Almost etale resolution of foliations

Michael MCQUILLAN and Daniel PANAZZOLO

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Novembre 2009

IHES/M/09/51

Almost étale resolution of foliations

Michael McQuillan & Daniel Panazzolo

November 16, 2009

Introduction

The ubiquity of canonical singularities goes far beyond the narrow confines in which they were first conceived in [R]. Quite generally, given the 'smooth objects' within a given category, those with canonical singularities could be defined as the largest sub-category containing the former for which the ramification is well defined and non-negative. As such, they bear a tight relation with the notion of isoperimetric dimension introduced in [G], and are the natural category in which theorems of "uniformisation type" take place. By way of examples,

- A complex-projective variety admits a metric of negative Ricci curvature -1 iff it is the canonical model (the original context of [R]) of a variety of general type, [E&].
- A complex projective variety foliated in curves admits a leafwise metric of negative Ricci curvature -1 and continuous non-trivial transverse variation iff it is the (foliated) canonical model of a foliation of general type, [M3].

Where, as a not unimportant precision, one should, more correctly, state the above, in either case, on the smallest algebraic stack, with moduli as given, on which the dualising sheaf is a bundle, *i.e.* the Gorenstein covering stack, I.ii.5.

Unfortunately, apart from the trivial foliation over a point (*i.e.* a variety) there is a paucity of existence results for canonical resolutions. The main theorems are: ambient dimension 2, [Se], co-dimension 1 in ambient dimension 3, [C2], local uniformisation of complete 3-dimensional local rings, [C1], and local uniformisation of Henselian local rings, [C&], with globalisation in the right direction, cf. op. cit. for the precise definition. Furthermore, already on surfaces, canonical resolution is best possible. In this context, a natural algebraic hypothesis when considering a singular vector field ∂ around a point, with maximal ideal $\mathfrak{m}$ on a variety would be to insist that the induced linearisation,

$$\overline{\partial} \in \mathrm{End}\left(\mathfrak{m}/\mathfrak{m}^2\right)$$

was non-nilpotent. Functorially with respect to the ideas this is equivalent, I.ii.3, to the foliation being Gorenstein and log-canonical, and, perhaps surprisingly, this is very close to being equivalent to being canonical, III.ii.1, while, again, perhaps surprisingly, terminal is synonymous with smooth, III.i.1.

Plainly, however, the non-nilpotence of the linearisation gives a firm handle on the local structure, and whence in [P] log-canonical singularities were called elementary. In particular, therefore, the problem of constructing log-canonical resolutions of 3-folds foliated by curves was essentially solved by op. cit.. More precisely, op. cit. constructed an (almost) étale local resolution strategy by way of weighted blow ups, and successfully applied this strategy to construct a global resolution of foliations by analytic curves on real 3-manifolds with boundary. The only caveat to applying these considerations in the complex case is that there is no suitable 1-category where weighted blowing up will preserve smoothness of the ambient space. Consequently, one either has to add a layer of complication to the proof to take account of quotient singularities, or work directly in the 2-category of algebraic stacks. As such, while it is a theorem, cf. I.4, that these two approaches are logically equivalent the flexibility of the étale site of a Deligne-Mumford stack permits the key invariants and calculations of op. cit. to be imported mutatis mutandis. It could also be true, since it's not wholly incorrect to think of a Deligne-Mumford stack as a cone manifold, that a pure general nonsense approach could be possible, but it's form for a general complex, rather than real, stack is not immediately event. In any case, it cannot, therefore, be emphasised sufficiently that the vast majority of the work which underlies the current article belongs properly to op. cit., and the present §II is simply an explanation of how to import it, with only an extremely minor change, which would not even have been necessary if one's only interest was a resolution procedure for foliated projective 3-folds. This said, the algorithm is not, however, étale local for foliated 3-folds, but étale local for foliated 3-folds with an additional structure of an axis, I.iii.1. In practice this is a rather minor condition to guarantee, in theory, it's quite delicate to produce a usable general criteria which doesn't involve a projective embedding in some way. Whence, to simplify this summary of the results let us have recourse to the hypothesis of projectivity, so that the main theorem becomes,

Theorem Let $(X, D, \mathcal{F})$ be a smooth complex projective foliated 3-fold with boundary then there is a sequence of *smoothed weighted blow ups*, I.iv.3, in the 2-category of smooth logarithmic Deligne-Mumford stacks,

$$(X, D, \mathcal{F}) = (\mathcal{X}_0, \mathcal{D}_0, \mathcal{F}_0) \leftarrow (\mathcal{X}_1, \mathcal{D}_1, \mathcal{F}_1) \leftarrow \ldots \leftarrow (\mathcal{X}_k, \mathcal{D}_k, \mathcal{F}_k) = (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$$

such that the resulting foliated 3-fold is smooth with simple normal crossing boundary and canonical singularities. Alternatively, in the 1-category of projective varieties with quotient singularities, there is a sequence of weighted blow ups,

$$(X, D, \mathcal{F}) = (X_0, D_0, \mathcal{F}_0) \leftarrow (X_1, D_1, \mathcal{F}_1) \leftarrow \ldots \leftarrow (X_k, D_k, \mathcal{F}_k) = (\tilde{X}, \tilde{D}, \tilde{\mathcal{F}})$$

such that the resulting foliated 3-fold has canonical singularities, and ambient singularities at worst quotient (infact, see below, at worst $\mathbb{Z}/2$ quotient if one wants). Better still, at each stage in the procedure,

- The relevant weighted filtration is invariant by the induced foliation at the given stage, and its centre is supported on the non-canonical locus at the said stage.

- The centre, in fact the weighted filtration, has, for an initial smooth boundary, in the 2-category of Deligne-Mumford stacks, normal crossings with the boundary at each stage.

It should be completely clear from [M1], [M2], & [M3] that such a final model is not only acceptable but to be encouraged, *e.g.* already for foliated surfaces the natural (infact in [M1] it's called canonical) resolution to employ will in general have ambient quotient singularities, and, the more generally employed resolutions with 'reduced singularities' are a non-functorial chimera, cf. III.iv. Consequently, it is purely a matter of curiousity to enquire to what extent passing to the 2-category of algebraic stacks is necessary to preserve a smooth ambient space, and to this there is a complete answer, viz:

Divertimento The above procedure can be continued with an additional operation of killing pseudo-reflecting monodromy, whence preserving ambient smoothness, so that the resulting resolution $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ has at singular points of the foliation no worse than $\mathbb{Z}/2$ monodromy, and is (foliated) terminal at all other non-scheme like points. In particular, it may be further continued by blowing up in the terminal non-scheme like points, thus, in general, not invariant by the induced foliation, so that the resulting foliated smooth stack has no worse than $\mathbb{Z}/2$ monodromy, and this occurs only at singular points of the foliation. This is, however, best possible, *i.e.* there exists, and in fact there is even a complete classification of the same, a germ of a foliated 3-fold $(U, \mathcal{F})$ such that the induced foliation on ANY smooth proper bi-rational modification $V \to U$ in the 1-category of varieties is NEVER canonical.

The theorem is proved in §III.2, and is a rather simple corollary of the importation of the log-canonical case from [P] in §II. The divertimento, §III.3, has it's roots in an original intuition of Felipe Cano based on his work [C1], and already brought to fruition by Fernando Sanz in a more particular form in [S]. Both authors are indebted to them for sharing this particular knowledge, and much more widely to Felipe Cano for having shared his entire expertise in the whole discipline. They are also indebted to the organisers Erwan Rousseau and Gianluca Pacienza of the conference algebraic varieties and hyperbolicity at IRMA, Strasbourg, which occasioned the fortunate meeting of the authors, and subsequent writing of this article. The first author, however, considers that this is in stark contrast to a long list of people from Dijon to Rio who were criminally negligent in drawing the work of the second author to his attention. The first author would be more than happy to name and shame those involved. Being late though, is better than never, so Jorge Vitório Pereira escapes this criticism (just) and the first author is happy to acknowledge his role in preparing the fortuitous meeting in Strasbourg by bringing the work of the second author to his attention a couple of months earlier. Fortunately, Cécile is radically more efficient than Jorge, and so many more thanks to her for the web copy.

I. Generalities

I.i Tagging

As in all resolution problems it will be necessary to introduce a slightly artificial variation of the principle object of interest in order to define a suitable invariant that will decrease under blowing up. To this end, we introduce,

I.i.1 Definition A *smooth foliated stack with tagged smooth boundary* is a 4-tuple $(\mathcal{X}, \mathcal{D}, \Upsilon, \mathcal{F})$ such that,

(1) $\mathcal{X}$ is a smooth connected Deligne Mumford stack of finite type over a field of characteristic zero, which for convenience will be supposed algebraically closed, and for even further convenience equal to the complex numbers $\mathbb{C}$.

(2) A simple normal crossing divisor $\mathcal{D}$ each of whose irreducible components $\mathcal{D}_i$ is a smooth co-dimension 1 sub-stack of $\mathcal{X}$.

(3) A reverse ordered list of natural numbers, Υ, *i.e.* $[i_1, \ldots, i_k]$, $i_1 > \ldots > i_k \in \mathbb{N}$, together with a bijection of sets,

$$\Upsilon \xrightarrow{\sim} \{\text{irreducible components of } \mathcal{D}\}$$

(4) A foliation by curves $\mathcal{F}$ leaving $\mathcal{D}$ invariant, *i.e.* a line bundle $T_{\mathcal{F}}$ on $\mathcal{X}$ together with an injection,

$$0 \longrightarrow T_{\mathcal{F}} \longrightarrow T_{\mathcal{X}}(-\log \mathcal{D})$$

with torsion free co-kernel.

Evidently, the somewhat artificial datum here is I.i.1.(3), and whence the more natural data of the 3-tuple $(\mathcal{X}, \mathcal{F}, \mathcal{D})$ will be referred to as a *smooth foliated stack with smooth boundary.* In the event that neither the smoothness of $\mathcal{X}$ nor of $\mathcal{D}$ nor even that $T_{\mathcal{F}}$ is anything better than reflexive rank 1 is supposed then we will call such a triple a *foliated logarithmic stack.* Consequently, the following remark is relevant,

I.i.2 Caution/Definitions A foliation is often defined as a saturated sub-sheaf of $T_{\mathcal{X}}$, and since $T_{\mathcal{X}}(-\log \mathcal{D}) \subset T_{\mathcal{X}}$ such a definition may well not be compatible with I.i.1.(4). Indeed, $T_{\mathcal{F}}$ in the sense of I.i.1.(4) is saturated in $T_{\mathcal{X}}$ if and only if it is saturated at each generic point of $\mathcal{D}$, which in turn is true if and only if the foliation considered as a saturated sub-sheaf of $T_{\mathcal{X}}$ fixes every generic point of $\mathcal{D}$. Consequently, and functorially with respect to the ideas, the dual $K_{\mathcal{F}}$ of $T_{\mathcal{F}}$ as defined in I.1.i.(4) is the *log canonical* bundle of the foliated logarithmic stack $(\mathcal{X}, \mathcal{D}, \mathcal{F})$. Furthermore, should a component of $\mathcal{D}$ remain invariant by the foliation viewed as a saturated sub-sheaf of $T_{\mathcal{X}}$ then it will be called *strictly invariant.*

I.ii Log canonical singularities

As has been suggested these are defined functorially with respect to the ideas in the usual way, *i.e.*

I.ii.1 Definition Let $(U, D, \mathcal{F})$ be an irreducible local germ of a $\mathbb{Q}$-Gorenstein foliated logarithmic normal variety, *i.e.* $U = \mathrm{Spec}\mathcal{O}_{X,Z}$, for Z a sub-variety of a normal variety X such that the log canonical bundle $K_{\mathcal{F}}$ is a $\mathbb{Q}$-divisor, then for v a divisorial valuation of $\mathbb{C}(U)$ centered on Z the *log discrepancy*, $a_{\mathcal{F}}(v)$ is defined as follows:

By hypothesis there is a normal modification $\pi : \tilde{U} \to U$ of finite type, together with a divisor E on $\tilde{U}$ such that $\mathcal{O}_{\tilde{U},E}$ is the valuation ring of v. In particular, bearing in mind I.i.2, there is an induced foliation $\tilde{\mathcal{F}}$ with log canonical bundle $K_{\tilde{\mathcal{F}}}$, *i.e.* whose dual is saturated in $T_{\tilde{U}}(-\log E)$, and,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} + a_{\mathcal{F}}(v) E$$

If furthermore we define $\epsilon(v)$ to be zero if E is strictly invariant, and 1 otherwise, then provided the following hold for all divisorial valuations centered on Z we say that the local germ $(U, D, \mathcal{F})$ is,

(1) *Terminal* if $a_{\mathcal{F}}(v) > \epsilon(v)$.

