# Improved resummation of post-Newtonian multipolar waveforms from circularized compact binaries

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# Two approaches for the gravitational self-force in black hole spacetime: Comparison of numerical results

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Recently, two independent calculations have been presented of finite-mass (''self-force'') effects on the orbit of a point mass around a Schwarzschild black hole. While both computations are based on the standard mode-sum method, they differ in several technical aspects, which makes comparison between their results difficult—but also interesting. Barack and Sago [Phys. Rev. D **75**, 064021 (2007)] invoke the notion of a self-accelerated motion in a background spacetime, and perform a direct calculation of the local self-force in the Lorenz gauge (using numerical evolution of the perturbation equations in the time domain); Detweiler [Phys. Rev. D **77**, 124026 (2008)] describes the motion in terms a geodesic orbit of a (smooth) perturbed spacetime, and calculates the metric perturbation in the Regge-Wheeler gauge (using frequency-domain numerical analysis). Here we establish a formal correspondence between the two analyses, and demonstrate the consistency of their numerical results. Specifically, we compare the value of the conservative $O(\mu)$ shift in $u^t$ (where $\mu$ is the particle's mass and $u^t$ is the Schwarzschild $t$ component of the particle's four-velocity), suitably mapped between the two orbital descriptions and adjusted for gauge. We find that the two analyses yield the same value for this shift within mere fractional differences of $\sim 10^{-5}$–$10^{-7}$ (depending on the orbital radius)—comparable with the estimated numerical error.

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## I. INTRODUCTION

Two recent works discussed the effects of gravitational self-force (GSF) on the motion of a point particle in a circular orbit around a Schwarzschild black hole. Barack and Sago [1] (hereafter BS) carried out a direct calculation of both dissipative and conservative components of the local GSF in the Lorenz gauge. Detweiler [2] (hereafter SD) focused on quantifying a number of gauge-invariant orbital effects associated with the finite-mass of the particle. Both analyses employ the mode-sum method [3,4], wherein the metric perturbation associated with the particle is first decomposed into multipole harmonics, and the contribution to the physical backreaction force is then calculated mode by mode. However, the two analyses use very difference languages in describing the motion of the particle. BS's description adheres to the original fundamental formulation of Mino, Sasaki and Tanaka [5] (also Quinn and Wald [6] and the recent Gralla and Wald [7]), wherein the GSF is viewed as accelerating the particle with respect to a background geodesic. SD, instead, depicts the particle's orbit as a geodesic of a smooth, perturbed spacetime—an interpretation first put forward by Detweiler and Whiting [8]. The two descriptions of the motion are known to be fundamentally equivalent [8] (see [9] for a pedagogical review), but the different language they use makes it rather difficult to compare between the results of the respective calculations.

The goal of this report is to establish a ''common language'' to facilitate comparison between the results of BS and SD, hence show that the two sets of results are in full agreement with each other. The motivation for such a comparison is threefold: First, it provides a reassuring confirmation for the equivalence between the two descriptions of the motion. Second, it demonstrates how the correspondence between the two descriptions is to be established in practice—a similar technique could then potentially be implemented in future calculations of the GSF. Third, the comparison provides a good overall check of both calculations. It is indeed a good check, because the two analyses are highly independent: Not only do they invoke different interpretations of the perturbed motion, they also use different gauges for the metric perturbation, and apply very different numerical methods to evaluate it.

The GSF itself is gauge dependent [10], and, of course, so is the metric perturbation. To compare the results of two calculations held in different gauges, one naturally seeks to consider gauge-invariant quantities which are affected by the finite mass of the particle in a nontrivial way. A list of such quantities, for circular orbits in Schwarzschild, was introduced by Detweiler in [11] (cf. SD). In discussing gauge-invariant GSF effects, it is useful to distinguish between effects which are *dissipative* and others which are *conservative*. Dissipative effects are manifest in a gradual, secular drift in the value of the intrinsic orbital parameters (such as the energy and angular momentum) due to the emission of gravitational waves. In the case of a circular-equatorial orbit in Schwarzschild, such effects arise from the $t$ and $\varphi$ components of the GSF (henceforth $t$, $r$, $\theta$, $\varphi$ denote the standard Schwarzschild coordinates), which are directly related to the rate of change of orbital energy and angular momentum, respectively. The deficit in energy and angular momentum is carried away by the gravitational waves, and can in principle be measured by a distant observer—it is therefore not surprising to find that, in the above orbital setup, the $t$ and $\varphi$ components of the GSF are *gauge invariant* (within a class of physically reasonable gauges). This makes it relatively easy to test the values of these components in numerical calculations. Indeed, both BS and SD were able to demonstrate that their numerical results for the $t$ and $\varphi$ components of the local GSF were in agreement with the values inferred from the fluxes of energy and angular momentum in the gravitational waves (the latter values were derived by BS and SD from their respective numerically calculated metric perturbations, and confirmed against values tabulated in the literature). Hence, BS and SD's results for the dissipative part of the GSF are already well tested, and we shall not consider them further in this work.

The situation with the conservative part of the GSF is more involved. Conservative GSF effects are manifest in a small shift in the instantaneous values of the orbital parameters away from their unperturbed values; such effects alter the time dependence of the orbital phases, and may

TABLE I. Comparison between numerical results from BS (Barack and Sago, Ref. [1]) and SD (Detweiler, Ref. [2]). The first column gives the radius of the circular orbit in terms of the Schwarzschild standard areal radial coordinate $r$ [or, equivalently at the relevant order, in terms of the gauge-invariant radius $R$—see the discussion around Eqs. (4.6) and (4.7)]. The second column displays the numerical value of the perturbation quantity $\tilde{H}$ [defined in Eq. (3.9)], as calculated in this paper using the BS Lorenz-gauge code. In the third column we tabulate the numerical values of the gauge-invariant $O(\mu)$ quantity $\Delta U$ [defined in Eq. (4.2)], as calculated here based on Eq. (4.7) and using the BS metric perturbation. The numbers in square brackets are the estimated fractional errors in the BS data. The fourth column displays the values of $\Delta U$ as derived using the SD code, to be compared with the BS values in the third column. In the fifth column we indicate the relative difference between the BS and SD results. This difference is found to be comparable in magnitude to the estimated numerical error in the BS data (which is larger than that in the SD data), providing a reassuring validation test for both analyses.

