Crystal melting on toric surfaces

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Abstract

We study the relationship between the statistical mechanics of crystal melting and instanton counting in $\mathcal{N} = 4$ supersymmetric $U(1)$ gauge theory on toric surfaces. We argue that, in contrast to their six-dimensional cousins, the two problems are related but not identical. We develop a vertex formalism for the crystal partition function, which calculates a generating function for the dimension 0 and 1 subschemes of the toric surface, and describe the modifications required to obtain the corresponding gauge theory partition function.

1 Introduction

The problem of computing instanton contributions to the partition functions of four-dimensional supersymmetric gauge theories has a multitude of applications in field theory, string theory, and black hole physics. The algebraic geometry of the corresponding moduli spaces has spawned much interest in mathematics. In this paper, we will consider a new related counting problem in the maximally supersymmetric case. Our approach is motivated by the six-dimensional cousin of the four-dimensional problem.

The vertex formalism [1, 2] allows for the computation of the topological string partition function on arbitrary toric (hence non-compact) Calabi-Yau threefolds. In ref. [3], this formalism was recast into an intuitive counting prescription for plane partitions (three-dimensional Young diagrams), and shown to have the interpretation of a simple statistical mechanics model of crystal melting. This reformulation was taken as the starting point for relating topological string theory on toric Calabi-Yau manifolds to a six-dimensional maximally supersymmetric $U(1)$ gauge theory in ref. [4]. In as far as the partition function of the gauge theory is the generating function for Donaldson-Thomas invariants, this relationship was proven in refs. [5, 6].

One now observes that in all examples which have been computed thus far, the $U(1)$ partition function of $\mathcal{N}=4$ Vafa-Witten twisted gauge theory in four dimensions [7] has as prefactor the Euler function $\hat{\eta}(q) = q^{-1/24}\,\eta(q)$, the generating function for ordinary partitions (Young tableaux), raised to the power of the Euler characteristic of the underlying four-manifold. This suggests that the $\mathcal{N}=4$ theory might be the four-dimensional analogue of the six-dimensional gauge theory underlying Donaldson-Thomas invariants, with its partition function computable from a melting crystal prescription.

Instantons of four-dimensional $U(N)$ gauge theories on toric surfaces (not necessarily Calabi-Yau) have been studied in refs. [8, 9, 10, 11, 12, 13, 14]. $U(1)$ instantons arise as building blocks in these works. In ref. [15], Nakajima studied $U(N)$ instantons on ALE spaces and showed (see Theorem 3.2 of that paper) that at the fixed points of an appropriately lifted toric action, they decompose into a sum of $U(1)$ instantons (see also refs. [12, 14] for related results). Based on this result, the authors of ref. [8] employed a localization calculation on the explicit ADHM instanton moduli space to argue that in the case of $\mathcal{N}=4$ Vafa-Witten twisted gauge theories on ALE spaces, the $U(N)$ partition function simply factorizes into N powers of the $U(1)$ partition function – the rigorous argument for the factorization of the combinatorial problem was provided in ref. [16]. They also indicate a heuristic argument as to why this factorization should hold in general,[1] which is supported by calculations in a two-dimensional reduction of the four-dimensional gauge theory on Hirzebruch-Jung spaces [9, 10]. In this paper, the $U(1)$ case will be the focus of attention.

In the following, we shall say that an enumerative problem has a melting crystal description if it can be recast in terms of a box counting prescription, analogous to that of ref. [3]. We shall see that in passing from the study of the Hilbert scheme of points, which features prominently in instanton calculations, to the Hilbert scheme of curves, we obtain a problem which is a close kin to the gauge theory problem and has a melting

[1] This is in contrast to the six-dimensional case where factorization does not generically hold. See ref. [17] for an explicit analysis of the Coulomb phase of the six-dimensional $U(N)$ gauge theory and its relationship to $U(1)$ instantons.

crystal description. We develop a vertex prescription for calculating the corresponding partition function, and describe the modifications necessary to arrive at the gauge theory partition function. In the process, we provide additional motivation for the conjectured forms of the partition functions of $\mathcal{N} = 4$ theory on Hirzebruch-Jung surfaces proposed in refs. [8, 9].

The organisation of this paper is as follows. In Section 2, we define the enumerative problems which will be addressed in the following, together with the underlying physical motivation, and introduce the corresponding generating functions. We proceed to compute the weights that enter in the definition of these generating function in Section 3. In Section 4, we set up the vertex formalism to compute the crystal melting partition function, and work out the explicit examples of the complex projective plane and Hirzebruch-Jung surfaces. Finally, we describe how these partition functions must be modified to arrive at the instanton partition function of gauge theory in Section 5. Four appendices at the end of the paper provide calculational details and background material. In Appendix A, we compute the Euler characteristics of torus invariant subschemes of a toric surface directly in Čech cohomology. We collect the facts we will need about toric surfaces in general and Hirzebruch-Jung surfaces in particular in Appendix B. Appendix C contains a brief review of characteristic classes of coherent sheaves. In Appendix D, we illustrate the factorization of the Hilbert scheme of curves into a divisorial and a punctual part, a result which plays a central role in the computations of this paper, for the example of the projective plane.

We have made an effort to include many intermediate steps and explanatory notes throughout, in the hope of rendering the exposition more accessible to the casual reader.

Unless otherwise noted, all schemes are defined over the field $\mathbb{C}$.

2 The enumeration problems

In six dimensions, the counting of closed 0 and 1 dimensional subschemes of a projective scheme X is closely related to a gauge theory problem on X. The crystal melting prescription of ref. [3], in hindsight, is most intuitive in this setting, as the boxes out of which the crystal is built correspond to sections of the structure sheaf $\mathcal{O}_Y$ of the corresponding closed subscheme Y.[2] In this section, we will explain the parametrization of subschemes and the gauge theory problem in turn. We then discuss why they coincide in six dimensions, and how they are related in four dimensions.

2.1 The Hilbert scheme

Grothendieck proved that the Hilbert functor of a projective scheme X is representable by a projective scheme called the Hilbert scheme $\mathrm{Hilb}^X_{P(t)}$ of X. The closed points of $\mathrm{Hilb}^X_{P(t)}$ correspond to closed subschemes Y of X with Hilbert polynomial $P^X_Y(t) = P(t)$. Recall that upon fixing a very ample line bundle $\mathcal{L}$, i.e. an embedding of X into projective space,

[2] This hindsight is based on the proof of the equivalence of the generating functions for Donaldson-Thomas and Gromov-Witten invariants on toric threefolds [5, 6].

the Hilbert polynomial $P_Y^X(t) \in \mathbb{Q}[t]$ is defined by the function

$$P_Y^X(t) = \chi\big(\mathcal{O}_Y \otimes \mathcal{L}^{\otimes t}\big) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \mathcal{O}_Y \otimes \mathcal{L}^{\otimes t})$$

for sufficiently large t.[3] The constant term of the Hilbert polynomial is hence the Euler characteristic of the subscheme Y. When $X = \mathbb{CP}^r$ is projective space, the leading term is given by

$$P_Y^{\mathbb{CP}^r}(t) = \frac{d}{n!}\, t^n + \dots\,,$$

with n the dimension of Y and d its degree.

