Vinberg Algebras and Combinatorics

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Abstract

Vinberg algebras are usually called pre-Lie algebras and were introduced long ago by Gerstenhaber. We propose to follow a different route by motivating these algebras by problems coming from differential geometry, and first studied in depth by Vinberg. We shall recall how the Lie bracket of vector fields can be obtained by skewsymmetrizing a more fundamental product. We shall then develop a combinatorial method for the higher order differential operators, quite similar to the procedure used in studying Runge–Kutta methods. We shall then move to nilpotent (or pronilpotent) Lie groups. In the last part of these lectures, I shall apply the previous methods in the renormalization theory of quantum fields (à la Connes–Kreimer).

Dedicated to Alain Connes,
on the occasion of his 60th birthday,
to witness our long-lasting friendship

Introduction

Vinberg (or pre-Lie) algebras have become important new tools in combinatorics and differential geometry. They generate a special class of Lie algebras. Our purpose in these notes is to describe them in some detail, and to apply them in the method of renormalization theory introduced by Alain Connes and Dirk Kreimer. These authors have introduced a Hopf algebra, here we consider a simpler algebraic tool, the Vinberg algebras. Such a Vinberg algebra gives rise to a Lie algebra, hence to a Lie group (or rather inverse limit of Lie groups). This provides an alternative route to the results of Connes and Kreimer.

*Notes of a short course in “Operads 2005”, Luminy 4–6 April 2005, written up by Jose Gracia-Bondia.

1 Vinberg (pre-Lie) algebras

1.1 The basic concept

Associative algebras and Lie algebras have been with us for a while. Historically, there have been many attempts to define other types of algebras. But the efforts were not systematic, or the right viewpoint was not reached, and only the theory of Jordan algebras was developed to a reasonable extent. Lie remarked that for commutators $[ab] := ab - ba$ the Jacobi identity,

$$J(a,b,c) := [a[bc]] + [b[ca]] + [c[ab]] = 0,$$

pertaining to the definition of Lie algebras, is implied by associativity. The computation is trivial, but it is useful for us to recall it. For a, b, c elements of any algebra:

$$\begin{aligned} [a[bc]] &= a(bc) - a(cb) - (bc)a + (cb)a, \\ [b[ca]] &= b(ca) - b(ac) - (ca)b + (ac)b, \\ [c[ab]] &= c(ab) - c(ba) - (ab)c + (ba)c. \end{aligned} \tag{1}$$

Define the *associator* $A(a,b,c)$ of three elements a, b, c as

$$A(a,b,c) = a(bc) - (ab)c.$$

An algebra is associative iff $A(a,b,c)$ vanishes identically. Then (1) tells us clearly that

$$J(a,b,c) = \text{total skewsymmetrization of } A(a,b,c).$$

Therefore, for J to vanish, it is not necessary that A vanish in turn; it is enough that

$$A(a,b,c) - A(b,a,c) = a(bc) - (ab)c - b(ac) + (ba)c = 0.$$

We call it *Vinberg identity*, and algebras with this property will be called *Vinberg algebras*. They were introduced with the name *pre-Lie algebras* by Gerstenhaber in 1962, and around the same epoch by Vinberg in relation with problems in differential geometry. A more general definition of a pre-Lie algebra would be that of an algebra with a product such that the corresponding commutators define a Lie algebra. In the previous definition, a Vinberg algebra is one in which $A(a,b,c)$ is symmetric in a, b; hence a more precise terminology would be *left-symmetric Vinberg algebra*. Similarly, if $A(a,b,c)$ is symmetric in b, c, we get a *right-symmetric Vinberg algebra*. For us, a pre-Lie algebra shall be a right-symmetric Vinberg algebra. We go to the examples at once.

1.2 Examples

1.2.1 Consider vector fields written in local coordinates $(x^1, \dots, x^n) \equiv (x^\alpha) \equiv x$, with $x \in \mathbb{R}^n$. To a vector function $X^\alpha(x)$ one can associate the Lie derivative $\mathcal{L}_X$ defined by

$$\mathcal{L}_X f = \sum_{\alpha=1}^{n} X^\alpha \frac{\partial f}{\partial x^\alpha} =: X^\alpha\, \partial_\alpha f;$$

we of course use Einstein's notation in this Einstein year![1] The little miracle is that

$$[\mathcal{L}_X \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

is again a first order differential operator, hence of the form $\mathcal{L}_{[X,Y]}$; and then the Jacobi identity for the Lie bracket $[X, Y]$ comes for free from associativity of the algebra of differential operators.

However, we can do things differently. We define $D_X Y$ by

$$(D_X Y)^\beta := X^\alpha \, \partial_\alpha Y^\beta. \tag{2}$$

This definition is not intrinsic, in the sense of not being consistent under general changes of coordinates. But allow us to go on. Suppose now we define a bracket $[X, Y]$ by

$$[X, Y] := D_X Y - D_Y X$$

and use also notations $X * Y$ and $X \rightarrow Y$ for $D_X Y$. Soon it will be apparent that the grafting notation $X \rightarrow Y$ for the product of X and Y given by $D_X Y$ is very pertinent; both notations $*$ and $\rightarrow$ are to be used in what follows. Now we check that $X \rightarrow Y$ satisfies a Vinberg identity:

$$\begin{aligned} X \rightarrow (Y \rightarrow Z) - (X \rightarrow Y) \rightarrow Z &= D_X D_Y Z - D_{D_X Y} Z \\ &= X^\alpha \, \partial_\alpha (Y^\beta \, \partial_\beta Z^\bullet) - (D_X Y)^\beta \, \partial_\beta Z^\bullet \\ &= X^\alpha \, \partial_\alpha (Y^\beta \, \partial_\beta Z^\bullet) - X^\alpha \, \partial_\alpha Y^\beta \, \partial_\beta Z^\bullet \\ &= X^\alpha Y^\beta \, \partial_\alpha \partial_\beta Z^\bullet. \end{aligned}$$

Because $\partial_\alpha \partial_\beta Z^\bullet$ is symmetric in α, β, we then see that $A(X, Y, Z) = A(Y, X, Z)$ with the operation $\rightarrow$, and this is all we need to establish the Jacobi identity. We shall eventually see (section 2.2) that this calculation can be given an intrinsic meaning, after all.