| $r_0/M$ | $\tilde{H}$ | $\Delta U$ (from BS) | $\Delta U$ (from SD) | Rel. diff. |
|---|---|---|---|---|
| 6.0 | −0.523602 | −0.296040244 [7e − 05] | −0.2960275 | 4e − 05 |
| 6.2 | −0.493483 | −0.276743089 [6e − 05] | −0.2767327 | 4e − 05 |
| 6.4 | −0.466941 | −0.260014908 [6e − 05] | −0.2600063 | 3e − 05 |
| 6.6 | −0.443349 | −0.245359714 [6e − 05] | −0.2453525 | 3e − 05 |
| 6.8 | −0.422222 | −0.232402084 [6e − 05] | −0.2323959 | 3e − 05 |
| 7.0 | −0.403177 | −0.220852781 [5e − 05] | −0.2208475 | 2e − 05 |
| 7.2 | −0.385906 | −0.210485427 [5e − 05] | −0.2104809 | 2e − 05 |
| 7.4 | −0.370163 | −0.201120318 [5e − 05] | −0.2011164 | 2e − 05 |
| 7.6 | −0.355745 | −0.192612971 [5e − 05] | −0.1926095 | 2e − 05 |
| 7.8 | −0.342483 | −0.184845874 [4e − 05] | −0.1848428 | 2e − 05 |
| 8.0 | −0.330239 | −0.177722443 [4e − 05] | −0.1777197 | 2e − 05 |
| 9.0 | −0.280717 | −0.149362192 [3e − 05] | −0.1493606 | 1e − 05 |
| 10.0 | −0.244630 | −0.129123253 [2e − 06] | −0.1291222 | 8e − 06 |
| 11.0 | −0.217039 | −0.113875315 [1e − 06] | −0.1138747 | 5e − 06 |
| 12.0 | −0.195196 | −0.101936046 [1e − 06] | −0.1019355 | 5e − 06 |
| 13.0 | −0.177441 | −0.092313661 [1e − 06] | −0.09231331 | 4e − 06 |
| 14.0 | −0.162705 | −0.084382221 [1e − 06] | −0.08438195 | 3e − 06 |
| 15.0 | −0.150267 | −0.077725527 [1e − 06] | −0.07772532 | 3e − 06 |
| 16.0 | −0.139621 | −0.072055223 [1e − 06] | −0.07205505 | 2e − 06 |
| 18.0 | −0.122337 | −0.062902026 [1e − 06] | −0.06290189 | 2e − 06 |
| 20.0 | −0.108893 | −0.055827795 [6e − 07] | −0.05582771 | 2e − 06 |
| 25.0 | −0.085479 | −0.043599881 [3e − 07] | −0.04359984 | 9e − 07 |
| 30.0 | −0.070380 | −0.035778334 [5e − 07] | −0.03577831 | 7e − 07 |
| 40.0 | −0.052029 | −0.026339690 [3e − 07] | −0.02633967 | 7e − 07 |
| 50.0 | −0.041277 | −0.020844661 [2e − 07] | −0.02084465 | 5e − 07 |
| 60.0 | −0.034211 | −0.017247596 [1e − 06] | −0.01724759 | 3e − 07 |
| 70.0 | −0.029211 | −0.014709648 [7e − 07] | −0.01470964 | 5e − 07 |
| 80.0 | −0.025487 | −0.012822962 [6e − 07] | −0.01282296 | 2e − 07 |
| 90.0 | −0.022605 | −0.011365317 [5e − 07] | −0.01136531 | 6e − 07 |
| 100.0 | −0.020309 | −0.010205285 [4e − 07] | −0.01020528 | 5e − 07 |
| 120.0 | −0.016880 | −0.008475253 [2e − 07] | −0.008475251 | 2e − 07 |
| 150.0 | −0.013469 | −0.006757093 [3e − 07] | −0.006757092 | 2e − 07 |
| 200.0 | −0.010076 | −0.005050643 [3e − 07] | −0.005050642 | 2e − 07 |

have important influence on the phasing of the emitted gravitational waves [12,13]. In the case of a circular orbit in Schwarzschild, which concerns us here, the conservative effects are entirely due to the radial ($r$) component of the GSF. This component is not gauge invariant, and its value depends on the gauge in which the associated metric perturbation is expressed. A meaningful, gauge-invariant description of the conservative effects requires knowledge of both the GSF and the metric perturbation associated with it. It is not possible to check the self-consistency of the numerical calculation of the conservative GSF's component using energy-momentum balance considerations as done with the dissipative components. It is possible to test this calculation against results from the post-Newtonian (PN) literature (as done in SD by examining gauge-invariant quantities; see below), but such tests are only applicable for orbits with a sufficiently large radius. The two independent analyses by BS and SD now provide us with an opportunity to test the calculation of the conservative GSF effects in the strong-field regime.

For our comparison we shall consider the two physically observable gauge-invariant quantities identified by Detweiler in [2,11]. (The meaning of "gauge invariance" here will be made precise below; it refers to a class of gauge transformations compatible with the helical symmetry of the perturbed spacetime.) The first quantity is the azimuthal orbital frequency $\Omega$ (defined with respect to Schwarzschild time $t$), from which one derives a "gauge-invariant" orbital radius $R$ [see Eq. (4.1) below]. The second gauge invariant is $U \equiv u^t$, the contravariant $t$ component of the particle's four-velocity. The functional form of the relation $U(R)$ is independent of the gauge, and thus provides a convenient handle with which to compare between two calculations held in different gauges.

The relation $U(R)$ (including finite-mass terms) has been calculated in SD and used there as a basis for comparison with results from PN calculations. Here we will calculate $U(R)$ based on the Lorenz-gauge results of BS. We shall see that a meaningful comparison between $[U(R)]_{\rm SD}$ and $[U(R)]_{\rm BS}$ requires two important adjustments: First, one needs to account for the fact that the orbits in the two analyses are formally defined in two different geometries: It is an accelerated trajectory on a background spacetime in BS, whereas in SD it is a geodesic in a perturbed spacetime. We shall see that a formal connection between the two descriptions can be established by suitably relating the proper-time parameters along the two orbits.

The second necessary adjustment is more subtle: The (perturbed) quantity $U$ was shown in SD to be gauge invariant, for circular orbits in Schwarzschild, within a class of "physically reasonable" gauges (we shall define this class more accurately in the next section). We will show, however, that the Regge-Wheeler-gauge perturbation of SD and the Lorenz-gauge perturbation of BS are related by a gauge transformation which does *not* leave $U$ invariant. We will then work out explicitly a gauge transformation which brings the BS perturbation into the same class of "physically reasonable" gauges as the SD perturbation, and use this to gauge-adjust the value of $U$ (the necessary gauge adjustment will have a monopole component only).

The structure of this work is as follows. In Sec. II we review BS's and SD's formulations of the orbital motion and introduce the necessary notation for our analysis. In Sec. III we establish a formal relation between $[U(R)]_{\rm SD}$ and $[U(R)]_{\rm BS}$ by performing the two adjustments mentioned above. In Sec. IV we describe the numerical derivation of $U(R)$ within the BS analysis, and compare the numerical values (adjusted for proper time and for gauge) with those derived in SD. This comparison is displayed in Table I.

Throughout this work we use standard geometrized units (with $c = G = 1$) and metric signature $(- + + +)$. $t$, $r$, $\theta$, $\varphi$ represent the standard Schwarzschild coordinates.