(2) *Canonical* if $a_{\mathcal{F}}(v) \geq \epsilon(v)$.

(3) *Log-Terminal* if $a_{\mathcal{F}}(v) > 0$.

(4) *Log-canonical* if $a_{\mathcal{F}}(v) \geq 0$.

Where the slightly unsettling shift of the definitions by $\epsilon(v)$ occurs as a result of the convention adopted in I.i.2 together with their correct functorial interpretation.

The discussion of definitions (1)-(3) will be postponed till §III.1, and a priori what is of relevance is the definition of log-canonical singularities. A priori this definition is not étale local. Observe, however, that if $U_h \to U$ is the strict Henselisation of U then at the cost of allowing v to be a divisorial valuation of $\mathbb{C}(U_h)$ the above definition continues to have perfect sense, and, unsurprisingly,

I.ii.2 Fact Notations as above then $(U, D, \mathcal{F})$ has a log-canonical singularity if and only if $(U_h, D_h, \mathcal{F}_h)$ has a log canonical singularity.

proof Plainly $K_{\mathcal{F}_h} = K_{\mathcal{F}}|_{U_h}$, so if v_h is a divisorial valuation lying over v, then $a_{\mathcal{F}_h}(v_h) = a_{\mathcal{F}}(v)$. $\square$

Rather more usefully, however, we can explicitly describe log-canonical singularities as soon as the germ $(U, D, \mathcal{F})$ is *Gorenstein*, *i.e.*, $K_{\mathcal{F}}$ (equivalently for U normal $T_{\mathcal{F}}$) is a line bundle. Consequently, the foliation is defined by a vector field ∂ and we have,

I.ii.3 Possibilities Exactly one of the following occurs,

(a) ∂ is smooth, *i.e.* there exists $f \in \mathcal{O}_{X,Z}$ such that $\partial(f) \neq 0 \in k(Z)$.

(b) Otherwise, so ∂ not only leaves $\mathfrak{m}_{X,Z}$ invariant but descends to a $k(Z)$ linear endomorphism of the Zariski tangent space,

$$\bar{\partial} : \frac{\mathfrak{m}_{X,Z}}{\mathfrak{m}^2_{X,Z}} \longrightarrow \frac{\mathfrak{m}_{X,Z}}{\mathfrak{m}^2_{X,Z}}$$

With this in mind, one observes,

I.ii.4 Fact Suppose the germ (U, D, F) is Gorenstein then it is log canonical if and only if a local generator ∂ is either smooth or $\bar{\partial}$ is a non-nilpotent endomorphism of the Zariski tangent space.

proof The only if definition is straightforward, since otherwise we're in I.ii.3.(b), and $\bar{\partial}$ is nilpotent. As such, it may very well be zero, in which case blowing up in Z, and normalising yields divisorial valuations with negative discrepancy, while for an arbitrary nilpotent field one can explicitly construct a valuation with negative discrepancy by way of blowing up, and normalisation, in suitable centres defined by the Jordan blocks of $\bar{\partial}$. Conversely, with the notation of I.ii.1, let $\pi : \tilde{U} \to U$ be a modification of U associated with a divisorial valuation, ∂, $\tilde{\partial}$ generators of the foliation on U, and $\tilde{U}$ respectively, with x a uniformising parameter of the divisor E, then for a the discrepancy of the valuation v,

$$\partial = x^{-a} u \tilde{\partial}$$

for u a unit. Now suppose that ∂ is smooth, then there is a function f on U such that, $v(\partial f) = 0$. As such,

$$av(x) = v(\tilde{\partial}(\pi^* f)) \geq 0$$

so $a \geq 0$. More generally, observe that since $\tilde{\partial}$ leaves E invariant, then for any $q \in \mathbb{N}$,

$$x^{qa} \partial^q$$

is a regular differential operator on $\tilde{U}$ leaving E invariant. Furthermore if ∂ is non-nilpotent then for any $n \in \mathbb{N}$ there is a $q \in \mathbb{N}$ and a function $f_n \in \mathfrak{m}_{X,Z} \backslash \mathfrak{m}^2_{X,Z}$ on the strict Henselisation, U_h, say, such that,

$$\partial^q (f_n) = \lambda f_n \bmod \mathfrak{m}^n_{X,Z}$$

for some unit λ. Indeed this is just a consequence of Jordan decomposition, and since v is divisorial no formal function has infinite order, so we can actually ensure that f_n is fixed modulo $\mathfrak{m}^2_{X,Z}$. for v_h any divisorial valuation of U_h lying over v, $v_h(f_n)$ is bounded independently of n. Whence for n sufficiently large,

$$v_h(f_n) = v(\partial^q f_n) \geq -aqv_h(x) + v_h(Df_n)$$

where D is a regular differential operator leaving E invariant, *i.e.* $v_h(Df_n) \geq v(f_n)$, so again $a \geq 0$. □

To cover the $\mathbb{Q}$-Gorenstein case, one observes,

I.ii.5 Fact/Definition Let $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ be a normal foliated $\mathbb{Q}$-Gorenstein log stack then there is a normal foliated Gorenstein log stack, $\nu : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to$

$(\mathcal{X}, \mathcal{D}, \mathcal{F})$ with $\nu : \tilde{\mathcal{X}} \to \mathcal{X}$ finite and étale in co-dimension 1, which is in fact universal for the said properties, and whence will be referred to as the *Gorenstein covering stack*. In particular, $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ has log canonical singularities if and only if it's Gorenstein covering stack does.

proof Let $U \to \mathcal{X}$ be an atlas such that for some $n \in \mathbb{N}$, $\mathcal{O}_U(nK_{\mathcal{F}})$ is trivial, and n is minimal. Consequently, we can define a finite covering $V \to U$ of degree at most n such that $\mathcal{O}_V(K_{\mathcal{F}})$ is trivial, and V is normal. By hypothesis the diagonal of $\mathcal{X}$ is representable so $V \times_{\mathcal{X}} V$ is a scheme, and by the construction of V, its normalisation R yields an étale groupoid,

$$R \rightrightarrows V$$

and the classifying stack $[V/R]$ is the required universal widget.

Certainly therefore, $\nu^* K_{\mathcal{F}} = K_{\tilde{\mathcal{F}}}$, while if $\tilde{\mathcal{E}}$ is any divisor on a normal modification of $\tilde{\mathcal{X}}$ lying over $\mathcal{E}$, then the log-discrepancy of $\tilde{\mathcal{F}}$ around $\tilde{\mathcal{E}}$ is just $\mathrm{ord}_{\tilde{\mathcal{E}}}(\mathcal{E})$ that of $\mathcal{F}$ around $\mathcal{E}$. $\square$

This leads to,

I.ii.6 Definition/Summary For $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ a normal $\mathbb{Q}$-Gorenstein foliated log stack with Gorenstein cover $\nu : \tilde{\mathcal{X}} \to \mathcal{X}$ the singular locus, $\mathrm{sing}(\mathcal{F})$, of $\mathcal{F}$ is the reduced image of the closed substack where a local generator of $\tilde{\mathcal{F}}$ is not smooth in the sense of I.ii.3.(b). In particular the locus of non-log canonical points, $\mathrm{NLC}(\mathcal{F})$, which, with reduced structure, we may identify with $\mathrm{NLC}(\tilde{\mathcal{F}})$, is a closed sub-stack of $\mathrm{sing}(\mathcal{F})$. Given I.i.2, however, $\mathrm{sing}(\mathcal{F})$, can be of co-dimension 1. Nevertheless, this only occurs at generic points of $\mathcal{D}$ which are not invariant by the foliation viewed as a saturated sub-sheaf of $T_{\mathcal{X}}$, and since $\mathcal{X}$ is regular in co-dimension 1, such points are log-canonical. Consequently, $\mathrm{NLC}(\mathcal{F})$ is closed, and of co-dimension at least 2.

I.iii Axes

Unfortunately, the need to work outwith the natural 2-category of foliated log-stacks is not limited to tagging, and will require a further, albeit very mild, hypothesis on the structure of $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ around $\mathrm{NLC}(\mathcal{F})$. Following [P], this structure will be called an axis and is defined as follows,

I.iii.1 Definition Let $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ be normal $\mathbb{Q}$-Gorenstein foliated log-stack, and $\mathfrak{N}$ its completion in $\mathrm{NLC}(\mathcal{F})$ then by an *axis* for $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ is to be understood a Gorenstein foliated formal log-stack, $(\mathfrak{N}, \mathcal{D}|_{\mathfrak{N}}, \mathcal{A})$, such that,

(a) The foliation $\mathcal{A}$ is smooth, *i.e.* as per I.ii.3.(a) defined everywhere locally by a non-vanishing vector field.

(b) The foliation is convergent, *i.e.* every point of the non-log canonical locus has an étale neighbourhood (in practice one may take this to be étale algebraic, but in principle étale analytic is a weaker condition) to which $\mathcal{A}$ extends.

(c) The foliation $\mathcal{A}$ is transverse to every generic point of $\mathrm{NLC}(\mathcal{F})$ contained in the boundary.

(d) If $f : \mathrm{Spf}\mathbb{C}[[x]] \to \mathfrak{N}$ is $\mathcal{A}$ invariant, with trace a non-boundary geometric point, then it is not $\mathcal{F}$ invariant.

Observe that in the definition of an axis $(\mathfrak{N}, \mathcal{D}|_{\mathfrak{N}}, \mathcal{A})$ comes, by definition, in the obvious extension of I.i.1.(3), with a foliation that leaves $\mathcal{D}|_{\mathfrak{N}}$ invariant.