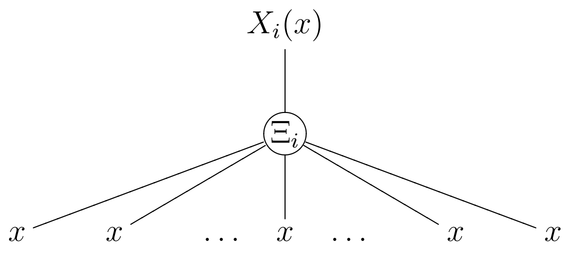
1.2.2 Next consider polynomial vector fields; that is, $X(x)$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that

$$X = X_0 + X_1 + X_2 + \cdots$$

where X_i is a (vector-valued) homogeneous polynomial of degree i in n variables. It is well known that any of those is of the form

$$X_i(x) = \Xi_i(x, \ldots, x),$$

for $\Xi_i(y_1, \ldots, y_i)$ a uniquely defined *symmetric* multilinear function. We represent the last identity by means of a graph:

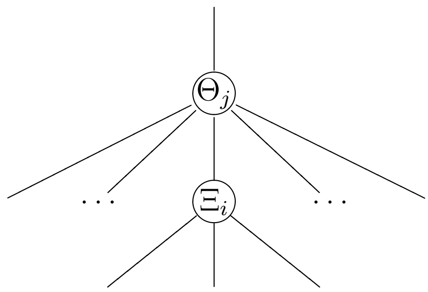


[1] This was written in 2005!

Here we have a rooted tree with the root on top and with unordered leaves.

Now we reconsider $D_X Y$ with X homogeneous of degree i and Y homogeneous of degree j. Let the symmetric function Θ_j with j entries correspond to Y as Ξ_i to X above. Then $Z = D_X Y$ has degree $i + j - 1$. Precisely, the Leibniz rule says that Z is obtained by considering the substitution for $\Xi_i(x, \ldots, x)$ of each variable argument in the symmetric function Θ_j, and summing on all the terms obtained. We define, for r between 1 and j,

$$\Theta_j \,|_r\, \Xi_i(x_1, \ldots, x_{i+j-1}) = \Theta_j(x_1, \ldots, x_{r-1}, \Xi_i(x_r, \ldots, x_{i+r-1}), x_{i+r}, \ldots, x_{i+j-1}),$$



and the sum

$$\Theta_j \circ \Xi_i = \sum_{r=1}^{j} \Theta_j \,|_r\, \Xi_i,$$

which is a symmetric function in $i + j - 1$ entries (also written as $\Xi_i \rightarrow \Theta_j$). Thereby Z is given as a sum of insertions or graftings:

$$Z(x) = \Theta_j \circ \Xi_i(x, \ldots, x).$$

We may look at the Vinberg property in the light of this graphical representation. Consider in turn $\Xi \rightarrow (\Theta \rightarrow \Lambda)$ in relation with $(\Xi \rightarrow \Theta) \rightarrow \Lambda$: when grafting Ξ on $\Theta \rightarrow \Lambda$, we can choose to do it on a Θ-part or on a Λ-part. Now, the insertions on a Θ-part are totally cancelled in $\Xi \rightarrow (\Theta \rightarrow \Lambda) - (\Xi \rightarrow \Theta) \rightarrow \Lambda$, and there remain the insertions on Λ-parts. But the latter are *symmetric* in Ξ, Θ: here $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$ is the rule of symmetry of the insertions! Thus the Vinberg property holds.

1.2.3 We look now at Poisson brackets, in two variables for simplicity of notation:

$$\{f, g\} = \partial_p f \, \partial_q g - \partial_p g \, \partial_q f.$$

Suppose we define a 'star' product by

$$f \star g := \partial_p f \, \partial_q g,$$

so

$$\{f, g\} = [f \star g].$$

Then
$$(f \star g) \star h = \partial_p (f \star g)\, \partial_q h = \partial^2_{pp} f\, \partial_q g\, \partial_q h + \partial_p f\, \partial^2_{pq} g\, \partial_q h$$
has two terms. The Jacobi identity certainly holds; it contains 24 summands. But $\star$ is a counterexample for the Vinberg property, since the six expressions obtained by permutation of f, g, h in $A(f, g, h)$ are distinct in general.