## II. REVIEW AND NOTATION

BS and SD use rather different notation. For our purpose it will be useful to introduce a unified notation, which is one of the aims of the following short review. Our convention will be that quantities arising from the BS analysis are denoted with a "tilded" symbol (e.g., $\tilde{X}$), while their SD counterparts are left "untilded":

$$\begin{aligned} &\tilde{X}\text{: quantities arising from BS,} \\ &X\text{: quantities arising from SD.} \end{aligned} \tag{2.1}$$

We will generally denote "background" quantities (i.e., ones obtained when the metric perturbation and all GSF effects are ignored) with a script "0" (as in $X_0$). To ensure that BS and SD correspond to identical physical setups, we will require that all background quantities attain the same values in both analyses:

$$X_0 = \tilde{X}_0\text{: 'background' quantities, GSF ignored.} \tag{2.2}$$

### A. Orbital setup and equation of motion

In both analyses we consider a pointlike particle of mass $\mu(= \tilde{\mu})$, in a circular orbit around a Schwarzschild black hole with mass $M(= \tilde{M}) \gg \mu$. The background Schwarzschild metric is denoted $g^{(0)}_{\alpha\beta}(= \tilde{g}^{(0)}_{\alpha\beta})$. At the limit $\mu \to 0$ (i.e., ignoring GSF effects) the particle moves on a geodesic of $g^{(0)}_{\alpha\beta}$, with a fixed Schwarzschild radius $r = r_0(= \tilde{r}_0)$. (Note that identifying $\mu = \tilde{\mu}$, $M = \tilde{M}$ and $r_0 = \tilde{r}_0$ is sufficient to guarantee that the underlying physical setups in both BS and SD are identical.) In both analyses we set up our Schwarzschild coordinate system such that the background geodesic is confined to the equatorial plane, $\theta = \pi/2$.

Now consider leading-order effects arising from the finite mass of the particle. Denote the retarded linear metric perturbations due to the particle by $h_{\alpha\beta}$ and $\tilde{h}_{\alpha\beta}$ (each $\propto \mu$), corresponding to SD and BS. The two perturbations are mathematically distinct, since $h_{\alpha\beta}$ is taken in the Regge-Wheeler gauge [2], whereas $\tilde{h}_{\alpha\beta}$ is taken in the Lorenz gauge [1]. In BS, the (regularized) backreaction from $\tilde{h}_{\alpha\beta}$ is seen as giving rise to a GSF $\tilde{F}^\alpha$, which accelerates the particle on the background spacetime $g^{(0)}_{\alpha\beta}$. SD, on the other hand, does not calculate the GSF directly. Instead, he constructs the smooth field $h^{\rm R}_{\alpha\beta}$ ("R" field), which is a particular vacuum solution of the perturbed Einstein equations (again, in the Regge-Wheeler gauge), with the property that the perturbed orbit is a geodesic of the total (smooth) perturbed metric $g_{\alpha\beta} \equiv g^{(0)}_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$. SD then attributes backreaction effects to the shift in the values of the orbital parameters with respect to their unperturbed values.

Let us parametrize the perturbed orbits in BS and SD by their respective proper times $\tilde{\tau}$ and $\tau$. We need here two different symbols, because the two orbits are formally defined in two different spacetimes: $g^{(0)}_{\alpha\beta}$ and $g_{\alpha\beta}$, correspondingly. Let events along the (accelerated) BS orbit have Schwarzschild coordinate values $\tilde{x}^\alpha_{\rm p}(\tilde{\tau})$, and events along the (geodesic) SD orbit have "perturbed" Schwarzschild coordinate values $x^\alpha_{\rm p}(\tau)$ [such that, for a given physical point along the orbit, $x^\alpha_{\rm p} - \tilde{x}^\alpha_{\rm p} = O(\mu)$]. Then define the corresponding four velocities $\tilde{u}^\alpha \equiv d\tilde{x}^\alpha_{\rm p}/d\tilde{\tau}$ and $u^\alpha = dx^\alpha_{\rm p}/d\tau$. (Note $\tilde{u}^\alpha$ and $u^\alpha$ are formally vectors in two different geometries.) In our setup we have $u^\theta = \tilde{u}^\theta = 0$, and we impose the circularity conditions $u^r$, $\tilde{u}^r = 0$ and $du^r/d\tau$, $d\tilde{u}^r/d\tilde{\tau} = 0$ [14]. A given physical event along the orbit will generally have $x^\alpha \neq \tilde{x}^\alpha$ and $u^\alpha \neq \tilde{u}^\alpha$, with equalities restored only at the limit $\mu \to 0$. An explicit relation between the corresponding tilded and nontilded perturbed orbital quantities will be established in the next section.

With the above notation, the equations of motion in BS and SD take the respective forms

$$\frac{d\tilde{u}^\alpha}{d\tilde{\tau}} + \Gamma^{\alpha(0)}_{\beta\gamma}\tilde{u}^\beta\tilde{u}^\gamma = \mu^{-1}\tilde{F}^\alpha \tag{2.3}$$

and

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0. \tag{2.4}$$

Here the connection coefficients $\Gamma^{\alpha(0)}_{\beta\gamma}$ correspond to the background metric $g^{(0)}_{\alpha\beta}$, and $\Gamma^\alpha_{\beta\gamma}$ correspond to the smooth perturbed metric $g_{\alpha\beta}$. Note that in BS the GSF effect is entirely accounted for by the acceleration term appearing on the right-hand side (RHS) of the equation of motion, whereas in SD finite-$\mu$ effects are encoded in the perturbed values of $u^\alpha$ and $\Gamma^\alpha_{\beta\gamma}$.

### B. Gauge-invariant conservative GSF effects

A gauge transformation in $g^{(0)}_{\alpha\beta} + h^{\rm R}_{\alpha\beta}$ of the form

$$x^\mu \to x'^\alpha = x^\alpha + \xi^\alpha, \tag{2.5}$$

where $\xi^\alpha$ is of $O(\mu)$, will change the SD vacuum perturbation $h^{\rm R}_{\alpha\beta}$ by an amount

$$\delta_\xi h^{\rm R}_{\alpha\beta} = -(\xi_{\alpha;\beta} + \xi_{\beta;\alpha}). \tag{2.6}$$

Under a similar gauge transformation in $g^{(0)}_{\alpha\beta} + \tilde{h}_{\alpha\beta}$, the GSF in Eq. (2.3) will vary by an amount [10]

$$\delta_\xi \tilde{F}^\alpha = \mu[(\delta^\alpha_\lambda + u^\alpha_0 u_{0\lambda})\ddot{\xi}^\lambda + R^{\alpha(0)}_{\mu\lambda\nu}u^\mu_0\xi^\lambda u^\nu_0], \tag{2.7}$$

where an overdot denotes covariant differentiation with respect to $\tilde{\tau}$, $u^\alpha_0$ is the four-velocity along the background geodesic, and $R^{\alpha(0)}_{\mu\lambda\nu}$ is the background Riemann tensor, evaluated at the particle [15]. The orbits $x^\alpha_{\rm p}(\tau)$ and $\tilde{x}^\alpha_{\rm p}(\tau)$ themselves will obviously change under (2.5) as $\delta_\xi x^\alpha_{\rm p} = \delta_\xi \tilde{x}^\alpha_{\rm p} = \xi^\alpha$, and, taking $\tau$ and $\tilde{\tau}$ derivatives, respectively, we find for the corresponding four-velocities

$$\delta_\xi u^\alpha = \delta_\xi \tilde{u}^\alpha = \frac{d\xi^\alpha}{d\tau}. \tag{2.8}$$

[Note that we may use $d\xi^\alpha/d\tau$ and $d\xi^\alpha/d\tilde{\tau}$ interchangeably, as the two quantities differ only by an amount of $O(\mu^2)$.]

