital A_∞-algebra A as a graded vector space for which a structure of DG-contramodule over the DG-coalgebra $\bigoplus_i A[1]^{\otimes i}$ is given on the free graded contramodule $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\, M)$ over the graded coalgebra $\bigoplus_i A[1]^{\otimes i}$. Since a contraderivation of a free graded contramodule $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i}, M)$ compatible with a given coderivation of the graded coalgebra $\bigoplus_i A[1]^{\otimes i}$ is determined by its restriction to the graded subspace $M\subset \mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i}, M)$, a nonunital left A_∞-module structure on M can be considered as a sequence of linear maps $p_i\colon M\longrightarrow \mathrm{Hom}_k(A^{\otimes i}, M)$, $i = 0, 1, \dots$ of degree $1 - i$. More presicely, define the maps p_i by the formula $p_i(x)(a_1\otimes\dots\otimes a_i) = (-1)^{i+i|x|+\sum_{s=1}^i(i-s)(|a_s|+1)}d(x)(a_1\otimes\dots\otimes a_i)$ for $a_s\in A$ and $x\in M$. Then the maps p_i are related to the above maps $n_i\colon A^{\otimes i}\otimes_k M\longrightarrow M$ by the rule $p_i(x)(a_1\otimes\dots\otimes a_i) = (-1)^{|x|\sum_{s=1}^i|a_s|}n_i(a_1\otimes\dots\otimes a_i\otimes x)$.

Furthermore, one can alternatively define the complex of morphisms between nonunital left A_∞-modules L and M over a nonunital A_∞-algebra A as the complex of morphisms between left DG-contramodules $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\, L)$ and $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\, M)$ over the DG-coalgebra $\bigoplus_i A[1]^{\otimes i}$. Thus a (not necessarily closed) morphism of nonunital left A_∞-modules $f\colon L\longrightarrow M$ of degree j is the same that a sequence of linear maps $f^i\colon L\longrightarrow \mathrm{Hom}_k(A^{\otimes i}, M)$, $i = 0, 1, \dots$ of degree $j - i$. More precisely, define the maps f^i by the formula $f^i(x)(a_1\otimes\dots\otimes a_i) = (-1)^{i+i|x|+\sum_{s=1}^i(i-s)(|a_s|+1)}f(x)(a_1\otimes\dots\otimes a_i)$ for $a_s\in A$ and $x\in L$, where L is considered as a graded subspace in $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\, L)$. Then the maps f^i are related to the above maps $f_i\colon A^{\otimes i}\otimes_k L\longrightarrow M$ by the rule $f^i(x)(a_1\otimes\dots\otimes a_i) = (-1)^{|x|\sum_{s=1}^i|a_s|}f_i(a_1\otimes\dots\otimes a_i\otimes x)$.

7.2. **Strictly unital A_∞-algebras.** Let A be a nonunital A_∞-algebra over a field k. An element $1\in A$ of degree 0 is called a *strict unit* if one has $m_2(1\otimes a) = a = m_2(a\otimes 1)$ for all $a\in A$ and $m_i(a_1\otimes\dots\otimes a_{s-1}\otimes 1\otimes a_{s+1}\otimes\dots\otimes a_i) = 0$ for all $i\neq 2$,

$1 \leqslant s \leqslant i$, and $a_t \in A$. Obviously, a strict unit is unique if it exists. A *strictly unital A_∞-algebra* is a nonunital A_∞-algebra that has a strict unit. A morphism of strictly unital A_∞-algebras $f\colon A \longrightarrow B$ is a morphism of nonunital A_∞-algebras such that $f_1(1_A) = 1_B$ and $f_i(a_1 \otimes \dots \otimes a_{s-1} \otimes 1_A \otimes a_{s+1} \otimes \dots \otimes a_i) = 0$ for all $i > 1$ and $a_t \in A$. Notice that for a strictly unital A_∞-algebra A with the unit 1_A one has $1_A = 0$ if and only if $A = 0$. We will assume our strictly unital A_∞-algebras to have nonzero units.

A *strictly unital left A_∞-module* M over a strictly unital A_∞-algebra A is a nonunital left A_∞-module such that $n_1(1 \otimes x) = x$ and $n_i(a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i \otimes x) = 0$ for all $i > 1$, $1 \leqslant s \leqslant i$, $a_t \in A$, and $x \in M$. Equivalently, one must have $p_1(x)(1) = 0$ and $p_i(x)(a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i) = 0$. Analogously, a *strictly unital right A_∞-module* N over A is a nonunital right A_∞-module such that $n_1(y \otimes 1) = y$ and $n_i(y \otimes a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i) = 0$ for all $i > 1$, $a_t \in A$, and $y \in N$.

The complex of morphisms between strictly unital left A_∞-modules L and M over a strictly unital A_∞-algebra A is the subcomplex of the complex of morphisms between L and M as nonunital A_∞-modules consisting of all morphisms $f\colon L \longrightarrow M$ such that $f_i(a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i \otimes x) = 0$ for all $i > 0$, $1 \leqslant s \leqslant i$, $a_t \in A$, and $x \in L$. Equivalently, one must have $f^i(x)(a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i) = 0$. Analogously, the complex of morphisms between strictly unital right A_∞-modules R and N over A is the subcomplex of the complex of morphisms between R and N as nonunital A_∞-modules consisting of all morphisms $g\colon R \longrightarrow N$ such that $g_i(y \otimes a_1 \otimes \dots \otimes a_{s-1} \otimes 1 \otimes a_{s+1} \otimes \dots \otimes a_i) = 0$ for all $i > 0$, $a_t \in A$, and $y \in R$.

Let A be a nonunital A_∞-algebra and $1_A \in A$ be a nonzero element of degree 0. Set $A_+ = A/k \cdot 1_A$. Then the graded tensor coalgebra $\bigoplus_i A_+[1]^{\otimes i}$ is a quotient coalgebra of the tensor coalgebra $\bigoplus_i A[1]^{\otimes i}$. Denote by K_A the kernel of the natural surjection $\bigoplus_i A[1]^{\otimes i} \longrightarrow \bigoplus_i A_+[1]^{\otimes i}$ and by $\kappa_A\colon K_A \longrightarrow k$ the homogeneous linear function of degree 1 sending $1_A \in K_A \cap A[1]$ to $1 \in k$ and annihilating $K_A \cap A[1]^{\otimes i}$ for all $i > 1$. Let $\lambda_A\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow k$ be any homogeneous linear function of degree 1 extending the linear function κ_A on K_A. Then the element $1_A \in A$ is a strict unit if and only if the odd coderivation $d'(c) = d(c) + \lambda_A * c - (-1)^{|c|} c * \lambda_A$ of degree 1 on the tensor coalgebra $\bigoplus_i A[1]^{\otimes i}$ preserves the subspace K_A and the linear function $h'(c) = \lambda_A(d(c)) + \lambda_A^2(c)$ of degree 2 on $\bigoplus_i A[1]^{\otimes i}$ annihilates K_A. This condition does not depend on the choice of λ_A. For strictly unital A_∞-algebras A and B, a morphism of nonunital A_∞-algebras $f\colon A \longrightarrow B$ is a morphism of strictly unital A_∞-algebras if and only if $f(K_A) \subset K_B$ and $\kappa_B \circ f|_{K_A} = \kappa_A$.

Let A be a strictly unital A_∞-algebra and M be a nonunital left A_∞-module over A. Then M is a strictly unital A_∞-module if and only if the odd coderivation $d'(z) = d(z) + \lambda_A * z$ of degree 1 on the cofree comodule $\bigoplus_i A[1]^{\otimes i} \otimes_k M$ compatible with the coderivation d' of the coalgebra $\bigoplus_i A[1]^{\otimes i}$ preserves the subspace $K_A \otimes_k M \subset \bigoplus_i A[1]^{\otimes i} \otimes_k M$. Equivalently, the odd contraderivation $d'(q) =$

$d(q) + \lambda_A * q$ of degree 1 on the free contramodule $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\ M)$ compatible with the coderivation d' of the coalgebra $\bigoplus_i A[1]^{\otimes i}$ must preserve the subspace $\mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M) \subset \mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\ M)$. Analogously, a nonunital right A_∞-module N over A is a strictly unital A_∞-module if and only if the odd coderivation $d'(z) = d(z) - (-1)^{|z|} z * \lambda_A$ of degree 1 on the cofree comodule $N \otimes_k \bigoplus_i A[1]^{\otimes i}$ compatible with the coderivation d' of the coalgebra $\bigoplus_i A[1]^{\otimes i}$ preserves the subspace $N \otimes_k K_A \subset N \otimes_k \bigoplus_i A[1]^{\otimes i}$. For strictly unital left A_∞-modules L and M over A, a (not necessarily closed) morphism of nonunital A_∞-modules $f\colon L \longrightarrow M$ is a morphism of strictly unital A_∞-modules if and only if one has $f(K_A \otimes_k L) \subset K_A \otimes_k M$, or equivalently, $f(\mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ L)) \subset \mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M)$. Analogously, for strictly unital right A_∞-modules R and N over A, a (not necessarily closed) morphism of nonunital A_∞-modules $g\colon R \longrightarrow N$ is a morphism of strictly unital A_∞-modules if and only if one has $g(R \otimes_k K_A) \subset N \otimes_k K_A$.

Let A be a strictly unital A_∞-algebra. Identify k with the subspace $k \cdot 1_A \subset A$ and choose a homogeneous k-linear retraction $v\colon A \longrightarrow k$. Define the homogeneous linear function $\lambda_A\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow k$ of degree 1 by the rules $\lambda_A(a) = v(a)$ and $\lambda_A(a_1 \otimes \cdots \otimes a_i) = 0$ for $i \ne 1$. Then the linear function λ_A is an extension of the linear function $\kappa_A\colon K_A \longrightarrow k$. Let $d\colon \bigoplus_i A_+[1]^{\otimes i} \longrightarrow \bigoplus_i A_+[1]^{\otimes i}$ be the map induced by the odd coderivation d' of $\bigoplus_i A[1]^{\otimes i}$ defined by the above formula, and let $h\colon \bigoplus_i A_+[1]^{\otimes i} \longrightarrow k$ be the linear function induced by the above linear function h'. Then $\mathrm{Bar}_v(A) = (\bigoplus_i A_+[1]^{\otimes i},\ d,\ h)$ is a coaugmented (and consequenly, conilpotent) CDG-coalgebra with the coaugmentation $k \simeq A_+[1]^{\otimes 0} \longrightarrow \bigoplus_i A_+[1]^{\otimes i}$. The CDG-coalgebra $\mathrm{Bar}_v(A)$ is called the *bar-construction* of a strictly unital A_∞-algebra A.