1.2.4 We come to Lie groups now. Let e be the neutral element of one of such G. Consider local coordinates $x \equiv (x^1, \ldots, x^n)$ on G with the property $x^i(e) = 0$. In general in those coordinates, the group product rule will have the form

$$z = F(x, y) = \sum_{p \geq 0,\, q \geq 0} F_{p,q}(x, y) \quad \text{if} \quad z = x \cdot y. \tag{3}$$

We trust that the reader will be able to distinguish when we refer to abstract elements of the group and when to their coordinates, in our notation. Here $F_{p,q}(x^1, \ldots, x^n; y^1, \ldots, y^n)$ is a polynomial in $2n$ variables, homogeneous of degree p in x, q in y; moreover $F_{0,0} = 0$ and we can take $F_{1,0}(x, y) = x$, $F_{0,1}(x, y) = y$. Let $\mathfrak{g}$ denote the tangent (Lie) algebra of G. Consider
$$(x, y) := xyx^{-1}y^{-1};$$
then
$$(x, y) = G(x, y) = \sum_{p \geq 0,\, q \geq 0} G_{p,q}(x, y),$$
where $G_{0,0} = G_{1,0} = G_{0,1} = 0$ and $G_{1,1}(x, y) = F_{1,1}(x, y) - F_{1,1}(y, x)$ is a bilinear function. If we make the identification with tangent vectors at the identity, then $G_{1,1}$ must satisfy the Jacobi identity. But we want a calculational explanation for this fact. The foregoing is obviously related to Poincaré's bilinear groups. Given an (associative, finite-dimensional, unital) algebra A, its units form a group $A^\times$ with $e = 1_A$. For x, y small enough, $1 + x$ and $1 + y$ lie in $A^\times$ with $(1 + x)(1 + y) = 1 + (x + y + xy)$ (hence the name "bilinear"), and the commutator $(1 + x, 1 + y)$ coincides, up to third order terms, with

$$(1 + x)(1 + y) - (1 + y)(1 + x) = xy - yx.$$

Hence, the Lie algebra corresponding to the Lie group $A^\times$ is the vector space A endowed with the bracket $[ab] = ab - ba$.

More generally, write $x * y$ for $F_{1,1}(x, y)$ and using polarization expand the product $x \cdot y$ as follows
$$F(x, y) = x + y + x * y + L(x, x, y) + M(x, y, y) + O_4(x, y),$$
where L and M are trilinear, satisfying

$$L(x, y, z) = L(y, x, z)$$

$$M(x, y, z) = M(x, z, y),$$

and O_4 contains no term of total degree < 4. By associativity of the group law, we obtain

$$F(x, F(y, z)) - F(F(x, y), z) = 0.$$

Keeping only the terms trilinear in x, y, z, we obtain the identity

$$A(x, y, z) = 2M(x, y, z) - 2L(x, y, z)$$

for the associator $A(x, y, z) = x * (y * z) - (x * y) * z$. From the symmetry properties of $L(x, y, z)$ and $M(x, y, z)$, it follows that the skew-symmetrization of $A(x, y, z)$ is 0, hence the bracket $[xy] = x * y - y * x = G_{1,1}(x, y)$ on $\mathbb{R}^d$ satisfies Jacobi identity.

If $F(x, y) - x$ is linear in y, then $M = 0$, hence the operation $x * y$ satisfies the left-symmetric Vinberg identity. The Lie algebra of G comes from a left-symmetric Vinberg algebra. Similarly, if $F(x, y) - y$ is linear in x, then the product $x * y$ defines on $\mathbb{R}^d$ a right-symmetric Vinberg algebra. More on this topic in section 2.3.

1.3 The Gerstenhaber approach and noncommutative polynomials

Gerstenhaber arrived at the concept of *pre-Lie algebra* when working on Hochschild cohomology. Let A be an associative algebra, and consider cochains:

$$c_p : A^{\otimes p} \to A; \qquad d_q : A^{\otimes q} \to A.$$

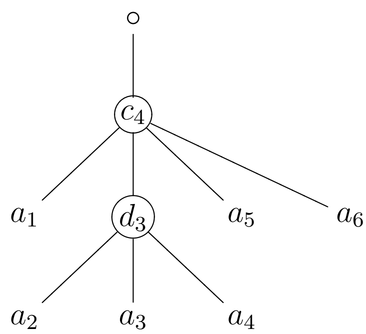
Then, for $i = 1, \ldots, p$, we form the cochain with $p + q - 1$ arguments given by:

$$c_p \mid_i d_q(a_1, \ldots, a_{p+q-1}) = c_p(a_1, \ldots, a_{i-1}, d_q(a_i, \ldots, a_{i+q-1}), a_{i+q}, \ldots, a_{p+q-1});$$

and we set

$$c_p \circ d_q := \sum_{i=1}^{p} c_p \mid_i d_q.$$

Here, for instance, is $c_4 \mid_2 d_3$:



The small miracle is that $[c_p\, d_q] := c_p \circ d_q - d_q \circ c_p$ satisfies the Jacobi identity! (This, by the way, is nowadays known as the Gerstenhaber bracket, and plays an important role for instance in theory of deformations.) We are going to see that the Jacobi identity holds because $c_p \circ d_q$ happens to satisfy the right-symmetric Vinberg property. Indeed, we can represent the cochains as we did with the symmetric functions in example 1.2.2; now d_q is grafted on c_p in all possible ways, and everything works the same, with the only difference that the leaves now have a natural order. But this does not affect the conclusion that $c_p \circ (d_q \circ e_r) - (c_p \circ d_q) \circ e_r$ is symmetric in d_q, e_r. Hence the Jacobi identity holds.

We have moreover gleaned an interpretation of the cochains as noncommutative vector fields.