Now plainly, the existence of an axis imposes conditions on $(\mathcal{X}, \mathcal{D}, \mathcal{F})$, for example even if $(\mathcal{X}, \mathcal{D})$ were as per I.i.1.(1) & (2) then as soon as a point of NLC$(\mathcal{F})$ meets dim$\mathcal{X}$ components of $\mathcal{D}$ an axis cannot exist. As such, although, modulo the obvious dimension restrictions, it's easy to impose conditions that an axis does exist at the initial stage of a resolution procedure, guaranteeing its continued existence becomes part of the problem. In any case, the basic existence criteria is rather evident, *viz:*

I.iii.2 Fact Let $(X, D, \mathcal{F})$ be a foliated log stack such that X is a smooth projective 3-fold, and D a strictly invariant simple normal crossing divisor such that no point of NLC$(\mathcal{F})$ meets the singularities of D then there exists an axis.

proof Put $N = NLC(\mathcal{F})$, and let $\mathcal{T}$ be the image of $T_X(-\log D)$ in T_X then for H sufficiently ample we have a diagram with exact rows and columns,

$$\begin{array}{ccccc} \Gamma(X,\, T_X(-\log D)\otimes_{\mathcal{O}_X} H) & \longrightarrow & \Gamma(N,\, T_X(-\log D)\otimes_{\mathcal{O}_X} H|_N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \Gamma(X,\, \mathcal{T}\otimes_{\mathcal{O}_X} H) & \longrightarrow & \Gamma(N,\, \mathcal{T}\otimes_{\mathcal{O}_X} H|_N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Now every stalk of $\mathcal{T}$ has image in T_X of rank at least 2, and N has dimension at most 1, so we need a variant of a lemma of Serre. Specifically, following the demonstration of the same in [F] B.9.1, we assert,

I.iii.2.bis Claim Let E be a vector bundle on a scheme, or better *fpqf* stack, X, and Γ a finite dimensional space of sections generating a sub-sheaf $\mathcal{E}$ of E such that at every geometric point x,

$$\dim_{k(x)} \mathrm{Im}\,(\mathcal{E}\otimes_{\mathcal{O}_X} k(x) \to E\otimes_{\mathcal{O}_X} k(x)) \geq r$$

then the locus where a generic section of Γ vanishes (considered as a section of E) has co-dimension at least r.

sub-proof Consider the composition of the canonical maps,

$$\mathcal{O}_{\mathcal{X}}[\Gamma^\vee] \leftarrow \mathrm{Sym}\mathcal{E}^\vee \leftarrow \mathrm{Sym}E^\vee$$

which in turn yields the natural map,

$$\pi : X \times \Gamma \to \mathbb{V}(E^\vee)$$

which has image a constructible set, whose geometric points may be identified with,

$$\coprod_x \mathrm{Im}\,(\mathcal{E}\otimes_{\mathcal{O}_X} k(x) \to E\otimes_{\mathcal{O}_X} k(x))$$

where the disjoint union is taken over geometric points of X. On the other hand Γ generates $\mathcal{E}$, so the pre-image of the zero section under π has co-dimension at least r, and whence the same for the co-dimension of the zero locus of a generic section in Γ. $\square$

Applying this to the case in hand, with and as ever H sufficiently ample, a generic section of the group in the bottom right hand corner defines a no-where vanishing section of $\Gamma(N, T_X \otimes_{\mathcal{O}_X} \mathcal{O}_N(H))$. Consequently, there is a Zariski neighbourhood U of N, such that we have an injection of bundles,

$$0 \longrightarrow \mathcal{O}_U(-H) \longrightarrow T_X(-\log D)|_U$$

which further extends to an injection of bundles,

$$0 \longrightarrow \mathcal{O}_U(-H) \longrightarrow T_X|_U$$

The foliations thus constructed, are, therefore, actually generically in the moduli,

$$\Gamma(X, T_X(-\log D) \otimes_{\mathcal{O}_X} H)$$

Consequently off $\mathrm{sing}(\mathcal{F})$ and $\mathcal{D}$ the tangency with $\mathcal{F}$ has co-dimension 3, and can be taken to miss $\mathrm{NLC}(\mathcal{F})$, while plainly, *e.g.* consider families of generic complete intersections, invariance of $\mathrm{NLC}(\mathcal{F})$ isn't a generic condition either. $\square$

I.iv Weighted Blowing Up

The weighted blow up of an algebraic stack $\mathcal{X}$ is defined in the obvious way, *viz:*
I.iv.1 Definition A weighted blow up of weight $\omega = (\omega_1, \ldots, \omega_r)$, $\omega_i \in \mathbb{N}$ without common divisor, with smooth centre, of co-dimension r, is the projectivisation,

$$\pi : \mathcal{X}' = \mathrm{Proj}(\mathcal{O}_{\mathcal{X}} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots) \to \mathcal{X}$$

where $\mathcal{I}_k$ are sheaves of ideals on $\mathcal{X}$ such that locally in the étale site of $\mathcal{X}$ there are smooth coordinate functions $x_1, \ldots, x_r$, defining a smooth co-dimension r centre, and $\mathcal{I}_k$ is generated by the monomials,

$$(x_1^{a_1} \ldots x_r^{a_r} | a_1\omega_1 + \ldots + a_r\omega_r \geq k)$$

Now, manifestly the minor difficulty with weighted blowing up, even in a smooth centres, is that as soon as some $\omega_i > 1$, it may very well lead to singularities. Nevertheless, these are no worse than quotient singularities, and so we may appeal to,
I.iv.2 Fact/Definition (characteristic 0) Let $\mathcal{X}$ be an algebraic stack with quotient singularities, so by definition normal, then there is a smooth stack $\mu : \tilde{\mathcal{X}} \to \mathcal{X}$, such that μ is finite and étale in co-dimension 1. Further μ may be taken as being universal with respect to the above properties, in which case we will refer to it as the *Vistoli covering stack.*
proof Following [V], let $U \to \mathcal{X}$ be an étale atlas, then refining U as necessary, we may realise each connected component U_i of U as the coarse moduli of some

V_i/G_i, where V_i is smooth, and G_i acts freely in co-dimension 1. As such, we put $V = \coprod_i V_i$, and R the normalisation of,

$$V \times_{\mathcal{X}} V$$

which as ever is a scheme, by the representability of the diagonal, so, by purity, we obtain an étale groupoid,

$$R \rightrightarrows \mathcal{X}$$

and the required universal widget is the classifying stack $[V/R]$. $\square$

Naturally, this leads to,

I.iv.3 Definition A *smoothed weighted blow up* $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ of weight ω in a smooth centre is the Vistoli covering stack $\tilde{\mathcal{X}} \to \mathcal{X}'$ of a weighted blow up $\mathcal{X}' \to \mathcal{X}$ of weight ω in a smooth centre.

An explicit description of charts for smoothed weighted blow ups of smooth stacks will be helpful, so to this end take a sufficiently small étale neighbourhood V of the moduli X such that $\mathcal{X} \times_X V = [U/G]$ for some finite group acting on a smooth affine U on which $\mathcal{I}_k$ admit a description as per I.iv.1. Thus, although the coordinates in this description may not be G invariant, we may, and will, identify the sheaves of ideals $\mathcal{I}_k|_{[U/G]}$ with G equivariant ideals of functions I_k on U. Consequently, the weighted blow up is, locally, a classifying stack of the form,

$$\mathcal{X}'_V = [\mathrm{Proj}(\Gamma(\mathcal{O}_U) \oplus I_1 \oplus I_2 \oplus \ldots)/G]$$

Now étale neighbourhoods V_p of a closed point, p, on a weighted Proj, can after a suitable re-ordering of the indices, be described as follows: there are non-zero constants $\eta_2, \ldots, \eta_s$, $s \leq r$, and coordinate functions $y_1, \ldots, y_n$ on a smooth affine W_p related to the original coordinate functions $x_1, \ldots, x_n$ by way of,

$$x_1 = y_1^{\omega_1},\ x_i = y_1^{\omega_i}(y_i + \eta_i),\ 2 \leq i \leq s,\ x_i = y_1^{\omega_j} y_j,\ s < j \leq r,\ x_k = y_k,\ r < k$$

together with an action of $\mathbb{Z}/d_p$ for d_p the gcd of $\omega_1, \ldots, \omega_s$ given for θ a primitive d_pth root of unity by,

$$y_1 \mapsto \theta y_1,\ y_i \mapsto \theta^{-\omega_i} y_i,\ 2 \leq i \leq r,\ y_k \mapsto y_k,\ r < k$$

such that the coarse moduli of $W_p/(\mathbb{Z}/d_p)$ may be identified with V_p. Now such a V_p may not be G stable, but for G_p the stabiliser of p, and further étale localisation as necessary, $[V_p/G_p] \to \mathcal{X}'_V$ is not only étale, but so too is the induced map on coarse moduli. Furthermore the strict Henselisation of W_p around p is unique with respect to the property that it is strictly local smooth connected and almost étale over the strict Henselisation of V_p, so, modulo further localisation we can lift the action of $\sigma \in G_p$ to an action of some $\tilde{\sigma}$ on W_p. Such liftings need not, however, respect the group structure of G_p, so we can, in general, do no better than,

I.iv.4 Summary Not only is W_p with coordinate functions as described above an étale neighbourhood of a smoothed weighted blow up, $\tilde{\mathcal{X}}$, of a smooth stack, $\mathcal{X}$, but there is an action of a finite group $\tilde{G}_p$, given as an extension,

$$1 \to \mathbb{Z}/d_p \to \tilde{G}_p \to G_p \to 1$$

such that both the map, $[W_p/\tilde{G}_p] \to \tilde{\mathcal{X}}$ and the induced map on moduli are open embeddings.

Furthermore the above explicit description also yields,

I.iv.5 Fact The exceptional divisor, $\mathcal{E}$ of a smoothed weighted blow up, $\tilde{\mathcal{X}}$, of a smooth stack $\mathcal{X}$ in a connected centre is itself smooth and connected.

I.v Modifying tags and axes

There remains to discuss how tags and axes will change during the resolution procedure. All steps of the procedure will be by smoothed weighted blow ups in smooth connected $\mathcal{F}$ invariant (albeit not necessarily strictly) centres. Indeed the stronger condition of being an *invariant weighted blow up, i.e.* the sheaf of graded algebras occurring in I.iv.1 is $\mathcal{F}$ invariant will even hold. As such,

I.v.1 Definition Let $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \Upsilon, \mathcal{F})$ be a smooth foliated stack with tagged smooth boundary and $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ a smoothed invariant blow up in a connected centre with exceptional divisor $\mathcal{E}$ then the associated smoothed weighted blow up,

$$\pi : \tilde{\mathbf{X}} \to \mathbf{X}$$

is the 4-tuple $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \widetilde{\Upsilon}, \tilde{\mathcal{F}})$, where

1. $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ is the aforesaid smoothed weighted blowing-up;

2. The list $\widetilde{\Upsilon}$ is given by $\Upsilon \cup [n]$, where $n := 1 + \max\{i \mid i \in \Upsilon\}$ if $\Upsilon \neq \varnothing$ and $n := 1$ if $\Upsilon = \varnothing$;

3. The divisor $\tilde{\mathcal{D}}$ is the total transform of $\mathcal{D}$, with the tagging

$$\widetilde{\Upsilon} \ni i \longrightarrow \begin{cases} \tilde{\mathcal{D}}_i\,, & \text{if } i \in \widetilde{\Upsilon} \setminus [n] \\ \\ \mathcal{E}\,, & \text{if } i = n \end{cases}$$

   where $\tilde{\mathcal{D}}_i$ is the strict transform of the corresponding divisor $\mathcal{D}_i$ on $\mathcal{X}$ (for each $i \in \Upsilon$). This of course, implicitly supposes, as will be true, that each resulting component of $\tilde{\mathcal{D}}$ is smooth and the total divisor has normal crossings;

4. The proper transform $\tilde{\mathcal{F}}$ of the foliation $\mathcal{F}$, *i.e.* by the hypothesis of the invariance of the weighted blow up, and the almost étale nature of the Vistoli cover the foliation lifts to a map,

$$\pi^* T_{\mathcal{F}} \to T_{\tilde{\mathcal{X}}}(-\log \tilde{\mathcal{D}})$$

   which a priori may not be saturated, and so we saturate it to a map,

$$\pi^* T_{\tilde{\mathcal{F}}} \to T_{\tilde{\mathcal{X}}}(-\log \tilde{\mathcal{D}})$$

   whence, in particular, $K_{\tilde{\mathcal{F}}} \le \pi^* K_{\mathcal{F}}$.