Let $f\colon A \longrightarrow B$ be a morphism of strictly unital A_∞-algebras. Let $v\colon A \longrightarrow k$ and $v\colon B \longrightarrow k$ be homogeneous k-linear retractions, and let λ_A and λ_B be the corresponding homogeneous linear functions of degree 1 on the graded tensor coalgebras. The morphism of tensor coalgebras $f\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow \bigoplus_i B[1]^{\otimes i}$ maps K_A into K_B, so it induces a morphism of graded tensor coalgebras $\bigoplus_i A_+[1]^{\otimes i} \longrightarrow \bigoplus_i B_+[1]^{\otimes i}$, which we will denote also by f. The linear function $\lambda_B \circ f - \lambda_A\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow k$ annihilates K_A, so it induces a linear function $\bigoplus_i A_+[1]^{\otimes i} \longrightarrow k$, which we will denote by ρ_f. Then the pair (f, ρ_f) is a morphism of CDG-coalgebras $\mathrm{Bar}_v(A) \longrightarrow \mathrm{Bar}_v(B)$. Thus the bar-construction $A \longmapsto \mathrm{Bar}_v(A)$ is a functor from the category of strictly unital A_∞-algebras with nonzero units to the category of coaugmented CDG-coalgebras whose underlying graded coalgebras are graded tensor coalgebras. One can easily see that this functor is an equivalence of categories. Alternatively, one can use any linear function λ_A of degree 1 extending the linear function κ_A in the construction of this equivalence of categories.

To obtain the inverse functor, assign to a conilpotent CDG-coalgebra (D, d_D, h_D) the conilpotent DG-coalgebra (C, d_C) constructed as follows. First, adjoin to D a single cofree cogenerator of degree -1, obtaining a conilpotent graded coalgebra C

endowed with a graded coalgebra morphism $C \longrightarrow D$ and a homogeneous linear function $\lambda\colon C \longrightarrow k$ of degree 1. Second, define the odd coderivation d'_C of degree 1 on the graded coalgebra C by the conditions that d'_C must preserve the kernel of the graded coalgebra morphism $C \longrightarrow D$ and induce the differential d_D on D, and that the equation $\lambda(d'_C(c)) = \lambda^2(c) + h_D(c)$ must hold for all $c \in C$, where h_D is considered as a linear function on C. Finally, set $d_C(c) = d'_C(c) - \lambda * c + (-1)^{|c|} c * \lambda$ for all $c \in C$.

Let A be a strictly unital A_∞-algebra, $v\colon A \longrightarrow k$ be a homogeneous k-linear retraction, and $\lambda_A\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow k$ be the corresponding homogeneous linear function of degree 1. Let M be a strictly unital left A_∞-module over A. Set $d\colon \bigoplus_i A_+[1]^{\otimes i} \otimes_k M \longrightarrow \bigoplus_i A_+[1]^{\otimes i} \otimes_k M$ to be the map induced by the differential d' on $\bigoplus_i A[1]^{\otimes i} \otimes_k M$ defined by the above formula. Then $\mathrm{Bar}_v(A, M) = (\bigoplus_i A_+[1]^{\otimes i} \otimes_k M,\ d)$ is a left CDG-comodule over the CDG-coalgebra $\mathrm{Bar}_v(A)$. Furthermore, set $d\colon \mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M) \longrightarrow \mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M)$ to be the restriction of the differential d' on $\mathrm{Hom}_k(\bigoplus_i A[1]^{\otimes i},\ M)$ defined above. Then $\mathrm{Cob}^v(A, M) = (\mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M),\ d)$ is a left CDG-contramodule over the CDG-coalgebra $\mathrm{Bar}_v(A)$. Analogously, for a strictly unital right A_∞-module N over A set $d\colon N \otimes_k \bigoplus_i A_+[1]^{\otimes i} \longrightarrow N \otimes_k \bigoplus_i A_+[1]^{\otimes i}$ to be the map induced by the differential d' on $N \otimes_k \bigoplus_i A[1]^{\otimes i}$ defined above. Then $\mathrm{Bar}_v(N, A) = (N \otimes_k \bigoplus_i A_+[1]^{\otimes i},\ d)$ is a right CDG-comodule over the CDG-coalgebra $\mathrm{Bar}_v(A)$.

To a (not necessarily closed) morphism of strictly unital left A_∞-modules $f\colon L \longrightarrow M$ over A one can assign the induced maps $\bigoplus_i A_+[1]^{\otimes i} \otimes_k L \longrightarrow \bigoplus_i A_+[1]^{\otimes i} \otimes_k M$ and $\mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ L) \longrightarrow \mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M)$. These are a (not necessarily closed) morphism of CDG-comodules $\mathrm{Bar}_v(A, L) \longrightarrow \mathrm{Bar}_v(A, M)$ and a (not necessarily closed) morphism of CDG-contramodules $\mathrm{Cob}^v(A, L) \longrightarrow \mathrm{Cob}^v(A, M)$ over the CDG-coalgebra $\mathrm{Bar}_v(A)$. So we obtain the DG-functor $M \longmapsto \mathrm{Bar}_v(A, M)$, which is an equivalence between the DG-category of strictly unital left A_∞-modules over A and the DG-category of left CDG-comodules over $\mathrm{Bar}_v(A)$ that are cofree as graded comodules, and the DG-functor $M \longmapsto \mathrm{Cob}^v(A, M)$, which is an equivalence between the DG-category of strictly unital left A_∞-modules over A and the DG-category of left CDG-contramodules over $\mathrm{Bar}_v(A)$ that are free as graded contramodules. These two equivalences of DG-categories form a commutative diagram with the equivalence between the DG-category of CDG-comodules that are cofree as graded comodules and the DG-category of CDG-contramodules that are free as graded contramodules provided by the functors $\Psi_{\mathrm{Bar}_v(A)}$ and $\Phi_{\mathrm{Bar}_v(A)}$. Analogously, to a (not necessarily closed) morphism of strictly unital right A_∞-modules $g\colon R \longrightarrow N$ over A one can assign the induced map $R \otimes_k \bigoplus_i A_+[1]^{\otimes i} \longrightarrow N \otimes_k \bigoplus_i A_+[1]^{\otimes i}$. This is a (not necessarily closed) morphism of CDG-comodules $\mathrm{Bar}_v(R, A) \longrightarrow \mathrm{Bar}_v(N, A)$. The DG-functor $N \longmapsto \mathrm{Bar}_v(N, A)$ is an equivalence between the DG-category of strictly unital right A_∞-modules over A and the DG-category of right CDG-comodules over $\mathrm{Bar}_v(A)$ that are cofree as graded comodules.

Now let A be a DG-algebra with nonzero unit, M be a left DG-module over A, and N be a right DG-module over A. Let $v\colon A \longrightarrow k$ be a homogeneous k-linear retraction. Define a strictly unital A_∞-algebra structure on A by the rules $m_1(a) = d(a)$, $m_2(a_1 \otimes a_2) = a_1a_2$, and $m_i = 0$ for $i > 2$. Define a structure of a strictly unital left A_∞-module over A on M by the rules $n_0(x) = d(x)$, $n_1(a \otimes x) = ax$, and $n_i = 0$ for $i > 1$, where $a \in A$ and $x \in M$. Analogously, define a structure of a strictly unital right A_∞-module over A on N by the rules $n_0(y) = y$, $n_1(y \otimes a) = ya$, and $n_i = 0$ for $i > 1$, where $a \in A$ and $y \in N$. Then the CDG-coalgebra structure $\mathrm{Bar}_v(A)$ on the graded tensor coalgebra $\bigoplus_i A_+[1]^{\otimes i}$ that was defined in 6.1 coincides with the CDG-coalgebra structure $\mathrm{Bar}_v(A)$ constructed above, so our notation is consistent. The left CDG-comodule structure $\mathrm{Bar}_v(A) \otimes^{\tau_{A,v}} M$ on the cofree graded comodule $\bigoplus_i A_+[1]^{\otimes i} \otimes_k M$ that was defined in 6.2 coincides with the left CDG-comodule structure $\mathrm{Bar}_v(A, M)$. The left CDG-contramodule structure $\mathrm{Hom}^{\tau_{A,v}}(\mathrm{Bar}_v(A), M)$ on the free graded contramodule $\mathrm{Hom}_k(\bigoplus_i A_+[1]^{\otimes i},\ M)$ coincides with the CDG-contramodule structure $\mathrm{Cob}^v(A, M)$. The right CDG-comodule structure $N \otimes^{\tau_{A,v}} \mathrm{Bar}_v(A)$ on the cofree graded comodule $N \otimes_k \bigoplus_i A_+[1]^{\otimes i}$ coincides with the right CDG-comodule structure $\mathrm{Bar}_v(N, A)$.

A morphism of strictly unital A_∞-algebras $f\colon A \longrightarrow B$ is called *strict* if $f_i = 0$ for all $i > 1$. An *augmented* strictly unital A_∞-algebra A is a strictly unital A_∞-algebra endowed with a morphism of strictly unital A_∞-algebras $A \longrightarrow k$, where the strictly unital A_∞-algebra structure on k comes from its structure of DG-algebra with zero differential. An augmented strictly unital A_∞-algebra is *strictly augmented* if the augmentation morphism is strict. A morphism of augmented or strictly augmented strictly unital A_∞-algebras is a morphism of strictly unital A_∞-algebras forming a commutative diagram with the augmentation morphisms. The categories of augmented strictly unital A_∞-algebras, strictly augmented strictly unital A_∞-algebras, and nonunital A_∞-algebras are equivalent. The equivalence of the latter two categories is provided by the functor of formal adjoining of the strict unit, and the equivalence of the former two categories can be deduced from the equivalence between the categories of DG-coalgebras C and CDG-coalgebras C endowed with a CDG-coalgebra morphism $C \longrightarrow k$. The DG-category of strictly unital A_∞-modules over an augmented strictly unital A_∞-algebra A is equivalent to the DG-category of nonunital A_∞-modules over the corresponding nonunital A_∞-algebra.

7.3. **Derived category of A_∞-modules.** Let A be a strictly unital A_∞-algebra over a field k. A (not necessarily closed) morphism of strictly unital left A_∞-modules $f\colon L \longrightarrow M$ over A is called *strict* if one has $f_i = 0$ and $(df)_i = 0$ for all $i > 0$, or equivalently, $f^i = 0$ and $(df)^i = 0$ for all $i > 0$. Strictly unital left A_∞-modules and strict morphisms between them form a DG-subcategory of the DG-category of strictly unital left A_∞-modules and their morphisms.

A closed strict morphism of strictly unital A_∞-modules is called a *strict homotopy equivalence* if it is a homotopy equivalence in the DG-category of strictly unital A_∞-modules and strict morphisms between them. A triple $K \longrightarrow L \longrightarrow M$ of strictly unital A_∞-modules with closed strict morphisms between them is said to be *exact* if $K \longrightarrow L \longrightarrow M$ is an exact triple of graded vector spaces. The total strictly unital A_∞-module of such an exact triple is defined in the obvious way.