2 Some good reasons to study Vinberg algebras

2.1 Operads

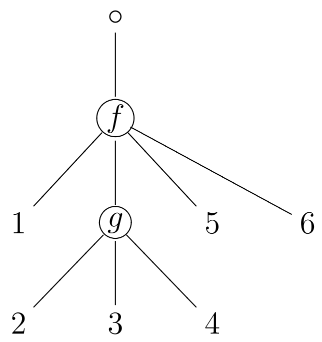
Let V be a vector space over a suitable field k of characteristic zero, and let us denote $E_V(1) := \operatorname{End} V$ and, for $n \geq 2$,

$$E_V(n) := \operatorname{Hom}(V^{\otimes n}, V); \qquad E_V = \bigoplus_{n\geq 1} E_V(n).$$

Consider then $f_n \in E_V(n)$, $g_p \in E_V(p)$ and, somewhat similarly to the previous section, define, for $i = 1, \dots, n$, an element of $E_V(n+p-1)$ by

$$f_n \mid_i g_p = f_n \circ (\mathrm{id}_V^{\otimes i-1} \otimes g_p \otimes \mathrm{id}_V^{\otimes n-i}).$$

This we represent by trees with half-edges (those branches that do not connect blobs) and edges (those that do connect blobs to effect the insertions). For example,



To each rooted planar tree does correspond an operation of this kind. In these operations there are numerous compatibility conditions of the associativity type in that

$$(f_m \mid_i g_n) \mid_j h_p = f_m \mid_i (g_n \mid_{j-i+1} h_p) \quad \text{when} \quad \begin{cases} 1 \leq i \leq m, \\ i \leq j \leq i+n-1; \end{cases}$$

and of the commutativity type:

$$(f_m \mid_i g_n) \mid_j h_p = (f_m \mid_{j-n+1} h_p) \mid_i g_n \quad \text{when} \quad \begin{cases} 1 \leq i \leq m, \\ i+n \leq j \leq n+m-1 \end{cases}$$

which is a kind of locality principle in the insertions. This construction gives rise to an *operad* P, that we regard as a collection of vector spaces $\mathsf{P}(n)$ indexed by the positive integers, together with bilinear maps:

$$\mathsf{P}(n) \otimes \mathsf{P}(p) \to \mathsf{P}(n+p-1) : f \otimes g \mapsto f \mid_i g, \quad \text{for each} \quad i = 1, \dots, n$$

satisfying the above-mentioned properties. In the standard definition, the $\mathsf{P}(n)$ are $k[S_n]$-modules, but we shall not employ that yet. In the example there are moreover maps

$$\xi_n : E_V(n) \otimes V^{\otimes n} \to V,$$

with obvious associativity and unity properties. In general, given the operad P, a P-algebra is a vector space A with maps

$$\mathsf{P}(n) \otimes A^{\otimes n} \to A$$

with the analogous properties. This gives rise to an 'operadic map' $\mathsf{P} \to E_A$.

What is the relation to Vinberg algebras? Take $f \circ g = \sum_{i=1}^n f \mid_i g$ and define θ_g by $\theta_g(f) = f \circ g$, for $f, g \in E$. Then, although $\circ$ is not associative, $\{\theta_g\}_{g \in E}$ is a Lie algebra of operators acting on E, since the bracket

$$[\theta_h \ \theta_g] := \theta_{g \circ h} - \theta_{h \circ g}, \tag{4}$$

by a reasoning like that of the previous section, is seen to possess the Jacobi property. This is the "half-adjoint representation" of Gerstenhaber. Hence $\mathsf{P} = \bigoplus_{n>0} \mathsf{P}(n)$ is a right-symmetric Vinberg algebra.

In general, given a category of algebras, we can associate to it an operad describing the "natural" operations that can be defined on it; if, for instance, we consider the category of associative algebras, all the trees of the same size in the construction above will determine the same operation in the corresponding operad. Thus there are different classes of operads, according to the basic properties of defining operations. Some are represented in the following table.

P	Operations	Relations
As	xy	$(xy)z = x(yz)$
Com	$xy = yx$	$(xy)z = x(yz)$
Lie	$[xy] = -[yx]$	Jacobi
Mag	xy	no relation
pre-Lie	xy	$x(yz) - (xy)z = x(zy) - (xz)y$
Zinb	xy	$(xy)z = x(yz) + x(zy)$
2-as	$x \cdot y$, $x * y$	both associative
Dend	$x \prec y$, $x \succ y$	
	$x * y = x \prec y + y \succ x$	(see below)

Dialgebras, that is, vector spaces with two multiplications, can be considered as well. A particularly interesting case of dialgebras are the dendriform dialgebras of Loday. They can be obtained as follows. Let $(D, *)$ be an associative algebra. Assume that D is a bimodule over itself, $D \equiv {}_D D_D$. We write $\succ$ and $\prec$ for the left and right actions, respectively. Assume moreover that, for all $a, b \in D$,

$$a * b = a \prec b + a \succ b. \tag{5}$$

Then by definition D is a dendriform dialgebra. In detail, the dendriform properties are

$$\begin{aligned} a \prec (b * c) &= a \prec (b \succ c + b \prec c) = (a \prec b) \prec c; \\ (a * b) \succ c &= (a \succ b + a \prec b) \succ c = a \succ (b \succ c); \\ (a \succ b) \prec c &= a \succ (b \prec c). \end{aligned}$$

Conversely, these last relations (on the right hand sides) are enough to establish associativity of $*$ defined by the equality in (5). Without changing the underlying linear structure, D gives rise not only to an associative algebra, but also to a pre-Lie algebra and, in two different ways, to the same Lie algebra. For that, consider the following:

$$x \rightarrow y := x \succ y - y \prec x.$$

This defines a pre-Lie algebra structure. Indeed,

$$x \rightarrow y - y \rightarrow x = x \succ y - y \prec x - y \succ x + x \prec y = x * y - y * x,$$

so the corresponding Lie algebra structures coincide. We have then the following quadrilateral of operads, with the same underlying vector structure:

$$\begin{array}{ccc} \mathsf{Dend} & \xrightarrow{\ \rightarrow\ } & \text{pre-}\mathsf{Lie} \\ {\scriptstyle *}\downarrow & & \downarrow{\scriptstyle [\cdot\,\cdot]} \\ \mathsf{As} & \xrightarrow{\ [\cdot\,\cdot]\ } & \mathsf{Lie} \end{array}$$