One can further stratify the Hilbert scheme, e.g. by considering the Hilbert scheme of curves with fixed homology class $\beta \in H_2(X, \mathbb{Z})$. In the case $X = \mathbb{CP}^2$ reviewed in Appendix D, the homology class is determined by the degree of the curve, and can hence be read off from the Hilbert polynomial. For space curves, i.e. $X = \mathbb{CP}^3$, this stratification has been studied in ref. [19]. For generic smooth projective surfaces, it is also used in ref. [20].

In general, Hilbert schemes are very complicated objects. The Hilbert scheme of points however, for which the Hilbert polynomial is a constant n, is well understood. It has already made various appearances in the physics literature. It is commonly denoted $X^{[n]} := \mathrm{Hilb}_n^X$. The Hilbert-Chow morphism

$$X^{[n]} \longrightarrow S^n X\,,$$

with $S^n X$ the n-th symmetric product of the scheme X, reflects the intuition that away from the locus at which points approach each other, the moduli space of n points on X is simply given by n copies of X modulo permutations.

The Hilbert scheme of curves generally exhibits much richer structure. On a smooth projective surface X, this structure simplifies: codimension 1 subschemes factorize into divisors, i.e. the multiples of integral codimension 1 subschemes, with multiplicity given by their degree, and sums of free and embedded points (see ref. [21, p. 514], and also ref. [22, Section 3]).[4] We will denote the Hilbert scheme of subschemes Y of X with $\beta = [Y] \in H_2(X, \mathbb{Z})$ and $n = \chi(\mathcal{O}_Y)$ as $I_n(X, \beta)$. Given a $Y \in I_n(X, \beta)$, $\beta \in H_2(X, \mathbb{Z})$ depends solely on the divisorial part D of Y. Its contribution to n is given by $n_\beta = -\frac{1}{2}\,\beta \cdot (\beta + K_X)$, with K_X the canonical class of X (see Section 3.1). $n - n_\beta$ is due to the free and embedded points Y_0 of Y. Thus,

$$I_n(X, \beta) \cong I_{n_\beta}(X, \beta) \times X^{[n - n_\beta]}\,. \tag{2.1}$$

To gain some intuition, we illustrate the factorization (2.1) of the Hilbert scheme at the level of the underlying topological spaces explicitly for the toric surface $X = \mathbb{CP}^2$ in Appendix D. Let us now consider the two factors contributing to eq. (2.1). The Hilbert scheme of points $X^{[n]}$ on a smooth projective surface X is non-singular and of dimension $2n$ [21, Theorem 2.4]. As for the moduli space of divisors $I_n(X, \beta)$, on a smooth projective surface X of any dimension with $H^1(X, \mathcal{O}_X) = 0$, it is a projective space. It follows

[3]This definition makes sense because the right-hand side of this equation is a polynomial for sufficiently large t, see e.g. [18, Theorem 7.5].

[4]We thank Richard Thomas for explanations concerning this point.

that $I_n(X,\beta)$ is non-singular [21, Corollary 2.7]. This will allow us to define generating functions involving integrals over the fundamental classes $[I_n(X,\beta)]$. Contrary to the six-dimensional case, recourse to virtual classes, as introduced on projective surfaces in ref. [20], will not be required.

2.2 Instantons, holomorphic bundles, and sheaves

In the physics literature, instantons are finite-energy minima of the action of a field theory. In the case of pure Yang-Mills theory in four dimensions, they are given by (anti)self-dual connections with appropriate boundary conditions at infinity. Vafa and Witten [7] demonstrated that $\mathcal{N}=4$ supersymmetric Yang-Mills theory can be modified, following Witten's prescription of topological twisting, such that its partition function computes, in favorable circumstances, the Euler characteristic of the moduli space of instantons. This is guaranteed if certain vanishing theorems for the geometry of the underlying four-manifold X and gauge bundle $E \to X$ are met [7, Section 2.4], e.g. if X is a compact Kähler manifold of positive curvature and the structure group of E is $SU(2)$. The twisted theory can be encountered in the wild (i.e. it can describe physical systems) in two situations; when the manifold X on which the gauge theory is defined is hyperkähler, so that the twisted theory coincides with a subsector of the physical theory, or in certain string theory embeddings, in which the twisting is induced by the background.

For the partition function to be well-defined, a smooth, compact instanton moduli space is required. One path towards this end is a compactification given by embedding bundles with irreducible anti-self-dual connection into the space of semistable sheaves. This is the Gieseker compactification. A semistable sheaf $\mathcal{F}$ is in particular torsion-free. Away from a codimension 2 locus, torsion-free sheaves are locally free, i.e. vector bundles. Hence, intuitively, in passing from bundles to torsion-free sheaves on surfaces, we are adding pointlike structures. The precise formulation of this statement is eq. (2.2) below.

In this paper, we consider only the gauge group $U(1)$. This simple case already merits study for two main reasons:

1. In six dimensions, this is the gauge group that has been related to the topological vertex formalism.
2. In the case of ALE spaces X, ref. [8] performs the calculation of the partition function $Z^{ALE}_{U(N)}(X)$ based on the description of the $U(N)$ instanton moduli space provided by ref. [23], and demonstrates that one has the factorization relation

$$Z^{ALE}_{U(N)}(X) = \left(Z^{ALE}_{U(1)}(X)\right)^N .$$

   The factorization follows from a localization argument which reduces the calculation to the fixed points of an appropriately chosen toric action, and the demonstration in ref. [15] that the instanton bundles on ALE spaces factorize at the fixed points.[5]

Rather than thinking about anti-self-dual connections, we can study holomorphic line bundles. The connection is established by the following well-known theorem.

[5] In fact, ref. [8] argues heuristically that this relation should hold in general, as the gauge symmetry of the $U(N)$ theory can be broken to the maximal torus $U(1)^N$ by giving vevs to the scalars, and the partition function is independent of these vevs. This argument is heuristic as the independence of the partition function from the scalar vevs needs to be established carefully.

Proposition 2.1. *([24, Proposition 2.2.6]) If $H^1(X,\mathbb{R}) = 0$ and $\mathcal{L}$ is a line bundle over the surface X, then for any 2-form ω representing $c_1(\mathcal{L})$ there is a unique gauge equivalence class of connections with curvature $-2\pi\,\mathrm{i}\,\omega$.*

Hence, in particular, in mapping line bundles to connections, we can choose the harmonic representative of $c_1(\mathcal{L})$ with respect to a chosen metric on X. The space of harmonic 2-forms $\mathcal{H}(X)$ on X has a decomposition into subspaces $\mathcal{H}^{\pm}(X)$ of self-dual and anti-self-dual 2-forms,

$$\mathcal{H}(X) = \mathcal{H}^+(X) \oplus \mathcal{H}^-(X)\,.$$

If the intersection matrix on $H^2(X,\mathbb{R})$ is well-defined (a condition we must impose for X non-compact) and negative definite, as will be the case for our main example, the Hirzebruch-Jung surfaces, it follows that $\mathcal{H}^+(X) = 0$, and hence every holomorphic line bundle on such a surface admits an anti-self-dual connection. Since exact forms cannot be anti-self-dual, this connection is unique.

Note finally that even in the case of $U(1)$ gauge theory, where the instanton moduli space is a lattice and in no need of regularization, we continue to consider the larger space of torsion-free sheaves, in accord with points 1 and 2 above.

