The QCD β-function from global solutions to Dyson-Schwinger equations

Guillaume VAN BAALEN, Dirk KREIMER, David UMINSKY and Karen YEATS

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Juin 2009

IHES/P/09/25

The QCD β-function from global solutions to Dyson-Schwinger equations

Guillaume van Baalen[*], Dirk Kreimer[†], David Uminsky[*], and Karen Yeats[*]

June 9, 2009

Abstract

We study quantum chromodynamics from the viewpoint of untruncated Dyson–Schwinger equations turned to an ordinary differential equation for the gluon anomalous dimension. This non-linear equation is parameterized by a function $P(x)$ which is unknown beyond perturbation theory. Still, very mild assumptions on $P(x)$ lead to stringent restrictions for possible solutions to Dyson–Schwinger equations.

We establish that the theory must have asymptotic freedom beyond perturbation theory and also investigate the low energy regime and the possibility for a mass gap in the asymptotically free theory.

Acknowledgments

D.K. and K.Y. were supported by NSF grant DMS-0603781. D.U. and G.v.B. are supported by NSF grant DMS-0405724 and thank C.E. Wayne for discussions. D.K. thanks Ivan Todorov for discussions.

1 Introduction

We study non-perturbative aspects of quantum chromodynamics (QCD). We do so by investigating Dyson–Schwinger equations. Instead of solving a truncated version of these a priori very intricate

[*]Department of Mathematics and Statistics, Boston University, 111 Cummington Street, Boston MA, 02215, USA

[†]CNRS-IHES 91440 Bures sur Yvette, France; and Center for Mathematical Physics Boston University, 111 Cummington Street, Boston MA, 02215, USA

equations [21], we use recent insight into the mathematical structure of quantum field theory to gain insight into the possible structure of solutions. This approach has been successfully applied to quantum electrodynamics in [5] and is here extended to QCD.

1.1 The method

We follow the methods employed in our work on QED [5], adopted to the study of QCD in the background field approach as developed by Abbott [1, 2]. Let us first reconsider the situation for quantum electrodynamics. There, thanks to the Ward identity, it suffices to consider the anomalous dimension $\gamma_1(x)$ of the photon which is essentially the β-function, $\beta(x) = x\gamma_1(x)$.

This anomalous dimension is obtained from the photon's self-energy, a two-point function which is determined non-perturbatively by the knowledge of a single Lorentz scalar function

$$G(x, L) = 1 - \sum_{k=1}^{\infty} \gamma_k(x) L^k \ ,$$

with x the fine structure constant and $L = \ln(-q^2/\mu^2)$, where the inverse photon-propagator is $(q^2 g_{\mu\nu} - q_\mu q_\nu)G(x, L)$.

Combining this with the combinatorial Dyson-Schwinger equations and using an expansion into suitable integral kernels which parametrize the corresponding integral equation, the Dyson-Schwinger equation combine with the renormalization group equation to give

i) a recursion for the γ_k:

$$\gamma_k(x) = -\frac{1}{k}\gamma_1(x)(1 - x\partial_x)\gamma_{k-1}(x), k \geq 2 \ , \tag{1}$$

ii) a differential equation for $\gamma_1(x)$:

$$\gamma_1(x)(1 - x\partial_x)\gamma_1(x) + \gamma_1(x) - P(x) = 0 \ . \tag{2}$$

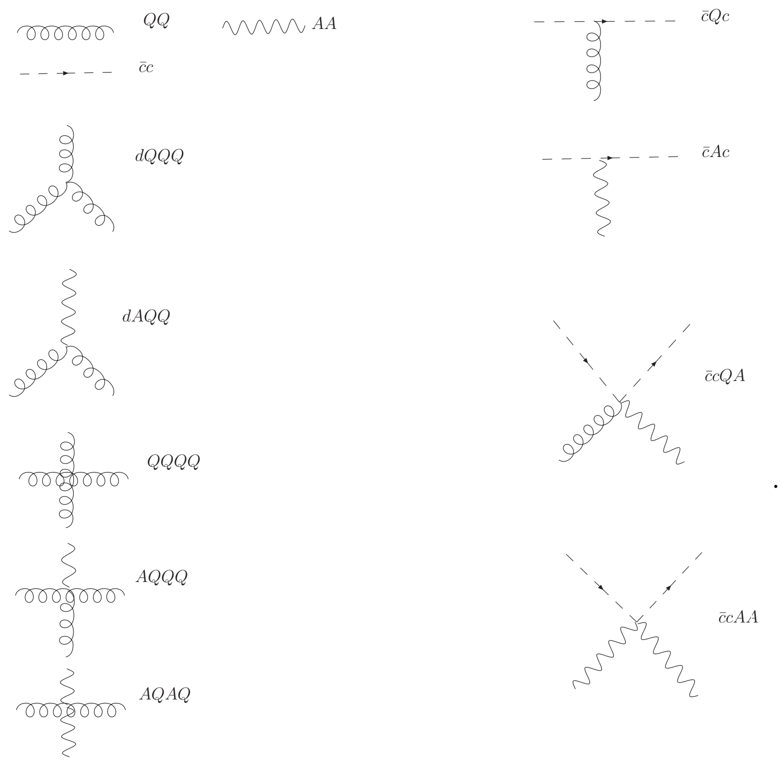
Here, i) comes from the renormalization group, ii) from the DSE, and $P(x)$ is a suitably constructed series over residues. The appearance of the operator $(1 \pm x\partial_x)$ is typical for a gauge theory.

We will now address a non-abelian gauge theory, resulting in a similar set-up (in particular, i) and ii) remain form-invariant) once we learned to make effective use of quantum gauge invariance to reduce again to a single ODE. Obviously, however, $P(x)$ will be a different function, in particular, it will change sign. We study the consequences of this fact with minimal knowledge on the behavior of $P(x)$. Still, as in QED, we will see that we can learn quite a bit regarding the non-perturbative sector of QCD.

To proceed, we turn to the background field method.

1.2 Background field method

We first have to consider the set of vertices and propagators in the background field gauge. They define a set $\mathcal{R}$ given as follows:



Here, names attached to the vertices and propagators are short hand memos of the corresponding monomials in the Lagrangian.

The field content is Q for the internal quantized gauge field, A for an external background gauge field, $\bar{c}, c$ for the anti-ghost and ghost field. Coupling of Fermionic matter will not change the ensuing discussion in any way and is omitted for convenience. See Abbott [2] for details.

With the set $\mathcal{R}$ comes an accompanying set of 1PI Feynman graphs naturally labeled by elements in this set according to their type and number of external legs.

We consider in particular Green functions for such graphs and adopt the results of [17], see also [18], which read as expected $\forall r \in \mathcal{R}$

$$X^r = \mathbb{I} \pm \sum_{k\geq 1} [g^2]^k \sum_{|\gamma|=k} \frac{1}{\mathrm{sym}(\gamma)} B_+^{\gamma;r} \left(\frac{\prod_{v\in\gamma^{[0]}} X^v}{\prod_{e\in\gamma^{[1]}} \sqrt{X^e}} \right) ,$$

with

$$B_+^{\gamma}(h) = \sum_{\Gamma\in<\Gamma>} \frac{\mathbf{bij}(\gamma, h, \Gamma)}{|h|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{(\gamma|h)} \Gamma \,, \qquad (3)$$

where $\mathrm{maxf}(\Gamma)$ is the number of maximal forests of Γ, $|h|_\vee$ is the number of distinct graphs obtainable by permuting edges of h, $\mathbf{bij}(\gamma, h, \Gamma)$ is the number of bijections of external edges of h with an insertion place in γ such that the result is Γ, and finally $(\gamma|h)$ is the number of insertion places for h in γ [17]. $\sum_{\Gamma\in<\Gamma>}$ indicates a sum over the linear span $<\Gamma>$ of generators of H.

Next, we divide by the ideal I which implements the Slavnov–Taylor identities which here is generated order in order in g^2 by

$$X^{\bar{c}Ac} = X^{\bar{c}Qc} = X^{\bar{c}c},\ X^{dQQQ} = X^{dAQQ} = X^{QQ} = X^{AA}\ ,$$
$$X^{AAQQ} = X^{AQAQ} = X^{AQQQ} = X^{QQQQ} = X^{dQQQ}\ .$$

On H/I we then get two independent Green functions which need renormalization

$$X^{AA},\ X^{\bar{c}c}\ ,$$

corresponding to a mere two-element set

$$\mathcal{R}_{H/I} = \{AA, \bar{c}c\}\ .$$

Also, we then find an combinatorial invariant charge uniquely defined as

$$C = \mathbb{I}/\sqrt{X^{AA}} = \mathbb{I}/\sqrt{X^{QQ}}\ ,$$

so that, as in QED, the β-function is just half the negative anomalous dimension of the gauge field.

Note that the addition of massless fermions would just add in H/I an element $\bar{\psi}\psi$ for the fermion self-energy but would not change the ideal or the invariant charge.

The system of combinatorial Dyson Schwinger equations is then

$$X^{AA} = \mathbb{I} - \sum_k [g^2]^k B_+^{k,AA}\left(X^{AA}\,[C]^{-2k}\right)$$
$$X^{\bar{c}c} = \mathbb{I} - \sum_k [g^2]^k B_+^{k,\bar{c}c}\left(X^{\bar{c}c}\,[C]^{-2k}\right)\ .$$

Note that this determines X^{AA} in terms of itself, while $X^{\bar{c}c}$ is a function of itself and X^{QQ}. We write $X^r = \mathbb{I} - \sum_{k\geq 1}[g^2]^k c_k^r$, with $c_k^r \in H/I$ the generators of a sub Hopf algebra [17, 9] given by all graphs which contribute to an amplitude r at a chosen order k. Note that the resolution of a Green functions into images of Hochschild one-cocycles B_+^r is the mathematical equivalent of a resolution of all overlapping divergences into non-overlapping integral kernels. That this is possible in a non-abelian theory was realized early by Baker and Lee [7].

See [17, 18, 24, 25] for explicit examples how Hochschild cohomology and Hopf algebras relate.

There hence is a quotient Hopf algebra H_{AA} spanned by generators c_k^{AA} of X^{AA}. Similarly, going temporarily to the quotient Hopf algebra $H_{\bar{c}c}$ defined by $X^{AA} = \mathbb{I}$, $c_k^{AA} = 0$, we find that this is a cocommutative Hopf algebra (which is obvious from setting $C = \mathbb{I}$) and hence we find a factorization of groups into an abelian subgroup $\mathrm{Spec}(H_{\bar{c}c})$ and a normal subgroup $\mathrm{Spec}(H_{AA})$,

$$\mathrm{Spec}(H/I) = \mathrm{Spec}(H_{AA}) \rtimes \mathrm{Spec}(H_{\bar{c}c}) \ ,$$

corresponding to a short exact sequence which splits

$$\mathbb{I} \to \mathrm{Spec}(H_{AA}) \to \mathrm{Spec}(H/I) \to \mathrm{Spec}(H_{\bar{c}c}) \to \mathbb{I} \ .$$

It is this factorization which allows us to compute the β-function of QCD by an ODE for a single equation below. Note that the situation is similar to QED: there, the Ward identity allows for a similar semi-direct product structure between photon amplitudes and Fermionic matter. Gauge invariance then allows to eliminate all short-distance singularities in the abelian subgroup thanks to the work of Baker, Johnson and Willey, and one is left with the photon propagation as the only source of renormalization.

Here, we can compute the β-function from $G^{AA}(x, L)$, but would have to consider the full coupled system to determine $G^{\bar{c}c}(x, L)$ (and $G^{\bar{\psi}\psi}(x, L)$), which we do not attempt here.

Furthermore, note that the simplification at $k = 1$

$$B_+^{1,AA}\left(X^{AA}\,[C]^{-2}\right) = B_+^{1,AA}(\mathbb{I}) \ .$$

This is typically for gauge theories and emphasizes that we are in a single equation situation with $s = 1$ [5]. Terms $B_+^r(\mathbb{I})$ always deliver a pure residue from their short-distance singularities, and these terms are intimately connected to fermion determinants. We will not pursue this connection any further here.

The background field method is then suited to our approach as it allows us to compute the QCD beta function from a single ordinary differential equation.

Indeed, a change of basis of primitives allows to reduce the application of Feynman rules to the study of one-variable Mellin transforms for the integral kernels for the above primitives $B_+^{k,r}(\mathbb{I})$, and from there we can strictly follow the techniques of [19, 20, 26] to get to an single ordinary differential equation:

$$\gamma_1(x) + \gamma_1(x)^2 - P(x) - x\gamma_1(x)\gamma_1'(x) = 0 \ .$$

Here, $P(x)$ is a suitable series over primitives. As always, we renormalize using a momentum scheme with subtractions at $q^2 = \mu^2$. That scheme is uniquely suited [14] to our gauge-invariant non-perturbative approach.

1.3 Qualitative properties of QCD

From perturbative computations, asymptotic freedom is firmly established. We will establish it beyond perturbation theory below. Perturbatively, this is mainly a self-consistency statement: assuming that the QCD coupling constant is small, we approximate the theory by its loop expansion to a few orders. The resulting polynomial approximation to the beta function supports the claim of asymptotic freedom in perturbation theory: $\beta(x) < 0$, $0 < x < 1$, hence at large momentum transfer $\lim_{-Q^2\to\infty} \alpha_s(L) \to 0$, $L = \ln(-Q^2/\mu^2)$. The coupling indeed becomes small in that limit.

As usual, perturbation theory agrees well with observations: asymptotic freedom is a well-established experimental fact.

Much more intricate is the study

$$-Q^2 \to 0_+ \ .$$

This is beyond the reach of perturbation theory. Nevertheless, different approaches point out that in that limit, the gluon propagator might turn to a constant, confirming an old suggestion of Cornwall that the free gluon develops in the interacting theory a momentum-dependent mass which vanishes at high energies, but turns to a non-vanishing constant in the limit $-Q^2 \to 0_+$. We study this behavior from our viewpoint in section 3.