Any strictly unital left A_∞-module M over A can be considered as a complex with the differential $n_0 = p_0\colon M \longrightarrow M$, since one has $n_0^2 = 0$. A strictly unital left A_∞-module M is called *acyclic* if it is acyclic as a complex with the differential n_0. For any closed morphism of strictly unital left A_∞-modules $f\colon L \longrightarrow M$ the map $f_0 = f^0\colon L \longrightarrow M$ is a morphism of complexes with respect to n_0. The morphism f is called a *quasi-isomorphism* if f_0 is a quasi-isomorphism of complexes.

Let $v\colon A \longrightarrow k$ be a homogeneous k-linear retraction and $C = \mathrm{Bar}_v(A)$ be the corresponding CDG-coalgebra structure on the graded tensor coalgebra $\bigoplus_i A_+[1]^{\otimes i}$.

Theorem 1. *The following five definitions of the* derived category $\mathsf{D}(A\text{–}\mathsf{mod})$ of strictly unital left A_∞-modules *over A are equivalent, i. e., lead to naturally isomorphic triangulated categories:*

(a) *the homotopy category of the DG-category of strictly unital left A_∞-modules over A and their morphisms;*

(b) *the localization of the category of strictly unital left A_∞-modules over A and their closed morphisms by the class of quasi-isomorphisms;*

(c) *the localization of the category of strictly unital left A_∞-modules over A and their closed morphisms by the class of strict homotopy equivalences;*

(d) *the quotient category of the homotopy category of the DG-category of strictly unital left A_∞-modules over A and strict morphisms between them by the thick subcategory of acyclic A_∞-modules;*

(e) *the localization of the category of stricly unital left A_∞-modules over A and their closed strict morphisms by the class of strict quasi-isomorphisms;*

(f) *the quotient category of the homotopy category of the DG-category of strictly unital left A_∞-modules over A and strict morphisms between them by its minimal triangulated subcategory containing all the total strictly unital A_∞-modules of exact triples of strictly unital A_∞-modules with closed strict morphisms between them.*

The derived category $\mathsf{D}(A\text{–}\mathsf{mod})$ *is also naturally equivalent to the following triangulated categories:*

(g) *the coderived category* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *of left CDG-comodules over C;*

(h) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *of left CDG-contramodules over C;*

(i) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(C\text{–}\mathsf{comod})$ *of left CDG-comodules over C;*

(j) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(C\text{–}\mathsf{contra})$ *of left CDG-contramodules over C.*

Proof. The equivalence of (a-g) holds in the generality of CDG-comodules and CDG-contramodules over an arbitrary conilpotent CDG-coalgebra C. More precisely, let us consider CDG-comodules over C that are cofree as graded comodules, or equivalently, CDG-contramodules over C that are free as graded contramodules, in place of strictly unital A_∞-modules. The equivalence of (a), (g), and (h) follows from (the proof of) Theorem 4.4.

There is a natural increasing filtration F on a conilpotent CDG-coalgebra C that was defined in 6.4, and there are induced increasing filtrations $F_nK = \nu^{-1}(F_nC \otimes_k K)$ on all CDG-comodules K over C and decreasing filtrations $F^nQ = \pi(\mathrm{Hom}_k(C/F_{n-1}C, Q))$ on all CDG-contramodules Q over C. In particular, from any CDG-comodule $C \otimes_k M$ that is cofree as a graded comodule and the corresponding CDG-contramodule $\mathrm{Hom}_k(C, M)$ that is free as a graded contramodule one can recover the complex M as $M \simeq F_0(C \otimes_k M) \simeq \mathrm{Hom}_k(C, M)/F^1 \mathrm{Hom}_k(C, M)$. A closed morphism of CDG-comodules $C \otimes_k L \longrightarrow C \otimes_k M$ and the corresponding closed morphism of CDG-contramodules $\mathrm{Hom}_k(C, L) \longrightarrow \mathrm{Hom}_k(C, M)$ are homotopy equivalences if and only if the corresponding morphism of complexes $L \longrightarrow M$ is a quasi-isomorphism. Indeed, let us pass to the cones and check that a cofree CDG-comodule $C \otimes_k M$ is contractible if and only if the complex M is acyclic. The "only if" is clear, and "if" follows from the fact that $C \otimes_k M$ is coacylic whenever M is acyclic. To check the latter, notice that the quotient CDG-comodules $F_i(C \otimes_k M)/F_{i-1}(C \otimes_k M)$ are just the tensor products of complexes of vector spaces $F_iC/F_{i-1}C \otimes_k M$ with the trivial CDG-comodule structures.

This proves the equivalence of (a) and (b), since for any DG-category DG with shifts and cones the homotopy category $H^0(\mathsf{DG})$ can be also obtained by inverting homotopy equivalences in the category of closed morphisms $Z^0(\mathsf{DG})$. The equivalence of (d) and (e) also follows from the latter result about DG-categories; and to prove the equivalence of (a) and (c) the following slightly stronger formulation of that result is sufficient. For any DG-category DG with shifts and cones consider the class of morphisms of the form $(\mathrm{id}_X, 0)\colon X \oplus \mathrm{cone}(\mathrm{id}_X) \longrightarrow X$. Then by formally inverting all the morphisms in this class one obtains the homotopy category $H^0(\mathsf{DG})$.

A morphism of CDG-comodules $f'\colon C \otimes_k L \longrightarrow C \otimes_k M$ and the corresponding morphism of CDG-contramodules $f''\colon \mathrm{Hom}_k(C, L) \longrightarrow \mathrm{Hom}_k(C, M)$ can be called strict if both f' and df' are as maps of graded vector spaces can be obtained by applying the functor $C \otimes -$ to certain maps $L \longrightarrow M$, or equivalently, both f'' and df'' as maps of graded vector spaces can be obtained by applying the functor $\mathrm{Hom}_k(C, -)$ to (the same) maps $L \longrightarrow M$. Let $w\colon k \longrightarrow C$ be the coaugmentation map; consider the DG-algebra $U = \mathrm{Cob}_w(C)$. When $C = \mathrm{Bar}_v(A)$ is the bar-construction of a strictly unital A_∞-algebra A, the DG-algebra U is called the *enveloping DG-algebra* of A. For any conilpotent CDG-coalgebra C, consider the DG-functors $C \otimes^{\tau_{C,w}} -$ and $\mathrm{Hom}^{\tau_{C,w}}(C, -)$ assinging CDG-comodules and CDG-contramodules over C to DG-modules over U. These two DG-functors are equivalences between the DG-categories of left DG-modules over A, left CDG-comodules over C that are cofree as graded comodules with strict morphisms between them, and left CDG-contramodules over C that are free as graded contramodules with strict morphisms between them. So the equivalence of (a) and (d) follows from Theorem 6.4, and the equivalence of (d) and (f) follows from Corollary 6.7.

Finally, the equivalences (g)$\Longleftrightarrow$(i) and (h)$\Longleftrightarrow$(j) for $C = \mathrm{Bar}_v(A)$ are provided by Theorem 4.5. $\square$

Let A be a DG-algebra over k; it can be considered as a strictly unital A_∞-algebra and left DG-modules over it can be considered as strictly unital A_∞-modules as expained in 7.2. It follows from Theorem 6.3 that the derived category of left DG-modules over A is equivalent to the derived category of left A_∞-modules, so our notation $\mathsf{D}(A\text{–}\mathsf{mod})$ is consistent.

Any strictly unital A_∞-algebra A can be considered as a complex with the differential $m_1\colon A \longrightarrow A$, since $m_1^2 = 0$. For any morphism of strictly unital A_∞-algebras $f\colon A \longrightarrow B$ the map $f_1\colon A \longrightarrow B$ is a morphism of complexes with respect to m_1. A morphism f of strictly unital A_∞-algebras is called a *quasi-isomorphism* if $f_1\colon A \longrightarrow B$ is a quasi-isomorphism of complexes, or equivalently, $f_{1,+}\colon A_+ \longrightarrow B_+$ is a quasi-isomorphism of complexes.

Let $f\colon A \longrightarrow B$ be a morphism of strictly unital A_∞-algebras and $g\colon \mathrm{Bar}_v(A) \longrightarrow \mathrm{Bar}_v(B)$ be the corresponding morphism of CDG-coalgebras. Any strictly unital left A_∞-module M over B can be considered as a strictly unital left A_∞-module over A; this corresponds to the extension-of-scalarars functors E_g on the level of CDG-comodules that are cofree as graded comodules and E^g on the level of CDG-contramodules that are free as graded contramodules. Denote the induced functor on derived categories by $\mathbb{I}R_f\colon \mathsf{D}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(A\text{–}\mathsf{mod})$. The functor $\mathbb{I}R_f$ has left and right adjoint functors $\mathbb{L}E_f$ and $\mathbb{R}E^f\colon \mathsf{D}(A\text{–}\mathsf{mod}) \longrightarrow \mathsf{D}(B\text{–}\mathsf{mod})$ that can be constructed as the functors $\mathbb{I}R_g$ and $\mathbb{I}R^g$ on the level of coderived categories of CDG-comodules and contraderived categories of CDG-contramodules (see 5.4).

Theorem 2. *The functor R_f is an equivalence of triangulated categories if and only if a morphism f of strictly unital A_∞-algebras is a quasi-isomorphism.*

Proof. The "if" part follows easily from Theorem 4.7. Both "if" and "only if" can be deduced from Theorem 1.7 in the following way. For any strictly unital A_∞-algebra A and the corresponding CDG-coalgebra $C = \mathrm{Bar}_v(A)$ with its coaugmentation w, the adjunction morphism $C \longrightarrow \mathrm{Bar}_v(\mathrm{Cob}_w(C))$ corresponds to a morphism of strict A_∞-algebras $u\colon A \longrightarrow U(A)$ from A to its the enveloping DG-algebra $U(A)$ (see the proof of Theorem 1). The morphism u is a quasi-isomorphism, as one can see by considering the increasing filtration F on A defined by the rules $F_0A = k$ and $F_1A = A$, and the induced filtration on $U(A)$. The functor $\mathbb{I}R_u$ is an equivalence of triangulated categories, as it follows from Theorems 6.3 and 6.4, or as we have just proved. It remains to apply Theorem 1.7 to the morphism of DG-algebras $U(f)\colon U(A) \longrightarrow U(B)$. $\square$

Let A be a strictly unital A_∞-algebra and $C = \mathrm{Bar}_v(A)$ be the corresponding CDG-coalgebra. All the above results about strictly unital left A_∞-modules over A apply to strictly unital right A_∞-modules as well, since one can pass to the opposite CDG-coalgebra C^{op} as defined in 4.6. In particular, the derived category of strictly unital right A_∞-modules $\mathsf{D}(\mathsf{mod}{-}A)$ is defined and naturally equivalent to the coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}{-}C)$.