2.3 Crystal melting vs. gauge theory

On a toric Calabi-Yau threefold X, the problem that Maulik, Nekrasov, Okounkov, and Pandharipande address in ref. [5] is the counting of subschemes Y of compact support with no component of codimension 1, and with holomorphic Euler characteristic n and second homology class β,

$$n = \chi(\mathcal{O}_Y)\,, \quad \beta = [Y] \in H_2(X,\mathbb{Z})\,.$$

They denote the corresponding moduli space of ideal sheaves by $I_n(X,\beta)$. They then compute the generating function

$$Z_{DT}(X;q,w) = \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{n\in\mathbb{Z}} \tilde{N}_{n,\beta}\; q^n\, w^\beta$$

for the Donaldson-Thomas invariants

$$\tilde{N}_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1\ ,$$

the lengths of the 0 dimensional virtual fundamental cycles. On a projective scheme X, $I_n(X,\beta)$ is the moduli space parametrizing isomorphism classes of torsion-free sheaves of rank 1 with trivial determinant, where the singularity sets of the torsion-free sheaves constitute the subschemes being counted.[6]

[6] The argument is the following (see e.g. ref. [25]). A torsion-free sheaf $\mathcal{T}$ injects into its double dual. The determinant sheaf of a rank r torsion-free sheaf is defined to be

$$\det\mathcal{T} = \left(\bigwedge^r \mathcal{T}\right)^{**}.$$

This is a line bundle, as the double dual of a sheaf is reflexive (i.e. isomorphic to its double dual), and reflexive sheaves of rank 1 are locally free. Rank 1 torsion-free sheaves with trivial determinant hence possess an injection into the structure sheaf, i.e. they are ideal sheaves. Note that the distinction

Shifting focus from 6 dimensional Calabi-Yau manifolds to toric surfaces, the counting problem that gives rise to a melting crystal interpretation and vertex formulation is again that of counting all subschemes of dimension 0 and 1 of compact support. In four dimensions, this does not quite map to a counting problem of torsion-free sheaves. Torsion-free sheaves of trivial determinant on a surface are the ideal sheaves merely of points, by the same argument as above. We can enlarge this space in two ways. We can add 1 dimensional subschemes of compact support by hand and consider the corresponding Hilbert scheme of compact curves, or equivalently the moduli space of ideal sheaves

$$I_n(X,\beta)$$

introduced above, now for X a surface. Alternatively, we can drop the trivial determinant condition, thus enlarging the space to involve factors of line bundles $\mathcal{L}$, given by the double duals of torsion-free sheaves $\mathcal{T}$, such that

$$\mathcal{T} = \mathcal{L} \otimes \mathcal{I}_Y \,. \tag{2.2}$$

Again dimension 1 subschemes come into play, by their relation to effective divisors, however now only up to linear equivalence. Indeed, the decomposition (2.2) corresponds to the gauge theory problem outlined in Section 2.2.

In the case of surfaces, we will hence be computing two different generating functions, defined as follows.

Crystal melting.

$$Z_{cm}(X;q,w) = \sum_{\beta\in H_2(X,\mathbb{Z})} \sum_{n\in\mathbb{Z}} N^{cm}_{n,\beta}\, q^n\, w^\beta \,, \tag{2.3}$$

where $n = \chi(\mathcal{O}_Y)$, $\beta = [Y] \in H_2(X,\mathbb{Z})$ and

$$N^{cm}_{n,\beta} = \int_{I_n(X,\beta)} e\big(TI_n(X,\beta)\big) \,. \tag{2.4}$$

Here and below, $e(E)$ denotes the Euler class of the bundle E. As announced above, no recourse to virtual fundamental classes is taken in these definitions.

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between trivializable determinant and trivial determinant is important here. The singularity set $S(\mathcal{T})$ of a torsion-free sheaf $\mathcal{T}$ occurs in codimension 2 or higher, hence the corresponding subscheme has no component in codimension 0 or 1. This argument can be summarized in the exact sequence of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}^{**} & \longrightarrow & S(\mathcal{T}) & \longrightarrow & 0 \\ & & & & \downarrow \cong & & & & \\ & & & & \mathcal{O}_X & & & & \end{array}$$

where the vertical isomorphism is a *fixed* trivialization.

Conversely, the ideal sheaf $\mathcal{I}_Y$ of a proper closed subscheme Y of a noetherian integral scheme X is a coherent sheaf of rank 1. As a subsheaf of the structure sheaf $\mathcal{O}_X$, it is torsion-free by the integrality assumption on X. If Y has no support in codimension 1, then the determinant of $\mathcal{I}_Y$ is trivial.

Gauge theory.

$$Z_{gt}(X;q,w)=\sum_{\beta\in H_2(X,\mathbb{Z})}\ \sum_{n\in\mathbb{Q}} N^{gt}_{n,\beta}\, q^{-n}\, w^{\beta}\,, \tag{2.5}$$

where $n=\mathrm{ch}_2(\mathcal{T})$, $\beta=\mathrm{ch}_1(\mathcal{T})\in H_2(X,\mathbb{Z})$ and

$$N^{gt}_{n,\beta}=\int_{\mathfrak{M}_X(\beta,n)} e\big(T\mathfrak{M}_X(\beta,n)\big)\,, \tag{2.6}$$

with $\mathfrak{M}_X(\beta,n)$ the moduli space parametrizing isomorphism classes of torsion-free sheaves $\mathcal{T}$ of the given Chern character. Note that for torsion-free sheaves on non-compact spaces, the second Chern characteristic class n can be fractional; we therefore take the summation in (2.5) over $\mathbb{Q}$, with $N^{gt}_{n,\beta}=0$ away from a fixed common denominator of n, depending on X.

Explaining the various ingredients in these formulae, and interpreting and evaluating the integrals (2.4) and (2.6), will occupy the rest of this paper.

3 The weights for the generating functions

Following the 6 dimensional discussion of ref. [5], we organize the crystal melting counting problem (2.3) on a toric surface X in terms of the holomorphic Euler characteristic $\chi(\mathcal{O}_Y)$ of the subschemes Y, which serves as a weight in the generating function, and their second homology class $[Y]\in H_2(X,\mathbb{Z})$. For the gauge theory partition function, the weight originates in the action, which for anti-self-dual connections in four dimensions evaluates to the Chern class of the vector bundle (locally free sheaf) via Chern-Weil theory. When we compactify the space of gauge connections by including pointlike instantons, it is natural to retain the Chern class as weight, as in (2.5). In this section, we will compute these two weights and find that on Calabi-Yau surfaces, they are equal up to sign, at least for torically invariant Y (this qualification arises due to the non-compactness of X, see point 3. in Subsection 3.2).

3.1 The Euler characteristic of subschemes

Based on the factorization (2.1), we can calculate the Euler characteristic of Y by adding the contributions from the divisorial and punctual parts, D and Y_0, of Y: the Euler characteristic of a 0 dimensional scheme enumerates its global sections,

$$\chi(\mathcal{O}_{Y_0})=h^0(Y_0,\mathcal{O}_{Y_0})\,,$$

while the Euler characteristic of a divisor D on a surface X, as is reviewed in Appendix C, is given by

$$\chi(\mathcal{O}_D)=-\tfrac{1}{2}\,D\cdot(D+K_X)\,, \tag{3.1}$$

with K_X the canonical class of the surface. The right-hand side of eq. (3.1) clearly only depends on the class of the divisors up to linear equivalence. We will denote this class by square brackets $[-]$. Altogether,

$$\chi(\mathcal{O}_Y)=-\tfrac{1}{2}\,D\cdot(D+K_X)+h^0(Y_0,\mathcal{O}_{Y_0})\,. \tag{3.2}$$

The derivation of formula (3.1) in Appendix C relies on the application of the Hirzebruch-Riemann-Roch theorem, valid for X projective. When relaxing the compactness condition, the terms on the left- and right-hand side of eq. (3.1) remain well-defined for D a divisor with compact support. By calculating the Euler characteristic directly in Čech cohomology in Appendix A, following ref. [5], we will see that at least in the case of torically invariant subschemes, eq. (3.1) remains valid for D of compact support also on a non-compact toric surface X. On restricting to torically invariant subschemes Y, the decomposition into a reduced subscheme D of dimension 1 and a 0 dimensional subscheme Y_0 is immediate.