Results to this effect were already obtained by
i) Lattice computations [4];

ii) numerical study of Dyson Schwinger equations truncated in a gauge invariant way [13];

iii) in the Gribov-Zwanziger formalism [16].

Below, we reconsider the problem from a study of the possible structure of solutions of Dyson Schwinger equations. We want to establish asymptotic freedom beyond perturbation theory, and want to discuss to what extent a solution which exhibits asymptotic freedom can also exhibit a mass gap.

Again, as in the case of QED, we find the most interesting solution to be a separatrix. In the case of QCD, that separatrix is the only solution which has asymptotic freedom.

2 Results

In QCD, the Dyson-Schwinger equation for γ_1 is

$$\frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} = f(\gamma_1(x), x) \equiv \frac{\gamma_1(x) + \gamma_1(x)^2 - P(x)}{x\gamma_1(x)} \ . \tag{4}$$

We will assume that the primitive skeleton function satisfies the following assumptions:

H1: P is a twice differentiable function on $\mathbf{R}^+$, with $P(0) = 0$, $P'(0) < 0$ and $P''(0) < 0$.

H2: There exist $x^\star$ such that $P(x) > -\frac{1}{4}$ and $P''(x) \leq 0$ (i.e. P is concave) on $[0, x^\star]$.

H3: The function $P(x)$ satisfies $P(x) < 0$ for all $x > 0$.

As in [5], we avoid the singularities of (4) at $\gamma_1 = 0$ and $x = 0$ by specifying with an initial condition at $x^\star$, namely,

$$\frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} = f(\gamma_1(x), x) \equiv \frac{\gamma_1(x) + \gamma_1(x)^2 - P(x)}{x\gamma_1(x)} \ , \quad \gamma_1(x^\star) = \gamma_0 \ . \tag{5}$$

This ensures that solutions of (5) exist at least locally around $x = x^\star$. Though we will mainly look for solutions or (4) with $\gamma_1(x^\star) < 0$ and $x \geq 0$, we will occasionally comment on the $\gamma_1(x^\star) > 0$ case.

In the QED case, we proved in [5] (see also Section 5 of the present paper) the existence of a unique value $\gamma_1^\star(x_0)$ (the separatrix) separating solutions that exist globally for all $x \geq x_0$ from those that can only be continued up to a finite $x_{\max} > x_0$. As shown in Section 5, all solutions in QED can be continued as $x \to 0$, differing there 'only' by a flat behavior $\sim \mathrm{e}^{-\frac{1}{x}}$. In QCD the situation is reversed: we will prove that all solutions starting at some appropriate $x^\star$ can be continued as $x \to \infty$, but that there is a unique value $\gamma_1^\star(x^\star)$ that separates solutions that cannot be continued as $x \to 0$ from those that can, which either satisfy $\gamma_1(0) = -1$ if $\gamma_1(x^\star) < \gamma_1^\star(x^\star)$ or $\gamma_1(0) = 0$ if $\gamma_1(x^\star) = \gamma_1^\star(x^\star)$. We call the solution $\gamma_1^\star(x)$ that satisfies $\gamma_1^\star(0) = 0$ the *asymptotically free* solution.

We will also use more speculative hypotheses on $P(x)$:

S1: There exist $p > 0$ such that $P(x) = -cx^p + o(x^p)$ as $x \to \infty$.

S2: There exist $x_c \geq -\frac{1}{P'(0)}$ such that $P''(x) \leq 0$ for all $x \in [0, x_c]$

S3: There exist a (finite) interval $[x_c, x_d]$ with $x_c > x^\star$ such that

$$-\int_{x_c}^{x_d} \frac{1 + 4P(z)}{2z} \mathrm{d}z \geq 1 \ .$$

S4: There exist finite x_l and x_r such that $P(x_l) = P(x_r) = -\frac{1}{4}$ and $P(x) > -\frac{1}{4}$ for all $0 \leq x < x_l$ and $x > x_r$. The function $P(x)$ does *not* satisfy S1, S2 and S3, but rather $\lim_{x\to\infty}(P(x) - P_\infty) = \lim_{x\to\infty} xP'(x) = 0$ for some $P_\infty > -\frac{1}{4}$.

Let us briefly comment on our logic here. The H1-H3 hypotheses are a bare minimum for the results we will present below. Within perturbation theory, we have

$$P(x) = \gamma_1(x) + \mathcal{O}(x^3) = -\beta_1 x - \beta_2 x^2 + \mathcal{O}(x^3) \tag{6}$$

as $x \to 0$, where $-\beta_1$ and $-\beta_2$ are the 1 and 2-loop coefficients of the β function, namely $\beta_1 = 9$ and $\beta_2 = 64$ for $n_f = 6$. As such, the hypotheses H1 and H2 are reasonable. While it also follows from (6) that $P(x) < 0$ at least for small values of x, extending this to all values of x is somewhat more speculative.

As we will show below, if $P(x) < -\frac{1}{4}$ on a 'sufficiently large interval', for instance if either S1, S2 or S3 hold, all solutions of (5) satisfy $\gamma_1(x) = -1$ for some $x \geq x^\star$. It then follows from H3 that they grow linearly as $x \to \infty$ if

$$\mathcal{D}(P) = -\int_{x^\star}^{\infty} \frac{P(z)}{z^3} \mathrm{d}z < \infty \ ,$$

and faster than linearly if $\mathcal{D}(P) = \infty$. Incidentally, we showed in [5] (see also Section 5 of the present paper), that the finiteness/infiniteness of $\mathcal{D}(P)$ was intimately linked with the existence/non-existence as $x \to \infty$ of solutions of the analogous of (5) for QED. It is striking to see that the *same criterion* distinguishes between different type of behavior in QCD as well[1].

Note that an anomalous dimension growing at least linearly as $x \to \infty$ leads to a Landau pole for the running coupling, and hence a serious obstacle to studying the infrared behavior of the Gluon propagator. Despite that, we will show in Section 3.1 that this pole (if present) can be removed using an unsubtracted dispersion relation, see e.g. [23], and the Gluon propagator can still be studied in the infrared limit.

In contrast, if $\gamma_1(x)$ is finite as $x \to \infty$, we avoid the Landau pole, and can study the Gluon propagator without using dispersion relations. Such constant asymptotics for γ_1 can only happen if $P(x)$ tends to a constant as $x \to \infty$. This motivates (part of) the hypothesis S4. Under that hypothesis, we will show that there is only one solution that satisfies

$$\lim_{x\to\infty} \gamma_1(x) = -\frac{1+\sqrt{1+4P_\infty}}{2} \equiv \gamma_\infty \ .$$

If $P_\infty = 0$, we call that solution the *confinement solution* $\gamma_1^c(x)$, and if $P_\infty > 0$, we call it a *strong confinement solution*. On physical grounds, the asymptotically free and (strong) confinement solutions need to be the same. Unfortunately, for generic $P(x)$ satisfying H1, H2 and S4, these two solutions are different. We did not succeed in finding a sufficient condition on $P(x)$ that guarantees both solutions are the same. Despite that, a necessary condition is certainly that $P(x)$ makes at least one (small) excursion below $-\frac{1}{4}$, while avoiding the S1-S3 conditions, see also figure 1 below or Section 4.4.

By standard folklore and heuristics [3], a nowhere vanishing β-function, $\beta(x) < 0,\ \forall x > 0$, which we will indeed establish below under assumption H1 above, implies a mass gap in QCD, and hence a confinement scenario following old ideas of Cornwall [15]. We will come back to that in Section 3 of this paper.

Remark 2.1 *If $P_\infty > 0$ in hypothesis S4, $P(x)$ has a further zero for a finite $x_1 > 0$, $P(x_1) = 0$. In such a case, using the running coupling formulation of (4), one sees that some solutions spiral around the zero of $P(x)$, themselves having infinitely many zeroes. The asymptotically free solution $\gamma_1^\star(x)$ may or may not spiral around $x = x_1$, depending on details of P. Should it be captured, we get a solution $\gamma_1(x)$ which has a UV fix-point at zero and an infrared fixpoint at x_1. This is the Banks-Zaks scenario [8].*

In the remainder of this section, we are going to state our main results. The proofs and technical details are postponed to Section 4 of this paper.

Our first main result gives a complete characterization of the behavior of solutions of (5) for $x < x^\star$. In particular, it establishes the uniqueness of the solution exhibiting asymptotic freedom.

[1] though of course the primitive skeleton functions P are different in both cases

Theorem 2.2 *Under the hypotheses* H1 *and* H2*, there is a unique value* $\gamma_1^\star(x^\star) < 0$ *such that the corresponding solution* $\gamma_1^\star(x)$ *of (5) exists for all* $x \in [0, x^\star]$ *and satisfies* $\lim_{x\to 0}\gamma_1^\star(x) = 0$*. Additionally, that solution satisfies*

$$\frac{\sqrt{1+4P(x)}-1}{2} \le \gamma_1^\star(x) \le P'(0)x\ , \tag{7}$$

If $\gamma_1(x^\star) < \gamma_1^\star(x^\star)$*, then the corresponding solution satisfies* $\lim_{x\to 0}\gamma_1(x) = -1$*. If* $\gamma_1^\star(x^\star) < \gamma_1(x^\star) < 0$*, then there exist* $x_{\min} > 0$ *such that the corresponding solution satisfies* $\gamma_1(x_{\min}) = 0$*.*

Note that since $\frac{\sqrt{1+4P(x)}-1}{2} = P'(0)x + \mathcal{O}(x^2)$ as $x \to 0$, (7) shows that $\gamma_1^\star(x) = P'(0)x + \mathcal{O}(x^2)$ as $x \to 0$.

As we will prove in Proposition 4.8) of Section 4 below, all solutions of Theorem 2.2 can be continued as $x \to \infty$ by only adding the H3 assumptions. In Proposition 4.6, we will show that solutions that satisfy $\gamma_1(x_{\min}) = 0$ for some $0 < x_{\min} < x^\star$ can be continued in the first quadrant (becoming double-valued) by reverting to the so-called 'running coupling' formulation of (5) (see also [5]). In particular these solutions will satisfy $\gamma_1(x_0) > 0$ for some $x_0 > x_{\min}$.

Our second main result concerns the asymptotic behavior as $x \to \infty$ of solutions that enter the first quadrant, or attain the value -1 somewhere. This last condition can be verified under additional assumptions on P such as S1, S2 or S3.

Proposition 2.3 *Assume* $P(x)$ *satisfies H1-H3 and that one of the two following statements holds:*

1. $-1 < \gamma_1(x^\star) < 0$ *and* $P(x)$ *satisfies S1 or S3,*
2. $\gamma_1(x^\star) \le \gamma_1^\star(x^\star)$ *and* $P(x)$ *satisfies S2.*

Then there exists $x_0 > x^\star$ *such that the corresponding solution* $\gamma_1(x)$ *satisfies* $\gamma_1(x_0) = -1$*.*

To be able to state our asymptotic result as $x \to \infty$, we need first to introduce the *slope function* $\mathrm{S}_P(x_0, x)$. This function is given by

$$\mathrm{S}_P(x_0, x) = \left(\frac{\gamma_1(x_0)^2}{x_0^2} + 2\int_{x_0}^{x} \frac{-P(z)}{z^3}\mathrm{d}z\right)^{\frac{1}{2}}\ ,$$

Note that, if $\mathcal{D}(P) < \infty$, the slope function $\mathrm{S}_P(x_0, x)$ goes to a finite value as $x \to \infty$ for any x_0.

We can now completely describe the asymptotic behavior as $x \to \infty$ of solutions of (5):

Theorem 2.4 *Assume $P(x)$ satisfies H1-H3. If there exists $x_0 > 0$ such that either $\gamma_1(x_0) > 0$ or $\gamma_1(x_0) \le -1$ then*

$$x\ S_P(x_0,x) \le \gamma_1(x) \le \quad x\left(S_P(x_0,x)+\frac{1}{x_0}\right) - 1 \quad \text{if} \quad \gamma_1(x_0) > 0\ ,$$
$$-x\ S_P(x_0,x) \le \gamma_1(x) \le -x\left(S_P(x_0,x)-\frac{1}{x_0}\right) - 1 \quad \text{if} \quad \gamma_1(x_0) \le -1\ .$$

Furthermore, if $\mathcal{D}(P) < \infty$ and $\gamma_1(x_0) \le -1$ or $\gamma_1(x_0) > 0$, there exists $s > 0$ such that

$$\lim_{x\to\infty} \frac{\gamma_1(x)}{x} = \begin{cases} -s < 0 & \text{if } \gamma_1(x_0) \le -1 \\ s > 0 & \text{if } \gamma_1(x_0) > 0 \end{cases} .$$

If $\gamma_1(x_0) \le -1$, the convergence towards the limit is given by

$$\left|\ \frac{\gamma_1(x)}{x} + s\ \right| \le C \int_x^\infty \frac{-P(z)}{z^3} \mathrm{d}z\ . \tag{8}$$

If $\mathcal{D}(P) < \infty$ and $\gamma_1(x_0) > 0$, then (8) also hold, with $-s$ replaced by s.

We want to stress here that the slope value s depends on the actual solution under consideration. Also, for solutions that eventually enter the first quadrant, the value of the slope s has no reason to be the same along the two branches of the solution (the one in the first quadrant, and the one in the fourth). Also, note that the result depends only on the assumption $\gamma_1(x_0) = -1$.

We now state the existence and uniqueness of *confinement solutions* under hypothesis S4.

Theorem 2.5 *Assume $P(x)$ satisfies H1, H2 and S4. Then there exist a unique solution $\gamma_1^c(x)$ of (4) such that*

$$\lim_{x\to\infty} \gamma_1^c(x) = \lim_{x\to\infty} -\frac{1+\sqrt{1+4P(x)}}{2} \equiv \gamma_\infty\ .$$

Furthermore, $\gamma_1^c(x)$ satisfies the usual trichotomy as x decreases: either $\gamma_1^c(0) = -1$, or $\gamma_1^c(0) = 0$, or $\gamma_1^c(x)$ cannot be continued for $x < x_{\min}$ for some $x_{\min} > 0$ where $\gamma_1^c(x_{\min}) = 0$. Finally, if there exists $x_{\max}$ such that $'P(x) > 0$ for all $x > x_{\max}$ (hence $P(x)$ is strictly increasing towards P_∞), then

$$-\frac{\sqrt{1+4P_\infty}+1}{2} \le \gamma_1^c(x) \le -\frac{\sqrt{1+4P(x)}+1}{2}$$

for all $x > x_{\max}$.