The functor $\mathrm{Tor}^A\colon \mathsf{D}(\mathsf{mod}{-}A) \times \mathsf{D}(A{-}\mathsf{mod}) \longrightarrow k{-}\mathsf{vect}^{\mathsf{gr}}$ can be constructed either by restricting the functor of cotensor product $\square_C\colon \mathsf{Hot}(\mathsf{comod}{-}C) \times \mathsf{Hot}(C{-}\mathsf{comod}) \longrightarrow \mathsf{Hot}(k{-}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-comodules that are cofree as graded comodules, or by restricting the functor of contratensor product $\odot_C\colon \mathsf{Hot}(\mathsf{comod}{-}C) \times \mathsf{Hot}(C{-}\mathsf{contra}) \longrightarrow \mathsf{Hot}(k{-}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-comodules that are cofree as graded comodules and CDG-contramodules that are free as graded contramodules. The functors one obtains in these two ways are naturally isomorphic by the result of 5.3. This definition of the functor Tor^A agrees with the definition of functor Tor^A for DG-algebras A by Theorem 6.8.1.

The functor $\mathrm{Ext}_A = \mathrm{Hom}_{\mathsf{D}(A{-}\mathsf{mod})}\colon \mathsf{D}(A{-}\mathsf{mod})^{\mathsf{op}} \times \mathsf{D}(A{-}\mathsf{mod}) \longrightarrow k{-}\mathsf{vect}^{\mathsf{gr}}$ can be computed in three ways. One can either restrict the functor $\mathrm{Hom}_C\colon \mathsf{Hot}(C{-}\mathsf{comod})^{\mathsf{op}} \times \mathsf{Hot}(C{-}\mathsf{comod}) \longrightarrow \mathsf{Hot}(k{-}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-comodules that are cofree as graded comodules, or restrict the functor $\mathrm{Hom}^C\colon \mathsf{Hot}(C{-}\mathsf{contra})^{\mathsf{op}} \times \mathsf{Hot}(C{-}\mathsf{contra}) \longrightarrow \mathsf{Hot}(k{-}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-contramodules that are free as graded contramodules, or restrict the functor $\mathrm{Cohom}_C\colon \mathsf{Hot}(C{-}\mathsf{comod})^{\mathsf{op}} \times \mathsf{Hot}(C{-}\mathsf{contra}) \longrightarrow \mathsf{Hot}(k{-}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-comodules that are cofree as graded comodules and CDG-contramodules that are free as graded contramodules. The functors one obtains in these three ways are naturally isomorphic by the result of 5.3, are isomorphic to the functor $\mathrm{Hom}_{\mathsf{D}(A{-}\mathsf{mod})}$ by Theorem 1 above, and agree with the functor Ext_A for DG-algebras A by Theorem 6.3 or Theorem 6.8.1.

Remark. One can define a nonunital curved A_∞-algebra A as a structure of not necessarily coaugmented DG-coalgebra on $\bigoplus_i A[1]^{\otimes i}$; such a structure is given by a sequence of linear maps $m_i\colon A^{\otimes i} \longrightarrow A$, $i = 0, 1, \dots$, where $m_0\colon k \longrightarrow A$ may be a nonzero map (corresponding to the curvature element of A). Any morphism of DG-coalgebras $f\colon \bigoplus_i A[1]^{\otimes i} \longrightarrow B[1]^{\otimes i}$ preserves the coaugmentations of the graded tensor coalgebras, though, so a morphism of nonunital curved A_∞-algebras $f\colon A \longrightarrow B$ is given by a sequence of linear maps $f_i\colon A^{\otimes i} \longrightarrow B$, $i = 1, 2, \dots$ All the definitions of 7.1–7.2 can be generalized straightforwardly to the curved situation, and all the results of 7.2 hold in this case. However, this theory is largely trivial. For any strictly unital curved A_∞-algebra A with $m_0 \neq 0$, every object of the DG-category of strictly unital curved A_∞-modules over A is contractible. In particular, the same applies to nonunital curved A_∞-modules over a nonunital curved A_∞-algebra. Two cases have to be considered separately, the case when the image of m_0 coincides with $k \cdot 1_A \subset A$, and the case when $m_0(1)$ and 1_A are linearly independent. The former case cannot occur in the $\mathbb{Z}$-graded situation for dimension reasons, but in the $\mathbb{Z}/2$-graded situation it is possible. In this case the differential on the CDG-coalgebra $C = \bigoplus_i A_+[1]^{\otimes i}$ is compatible with the coaugmentation $w\colon k \simeq A_+[1]^{\otimes 0} \longrightarrow C$, but the curvature linear function $h\colon C \longrightarrow k$ is not, i. e., $h \circ w \neq 0$. Let $C \otimes_k M$ be a left CDG-comodule over C that is cofree as a graded comodule and $F_nC \otimes_k M$ be its natural increasing filtration induced by the natural increasing filtration F of the conilpotent graded coalgebra C. Then the filtrations F on both C and $C \otimes_k M$ are preserved by the differentials, since the differential on C in compatible with the coaugmentations. The induced differential $n_0\colon M \longrightarrow M$ on $M = F_0C \otimes_k M$ has the square equal to a nonzero constant from k times the identity endomorphism of M. The CDG-comodule (M, n_0) over the CDG-coalgebra F_0C is clearly contractible; let t_0 be its contracting homotopy. Set $t = \mathrm{id} \otimes t_0\colon C \otimes_k M \longrightarrow C \otimes_k M$; then t is a nonclosed endomorphism of M of degree -1 and the endomorphism $d(t) = dt + td$ is invertible, hence $C \otimes_k M$ is contractible. This argument is applicable to any conilpotent graded coalgebra C. In the case when $m_0(1)$ and 1_A are linearly independent, the theory trivializes even further. The author learned the idea of the following arguments from M. Kontsevich. All strictly unital curved A_∞-algebra structures with $m_0(1)$ and 1_A linearly independent on a given graded vector space A are isomorphic, and all structures of a strictly unital curved A_∞-module over A on a given graded vector space M are isomorphic. Indeed, consider the component of tensor degree $i = 1$ of the differential on $C = \bigoplus_i A_+[1]^{\otimes i}$; it is determined by m_0. This differential makes $\bigoplus_i A_+[1]$ into a complex, and this complex is acyclic. Taking this fact into account, one can first find a CDG-coalgebra isomorphism of the form (id, a), $\ a\colon C \longrightarrow k$ between a given CDG-coalgebra structure on C and a certain CDG-coalgebra structure with $h = 0$, i. e., a DG-coalgebra structure. One proceeds step by step, killing the component $h_i\colon A_+^{\otimes i} \longrightarrow k$ of the linear function h using a linear function a with the only component $a_{i+1}\colon A_+^{\otimes i+1} \longrightarrow k$. Having obtained a DG-coalgebra structure on C, one subsequently kills all the components $m_i\colon A_+^{\otimes i} \longrightarrow A$ of the differential d with $i > 0$ using graded tensor coalgebra automorphisms f of C with the only nonzero components $f_1 = \mathrm{id}_{A_+}$ and $f_{i+1}\colon A_+^{\otimes i+1} \longrightarrow A_+$. Analogously one shows that any DG-comodule over C is isomorphic to a direct sum of shifts of the DG-comodule C. Since C is acyclic, such DG-comodules are clearly contractible. Alternatively, one could consider the DG-category of strictly unital curved A_∞-modules over a strictly unital curved A_∞-algebra A with strict morphisms between the curved

A_∞-modules. This DG-category is equivalent to the DG-category of CDG-modules over the CDG-coalgebra $U = \mathrm{Cob}_w(C)$. Its homotopy category $\mathsf{Hot}(U\text{–}\mathsf{mod})$ can well be nonzero, but the corresponding absolute derived category $\mathsf{D}^{\mathsf{abs}}(U\text{–}\mathsf{mod})$ is zero by Theorem 6.7. So in the category of (strictly unital or nonunital) curved A_∞-algebras over a field there are too few and too many morphisms at the same time: there are no "change-of-connection" morphisms, and in particular no morphisms corresponding to nonstrict morphisms of CDG-algebras, and still there are enough morphisms to trivialize the theory in almost all cases.

7.4. **Noncounital curved A_∞-coalgebras.** Let C be a graded vector space over a field k. Consider the graded tensor algebra (free associative graded algebra) $\bigoplus_{i=0}^\infty C[-1]^{\otimes i}$ generated by the graded vector space $C[-1]$. A *noncounital curved A_∞-coalgebra* structure on C is, by the definition, a DG-algebra structure on $\bigoplus_i C[-1]^{\otimes i}$, i. e., an odd derivation d of degree 1 on $\bigoplus_i C[-1]^{\otimes i}$ such that $d^2 = 0$. Since a derivation of $\bigoplus_i C[-1]^{\otimes i}$ is uniquely determined by its restriction to $C[-1] \simeq C[-1]^{\otimes 1} \subset \bigoplus_i C[-1]^{\otimes i}$, a noncounital curved A_∞-coalgebra structure on C can be considered as a sequence of linear maps $\mu_i\colon C \longrightarrow C^{\otimes i}$, $i = 0, 1, \dots$ of degree $2 - i$. More precisely, define the maps μ_i by the formula $d(c) = \sum_{i=0}^\infty (-1)^{i+\sum_{s=1}^i (i-s)(|\mu_{i,s}(c)|+1)} \mu_{i,1}(c) \otimes \dots \otimes \mu_{i,i}(c)$, where $c \in C$ and $\mu_i(c) = \mu_{i,1}(c) \otimes \dots \otimes \mu_{i,i}(c)$ is a symbolic notation for the tensor $\mu_i(c) \in C^{\otimes i}$. A *convergence condition* must be satisfied: for any $c \in C$ one must have $\mu_i(c) = 0$ for all but a finite number of the degrees i. Furthermore, the sequence of maps μ_i must satisfy a sequence of quadratic equations corresponding to the equation $d^2 = 0$ on the derivation d.

A morphism of noncounital curved A_∞-coalgebras $f\colon C \longrightarrow D$ over a field k is, by the definition, a morphism of DG-algebras $\bigoplus_i C[-1]^{\otimes i} \longrightarrow \bigoplus_i D[-1]^{\otimes i}$ over k. Since the graded algebra morphism from a graded tensor algebra $\bigoplus_i C[-1]^{\otimes i}$ is determined by its restriction to $C[-1] \simeq C[-1]^{\otimes 1} \subset \bigoplus_i C[-1]^{\otimes i}$, a morphism of noncounital curved A_∞-coalgebras $f\colon C \longrightarrow D$ can be considered as a sequence of linear maps $f_i\colon C \longrightarrow D^{\otimes i}$, $i = 0, 1, \dots$ of degree $1 - i$. More precisely, define the maps f_i by the formula $f(c) = \sum_{i=0}^\infty (-1)^{i-1+\sum_{s=1}^i (i-s)(|f_{i,s}(c)|+1)} f_{i,1}(c) \otimes \dots \otimes f_{i,i}(c)$, where $c \in C$ and $f_i(c) = f_{i,1}(c) \otimes \dots \otimes f_{i,i}(c) \in D^{\otimes i}$. A convergence condition must be satisfied: for any $c \in C$ one must have $f_i(c) = 0$ for all but a finite number of the degrees i. Furthermore, the sequence of maps f_i must satisfy a sequence of polynomial equations corresponding to the equation $d \circ f = f \circ d$ on the morphism f.