What quantities can we construct, in our circular-equatorial orbit case, which are independent of the gauge? Apart from the obvious $\mu$ and $\tau$, two familiar gauge-invariant quantities are $-du_t/d\tau$ and $du_\varphi/d\tau$ (and their "tilded" counterparts)—the rate of change of the geodesic energy and angular momentum parameters, respectively. These quantities vanish at $\mu \to 0$, and at the leading order $[O(\mu)]$ are entirely due to the *dissipative* effect of the GSF.

For our BS/SD comparison we shall be interested in orbital gauge invariants which are affected by the *conservative* component of the GSF. Let us restrict attention to a class of gauge transformations whose members, denoted $\bar{\xi}^\alpha$, have $\bar{\xi}^\theta = 0$ as well as the helical symmetry

$$(\partial_t + \Omega_0\partial_\phi)\bar{\xi}^\alpha = 0, \tag{2.9}$$

where $\Omega_0 = u^\varphi_0/u^t_0 = \sqrt{M/r_0^3}$ is the orbital frequency of the background geodesic. This choice is motivated as follows: If the original metric perturbation respects the helical symmetry of the physical (black hole + particle) configuration, and is also symmetric under reflection through the equatorial plane—then a gauge transformation within the family $\bar{\xi}$ would retain these symmetries. The condition (2.9) implies that, along the orbit, $d\bar{\xi}^\alpha/d\tau = d\bar{\xi}^\alpha/d\tilde{\tau} = 0$ through $O(\mu)$. Equation (2.8) therefore tells us that all components $u^\alpha$ (and $\tilde{u}^\alpha$) are invariant under $\bar{\xi}$. Of these four components, only $u^t$ and $u^\varphi$ exhibit nontrivial conservative $O(\mu)$ effects, and so could serve usefully as gauge invariants for our comparison. In fact, we shall follow here SD, and utilize the alternative pair $U \equiv u^t$ and

$$\Omega \equiv u^\varphi/u^t = d\varphi_{\rm p}/dt_{\rm p}, \tag{2.10}$$

along with their "tilded" counterparts $\tilde{U} \equiv \tilde{u}^t$ and $\tilde{\Omega}$. The physical interpretation of the gauge-invariant $U$ is somewhat less obvious than that of the orbital frequency $\Omega$; it is discussed in SD. Here we shall refer to $U$ (or $\tilde{U}$) as the *time function* along the orbit.

The orbital frequency and time function are given in BS, through $O(\mu)$, by

$$\tilde{\Omega} = \Omega_0\left[1 - \frac{r_0(r_0 - 3M)}{2\mu M}\tilde{F}_r\right], \tag{2.11}$$

$$\tilde{U} = U_0\left(1 - \frac{r_0}{2\mu}\tilde{F}_r\right), \tag{2.12}$$

where $U_0 \equiv (1 - 3M/r_0)^{-1/2}$ is the time function along the background geodesic. The corresponding SD quantities are given by

$$\Omega = \Omega_0\left[1 - \frac{r_0(r_0 - 3M)}{4M}u^\alpha u^\beta h^{\rm R}_{\alpha\beta,r}\right], \tag{2.13}$$

$$U = U_0\left(1 + \frac{1}{2}u^\alpha u^\beta h^{\rm R}_{\alpha\beta} - \frac{r_0}{4}u^\alpha u^\beta h^{\rm R}_{\alpha\beta,r}\right), \tag{2.14}$$

where $h^{\rm R}_{\alpha\beta}$ and $h^{\rm R}_{\alpha\beta,r}$ are, of course, evaluated on the circular orbit. That the quantities in Eqs. (2.11), (2.12), (2.13), and (2.14) are gauge invariant under $\bar{\xi}$ can be readily verified with an explicit calculation, using $\delta_{\bar{\xi}} r_0 = \bar{\xi}^r$ and $\delta_{\bar{\xi}}\tilde{F}_r = -3\mu M\bar{\xi}^r/[r_0^2(r_0 - 3M)]$ (see Ref. [1]), along with Eq. (2.6).

## III. RELATION BETWEEN BS AND SD GAUGE INVARIANTS

We now seek to obtain an explicit relation between $\Omega$ and $\tilde{\Omega}$ and between $U$ and $\tilde{U}$. We achieve this in two steps. In the first step we map the BS trajectory onto a geodesic $\tilde{y}(\tau)$ defined à la SD in a perturbed spacetime (but in the Lorenz gauge). We define the quantities $\tilde{\Omega}_y$ and $\tilde{U}_y$ associated with this geodesic, and relate them to the original $\tilde{\Omega}$ and $\tilde{U}$. We then discuss the (somewhat unexpected) fact that the BS and SD metric perturbations are related by a gauge transformation which is *not* within the family $\bar{\xi}$. In the second step we therefore work out a gauge transformation which brings the two perturbations within a $\bar{\xi}$ transformation, and use this to establish an explicit relation between $\tilde{\Omega}_y$, $\tilde{U}_y$ and the SD quantities $\Omega$, $U$.

### A. Mapping between BS and SD trajectories

In BS, the orbit $\tilde{x}^\alpha_{\rm p}(\tilde{\tau})$ is an (accelerated) trajectory in the Schwarzschild background $g^{(0)}_{\alpha\beta}$. However, one can reinterpret this orbit, in the spirit of Detweiler and Whiting [8], as a geodesic of a smooth perturbed geometry $\tilde{g}_{\alpha\beta} \equiv g^{(0)}_{\alpha\beta} + \tilde{h}^{\rm R}_{\alpha\beta}$, where $\tilde{h}^{\rm R}_{\alpha\beta}$ is the R part of the BS (Lorenz-gauge) metric perturbation $\tilde{h}_{\alpha\beta}$. (The field $\tilde{h}^{\rm R}_{\alpha\beta}$ has not been constructed explicitly in Ref. [1] as it is not needed for calculating the GSF in the BS approach. This field can, in principle, be constructed following the prescription of SD but working in the Lorenz gauge.) This geodesic is physically identical with the SD geodesic $x^\alpha_{\rm p}(\tau)$, since $h^{\rm R}_{\alpha\beta}$ and $\tilde{h}^{\rm R}_{\alpha\beta}$ represent the same physical perturbation (merely expressed in different gauges). We may therefore parametrize it by the SD proper-time $\tau$. Let us then denote this geodesic by $\tilde{y}^\alpha_{\rm p}(\tau)$, with associated four-velocity $\tilde{v}^\alpha(\tau) \equiv d\tilde{y}^\alpha_{\rm p}/d\tau$. Here the "tildes" are meant to remind us that these quantities are described in the Lorenz gauge of BS, but one should bear in mind that both $\tilde{y}^\alpha_{\rm p}$ and $\tilde{v}^\alpha$ are defined in the *perturbed* geometry $\tilde{g}_{\alpha\beta}$.