To conclude this part, we remark the affinity between the notion of operad and the functor of extension of scalars. Let $\mathcal{A}$ and $\mathcal{B}$ be rings with unit, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital ring homomorphism. Then any (say) right $\mathcal{B}$-module $\mathcal{F}$ becomes a right $\mathcal{A}$-module by defining

$$t \cdot a := t\,\phi(a) \quad \text{for} \quad t \in \mathcal{F},\ a \in \mathcal{A}.$$

If $\mathcal{G}$ is another right $\mathcal{B}$-module and $\psi \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}, \mathcal{G})$, then $\psi(t \cdot a) = \psi(t\phi(a)) = \psi(t)\,\phi(a) = \psi(t) \cdot a$, so ψ can also be regarded as a member of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{G})$. In this way, ϕ defines a functor R_ϕ from the category of right $\mathcal{B}$-modules to the category of right $\mathcal{A}$-modules. This functor is called *restriction of scalars* in regard of the case ϕ is the inclusion map of a subring $\mathcal{A}$ into a larger ring $\mathcal{B}$, for obvious reasons. The adjoint functor from right $\mathcal{A}$-modules to right $\mathcal{B}$-modules is the *extension of scalars*. Similarly, going from an As-algebra (associative algebra) to the corresponding Lie-algebra, is a kind of restriction corresponding to an operad-homomorphism from Lie to As.

2.2 More on Vinberg algebras in differential geometry

Let M^d denote a manifold with $\dim M = d$, and let TM be its tangent bundle. We define a linear connection as an $\mathbb{R}$-bilinear operation ∇ that, given two vector fields $X, Y \in \Gamma(M, TM)$, produces a new vector field $\nabla_X Y$ with the properties

$$\nabla_{fX} Y = f\nabla_X Y \quad (C^\infty\text{-linearity on } X);$$
$$\nabla_X fY = (\mathcal{L}_X f)Y + f\nabla_X Y \quad \text{(Leibniz rule).}$$

Here $\mathcal{L}_X$ denotes the Lie derivative with respect to the vector field X. Now define the *torsion* T and *curvature* R of ∇ respectively by

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y];$$
$$R(X, Y)Z := \big(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\big)Z,$$

with $Z \in \Gamma(M, TM)$, too. One checks without too much difficulty C^∞-bilinearity of T,

$$T(fX, Y) = T(X, fY) = fT(X, Y).$$

This means that the map $T : \Gamma(M, TM)^{\otimes 2} \to \Gamma(M, TM)$ descends to a map $T_xM \otimes T_xM \to T_xM$, for all $x \in M$. Similarly for $R(\cdot, \cdot)Z$. This allows the definition of the torsion and curvature as tensors, familiar from Riemannian geometry. Suppose now $R = 0$, $T = 0$. That is:

$$[X, Y] := \nabla_X Y - \nabla_Y X; \quad \text{and thus}$$
$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{[X,Y]} Z = \nabla_{\nabla_X Y - \nabla_Y X} Z.$$

We can see the Vinberg relation here; in fact this is how he came by it, when dealing with simply transitive affine actions of groups on $\mathbb{R}^d$.

The covariant derivative map $\nabla X : Y \mapsto \nabla_Y X$ is a tensor map from TM to TM. In general we envisage $\mathrm{Hom}(TM^{\otimes p}, TM^{\otimes q}) =: \mathcal{T}_{p,q}$, where the maps considered are fibrewise linear. The sections of these bundles are called tensor fields. We have $\mathcal{T}_{0,1} \simeq TM$; $\mathcal{T}_{1,1} \simeq \mathrm{End}\, TM$, and so on. We can look at ∇ as a map

$$\nabla : \Gamma(M, \mathcal{T}_{0,1}) \ni X \mapsto \nabla X \in \Gamma(M, \mathcal{T}_{1,1}),$$

with the property $\nabla(fX) = df \otimes X + f\nabla X$. This map can be extended to all tensor fields:

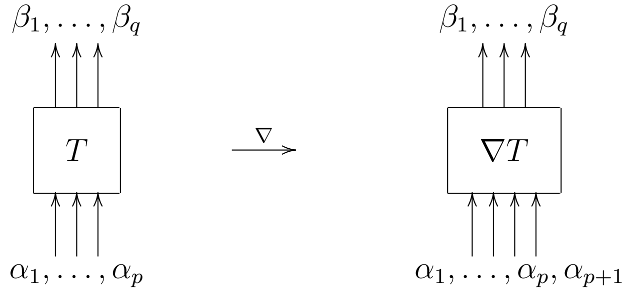
$$\nabla : \Gamma(M, \mathcal{T}_{p,q}) \ni X \mapsto \nabla X \in \Gamma(M, \mathcal{T}_{p+1,q}),$$

by

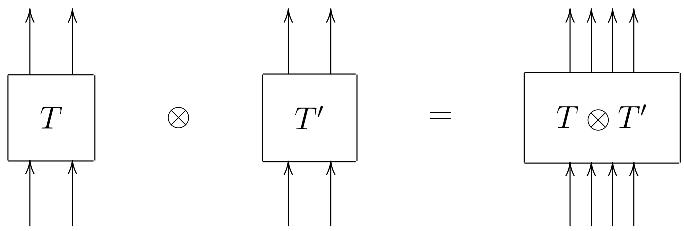
$$\nabla(T \otimes T') = \nabla T \otimes T' + T \otimes \nabla T'.$$

We have then $\nabla f = df$, for $f \in \mathcal{T}_{0,0}$.