Let C be a noncounital curved A_∞-coalgebra over a field k and M be a graded vector space over k. A structure of *noncounital left curved A_∞-comodule* over C on M is, by the definition, a structure of DG-module over the DG-algebra $\bigoplus_i C[-1]^{\otimes i}$ on the free graded left module $\bigoplus_i C[-1]^{\otimes i} \otimes_k M$ over the graded algebra $\bigoplus_i C[-1]^{\otimes i}$. Analogously, a structure of *noncounital right curved A_∞-comodule* over C on a graded vector space N is defined as a structure of DG-module over $\bigoplus_i C[-1]^{\otimes i}$ on the free graded right module $N\otimes_k\bigoplus_i C[-1]^{\otimes i}$. Since a derivation of a free graded module $\bigoplus_i C[-1]^{\otimes i}\otimes_k M$ compatible with a given derivation of the graded algebra $\bigoplus_i C[-1]^{\otimes i}$ is determined by its restriction to the subspace of generators $M\simeq C[-1]^{\otimes 0}\otimes_k M\subset\bigoplus_i C[-1]^{\otimes i}\otimes_k M$, a noncounital left curved A_∞-comodule structure on M can be considered as a sequence of linear maps $\nu_i\colon M\longrightarrow C^{\otimes i}\otimes_k M$, $i=0,\,1,\,\dots$ of degree $1-i$. More precisely, define the maps ν_i by the formula $d(x)=\sum_{i=0}^\infty(-1)^{i+\sum_{s=-i}^{-1}(i+1)(|\nu_{i,s}(x)|+1)}\nu_{i,-i}(x)\otimes\dots\otimes\nu_{i,-1}(x)\otimes\nu_{i,0}(x)$, where $x\in M$ and $\nu_i(x)=\nu_{i,-i}(x)\otimes\dots\otimes\nu_{i,-1}(x)\otimes\nu_{i,0}(x)\in C^{\otimes i}\otimes_k M$. A convergence condition must be satisfied: for any $x\in M$ one must have $\nu_i(x)=0$ for all but a finite number of the degrees i. Furthermore, the sequence of maps ν_i must satisfy a sequence of nonhomogeneous quadratic equations corresponding to the equation $d^2=0$ on the derivation d on $\bigoplus_i C[-1]^{\otimes i}\otimes_k M$. Analogously, a noncounital right curved A_∞-comodule structure on N can be considered as a sequence of linear maps $\nu_i\colon N\longrightarrow N\otimes_k C^{\otimes i}$ defined by the formula $d(c)=\sum_{i=0}^\infty(-1)^{i|\nu_{i,0}(y)|+\sum_{s=1}^{i}(i-s)(|\nu_{i,s}(y)|+1)}\nu_{i,0}(y)\otimes\nu_{i,1}(y)\otimes\dots\otimes\nu_{i,i}(y)$, where $y\in N$ and $\nu_i(y)=\nu_{i,0}(y)\otimes\nu_{i,1}(y)\otimes\dots\otimes\nu_{i,i}(y)\in N\otimes_k C^{\otimes i}$.

The complex of morphisms between noncounital left curved A_∞-comodules L and M over a noncounital curved A_∞-coalgebra C is, by the definition, the complex of morphisms between left DG-modules $\bigoplus_i C[-1]^{\otimes i}\otimes_k L$ and $\bigoplus_i C[-1]^{\otimes i}\otimes_k M$ over the DG-algebra $\bigoplus_i C[-1]^{\otimes i}$. Analogously, the complex of morphisms between noncounital right curved A_∞-comodules R and N over C is, by the definition, the complex of morphisms between right DG-modules $R\otimes_k\bigoplus_i C[-1]^{\otimes i}$ and $N\otimes_k\bigoplus_i C[-1]^{\otimes i}$ over the DG-algebra $\bigoplus_i C[-1]^{\otimes}$. A morphism of noncounital left curved A_∞-comodules $f\colon L\longrightarrow M$ of degree j is the same that a sequence of linear maps $f_i\colon L\longrightarrow C^{\otimes i}\otimes_k M$, $i=0,\,1,\,\dots$ of degree $j-i$ satisfying the convergence condition: for any $x\in L$ one must have $f_i(x)=0$ for all but a finite number of degrees i. More precisely, define the maps f_i by the formula $f(x)=\sum_{i=0}^\infty(-1)^{i+\sum_{s=-i}^{-1}(i+1)(|f_{i,s}(x)|+1)}f_{i,-i}(x)\otimes\dots\otimes f_{i,-1}(x)\otimes f_{i,0}(x)$, where $x\in L$ and $f_i(x)=f_{i,-i}(x)\otimes\dots\otimes f_{i,-1}(x)\otimes f_{i,0}(x)\in C^{\otimes i}\otimes_k M$. Any sequence of linear maps f_i satisfying the convergence condition corresponds to a (not necessarily closed) morphism of noncounital curved A_∞-comodules f. Analogously, a morphism of noncounital right curved A_∞-comodules $f\colon R\longrightarrow N$ is the same that a sequence of linear maps $f_i\colon R\longrightarrow N\otimes_k C^{\otimes i}$, $i=0,\,1,\,\dots$ of degree $j-i$ satisfying the convergence condition. More precisely, define the maps f_i by the formula $f(y)=\sum_{i=0}^\infty(-1)^{i|f_{i,0}(y)|+\sum_{s=1}^{i}(i-s)(|f_{i,s}(y)|+1)}f_{i,0}(y)\otimes f_{i,1}(y)\otimes\dots\otimes f_{i,i}(y)$, where $y\in R$ and $f_i(y)=f_{i,0}(y)\otimes f_{i,1}(y)\otimes\dots\otimes f_{i,i}(y)\in N\otimes_k C^{\otimes i}$.

Let C be a noncounital curved A_∞-coalgebra over a field k and P be a graded vector space over k. A structure of *noncounital left curved A_∞-contramodule* over C on P is, by the definition, a structure of DG-module over the DG-algebra $\bigoplus_i C[-1]^{\otimes i}$

on the cofree graded left module $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ over the graded algebra $\bigoplus_i C[-1]^{\otimes i}$. The action of $\bigoplus_i C[-1]^{\otimes i}$ in $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ is induced by the right action of P in itself as explained in 1.5 and 1.7. Since a derivation of a cofree graded module $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ compatible with a given derivation of the graded algebra $\bigoplus_i C[-1]^{\otimes i}$ is determined by its composition with the projection $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P) \longrightarrow P$ induced by the unit map $k \longrightarrow \bigoplus_i C[-1]^{\otimes i}$, a noncounital left curved A_∞-contramodule structure on P can be considered as a linear map $\pi\colon \prod_{i=0}^\infty \mathrm{Hom}_k(C^{\otimes i}, P)[i-1] \longrightarrow P$ of degree 0. More precisely, define the map π by the rule that the image of the element $d(g)$ under the projection to P equals $\pi((g_i)_{i=0}^\infty)$, where a map $g\colon \bigoplus_i C[-1]^{\otimes i} \longrightarrow P$ and a sequence of maps $g_i\colon C^{\otimes i} \longrightarrow P$ are related by the formula $g(c_1\otimes\dots\otimes c_i) = (-1)^{i|g_i|+\sum_{s=1}^i(i-s)(|c_s|+1)}g_i(c_1\otimes\dots\otimes c_i)$ for $c_s \in C$. The map π must satisfy a system of nonhomogeneous quadratic equations corresponding to the equation $d^2=0$ on the derivation d on $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$.

The complex of morphisms between noncounital left curved A_∞-contramodules P and Q over a noncounital curved A_∞-coalgebra C is, by the definition, the complex of morphisms between left DG-modules $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ and $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ Q)$ over the DG-algebra $\bigoplus_i C[-1]^{\otimes i}$. A morphism of noncounital left curved A_∞-contramodules $f\colon P \longrightarrow Q$ of degree j is the same that a linear map $f_\sqcap\colon \prod_{i=0}^\infty \mathrm{Hom}_k(C^{\otimes i}, P)[i] \longrightarrow Q$ of degree j. More precisely, define the map $f_\sqcap$ by the rule that the image of the element $f(g)$ under the projection to Q equals $f_\sqcap((g_i)_{i=0}^\infty)$, where a map $g\colon \bigoplus_i C[-1]^{\otimes i} \longrightarrow P$ and a sequence of maps $g_i\colon C^{\otimes i} \longrightarrow P$ are related by the above formula. Any linear map $f_\sqcap$ corresponds to a (not necessarily closed) morphism of noncounital curved A_∞-contramodules f.

7.5. **Strictly counital curved A_∞-coalgebras.** Let C be a noncounital curved A_∞-coalgebra over a field k. A homogeneous linear function $\varepsilon\colon C \longrightarrow k$ of degree 0 is called a *strict counit* if one has $\varepsilon(\mu_{2,1}(c))\mu_{2,2}(c) = c = \varepsilon(\mu_{2,2}(c))\mu_{2,1}(c)$ and $\varepsilon(\mu_{i,s}(c))\mu_{i,1}(c)\otimes\dots\otimes\mu_{i,s-1}(c)\otimes\mu_{i,s+1}(c)\otimes\dots\otimes\mu_{i,i}(c) = 0$ for all $c\in C$, $1\leqslant s\leqslant i$, and $i\neq 2$. A strict counit is unique if it exists. A *strictly counital curved A_∞-coalgebra* is a noncounital curved A_∞-coalgebra that admits a strict counit. A morphism of strictly counital curved A_∞-coalgebras $f\colon C \longrightarrow D$ is a morphism of noncounital curved A_∞-coalgebras such that $\varepsilon_D\circ f_1 = \varepsilon_C$ and $\varepsilon(f_{i,s}(c))f_{i,1}(c)\otimes\dots\otimes f_{i,s-1}(c)\otimes f_{i,s+1}(c)\otimes\dots\otimes f_{i,i}(c) = 0$ for all $c\in C$, $1\leqslant s\leqslant i$, and $i>1$. Notice that for a strictly counital curved A_∞-coalgebra C one has $\varepsilon_C = 0$ if and only if $C=0$. We will assume our strictly counital curved A_∞-coalgebras to have nonzero counits.