Our goal now will be to relate the vector $\tilde{v}^\alpha(\tau)$ to the BS four-velocity $\tilde{u}^\alpha(\tilde{\tau})$. This task is somewhat delicate, because the two vectors are defined in two different spacetimes, and so one first needs to establish a concrete mapping between the two trajectories $\tilde{y}^\alpha_{\rm p}(\tau)$ and $\tilde{x}^\alpha_{\rm p}(\tilde{\tau})$. The mapping procedure (and its intimate relation with the gauge freedom of the GSF) has been discussed in detail by Barack and Ori [10]. The following derivation is largely inspired by that work.

We begin by obtaining a relation between the proper-time parameters $\tau$ and $\tilde{\tau}$ for a given physical point along the orbit. The $\tau$-interval along the geodesic $\tilde{y}^\alpha_{\rm p}(\tau)$ in $\tilde{g}_{\alpha\beta}$ satisfies

$$d\tau^2 = -(g^{(0)}_{\alpha\beta} + \tilde{h}^{\rm R}_{\alpha\beta})d\tilde{y}^\alpha_{\rm p}d\tilde{y}^\beta_{\rm p}. \tag{3.1}$$

The four-velocity $\tilde{v}^\alpha(\tau)$ along this geodesic satisfies

$$\frac{d\tilde{v}^\alpha}{d\tau} + \tilde{\Gamma}^\alpha_{\beta\gamma}(\tilde{y}_{\rm p})\tilde{v}^\beta\tilde{v}^\gamma = 0, \tag{3.2}$$

where the connections $\tilde{\Gamma}^\alpha_{\beta\gamma}$ are those associated with the full metric $\tilde{g}_{\alpha\beta}$, and are here evaluated at $\tilde{y}^\alpha_{\rm p}(\tau)$. We now wish to think of $g^{(0)}_{\alpha\beta}$ and $\tilde{h}^{\rm R}_{\alpha\beta}$ as separate tensor fields in the geometry $\tilde{g}_{\alpha\beta}$. In general, of course, the splitting of $\tilde{g}_{\alpha\beta}$ into a background field $g^{(0)}_{\alpha\beta}$ and a "perturbation" field $\tilde{h}^{\rm R}_{\alpha\beta}$ depends on the choice of gauge (indeed, it is the *origin* of the gauge freedom). Here, however, we fix both the background coordinates (Schwarzschild) and the gauge (Lorenz), and so both $g^{(0)}_{\alpha\beta}$ and $\tilde{h}^{\rm R}_{\alpha\beta}$ are defined unambiguously. Let $\tilde{\Gamma}^{\alpha(0)}_{\beta\gamma}$ be the connections associated with the field $g^{(0)}_{\alpha\beta}$, and let $\Delta\tilde{\Gamma}^\alpha_{\beta\gamma} \equiv \tilde{\Gamma}^\alpha_{\beta\gamma} - \tilde{\Gamma}^{\alpha(0)}_{\beta\gamma}$. Then Eq. (3.2) can be written in the form

$$\frac{d\tilde{v}^\alpha}{d\tau} + \tilde{\Gamma}^{\alpha(0)}_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{v}^\beta\tilde{v}^\gamma = -\Delta\tilde{\Gamma}^\alpha_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{v}^\beta\tilde{v}^\gamma. \tag{3.3}$$

Next, *define* a new parameter $\tilde{\tau}$ along the geodesic $\tilde{y}^\alpha_\mathrm{p}(\tau)$ using

$$d\tilde{\tau}^2 = -g^{(0)}_{\alpha\beta}d\tilde{y}^\alpha_\mathrm{p}d\tilde{y}^\beta_\mathrm{p} \tag{3.4}$$

[with the requirement that the "zero point" of $\tilde{\tau}$ is chosen such that $\tilde{\tau} - \tau = O(\mu)$]; and *define* the new tangent vector

$$\tilde{u}^\alpha = \frac{d\tilde{y}^\alpha_\mathrm{p}}{d\tilde{\tau}} = \tilde{v}^\alpha\frac{d\tau}{d\tilde{\tau}}. \tag{3.5}$$

We use here the symbols $\tilde{\tau}$ and $\tilde{u}^\alpha$ (in a slight abuse of notation) as we are soon to interpret these as the BS proper time and four-velocity. For now, however, recall that $\tilde{\tau}$ and $\tilde{u}^\alpha$ are defined along the geodesic $\tilde{y}^\alpha_\mathrm{p}$ in the *perturbed* spacetime $\tilde{g}_{\alpha\beta}$ (not in $g^{(0)}_{\alpha\beta}$); but note $\tilde{\tau}$ and $\tilde{u}^\alpha$ are *not* proper time and four-velocity along this geodesic. Substituting $\tilde{v}^\alpha = (d\tilde{\tau}/d\tau)\tilde{u}^\alpha$ and $d/d\tau = (d\tilde{\tau}/d\tau)d/d\tilde{\tau}$ in Eq. (3.3), we now obtain

$$\frac{d\tilde{u}^\alpha}{d\tilde{\tau}} + \tilde{\Gamma}^{\alpha(0)}_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{u}^\beta\tilde{u}^\gamma = -\Delta\tilde{\Gamma}^\alpha_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{u}^\beta\tilde{u}^\gamma - \beta\tilde{u}^\alpha, \tag{3.6}$$

where $\beta \equiv (d\tau/d\tilde{\tau})^2(d^2\tilde{\tau}/d\tau^2)$. The expression on the left-hand side here can be *interpreted* as the acceleration of a trajectory $\tilde{y}_\mathrm{p}(\tilde{\tau})$ (with proper-time $\tilde{\tau}$ and four-velocity $\tilde{u}^\alpha$) in a spacetime $g^{(0)}_{\alpha\beta}$. This acceleration is orthogonal to $\tilde{u}^\alpha$ in the spacetime $g^{(0)}_{\alpha\beta}$ [by virtue of $g^{(0)}_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta = -1$, which follows from Eq. (3.4)], so formally projecting both sides of Eq. (3.6) orthogonally to $\tilde{u}^\alpha$ (in the metric $g^{(0)}_{\alpha\beta}$) finally gives

$$\frac{d\tilde{u}^\alpha}{d\tilde{\tau}} + \tilde{\Gamma}^{\alpha(0)}_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{u}^\beta\tilde{u}^\gamma = -(\delta^\alpha_\mu + \tilde{u}^\alpha\tilde{u}^\nu g^{(0)}_{\nu\mu})\Delta\tilde{\Gamma}^\mu_{\beta\gamma}(\tilde{y}_\mathrm{p})\tilde{u}^\beta\tilde{u}^\gamma. \tag{3.7}$$

Now, Detweiler and Whiting have shown [8] that the expression on the RHS of Eq. (3.7) (where, recall, $\Delta\tilde{\Gamma}^\mu_{\beta\gamma}$ is derived from the R-field $\tilde{h}^\mathrm{R}_{\alpha\beta}$) *is precisely the self-acceleration* $\mu^{-1}\tilde{F}^\alpha$, if one interprets this quantity as a vector in $g^{(0)}_{\alpha\beta}$. Comparing then the forms of Eqs. (2.3) and (3.7), we arrive at the following conclusion: If each point along the geodesic $\tilde{y}_\mathrm{p}(\tau)$ is associated with a point along the BS trajectory having *the same coordinate value* (i.e., $\tilde{x}_\mathrm{p} = \tilde{y}_\mathrm{p}$), then the quantities $\tilde{\tau}$ and $\tilde{u}^\alpha$ defined in the perturbed spacetime $\tilde{g}^{\alpha\beta}$ in Eqs. (3.4) and (3.5) can be interpreted—and would have the same values—as the corresponding BS quantities $\tilde{\tau}$ and $\tilde{u}^\alpha = d\tilde{x}^\alpha_\mathrm{p}/d\tilde{\tau}$.