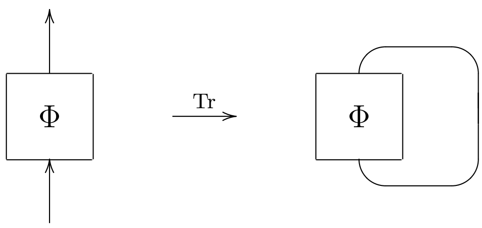
In the index notation, we go from $T^{\beta_1,\ldots,\beta_q}_{\alpha_1,\ldots,\alpha_p}$ to $T^{\beta_1,\ldots,\beta_q}_{\alpha_1,\ldots,\alpha_p;\alpha_{p+1}}$. We can think of T as a 'box' that takes the 'input' $(\alpha_1,\ldots,\alpha_p)$ and transforms it into the 'output' $(\beta_1,\ldots,\beta_q)$. See this in the next figures, as the action of ∇:



and the tensor product as a 'glueing' or juxtaposing operation:



Another operation is contraction: given $\Phi \in \operatorname{End} TM$, we can consider $\operatorname{Tr}\Phi = \sum \Phi^\alpha_\alpha$. This is represented by 'sewing':



The graphical representation makes the point, underlined by Penrose often, that this apparently coordinate-dependent definition, and the tensor index notation in general, are actually intrinsic.

Let now assume that on M^d we have given a connection with vanishing torsion and curvature (so the manifold is "locally flat"). Consider $\nabla^2 f \in \Gamma(M, \operatorname{Sym}^2 T^*M) \subset \mathcal{T}_{2,0}$, and assume $\nabla^2 f = 0$. Then there exist local coordinates $(x^1,\ldots,x^d)$ such that f is affine:

$$f = c_0 + c_1 x^1 + \cdots + c_d x^d$$

with real constants $c_1,\ldots,c_d$. Moreover, $\nabla_X Y = D_X Y$ in those coordinates; and so we come to see that

$$[X,Y] := D_X Y - D_Y X$$

is a special form of an intrinsic object! Under the present hypothesis, we can further consider the maps

$$\nabla^m X : \mathrm{Sym}^m TM \to TM,$$

where $\nabla^m X$ is of the form $t^{\beta}_{\alpha_1,\dots,\alpha_m}$, with symmetry in the indices; symmetry comes from torsion-freeness, of course. If we put

$$\{X \mid Y_1, \dots, Y_m\} := \nabla^m X(Y_1, \dots, Y_m),$$

we find the properties

$$\{X \mid Y_1, \dots, Y_m, Y_{m+1}\} = \nabla_{Y_{m+1}}\{X \mid Y_1, \dots, Y_m\} - \sum_{i=1}^{m}\{X \mid Y_1, \dots, \nabla_{Y_{m+1}} Y_i, \dots, Y_m\}, \quad (6)$$

characteristic of the 'brace' operation in Vinberg algebras.

2.3 Lie groups with affine coordinates

Consider a real pre-Lie algebra $\mathfrak{g}$ of finite dimension, with product $a \triangleright b$, giving rise to a Lie algebra by $[ab] = a \triangleright b - b \triangleright a$. We can regard the Lie algebra $\mathfrak{g}$ as the tangent algebra of a (simply connected, if you wish) Lie group G. We can identify $\mathfrak{g}$ with the Lie subalgebra of right-invariant vector fields on the group: $\mathfrak{g} \ni a \leftrightarrow X_a \in \mathfrak{X}^R(G)$. It is possible to define on G a (necessarily unique) right-invariant connection with vanishing torsion and curvature, whose covariant derivative ∇ is right-invariant with $X_{a \triangleright b}$ corresponding to $\nabla_{X_a} X_b$. We can choose on G linear coordinates around the origin, and if $z = xy$, then the coordinates z^i of z are affine linear in the coordinates of x:

$$z^i = a^i(y) + \sum_j x^j b^i_j(y),$$

with an obvious notation. The group Diff of formal local diffeomorphisms provides an example, as if we represent one (tangent to the identity) such by a series

$$g(x) = x + g_2 x^2 + g_3 x^3 + \cdots,$$

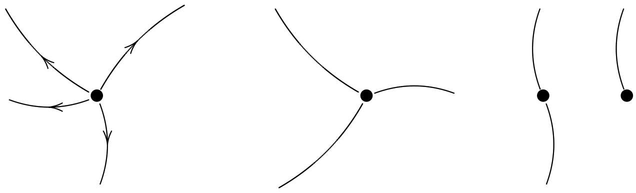
then $f(g(x))$, as given by the Faà di Bruno formula —which preserves its validity for analytic functions on the appropriate domains [1]— has this property of being linear in the coordinates of f. Also, we can regard Diff as the inverse limit of finite dimensional matrix groups.

The corresponding Lie algebra can be identified with power series $u(x) = \sum_{k \geq 2} u_k\, x^k$, with a pre-Lie operation uv' (where v' is the derivation of v, and the well-known bracket $[uv] = uv' - u'v$.

3 The Connes–Kreimer paradigm

3.1 Graph combinatorics in physics

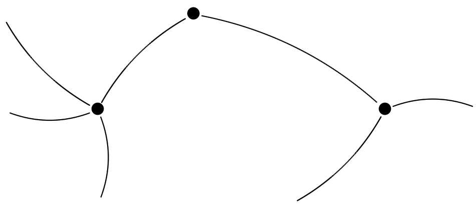
We begin by some Feynman-graphology. Feynman diagrams are constructed out of sets of vertices $v \in V$ and sets of rays R_v originating in each vertex. This gives rise to 'stars' or 'corollas'.