Now suppose the foliated log-stack $\mathbf{X} = (\mathcal{X}, \mathcal{D}, \mathcal{F})$ admits an axis $\mathcal{A}$. Plainly, associated to the modified data $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ there is a new non-log canonical locus, together with the completion $\tilde{\mathfrak{N}}$ of $\tilde{\mathcal{X}}$ at the same. Furthermore, since the weighted blow up is invariant, $\mathrm{NLC}(\tilde{\mathcal{F}}) \subset \pi^{-1}(NLC(\mathcal{F}))$, so there is an induced map of formal schemes, $\pi : \tilde{\mathfrak{N}} \to \mathfrak{N}$ factoring through a projective modification of $\mathfrak{N}$, so at the price of correcting for poles, the original axis yields a saturated rank 1 sub-sheaf of $T_{\tilde{\mathfrak{N}}}(-\log \tilde{\mathcal{D}})$, *i.e.* we have an induced foliated log-stack $(\tilde{\mathfrak{N}}, \tilde{\mathcal{D}}|_{\tilde{\mathfrak{N}}}, \tilde{\mathcal{A}})$, and so:

I.v.2 Definition Notations as above, then should it occur that $\tilde{\mathcal{A}}$ is an axis for the foliated log stack $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}})$ then it will be called the proper transform of the axis. Consequently, should all of the above hold, so that $(\mathbf{X}, \mathrm{Ax}) = (\mathcal{X}, \mathcal{D}, \mathcal{F}, \mathcal{A})$ is a smooth foliated stack with smooth tagged boundary admitting an axis, and, as such, will be referred to as a *controlled foliated log-stack*, then $\pi : (\tilde{\mathbf{X}}, \widetilde{\mathrm{Ax}}) \to (\mathbf{X}, \mathrm{Ax})$ with the proper transform axis, and $\tilde{\mathbf{X}}$ as per I.v.1. will be referred to as the proper transform of the controlled foliated log-stack. In particular,

I.v.3 WARNING In what follows we will never speak of the proper transform of a controlled foliated log-stack unless this further non-trivial condition on how the axis transforms is satisfied.

II. The Algorithm

II.i. The Local Invariants

Let $(\mathbf{X}, \mathrm{Ax})$ be a controlled foliated log-stack of dimension 3, with non-empty boundary, and let p be a closed point of the smooth stack $\mathcal{X}$. Suppose further that $p \in \mathcal{D} \cap \mathrm{NLC}(\mathcal{F})$ then in [P] 4.8 there has been defined,

(a) The *Newton Invariant* at p,

$$\mathrm{inv}(\mathbf{X}, \mathrm{Ax}, p) \in \mathbb{Z}^6_{\geq 0}$$

where the latter is to be understood as an ordered additive semi-group in the standard lexiographic ordering.

(b) The *weight vector* at p,

$$\omega_{\mathbf{p}} = (\omega_1, \omega_2, \omega_3) \in \mathbb{Z}^3_{\geq 0}$$

where ω_1 or ω_2, but not ω_3, may be zero, and the greatest common divisor of the non-zero ω_i's is 1.

(c) The *face order* at p,

$$\mu_p \in \mathbb{Z}.$$

More precisely the invariant was originally defined in the category of real manifolds with corners, and may be read from the Newton polygon associated to a vector field generating the foliation in an analytic neighbourhood of p, provided that the polygon is computed in an *adapted* coordinate system, §II.ii, infra, & op. cit. §3.1, and the Newton data is *stable*, op. cit. §4.1. Consequently, provided that the twin hypothesis of being adapted and stable are satisfied, the definition on any sufficiently small analytic étale neighbourhood of p proceeds mutatis mutandis on changing the ground field from $\mathbb{R}$ to $\mathbb{C}$. The fact that it is possible, at least on sufficiently small analytic neighbourhoods of p, to satisfy the said twin hypothesis on real manifolds with corners is op. cit. 4.22. However, the proof only uses properties of rings of (analytically) convergent power series, and so, again, is valid mutatis mutandis on changing the ground field from $\mathbb{R}$ to $\mathbb{C}$. A prior the definitions might depend on coordinate patches, but this is exactly what the stable condition avoids, and we have,

II.i.1 Fact Notations as above, then the Newton Invariant, the weight vector, and the face order at p depend only on the germ of $(\mathbf{X}, \mathrm{Ax})$ at p in the analytic étale site of $\mathcal{X}$. In particular, and a fortiori, it is an étale local invariant in the (algebraic) étale site of $\mathcal{X}$.

proof Again, up to changing the ground field from $\mathbb{R}$ to $\mathbb{C}$, this proceeds mutatis mutandis à la op. cit. 4.20, since in the notation of op. cit. a change of adapted coordinates for stable Newton data must lie in the group $\hat{G}^i$ occurring in the aforesaid proof. This gives dependence only on the analytic germ at p which is better than dependence on only the strictly Henselian germ, since isomorphic strict Henselisations of local rings of complex schemes of finite type have isomorphic analytic local rings, but not conversely. □

II.ii Local Resolution

Again let $p \in \mathcal{D}$ be a closed point of the non-log canonical locus of a 3-dimensional controlled foliated log-stack $(\mathbf{X}, \mathrm{Ax})$ with non-empty boundary, then the definition of an adapted coordinate system, x, y, z on an étale neighbourhood $\Delta \to \mathcal{X}$ previously alluded to respects the tagging and the axis, *i.e.*,

- $\frac{\partial}{\partial z}$ is a local generator of the axis;
- If $p \in \mathcal{D}$ and $\iota_p = [i]$ then $\mathcal{D}_i = \{x = 0\}$;
- If $p \in \mathcal{D}$ and $\iota_p = [i, j]$ (with $i > j$) then $\mathcal{D}_i = \{x = 0\}$ and $\mathcal{D}_j = \{y = 0\}$.

where ι_p is the support of the tagging at p, which necessarily has cardinality at most 2 by the definition of an axis, and, of course, x, y, z yield an isomorphism between Δ and its image in $\mathbb{C}^3$. As such, and in a manner which a priori depends not just on the coordinate system but even the neighbourhood Δ, we have weighted blow ups defined by filtrations,

$$F^k \mathcal{O}_{\Delta,p} = (x^a y^b z^c : a\omega_1 + b\omega_2 + c\omega_3 \geq k), k \in \mathbb{Z}_{\geq 0}$$

where $(\omega_1, \omega_2, \omega_3) = \omega_{\mathbf{p}}$. Nevertheless, as per II.i.1,

II.ii.1 Fact Let $\Delta \to \mathcal{X}$ and $\Delta' \to \mathcal{X}$ be étale neighbourhoods of p and, s, t the projections of $U = \Delta \times_{\mathcal{X}} \Delta'$ onto Δ and Δ' respectively, with F^k, and G^k

the filtrations associated to some, potentially wholly different, systems of stable adapted coordinates on Δ and Δ', then,

$$s^*F^k = t^*G^k.$$

proof The projections are étale, and U is a space so at any point $q \in U$ we can compare the coordinate systems (s^*x, s^*y, s^*z) and (t^*x', t^*y', t^*z') in the usual way. This change, however, is not arbitrary since both systems are stable, and, so, around q it takes the form,

$$t^*x' = s^*x\,u, \quad t^*y' = g(sx) + s^*y\,v, \quad t^*z' = f(s^*x, s^*y) + s^*z\,w$$

where the mapping associated to the quintuple (f, g, u, v, w) has exactly the properties of the same map with the same notations occurring in [P] 4.28, and, so, as ever proceeding mutatis mutandis over $\mathbb{C}$ rather than $\mathbb{R}$, we conclude from op. cit. $\square$

Manifestly, II.ii.1, allows us, at least in the analytic étale site of $\mathcal{X}$, to speak un-ambiguously of *the weighted blow up* of $(\mathbf{X}, \mathrm{Ax})$ at p. As it happens, by a systematic use of Henselian local rings, and appropriate algebraic integrability of the axis, this could have been achieved un-ambiguously in the (algebraic) étale site of $\mathcal{X}$, nevertheless, this was not done, so, in order to follow op. cit. as closely as possible, we will, for the moment, allow the possibility that such a weighted blow up may only be (convergent) analytic, and possibly not algebraic. Consequently we extend the definitions I.i.1, I.iii.1, I.iv.3, and I.v.2 to the analytic topology in the obvious way, so that we can even speak unambiguously of the smoothed weighted blow up at p and observe,

II.ii.2 Proposition For p a closed point of a controlled singularly foliated log-stack $(\mathbf{X}, \mathrm{Ax})$ of dimension 3 there exists an embedded open analytic sub-stack $\mathcal{U}$ around p such that if $(\mathbf{U}, \mathrm{Ax}) = (\mathbf{X}, \mathrm{Ax}) \times_{\mathcal{X}} \mathcal{U}$ is the restriction, *i.e.* base change every element of the quintuple to $\mathcal{U}$, then for p a non-log canonical point of the boundary there is a well defined smoothed weighted blow up,

$$\pi : \widetilde{\mathbf{U}} \to \mathbf{U}$$

Better still, for every closed point q of $\tilde{\mathcal{U}}$ lying over p which is not log-canonical (and necessarily lying in the boundary),

$$\mathrm{inv}(\widetilde{\mathbf{U}}, \widetilde{\mathrm{Ax}}, q) \ <_{\mathrm{lex}} \ \mathrm{inv}(\mathbf{U}, \mathrm{Ax}, p)$$

proof By the main theorem of [KM], p factors through an embedded open sub-stack of the form $[\Delta/G]$ for G a finite group acting, possibly with generic stabiliser, on an analytic polydisc Δ, so existence and well definedness are II.ii.1 and [P] 4.22, at least in so much as they refer to stable coordinates and weighted blow ups. The harder, and better still part, follows from the description of local coordinate patches on smoothed weighted blow ups in I.iv.4, and [P] 4.29 et sequel. Indeed, each closed point q of $\tilde{\mathcal{U}}$ has an étale neighbourhood of the form described in I.iv.4 with x_1 one of x, y, z as appropriate, and x_2, x_3 the other 2.

Such étale neighbourhoods correspond precisely, to the x, y, and z directional blow ups of [P] 4.9, and consequently the fact the the invariant decreases according to the lexiographic ordering in $\mathbb{Z}^6_{\geq 0}$ on these étale neighbourhoods again proceeds verbatim as per op. cit. The invariant, however, is, II.ii.1, étale local, so we are done. □

II.iii Equireducibility

Proposition II.ii.2 is more than adequate to yield a global resolution theorem, but in order to achieve one which is dynamically optimal we introduce some more definitions in dimension 3, to wit:

II.iii.1 Definition Let $\Delta \to \mathcal{X}$ be an étale neighbourhood of a closed non-log canonical point p of a controlled foliated log-stack $(\mathbf{X}, \mathrm{Ax})$ with x, y, z an adapted coordinate system on Δ at p, then we will say that p is smooth if $\mathrm{NLC}(\mathcal{F})$ is *smooth* at p, and in addition we will say that the coordinate system is *smoothly adapted* at p if $y = z = 0$ cuts out $\mathrm{NLC}(\mathcal{F})$ at p. Finally we define a point $p \in \mathrm{NLC}(\mathcal{F})\backslash\mathcal{D}$ to be equireducible if there is a smoothly adapted coordinate system at p with respect to which the Newton polyhedra of a local generator of the field is generic, and refer to [P] §5.3 for the precise sense in which the behaviour of the Newton polyhedra is to be considered generic.