To explicitly determine the Euler characteristic of a given divisor on a toric surface X, we benefit from the property that the Chow ring $A_1(X)$ is generated by the classes of torically invariant divisors D_i (this is in fact true in arbitrary dimensions). We will enumerate the D_i via $i = 0, \ldots, n+1$, reserving the indices $i = 0$ and $i = n+1$ for non-compact toric divisors if these are present, otherwise setting $D_0 = D_{n+1} = 0$. This notation allows for the simultaneous treatment of compact and non-compact toric surfaces.

Expanding $[D]$ in classes $[D_i]$ generated by compactly supported divisors,

$$[D] = \sum_{i=1}^{n} \lambda_i \, [D_i] \,,$$

with λ_i non-negative integers, and with the intersection matrix as given in eq. (B.2) of Appendix B, the calculation of the Euler characteristic in Appendix A yields

$$\chi(\mathcal{O}_D) = \sum_{i=1}^{n} \left(a_i \, \frac{\lambda_i \, (\lambda_i - 1)}{2} + \lambda_i - \lambda_i \, \lambda_{i+1} \right) , \qquad (3.3)$$

where the a_i denote the negative self-intersection numbers $a_i = -D_i^2$, and $\lambda_{n+1} = 0$ is introduced for notational convenience.

To compare with eq. (3.1), note that the total Chern class of a non-singular toric variety X is given by

$$c_t(X) = \prod_{i=0}^{n+1} (1 + [D_i]) \,.$$

It follows that

$$K_X = - \sum_{i=0}^{n+1} [D_i] \,,$$

and hence

$$\chi(\mathcal{O}_D) = -\frac{1}{2} \sum_{i,j=1}^{n} \lambda_i (\lambda_j - 1) \, D_i \cdot D_j + \frac{1}{2} \sum_{i=1}^{n} \lambda_i \, D_i \cdot (D_0 + D_{n+1}) \,. \qquad (3.4)$$

Borrowing the result (5.5) from Section 5, in which $[D_0]$ and $[D_{n+1}]$ are expressed as linear combinations of $[D_1], \ldots, [D_n]$, we conclude that eq. (3.1) reproduces the Čech cohomology result (3.3).

3.2 The Chern characteristic of sheaves

In the gauge theory setup, we wish to weigh all sheaves via the degree of their second Chern character. This is the natural extension of the notion of instanton number beyond locally free sheaves (i.e. vector bundles), see Appendix C. The Chern character satisfies the multiplicative property

$$\mathrm{ch}(\mathcal{E}\otimes\mathcal{F}) = \mathrm{ch}(\mathcal{E})\cdot\mathrm{ch}(\mathcal{F})\,.$$

For the ideal sheaf $\mathcal{I}_Z$ of a cycle Z, an application of the Grothendieck-Riemann-Roch theorem yields

$$\mathrm{ch}(\mathcal{I}_Z) = 1-\eta_Z\,,$$

with η_Z the class of the cycle (see e.g. ref. [26, p. 159]). Due to the relation $\mathrm{ch}_0(\mathcal{L}) = \mathrm{rk}\,(\mathcal{L}) = 1$ for a line bundle $\mathcal{L}$, we have

$$\mathrm{ch}_2(\mathcal{L}\otimes\mathcal{I}_Z) = \mathrm{ch}_2(\mathcal{L}) - \eta_Z\,.$$

We can therefore consider the two factors contributing to the weight of a given torsion-free sheaf separately.

As we show in Lemma C.1 of appendix C, $\deg(\eta_Z) = \chi(\mathcal{O}_Z)$.

For line bundles, we can work in the more familiar cohomological setup. On a compact surface X, the instanton number evaluates to the intersection pairing of the corresponding divisors,

$$\frac{1}{2}\int_X c_1\big(\mathcal{O}_X(D)\big)\wedge c_1\big(\mathcal{O}_X(D)\big) = \frac{1}{2}\,D\cdot D\,. \tag{3.5}$$

For a divisor D with compact support, this relation in fact continues to hold on arbitrary toric manifolds. The argument consists of three parts:

1. Also on non-compact manifolds, the first Chern class $c_1(\mathcal{O}_X(D))$ of the line bundle associated to a divisor and the closed Poincaré dual η_D of the support of the divisor are cohomologous, $c_1(\mathcal{O}_X(D))\sim\eta_D$.[7]
2. Since the support $|D|$ is compact, we can replace the closed Poincaré dual by the compact Poincaré dual. By localization of this class (in the sense of e.g. ref. [27]), we know that we can choose its support to be contained in an arbitrary open set containing $|D|$, such that the integral in eq. (3.5) is well-defined.
3. We verify the relation (3.5) by performing the calculation on a toric compactification $\bar{X}$ of X for which the compactification divisor does not intersect the image of the support of D. This property guarantees that the evaluations of the integrals over X and $\bar{X}$ coincide. An example of such a compactification, which always exists for smooth toric surfaces, is given in Figure 1.

[7] Recall that the closed Poincaré dual is integrated against forms of compact support, in contrast to the compact Poincaré dual which itself has compact support (see e.g. ref. [27, pp. 51–53]). Hence, the closed Poincaré dual η_Σ of a cycle Σ satisfies

$$\int_\Sigma\psi = \int_X \eta_\Sigma\wedge\psi$$

for any $\psi\in H^*_c(X,\mathbb{R})$ (defining both sides to vanish if the form degrees are not appropriate). By restricting the integration to the support of ψ, the argument establishing $c_1(\mathcal{O}_X(-D))\sim\eta_D$ on compact manifolds (see e.g. ref. [28, p. 143]) goes through in the non-compact case.

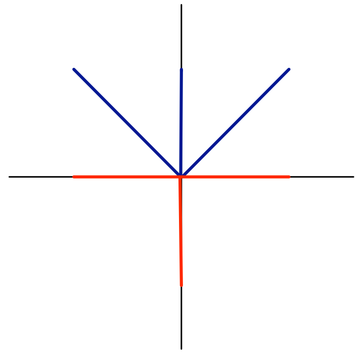

Figure 1: A toric compactification of the resolved A_1 singularity (in blue) to $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at two points.

The toric assumption can be replaced by the requirement that a compactification with the requisite properties exists.

We thus arrive at

$$\mathrm{ch}_2\big(\mathcal{O}_X(D) \otimes \mathcal{I}_Z\big) = \tfrac{1}{2}\, D \cdot D - \chi(\mathcal{O}_Z)\,.$$

For $K_X = 0$ this agrees with eq. (3.2) up to sign, as announced at the beginning of this section.