As already noted above, we cannot show that $\gamma_1^\star(x) = \gamma_1^c(x)$ without additional hypotheses on $P(x)$. We can however note that the two types of solutions are compatible: the confinement solution satisfies the integral equation

$$\gamma_1^c(x) = -1 + \int_1^\infty \frac{P(xt)}{t^2\gamma_1^c(xt)} \mathrm{d}t\ , \tag{9}$$

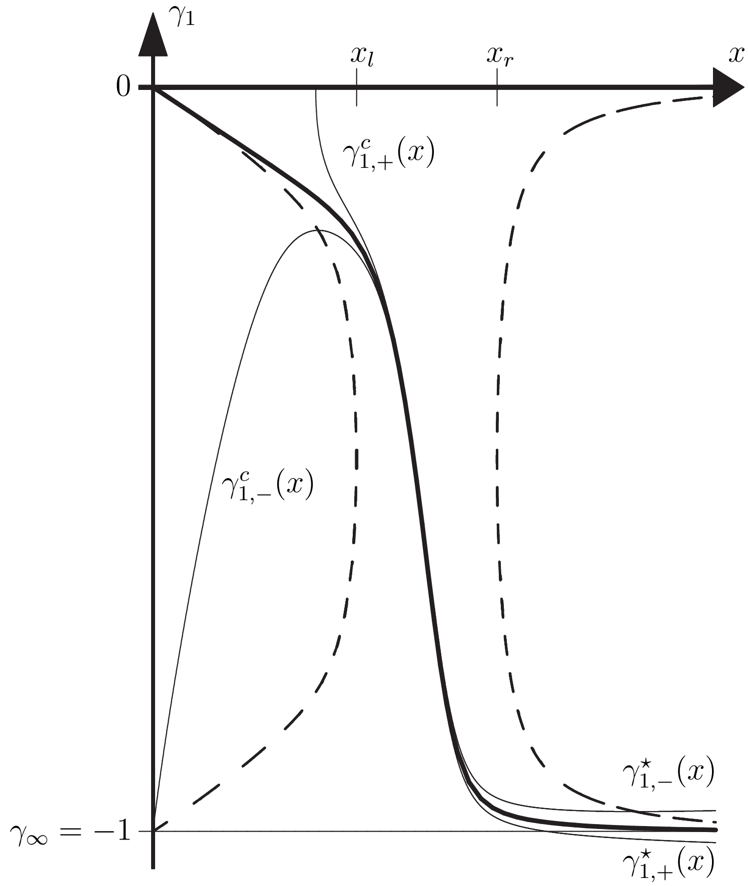


Figure 1: A 'fragile' solution: the solid bold curve is the solution combining asymptotic freedom and confinement for some artificial $P(x)$ with $P_\infty = 0$. The dashed curves are the nullclines $\gamma_1 + \gamma_1^2 - P(x) = 0$. The solid curves $\gamma^c_{1,\pm}$ and $\gamma^\star_{1,\pm}$ correspond to confinement and asymptotically free solutions with $P(x)$ perturbed so that the gap $|x_r - x_l|$ is slightly larger ('$+$' subscripts) or smaller ('$-$' subscripts). In both cases, the asymptotically free solution and the confinement solution do not match.

whose r.h.s. converges to 0 as $x \to 0$ if $P(x)/\gamma^c_1(x) \to 1$ as $x \to 0$. Characterizing the set of functions $P(x)$ for which the confinement solution and the asymptotically free one are the same is a difficult problem, see e.g. figure 1 for an example with an 'artificial' $P(x)$. A necessary condition is that $P(x)$ makes an excursion below $-\frac{1}{4}$ on (at least) one interval so that the nullclines, i.e. the location in (x, γ_1)-plane where $\gamma_1'(x) = 0$,

$$\gamma_\pm(x) = \frac{\pm\sqrt{1 + 4P(x)} - 1}{2}$$

show a gap as in figure 1. However if the gap is 'too wide', we have $\gamma^\star_1(x) = \gamma_\infty$ for some finite x, and $\gamma^c_1(x_{\min}) = 0$ for some $x_{\min} > 0$, whereas if the gap is 'too small', $\gamma^\star_1(x)$ intersects the nullcline at some $x > x_r$, and hence cannot reach γ_∞ as $x \to \infty$, while $\gamma^c_1(x)$ intersects the nullcline at some $x < x_l$, and hence cannot reach 0 as $x \to 0$.

We conclude this section by explaining the terminology *asymptotically free*, *confinement* and *strong confinement* for our solutions. These come from the study of the inverse Gluon propagator,

$$P^{-1}(x,Q^2) = Q^2 G(x,L) \quad \text{with} \quad L = \ln(Q^2/\mu^2) \ .$$

In Section (3.2) below, we solve the RGE equation expressing the scale invariance of $G(x,L)$. In particular, in Theorem 3.2, we show that if $\gamma_1(x) \to \gamma_\infty$ as $x \to \infty$, then

$$G(x,L) = \frac{X(L,x)}{x} \ ,$$

where $X(t,x)$ is the running coupling, i.e. the solution of

$$\frac{\mathrm{d}X(t,x)}{\mathrm{d}t} = X(t,x)\gamma_1(X(t,x)) \quad \text{with} \quad X(t=0,x) = x \ .$$

The possible large $|t|$ behavior of $X(t,x)$ are given by

$$X(t,x) \simeq \begin{cases} x_\infty \mathrm{e}^{\gamma_\infty t} & \text{as } \ t \to -\infty \quad \text{if} \ \lim\limits_{x\to\infty} \gamma_1(x) = \gamma_\infty \ , \\ -\frac{1}{P'(0)t} & \text{as } \ t \to \infty \quad \text{if} \ \lim\limits_{x\to 0} \frac{\gamma_1(x)}{x} = P'(0) \ , \\ x_0 \mathrm{e}^{-t} & \text{as } \ t \to \infty \quad \text{if} \ \lim\limits_{x\to 0} \gamma_1(x) = -1 \ , \end{cases}$$

where x_∞ and x_0 are some positive functions of x. Thus, in all cases where $\gamma_1(x)$ can be continued to $x = 0$, $G(x,L) \to 0$ in the ultraviolet regime $L \to \infty$, and moreover

$$P^{-1}(x,Q^2) \to \begin{cases} 0 & \text{as } \ L \to \infty \quad \text{if} \ \lim\limits_{x\to 0} \frac{\gamma_1(x)}{x} = P'(0) \ , \\ +x_0 & \text{as } \ L \to \infty \quad \text{if} \ \lim\limits_{x\to 0} \gamma_1(x) = -1 \ , \end{cases}$$

hence the terminology *asymptotic freedom* for the solution that satisfies $\gamma_1(x) = xP'(0) + \mathcal{O}(x^2)$ as $x \to 0$. In the infrared regime, however,

$$G(x,L) \simeq \frac{x_\infty}{x} \mathrm{e}^{\gamma_\infty L} = \frac{x_\infty}{x} \left(\frac{Q^2}{\mu^2}\right)^{\gamma_\infty} \quad \text{as} \quad L \to -\infty \ ,$$

and hence we find

$$\lim_{L\to-\infty} P^{-1}(x,Q^2) = \begin{cases} \mu^2 \ \frac{x_\infty}{x} & \text{if} \quad \lim\limits_{x\to\infty} \gamma_1(x) = -1 \\ \infty & \text{if} \quad \lim\limits_{x\to\infty} \gamma_1(x) < -1 \end{cases} \ , \tag{10}$$

a finite mass gap if $\gamma_\infty = -1$.

3 The Gluon propagator, confinement and mass gaps in QCD

3.1 Corrections from dispersion relations

In QED, the coupling is weak at low energy or momentum transfer. In our previous work [5], we could hence define boundary conditions at low energy, and studied the behavior of $\sum_k \gamma_k L^k$ for $L > 0$. In particular, for $\gamma_1 = \gamma_1(\bar{x}(L))$, a continuation to $L < 0$ was never needed by the choice of our renormalization conditions. On the other hand, for large $L \gg 0$ we could establish a separatrix, which possibly avoids a Landau pole at any finite positive L. We conjectured that this might be the solution chosen by Nature, and further detailed analysis of its properties awaits more analysis of the function $P(x)$. Should it turn out that $P(x)$ is such that the separatrix will not avoid a Landau pole (*i.e.* turns to infinity at finite L), we will have to turn to dispersion relations to understand the non-perturbative corrections coming with such a pole, as recognized by Shirkov and collaborators early on [11].

For QCD, we again fix a small coupling, but this time large momentum transfer, $L \gg 0$ for our boundary conditions. We are now interested in a continuation to $L \ll 0$, in particular we are interested in $L \to -\infty$. Under very mild assumptions on $P(x)$, and certainly by any experience from perturbative approximations of the theory, we expect the anomalous dimension γ_1 to go below the value -1 at some finite coupling x, and hence $\bar{x}(L)$ to turn to infinity at some finite negative L. Shifting that L to zero essentially defines the scale Λ_{QCD}, and we are interested for that shifted L_Λ to study the regime $L_\Lambda < 0$, in particular $L_\Lambda \to -\infty$. To consider such a limit based from an approach formulated for $L_\Lambda > 0$, we will use dispersion relations. Our approach is motivated again by Shirkov and collaborators work [23].

On general grounds, we know that $\bar{x}(L_\Lambda)$ and $G(x, L_\Lambda)$ can be treated by an unsubtracted dispersion relation [22]:

$$f_{\mathrm{disp}}(Q^2) = \int_0^\infty \frac{\Im(f(\sigma))}{\sigma + Q^2 - i\eta} \mathrm{d}\sigma \ .$$

The inverse propagator needs a subtracted dispersion relation, which leads back to an unsubtracted dispersion relation for G [22].

The Dyson–Schwinger equations themselves are supposed to hold for the whole theory regardless of the sign of L_Λ. Similar, the renormalization group equations for the running coupling are supposed to hold. Our derivation which turned the Dyson–Schwinger equations into a ODE was valid for $L > -L_\Lambda$, hence remain valid, after shifting, for $L_\Lambda > 0$.

Continuing to $L_\Lambda < 0$ will generate non-perturbative corrections to $\gamma_1(x)$, and hence $\beta(x)$, determined from the requirement that equations of motion, renormalization group flow and analyticity properties of field theory are what they are supposed to be.

Any perturbative approximation is in accordance with these properties of field theory only up to the order considered. When we study solutions of Dyson–Schwinger equations, we demand *accord* with these properties as a guide to find the necessary non-perturbative corrections in the region $L_\Lambda < 0$.

We hence will start by first applying a dispersion relation to analyze $\bar{x}(L_\Lambda)$. That leads to a corrected $\gamma_{1,\mathrm{disp}}$ due to the fact that at $L_\Lambda = 0$ we find that $\bar{x} = \infty$. Combining this with the assumption that the uncorrected $\gamma_1(x)$ is driven below -1 at finite x allows an easy estimate of the corrected $\gamma_{1,\mathrm{disp}}$ and also allows for consistency with the direct analysis of G by dispersion methods.

Let us now start with a study of $\bar{x}(L_\Lambda)$. We start our considerations by boundary conditions such that $G(x, L) = 1$ at some very high momentum transfer $\mu^2 \gg 0$, $L = \ln(Q^2/\mu^2)$, which determines a suitably small x in agreement with say deep inelastic scattering experiments [10].

We have

$$\frac{\mathrm{d}\bar{x}(x,L)}{\mathrm{d}L} = \bar{x}\gamma_1(\bar{x}) \Leftrightarrow \int_x^{\bar{x}(x,L)} \frac{1}{u\gamma_1(u)} du = L \; ,$$

and assume that the integral

$$\int_x^\infty \frac{1}{u\gamma_1(u)} \mathrm{d}u < \infty \; .$$

In particular, we assume that $\gamma_1(x) < -1$ for some positive finite x, in accordance with our previous discussions and experimental evidence. We thus define

$$\Lambda^2_{\mathrm{QCD}} = \mu^2 + e^{\left\{\int_x^\infty \frac{1}{u\gamma_1(u)} \mathrm{d}u\right\}} \; .$$

Setting $L_\Lambda = \ln(Q^2/\Lambda^2)$ to absorb the dependence on x, μ^2, we get

$$-\int_{\bar{x}(L_\Lambda)}^\infty \frac{1}{u\gamma_1(u)} \mathrm{d}u = L_\Lambda \; . \tag{11}$$

This equation defines $L_\Lambda(\bar{x})$ as well as the inverse function $\bar{x}(L_\Lambda)$. Since $\bar{x}(0) = +\infty$, we cannot trust our solution for $Q^2 < \Lambda^2$. To remedy this, we use our previous result that for large x, $\gamma_1(x) \to -sx$, under our present assumptions.

Following the conventions of Shirkov [23], we define a running coupling in accordance with the expected analytic behavior of field theory using a dispersion relation:

$$\bar{x}_{\mathrm{disp}}(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\Im(\bar{x}(\ln(\sigma/\Lambda^2)))}{\sigma + Q^2 - i\eta} \mathrm{d}\sigma \; . \tag{12}$$

The pole at $-Q^2$ gives us back the uncorrected $\bar{x}(L_\Lambda)$. But by assumption, there is a further pole in the complex σ-plane, located at $L_\Lambda(\infty) = 0$. To study the contribution from that pole, we first note that as $\bar{x} \to \infty$, we have

$$L_\Lambda = -\int_{\bar{x}(L_\Lambda)}^\infty \frac{1}{u\gamma_1(u)} \mathrm{d}u \simeq \int_{\bar{x}(L_\Lambda)}^\infty \frac{1}{su^2} \, \mathrm{d}u = \frac{1}{s\bar{x}(L_\Lambda)} \; ,$$

and hence $\bar{x}(L_\Lambda) \simeq \frac{1}{sL_\Lambda}$ near $L_\Lambda = 0$. Feeding this relation into (12) gives (see [23])

$$\bar{x}_{\mathrm{disp}}(L_\Lambda) = \bar{x}(L_\Lambda) + \frac{1}{s(1 - \mathrm{e}^{L_\Lambda})} \; .$$

If we were to identify s with the one-loop coefficient of the β-function, this would reproduce Shirkov's analysis for one-loop QCD, see [23], where Shirkov also notes that s seem not to vary much at low loop orders. Note that the correction to $\bar{x}(L_\Lambda)$ goes to the finite value $1/s$ in the infrared limit $L_\Lambda \to -\infty$.