A *strictly counital left curved A_∞-comodule* M over a strictly counital curved A_∞-coalgebra C is a noncounital left curved A_∞-comodule such that $\varepsilon(\nu_{1,-1}(x))\nu_{1,0}(x) = x$ and $\varepsilon(\nu_{i,s}(x))\nu_{i,-i}(x)\otimes\dots\otimes\nu_{i,s-1}(x)\otimes\nu_{i,s+1}(x)\otimes\dots\otimes\nu_{i,-1}(x)\otimes\nu_{i,0}(x) = 0$ for all $x\in M$, $-i\leqslant s\leqslant -1$, and $i>1$. Analogously, a *strictly counital* *right curved A_∞-comodule* N over C is a noncounital right curved A_∞-comodule such that $\varepsilon(\nu_{1,1}(y))\nu_{1,0}(y) = y$ and $\varepsilon(\nu_{i,s}(y))\nu_{i,0}(y) \otimes \nu_{i,1}(y) \otimes \cdots \otimes \nu_{i,s-1}(y) \otimes \nu_{i,s+1}(y) \otimes \cdots \otimes \nu_{i,i}(y) = 0$ for all $y \in N$, $1 \leqslant s \leqslant i$, and $i > 1$. Finally, a *strictly counital left curved A_∞-contramodule* P over C is a noncounital left curved A_∞-contramodule satisfying the following condition. For any sequence of linear maps $g_i\colon C^{\otimes i} \longrightarrow P$, $i = 0, 1, \ldots$ of degree $n + i - 1$ for which there exists a double sequence of linear maps $g'_{i,s}\colon C^{\otimes i-1} \longrightarrow P$, $1 \leqslant s \leqslant i$ such that $g_i(c_1 \otimes \cdots \otimes c_i) = \sum_{i=1}^{s} \varepsilon(c_s)g'_i(c_1 \otimes \cdots \otimes c_{s-1} \otimes c_{s+1} \otimes \cdots \otimes c_i)$ for all i and $c_t \in C$ the equation $\pi((g_i)_{i=0}^{\infty}) = g'_{1,1}(1)$ must hold in P^n.

The complex of morphisms between strictly counital left curved A_∞-comodules L and M over a strictly counital left curved A_∞-coalgebra C is the subcomplex of the complex of morphisms between L and M as noncounital A_∞-comodules consisting of all morphisms $f\colon L \longrightarrow M$ such that $\varepsilon(f_{i,s}(x))f_{i,-i}(x) \otimes \cdots \otimes f_{i,s-1}(x) \otimes f_{i,s+1}(x) \otimes \cdots \otimes f_{i,-1}(x) \otimes f_{i,0}(x) = 0$ for all $x \in L$, $-i \leqslant s \leqslant 1$, and $i > 0$. Analogously, the complex of morphisms between strictly counital right curved A_∞-comodules R and N over C is the subcomplex of the complex of morphisms between R and N as noncounital A_∞-comodules consisting of all morphisms $f\colon R \longrightarrow N$ such that $\varepsilon(f_{i,s}(y))f_{i,0}(y) \otimes f_{i,1}(y) \otimes \cdots \otimes f_{i,s-1}(y) \otimes f_{i,s+1}(y) \otimes \cdots \otimes f_{i,i}(y) = 0$ for all $y \in R$, $1 \leqslant s \leqslant i$, and $i > 0$. Finally, the complex of morphisms between strictly counital left curved A_∞-contramodules P and Q over C is the subcomplex of the complex of morphisms between P and Q as noncounital A_∞-contramodules consisting of all morphisms $f\colon P \longrightarrow Q$ satisfying the following condition. For any sequence of linear maps $g_i\colon C^{\otimes i} \longrightarrow P$, $i = 0, 1, \ldots$ of degree $n + i - j$ for which there exists a double sequence of linear maps $g'_{i,s}\colon C^{\otimes i-1} \longrightarrow P$, $1 \leqslant s \leqslant i$ such that $g_i(c_1 \otimes \cdots \otimes c_i) = \sum_{i=1}^{s} \varepsilon(c_s)g'_i(c_1 \otimes \cdots \otimes c_{s-1} \otimes c_{s+1} \otimes \cdots \otimes c_i)$ for all i and $c_t \in C$ the equation $f_{\sqcap}((g_i)_{i=0}^{\infty}) = 0$ must hold in Q^n.

Let C be a noncounital curved A_∞-coalgebra and $\varepsilon_C\colon C \longrightarrow k$ be a homogeneous linear function of degree 0. Set $C_+ = \ker\varepsilon$. Then the graded tensor algebra $\bigoplus_i C_+[-1]^{\otimes i}$ is a subalgebra of the tensor algebra $\bigoplus_i C[-1]^{\otimes i}$. Denote by K_C the cokernel of the embedding $\bigoplus_i C_+[-1]^{\otimes i} \longrightarrow \bigoplus_i C[-1]^{\otimes i}$ and by $\kappa_C \in K_C$ the element of $C/C_+[-1] \subset K_C$ for which $\varepsilon(\kappa_C) = 1$. Let $\lambda_C \in \bigoplus_i C[-1]^{\otimes i}$ be any element of degree 1 whose image in K_C is equal to κ_C. Then the linear function $\varepsilon_C\colon C \longrightarrow k$ is a strict counit if and only if the odd derivation $d'(a) = d(a) + [\lambda_C, a]$ of degree 1 on the tensor algebra $\bigoplus_i C[-1]^{\otimes i}$ preserves the subalgebra $\bigoplus_i C_+[-1]^{\otimes i}$ and the element $h = d(\lambda_C) + \lambda_C^2$ belongs to $\bigoplus_i C_+[-1]^{\otimes i}$. This condition does not depend on the choice of λ_C. For strictly counital curved A_∞-coalgebras C and D, a morphism of noncounital curved A_∞-coalgebras $f\colon C \longrightarrow D$ is a morphism of strictly counital curved A_∞-coalgebras if and only if $f(\bigoplus_i C_+[-1]^{\otimes i}) \subset \bigoplus_i D_+[-1]^{\otimes i}$ and $f(\kappa_C) = \kappa_D$.

Let C be a strictly counital curved A_∞-coalgebra and M be a noncounital left curved A_∞-comodule over C. Then M is a strictly counital curved A_∞-comodule if and only if the odd derivation $d'(z) = d(z) + \lambda_C z$ of degree 1 on the free module $\bigoplus_i C[-1]^{\otimes i} \otimes_k M$ compatible with the derivation d' of the algebra $\bigoplus_i C[-1]^{\otimes i}$ preserves the subspace $\bigoplus_i C_+[-1]^{\otimes i} \otimes_k M \subset \bigoplus_i C[-1]^{\otimes i} \otimes_k M$. Analogously, a noncounital right curved A_∞-comodule N over C is a strictly counital curved A_∞-comodule if and only if the odd derivation $d'(z) = d(z) - (-1)^{|z|} z\lambda_C$ of degree 1 on the free module $N \otimes_k \bigoplus_i C[-1]^{\otimes i}$ compatible with the derivation d' of the algebra $\bigoplus_i C[-1]^{\otimes i}$ preserves the subspace $N \otimes_k \bigoplus_i C_+[-1]^{\otimes i} \subset N \otimes_k \bigoplus_i C[-1]^{\otimes i}$. Finally, a noncounital left curved A_∞-contramodule P over C is a strictly counital curved A_∞-contramodule if and only if the odd derivation $d'(q) = d(q) + \lambda_C q$ of degree 1 on the cofree module $\operatorname{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ compatible with the derivation d' of the algebra $\bigoplus_i C[-1]^{\otimes i}$ preserves the subspace $\operatorname{Hom}_k(K_C, P) \subset \operatorname{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$.

For strictly counital left curved A_∞-comodules L and M over a strictly counital curved A_∞-coalgebra C, a (not necessarily closed) morphism of noncounital A_∞-comodules $f\colon L \longrightarrow M$ is a morphism of strictly counital A_∞-comodules if and only if one has $f(\bigoplus_i C_+[-1]^{\otimes i} \otimes_k L) \subset \bigoplus_i C_+[-1]^{\otimes i} \otimes_k M$. Analogously, for strictly counital right curved A_∞-comodules R and N over C, a (not necessarily closed) morphism of noncounital A_∞-comodules $f\colon R \longrightarrow N$ is a morphism of strictly counital A_∞-comodules if and only if one has $f(R \otimes_k \bigoplus_i C_+[-1]^{\otimes i}) \subset N \otimes_k \bigoplus_i C_+[-1]^{\otimes i}$. Finally, for strictly counital left curved A_∞-contramodules P and Q over C, a (not necessarily closed) morphism of noncounital A_∞-contramodules $f\colon P \longrightarrow Q$ is a morphism of strictly counital A_∞-contramodules if and only if one has $f(\operatorname{Hom}_k(K_C, P)) \subset \operatorname{Hom}_k(K_C, Q)$.

Let C be a strictly counital curved A_∞-coalgebra. Choose a homogeneous k-linear section $w\colon k \longrightarrow C$ of the strict counit map $\varepsilon\colon C \longrightarrow k$. Define the element $\lambda_C \in \bigoplus_i C[-1]^{\otimes i}$ as $\lambda_C = w(1) \in C[-1] \subset \bigoplus_i C[-1]^{\otimes i}$. Then λ_C is an element of degree 1 whose image in K_C is equal to κ_C. Let $d\colon \bigoplus_i C_+[-1]^{\otimes i} \longrightarrow \bigoplus_i C_+[-1]^{\otimes i}$ be the restriction of the odd derivation d' of $\bigoplus_i C[-1]^{\otimes i}$ defined by the above formula, and let $h \in \bigoplus C_+[-1]^{\otimes i}$ be the element defined above. Then $\operatorname{Cob}_w(C) = (\bigoplus_i C_+[-1]^{\otimes i}, d, h)$ is a CDG-algebra. The CDG-algebra $\operatorname{Cob}_w(C)$ is called the *cobar-construction* of a strictly counital curved A_∞-coalgebra C.

Let $f\colon C \longrightarrow D$ be a morphism of strictly counital curved A_∞-coalgebras. Let $w\colon k \longrightarrow C$ and $w\colon k \longrightarrow D$ be homogeneous k-linear sections, and let λ_C and λ_D be the corresponding elements of degree 1 in the graded tensor algebras. The morphism of tensor algebras $f\colon \bigoplus_i C[-1]^{\otimes i} \longrightarrow \bigoplus_i D[-1]^{\otimes i}$ induces a morphism of graded tensor algebras $\bigoplus_i C_+[-1]^{\otimes i} \longrightarrow \bigoplus_i D_+[-1]^{\otimes i}$, which we will denote also by f. The element $\rho_f = f(\lambda_C) - \lambda_D \in \bigoplus_i D[-1]^{\otimes i}$ has a zero image in K_D, so it belongs to $\bigoplus_i D_+[-1]^{\otimes i}$. Then the pair (f, ρ_f) is a morphism of CDG-algebras $\operatorname{Cob}_w(C) \longrightarrow \operatorname{Cob}_w(D)$. Thus the cobar-construction $C \longmapsto \operatorname{Cob}_w(C)$ is a functor from the category of strictly counital curved A_∞-coalgebras with nonzero counits to the category of CDG-algebras whose underlying graded algebras are graded tensor algebras. One can easily see that this functor is an equivalence of categories. Alternatively, one can use any element λ_C of degree 1 whose image in K_C is equal to κ_C in the construction of this equivalence of categories.