It turns out that all non-compact divisors in the geometries that we will consider are linearly equivalent to divisors with compact support. This thus allows us to compute the instanton number on the full Picard group. Note that the instanton numbers of line bundles associated to prime divisors with non-compact support are no longer necessarily integral.

4 Crystal melting: counting subschemes

In the previous sections, we have introduced two closely related enumerative problems: counting subschemes vs. counting torsion-free sheaves on a toric surface. The line bundle factor in eq. (2.2) requires invoking linear equivalence between toric divisors, and hence cannot be straightforwardly implemented within a vertex formalism that essentially only allows for nearest neighbor interactions (in terms of the 2-cones of the toric fan, or the vertices of the dual web diagram).[8] The problem of counting, in an appropriate sense, the 0 and 1 dimensional compactly supported subschemes of a toric surface does however have a melting crystal implementation, as we demonstrate in this section.

The factorization (2.1) of the Hilbert scheme discussed in Section 2.1 results in a factor-

[8]We say 'essentially' as even the vertex formalism in six dimensions requires identifying homologous curve classes by hand. In fact, when $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$, $\mathrm{Pic}\,(X) \cong H^2(X, \mathbb{Z})$, so 'linearly equivalent' and 'homologous' are the same notion on smooth compact toric surfaces (in fact, this holds in any dimension: all cohomology classes on toric manifolds are analytic, hence of pure type (p,p)). Even so, in Z_{cm}, all curves are counted and only the weight w^β invokes homological equivalence, whereas in Z_{gt}, the enumeration itself proceeds over equivalence classes.

ization of the partition function Z_{cm} defined in eq. (2.3),

$$Z_{cm}(X;q,w) = \sum_{\beta\in H_2(X,\mathbb{Z})} \int_{I_{n_\beta}(X,\beta)} e\big(TI_{n_\beta}(X,\beta)\big)\; q^{n_\beta}\, w^\beta \sum_{n\geq 0} \int_{X^{[n]}} e\big(TX^{[n]}\big)\; q^n\,. \tag{4.1}$$

We begin by considering the contribution of the free and embedded 0 dimensional subschemes. The corresponding generating function for smooth projective surfaces has been calculated by Göttsche in ref. [29]. We reproduce his result in the case of toric surfaces via a localization calculation, which then permits an extension to the non-compact case. This calculation will also be relevant in the gauge theory context of Section 5. We next apply a similar localization argument to the divisorial contribution to the partition function. Finally, we show how the computation of Z_{cm} can be encapsulated in a small set of diagrammatic rules, and illustrate these in the examples of projective space and Hirzebruch-Jung spaces. Localization arguments lie at the heart of the calculations in this section.

4.1 0 dimensional subschemes

For X a smooth projective surface, the moduli space $X^{[n]}$ of 0 dimensional subschemes of length n is non-singular of dimension $2n$ [21, Theorem 2.4]. The generating function we are after was computed by Göttsche for smooth projective surfaces as [29]

$$\sum_{n\geq 0} \chi\big(X^{[n]}\big)\; q^n = \big(\hat\eta(q)^{-1}\big)^{\chi(X)}\,, \tag{4.2}$$

with

$$\hat\eta(q) = \prod_{k=1}^{\infty} \big(1-q^k\big)$$

the generating function of partitions. The computation of ref. [29] does not require a torus action on the surface. If such an action exists, i.e. in the case of a toric surface X, we can reproduce Göttsche's formula (4.2) by a localization computation (see also ref. [30] and ref. [16, appendix A]). As the Hilbert scheme $X^{[n]}$ is smooth, we can use conventional Atiyah-Bott localization [31] in equivariant cohomology. We quote here the more general integration formula of Edidin and Graham [32] in equivariant Chow theory; this is the framework that generalizes beyond smooth varieties.

Theorem 4.1. *([32, Proposition 5]) Let M be a smooth and complete scheme with the action of a torus $T=(\mathbb{C}^*)^k$. Denote the fixed point locus of the T-action by M^T, with embedding*

$$i : M^T \;\hookrightarrow\; M\,.$$

Let $a\in A_0(M)$ descend from an equivariant class $\alpha\in A_0^T(M)$, i.e. $a=i^*\alpha$. Then

$$\deg(a) = \sum_{F\subset M^T} \pi_{F*}\Big(\frac{i_F^*\alpha}{e_T(N_F M)}\Big)\,, \tag{4.3}$$

where the sum runs through the connected components of the fixed point locus, $N_F M$ denotes the normal bundle over F in M, i_F the embedding of F into M, and π_F the projection of F to a point. $e_T(N_F M)$ is the T-equivariant Euler class which is invertible in $A_*^T(F) \otimes_{\mathbb{Q}[t_1,\ldots,t_k]} \mathbb{Q}[t_1,\ldots,t_k]_{\mathfrak{m}}$, where $\mathbb{Q}[t_1,\ldots,t_k]_{\mathfrak{m}}$ is the localization of the ring $\mathbb{Q}[t_1,\ldots,t_k]$ at the maximal ideal $\mathfrak{m}$ spanned by the generators $t_1,\ldots,t_k$ of the equivariant ring of T.

When the set of T-fixed points is a union of isolated points, the tangent bundle to each F is trivial, and thus

$$e_T(N_F M) = e_T(TM|_F)\,.$$

With $a = e(TM)$, we have $\alpha = e_T(TM)$, and hence $i_F^* \alpha = e_T(TM|_F)$ cancels the denominator in the integrand on the right-hand side of the localization formula (4.3). In this case, the integral (4.3) can simply be evaluated by counting the fixed points of the T-action on M.

Note that characteristic classes of equivariant vector bundles can be extended to equivariant classes. Theorem 4.1 hence applies to our case of interest, with $M = X^{[n]}$ and $a = e(TX^{[n]})$ the Euler class of the tangent bundle of the Hilbert scheme $X^{[n]}$.

The fixed points of $X^{[n]}$ parametrize the torically invariant 0 dimensional subschemes of X with holomorphic Euler characteristic n. They can be enumerated by considering an affine patch $\mathbb{C}[x,y]$ around each set theoretic fixed point, and ideals $I \subset \mathbb{C}[x,y]$ generated by monomials $x^m\, y^n$ giving rise to non-reduced schemes with support at this point. The ideals I are in one-to-one correspondence with Young tableaux π_I, as illustrated in Figure 2. The boxes of the Young tableau map to a basis of global sections of the corresponding 0

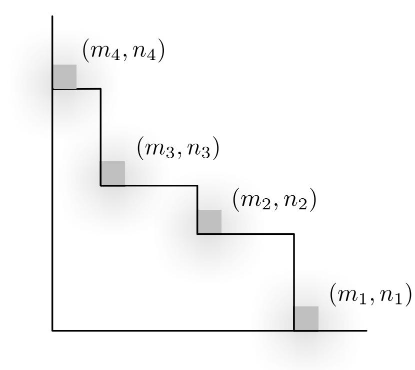

Figure 2: The Young tableau encoding the ideal generated by the monomials $x^{m_i}\, y^{n_i}$, $i = 1,\ldots,4$, with (m_i, n_i) the coordinates of the shaded boxes.

dimensional subscheme Y. Its Euler characteristic is hence equal to

$$\chi(\mathcal{O}_Y) = h^0(Y, \mathcal{O}_Y) = \dim_{\mathbb{C}} \big(\mathbb{C}[x,y]\,/\,I\big) = |\pi_I|\ ,$$

the number of boxes in the Young tableau π_I. The contribution to the partition function per geometric fixed point is hence

$$\sum_\pi q^{|\pi|} = \hat{\eta}(q)^{-1}\,.$$

The toric fixed points correspond to the maximal cones of the toric fan of X. Since the Euler characteristic $\chi(X)$ of a toric manifold X is given by the number of maximal cones of X, this reproduces the formula (4.2).