Now, γ_1 also obeys an unsubtracted dispersion relation. As $\gamma_1(x)$ is finite for all finite x and depends on L only through $\bar{x}$, the dispersion integral will correct $\gamma_1(\bar{x}(L)) \simeq -s\bar{x}(L_\Lambda)$ by

$$\gamma_{1,\mathrm{disp}}(\bar{x}(L_\Lambda)) = \gamma_1(\bar{x}(L_\Lambda)) - \frac{1}{1-\mathrm{e}^{L_\Lambda}} \, .$$

Note now that the correction to γ_1 goes to -1 as $L_\Lambda \to -\infty$. Using (1) and $x\gamma_1(x)\partial_x = \partial_L$, we get

$$\gamma_{k,\mathrm{disp}}(\bar{x}(L_\Lambda)) = \gamma_k(\bar{x}(L_\Lambda)) + \gamma_{k,\mathrm{corr}}(L_\Lambda) \quad \text{with} \quad \gamma_{k,\mathrm{corr}}(L_\Lambda) \to \frac{(-1)^k}{k!} \quad \text{as} \quad L_\Lambda \to -\infty \, .$$

As such, the correction from the dispersion relation to the function $G(x, L_\Lambda)$ satisfies

$$G_{\mathrm{corr}}(x, L_\Lambda) \to \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L_\Lambda^k = \mathrm{e}^{-L_\Lambda} = \frac{\Lambda^2}{Q^2} \quad \text{as} \quad L_\Lambda \to -\infty \, ,$$

which gives an inverse propagator satisfying

$$\lim_{L_\Lambda \to -\infty} P^{-1}(x, Q^2) = -\Lambda^2 \, .$$

This gives a finite and renormalization group invariant positivity-violating mass gap. This compares nicely with the results of Gracey et.al. [16].

Note that we rely completely on the assumption that physical quantities in massless field theory have neither poles nor branch cuts off the negative (in our conventions) real axis and the result that the solutions we get from our ODEs for the uncorrected $\bar{x}, \gamma_k, G(x, L_\Lambda)$ are in accordance with these requirements but for the isolated pole at $L_\Lambda = 0$.

If a future analysis of $P(x)$ justifies the trust in field theory expressed in this section remains to be seen. We are content having identified a clean mechanism for the generation of a mass gap, based on the consequences of the Hopf algebra structure underlying local field theory, and assuming analyticity properties in accordance with the underlying axiomatic structure of local quantum fields.

3.2 Solving The Renormalization Group Equation for $G(x, L)$

Our goal in this section is to analyze the dressing function $G(x, L)$ that modulates the free inverse propagator. On general grounds, this function solves the Renormalization Group equation

$$\Big(-\partial_L + x\gamma_1(x)\partial_x - s\gamma_1(x)\Big)G(x, L) = 0 \quad \text{and} \quad G(x, 0) = 1 \quad \text{for} \quad x \geq 0 \, , \tag{13}$$

where $s = \pm 1$ distinguishes between writing the propagator as

$$P(x, Q^2) = \frac{G(x, \ln(Q^2/\mu^2))^s}{Q^2}$$

with $s = 1$ or $s = -1$. Note that the difference of (13) from the usual RGE equation

$$\Big(\mu\partial_\mu + \tilde{\beta}(g)\partial_g + \tilde{\gamma}(g)\Big)G = 0$$

is merely a matter of convention on the definition of β, γ_1 and x. In particular, (13) agrees with two-loop computation of the Gluon propagator.

We will here make the following (minimal) hypothesis on $\gamma_1(x)$:

Hypothesis 3.1 *The function $\gamma_1(x)$ is a negative $\mathcal{C}^1([0,\infty),(-\infty,0])$ function whose only possible zero is at $x = 0$, where $\gamma_1(x) = -dx^{q_0} + \mathcal{O}(x^{q_0+1})$ with $q_0 \geq 0$ and $d > 0$.*

Note that the results of Section 2 give $q_0 = 1$ and $d = -P'(0)$ for the asymptotically free solution $\gamma_1(x)$, in accordance with perturbation theory. For the solutions satisfying $\gamma_1(0) = -1$ however, we get $q_0 = 0$ and $d = 1$.

We first note that one can attempt to solve (13) by writing

$$G(x, L) = 1 - s\sum_{k=1}^{\infty}\gamma_k(x)L^k \quad \text{with} \quad \gamma_k(x) = \frac{\gamma_1(x)}{k}\Big(x\partial_x - s\Big)\gamma_{k-1}(x) \quad k \geq 2\;, \tag{14}$$

since one has (at least formally) that

$$\begin{aligned}\Big(-\partial_L + x\gamma_1(x)\partial_x - s\gamma_1(x)\Big)G(x,L) &= -s\gamma_1(x) + s\sum_{k=1}^{\infty}k\gamma_k(x)L^{k-1} - \gamma_1(x)\Big(x\partial_x - s\Big)\gamma_k(x)L^k \\ &= s\sum_{k=2}^{\infty}\bigg(\gamma_k(x) - \frac{\gamma_1(x)}{k}\Big(x\partial_x - s\Big)\gamma_{k-1}(x)\bigg)\,kL^k = 0\;. \end{aligned} \tag{15}$$

This approach naturally raises the (difficult) question of convergence of the series in (14) and (15). We will use instead an alternative way of solving (13) that avoids these convergence problems. In particular, while the series (14) necessarily converge on a symmetric interval of the form $(-L_0, L_0)$ for some (possibly infinite) $L_0 \geq 0$, our approach will give a solution to (13) that is defined on an interval of the form $(-L_0, \infty)$ for the same L_0. As such, the approach below shows that the limit $L \to \infty$ of $G(x, L)$ makes sense also if the series solution converges only on a finite interval.

Our method is based on the fact that (13) can be transformed into a *linear transport equation* by appropriately factorizing $G(x, L)$ into one part that cancels the term involving no derivatives and one part that solves a genuine transport equation of the form

$$(-\partial_L + x\gamma_1(x)\partial_x)H(x, L) = 0\;. \tag{16}$$

As such, it is important at first to consider the characteristics curve of (16). For each fixed $x > 0$, we first define $X(t,x)$ as the solution of the running coupling equation

$$\frac{\mathrm{d}X(t,x)}{\mathrm{d}t} = X(t,x)\gamma_1(X(t,x)) \quad \text{with} \quad X(t=0,x) = x \; . \tag{17}$$

Since $\gamma_1(x)$ is assumed to be $\mathcal{C}^1$, solutions of this equation exist at least locally around $t = 0$. For further reference, we denote by $\mathcal{D}(x)$ the maximal interval of existence of the solution of (17) for a fixed x.

Then for each fixed $(x, L) \in \mathbf{R}^+ \times \mathbf{R}$, we define the characteristic curve $\mathcal{C}(x, L)$ as

$$\begin{aligned} \mathcal{C}(x,L) &= \Big\{ \; (X(t,x), L-t) \quad \text{with} \quad t \in \mathcal{D}(x) \Big\} \; , \\ &= \left\{ \; \left(X, L - \int_x^X \frac{\mathrm{d}z}{z\gamma_1(z)} \right) \quad \text{with} \quad X \in \mathbf{R}^+ \right\} \; . \end{aligned} \tag{18}$$

Note that both above formulations of the characteristic curve $\mathcal{C}(x, L)$ are equivalent. The characteristics corresponding to different values of L are *vertical* translations of the same curve in the (x, L)-plane. By hypothesis 3.1, we get that the characteristics are all asymptotically vertical as $X \to 0$, and, as a function of t, we have

$$X(t,x) \simeq \begin{cases} \mathrm{e}^{-ct} & \text{as} \quad t \to \infty \qquad \text{if} \quad q_0 = 0 \\ (ct)^{\frac{1}{-q_0}} & \text{as} \quad t \to \infty \qquad \text{if} \quad q_0 > 0 \end{cases} \; . \tag{19}$$

The behavior of the characteristics as $X \to \infty$ depends on the asymptotic behavior of $\gamma_1(x)$ as $x \to \infty$. In all cases, $\mathcal{C}(x, L)$ approaches $(\infty, L - L_\infty(x))$ as $X \to \infty$, where $L_\infty(x)$ is the *possibly infinite* quantity defined by

$$L_\infty(x) = \int_x^\infty \frac{\mathrm{d}z}{z\gamma_1(z)} \; .$$

This shows that the maximal interval of existence for solutions of (17) is $\mathcal{D}(x) = (L_\infty(x), \infty)$. The results of Section 2 show that $L_\infty(x)$ is finite if there exists $x_0 > 0$ where $\gamma_1(x_0) = -1$, and infinite otherwise. If $L_\infty(x) > -\infty$, the characteristic $\mathcal{C}(x, L)$ intersects the line $L = 0$ if and only if $L > L_\infty(x)$. If $-1 < \gamma_1(x) < 0$ for all $x > 0$, then $L_\infty(x) = -\infty$ for all $x > 0$, and *all* characteristic curves cross the line $L = 0$. The generic shape of the characteristics curves is displayed in figure 2.

We can now give the solution of (13) in accordance (for $s = -1$, as expected) with Bogoliubov-Shirkov ([12], App.IX, eq.(27)):

Theorem 3.2 *Assume $\gamma_1(x)$ satisfies Hypothesis 3.1. Then the solution of (13) is given by*

$$G(x,L) = \left(\frac{x}{X(L,x)} \right)^s \tag{20}$$

for all (x, L) such that $L_\infty(x) < L < \infty$ and $x > 0$.

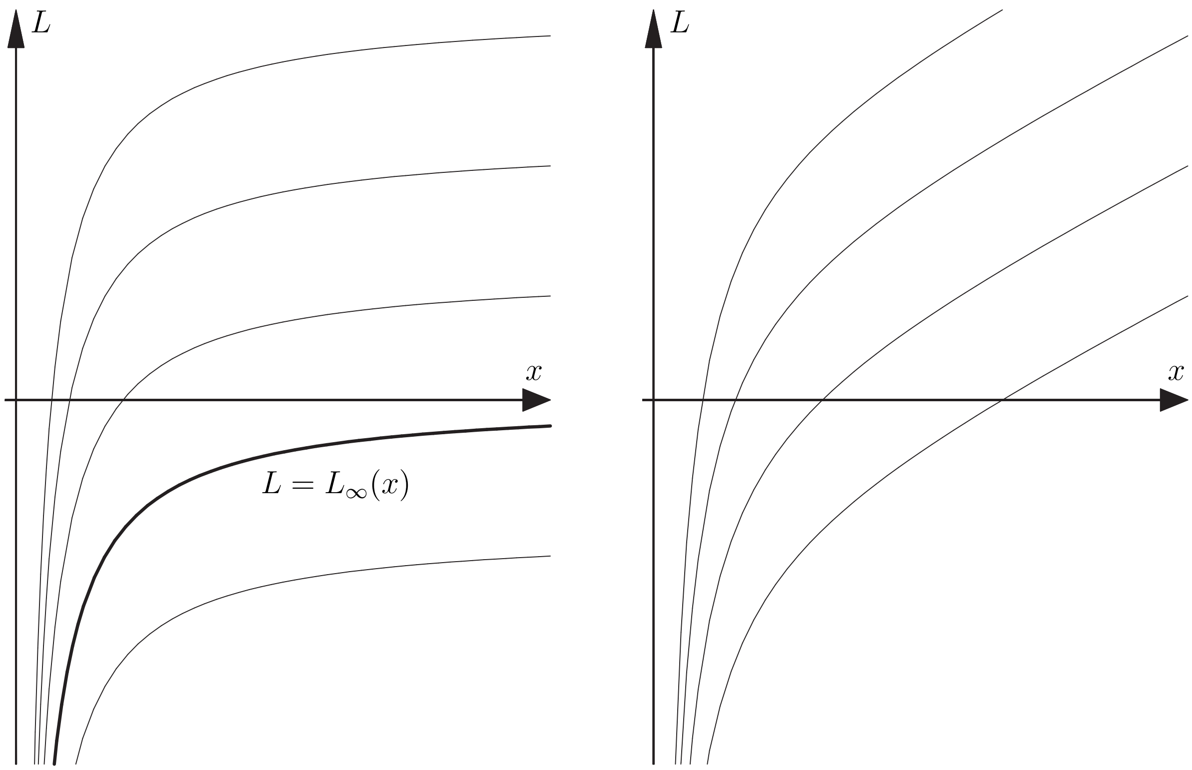


Figure 2: Generic shape of characteristics curves in (x, L)-plane. The left panel shows the case where $L_\infty(x)$ is finite, the right panel when it is infinite. In the left panel, the characteristics cross the line $L = 0$ if and only if they are above the (bold) curve $L = L_\infty(x)$, and all characteristics are asymptotically horizontal as $x \to \infty$. In the right panel, all characteristics cross the line $L = 0$, as they link $(0, -\infty)$ to (∞, ∞).

We want to stress here that the relation (20) hold only for pair of values of (x,L) satisfying $L_\infty(x) < L < \infty$. For pairs (x,L) with $L \leq L_\infty(x)$, $G(x,L)$ can only be determined from $G(x,0)=1$ if one specifies $\gamma_1(x)$ for $x<0$ as well.