Let C be a strictly counital curved A_∞-coalgebra, $w\colon k \longrightarrow C$ be a homogeneous k-linear section, and $\lambda_C \in \bigoplus_i C[-1]^{\otimes i}$ be the corresponding element of degree 1. Let M be a strictly counital curved A_∞-comodule over C. Set $d\colon \bigoplus_i C_+[-1]^{\otimes i} \otimes_k M \longrightarrow \bigoplus_i C_+[-1]^{\otimes i} \otimes_k M$ to be the restriction of the differential d' on $\bigoplus_i C[-1]^{\otimes i}$ defined by the above formula. Then $\mathrm{Cob}_w(C,M) = (\bigoplus_i C_+[-1]^{\otimes i} \otimes_k M,\ d)$ is a left CDG-module over the CDG-algebra $\mathrm{Cob}_w(C)$. Analogously, for a strictly counital curved A_∞-comodule N over C set $d\colon N \otimes_k \bigoplus_i C_+[-1]^{\otimes i} \longrightarrow N \otimes_k \bigoplus_i C_+[-1]^{\otimes i}$ to be the restriction of the differential d' on $N \otimes_k \bigoplus_i C[-1]^{\otimes i}$ defined above. Then $\mathrm{Cob}_w(N,C) = (N \otimes_k \bigoplus_i C_+[-1]^{\otimes i},\ d)$ is a right CDG-module over the CDG-algebra $\mathrm{Cob}_w(C)$. Finally, let P be a strictly counital curved A_∞-contramodule over C. Set $d\colon \mathrm{Hom}_k(\bigoplus_i C_+[-1]^{\otimes i},\ P) \longrightarrow \mathrm{Hom}_k(\bigoplus_i C_+[-1]^{\otimes i},\ P)$ to be the map induced by the differential d' on $\mathrm{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ defined above. Then $\mathrm{Bar}^w(C,P) = (\mathrm{Hom}_k(\bigoplus_i C_+[-1]^{\otimes i},\ P),\ d)$ is a left CDG-module over the CDG-algebra $\mathrm{Cob}_w(C)$.

To a (not necessarily closed) morphism of strictly counital left curved A_∞-comodules $f\colon L \longrightarrow M$ over C one can assign the induced map $\bigoplus_i C_+[-1]^{\otimes i} \otimes_k L \longrightarrow \bigoplus_i C_+[-1]^{\otimes i} \otimes_k M$. So we obtain the DG-functor $M \longmapsto \mathrm{Cob}_w(C,M)$, which is an equivalence between the DG-category of strictly counital left curved A_∞-comodules over C and the DG-category of left CDG-modules over $\mathrm{Cob}_w(C)$ that are free as graded modules. Analogously, to a (not necessarily closed) morphism of strictly counital right curved A_∞-comodules $f\colon R \longrightarrow N$ over C one can assign the induced map $R \otimes_k \bigoplus_i C_+[-1]^{\otimes i} \longrightarrow N \otimes_k \bigoplus_i C_+[-1]^{\otimes i}$. So we obtain the DG-functor $N \longmapsto \mathrm{Cob}_w(N,C)$, which is an equivalence between the DG-categories of strictly counital right curved A_∞-comodules over C and right CDG-modules over $\mathrm{Cob}_w(C)$ that are free as graded modules. Finally, to a (not necessarily closed) morphism of strictly counital left curved A_∞-contramodules $f\colon P \longrightarrow Q$ over C one can assign the induced map $\mathrm{Hom}_k(\bigoplus_i C_+^{\otimes i},\ P) \longrightarrow \mathrm{Hom}_k(\bigoplus_i C_+^{\otimes i},\ Q)$. So we obtain the DG-functor $P \longmapsto \mathrm{Bar}^w(C,P)$, which is an equivalence between the DG-category of strictly counital left curved A_∞-contramodules over C and the DG-category of left CDG-modules over $\mathrm{Cob}_w(C)$ that are cofree as graded modules.

Now let C be a CDG-coalgebra with a nonzero counit, M be a left CDG-comodule over C, N be a right CDG-comodule over C, and P be a left CDG-contramodule over C. Let $w\colon k \longrightarrow C$ be a homogeneous k-linear section. Define a strictly counital curved A_∞-coalgebra structure on C by the rules $\mu_0(c) = h(c)$, $\mu_1(c) = d(c)$, $\mu_2(c) = c_{(1)} \otimes c_{(2)}$, and $\mu_i(c) = 0$ for $i > 2$. Define a structure of strictly counital left curved A_∞-comodule over C on M by the rules $\nu_0(x) = d(x)$, $\nu_1(x) = x_{(-1)} \otimes x_{(0)}$, and $\nu_i(x) = 0$ for $i > 1$, where $x \in M$. Define a structure of strictly counital right curved A_∞-comodule over C on N by the rules $\nu_0(y) = d(y)$, $\nu_1(y) = y_{(0)} \otimes y_{(1)}$, and $\nu_i(x) = 0$ for $i > 1$, where $y \in N$. Finally, define a structure of strictly counital left curved A_∞-contramodule over C on P by the rule $\pi((g_i)_{i=0}^\infty) = d(g_0) + \pi_P(g_1)$, where $d\colon P \longrightarrow P$ is the differential on P and $\pi_P\colon \operatorname{Hom}_k(C,P) \longrightarrow P$ is the contraaction map. Then the CDG-algebra structure $\operatorname{Cob}_w(C)$ on the graded tensor algebra $\bigoplus_i C[-1]^{\otimes i}$ that was defined in 6.1 coincides with the CDG-algebra structure $\operatorname{Cob}_w(C)$ constructed above, so our notation is consistent. The left CDG-module structure $\operatorname{Cob}_w(C) \otimes^{\tau_{C,w}} M$ on the free graded module $\bigoplus_i C[-1]^{\otimes i} \otimes_k M$ that was defined in 6.2 coincides with the left CDG-module structure $\operatorname{Cob}_w(C,M)$. The right CDG-module structure $N \otimes^{\tau_{C,w}} \operatorname{Cob}_w(C)$ on the free graded module $N \otimes_k \bigoplus_i C[-1]^{\otimes i}$ coincides with the right CDG-module structure $\operatorname{Cob}_w(N,C)$. The left CDG-module structure $\operatorname{Hom}^{\tau_{C,w}}(\operatorname{Cob}_w(C),P)$ on the cofree graded module $\operatorname{Hom}_k(\bigoplus_i C[-1]^{\otimes i},\ P)$ coincides with the left CDG-module structure $\operatorname{Bar}^w(C,P)$.

A morphism of strictly counital curved A_∞-coalgebras $f\colon C \longrightarrow D$ is called *strict* if $f_i = 0$ for all $i \neq 1$. A *coaugmented* strictly counital curved A_∞-coalgebra C is a strictly counital curved A_∞-coalgebra endowed with a morphism of strictly counital curved A_∞-coalgebras $k \longrightarrow C$, where the strictly counital curved A_∞-coalgebra structure on k comes from its structure of CDG-coalgebra with zero differential and curvature linear function. A coaugmented strictly counital curved A_∞-coalgebra is *strictly coaugmented* if the coaugmentation morphism is strict. A morphism of coaugmented or strictly coaugmented strictly counital curved A_∞-coalgebras is a morphism of strictly counital curved A_∞-coalgebras forming a commutative diagram with the coaugmentation morphisms. The categories of coaugmented strictly counital curved A_∞-coalgebras, strictly coaugmented strictly counital curved A_∞-coalgebras, and noncounital curved A_∞-coalgebras are equivalent. The DG-category of strictly counital curved A_∞-comodules or A_∞-contramodules over a coaugmented strictly counital curved A_∞-coalgebra C is equivalent to the DG-category of noncounital curved A_∞-comodules or A_∞-contramodules over the corresponding noncounital curved A_∞-coalgebra C. If C is a strictly coaugmented strictly counital curved A_∞-coalgebra and $w\colon k \longrightarrow C$ is the coaugmentation map, then the CDG-algebra $\operatorname{Cob}_w(C)$ is in fact a DG-algebra.

7.6. **Coderived category of curved A_∞-comodules and contraderived category of curved A_∞-contramodules.** Let C be a strictly counital curved A_∞-coalgebra over a field k. Let $w\colon k \longrightarrow C$ be a homogeneous k-linear section and $B = \operatorname{Cob}_w(C)$ be the corresponding CDG-algebra structure on $\bigoplus_i C_+[-1]^{\otimes i}$.

The *coderived category* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$ *of strictly counital left curved A_∞-comodules* over C is defined as the homotopy category of the DG-category of strictly counital left curved A_∞-comodules over C. The coderived category $\mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C)$ of strictly counital right curved A_∞-comodules over C is defined in the analogous way. The *contraderived category* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$ *of strictly counital right curved* A_∞*-contramodules* over C is defined as the homotopy category of the DG-category of strictly counital left curved A_∞-contramodules over C.

Theorem. *The following five triangulated categories are naturally equivalent:*
(a) *the coderived category* $\mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})$*;*
(b) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra})$*;*
(c) *the coderived category* $\mathsf{D}^{\mathsf{co}}(B\text{–}\mathsf{mod})$*;*
(d) *the contraderived category* $\mathsf{D}^{\mathsf{ctr}}(B\text{–}\mathsf{mod})$*;*
(e) *the absolute derived category* $\mathsf{D}^{\mathsf{abs}}(B\text{–}\mathsf{mod})$.

Proof. The isomorphism of triangulated categories (c–e) is provided by Theorem 3.7(a), and the equivalence of triangulated categories (a), (b), and (e) is the assertion of Theorem 3.7(b) with projective and injective graded modules replaced by free and cofree ones. It suffices to find for any left CDG-module M over B a closed injection from M to a CDG-module J such that both J and J/M are cofree as graded B-modules, and a closed surjection onto M from a CDG-module F such that both CDG-modules M and $\ker(F \to M)$ are free as graded B-modules. This can be easily accomplished with either of the constructions of Theorems 3.5–3.6 or Theorem 4.4. One only has to notice that for any graded module M over a graded tensor algebra B the kernel of the map $B \otimes_k M \longrightarrow M$ is a free graded B-module and the cokernel of the map $M \longrightarrow \mathrm{Hom}_k(B, M)$ is a cofree graded B-module. □

Let C be a CDG-coalgebra over k; it can be considered as a strictly counital curved A_∞-coalgebra, and CDG-comodules and CDG-contramodules over it can be considered as strictly counital curved A_∞-comodules and A_∞-contramodules as explained in 7.5. It follows from Theorem 6.7 that the coderived category of left CDG-comodules over C is equivalent to the coderived category of strictly counital left curved A_∞-comodules and the contraderived category of left CDG-contramodules over C is equivalent to the contraderived category of strictly counital left curved A_∞-contramodules, so our notation is consistent. Thus the above Theorem provides the *comodule-contramodule correspondence for strictly counital curved* A_∞*-coalgebras.* By Theorem 6.7(c), the comodule-contramodule correspondence functors in the CDG-coalgebra case agree with the comodule-contramodule correspondence functors we have construced in the strictly counital curved A_∞-coalgebra case.