The local invariants of II.i, most importantly the Newton Invariant and the face order, may then be extended to equireducible points following [P] §5.5. In particular the weight vector is of the form, $(0, \omega_2, \omega_3)$, and, at least locally yields a smoothed weighted blow up $\tilde{\Delta} \to \Delta$. Again, however, following the schema of II.ii.1 one checks that the proof of the independence from the étale neighbourhood and the coordinate system of the weighted blow up in op. cit. 5.9 continues to hold, and so we conclude,

II.iii.2 Fact For p an equireducible point (so, implicitly the dimension is 3) of a controlled singularly foliated log-stack $(\mathbf{X}, \mathrm{Ax})$ there exists an embedded open analytic sub-stack $\mathcal{U}$ around p such that if $(\mathbf{U}, \mathrm{Ax}) = (\mathbf{X}, \mathrm{Ax}) \times_{\mathcal{X}} \mathcal{U}$ is the restriction, then there is a well defined smoothed weighted blow up,

$$\pi : \widetilde{\mathbf{U}} \to \mathbf{U}$$

Such that for every closed point q of $\tilde{\mathcal{U}}$ lying over p which is not log-canonical (and necessarily lying in the boundary),

$$\mathrm{inv}(\widetilde{\mathbf{U}}, \widetilde{\mathrm{Ax}}, q\,) \;<_{\mathrm{lex}}\; \mathrm{inv}(\mathbf{U}, \mathrm{Ax}, p)$$

II.iv Distinguished vertices

We continue to study closed non-log canonical points p outwith the boundary of a controlled foliated log-stack of dimension 3. By way of property I.iii.1.(d) of an axis, and for ∂, A, generators of the foliation and the axis we can define an invariant,

$$h := \min\{k : A^k.\partial \neq 0 \ (\mathrm{mod}\ T_{\mathcal{A}})\}$$

where $A._-$ is Lie derivation. On the other hand, and quite generally, the possible changes of coordinates which preserve an adapted coordinate system x, y, z are of the form,

$$x' = f(x, y),\ y' = g(x, y),\ z' = uz + h(x, y)$$

for u a unit, and f, g with non-vanishing Jacobian. As such, for any integer $n \in \mathbb{N}$,

$$\mathcal{I}_k^{(n)} = (x^p y^q z^r | pn + qn + r \geq k),\ k \in \mathbb{N}$$

is always a well defined filtration of ideal sheaves in the étale topus of $\mathcal{X}$. With the particular choice of $n = h$, however, this filtration is actually $\mathcal{F}$ invariant, and defines a smoothed weighted blow up $\tilde{\mathcal{X}} \to \mathcal{X}$, *the distinguished vertex blowing up*, of log-discrepancy zero, and we assert,

II.iv.1 Fact Let $(\widetilde{\mathbf{X}}, \widetilde{\mathrm{Ax}}) \to (\mathbf{X}, \mathrm{Ax})$ be the above distinguished vertex blowing up of a controlled singularly foliated log stack of dimension 3 in a point p outwith the boundary divisor, then at every point of the exceptional divisor divisor the proper transform of the axis is again an axis.

proof This is basically [P] proposition 5.16, but since a slight change has occurred to accommodate the possibility of non-scheme like points at the initial stage of the algorithm, we give the details. In the first place, observe, more or less by construction, that at the unique singular point of the axis on the exceptional divisor the induced foliation is smooth, so, without loss of generality, we are reduced to considering the $x \neq 0$ patch of the blow up. Now write a generator of the foliation as,

$$\partial = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$$

then each of a, b, c has some order, say, α, β, γ along the exceptional divisor, and by a linear change of coordinates we can arrange that exactly one of α, β is h, and the other strictly greater. Now take coordinates at an étale patch around a closed point q of the exceptional divisor of the form,

$$x = \xi^h,\ y = \xi^h(\eta + Y),\ z = \xi(\zeta + Z)$$

for $Y, Z \in \mathbb{C}$, then the induced foliation in the exceptional divisor, around q, takes the form,

$$h(\tilde{b}\xi^{\beta-h} - \tilde{a}\xi^{\alpha-h}(\eta + Y))\frac{\partial}{\partial \eta} + (h\tilde{c}\xi^{\gamma-1} - \tilde{a}\xi^{\alpha-h}(\zeta + Z))\frac{\partial}{\partial \zeta}$$

where $a = \tilde{a}\xi^\alpha$, *etc.* As such, any positive dimensional component of the non-log-canonical locus of the induced foliation is necessarily a component of $\tilde{a} = 0$ or $\tilde{b} = 0$ according as $h = \alpha$ or $h = \beta$. Plainly, however, such a component cannot be left invariant by the foliation induced by $\mathcal{A}$, so we conclude. $\Box$

Now, while we have eschewed giving the definition of the sense in which the Newton polyhedron should be considered generic at equireducible points, it should be unsurprising that this condition is upper semi-continuous in the étale

site (be it algebraic or analytic) of our stack, albeit cf. [P]§5.7 for a proof in the étale analytic site. Equally unsurprisingly, any generic non-log canonical point outwith the boundary is equireducible, so there are only finitely many non-equireducible ones to which we may apply II.iv.1 sequentially, with a random choice of ordering of the non-equireducible points, to obtain as per op. cit. 5.22,
II.iv.2 Corollary Let $(\mathbf{X}, \mathrm{Ax})$ be a controlled singularly foliated log-stack of dimension 3, then there is a sequence of invariant smoothed weighted blow ups,

$$\mathbf{X} = \mathbf{X}_0 \leftarrow \mathbf{X}_1 \leftarrow \ldots \leftarrow \mathbf{X}_k = \widetilde{\mathbf{X}}$$

such that the foliation induced on $\tilde{\mathcal{X}}$ by the axis of $\mathcal{X}$ is again an axis, and every non-log canonical point outwith the boundary of $\widetilde{\mathbf{X}}$ is equireducible.

II.v Final assembly

All the tools are now in place for the final assembly of the key resolution from which all subsequent resolutions will be derived, *i.e.*
II.v.1 Theorem Let $(\mathbf{X}, \mathrm{Ax})$ be a controlled singularly foliated log-stack of dimension 3, then there is a sequence of smoothed weighted blow ups,

$$(\mathbf{X}, \mathrm{Ax}) = (\mathbf{X}_0, \mathrm{Ax}_0) \leftarrow (\mathbf{X}_1, \mathrm{Ax}_1) \leftarrow \ldots \leftarrow (\mathbf{X}_k, \mathrm{Ax}_k) = (\widetilde{\mathbf{X}}, \widetilde{\mathrm{Ax}})$$

such that each weighted centre is invariant by the induced foliation at each stage with its support contained in the non-log canonical case of the said stage, satisfies warning I.v.3, and the foliation singularities of $\widetilde{\mathbf{X}}$ are everywhere log-canonical.

The proof, or more accurately an appropriate review of [P]§5.9-5.12, will occupy the rest of this section. By II.iv.2, we may suppose that all points of the non-log canonical locus are either contained in the boundary, or are equireducible. In particular the local invariant,

$$\mathrm{inv}(\widetilde{\mathbf{X}}, \widetilde{\mathrm{Ax}}, p\,)$$

is well defined at every closed non-log canonical point p. Consequently, there is an embedded open substack $\mathcal{U}_p$ containing p, and a closed substack Y_p of $\mathcal{U}_p$ supporting a smoothed weighted blow up as found in II.ii.2 or II.iii.2. This data depends only on the closed point, so there is no additional global patching problem beyond those already faced in op. cit., *i.e.*,

- Extending to a smoothed weighted blow up over the Zariski closure of Y_p.
- Ensuring that this extension to $q \in \overline{Y}_p \backslash Y_p$ coincides with the centre previously defined at q.

To address these problems, we should, plainly, place ourselves in the stratum where the local invariant is maximal. Necessarily this is a closed substack by way of the upper semi-continuity of the invariant, op. cit. propositions 5.5 & 5.25, so that at the finite set, $\mathrm{Bad}(p)$ of closed points $q \in \overline{Y}_p \backslash Y_p$ where the centre at q may fail to coincide with that at p, the local invariant is again maximal. Regardless $\mathrm{Bad}(p)$ is well defined irrespectively of whether we are in the maximal strata or not, and we organise the combinatorics by way of a directed graph $T_P = (V, E)$ associated to a finite subset P, to be chosen, within the non-log canonical locus, where,

- The vertices v correspond to closed points which can be obtained by way of a finite chain p_i starting at $p_o \in P$, and $p_{i+1} \in \mathrm{Bad}(p_i)$.
- There is a directed edge $v \to w$ if for a chain p_i as above, there is some j such that $v = p_j$ and $w = p_{j+1}$.

Now the key point is,
II.v.2 Fact For any finite subset P the associated directed graph T_P is a finite directed tree.
proof As ever, we appeal to op. cit., this time lemma 5.26, corollary 5.30, lemma 5.33, and lemma 5.35. □
By way of clarification, let us make,
II.v.3 Remark The proof of II.v.2 is where the careful tagging introduced in I.i makes an essential appearance. Nevertheless, this would not have dealt with patching issues arising from local monodromy which have been treated by II.ii.2, II.iii.2, and II.iv.1.

From here, we conclude as follows: associated to the tree T_P there is the length L_P of the longest chain, the lexiographic maximum, $\mathrm{inv}(T_P)$, of the local invariant, inv, at vertices without descendants, and the set of closed points, $\mathrm{Loc}(T_P)$, corresponding to the vertices where this maximum is attained. All of which may be organised into a lexiographic invariant,

$$\mathrm{Mult}(T_P) = (L_P, \mathrm{inv}(T_P), \mathrm{Loc}(T_P)) \in \mathbb{Z}^8_{\geq 0}$$

It therefore remains to make an astute choice of P, and this is done by a close examination of the possible relations between components of the locus where the local invariant is maximal and the structure of the boundary divisor. This is carried out in op. cit. §5.10, and leads to a unique definition of an appropriate initial set $P = P_{\max}$ if the locus where the local invariant is maximal meets 3 components of the boundary. Otherwise one may have to choose a point from within the smooth locus of the maximal strata, albeit that all choices are equally good, and again this leads to the definition of $P = P_{\max}$. As such, we obtain our final lexiographic invariant,

$$\mathrm{Mult}(\mathbf{X}, \mathrm{Ax}) = (\max\{\mathrm{inv}(\mathbf{X}, \mathrm{Ax}, p) : p \in \mathrm{NLC}(\mathcal{F})\}, \mathrm{Mult}(T_{P_{\max}}))$$

Now, by construction, this invariant is associated to vertices of a tree without descendants, so it comes equipped with a globally well defined smoothed invariant weighted blow up,

$$(\widetilde{\mathbf{X}}, \widetilde{\mathrm{Ax}}) \to (\mathbf{X}, \mathrm{Ax})$$

albeit in a possibly non-connected centre, and we conclude by way of,

II.v.4 Fact Let things be as above then,

$$\mathrm{Mult}(\widetilde{\mathbf{X}}, \widetilde{\mathrm{Ax}}) < \mathrm{Mult}(\mathbf{X}, \mathrm{Ax})$$

proof Apply the above as per op. cit. Theorem 5.44. □.

III. Complements

III.i Canonical singularities

The deepest known applications of the main theorem II.v.1 arise from a better understanding of the relation between canonical and log-canonical singularities. As such, we re-visit §I.ii, with the notations therein, and observe,

III.i.1 Fact Let $(U, D, \mathcal{F})$ be a foliated germ of a smooth variety, then the following are equivalent,

(1) $(U, D, \mathcal{F})$ is terminal.

(2) $(U, D, \mathcal{F})$ is log-terminal.