The main practical outcome from the above discussion are formulas relating the BS quantities $\tilde{\tau}$ and $\tilde{u}^\alpha$ to their counterparts $\tau$ and $\tilde{v}^\alpha$ defined à la SD in the perturbed spacetime. From Eqs. (3.1) and (3.4) we obtain

$$\frac{d\tilde{\tau}}{d\tau} = 1 + \tilde{H} \tag{3.8}$$

[through $O(\mu)$], where

$$\tilde{H} \equiv \frac{1}{2}\tilde{h}^\mathrm{R}_{\alpha\beta}(x_\mathrm{p})\tilde{u}^\alpha\tilde{u}^\beta. \tag{3.9}$$

Equation (3.8) describes the relation between the BS and SD proper-time parameters, assuming the "same-coordinate-value" mapping between the respective trajectories. The relation between the four-velocities then follows immediately from Eq. (3.5):

$$\tilde{v}^\alpha = (1 + \tilde{H})\tilde{u}^\alpha \tag{3.10}$$

[again, through $O(\mu)$].

Let us finally define the frequency $\tilde{\Omega}_y \equiv \tilde{v}^\varphi/\tilde{v}^t$ and time function $\tilde{U}_y \equiv \tilde{v}^t$ associated with the trajectory $\tilde{y}^\alpha(\tau)$, and relate these to the BS quantities $\tilde{\Omega}$ and $\tilde{U}$. For the frequency we have, using Eq. (3.10),

$$\tilde{\Omega}_y = \tilde{v}^\varphi/\tilde{v}^t = \tilde{u}^\varphi/\tilde{u}^t = \tilde{\Omega}. \tag{3.11}$$

For the time function, Eq. (3.10) immediately gives

$$\tilde{U}_y = (1 + \tilde{H})\tilde{U}. \tag{3.12}$$

### B. Adjusting the gauge

If the BS perturbation $\tilde{h}^\mathrm{R}_{\alpha\beta}$ and the SD perturbation $h^\mathrm{R}_{\alpha\beta}$ were related through a gauge transformation within the family $\bar{\xi}$, then the quantities $\tilde{\Omega}_y$ and $\tilde{U}_y$ would have to be equal to their SD counterparts $\Omega$ and $U$, since, recall, $\Omega$ and $U$ are gauge invariant under $\bar{\xi}$. As we now discuss, it turns out that this is not the case: The two perturbations differ by a gauge transformation *not* belonging to the family $\bar{\xi}$.

To see this, it suffices to examine the asymptotic form of the $tt$ components of the BS and SD metric perturbations at $r \to \infty$. In SD a gauge is chosen (within the class of Regge-Wheeler gauges) such that the perturbed metric is asymptotically Minkowskian:

$$g^{(0)}_{tt} + h^\mathrm{R}_{tt} \to -1 \qquad (r \to \infty). \tag{3.13}$$

This is impossible to achieve with the BS Lorenz-gauge perturbation, which has a monople contribution that fails to vanish at $r \to \infty$. This monopole contribution was first derived by Detweiler and Poisson [16], who showed that it *uniquely* describes the correct mass perturbation due to the particle: Any other Lorenz-gauge monopole solution would either diverge at the event horizon or at infinity, and/or fail to describe the correct mass perturbation. The field $\tilde{h}^\mathrm{R}_{\alpha\beta}$ inherits the large-$r$ asymptotic form of the full perturbation $\tilde{h}_{\alpha\beta}$, which is dominated by the above monopole term. The perturbed BS metric has the asymptotic form

$$g^{(0)}_{tt} + \tilde{h}^\mathrm{R}_{tt} \to -1 - 2\alpha \qquad (r \to \infty), \tag{3.14}$$

where

$$\alpha = \frac{\mu}{\sqrt{r_0(r_0 - 3M)}}, \tag{3.15}$$

and where the term $-2\alpha$ comes entirely from the monopole perturbation. (The other tensorial components of the BS metric attain their Minkowski values at $r \to \infty$, just like the SD metric.) The difference between the asymptotic forms in Eqs. (3.13) and (3.14) must be accounted for by a monopole gauge transformation. From Eq. (2.6) we find that the generator of this gauge transformation must satisfy, asymptotically, $\xi_{t,t} = \alpha$ (as well as $\xi_{t,\varphi} = 0$, since a monople transformation cannot depend on $\varphi$), which necessarily violates the condition of Eq. (2.9). Hence, the BS and SD perturbations differ by a gauge transformation *outside* the family $\bar{\xi}$.

The above conclusion by no means suggests that either of BS/SD's metric perturbations violates the helical symmetry of the physical spacetime (they both respect it, in fact). Rather, it calls for a more careful inspection of the class of gauge transformations which maintain helical symmetry. Consider a gauge transformation $\xi^\alpha = at\delta^\alpha_t$, where $a(\propto \mu)$ is a constant. This transformation affects only the $tt$ component of the metric perturbation, which shifts by an amount $\delta_\xi h^R_{tt} = 2(1 - 2M/r)a$. If the original perturbation is helically symmetric [i.e., satisfying $(\partial_t + \Omega_0\partial_\phi)h^R_{\alpha\beta} = 0$], then so will be the gauge-transformed perturbation, even though the generator $\xi^\alpha$ itself does not respect the helical symmetry of the geometry. This suggests a more general class of "physically reasonable" gauge transformations, with generators given by

$$\hat{\xi}^\alpha = \bar{\xi}^\alpha + at\delta^\alpha_t, \tag{3.16}$$

where $\bar{\xi}^\alpha$ is any vector satisfying Eq. (2.9), and $a$ is any constant ($\propto \mu$). While the generators $\hat{\xi}$ themselves are generally not helically symmetric, they do not interfere with the helical symmetry of the metric perturbation they act upon. It is not difficult to convince oneself that the class $\hat{\xi}$ is the most general class with this property [17].