The application of standard theorems is complicated when the surface X is non-compact. We will proceed by applying the localization formula to a toric compactification $\bar{X}$ of X, and then restrict to the fixed points lying in X. This procedure is clearly independent of the choice of compactification.

4.2 1 dimensional subschemes

We now want to apply Theorem 4.1 to the integral over $[I_{n_\beta}(X,\beta)]$. For X a smooth projective surface, this class exists as the corresponding Hilbert scheme of curves is smooth, as argued in Section 2.1. For X non-compact, we will again consider a toric compactification $\bar{X}$ of X, as illustrated in Figure 1. This compactification is obtained by gluing in a set of torically invariant divisors which have vanishing intersection with the compactly supported divisors of X. For β the class of a compactly supported divisor, it follows that

$$I_{n_\beta}(X,\beta) \cong I_{n_\beta}(\bar{X},\beta)\,,$$

as for D such that $[D]=\beta$, all divisors linearly equivalent to D will lie within X. We conclude that for β the class of a compactly supported divisor, $I_{n_\beta}(X,\beta)$ is smooth on a smooth quasi-projective toric surface as well.

4.2.1 The toric fixed points

Above, we considered ideal sheaves corresponding to 0 dimensional subschemes. In general, ideal sheaves invariant under the torus action are monomial, i.e. locally generated by monomials. We will describe such an ideal sheaf $\mathcal{I}$ by specifying it locally on the torically invariant open sets U_i of the surface X, $I_i = \mathcal{I}(U_i) \subset \mathbb{C}[x,y]$, such that restrictions to overlaps coincide. The monomial ideals I_i are in a one-to-one relation to Young tableaux which in distinction to the 0 dimensional case may be infinite, i.e. the generators do not necessarily include monomials of the form x^m or y^n.

The factorization (2.1) of the Hilbert scheme into a divisorial and a punctual part is immediate when restricting to the toric fixed points: the possible associated primes to I_i are (x), (y), and (x,y) (see e.g. ref. [33] for an explanation of this notion), the latter implying the existence of an embedded point. It is easy to see that all Young diagrams other than Hook diagrams correspond to closed subschemes with embedded points. The decomposition

$$\{\text{infinite Young tableau}\} \;\longleftrightarrow\; \big(\mathbb{N}\cup\{0\}\big)^2 \times \{\text{finite Young tableau}\}$$

illustrated in Figure 3 hence corresponds to the decomposition of the fixed point into an effective divisor, and free and embedded point contributions (free in case $\lambda_i = \lambda_{i+1} = 0$).

We now consider the two factors contributing to Z_{cm} as given in eq. (4.1) in turn.

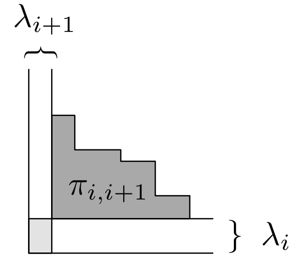

Figure 3: Decomposition of a subscheme into a reduced and a 0 dimensional component.

Points. The torically invariant ideal sheaves of points were discussed in Section 4.1. They are in one-to-one correspondence with vectors of finite Young tableaux

$$\pi = (\pi_1, \ldots, \pi_n)\,,$$

one diagram assigned to each toric fixed point of the surface. The Euler characteristic of the corresponding subscheme Y_0 is given by

$$\chi(\mathcal{O}_{Y_0}) = \sum_{i=1}^{n} |\pi_i|\,.$$

Below we will also identify toric fixed points, which correspond to 2-cones of the toric fan, by the two bounding 1-cones, and whence use the notation $\pi_{i,i+1} := \pi_i$ as e.g. in Figure 3.

Divisors. As reviewed in appendix B, the torically invariant divisors are in one-to-one correspondence with the 1-cones of the toric fan. Labeling the compactly supported torically invariant divisors as D_i, $i = 1, \ldots, n$ (i.e. disregarding the two outermost 1-cones in the case of non-compact surfaces), a general effective divisor D kept fixed by the toric action is parametrized by n non-negative integers λ_i,

$$D = \sum_{i=1}^{n} \lambda_i\, D_i\,.$$

The Euler characteristic of D is then computed via (3.4),

$$\chi(\mathcal{O}_D) = \sum_{i=1}^{n} \left(a_i\, \frac{\lambda_i\,(\lambda_i - 1)}{2} + \lambda_i - \lambda_i\, \lambda_{i+1} \right).$$

The self-intersection numbers are here denoted $D_i^2 = -a_i$. We have furthermore set $\lambda_{n+1} = \lambda_1$ in the compact case and $\lambda_{n+1} = 0$ in the non-compact case. We discuss how to determine the intersection matrix of the compactly supported prime divisors in appendix B.

4.2.2 Crystal melting

To emphasize the similarity with melting crystal combinatorics in six dimensions [3], we can express the Euler characteristic $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{Y_0}) + \chi(\mathcal{O}_D)$ in terms of the infinite

Young tableau $\tilde{\pi}_{i,i+1}$ as

$$\chi(\mathcal{O}_Y) = \sum_{i=1}^{n+1} |\tilde{\pi}_{i,i+1}| + \sum_{i=1}^{n} \Big(a_i \, \frac{\lambda_i\,(\lambda_i - 1)}{2} + \lambda_i \Big) \, ,$$

where the box count of the infinite Young tableau is defined as (see Figure 3)

$$\begin{aligned} |\tilde{\pi}_{i,i+1}| \ &:= \ \Big(\sum_{(I,J)\in\tilde{\pi}_{i,i+1}\cap[0,1,\dots,N]^2} 1 \Big) - (N+1)\,\lambda_i - (N+1)\,\lambda_{i+1} \ , \qquad N \gg 0 \\ &= \ \Big(\sum_{(I,J)\in\pi_{i,i+1}} 1 \Big) - \lambda_i\,\lambda_{i+1} \ . \end{aligned}$$

This determines the Boltzmann weight for dissolved atoms in a crystal described by the combinatorial quantities $\tilde{\pi}_{i,i+1}$ and λ_i. Note that for the Calabi-Yau case, the self-intersection numbers are all given by $a_i = 2$ and the Euler characteristic simplifies to

$$\chi(\mathcal{O}_Y) = \sum_{i=1}^{n+1} |\tilde{\pi}_{i,i+1}| + \sum_{i=1}^{n} \lambda_i^2 \, .$$

4.3 The vertex formalism for toric surfaces

We can now easily summarize the computation of the partition function $Z_{cm}(X;q,w)$ in terms of a simple set of vertex rules:

1. Draw the dual web diagram of the toric fan. 2-cones are dual to vertices, and 1-cones are dual to legs.
2. Each vertex i carries two positive integer labels λ_i and λ_{i+1} (“one-dimensional Young diagrams”), one assigned to each emanating leg, and contributes a vertex factor

   $$V_{\lambda_i,\lambda_{i+1}}(q) = \frac{1}{\hat{\eta}(q)} \ q^{-\lambda_i\,\lambda_{i+1}}$$

   to the partition function.
3. Vertices are glued along legs carrying the same integer label λ_i with a gluing factor

   $$G_{\lambda_i}(q,w_i) = q^{a_i\,\frac{\lambda_i\,(\lambda_i-1)}{2}+\lambda_i} \ w_i^{\lambda_i} \ ,$$

   where the self-intersection numbers $-a_i$ are determined graphically as described below. w_i labels the homology class of the curve corresponding to the leg along which the vertices are glued.
4. Multiplying the vertex and gluing factors together, summing the λ_i on internal legs over all non-negative integers while setting those on external legs to zero then yields the melting crystal partition function $Z_{cm}(X;q,w)$.