(3) D is strictly invariant and $\mathcal{F}$ is smooth transverse to the generic point of Z.

proof The implication (1) ⇒ (2) is trivial. Next let I be the ideal of Z, ∂ a local generator of the foliation, and m the multiplicity of $\partial(I)$ along Z. As such, bearing in mind I.i.2, with, $\pi : \tilde{U} \to U$ the blow up in Z and v the valuation associated to the exceptional divisor, E, the canonical bundle of the induced foliation $\tilde{\mathcal{F}}$ satisfies,

$$K_{\tilde{\mathcal{F}}} = \pi^* K_{\mathcal{F}} - (m-1)E$$

Whence, if this were log-terminal, then, $m < 1$, so, infact $m = 0$, so, indeed (2) ⇒ (3). For the remaining implication, notice that by the convention I.i.2 we may, without loss of generality, suppose that D is empty, and we take x to be a function with $\partial(x) = 1$. Now, let E be the exceptional divisor associated to some valuation v centered on Z, with π a uniformiser, and $\tilde{\partial}$ a generator of the induced foliation around E in the usual sense, *i.e.* not following the convention I.i.2. Consequently, after multiplication by units, we may suppose, étale locally, that for some $n \in \mathbb{Z}$,

$$\partial = \pi^n \tilde{\partial}$$

while for an appropriate $m \in \mathbb{N}$, $x = \pi^m$. Whence,

$$0 \le v(\tilde{\partial}(\pi) = 1 - m - n$$

On the other hand since v is divisorial, it is also a valuation of the local ring completed in Z. Equivalently, no element of the completion in Z has infinite valuation with respect to v. The Frobenius theorem (or more correctly its proof) yields, however, that there is a non-zero, non-unital, function y in the completion with $\partial(y) = 0$. As such, if l is the valuation of this function, there is a unit, u, around E, such that,

$$0 = \pi^l \tilde{\partial}(u) + l\pi^{l-1} u \tilde{\partial}(\pi)$$

So infact, $v(\tilde{\partial}(\pi)) \geq 1$, $\epsilon(v) = 0$, and $n \leq -m \leq -1$. From, which,

$$a_{\mathcal{F}}(v) = -n + \epsilon(v) \geq 1 > \epsilon(v) = 0 \qquad \Box$$

As it happens, and rather unsurprisingly, this proposition remains wholly true should the germ only be $\mathbb{Q}$-foliated Gorenstein, and normal. Nevertheless, this isn't relevant to the applications of the main theorem, but rather the analogue of the above for canonical singularities. To this end, suppose we have a log-canonical singularity which is not canonical. As such, there is a divisorial valuation, v, of nil discrepancy, with $\epsilon(v) = 1$. Now consider the linearisation of a local generator ∂ of the foliation in,

$$\mathrm{End}\left(\mathfrak{m}_{U,Z}/\mathfrak{m}_{U,Z}^2\right)$$

As per I.ii.2, we can for convenience, pass to a strictly Henselian germ, and whence the linearisation possesses a Jordan decomposition $\partial_S + \partial_N$ into semi-simple and nilpotent parts. Better still, the filtration of the Zariski tangent space at Z by the order along v is, by hypothesis, ∂ invariant, so it is ∂_S and ∂_N invariant too. Furthermore, if F^n is the corresponding filtration of the strict local ring, inducing the aforesaid filtration $\bar{F}^n$ of the tangent space, then we have an exact sequence,

$$0 \to \left(F^{n+1} + \mathfrak{m}_{U,Z}^2\right)/F^{n+1} \to F^n/F^{n+1} \to \bar{F}^n/\bar{F}^{n+1} \to 0$$

together with an injection,

$$F^n/F^{n+1} \hookrightarrow \mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$$

where R is the valuation ring in the function field of the strict localisation, and $\mathfrak{m}_R$ its maximal ideal. Now, since $\epsilon(v) = 1$, ∂ is a $k(v)$ linear homothety of every, $\mathfrak{m}_R^n/\mathfrak{m}_R^{n+1}$, of the form $n\lambda$ for some fixed $\lambda \in k(v)$ independent of n. Better still, ∂ also induces a $k(Z)$ linear map of each F^n/F^{n+1}, so in fact, a homothety, whence on each $\bar{F}^n/\bar{F}^{n+1}$, ∂ is a homothety. Up to multiplication by a unit, however, these homotheties are multiplication by n, so we deduce,

III.i.2 Fact If $(U, D, \mathcal{F})$ is a foliated germ of a smooth variety with a log-canonical singularity which is not canonical then the linearised action of a local generator of the foliation on the Zariski tangent space at Z is semi-simple, with, after multiplication by a unit, positive integer eigenvalues. In particular D must be strictly invariant, so, without loss of generality empty.

Now, while this condition is necessary, and whence a wholly adequate criteria to pass from a log-canonical resolution to a canonical one, it is not actually sufficient since there are finitely many possibilities for resonances amongst the eigenvalues, and in fact,

III.i.3 Fact Let $(U, D, \mathcal{F})$ be a foliated germ of a smooth variety with a log-canonical singularity then, in fact, it is canonical unless it is *radial*, *i.e.* in the completion of the local ring at Z, there are coordinates $x_1, \ldots, x_r$ cutting out Z, such that the foliation has a generator of the form,

$$n_1 x_1 \frac{\partial}{\partial x_1} + \ldots + n_r x_r \frac{\partial}{\partial x_r} + D, \; n_i \in \mathbb{N}$$

where D is a derivation of a quasi-coefficient field with coefficients in I_Z.
proof If all the eigenvalues were 1, then radial is equivalent to semi-simplicity of the linearisation, while blowing up in Z yields a valuation of nil discrepancy and $\epsilon(v) = 1$. In general one reduces to this case by induction on the height of the eigenvalues and the Euclidean algorithm, cf. III.ii.2 □

These explicit descriptions of terminal, log-terminal, *etc* singularities have an important corollary,
III.1.4 Corollary Let $\nu : (\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{\mathcal{F}}) \to (\mathcal{X}, \mathcal{D}, \mathcal{F})$ be the Gorenstein covering stack of a normal $\mathbb{Q}$-Gorenstein foliated log-stack then the singularities of the former are terminal, log-terminal, canonical, respectively log-canonical iff they are so of the latter.
proof The if direction is general nonsense valid in any remotely sane category. To prove the only if direction, we require,
III.1.5 Claim Let $(X', \mathcal{F}', E') \to (X, \mathcal{F}, E)$ be a galois covering of foliated normal divisorial germs, then $\epsilon(E') = \epsilon(E)$.
which, indeed, follows by direct calculation. Of itself this implies the equivalence of the conditions for canonical, and log-canonical, while the terminal and log-terminal cases follow by observing that in the proof of III.i.1 we've established that ϵ is zero on every valuation centered on a terminal, equivalently log-terminal, singularity. □

At which point, we may usefully note that we've proved for terminal singularities,
III.1.6 Sub-corollary If $(U, D, \mathcal{F})$ is a foliated germ with canonical singularities, then for every divisorial valuation, v, centred on it, $\epsilon(v) = 0$.
proof Quite generally a divisorial valuation may be resolved by a chain of blow ups in its centres on the successive elements of the chain. Now either such centres remain invariant, and we're done since the singularity is canonical, or they're generically transverse, and we conclude by the terminal case. □

III.ii Canonical resolutions

We will require to extend III.1.2 from generic points of log-canonical singularities to their closure. To this end let I be any sheaf of ideals containing the singular ideal of the foliation, then any local generator of the foliation lies in,

$$I \otimes_{\mathcal{O}_{\mathcal{X}}} T_{\mathcal{X}}$$

Furthermore any local derivation, D, defines an $\mathcal{O}_{\mathcal{X}}/I$ liner mapping,

$$D : I/I^2 \to \mathcal{O}_{\mathcal{X}}/I$$

Whence a local generator of the foliation defines a tensor,

$$I/I^2 \otimes_{\mathcal{O}_{\mathcal{X}}/I} \mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}/I}(I/I^2,\, \mathcal{O}_{\mathcal{X}}/I)$$

and so, in particular, we may take symmetric functions to obtain sections,

$$\sigma_n \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/I \otimes_{\mathcal{O}_{\mathcal{X}}} K_{\mathcal{F}}^{\otimes n})$$

for n at most the dimension of $\mathcal{X}$. Consequently, while it's functorially slightly incorrect to speak about the variation of eigenvalues of linearisations of the foliation along the singular locus, we may do so by abuse of language by identifying this notion with the linear system on an appropriate power of $K_{\mathcal{F}}$, which, for log canonical singularities is well defined on the whole singular locus, and we observe,

III.ii.1 Fact Let Z be a component of the singular locus of a foliated log-stack $(\mathcal{X}, \mathcal{D}, \mathcal{F})$ with log-canonical singularities such that the generic point of Z is not a canonical singularity then Z is a smooth connected component of the singular locus with constant eigenvalues.

proof By III.i.3 the linear system given by suitable powers of symmetric functions is generically constant on Z, and whence constant everywhere. Now take a closed point z of Z, and observe that the above tensor naturally specialises to an endomorphism of,

$$I/I + \mathfrak{m}_{\mathcal{X}}(z)^2$$

which fits into an exact sequence,

$$0 \to I/I + \mathfrak{m}_{\mathcal{X}}(z)^2 \to \mathfrak{m}_{\mathcal{X}}(z)/\mathfrak{m}_{\mathcal{X}}(z)^2 \to \mathfrak{m}_Z(z)/\mathfrak{m}_Z(z)^2 \to 0$$

As such, if we profit from the log-canonical nature of the singularity at z to fix some eigenvalue equal to 1, then the characteristic polynomial of the linearised foliation at the tangent space of z has exactly the co-dimension of Z non-zero roots. Furthermore if C is a generic curve through z, then we may form the saturated sub-sheaves of $I/I^2|_C$ which are the eigenspaces at the generic point of C, and since C is a curve, these are actually sub-vector bundles of $I/I^2|_C$. Consequently the linearisation of the foliation at the tangent space of z is semi-simple, and the classical theory of Jordan forms of vector fields, cf. III.iii.1, shows that we have semi-simplicity along all of Z. $\square$

Now we can apply this in the obvious way to obtain,

III.ii.2 Resolution Let $(X, \mathcal{F})$ be a smooth projective foliated 3-fold then there is a sequence of smoothed weighted blow ups,

$$X = \mathcal{X}_0 \leftarrow \mathcal{X}_1 \leftarrow \ldots \leftarrow \mathcal{X}_k = \tilde{\mathcal{X}}$$

in the 2-category of algebraic stacks (with projective moduli) such that at each stage the weighted centres are invariant by the induced logarithmic foliation, and the foliation $\tilde{\mathcal{F}}$ on $(\tilde{\mathcal{X}}, \tilde{\mathcal{E}})$, for $\tilde{\mathcal{E}}$ the simple normal crossing total exceptional divisor, has canonical foliation singularities.

proof For log-canonical resolution this follows from I.iii.2 and the main theorem II.v.1. As such, without loss of generality, let us say that $\mathcal{X}_0$ has log-canonical singularities, and we may, by III.ii.1, identify the locus which is not canonical with appropriate (stack) smooth connected components of the singular locus which are disconnected from any exceptional divisor which may have been introduced, and which is not strictly invariant. Consequently, we may for free (more accurately, applying the algorithm of [BM]) suppose that the irreducible components of the exceptional divisor not vanishing on a given component of the non-canonical locus form a system of coordinate when they intersect. As such, if Z is one such component we may blow up in it without disturbing the normal crossing condition on the exceptional divisor. On the other hand, if we normalise the eigenvalues along it so that they are positive integers n_i without common denominators, such that on a local patch the corresponding eigenvectors were given by x_i, then after blowing up in Z the only candidate for a non-canonical singularity occurs where the proper transforms of the x_j with $n_j = m$, say, minimal, amongst the n_i's, cross the exceptional divisor. Necessarily we're done unless this locus again satisfies III.ii.1, and should it do so the eigenvalues, without multiplicity, are $n_i - m$, for $n_i > m$, and m itself. Consequently, the sum,

$$\sum_i n_i$$

of the eigenvalues must go down, and this eventually terminates when the resulting foliation is singular along the entire exceptional divisor. Better still not only is this exceptional divisor of discrepancy -1, but the above considerations with $m = 1$ even show that it is everywhere transverse to the induced and saturated in co-dimension 2 foliation around it. □