The gauge transformation between the BS and SD perturbations belongs to the class $\hat{\xi}$, with $a = \alpha \neq 0$. Crucially for our analysis, the two quantities $\Omega$ and $U$, which are invariant under $\bar{\xi}$, are not invariant under $\hat{\xi}$ (for any $a \neq 0$). More precisely, we have (for $a = \alpha$)

$$\delta_{\hat{\xi}}\Omega = -\alpha\Omega_0, \qquad \delta_{\hat{\xi}}U = \alpha U_0, \tag{3.17}$$

through $O(\mu)$. These expressions are easily derived using Eq. (2.8), noting $d\bar{\xi}^\alpha/d\tau = 0$ and hence $d\hat{\xi}^\alpha/d\tau = \alpha U\delta^\alpha_t$ along the worldline. Applying the gauge transformation $\hat{\xi}^\alpha = \alpha t\delta^\alpha_t$, the quantities $\tilde{\Omega}_y$ and $\tilde{U}_y$ of Eqs. (3.11) and (3.12) transform as

$$\tilde{\Omega}_y \to \tilde{\Omega}_y - \alpha\Omega_0 = \tilde{\Omega} - \alpha\Omega_0 = \tilde{\Omega}(1 - \alpha) + O(\mu^2), \tag{3.18}$$

and

$$\tilde{U}_y \to \tilde{U}_y + \alpha U_0 = (1 + \tilde{H})\tilde{U} + \alpha U_0 = \tilde{U}(1 + \alpha + \tilde{H}) + O(\mu^2). \tag{3.19}$$

Since this gauge transformation brings the BS and SD perturbations to within a $\bar{\xi}$ transformation from one other, the RHS expressions in Eqs. (3.18) and (3.19) must be equal to the SD quantities $\Omega$ and $U$, respectively, (as, recall, the frequency and time function are invariant under $\bar{\xi}$ transformations). We hence arrive at the desired relations

$$\Omega = \tilde{\Omega}(1 - \alpha), \tag{3.20}$$

$$U = \tilde{U}(1 + \alpha + \tilde{H}), \tag{3.21}$$

holding through $O(\mu)$.

Equations (3.20) and (3.21) explicitly relate between the BS and SD values of the perturbed frequency and time function. These relations do not involve the GSF, but they do require knowledge of the quantity $\tilde{H} \equiv \tilde{h}^R_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta/2$, which is to be constructed from the BS perturbation. A physical interpretation of the quantity $\tilde{H}$ is suggested from Eq. (3.8): It describes how proper-time intervals relate to each other in the different orbital representations of BS and SD. It is straightforward to check that $\tilde{H}$ is invariant, for circular orbits, within the class of gauge transformations $\bar{\xi}$ (though not within $\hat{\xi}$ for $a \neq 0$).

## IV. COMPARISON OF NUMERICAL RESULTS

### A. Gauge-invariant comparison formula

The relations (3.20) and (3.21), as they stand, do not quite yet offer a practical means by which to test the BS/SD numerical results against each other. The reason is as follows: The equalities expressed in these relations hold, through $O(\mu)$, for a given physical orbit with (perturbed) SD radius $r_p$ and BS radius $\tilde{r}_p$. In these equalities, $\Omega$ and $U$ are to be evaluated at $r_p$, while $\tilde{\Omega}$ and $\tilde{U}$ are to be evaluated at $\tilde{r}_p$. Alas, the relation between $r_p$ and $\tilde{r}_p$ is not known to us at $O(\mu)$: It depends on the precise gauge transformation between the BS and SD perturbations, which we have not solved for. Without knowledge of how $r_p$ relates to $\tilde{r}_p$ at $O(\mu)$, it is not possible to extract the $O(\mu)$ parts of Eqs. (3.20) and (3.21). Stated differently, we have the following problem: While the quantities $\Omega$ and $U$ are gauge invariant (under $\bar{\xi}$), the finite-mass differences $\Omega - \Omega_0$ and $U - U_0$ (which are the quantities whose numerical values we wish to test) are *not* gauge invariant, since $\Omega_0$ and $U_0$ depend on the gauge (through $r_0$). We should instead be looking at finite-$\mu$ corrections which are gauge independent.

A standard solution to this problem is to express one gauge invariant in terms of the other, and, following SD, we shall pursue this direction here. SD introduced the gauge-invariant radius

$$R \equiv (M/\Omega^2)^{1/3} \tag{4.1}$$

(denoted $R_\Omega$ in Ref. [2]), and then expressed $U$ in terms of $R$. The difference

$$\Delta U(R) \equiv U(R) - (1 - 3M/R)^{-1/2} \tag{4.2}$$

[which is of $O(\mu)$, since $(1 - 3M/R)^{-1/2} = U_0 + O(\mu)$] is then a genuinely gauge-invariant measure of the conservative finite-$\mu$ effect. SD derived the relation $\Delta U(R)$ numerically, and utilized it for comparison with results from PN theory. Here we shall reconstruct the relation $\Delta U(R)$ from BS quantities, and use it to compare with SD.

Using Eq. (2.12) to substitute for $\tilde{U}$ in Eq. (3.21), and then substituting the result for $U$ in Eq. (4.2), we obtain

$$\Delta U = (1 - 3M/r_0)^{-1/2}\left(1 + \alpha + \tilde{H} - \frac{r_0}{2\mu}\tilde{F}_r\right) - (1 - 3M/R)^{-1/2} + O(\mu^2). \tag{4.3}$$

Our goal is to express the RHS here entirely in terms of the gauge-invariant radius $R$. For this, we need first to express $r_0$ in terms of $R$ up through $O(\mu)$. Starting from the definition (4.1), then using Eqs. (2.11) and (3.20) in succession, and finally substituting $\Omega_0^{-2} = r_0^3/M$, we obtain

$$R^3 = M\Omega^{-2} = M\tilde{\Omega}^{-2}(1-\alpha)^{-2} = r_0^3\left[1 + 2\alpha + \frac{r_0(r_0 - 3M)}{\mu M}\tilde{F}_r\right] + O(\mu^2). \tag{4.4}$$

Hence,

$$r_0 = R\left[1 - \frac{2}{3}\alpha + \frac{R(R-3M)}{3\mu M}\tilde{F}_r\right] + O(\mu^2), \tag{4.5}$$

where the quantities $\alpha$ and $\tilde{F}_r/\mu$, which are already $O(\mu)$, are evaluated at $R$. Substituting for $r_0(R)$ in Eq. (4.3) and expanding through $O(\mu)$, we obtain

$$\Delta U = (1 - 3M/R)^{-1/2}\left(\frac{R - 2M}{R - 3M}\alpha + \tilde{H}\right) + O(\mu^2). \tag{4.6}$$

Finally, we note that on the RHS we can replace $R \to r_0$ without affecting $\Delta U$ at leading order [since $r_0 - R \sim O(\mu)$, and $\alpha$ and $\tilde{H}$ are already $O(\mu)$]. This allows us to recast Eq. (4.6) in a more practical form:

$$\Delta U = (1 - 3M/r_0)^{-1/2}\left(\frac{r_0 - 2M}{r_0 - 3M}\alpha + \tilde{H}\right), \tag{4.7}$$

with corrections of $O(\mu^2)$. Recall $\alpha = \mu[r_0(r_0 - 3M)]^{-1/2}$, and $\tilde{H}$ is the quantity constructed from the BS metric perturbation through Eq. (3.9).

It is interesting to point out that our final expression, Eq. (4.7), bears no direct reference to the GSF. This suggests that the $O(\mu)$ coordinate change $r \to R$, in fact, amounts to a gauge transformation which sets the conservative piece of the BS GSF to zero. This is further suggested by noticing $U_0(r_0(R)) = \tilde{U}(r_0 \to R)$ (up to terms $\propto \alpha$)—which one can readily verify starting with $U_0 = (1 - 3M/r_0)^{-1/2}$ and using Eqs. (2.12) and (4.5).