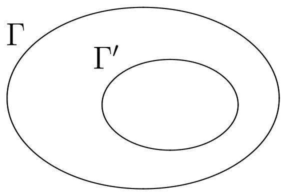
We sew them by choosing sets of disjoint pairs in $R := \bigsqcup_v R_v$ and involutive maps $s : R \to R$ that interchange elements in a chosen pair

$$s(a) = b; \quad s(b) = a.$$

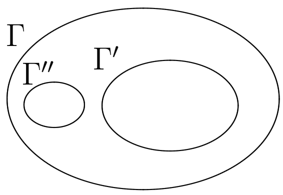
The pair (a, b) becomes an *edge*.



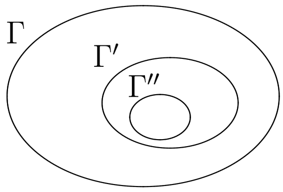
Graphical insertion of middle points to keep track of edges is useful in some circumstances. Rays that do not become edges may be called legs or half-edges. On the graphs we define a grafting operation, that slightly generalizes the grafting of trees. We graft a connected diagram Γ' into another connected diagram Γ by choosing a vertex $v \in \Gamma$ and establishing a bijection φ between R_v and the legs of Γ'. Grafting in the middle of edges is accepted. Obviously one needs the cardinality of those sets to coincide to proceed; but the order implied in the bijection is also important. The result is called $\Gamma \circ_{v,\varphi} \Gamma'$.



As before, the insertion of Γ'' into $\Gamma \circ_{v,\varphi} \Gamma'$ can be done like this:



which has the commutativity property, or like this:



which has the associativity property.

Then one sums (in the sense of formal linear combinations) over the vertices and possible bijections:

$$\Gamma' \twoheadrightarrow \Gamma := \Gamma \circ \Gamma' := \sum_{v,\varphi} \Gamma \circ_{v,\varphi} \Gamma'.$$

Some of the graphs we sum over might be isomorphic; we disregard 'symmetry factors'. *This grafting product* $\Gamma' \twoheadrightarrow \Gamma$ *satisfies the (left-symmetric) Vinberg identity.*

The set of graphs is endowed with a weight function, given by the number n of external legs, and a grading, given by the number L of loops. Grafting does not change the weight of a diagram, whereas the grade of $\Gamma' \twoheadrightarrow \Gamma$ is $L(\Gamma) + L(\Gamma')$. If π denotes a Feynman rule (an appropriate linear map of the space of linear combinations of graphs into the complex numbers), then the operator $\theta_{\Gamma'}$ is defined by

$$\theta_{\Gamma'}\pi(\Gamma) = \pi(\Gamma' \twoheadrightarrow \Gamma).$$

In view of preceding considerations,

$$[\theta_{\Gamma_1}\, \theta_{\Gamma_2}] := \theta_{\Gamma_2 \twoheadrightarrow \Gamma_1} - \theta_{\Gamma_1 \twoheadrightarrow \Gamma_2},$$

just like in (4), is a Lie bracket. Consider now truncated spaces of graphs $W_{n,\leq L}$ with fixed number n of external legs and bounded loop grading. We can regard W_n as an algebraic unipotent group $\lim_{L\to\infty} W_{n,\leq L}$, an inverse limit of groups of triangular matrices. The situation is similar to the case of Diff with its standard affine coordinates.

The important point is that *the Lie algebra corresponding to the group* W_n (or its truncation $W_{n,\leq L}$) *is a vector space whose basis consists of diagrams, with bracket* $[\Gamma, \Gamma'] = \Gamma' \twoheadrightarrow \Gamma - \Gamma \twoheadrightarrow \Gamma'$ *derived from the Vinberg bracket defined above.*

3.2 Regularization scheme according to Connes and Kreimer

Alain Connes and Dirk Kreimer arrive substantially at amplitudes of the form

$$g(\epsilon) \in T_n := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}.$$

The elements $g_{ij}(\epsilon)$ belong to $\mathbb{C}[[\epsilon, \epsilon^{-1}]$: this notation means of course that they are formal Laurent series in ϵ.

When a diagram is divergent, but has no subdivergences, the corresponding amplitude is of the form

$$g(\epsilon) = \begin{pmatrix} 1 & a(\epsilon) \\ 0 & 1 \end{pmatrix}.$$

Here

$$a(\epsilon) = a_-(\epsilon^{-1}) + a_+(\epsilon),$$

where a_- is a polynomial and a_+ is a series, and the ambiguity in their definition is solved by deciding $a_-(0) = 0$. Then the famous 'Birkhoff decomposition' by Connes and Kreimer is simply given in this case by

$$g(\epsilon) = \begin{pmatrix} 1 & a_-(\epsilon^{-1}) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_+(\epsilon) \\ 0 & 1 \end{pmatrix} =: g_-(\epsilon^{-1})\, g_+(\epsilon).$$

Suppose now the diagram is divergent overall, and has a subdivergence; then the corresponding amplitude is typically of the form

$$g(\epsilon) = \begin{pmatrix} 1 & a(\epsilon) & b(\epsilon) \\ 0 & 1 & c(\epsilon) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_-(\epsilon^{-1}) + a_+(\epsilon) & b_-(\epsilon^{-1}) + b_+(\epsilon) \\ 0 & 1 & c_-(\epsilon^{-1}) + c_+(\epsilon) \\ 0 & 0 & 1 \end{pmatrix}.$$

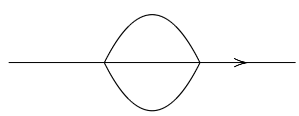
We are going to factorize this in the form $g(\epsilon) = g_-(\epsilon^{-1})g_+(\epsilon)$ again. We obtain

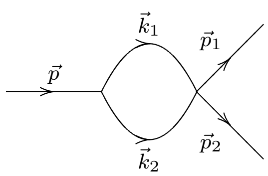
$$\begin{aligned} g(\epsilon) &= \begin{pmatrix} 1 & a_-(\epsilon^{-1}) & b_-(\epsilon^{-1}) - (a_-c_+)_-(\epsilon^{-1}) \\ 0 & 1 & c_-(\epsilon^{-1}) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_+(\epsilon) & b_+(\epsilon) - (a_-c_+)_+(\epsilon) \\ 0 & 1 & c_+(\epsilon) \\ 0 & 0 & 1 \end{pmatrix} \\ &=: g_-(\epsilon^{-1})g_+(\epsilon). \end{aligned}$$

That is, one proceeds subdiagonal by subdiagonal, effecting the previous renormalization of the subdivergence —this Kurusch Ebrahimi-Fard would do by use of abstract Rota–Baxter operator properties.