III.ii.3 Log-Resolution Let $(X, D, \mathcal{F})$ be a smooth projective foliated 3-fold with boundary then there is a sequence of smoothed weighted blow ups,

$$(X, D) = (\mathcal{X}_0, \mathcal{D}_0) \leftarrow (\mathcal{X}_1, \mathcal{D}_1) \leftarrow \ldots \leftarrow (\mathcal{X}_k, \mathcal{D}_k) = (\tilde{\mathcal{X}}, \tilde{\mathcal{D}})$$

in the 2-category of algebraic log-stacks (with projective moduli) such that at each stage the weighted centres are invariant by the induced logarithmic foliation with support in the non-canonical locus, and the foliation $\tilde{\mathcal{F}}$ on $\tilde{\mathcal{X}}$ has canonical foliation singularities

proof Observe that only the non-invariant components are relevant to the statement, and we proceed by induction starting from the empty initial boundary case III.ii.2, so, without loss of generality we may suppose that $(\mathcal{X}_0, \mathcal{F}_0)$ has canonical singularities. Now for f a local equation of a divisor without invariant components, and ∂ a generator of a foliation with canonical singularities, the locus where we fail to have an induced log-foliation with canonical singularities is cut out by the ideal,

$$(f, \partial(f))$$

or, equivalently, the tangencies of $\mathcal{F}$ to D, so, inter alia the singularities of D. Consequently, in the first instance consider trying to smooth D, respecting, and including any non-strictly invariant components of the exceptional divisor that may have been introduced in III.ii.3. To do this, independently of $\mathcal{F}$, [BM] provides a perfectly good étale local algorithm to yield a total transform which is simple normal crossing. On the other hand by III.1.6, all exceptional divisors introduced by such a resolution are strictly invariant, so if the centres of such a procedure are admissible in the sense of the above log-resolution claim, then such an algorithm will actually smooth the boundary. The centres, however, are singularities of the boundary, so a fortiori tangencies with the non-invariant components, unless they are a singularity of the proper transform of D together with the exceptional divisor. Now, blow ups in points are always admissible, so by the corresponding result in dimension 2, we can harmlessly do some extra blow ups to render such centres admissible as well. Consequently, we reduce to the case that every non-invariant component of the boundary is smooth, so we can appeal to I.iii.2, and II.v.1 again. □

III.iii Reduction of Monodromy

We will require a classification of $\mathbb{Q}$-Gorenstein log-canonical singularities of foliations which are not actually Gorenstein. To this end, recall,

III.iii.1 Revision Let A be a complete local ring with algebraically closed residue field k and maximal ideal $\mathfrak{m}$ such that, $\dim_k(\mathfrak{m}/\mathfrak{m}^2) < \infty$ then for $\partial \in \mathrm{Der}_k(A)$ singular there is a Jordan decomposition,

$$\partial = \partial_S + \partial_N$$

into semi-simple and nilpotent parts. In particular, if A is regular, there are functions $x_1, \ldots, x_r \in \mathfrak{m}$ forming a k-basis mod $\mathfrak{m}^2$ such that,

- $\exists \Lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$ for which,

$$\partial_S = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \ldots + \lambda_r x_r \frac{\partial}{\partial x_r}$$

  .

- The nilpotent field may be written as,

$$\sum_{1 \leq i \leq r} x_i \frac{\partial}{\partial x_i} \sum_{Q_i = (q_{i1}, \ldots, q_{ir})} a_{Q_i} x_1^{q_{i1}} \ldots x_r^{q_{ir}}$$

  where in the inner sum, $q_{ij} \in \mathbb{Z}_{\geq 0}$ unless $j = i$ in which case -1 is also permitted, while for the standard inner product, $\Lambda . Q_i = 0$, and all i.

Now suppose a finite group G acts on A, fixing k together with the foliation defined by a singular field ∂, and that the characteristic is zero (or, more generally, the cardinality of G is prime to the characteristic of k) then, without loss of generality, we may suppose that the group acts as,

$$\partial^\sigma := \sigma \partial \sigma^{-1} = \chi(\sigma) \partial, \ \sigma \in G$$

for χ a character of G. Consequently, with the notations of III.iii.1, $\partial_S^\sigma + \partial_N^\sigma$ is a Jordan decomposition of ∂^σ with eigenvectors x_i^σ, and, indeed,

$$\partial_S^\sigma = \chi(\sigma)\partial_S, \text{ and } \partial_N^\sigma = \chi(\sigma)\partial_N$$

for all $\sigma \in G$. From which, x_i^σ is again an eigenvalue of ∂_S but with eigenvector $\chi(\sigma)^{-1}\lambda_i$, and we may, without loss of generality, suppose that we have an induced permutation action by a cyclic group on a set of distinct linear eigenspaces, with non-zero eigenvalues. As such, the order of the character is at most r. Furthermore, changing basis as necessary, we may equally suppose that the action of G is a cyclic permutation action on the eigenvectors with non-nil eigenvalue. Plainly, there is absolutely no difficulty in proceeding from here to a complete enumeration in all dimensions, however, let us concentrate on the immediate case of interest, *viz:* 3-folds, where the number of possibilities are limited to,

III.iii.2 Case A The character has order 3, and should it be faithfull the group action is given by,

$$(x, y, z) \mapsto (z, x, y)$$

while the semi-simple part of a generator of the foliation may be taken as,

$$\partial_S = x\frac{\partial}{\partial x} + \zeta y\frac{\partial}{\partial y} + \zeta^2 z\frac{\partial}{\partial z}$$

for ζ a primitive cube root of unity, together with a nilpotent part of the form,

$$\partial_N = a(xyz)x\frac{\partial}{\partial x} + a(xyz)\zeta y\frac{\partial}{\partial y} + a(xyz)\zeta^2 z\frac{\partial}{\partial z}$$

for a a formal function of one variable.

proof Just apply the above considerations in conjunction with III.iii.1. $\square$

As a consequence, observe,

III.iii.2.bis Corollary Let $\tilde{\mathfrak{X}} \to \mathfrak{X} = \mathrm{Spf}A$ be the blow up in the origin, then the induced $\mathbb{Z}/3$ action for a faithfull character is transitive at the foliation singularities of the modification. In fact it cyclically permutes the singularities among themselves.

proof In the standard x, y, z coordinates on $\mathbb{P}^2$, the foliation singularities are at $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$, and the fixed points at $[1, \theta, \theta^2]$, for $\theta^3 = 1$. $\square$

The remaining case, however, is rather more interesting, *viz:*

III.iii.2 Case B The character has order 2, and if the charcter is faithfull the group action may be written as,

$$(x, y) \mapsto (y, x), \ z \mapsto -z$$

so that the semi-simple part of a generator of the foliation may be taken as,

$$\partial_S = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

together with a nilpotent part of the form,

$$\partial_N = a(xy, z)x\frac{\partial}{\partial x} - a(xy, -z)y\frac{\partial}{\partial y} + c(xy, z)\frac{\partial}{\partial z}$$

where a, c are formal functions of two variables with a arbitrary, and c even, *i.e.* $c(xy, -z) = c(xy, z)$, in z, non-unital.
proof Again, apply the previous considerations in conjunction with III.iii.1. □

Manifestly this singularity comes equipped with an invariant curve, $x = y = 0$, and if A were the completion of a regular ring of essentially finite type, it may very well occur that we have,
III.iii.3.bis Possibility A is the completion in a closed point of the germ of a foliated algebraic variety $(X, \mathcal{F})$ with $\mathbb{Z}/2$ action, and the curve $x = y = 0$ is not algebraic, or even just not analytically convergent. In particular, should this occur, the singularity is isolated, and for $(Y, \mathcal{G})$ the foliated quotient variety under the $\mathbb{Z}/2$ action, there is no birational modification which is Gorenstein and canonical around the proper transform of the curve.
proof The singular locus of the completion is the completion of the singular locus, so the singularity must be isolated. As such, suppose to the contrary there were such a modification $\pi : \tilde{Y} \to Y$, so that supposing for the moment that the formal curve actually defines a valuation v of the function field, we let ∂ be a local generator of the foliation around the centre of v on X. Whence for every meromorphic function f on X,

$$v(\partial f) > v(f)$$

On the other hand, if I is any ideal in a neighbourhood of the centre, then for some sufficiently large n,

$$J := (I, \partial(I), \partial^2(I), \ldots, \partial^n(I))$$

is a ∂ invariant ideal, so by [BM], there is a modification $\rho : Z \to X$ by a sequence of blow ups in ∂ smooth invariant centres, and we assert,
Claim The above resolution $Z \to X$ is a resolution of I around the centre of v on Z.
sub-proof Indeed, $\rho^{-1}(J)$ is a Cartier divisor with simple normal crossings, defined by $f = 0$ for some $\rho^* f$, and $f \in \partial^n(I)$ for some n. However, $v(\partial(f)) > v(f)$ for all meromorphic functions, so $n = 0$. □

Now if we apply this to the situation in hand, we may construct by a sequence of blow ups in smooth invariant centres a modification,

$$\rho : \tilde{X} \to X$$

such that around the centres of v we have a diagram,

$$\begin{array}{ccc} (X, \mathcal{F}) & \xleftarrow{\rho} & (\tilde{X}, \tilde{\mathcal{F}}) \\ \downarrow \lambda & & \downarrow \lambda \\ (Y, \mathcal{G}) & \xleftarrow{\pi} & (\tilde{Y}, \tilde{\mathcal{G}}) \end{array}$$

in which every entry has canonical singularities, while the left and upper arrows are unramified in the foliation direction, whence so are the other two. In particular,

$$K_{\tilde{\mathcal{G}}} = \pi^* K_{\mathcal{G}}$$

This would, however, for example, for a faithfull character force the function c of III.iii.2 case B to be odd in z, and more generally descend $K_{\mathcal{F}}$ at v to Y, which is nonsense.

It may, of course, happen that the curve doesn't define a valuation, but since it's formal, it factors through a unique irreducible invariant divisor D on which it does define a valuation, v_D, say. As such, we have an invariant discrete rank 2 valuation, w, with valuation ring,

$$R = \{f \in k(X) | ord_D(f) > 0, \text{ or, } ord_D(f) = 0, \text{ and, } v_D(f) \geq 0\}$$

and replacing v by w, we may proceed as above. □

Now let us apply this to obtain the main result of this section, *viz:*

III.iii.4 Fact Let $(X, D, \mathcal{F})$ be a foliated smooth log-3-fold then in the 1-category of foliated log-3-folds with quotient singularities there is a sequence of modifications,

$$(X, D, \mathcal{F}) = (X_0, D_0, \mathcal{F}_0) \leftarrow (X_1, D_1, \mathcal{F}_1) \leftarrow \ldots \leftarrow (X_k, D_k, \mathcal{F}_k) = (\tilde{X}, \tilde{D}, \tilde{\mathcal{F}})$$

where each modification is a weighted blow up in a foliated log-invariant centre, and the final model has canonical singularities. In addition, at points of the final model where the ambient space is not smooth, we have exactly one of the following possibilities,

- The foliation singularity is terminal, and for the Vistoli covering stack $\tilde{\mathcal{X}} \to \tilde{X}$ the induced foliation is everywhere transverse to the corresponding non-scheme like points.
- The singular point is precisely the $\mathbb{Z}/2$ quotient singularity of III.iii.3.bis.

proof At the price of allowing quotient singularities on X_0 we may by III.ii.3 and III.i.4 suppose $(X, D, \mathcal{F})$ has canonical singularities, and we denote by $\mathcal{X}^g \to X_0$, and $\mathcal{X}^v \to X_0$ its Gorenstein and Vistoli covering stack respectively. To obtain the former from the latter around a closed point x, observe that the monodromy of the former is infact a character of the latter. Indeed, it is precisely the character occurring in III.iii.1 at singular points of $\mathcal{F}$, since being an eigenfunction for the group action is a functor of finite type, and everything is étale local. Now by [BM] $\mathcal{X}^g$ admits a smooth strictly invariant resolution, so, without loss of generality, we may suppose that the Vistoli covering stack has at most cyclic monodromies. Better still, if,

$$x \mapsto g(x), \; x \mapsto v(x)$$

are the upper semi-continuous functions on closed points (identified with the same on the moduli) corresponding to the orders of the Gorenstein and Vistoli covering stack then at every point $\tilde{x}$ of the above resolution of $\mathcal{X}^g$ over x,

$$g(\tilde{x}) \le v(\tilde{x}) \le g(x)$$

Consequently, we may by sufficient repetition of the above, eventually suppose that the Vistoli and Gorenstein covering stacks coincide.