The functor $\mathrm{Cotor}^C\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \times \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod}) \longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$ is constructed by restricting the functor of tensor product $\otimes_B\colon \mathsf{Hot}(\mathsf{mod}\text{–}B) \times \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(k\text{–}\mathsf{vect})$ to the Carthesian product of the homotopy categories of CDG-modules that are free as graded modules. The functor $\mathrm{Coext}_C\colon \mathsf{D}^{\mathsf{co}}(C\text{–}\mathsf{comod})^{\mathsf{op}} \times \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$ is constructed by restricting the functor of homomorphisms $\mathrm{Hom}_B\colon \mathsf{Hot}(B\text{–}\mathsf{mod})^{\mathsf{op}} \times \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(k\text{–}\mathsf{vect})$ to the Carthesian product of the homotopy category of CDG-modules that are free as graded modules and the homotopy category of CDG-modules that are cofree as graded modules. The functor $\mathrm{Ctrtor}^C\colon \mathsf{D}^{\mathsf{co}}(\mathsf{comod}\text{–}C) \times \mathsf{D}^{\mathsf{ctr}}(C\text{–}\mathsf{contra}) \longrightarrow k\text{–}\mathsf{vect}^{\mathsf{gr}}$ is constructed by restricting the functor of tensor product $\otimes_B\colon \mathsf{Hot}(\mathsf{mod}\text{–}B) \times \mathsf{Hot}(B\text{–}\mathsf{mod}) \longrightarrow \mathsf{Hot}(k\text{–}\mathsf{vect})$ to the Carthesian product of the homotopy category of CDG-modules that are free as graded modules and the homotopy category of CDG-modules that are cofree as graded modules. These definitions of Cotor^C, Coext_C, and Ctrtor^C agree with the definitions of the functors Cotor^C, Coext_C, and Ctrtor^C for CDG-coalgebras C by Theorem 6.8.2.

Remark. For any graded vector space C, consider the topological graded tensor algebra $\prod_i C[-1]^{\otimes i} = \varprojlim_i \bigoplus_{s=0}^i C[-1]^{\otimes s}$ (see Remark 4.4 for the relevant general definitions). One can define a noncounital ("uncurved") A_∞-coalgebra C as a structure of augmented DG-algebra with a continuous differential on $\prod_i C[-1]^{\otimes i}$. Such a structure is given by a sequence of linear maps $\mu_i\colon C \longrightarrow C^{\otimes i}$, $i = 1, 2, \ldots$ without any convergence condition imposed on them (but satisfying a sequence of quadratic equations corresponding to the equation $d^2 = 0$). A morphism of noncounital A_∞-coalgebras is a continuous morphism of the corresponding topological DG-algebras (which always preserves the augmentations). The notion of a noncounital A_∞-coalgebra is neither more nor less general than that of a noncounital curved A_∞-coalgebra. Still, any noncounital curved A_∞-coalgebra C with $\mu_0 = 0$ can be considered as a noncounital A_∞-coalgebra. A morphism f of noncounital curved A_∞-coalgebras with $\mu_0 = 0$ can be considered as a morphism of noncounital A_∞-coalgebras provided that $f_0 = 0$. A noncounital left A_∞-comodule M over C is a structure of DG-module with a continuous differential over the topological DG-algebra $\prod_i C[-1]^{\otimes i}$ on the free topological graded module $\prod_i C[-1]^{\otimes i} \otimes_k M$. Such a structure is given by a sequence of linear maps $\nu_i\colon M \longrightarrow C^{\otimes i} \otimes_k M$, $i = 0, 1, \ldots$ without any convergence conditions imposed. A noncounital left A_∞-contramodule P over C is a structure of DG-module over $\prod_i C[-1]^{\otimes i}$ on the cofree discrete graded module $\bigoplus_i \mathrm{Hom}_k(C[-1]^{\otimes i}, P)$ of continuous homogeneous linear maps $\prod_i C[-1]^{\otimes i} \longrightarrow P$, where P is discrete. Such a structure is given by a sequence of linear maps $\pi_i\colon \mathrm{Hom}_k(C^{\otimes i}, P) \longrightarrow P$, $i = 0, 1, \ldots$ All the definitions of 7.4–7.5 are applicable in this situation, and all the results of 7.5 hold in this case. For a (strictly counital or noncounital) curved A_∞-coalgebra C with $\mu_0 = 0$ there are forgetful functors from the DG-categories of (strictly counital or noncounital) curved A_∞-comodules and A_∞-contramodules to the corresponding DG-categories of uncurved A_∞-comodules and A_∞-contramodules. Furthermore, let C be an (uncurved) strictly counital A_∞-coalgebra. Define the derived categories of strictly counital A_∞-comodules and A_∞-contramodules as the quotient categories of the homotopy categories corresponding to the DG-categories of strictly counital A_∞-comodules and

A_∞-contramodules by the thick subcategories formed by all the A_∞-comodules and A_∞-contramodules that are acyclic with respect to ν_0 and π_0. Then one can define the functors $\mathrm{Cotor}^{C,I}$, Coext_C^I, $\mathrm{Ctrtor}^{C,I}$, Ext_C^I, and $\mathrm{Ext}^{C,I}$ on the Carthesian products of the derived categories of strictly counital A_∞-comodules and A_∞-contramodules by applying the functors of topological tensor product and continuous homomorphisms over $\prod_i C_+[-1]^{\otimes i}$ to the corresponding (topological or discrete) CDG-modules over $\prod_i C_+[-1]^{\otimes i}$. These functors are even preserved by the restrictions of scalars corresponding to quasi-isomorphisms of strictly counital A_∞-coalgebras (i. e., morphisms f such that f_1 is a quasi-isomorphism of complexes with respect to μ_1). In the case of a strictly counital A_∞-coalgebra coming from a DG-coalgebra C, these functors agree with the derived functors $\mathrm{Cotor}^{C,I}$, Coext_C^I, etc., defined in 2.5.

References

[1] H. Bass. Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.* **95**, p. 466–488, 1960.

[2] A. A. Beilinson, J. N. Bernstein, P. Deligne. Faisceaux pervers. *Astérisque* **100**, p. 5–171, 1982.

[3] A. A. Beilinson, V. A. Ginzburg, V. V. Schechtman. Koszul duality. *Journ. Geom. Phys.* **5**, #3, p. 317–350, 1988.

[4] A. Beilinson, V. Ginzburg, W. Soergel. Koszul duality patterns in representation theory. *Journ. Amer. Math. Soc.* **9**, #2, p. 473–527, 1996.

[5] J. N. Bernstein, I. M. Gelfand, S. I. Gelfand. Algebraic vector bundles on P^n and problems of linear algebra. *Funct. Anal. Appl.* **12**, #3, p. 214–216, 1978.

[6] J. Bernstein, V. Lunts. Equivariant sheaves and functors. *Lecture Notes Math.* **1578**, Springer-Verlag, Berlin, 1994.

[7] A. I. Bondal. Representations of associative algebras and coherent sheaves. *Math. USSR Izvestiya* **34**, #1, p. 23–42, 1990.

[8] A. Bondal, M. Kapranov. Enhanced triangulated categories. *Math. USSR Sbornik* **70**, #1, p. 93–107, 1991.

[9] S. U. Chase. Direct products of modules. *Trans. Amer. Math. Soc.* **97**, p. 457–473, 1960.

[10] S. Eilenberg, J. C. Moore. Limits and spectral sequences. *Topology* **1**, p. 1–23, 1962.

[11] S. Eilenberg, J. C. Moore. Foundations of relative homological algebra. *Memoirs Amer. Math. Soc.* **55**, 1965.

[12] G. Fløystad. Koszul duality and equivalences of categories. *Trans. Amer. Math. Soc.* **358**, #6, p. 2373–2398, 2006. `arXiv:math.RA/0012264`

[13] V. Hinich. DG coalgebras as formal stacks. *Journ. Pure Appl. Algebra* **162**, #2–3, p. 209–250, 2001. `arXiv:math.AG/9812034`

[14] J. Huebschmann. Homological perturbations, equivariant cohomology, and Koszul duality. Electronic preprint `arXiv:math/0401160 [math.AT]`.

[15] D. Husemoller, J. C. Moore, J. Stasheff. Differential homological algebra and homogeneous spaces. *Journ. Pure Appl. Algebra* **5**, p. 113–185, 1974.

[16] B. Keller. Deriving DG-categories. *Ann. Sci. École Norm. Sup. (4)* **27**, #1, p. 63–102, 1994.

[17] B. Keller. Koszul duality and coderived categories (after K. Lefèvre). October 2003. Available from `http://www.math.jussieu.fr/~keller/publ/index.html`.

[18] H. Krause. Localization for triangulated categories. Electronic preprint `arXiv:0806.1324 [math.CT]`.
[19] K. Lefèvre-Hasegawa. Sur les A_∞-catégories. Thèse de doctorat, Université Denis Diderot – Paris 7, November 2003. `arXiv:math.CT/0310337`. Corrections, by B. Keller. Available from `http://people.math.jussieu.fr/~keller/lefevre/publ.html`.
[20] A. Neeman. The derived category of an exact category. *Journ. Algebra* **135**, #2, p. 388–394, 1990.
[21] M. Boekstedt, A. Neeman. Homotopy limits in triangulated categories. *Compositio Math.* **86**, #2, p. 209–234, 1993.
[22] P. Nicolas. The bar derived category of a curved dg algebra. Electronic preprint `arXiv:math/0702449 [math.RT]`.
[23] A. Polishchuk, L. Positselski. Quadratic Algebras. University Lecture Series, 37. American Mathematical Society, Providence, RI, 2005. xii+159 pp.
[24] L. Positselski. Koszul property and Bogomolov's conjecture. *Intern. Math. Research Notices* **2005**, #31, p. 1901–1936.
[25] L. Positselski. Homological algebra of semimodules and semicontramodules. With appendices coathored by D. Rumynin and S. Arkhipov. Electronic preprint `arXiv:0708.3398 [math.CT]`.
[26] N. Spaltenstein. Resolutions of unbounded complexes. *Compositio Math.* **65**, #2, p.121–154, 1988.
[27] J.-L. Verdier. Catégories dérivées, état 0. SGA 4 1/2. *Lect. Notes Math.* **569**, p. 262–311, 1977.
[28] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque* **239**, 1996.

Sector of Algebra and Number Theory, Institute for Information Transmission Problems, 19 Bolshoj Karetnyj per., Moscow, Russia
E-mail address: `posic@mccme.ru`