We can determine the self-intersection numbers $-a_i$ graphically as follows. Recall that they are given by the relation

$$a_i\,v_i = v_{i-1} + v_{i+1}$$

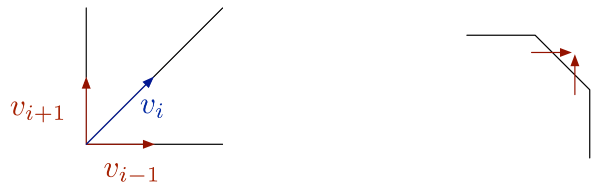

Figure 4: An example of a curve with self-intersection number -1 (the exceptional divisor of the blow-up of $\mathbb{C}^2$ at the origin). On the right-hand side, we have indicated the dual web diagram with the analogue of the framing vectors of the six-dimensional vertex formalism [1, 2].

between the generator v_i of the 1-cone associated to D_i and those of the two neighboring 1-cones, see Figure 4. Note that since the surface X is non-singular by assumption, one has

$$v_{i-1} \times v_i = v_i \times v_{i+1} = 1 \,,$$

where the $\times$ product computes the volume of the cell spanned by the generators. Hence

$$a_i = v_{i-1} \times v_{i+1} \,.$$

4.4 Examples

4.4.1 Projective plane

The toric fan and web diagram of the projective space $\mathbb{CP}^2$ are depicted in Figure 5. The

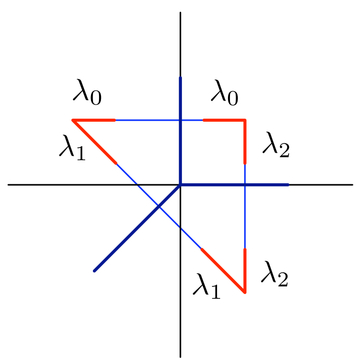

Figure 5: The toric fan for $\mathbb{CP}^2$, and the corresponding web diagram, with the legs of the vertices labelled.

self-intersection numbers $-a_i$ of the torically invariant divisors are $a_i = -1$. The partition function is thus

$$\begin{aligned} Z_{cm}(\mathbb{CP}^2; q, w) &= \sum_{\lambda_0,\lambda_1,\lambda_2=0}^{\infty} \frac{1}{\hat\eta(q)}\, q^{-\lambda_0\,\lambda_1}\; q^{-\frac{\lambda_1^2}{2}+\frac{3}{2}\,\lambda_1}\, w^{\lambda_1}\, \frac{1}{\hat\eta(q)}\, q^{-\lambda_1\,\lambda_2}\; q^{-\frac{\lambda_2^2}{2}+\frac{3}{2}\,\lambda_2}\, w^{\lambda_2} \\ &\qquad \times \frac{1}{\hat\eta(q)}\, q^{-\lambda_2\,\lambda_0}\; q^{-\frac{\lambda_0^2}{2}+\frac{3}{2}\,\lambda_0}\, w^{\lambda_0} \\ &= \frac{1}{\hat\eta(q)^3} \sum_{\lambda_0,\lambda_1,\lambda_2=0}^{\infty} q^{-\frac{1}{2}\,(\lambda_0+\lambda_1+\lambda_2)^2+\frac{3}{2}\,(\lambda_0+\lambda_1+\lambda_2)}\, w^{\lambda_0+\lambda_1+\lambda_2} \;, \end{aligned}$$

with w labeling the hyperplane class.

As a check, we can use this formula to extract the Euler characteristic $\chi\left(I_{n_\beta}(X,d)\right)$ of the moduli space of degree d divisorial curves on $X = \mathbb{CP}^2$. By our result for the partition function, it is given by the number of ways to obtain d as the sum of three non-negative integers,

$$d = \lambda_0 + \lambda_1 + \lambda_2 \,.$$

As $|I_{n_\beta}(X,d)| = \mathbb{P}(H^0(X, \mathcal{O}_X(D))$ for a choice of divisor D with $[D] = \beta$, (see e.g. [28, p. 137]), and $\chi(\mathbb{CP}^n) = n+1$, this is indeed the correct result.

4.4.2 Hirzebruch-Jung surfaces

A choice of 1-cones describing the Hirzebruch-Jung surfaces $Y_{p,q}$ is given in Appendix B. The corresponding fan for the example $Y_{3,2} = A_2$ is depicted in Figure 6, together with the dual web diagram.

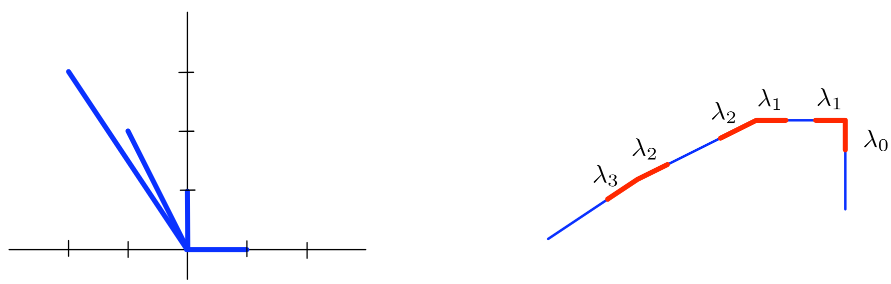

Figure 6: The toric fan for A_2, and the corresponding web diagram, with the legs of the vertices labelled.

The vertex rules yield the partition function

$$\begin{aligned} Z_{cm}(Y_{p,q};q,w) &= \sum_{\lambda_1,\ldots,\lambda_n=0}^{\infty} \frac{1}{\hat{\eta}(q)} q^{-\lambda_0\,\lambda_1}\; q^{\frac{1}{2}a_1\lambda_1^2+\lambda_1(1-\frac{1}{2}a_i)}\, w_1^{\lambda_1}\; \frac{1}{\hat{\eta}(q)} q^{-\lambda_1\,\lambda_2} \;\cdots \\ &\qquad \times q^{\frac{1}{2}a_n\lambda_n^2+\lambda_n(1-\frac{1}{2}a_n)}\, w_n^{\lambda_n}\; \frac{1}{\hat{\eta}(q)} q^{-\lambda_n\,\lambda_{n+1}} \\ &= \frac{1}{\hat{\eta}(q)^{n+1}} \sum_{\lambda_1,\ldots,\lambda_n=0}^{\infty} q^{\frac{1}{2}\lambda\cdot C\lambda-\frac{1}{2}\lambda\cdot Ce-\frac{1}{2}\lambda_1-\frac{1}{2}\lambda_n}\, w^{\lambda}\,, \end{aligned}$$

where we have defined $e := (1,\ldots,1)$, $\lambda := (\lambda_1,\ldots,\lambda_n)$, and $w^\lambda := w_1^{\lambda_1}\cdots w_n^{\lambda_n}$. The negative of C is the intersection matrix (B.2) of the compact divisors given in appendix B, where we also review how to determine the self-intersection numbers $-a_i$.