Applying this at singular points, we may therefore suppose that the characters occuring in III.iii.2 Cases A & B are faithfull. Whence, we may apply III.iii.2.bis to eliminate $\mathbb{Z}/3$ monodromy at singular points, while non-isolated $\mathbb{Z}/2$ monodromies follow similarly, *i.e.* blow up in the non-scheme like locus (infact, if the reader is paying attention the latter is really the surface case of the former, and the general structure of $\mathbb{Z}/n$ monodromies in dimension n). The case of isolated $\mathbb{Z}/2$ mondoromies, with the curve $x = y = 0$ of III.iii.3.bis formal has already been discussed, and, should it occur, which, by the way, III.iv.1, it does, it has been proved that the monodromy cannot be reduced. Otherwise, the curve in question is algebraic, and since it is foliation and $\mathbb{Z}/2$ invariant, we may, after globally smoothing it, legitimately blow up in it to kill the monodromy at the foliation singularities.

There may also be further monodromy at smooth points. Here the coincidence of the Vistoli and Gorenstein covering stacks about a geometric point x yields a local generator, ∂, such that for G_x the local monodromy,

$$\partial^\sigma = \chi(\sigma)\partial,\ \sigma \in G_x$$

for some faithful character χ. On the other hand, we can find a function ξ on an étale analytic neighbourhood of x such that $\partial(\xi) = 1$, so, G_x also acts faithfully on the divisor $\xi = 0$. Whence, ξ pertains to the ideal of non-scheme like points, and so the foliation is, indeed, everywhere transverse to the same. □

III.iv Optimality and Minimality

To discuss the optimality of III.iii.4 we first translate the possibility III.iii.3.bis into its manifestation on a smooth model of the $\mathbb{Z}/2$ quotient variety. Plainly we only need to do this about the proper transform of the curve $x = y = 0$ that occurs therein, which we'll slightly abusively denote by v even though we haven't yet decided whether it's an honest valuation, or factors through a divisor on which it is a valuation, or is algebraic, so that in the latter two possibilities we would have to allow the value infinity on functions. In any case, by a single blow up in the fixed locus of the group action we resolve the quotient singularity, and we find a priori formal coordinates ξ, η, ζ around the centre of v on the smoothed quotient such that the foliation is given by a generator of the form,

$$D = (\zeta\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}) + B(w,\zeta)\zeta(\xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta}) + C(w,\zeta)(2\zeta\frac{\partial}{\partial\zeta} - \eta\frac{\partial}{\partial\eta})$$

where $\zeta = 0$ is the equation of the exceptional divisor, $w = \xi^2 - \eta^2\zeta$ is the defining equation of a Whitney umbrella, $\xi = \eta = 0$ the defining equations of v, and B, C are arbitrary formal functions of two variables, except that C is a non-unit not divisible by w. Indeed in the original notation of III.iii.2 Case B, and up to diving w by 4, put $a = \alpha + z\beta$, for α and β even in z, then $B(1+\alpha) = \beta$, and $C(1+\alpha) = c$. Now consider the particular choice of $B = \beta \in \mathbb{C}\backslash(1/2)\mathbb{Z}_{<0}$, $C = -\zeta$, giving rise to a family of fields, D_β, and suppose, as we quite legitimately may that the coordinate system ξ, η, ζ is defined in the Henselisation of the local ring, or even just analytically convergent, so that we have a perfectly algebraic perturbation,

$$D_{\beta,\lambda} = D_\beta + \lambda\zeta\frac{\partial}{\partial\eta}$$

for $\lambda \in \mathbb{C}$. To such a field with $\lambda \neq 0$ there is a purely formal coordinate change whereby the perturbation term disappears. Specifically, $\hat{\xi} = \xi - \lambda X(\zeta)$, and $\hat{\eta} = \eta - \lambda Y(\zeta)$, where,

$$X(\zeta) = \sum_{n\geq 1} c_n\zeta^n,\ Y(\zeta) = \sum_{n\geq 1}(\beta+n)c_n\zeta^n,\ c_n = \prod_{i=0}^{n}(\beta+i-1)(\beta+i-\frac{1}{2})$$

gives $D_{\beta,\lambda}$ as a field of the form D_β in $\hat{\xi}, \hat{\eta}, \zeta$ coordinates, so, perhaps better $\hat{D}_\beta$. Consequently, to summarise,

III.iv.1 Fact Possibility III.iii.3.bis really occurs, so in particular it is in general impossible to have a canonical or even log-canonical resolution of a foliated 3-fold in the 1-category of varieties or algebraic spaces without the $\mathbb{Z}/2$ quotient singularity described therein, equivalently for the ambient object to be smooth one must work in the 2-category of algebraic stacks with $\mathbb{Z}/2$ monodromy, so that at foliation singularities III.iii.4 is absolutely optimal from the point of view of reduction of monodromy.

All of which calls for,

III.iv.2 Historical remarks The above example, and indeed the entire discussion of the last two sections traces itself to F. Sanz, [S], prompted by an intution of F. Cano resulting from his formal local uniformisation theorem, [C1]. Indeed, while lacking the general statement of III.iii.3.bis, [S] describes a 3-complex parameter family of examples that cannot be resolved by blowing up in smooth convergent invariant centres, of which the above chosen example is a co-dimension 1 subspace. The generality of III.iii.3.bis, however, should leave little doubt that the family of examples in question is infinite dimensional.

III.iv.2.bis Mathematical remarks The omni-presence of the Whitney umbrella in the above general formula is rather curious. Indeed, not only does it occur as an argument of the functions U, B and C, but each of the fields that these functions multiply leave the umbrella invariant. It is, therefore, tempting to conclude that a monodromy free log-canonical resolution of possibility III.iii.3.bis must necessarily also resolve the umbrella, and whence eventually blow up in the in-admissible centre corresponding to our purely formal curve. Nevertheless, there doesn't appear to be a mechanism to force this beyond the general monodromy considerations of §III.iii, which indeed provides a complete theory of such examples for $\mathbb{Z}/m$ monodromies in all dimension for any m less than the ambient dimension.

As such, there remains to discuss the optimality of III.iii.4 at the terminal but non-scheme like points of the necessarily smooth gorenstein covering stack. Now plainly, one cannot kill the monodromy at terminal everywhere non-scheme like points by way of an invariant modification. Equally plainly, it's a trivial thing to do by way of non-invariant blowing up. Indeed it's simply a matter of resolving the quotient, or better reducing the monodromy, of foliated classifying stacks described by a generator and action of the form,

$$\partial = \frac{\partial}{\partial x},\ x \mapsto \zeta x,\ y \mapsto \zeta^b y,\ z \mapsto \zeta^c z$$

for ζ a primitive ath root of unity, and a, b, c without common divisor. Somewhat less plainly, perhaps, this is an absolutely mindless thing to do since for a negligble improvement in the ambient space, it replaces smooth foliated points by singular ones. Infact, and increaingly less plainly, even carrying out the monodromy reduction steps of §III.iii is mindless. As such, the purpose of this section is really to illustrate the necessity for working in the 2-category of algebraic stacks. Indeed, for a truly optimal resolution, what one should really do is run the minimal model algorithm of [M2], and thus, almost certainly create more monodoromy. For example, if one were to kill the monodromy at terminal everywhere transverse non-scheme like points, then running the minimal model algorithm would create at least as much monodromy as the original situation, which would, in fact, be the exact result of running the algorithm relative to the said situation, while, quite generally, the minimal model algorithm preserves, and creates, the terminal everywhere transverse condition at non-scheme like points. In any case, after achieving II.5.4, III.ii.2, or III.ii.3 as appropriate to the given situation, the algorithm should be run relatively or absolutely, again, according to the situation, so for example one has,

III.iv.3 Fact Let $(X, D, \mathcal{F})$ be a smooth projective foliated 3-fold, then there is a modification $\pi : (\tilde{\mathcal{X}}, \tilde{\mathcal{F}}) \to (X, \mathcal{F})$ in the 2-category of smooth algebraic stacks, isomorphic outwith the non-canonical locus of $(X, D, \mathcal{F})$, such that $K_{\tilde{\mathcal{F}}} - \pi^* K_{\mathcal{F}}$ is relatively nef.

We will, however, refrain from describing this as optimal, since [M3] has defined a notion of a 'canonical', or better in the current context to avoid confusion, absolutely minimal, resolution, which can actually have an ambient space whose singularities aren't even quotient singularities. Currently, however, a canonical as opposed to minimal model theorem hasn't been proved since in one of life's ironies foliated flops are actually harder than foliated flips. In any case, it should be clear that trying to reduce the monodromy of the resolution is the opposite of help, and if one doubts this, a short reflection of what it means to do this in the rather simple case of the moduli stack of curves should clear up any doubt.

References

[BM] Bierstone, E. & Milman, P. Canonical desingularisation in characterictic zero by blowing up the maximum strata of a local invariant, *Invent. Math.* **128** (1997), 207-302.

[C1] Cano, F. Desingularization strategies for three-dimensional vector fields *Lecture Notes in Mathematics* **1259**, Springer-Verlag, Berlin, 1987.

[C2] Cano, F. Reduction of the singularities of codimension one singular foliations in dimension three, *Ann. of Math.* **160** (2004) 907-1011.

[C&] Cano, F., Roche, C., Spivakovsky, M. Birational reductions of singularities of vector fields in dimension 3, pre-print, Universidad de Valladolid, to appear.

[E&] Eyssidieux, P., Guedj, V., & Zeriahi, A. Singular Kähler-Einstein metrics, *J. Amer. Math. Soc.* **22** (2009) 607-639.

[F] Fulton, W. Intersection Theory, Springer-Verlag, Berlin, 1984.

[G] Gromov, M. Metric Structures for Riemannian and Non-Riemannian Spaces, Birkhauser.

[KM] Keel, S. & Mori, S. Quotients by groupoids, *Ann. of Math.* **145** (1997), 193-213.

[M1] McQuillan, M. Canonical models of foliations, *Pure and Appl. Math. Quarterly* **4** (2008), 877-1012.

[M2] McQuillan, M. Semi-stable reduction of foliations *IHES pre-print* (2005) IHES/M/05/02.

[M3] McQuillan, M. Uniform Uniformisation *IHES pre-print* (2005) IHES/M/05/03.

[P] Panazzolo, D. Resolution of singularities of real analytic vector fields in dimension three, *Acta Math.* **197** (2006) 167-289.

[R] Reid, M. Young person's guide to canonical singularities, in Algebraic Geometry, Bowdoin 1985, ed. S. Bloch, Proc. of Symposia in Pure Math. 46, A.M.S. (1987), vol. 1, 345–414.

[S] Sanz, F. Personal communication.

[Se] Seidenberg, A. Reduction of the singularities of the differential equation Ady = Bdx, *Am. J. of Math.* **90** (1968) 248-269.

[V] Vistoli, A. Intersection theory on algebraic stacks and on their moduli spaces, *Invent. Math.* **97** (1989) 613-670.