Assuming $X(t,x)$ to be analytic for all $t \in \mathcal{D}(x)$, it is a straightforward computation (see Section (3.3)) to show that (14) is the Taylor series expansion of (20) at $L=0$. As such, the series solution (14) is expected *not* to converge for $|L| > L_\infty(x)$. However, the function provided by (20) is defined for unbounded positive L and do solve (13) for all such L. If $L_\infty(x) > -\infty$, Theorem 3.2 thus gives $G(x,L)$ for values of (x,L) for which the series formulation fails.

proof of Theorem 3.2. We first set

$$G(x,L) = x^s H(x,L) \tag{21}$$

for all $x>0$. Substitution into (13) gives

$$\Big(-\partial_L + x\gamma_1(x)\partial_x\Big)H(x,L) = 0 \quad \text{and} \quad H(x,0) = x^{-s} \quad \text{if} \quad x>0\ . \tag{22}$$

We then note that $H(x,L)$ is constant along the characteristic curve $\mathcal{C}(x,L)$, for we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(H(X(t,x),L-t)\Big) &= -D_2 H(X(t,x),L-t) + D_1 H(X(t,x),L-t)\frac{\mathrm{d}X(t,x)}{\mathrm{d}t}\\ &= \Big[\Big(-\partial_L + x\gamma_1(x)\partial_x\Big)H(x,L)\Big]_{x=X(t,x),L=L-t} = 0\ .\end{aligned}$$

If the curve $\mathcal{C}(x,L)$ crosses the line $L=0$ in the (x,L)-plane, it does so at $t=L$, and we get

$$H(x,L) = H(X(L,x),0) = X(L,x)^{-s}\ ,$$

which completes the proof. ■

3.3 The series solution for $G(x,L)$

Throughout this section, we consider γ_1 to be a fixed solution of (5) that exists for all $x \geq 0$. We first pick $x_0 > 0$ and $t_0 \in \mathbf{R}$. These values are arbitrary. We then introduce

$$T(x) = t_0 + \int_{x_0}^{x} \frac{\mathrm{d}z}{\gamma_1(z) z}\ . \tag{23}$$

We then note that

$$\lim_{x\to\infty} T(x) = \begin{cases} T_\infty = -\infty & \text{if } -1 < \gamma_1(x) \leq 0 \ \ \forall x \geq 0\ , \\ T_\infty > -\infty & \text{if } \exists x_0 > 0 \ \text{ with } \ \gamma_1(x_0) = -1 \end{cases}\ . \tag{24}$$

Since $T'(x) = \frac{1}{\gamma_1(x)x} < 0$, $x \mapsto T(x)$ is an invertible map from $\mathbf{R}^+$ to $[T_\infty, \infty)$. Moreover, we have $T(0) = \infty$ and $T(\infty) = T_\infty$. The inverse map $\tilde{x}(t)$ (the solution of $t = T(\tilde{x}(t))$) satisfies the 'running coupling equations'

$$\frac{\mathrm{d}\tilde{\gamma}_1(t)}{\mathrm{d}t} = \tilde{\gamma}_1(t) + \tilde{\gamma}_1(t)^2 - P(\tilde{x}(t)) \quad \text{and} \quad \frac{\mathrm{d}\tilde{x}(t)}{\mathrm{d}t} = \tilde{x}(t)\tilde{\gamma}_1(t) \ , \tag{25}$$

where $\tilde{\gamma}_1(t) = \gamma_1(\tilde{x}(t))$. Since $T(\infty) = T_\infty$, $\tilde{x}(t)$ diverges as $t \to T_\infty$ (and so does $\tilde{\gamma}_1(t)$ if $\gamma_1(x) = -1$ for some $x > 0$).

Consider now the series solution (14). We first introduce the functions S_k such that

$$\gamma_k(x) = \frac{1}{k!}\, x^s\, S_k(T(x))$$

for $k \geq 1$. Substitution into (14) gives

$$S_k(T(x)) = S'_{k-1}(T(x)) = \left(\frac{\mathrm{d}}{\mathrm{d}t}S_{k-1}(t)\right)\bigg|_{t=T(x)} \ .$$

Since $s = \pm 1$, we find from (25) that $S_1(t) = \frac{\tilde{\gamma}_1(t)}{\tilde{x}(t)^s} = -s\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}(t)^{-s})$. Using $s^2 = 1$, we find

$$-s\gamma_k(x) = \frac{x^s}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\frac{1}{\tilde{x}(t)^s}\Big)\right)\bigg|_{t=T(x)} \tag{26}$$

for all $k \geq 1$. Since $\tilde{x}(T(x)) = x$, the r.h.s. of (26) is equal to 1 when $k = 0$, and so

$$G(x,L) = 1 + \sum_{k=1}^{\infty} -s\gamma_k(x)L^k = x^s \sum_{k=0}^{\infty} \frac{L^k}{k!}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\Big(\frac{1}{\tilde{x}(t)^s}\Big)\right)\bigg|_{t=T(x)} \ . \tag{27}$$

We now fix $0 < x < \infty$. Since x is finite, $T(x) > T_\infty$, and since the above series is a Taylor series of $\tilde{x}(t)^{-s}$ at $t = T(x)$, the series converges and takes the value $\tilde{x}(t+L)^{-s}$ for small values of L *if we assume $z(t)$ to be analytic at $t = T(x)$*. We thus find

$$G(x,L) = \left(\frac{x}{\tilde{x}(T(x)+L)}\right)^s \ , \tag{28}$$

at least for sufficiently small L. This formula is the same as the one of Theorem 3.2: $\tilde{x}(T(x)+t)$ solves

$$\frac{\mathrm{d}\tilde{x}(T(x)+t)}{\mathrm{d}t} = \tilde{x}(T(x)+t)\gamma_1(\tilde{x}(T(x)+t)) \ , \tag{29}$$

with initial condition $\tilde{x}(T(x)) = x$ at $t = 0$, and so $\tilde{x}(T(x)+t) = X(t,x)$ by uniqueness of solutions of (17) and (29).

The question of convergence of (27) for large L depends on the (in-)finiteness of T_∞. If T_∞ is finite, $\tilde{x}(t)$ has a pole as $t \to T_\infty$, and (27) must diverge for

$$L \leq T_\infty - T(x) = \int_x^\infty \frac{\mathrm{d}z}{z\gamma_1(z)} = L_\infty(x)$$

as in Section 3.2. However, since the series (27) can only converge for L on symmetric intervals, it is also divergent for $L \geq -L_\infty(x) > 0$. In that case, we have to revert to Section 3.2 for the validity of (28).

4 Technical details for QCD

In this section, we present the technical details proving the statements of Section 2 on QCD. In our analysis of (5), we use mainly two tools: integral representations of the solutions and null-clines. Our strategy is directly inspired from [6] and our previous work [5] on QED.

There are two types of integral equations one can write for (5). The first one (see also [5]) reads

$$\gamma_1(x) = \frac{x(1+\gamma_1(x^\star))}{x^\star} - 1 - x\int_{x^\star}^{x} \frac{P(z)}{z^2\gamma_1(z)}\mathrm{d}z\ . \tag{30}$$

For the second one, we let $\gamma_1(x)$ be a solution of (5) and $\gamma_2(x)$ be *any* function. Then for all $x_0, x \in \mathcal{I}$, where $\mathcal{I}$ is the common interval where γ_1 and γ_2 are defined, we have

$$\gamma_1(x) - \gamma_2(x) = \Big(\gamma_1(x_0) - \gamma_2(x_0)\Big)\ K[\gamma_1,\gamma_2](x_0,x) + \int_x^{x_0} R[\gamma_2](y)\ K[\gamma_1,\gamma_2](y,x)\ \mathrm{d}y\ , \tag{31}$$

where

$$R[\gamma_2](x) \equiv \frac{\mathrm{d}\gamma_2(x)}{\mathrm{d}x} - \frac{\gamma_2(x)+\gamma_2(x)^2 - P(x)}{x\gamma_2(x)}\ ,$$
$$K[\gamma_1,\gamma_2](x_0,x) \equiv \left(\frac{x}{x_0}\right)\exp\left(-\int_x^{x_0}\frac{P(z)}{z\gamma_1(z)\gamma_2(z)}\mathrm{d}z\right) .$$

The null-clines are defined as the locations in (x,γ_1)-plane where solutions satisfy $\gamma_1'(x) = 0$). These are given by the graph of the two functions

$$\gamma_c^{\pm}(x) = \frac{\pm\sqrt{1+4P(x)}-1}{2}\ .$$

In particular, as $P(0) = 0$ by hypothesis H1, and $P(x) > -\frac{1}{4}$ for $x \in [0, x^\star]$, the null-clines extend at least up to the line $x = x^\star$ in the (x,γ_1)-plane. On the null-clines, the second derivative of γ_1 is given by

$$\gamma_1''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(\gamma_c^{\pm}(x),x)\right) = \frac{P'(x)}{|x\gamma_c^{\pm}(x)|}\ . \tag{32}$$

By hypotheses H1 and H2, we have

$$P'(x) = P'(0) + \int_0^x P''(z)\mathrm{d}z \le P'(0) < 0 \quad \forall x \in [0,x^\star]\ . \tag{33}$$

Hence by (32), solutions of (5) can have at most one local maximum in the interval $x \in [0, x^\star]$, and no local minimum. For further reference, we also note that by hypotheses H1 and H2,

$$P(x) = P'(0)x + \int_0^x\left(\int_0^y P''(z)\mathrm{d}z\right)\mathrm{d}y \le P'(0)x \quad \forall x \in [0,x^\star]\ . \tag{34}$$

In particular, we have

$$\min_{x\in[0,x^\star]}\frac{-P(x)}{x} = -P'(0) = |P'(0)| > 0\ , \tag{35}$$

and since $P(x^\star) > -\frac{1}{4}$ by hypothesis on $x^\star$, we have $x^\star < -\frac{1}{4P'(0)} < \infty$.

4.1 Existence, uniqueness and properties of the asymptotic freedom solution

We can now establish the existence, uniqueness and properties of the solution with asymptotic freedom.

Theorem 4.1 *Under the hypotheses* H1 *and* H2*, there exists a unique value* $\gamma_1^\star(x^\star)$ *such that the corresponding solution* $\gamma_1^\star(x)$ *of (5) exists for all* $x \in [0, x^\star]$ *and satisfies* $\lim\limits_{x\to 0}\gamma_1^\star(x) = 0$.

Proof. We first prove that the solution is unique. Namely, assume *ab absurdum* that γ_1 and γ_2 are two solutions of (5) on $[0, x^\star]$ satisfying $\lim\limits_{x\to 0}\gamma_i(x) = 0$. Since $\gamma_i(x^\star) < 0$ and $\lim\limits_{x\to 0}\gamma_i(x) = 0$, we necessarily have $\gamma_i(x) \geq \gamma_c^+(x)$ for all $x \in [0, x^\star]$. Since γ_2 is a solution, we can apply (31) with $R[\gamma_2] = 0$, and using (35), we get

$$|\gamma_1(x) - \gamma_2(x)| \geq |\gamma_1(x^\star) - \gamma_2(x^\star)| \frac{x}{x^\star} \exp\left(\int_x^{x^\star} \frac{|P'(0)|}{\gamma_c^+(z)^2} \mathrm{d}z \right) . \tag{36}$$

Since $\gamma_c^+(z) = P'(0)z + \mathcal{O}(z^2)$ as $z \to 0$, the r.h.s. of (36) diverges as $x \to 0$, which contradicts $\lim\limits_{x\to 0}\gamma_i(x) = 0$. This shows that there can be at most one solution of (5) satisfying $\lim\limits_{x\to 0}\gamma_1(x) = 0$.

To prove the existence of this solution, we define the two sets

$$\begin{aligned} \mathrm{I}_1 &= \{\gamma_1(x^\star) \in]\gamma_c^+(x^\star), 0[\text{ s.t. } \exists x_{\min} \in]0, x^\star[\text{ with } \gamma_1(x_{\min}) = 0\} , \\ \mathrm{I}_2 &= \{\gamma_1(x^\star) \in]\gamma_c^+(x^\star), 0[\text{ s.t. } \exists x_1 \in]0, x^\star[\text{ with } \gamma_1(x_1) = \gamma_c^+(x_1)\} . \end{aligned}$$

We will prove in Propositions 4.2 and 4.4 below that these sets are non-empty. Continuity of solutions w.r.t. initial conditions imply that they are open, while (31) with $R[\gamma_2] = 0$ shows that solutions are *ordered*, which imply that each I_i is a single interval. Now since I_1 and I_2 are disjoint open intervals, there exist at least one initial condition $\gamma_1^\star(x^\star)$ that is in neither sets, and hence the corresponding solution $\gamma_1^\star(x)$ satisfies $\lim_{x\to 0} \gamma_1^\star(x) = 0$. ■

To establish that I_1 is non-empty, we show that initial conditions at $x = x^\star$ sufficiently close to the x-axis necessary cross it at some $x_{\min} < x^\star$.

Proposition 4.2 *For all* $\gamma_0 \in [P'(0)x^\star, 0[$*, there exists* $x_{\min} \in]0, x^\star[$ *such that the solution of (5) only exists on* $x \in [x_{\min}, x^\star]$ *and* $\gamma_1(x_{\min}) = 0$.

Proof. Pick $\gamma_2(x) = P'(0)x$. Then by (34), we have

$$R[\gamma_2](x) = \frac{1}{x^2 P'(0)} \int_0^x \left(\int_0^y P''(z)\mathrm{d}z \right) \mathrm{d}y > 0 \quad \forall x \in [0, x^\star].$$

Applying (31), we get

$$\gamma_1(x) = P'(0)x + \int_x^{x^\star} R[\gamma_2](y)\ K[\gamma_1, \gamma_2](y, x)\ \mathrm{d}y > P'(0)x \quad \forall x < x^\star . \tag{37}$$

In particular, γ_1 cannot cease to exist by diverging to $-\infty$ at a finite $x < x^\star$. We also see from (37) that there exists $x_0 < x^\star$ with $\gamma_1(x_0) > P'(0)x_0$. Using $R[\gamma_2] > 0$, (35) and $\gamma_1(x) \geq P'(0)x$ in (31), we find

$$\begin{aligned}\gamma_1(x) &\geq P'(0)x + (\gamma_1(x_0) - P'(0)x_0)\,\frac{x}{x_0}\exp\left(\int_x^{x_0} \frac{-1}{P'(0)z^2}\mathrm{d}z\right) \\ &\geq \frac{x}{x_0}\left(P'(0)x_0 + (\gamma_1(x_0) - P'(0)x_0)\mathrm{e}^{\frac{1}{|P'(0)|x} - \frac{1}{|P'(0)|x_0}}\right) \ .\end{aligned}$$

The proof is completed since $\gamma_1(x_0) - P'(0)x_0 > 0$. ■

Before proving that the interval I_2 is non-empty, we can establish an additional property of the asymptotically free solution $\gamma_1^\star$:

Corollary 4.3 *The solution $\gamma_1^\star$ of Theorem 4.1 satisfies*

$$\gamma_c^+(x) < \gamma_1(x) < P'(0)x \ . \tag{38}$$

for all $x \in]0, x^\star]$.