3.3 Analytical considerations

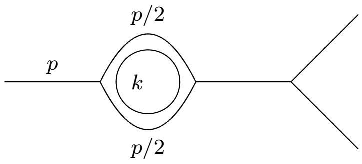
Consider now a diagram Γ with n external legs (corresponding to high-energy reactions involving n particles). The figure shows two examples, respectively with $n = L = 2$; and with $n = 3$ and $L = 1$.





In the figure on the right, we must have $\vec{p} = \vec{p}_1 + \vec{p}_2$, by momentum conservation. In fact, one considers $p \in M_4$, the relativistic four-momentum, to include conservation of energy.

(By the way, possible symmetries here, like the one implemented by change of the time orientation $p \mapsto -p$ correspond to deep conservation principles in physics.) The result of the calculation of the scattering amplitude defined by the diagram will depend on p, obviously. In the diagram



we put $k_1 = p/2 + k$, $k_2 = p/2 - k$; to it does correspond the integral

$$I(p) = \int \frac{d^4k}{(p/2+k)^2(p/2-k)^2}. \tag{7}$$

By means of analytic continuation we have gone over to "Euclidean" integrals; so the squares in this formula have their ordinary meaning. By power counting, it is clear that $I(p)$ diverges. We can make a cutoff $|k| < \Lambda$ (physically this is justified, as we have no information on ultra-high frequencies) and study the asymptotic behaviour of the integral with Λ. This method is however awkward for diagrams with subdivergences. Following Schwinger, Feynman, Symanzik, Nambu, Nakanishi and many others since the 50's and 60's, we can pass to the *effective action* integral, mathematically given by the identity:

$$\frac{1}{p^2+m^2} = \int_0^\infty d\alpha\, e^{-\alpha(p^2+m^2)}.$$

If we now substitute these integrals for the fractions in (7), after doing the easy Gaussian integral, we are left with an integral of the form

$$\int_0^\infty \int_0^\infty d\alpha_1\, d\alpha_2\, \frac{e^{-(\alpha_1+\alpha_2)m^2}\, e^{-\frac{\alpha_1\alpha_2}{2(\alpha_1+\alpha_2)}p^2}}{(\alpha_1+\alpha_2)^2}. \tag{8}$$

Whereas Λ had dimensions of energy, the α's behave as the square of a minimal length. In order to avoid the divergence of this integral near zero, one can regularize by choosing $\alpha_1, \alpha_2 \geq \epsilon$; a harmless alternative is to cut the corner out, $\alpha_1 + \alpha_2 \geq \epsilon$. In case of having three denominators, we would get numerators of the type $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$ in the exponentials; and so on. These integrals can be attacked by blow-up methods. Let us consider the relatively simple related integral:

$$\int_0^1 \int_0^1 \frac{d\alpha_1\, d\alpha_2}{1-\alpha_1\alpha_2}.$$

This is finite, as the limit of the integral with the corner cut out (yielding the Stasheff polyhedron $\mathcal{P}_4$ [2]) exists and is equal to $\zeta(2) = \pi^2/6$ —by the way, as an exercise the reader is challenged to find an elementary proof of this last fact, discovered by Euler. One can practice a blow-up at the point $\alpha_1 = \alpha_2 = 1$ with the introduction of a new coordinate

$\alpha_3 := \frac{1-\alpha_1}{1-\alpha_1\alpha_2}$, to obtain a surface in the real three-space as domain of integration; then the singularity disappears.

We make a final comment on dimensional renormalization. This is a form of analytic regularization, in turn developed in the last century by Hadamard, Riesz, Gelfand, Bernstein,... and is concerned with integrals of the form

$$\int_{\mathcal{D}} \frac{\varphi(x)}{[P(x)]^s}\, dx$$

with P a polynomial and s a complex parameter. The denominator in (8) that comes from the Gaussian integral is in fact $(\alpha_1 + \alpha_2)^{d/2}$, where d is the dimension. The idea is to make $d = 4 - \epsilon$ a complex variable. Feynman amplitudes, always for a fixed number of external legs, become integrals $I(\Gamma, p, \epsilon)$, that give rise to Laurent expansions in ϵ. The elements of the Connes–Kreimer group of diffeographisms that 'kill' the divergencies are of the form

$$g_-(\epsilon^{-1}) = \exp\left(\frac{\beta_{-1}}{\epsilon} + \frac{\beta_{-2}}{\epsilon^2} + \cdots\right),$$

where the β_{-i} live in the Lie algebra of that group. The same is related to the notion of the motivic Galois group, investigated at present by Alain Connes, Matilde Marcolli and myself.

References

[1] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser, Boston, 1992.

[2] P. Cartier, "Combinatorics of trees", in *Surveys in Modern Mathematics*, V. Prasolov and Y. Ilyashenko, eds., Cambridge University Press, Cambridge, 2005; pp. 274–282.