## B. Numerical results

We have reconstructed the relation $\Delta U(r_0)$ numerically, using the BS Lorenz-gauge code, based on Eq. (4.7). The calculation of the full (retarded) Lorenz-gauge perturbation is describe in detail in Ref. [1]. In short, the method relies on the perturbation formalism of Barack and Lousto [18], in which the Lorenz-gauge perturbation equations are decomposed into tensor harmonics, resulting in a set of 10 partial differential equations (in $t$ and $r$) for each $l$, $m$-harmonic of the perturbation. These equations couple between different tensorial components of the perturbation, but are conveniently written in a form where no coupling occurs in the principal part. In BS these equations are then solved in the time domain (for each given $l$, $m$) using finite differentiation on a characteristic mesh. The monopole and dipole modes ($l = 0, 1$) are calculated separately, based essentially on the semianalytical results of Detweiler and Poisson [16]. The full perturbation is finally obtained by summing over a sufficiently large number of modes $l$, $m$.

To construct the quantity $\tilde{H}$ in Eq. (4.7) next requires us to obtain the R-part of our numerical solutions, evaluated at the particle. Conveniently, it is not strictly the R-part $\tilde{h}^{\rm R}_{\alpha\beta}$ that we need, but rather the contracted quantity $\tilde{h}^{\rm R}_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta$. The construction of this quantity, through mode-by-mode regularization, is prescribed by SD in Sec. IV.B of [2] for any gauge related to the SD gauge through a $\bar{\xi}$-type transformation, and we employ it here in a direct manner. (The regular gauge transformation $\hat{\xi}^\alpha = at\delta^\alpha_t$ does not affect the singular part of the perturbation, hence the regularization procedure described in [2] is applicable within the broader $\hat{\xi}$ family.) We construct $\tilde{H}$ for a series of orbital radii $r_0$, and then use Eq. (4.7) to obtain $\Delta U$ for these radii. We tabulate the values thus obtained in Table I, alongside the corresponding SD values from Ref. [2]. The fractional difference between the BS and SD results is found to be similar in magnitude to the estimated fractional numerical error in the BS data. (How this numerical error is estimated is described in detail in Ref. [1]; the numerical error in the corresponding SD data is in all cases much smaller.) We conclude that the two calculations are in agreement with each other.

## V. DISCUSSION

The agreement established here between the numerical results of BS and SD not only provides an important validation test for both analyses, but it also illustrates (and confirms) a few fundamental results from GSF theory. The following list highlights these results. (i) The GSF, as defined and calculated by Mino, Sasaki and Tanaka [5] and

Quinn and Wald [6], can also be derived from the Detweiler–Whiting R-field $\tilde{h}^{\rm R}_{\alpha\beta}$ [8] using the formula on the RHS of Eq. (3.7). (ii) The GSF can be defined and calculated through a "same-coordinate-value" mapping of the orbit from the physical perturbed spacetime onto a background spacetime; relaxing the "same-coordinate-value" rule gives rise to the gauge ambiguity in the GSF [10]. (iii) A suitable gauge transformation can be made (here $r \to R$) which nullifies the GSF (here, the conservative piece thereof). However, the information about the physical finite-$\mu$ effect can then still be retrieved in full from the metric perturbation in the new gauge—cf. Eq. (4.7). This is a particular example of the general statement that the full information about the finite-$\mu$ effect is contained in the combination of both the GSF and the metric perturbation [10].

As an aside in this work, we discussed what one may mean by a "physically reasonable" gauge transformation in the context of circular orbits. Sensibly, a "physically reasonable" gauge is one in which the metric perturbation admits the approximate helical symmetry of the black hole + particle system, i.e., it satisfies $(\partial_t + \Omega_0 \partial_\phi) h_{\alpha\beta} = 0$. However, it seems unnecessary to require that the gauge transformation generators $\xi^\alpha$ connecting any two such physically-reasonable gauges be themselves helically symmetric. If fact, the general class of such generators, denoted here $\hat{\xi}$, includes members $\hat{\xi}(a \neq 0)$ which are *not* helically symmetric. The gauge transformation between the BS and SD perturbations—both of which being "physically reasonable" in the above sense—is indeed generated by a vector $\hat{\xi}$ which is *not* helically symmetric.

We anticipate that comparisons similar to the one discussed in this work will allow robust tests of self-force calculations for other orbits and other spacetimes (e.g., Kerr) when such calculations are available. The essential elements of our formal discussion are directly applicable to other orbits and geometries. Most important, Eqs. (3.8) and (3.10), which describe the mapping of the orbital elements from a BS-type background spacetime to an SD-type perturbed spacetime, hold quite generally for any orbit in any black hole spacetime, and could form a basis for future comparisons. The major challenge in any such future comparison would remain to devise a suitable set of gauge-invariant quantities.

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[1] L. Barack and N. Sago, Phys. Rev. D **75**, 064021 (2007).
[2] S. Detweiler, Phys. Rev. D **77**, 124026 (2008).
[3] L. Barack and A. Ori, Phys. Rev. D **61**, 061502 (2000).
[4] L. Barack, Y. Mino, H. Nakano, A. Ori, and M. Sasaki, Phys. Rev. Lett. **88**, 091101 (2002).
[5] Y. Mino, M. Sasaki, and T. Tanaka, Phys. Rev. D **55**, 3457 (1997).
[6] T.C. Quinn and R.M. Wald, Phys. Rev. D **56**, 3381 (1997).
[7] S.E. Gralla and R.M. Wald, Classical Quantum Gravity **25**, 205009 (2008).
[8] S. Detweiler and B.F. Whiting, Phys. Rev. D **67**, 024025 (2003).
[9] E. Poisson, Living Rev. Relativity **7**, 6 (2004).
[10] L. Barack and A. Ori, Phys. Rev. D **64**, 124003 (2001).
[11] S. Detweiler, Classical Quantum Gravity **22**, S681 (2005).
[12] A. Pound and E. Poisson, Phys. Rev. D **77**, 044012 (2008).
[13] A. Pound and E. Poisson, Phys. Rev. D **77**, 044013 (2008).
[14] These strict circularity conditions effectively "switch off" the dissipative component of the GSF. Such an exercise enables us to study the instantaneous conservative effect in isolation. Of course, a consistent treatment of the long-term evolution would have to account for the dissipative orbital decay.
[15] Equation (2.7) only makes sense when $\xi^\alpha$ and $\ddot{\xi}^\alpha$ are well defined on the worldline. It is not immediately clear how to define the GSF in gauges related to the Lorenz gauge through transformations $\xi^\alpha$ which do not satisfy these conditions. See Ref. [10] for a discussion of this issue.
[16] S. Detweiler and E. Poisson, Phys. Rev. D **69**, 084019 (2004).
[17] The class $\hat{\xi}^\alpha$ has been introduced before in the literature—see Appendix A of Ref. [16].
[18] L. Barack and C.O. Lousto, Phys. Rev. D **72**, 104026 (2005).