Proof. To get the lower bound in (38), we recall that solutions can have at most one local maximum in $]0, x^\star]$ and no local minimum. Since $\lim_{x\to 0}\gamma_1^\star(x) = 0$, $\gamma_1(x) > \gamma_c^+(x)$ for all $x \in]0, x^\star]$. The upper bound follows immediately from Proposition 4.2. ■

We now show that initial conditions sufficiently close (but above) the null-cline $\gamma_c^+(x^\star)$ necessarily cross it at some $x < x^\star$.

Proposition 4.4 *There exist $\epsilon_1 \ll 1$ sufficiently small such that the solution of (5) with $\gamma_1(x^\star) = \gamma_c^+(x^\star)(1-\epsilon_1)$ satisfies $\gamma_1(x) = \gamma_c(x)$ for some $0 < x < x^\star$.*

Proof. Let $0 < \epsilon_1 < \frac{1}{2}$ and $\gamma_1(x^\star) = \gamma_c^+(x^\star)(1-\epsilon_1)$. By continuity of solutions and since $\gamma_c^+(x^\star) < \gamma_1(x^\star) < \gamma_1(x^\star)(1-2\epsilon_1)$, there exists $0 < x_1 < x^\star$ such that

$$\begin{cases} \gamma_c^+(x^\star) < \gamma_1(x) < \gamma_c^+(x^\star)(1-2\epsilon_1) \quad \forall x \in]x_1, x^\star] \quad \text{and} \\ \gamma_1(x_1) = \gamma_c^+(x^\star) \quad \text{or} \quad \gamma_1(x_1) = \gamma_c^+(x^\star)(1-2\epsilon_1) \end{cases} \ . \tag{39}$$

Using these inequalities and hypothesis H2, we get

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}x} \geq -\frac{2\epsilon_1}{1-2\epsilon_1}\frac{P(x^\star)}{\gamma_c^+(x^\star)} \quad \forall x \in [x_1, x^\star] \ ,$$

which, upon integration, gives

$$\gamma_1(x) \leq \gamma_c^+(x^\star)(1-\epsilon_1) + \frac{2\epsilon_1}{1-2\epsilon_1}\frac{P(x^\star)}{\gamma_c^+(x^\star)} \ln\left(\frac{x^\star}{x}\right) \quad \forall x \in [x_1, x^\star] \ . \tag{40}$$

Let now x_2 be the value at which the r.h.s. of (40) attains $\gamma_c^+(x^\star)(1-2\epsilon_1)$, namely

$$x_2 = x^\star \exp\left(\frac{-\gamma_c^+(x^\star)^2(1-2\epsilon_1)}{2|P(x^\star)|}\right) < x^\star \ .$$

We now consider the two alternatives $x_2 < x_1$ and $x_2 \geq x_1$.

Assume first that $x_2 < x_1$. Then (40) shows that $\gamma_1(x_1) < \gamma_c^+(x^\star)(1-2\epsilon_1)$, and thus by definition of x_1 (see (39)), we have $\gamma_1(x_1) = \gamma_c^+(x^\star)$ and since by H2, $\gamma_c^+(x)$ increases as $x \to 0$, there exists an $x \in [x_1, x^\star]$ with $\gamma_1(x) = \gamma_c^+(x)$.

Consider now the other possible case, namely $x_2 \geq x_1$. Since now $\gamma_c^+(x^\star) \leq \gamma_1(x) \leq \gamma_c^+(x^\star)(1-2\epsilon_1)$ for all $x \in [x_2, x^\star]$, and we conclude from (40) that

$$\gamma_1(x) - \gamma_c^+(x) \leq \gamma_c^+(x^\star) - \gamma_c^+(x) - \epsilon_1\gamma_c^+(x^\star) + \frac{2\epsilon_1}{1-2\epsilon_1}\frac{P(x^\star)}{\gamma_c^+(x^\star)}\ln\left(\frac{x^\star}{x}\right) \ . \tag{41}$$

The proof is completed by noting that for x very close to (but strictly less than) $x^\star$, $\gamma_c^+(x^\star) - \gamma_c^+(x)$ is negative, while the last two (positive terms) in (41) can be made arbitrarily small by picking ϵ_1 small enough. ■

4.2 Behavior towards $x = 0$ of non-asymptotically free solutions

We first consider solutions of (5) corresponding to initial conditions $\gamma_1(x^\star) < \gamma_1^\star(x^\star)$. We have the following result.

Proposition 4.5 *Let $\gamma_0 < \gamma_1^\star(x^\star)$. The corresponding solution $\gamma_1(x)$ of (5) exists for all $x \in [0, x^\star]$ and satisfies $\gamma_1(x) = -1 + \mathcal{O}(x\ln(x))$ as $x \to 0$.*

Proof. Note first that since $\gamma_0 < \gamma_1^\star(x^\star)$, there always exists $x_0 \leq x^\star$ such that $\gamma_1(x_0) \leq \gamma_c^+(x_0)$. Namely, if this does not already hold at $x^\star$, the proof of Theorem 4.1 shows that it will eventually hold at some smaller value of x. Since solutions can have at most one local maximum and no local minimum in $[0, x^\star]$, we then have

$$\min(-1, \gamma_1(x_0)) \leq \gamma_1(x) \leq \gamma_c^+(x_0) \quad \forall x \in [0, x_0] \ .$$

In particular, these solutions exist for all values of $x \in [0, x^\star]$. We then apply (30) and get

$$-1 + Cx + \frac{x}{|\gamma_c^+(x_0)|}\int_x^{x_0}\frac{-P(z)}{z^2}\mathrm{d}z \leq \gamma_1(x) \leq -1 + Cx + \frac{x}{\max(1, |\gamma_c^+(x_0)|)}\int_x^{x_0}\frac{-P(z)}{z^2}\mathrm{d}z \ ,$$

where $C = \frac{1+\gamma_1(x_0)}{x_0}$. By hypotheses H1 and H2 (see also (34), $P(x) = P'(0)x + \mathcal{O}(x^2)$ it is then straightforward to prove that $\gamma_1(x) = -1 + \mathcal{O}(x(1+\ln(x)))$ as $x \to 0$, which completes the proof. ■

We now consider solutions of (5) corresponding to initial conditions $\gamma_1(x^\star)$ in the interval $]\gamma_1^\star(x^\star), 0[$. We have the following result.

Proposition 4.6 *Let* $\gamma_0 \in]\gamma_1^\star(x^\star), 0[$. *The corresponding solution* $\gamma_1(x)$ *of (5) satisfies* $\gamma_1(x_{\min}) = 0$ *for some* $x_{\min} \in]0, x^\star[$. *It can then be continued and enters the first quadrant (becoming double-valued), and thus satisfies* $\gamma_1(x_0) > 0$ *for* $x > x_{\min}$.

Proof. Let $x_{\min} \geq 0$ be the minimal value such that $\gamma_1(x)$ exists $\forall x \in [x_{\min}, x^\star]$. We claim that $x_{\min} > 0$ and $\gamma_1(x_{\min}) = 0$. Namely, since $\gamma_1(x^\star) > \gamma_1^\star(x^\star)$, we have by (31) with $R[\gamma_2] = 0$ that

$$\gamma_1(x) = \gamma_1^\star(x) + (\gamma_1(x^\star) - \gamma_1^\star(x^\star))\frac{x}{x^\star}\exp\biggl(\int_x^{x^\star} \frac{-P(z)}{z\gamma_1^\star(z)\gamma_1(z)}\mathrm{d}z\biggr) > \gamma_1^\star(x) \tag{42}$$

for all $x \in [x_{\min}, x^\star]$. Assuming *ab absurdum* that $x_{\min} = 0$ and $\gamma_1(x) < 0$ for all $x \in [0, x^\star]$ leads to a contradiction, for then we would have $\gamma_1(x) > \gamma_1^\star(x) \geq \gamma_c^+(x)$ for all $x \in [0, x^\star]$, and using (35) and (42), we get

$$\gamma_1(x) \geq \gamma_1^\star(x) + (\gamma_1(x^\star) - \gamma_1^\star(x^\star))\frac{x}{x^\star}\exp\biggl(\int_x^{x^\star} \frac{|P'(0)|}{\gamma_c^+(z)^2}\mathrm{d}z\biggr)$$

which goes to $+\infty$ as $x \to 0$. So $x_{\min} > 0$ and $\gamma_1(x_{\min}) = 0$. Although (5) is singular at $\gamma_1(x_{\min}) = 0$, these solutions can be continued in the first quadrant by reverting to the so-called 'running coupling' formulation of (5) (see also (25) and [5]). Namely, we introduce a new independent variable t, and write $x = X(t)$ and $\gamma_1(X(t)) = \tilde{\gamma}_1(t)$, getting

$$\begin{aligned}
\frac{\mathrm{d}\tilde{\gamma}_1(t)}{\mathrm{d}t} &= \tilde{\gamma}_1(t) + \tilde{\gamma}_1(t)^2 - P(X(t)) \qquad && \tilde{\gamma}_1(t_0) = 0\,, \\
\frac{\mathrm{d}X(t)}{\mathrm{d}t} &= X(t)\tilde{\gamma}_1(t) && X(t_0) = x_{\min}\,.
\end{aligned}$$

These equations are *not* singular at $\tilde{\gamma}_1 = 0$, and thus solutions will exist (at least locally around $t = t_0$). Since $P(x) < 0$, the solution to these equations will satisfy $\tilde{\gamma}_1(t) = \gamma_0 > 0$ and $X(t) = x_0 > x_{\min}$ for some finite $t > t_0$. ■

4.3 Behavior as $x \to \infty$

We first show that solutions in the first quadrant are global, and satisfy appropriate estimates as $x \to \infty$.

Proposition 4.7 *Let* $\gamma_1(x_0) > 0$*, and assume* $P(x)$ *satisfies* H3. *The corresponding solution* $\gamma_1(x)$ *exists for all* $x \geq x_0$*, and satisfies*

$$0 < xS_P(x_0, x) \leq \gamma_1(x) \leq xS_P(x_0, x) + \frac{x}{x_0} - 1$$

for all $x \geq x_0$.

Proof. We first note that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\gamma_1(x)^2\Big) = \gamma_1(x)\frac{\mathrm{d}\gamma_1}{\mathrm{d}x} = \frac{\gamma_1(x)}{x} + \frac{\gamma_1(x)^2 - P(x)}{x} \geq \frac{\gamma_1(x)^2 - P(x)}{x} \ .$$

By integration, we find

$$\gamma_1(x) \geq x\sqrt{\frac{\gamma_1(x_0)^2}{x_0^2} + 2\int_{x_0}^{x} \frac{-P(z)}{z^3}\mathrm{d}z} = xS_P(x_0,x) > 0 \ . \tag{43}$$

This shows that solutions cannot cease to exist by reaching $\gamma_1(x) = 0$ at some $x > x_0$. Inserting (43) into (30) gives

$$\gamma_1(x) \leq \frac{x(1+\gamma_1(x_0))}{x_0} - 1 + x\int_{x_0}^{x} \frac{-P(z)}{z^3 S_P(x_0,z)}\mathrm{d}z = xS_P(x_0,x) + \frac{x}{x_0} - 1 \ , \tag{44}$$

since

$$\frac{\mathrm{d}S_P(x_0,x)}{\mathrm{d}x} = -\frac{P(x)}{x^3 S_P(x_0,x)} \ .$$

The proof is completed since (44) shows that solutions cannot cease to exist by diverging to ∞ at a finite $x \geq x_0$ either. ■

We now turn to the fate of any type of solutions of (5) with $\gamma_1(x^\star) < 0$ as $x \to \infty$. Our first result is that these solutions are global, *i.e.*, they can be extended as $x \to \infty$. In particular, the asymptotically free $\gamma_1^\star(x)$ is global.

Proposition 4.8 *Let* $\gamma_1(x^\star) < 0$ *and assume* $P(x)$ *satisfies* H3. *The corresponding solution of (5) exists for all* $x \geq x^\star$*, and satisfies*

$$-xS_P(x^\star,x) \leq \gamma_1(x) \leq \max\left(\gamma_1(x^\star), \sup_{z\in[x^\star,x]} \frac{\sqrt{1+4P(z)}-1}{4}\right) < 0$$

for all $x \geq x^\star$*. In particular, if* $\mathcal{D}(P) < \infty$*, solutions grow at most linearly as* $x \to \infty$*.*

Proof. For the lower bound, note first that, as in the proof of Proposition 4.6, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\Big(\gamma_1(x)^2\Big) = \gamma_1(x)\frac{\mathrm{d}\gamma_1}{\mathrm{d}x} = \frac{\gamma_1(x)}{x} + \frac{\gamma_1(x)^2 - P(x)}{x} \leq \frac{\gamma_1(x)^2 - P(x)}{x} \ ,$$

which gives $\gamma_1(x) \geq -xS_P(x^\star,x)$ upon integration. This shows that solutions cannot diverge to $-\infty$ at a finite $x > x^\star$. Now suppose *ab absurdum* that there exists $x_{\max} < \infty$ such that $\gamma_1(x_{\max}) = 0$. By hypothesis H1-H3, we have $-\frac{1}{4} < \sup_{x\in[x^\star,x_{\max}]} P(x) < 0$, and thus

$$\gamma_{\max} = \max\left(\gamma_1(x^\star), \sup_{z\in[x^\star,x_{\max}]} \frac{\sqrt{1+4P(z)}-1}{4}\right)$$

satisfies $-\frac{1}{4} < \gamma_{\max} < 0$. Note then that $f(\gamma_1, x)$ is strictly negative along the $\gamma_1 = \gamma_{\max}$ line since

$$\sup_{x \in [x^\star, x_{\max}]} x f(\gamma_{\max}, x) \leq \left(1 + \gamma_{\max} - \frac{\delta}{\gamma_{\max}}\right) \leq -\frac{1}{4} .$$

Since $\gamma_1(x^\star) \leq \gamma_{\max}$, this shows that $\gamma_1(x) \leq \gamma_{\max}$ for all $x \in [x^\star, x_{\max}]$, contradicting the *ab absurdum* assumption. Hence solutions exist globally as $x \to \infty$. ■

Our second result concern the asymptotics of some of these solutions as $x \to \infty$. Namely, we can estimate the growth of solutions that are somewhere less than -1.

Proposition 4.9 *Assume $P(x)$ satisfies* H3 *and $\gamma_1(x^\star) < 0$. If the corresponding solution of (5) satisfies $\gamma_1(x_0) \leq -1$ for some $x_0 \geq x^\star$, then*

$$-xS_P(x_0, x) \leq \gamma_1(x) \leq -xS_P(x_0, x) + \frac{x}{x_0} - 1 < 0$$

for all $x \geq x_0$.

Proof. The lower bound is already contained in Proposition 4.8. The upper bound then follows immediately from the lower bound and the integral formulation (30). ■

The condition $\gamma_1(x_0) \leq -1$ is essential in Proposition 4.9 to guarantee that the upper bound is indeed negative. We now give possible scenarios that guarantee solutions indeed reach $\gamma_1 = -1$.

Proposition 4.10 *Assume one of the two following statements holds:*

1. *$-1 < \gamma_1(x^\star) < 0$ and $P(x)$ satisfies S1 or S3,*
2. *$-1 < \gamma_1(x^\star) \leq \gamma_1^\star(x^\star)$ and $P(x)$ satisfies S2.*

Then there exists $x_0 > x^\star$ such that the corresponding solution $\gamma_1(x)$ satisfies $\gamma_1(x_0) = -1$.

Proof. We consider the alternative 1. first. Note that S1 implies S3, so we can use hypothesis S3 only. As shown in Proposition 4.8, any solution starting at $\gamma_1(x^\star) < 0$ exists for all values of $x \geq x^\star$. In particular, $\gamma_1(x_c) = \gamma_{\min} < 0$. If $\gamma_{\min} \leq -1$, the proof is completed. If $-1 < \gamma_{\min} < 0$, we assume *ab absurdum* that $\gamma_1(x) > -1$ for all $x \in [x_c, x_d]$. Note that there cannot be an $x \in [x_c, x_d]$ such that $\gamma_1(x) = 0$, hence $-1 < \gamma_1(x) < 0$ for all $x \in [x_c, x_d]$, and we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\gamma_1(x)^2) = 2\frac{\gamma_1(x) + \gamma_1(x)^2 + \frac{1}{4}}{x} - \frac{4P(x)+1}{2x} \geq -\frac{4P(x)+1}{2x}$$

for all $x \in [x_c, x_d]$. Upon integration, we thus find that

$$\gamma_1(x) \leq -\sqrt{\gamma_{\min}^2 - \int_{x_c}^{x} \frac{1+4P(z)}{2z}\mathrm{d}z} \leq -\sqrt{1+\gamma_{\min}^2} < -1$$

by hypothesis S3, which is a contradiction.

Consider then the alternative 2. Under hypothesis S2, we can extend (35) to get $P(x) \leq P'(0)x$ for all $x \in [0, x_c]$. Consider now $\gamma_2(x) = P'(0)x$. We have $R[\gamma_2](x) > 0$ for all $x \in [0, x_c]$. Thus, since $\gamma_1(x^\star) \leq \gamma_1^\star(x^\star)$ and $\gamma_1^\star(x^\star) < P'(0)x^\star$ by Corollary 4.3, we find from (31) that

$$\gamma_1(x) = P'(0)x - \Big|\ \gamma_1(x^\star) - \gamma_2(x^\star)\ \Big|\ \ K[\gamma_1,\gamma_2](x^\star, x) - \int_{x^\star}^{x} R[\gamma_2](y)\ K[\gamma_1,\gamma_2](y,x)\ \mathrm{d}y \leq P'(0)x$$

for all $x \in [x^\star, x_c]$. The proof is completed since $x_c > -\frac{1}{P'(0)}$ by hypothesis S2, and thus $\gamma_1(x_c) \leq P'(0)x_c \leq -1$. ■

We conclude this section by showing that $\mathcal{D}(P) < \infty$ implies that all solutions that are either positive or go below $\gamma_1 = -1$ have a finite slope as $x \to \infty$.

Proposition 4.11 *Assume $\mathcal{D}(P) < \infty$ and $\gamma_1(x_0) \leq -1$ or $\gamma_1(x_0) > 0$, there exists $s > 0$ such that*

$$\lim_{x\to\infty} \frac{\gamma_1(x)}{x} = \begin{cases} -s < 0 & \textit{if } \gamma_1(x_0) \leq -1 \\ s > 0 & \textit{if } \gamma_1(x_0) > 0 \end{cases} .$$

If $\gamma_1(x_0) \leq -1$, the convergence towards the limit is given by

$$\Big|\ \frac{\gamma_1(x)}{x} + s\ \Big| \leq C \int_x^\infty \frac{-P(z)}{z^3}\mathrm{d}z\ . \tag{45}$$

If $\mathcal{D}(P) < \infty$ and $\gamma_1(x_0) > 0$, then (8) also hold, with $-s$ replaced by s.

Proof. Note that from Proposition 4.7 and 4.9, the hypothesis $\mathcal{D}(P) < \infty$ implies that all solutions under consideration here satisfy

$$c_1\, x \leq |\gamma_1(x)| \leq c_2\, x \tag{46}$$

for some $c_1, c_2 > 0$ and all $x \geq x_0$. The integral formulation (30) then gives

$$\frac{\gamma_1(x)}{x} = \frac{1+\gamma_1(x_0)}{x_0} - \frac{1}{x} - \int_{x_0}^{x} \frac{P(z)}{z^2\gamma_1(z)}\mathrm{d}z\ , \tag{47}$$

from which the proof follows immediately, since the r.h.s. of (47) converges by (46) and the hypothesis $\mathcal{D}(P) < \infty$. ■

4.4 The confinement solution

In this section, we consider $P(x)$ satisfying the hypotheses H1,H2 and S4. In particular, recall that we assume the existence of $x_r > 0$ such that $P(x_r) = -\frac{1}{4}$ and $P(x) > -\frac{1}{4}$ for all $x > x_r$ and $P(x)$ tending to a finite limit as $x \to \infty$. Any solution of (5) that satisfies

$$\lim_{x\to\infty} \gamma_1(x) = \gamma_\infty \equiv -\frac{1+\sqrt{1+4P_\infty}}{2} \,, \tag{48}$$

needs to solve the integral equation obtained by taking the (formal) limit $x^\star \to \infty$ in (30), namely

$$\gamma_1^c(x) = -1 + x\int_x^\infty \frac{P(z)}{z^2\gamma_1^c(z)}\mathrm{d}z = -1 + \int_1^\infty \frac{P(xt)}{t^2\gamma_1^c(xt)}\mathrm{d}t \,. \tag{49}$$

Defining

$$\mathcal{T}[h](x) = \int_1^\infty \frac{P(xt)-P_\infty}{t^2(\gamma_\infty + h(xt))}\mathrm{d}t - \frac{P_\infty}{\gamma_\infty}\int_1^\infty \frac{h(xt)}{t^2(\gamma_\infty + h(xt))}\mathrm{d}t \,,$$

we see that any confinement solution can be written as $\gamma_1^c(x) = \gamma_\infty + h(x)$ where $h(x)$ satisfies $h(x) = \mathcal{T}[h](x)$. Consider then $\mathcal{B}_{x_0}$ the Banach space obtained by completing the space of $\mathcal{C}_0^\infty([x_0,\infty),\mathbf{R})$ functions under the norm

$$\|f\|_{x_0} \equiv \sup_{x\geq x_0} |f(x)| + x|f'(x)| \,.$$

Since $\lim\limits_{x\to\infty} P(x) - P_\infty = \lim\limits_{x\to\infty} xP'(x) = 0$, $\|P-P_\infty\|_{x_0}$ can be made as small as one likes by taking $x_0 > x_r$ large enough. Standard arguments then show that $\mathcal{T}$ is a *contraction* in a ball of positive radius $\rho < 1$ centered at 0 in $\mathcal{B}$, which shows there exists a unique $h \in \mathcal{B}$ solving $h = \mathcal{T}[h]$. Since $h(x)$ is regular, $\gamma_1^c(x) = \gamma_\infty + h(x)$ solves (5) for all $x \geq x_0$. We now remark that $\gamma_1^c(x)$ will satisfy (49) as long as it exists when x decreases below x_0. However, it can only cease to exist if it satisfies $\gamma_1(x_{\min}) = 0$ for some $x_{\min} > 0$, for the r.h.s. of (5) is negative for large negative γ_1. Assuming it can be continued up to $x = 0$, we can conclude from Theorem (2.2) that either $\gamma_1^c(0) = 0$ or $\gamma_1^c(0) = -1$. Finally, if $P'(x) > 0$ for all $x > x_{\max}$, then the lower nullcline

$$\gamma_1^-(x) = -\frac{\sqrt{1+4P(x)}+1}{2}$$

is decreasing towards γ_∞. If there was an $x_0 > x_{\max}$ such that $\gamma_1^c(x_0) = \gamma_1^-(x_0)$, then $\gamma_1^c(x)$ would enter a region of strictly positive derivatives w.r.t. x, and hence we would get $\gamma_1^c(x) > \gamma_1^-(x_0) > \gamma_\infty$, contradicting (48). Similarly, if there was an $x_0 > x_{\max}$ such that $\gamma_1^c(x_0) = \gamma_\infty$, then $\gamma_1^c(x)$ would enter a region of strictly negative derivative w.r.t. x, and hence would satisfy $\gamma_1^c(x) < \gamma_\infty$ for all $x > x_0$, contradicting (48) again. We thus find that under the monotonicity assumption on $P(x)$, we have

$$\gamma_\infty \equiv -\frac{1+\sqrt{1+4P_\infty}}{2} < \gamma_1(x) < -\frac{1+\sqrt{1+4P(x)}}{2}$$

for all (finite) $x > x_{\max}$. This concludes the proof of Theorem 2.5.

5 Post Scriptum: QED[5] revisited

In a previous publication (see [5]), we have considered the initial value problem

$$\frac{\mathrm{d}\gamma_1(x)}{\mathrm{d}x} = f_s(\gamma_1(x), x) \equiv \frac{\gamma_1(x) + \gamma_1(x)^2 - P(x)}{sx\gamma_1(x)} , \quad \gamma_1(x_0) = \gamma_0 > 0 . \tag{50}$$

with $0 < x_0 < 1$ fixed and P a $\mathcal{C}^2$ function on $[0, \infty)$, positive on $(0, \infty)$, with $P(0) = 0$ and $P'(0) \neq 0$. Defining the following (possibly infinite) quantity

$$\mathcal{D}(P) = \int_{x_0}^{\infty} \frac{-P(z)}{z^3} \mathrm{d}z ,$$

we showed that $\mathcal{D}(P)$ was intimately linked to the behavior/existence as $x \to \infty$ of solutions starting with $\gamma_1(x_0) > 0$. Namely, if $\mathcal{D}(P) < \infty$, we showed that there was a smallest (non-zero) value $\gamma_1^\star(x_0)$ separating global solutions (with $\gamma_1(x_0) \geq \gamma_1^\star(x_0)$) from solutions that cannot be continued to $x = \infty$. We also showed that despite the singular nature of (50) as $x \to 0$ and/or $\gamma_1 \to 0$, all solutions of (50) could be continued for all $x \in [0, x_0]$, and would approach $(0, 0)$ while satisfying for all $x \in [0, x_0]$ the bound

$$\gamma_1(x) \leq \begin{cases} C_b \, x & \text{if} \quad s < 1 \\ C_b \, x \, |\ln(x)| & \text{if} \quad s = 1 \\ C_b \, x^{1/s} & \text{if} \quad s > 1 \end{cases} , \tag{51}$$

for some constant $C_b = C_b(s, \gamma_0)$. In Lemma 5.2 below, we will improve the above to get linear bounds in all cases (i.e. for $s \geq 1$ as well). In the mean time, we define, for any (finite) integer $p \geq 2$, the truncation of the (divergent) series solution:

$$\gamma_{2,p}(x) = P'(0)x + \sum_{n=2}^{p} a_n \, x^n .$$

The coefficients $\{a_n\}_{n=2 \dots p}$ can be found recursively by imposing $|R[\gamma_{2,p}](x)| \leq Cx^{p-1}$ as $x \to 0$, where the remainder map R is defined by

$$R[\gamma_2](x) \equiv \frac{\mathrm{d}\gamma_2(x)}{\mathrm{d}x} - \frac{\gamma_2(x) + \gamma_2(x)^2 - P(x)}{sx\gamma_2(x)}$$

for $\gamma_2(x)$ *any* function on $[0, x_0]$. Note that by choosing x_0 sufficiently small, we can ensure that $\gamma_{2,p}(x) > 0$ for all $x \in [0, x_0]$. Also, since $P'(0) \neq 0$ and $\gamma_{2,p}$ is continuous, there exists a constant $C > 0$ such that $\gamma_2(x) \leq Cx$ for all $x \in [0, x_0]$.

Finally, we also note that if γ_1 solves (50) and $\gamma_2(x)$ is *any* positive function on $[0, x_0]$, then for all $x \in [0, x_0]$, we have

$$\gamma_1(x) - \gamma_2(x) = \Big(\gamma_1(x_0) - \gamma_2(x_0)\Big) \, K[\gamma_1, \gamma_2](x_0, x) + \int_x^{x_0} R[\gamma_2](y) \, K[\gamma_1, \gamma_2](y, x) \, \mathrm{d}y , \tag{52}$$

where

$$K[\gamma_1,\gamma_2](x_0,x) \equiv \exp\Big(\int_{x_0}^{x} \frac{1}{sz} + \frac{P(z)}{sz\gamma_1(z)\gamma_2(z)}\mathrm{d}z\Big) = \Big(\frac{x}{x_0}\Big)^{\frac{1}{s}} \exp\Big(-\int_{x}^{x_0} \frac{P(z)}{sz\gamma_1(z)\gamma_2(z)}\mathrm{d}z\Big) .$$

We are now in position to show that all solutions to (50) agree to all orders in perturbation theory with the series solution.

Theorem 5.1 *Let γ_1 and γ_2 be two solutions of (50). Fix $p \geq 2$ and let $\gamma_{2,p}$ as above. Then there exist constants C and C_p such that*

$$\Big|\ \gamma_1(x) - \gamma_{2,p}(x)\ \Big| \leq C_p x^p \ , \tag{53}$$

$$\Big|\ \gamma_1(x) - \gamma_2(x)\ \Big| \leq \Big|\ \gamma_1(x_0) - \gamma_2(x_0)\ \Big| \Big(\frac{x}{x_0}\Big)^{\frac{1}{s}} \exp\Big(C\Big(\frac{1}{x_0} - \frac{1}{x}\Big)\Big) , \tag{54}$$

where (53) holds for all $x \in [0, x_0/2]$ and (54) for all $x \in [0, x_0]$.

In other words, for any solution of (50), any derivative (of finite order) converges as $x \to 0$ to the corresponding derivative of $\gamma_{2,p}$. Hence a (truncated) power series expansion at $x = 0$ of any solution agrees to any (finite) order with the truncated divergent series of the same order. Also, the difference between any two solutions of (50) decays faster than $\mathrm{e}^{-C/x}$ as $x \to 0$.

Proof. In Lemma 5.2 below, we prove that there exist constants c_1 and c_2 such that $\gamma_1(x) \leq c_1 x$ and $\gamma_2(x) \leq c_2 x$ for all $x \in [0, x_0]$ if γ_1 and γ_2 are solutions of (50). In particular, there exist a constant $C > 0$ such that

$$K[\gamma_1,\gamma_2](y,x) \leq \Big(\frac{x}{y}\Big)^{\frac{1}{s}} \exp\Big(-\int_{x}^{y} \frac{C}{z^2}\mathrm{d}z\Big) = \Big(\frac{x}{y}\Big)^{\frac{1}{s}} \exp\Big(C\Big(\frac{1}{y} - \frac{1}{x}\Big)\Big) \tag{55}$$

for all $0 \leq x \leq y \leq x_0$. Since $R[\gamma_2](x) = 0$ if γ_2 is a solution of (50), we have

$$\gamma_1(x) - \gamma_2(x) = \Big(\gamma_1(x) - \gamma_2(x)\Big) K[\gamma_1,\gamma_2](x_0,x) \ ,$$

from which we get (54) immediately.

On the other hand, $\gamma_{2,p}$ also satisfies $\gamma_{2,p}(x) \leq Cx$ for all $x \in [0, x_0]$, and so $K[\gamma_1, \gamma_{2,p}]$ also satisfies

$$K[\gamma_1,\gamma_{2,p}](y,x) \leq \Big(\frac{x}{y}\Big)^{\frac{1}{s}} \exp\Big(C\Big(\frac{1}{y} - \frac{1}{x}\Big)\Big)$$

for all $0 \leq x \leq y \leq x_0$. In particular, for fixed x_0, $K[\gamma_1, \gamma_{2,p}](x_0, x)$ decays faster than any power law as $x \to 0$. Assume now that $x \leq x_0/2$, and note that

$$\Big|\ \gamma_1(x) - \gamma_{2,p}(x)\ \Big| \leq \Big|\ \gamma_1(x_0) - \gamma_{2,p}(x_0)\ \Big|\ \ K[\gamma_1,\gamma_{2,p}](x_0,x) + C\int_{x}^{x_0} y^{p-1}\, K[\gamma_1,\gamma_{2,p}](y,x)\, \mathrm{d}y \ .$$

Splitting the integral over $[x, x_0]$ into $[x, 2x]$ and $[2x, x_0]$, and using that $K[\gamma_1, \gamma_{2,p}](y, x)$ is a decreasing function of y, we find

$$\int_x^{2x} y^{p-1}\, K[\gamma_1, \gamma_{2,p}](y, x)\, \mathrm{d}y \le \int_x^{2x} y^{p-1}\, K[\gamma_1, \gamma_{2,p}](x, x)\, \mathrm{d}y = x^p \big(\tfrac{2^p - 1}{p}\big)$$
$$\int_{2x}^{x_0} y^{p-1}\, K[\gamma_1, \gamma_{2,p}](y, x)\, \mathrm{d}y \le K[\gamma_1, \gamma_{2,p}](2x, x) \int_{2x}^{x_0} y^{p-1}\, \mathrm{d}y = K[\gamma_1, \gamma_{2,p}](2x, x) \big(\tfrac{x_0^p - x^p}{p}\big)\ .$$

The proof of (53) is completed by noting that by (55), $K[\gamma_1, \gamma_{2,p}](2x, x)$ and $K[\gamma_1, \gamma_{2,p}](x_0, x)$ decay faster than any power law as $x \to 0$. ■

Lemma 5.2 *For any $\gamma_0 > 0$, the solution of (50) exists for all $x \in [0, x_0]$ and*

$$\gamma_1(x) \le C\, x \quad \forall\, x \in [0, x_0]$$

for some $C = C(x_0, \gamma_0) > 0$.

Proof. We first note for future reference that $\frac{P(x)}{x}$ is continuous for all $x \in [0, x_0]$, and that there exists constants $C_\pm > 0$ such that $C_- x \le P(x) \le C_+ x$ for all $x \in [0, x_0]$. By (51), we only have to consider $s \ge 1$. We first choose $\alpha > 0$ such that

$$\alpha \ge \begin{cases} \max(2C_+, C_b |\ln(x_0)|) & \text{if} \quad s = 1 \\ \max\Big(2C_+, \frac{1}{4x_0(s-1)}, C_b^s 4^{s-1}(s-1)^{s-1}\Big) & \text{if} \quad s > 1 \end{cases} .$$

We then note that

$$f_s(\alpha\, x, x) - \alpha = \frac{1}{sx} - \frac{P(x)}{sx^2\alpha} + \alpha\big(\tfrac{1}{s} - 1\big) \ge \frac{1}{sx} - \frac{C_+}{sx\alpha} + \alpha\big(\tfrac{1}{s} - 1\big) \ge \frac{1}{2sx} + \alpha\big(\tfrac{1}{s} - 1\big)\ ,$$

from which it follows that

$$f_s(\alpha\, x, x) > \alpha \quad \forall\, x \in [0, X(s, \alpha)] \quad \text{where} \quad X(s, \alpha) = \begin{cases} x_0 & \text{if} \quad s = 1 \\ \frac{1}{4\alpha(s-1)} & \text{if} \quad s > 1 \end{cases} . \qquad (56)$$

We then note that

$$\begin{aligned} C_b\, X(1, \alpha) |\ln(X(1, \alpha))| &= C_b\, x_0 |\ln(x_0)| \le \alpha x_0 = \alpha X(1, \alpha)\ , \\ C_b\, X(s, \alpha)^{1/s} &= \frac{C_b}{(4\alpha(s-1))^{1/s}} \le \frac{1}{4(s-1)} = \alpha X(s, \alpha) \quad \text{if} \quad s > 1\ . \end{aligned} \qquad (57)$$

In other words, (51) and (57) imply that the solution of (50) satisfies $\gamma_1(X(s, \alpha)) \le \alpha X(s, \alpha)$. Since $X(s, \alpha) \le x_0$, this means that as x decreases, solutions enter the triangle

$$\Delta_{s,\alpha} = \big\{\ (x, \gamma_1) \mid x \in [0, X(s, \alpha)] \text{ and } 0 \le \gamma_1 \le \alpha\, x\ \big\}$$

through its right boundary (i.e. at $x = X(s, \alpha)$).

Solutions that enter $\Delta_{s,\alpha}$ at $x = X(s, \alpha)$ cannot satisfy $\gamma_1(x^\star) = \alpha x^\star$ at some $x^\star < X(s, \alpha)$, for by (56), we would have $\gamma_1(x) > \alpha x$ for all $x \in (x^\star, X(s, \alpha)]$, a contradiction with $\gamma_1(X(s, \alpha)) \le \alpha X(s, \alpha)$. Hence $\gamma_1(x) \le \alpha x$ for all $x \in [0, X(s, \alpha)]$, and the proof is completed by noting that the bound (51) is stronger than $\gamma_1(x) \le \alpha x$ for $x \in (X(s, \alpha), x_0]$. ■

References

[1] L. F. Abbott, *Introduction To The Background Field Method*, Acta Phys. Polon. B **13** (1982) 33.

[2] L. F. Abbott, *The Background Field Method Beyond One Loop*, Nucl. Phys. B **185** (1981) 189.

[3] R. Acharya, *Mass gap in quantum chromodynamics*, arXiv:hep-th/0611220.

[4] A. C. Aguilar, D. Binosi and J. Papavassiliou, *Gluon and ghost propagators in the Landau gauge: Deriving lattice results from Schwinger-Dyson equations*, Phys. Rev. D **78** (2008) 025010 [arXiv:0802.1870 [hep-ph]].

[5] G. van Baalen, D. Kreimer, D. Uminsky and K. Yeats, *The QED beta-function from global solutions to Dyson-Schwinger equations*, Annals Phys. **324** (2009) 205 [arXiv:0805.0826 [hep-th]].

[6] G. van Baalen, A. Schenkel and P. Wittwer, *Asymptotics of solutions in $nA + nB \to C$ reaction-diffusion systems*, Comm. Math. Phys. **210** (2000) 145–176.

[7] M. Baker and C. k. Lee, *Overlapping Divergence Free Skeleton Expansion In Nonabelian Gauge Theories*, Phys. Rev. D **15** (1977) 2201 [Erratum-ibid. D **17** (1978) 2182].

[8] T. Banks and A. Zaks, *On The Phase Structure Of Vector-Like Gauge Theories With Massless Fermions*, Nucl. Phys. B **196** (1982) 189.

[9] C. Bergbauer and D. Kreimer. *Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology*. IRMA Lect. Math. Theor. Phys. **10** (2006) 133-164.

[10] S. Bethke, *Experimental tests of asymptotic freedom*, Prog. Part. Nucl. Phys. **58** (2007) 351 [arXiv:hep-ex/0606035].

[11] N.N. Bogoliubov, A.A. Logunov, and D.V. Shirkov, Sov. Phys. JETP 37, 574 (1960).

[12] N.N. Bogoliubov, D.V. Shirkov, *Quantum Fields*, Benjamin/Cummings (1983).

[13] Ph. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pene and J. Rodriguez-Quintero, *On the IR behavior of the Landau-gauge ghost propagator*, JHEP **0806** (2008) 099 [arXiv:0803.2161 [hep-ph]].

[14] W. Celmaster and R. J. Gonsalves. *The Renormalization Prescription Dependence Of The QCD Coupling Constant*. Phys. Rev. D **20** (1979) 1420.

[15] J. M. Cornwall, *Dynamical Mass Generation In Continuum QCD*, Phys. Rev. D **26** (1982) 1453.

[16] D. Dudal, J.A. Gracey, S.P. Sorella, N. Vandersickel and H. Verschelde, *A refinement of the Gribov-Zwanziger approach in the Landau gauge: infrared propagators in harmony with the lattice results*, Phys. Rev. D **78** (2008) 065047, 0806.4348(hep-th).

[17] D. Kreimer. Anatomy of a gauge theory. *Ann. Phys.* 321 (2006) 2757–2781.

[18] D. Kreimer, W.D. van Suijlekom, *Recursive relations in the core Hopf algebra,* arXiv:0903.2849 [hep-th], Nucl. Phys. **B** (2009), in press, DOI information: 10.1016/j.nuclphysb.2009.04.025.

[19] D. Kreimer and K. Yeats. *An étude in non-linear Dyson-Schwinger equations.* Nuclear Phys. B Proc. Suppl. **160** (2006) 116–121.

[20] D. Kreimer and K. Yeats, *Recursion and growth estimates in renormalizable quantum field theory,* Commun. Math. Phys. **279** (2008) 401 [arXiv:hep-th/0612179].

[21] W. J. Marciano and H. Pagels, *Quantum Chromodynamics, In *Jacob, M. (Ed.): Perturbative Quantum Chromodynamics*, 59-198. (Phys. Rep. 36C (1978) 137-276) and Rockefeller Univ. New York - COO- 22328-130 (REC.JUN.77) 310 P. (706581)*

[22] R. Oehme, W. Zimmermann, *Quark and gluon propagators in quantum chromodynamics,* Phys. Rev. **D21** (1980) 471-484.

[23] D. V. Shirkov and I. L. Solovtsov, *Analytic model for the QCD running coupling with universal alpha(s)-bar(0) value,* Phys. Rev. Lett. **79** (1997) 1209 [arXiv:hep-ph/9704333].

[24] W. D. van Suijlekom. *Renormalization of gauge fields: A Hopf algebra approach.* Commun. Math. Phys. **276** (2007) 773–798.

[25] W. D. van Suijlekom. *The structure of renormalization Hopf algebras for gauge theories I: Representing Feynman graphs on BV-algebras.* To appear in Commun. Math. Phys. arXiv:0807.0999.

[26] K. A. Yeats, *Growth estimates for Dyson-Schwinger equations,* Thesis, Boston U., arXiv:0810.2249 [math-ph].