ALE spaces have vanishing canonical class. By specializing the above formula to this case, with all $a_i = 2$, we observe the ensuing simplification to

$$Z_{cm}(A_n;q,w) \;=\; \frac{1}{\hat{\eta}(q)^{n+1}} \sum_{\lambda_1,\ldots,\lambda_n=0}^{\infty} q^{\frac{1}{2}\lambda\cdot C\lambda}\, w^{\lambda}\,.$$

5 Gauge theory

On non-compact surfaces, the vertex rules from the previous section only capture part of the complete gauge theory partition function. To obtain the full partition function, we need to include contributions from both negative and non-compact divisors. In the compact case, linear equivalence will furthermore identify divisors in the gauge theory that correspond to distinct fixed points of the torus action, changing the combinatorics of the problem.

The factorization (2.2) of rank 1 torsion-free sheaves $\mathcal{T}$,

$$\mathcal{T} = \mathcal{L} \otimes \mathcal{I}_Z \, ,$$

implies that the moduli space of isomorphism classes of torsion-free sheaves factorizes

$$\mathfrak{M}_X(\beta, n) = \mathrm{Pic}\,^X_\beta \times X^{[n-n_\beta]} \, ,$$

with the Picard group $\mathrm{Pic}\,^X_\beta$ parametrizing line bundles which contribute $n_\beta = -\frac{1}{2}\beta \cdot (\beta + K_X)$ to the Euler characteristic of the torsion-free sheaf. It follows that the generating function (2.5) for the counting problem splits into a discrete and a continuous part,

$$Z_{gt}(X; q, w) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{\mathcal{L} \in \mathrm{Pic}\,^X_\beta} q^{-n_\beta} \, w^\beta \sum_{n \geq 0} \int_{X^{[n]}} e\big(TX^{[n]}\big) \; q^n \; .$$

For the continuous part, we need to count 0 dimensional subschemes. These contribute identically to Z_{gt} and Z_{cm}, by the factor

$$\frac{1}{\hat{\eta}(q)^{\chi(X)}} \, ,$$

as determined in Section 4.1.

It remains to enumerate the holomorphic line bundles $\mathcal{L} \in \mathrm{Pic}\,^X_\beta$. In fact, on a toric manifold, the Picard group is spanned by the classes of torically invariant divisors. Our task will be to determine an integral generating set among these. We do this for each of the examples considered in Section 4.4 in turn.[9]

5.1 Projective plane

The homology of $\mathbb{CP}^2$ is spanned by the hyperplane class, with self-intersection number 1. Holomorphic line bundles on $\mathbb{CP}^2$ hence permit self-dual, but not anti-self-dual connections. Of course, the two conditions are interchanged upon reversing the orientation of the surface. Let us proceed to determine the gauge theory partition function, in the sense developed in Section 2.2, upon replacing anti-self-duality by self-duality.

The three torically invariant divisors of the complex projective plane are of course linearly equivalent (the Picard group of complex projective space in any dimension is spanned by

[9]Note the deceptive similarities to the case of counting subschemes. The sum over torically invariant divisors there was due to localization. Furthermore, we will start off here by considering two additional toric divisors (those of non-compact support), but taking linear equivalence into account will result in the same number of summations as previously.

the hyperplane divisor). In contrast to the non-compact examples discussed below, we obtain the gauge theory partition function from the melting crystal partition function simply by dropping the sum over equivalent bundles and the restriction to effective divisors, and by taking into account the change in weight due to $K_{\mathbb{CP}^2} \neq 0$. One thereby finds

$$Z_{gt}(\mathbb{CP}^2; q, w) = \frac{1}{\hat{\eta}(q)^3} \sum_{u=-\infty}^{\infty} q^{-\frac{1}{2}u^2}\, w^u \,.$$

5.2 Hirzebruch-Jung surfaces

When including non-compact prime divisors, the full set of divisors associated to the 1-cones of the toric fan of a $Y_{p,q}$ space become linearly dependent. We will now determine an integer generating set for the Picard group. For clarity, we will first treat the simpler case of ALE spaces A_n, though they are of course encompassed by the subsequent treatment of general Hirzebruch-Jung surfaces $Y_{p,q}$ with $(p, q) = (n + 1, n)$.

ALE spaces. Consider the vectors $(1, 0)$ and $(0, 1)$ in the toric fan of the resolved geometry of $A_n = \mathbb{C}^2/\mathbb{Z}_{n+1}$ introduced in appendix B.2. They correspond to the two principal divisors[10]

$$\begin{aligned} \mathrm{div}\left(\chi^{(1,0)}\right) &= D_0 - D_2 - 2\,D_3 - \ldots - n\,D_{n+1}\,, \\ \mathrm{div}\left(\chi^{(0,1)}\right) &= D_1 + 2\,D_2 + \ldots + (n+1)\,D_{n+1}\,, \end{aligned} \qquad (5.1)$$

where we have labelled toric divisors in clockwise order. The two divisors corresponding to the outermost 1-cones, D_0 and D_{n+1}, have non-compact support. Based on the relations of linear equivalence induced by eq. (5.1), we now demonstrate that the classes

$$e^i := -\sum_{j=1}^{n} \left(C^{-1}\right)^{ij} [D_j]\;, \qquad i = 1, \ldots, n\,, \qquad (5.2)$$

with $-C$ the intersection matrix of the compact divisors as given in eq. (B.2) of Appendix B, constitute an integral generating set for the Picard group $A_1(X)$.[11] As the entries of C^{-1} are fractional, we need to demonstrate both that the elements e^i are generators, and that they are integer linear combinations of the toric divisors (including $[D_0]$ and $[D_{n+1}]$). Both properties follow upon providing the following recursive presentation of the e^i (for $n > 1$; the case $n = 1$ is trivial, with a single generator $[D_0] = [D_2]$). It is easy to verify that $e^1 = [D_0]$ and $e^n = [D_{n+1}]$. For $i = 2, \ldots, n-1$, the e^i satisfy

$$e^i = e^{i-1} - e^n - [D_i] - \ldots - [D_{n+1}]\,.$$

It follows that $\{e^i\}$ represents an extension of the set of non-compact torically invariant divisors $\{[D_0], [D_{n+1}]\}$ to an integral generating set for the Picard group.

[10] See e.g. ref. [34] for the notation χ^u, which assigns a function to the lattice vector u.

[11] Such a generating set is of course not unique. Our choice provides a dual set, via the intersection product linearly extended to non-compact divisors, to the compactly supported divisors $D_1, \ldots, D_n$, and as such corresponds to the basis of bundles constructed by Kronheimer and Nakajima in ref. [23].

Parametrizing the class of a divisor $D = D_u$ in terms of the generators e^i,

$$[D_u] = \sum_{i=1}^{n} u_i \, e^i$$

with $u = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, we can now compute its second Chern character. Note that the Chern classes corresponding to divisor classes are integral, irrespective of the support of the divisor. At the level of Chern classes, we can hence invoke the presentation of $e^1 = [D_0]$ and $e^n = [D_{n+1}]$ in eq. (5.2) to solve for e^1 and e^n, and the right-hand side, despite the appearance of the fraction $\frac{1}{n+1}$, maps into integral cohomology. With the intersection pairing (3.5), we thus arrive at

$$\mathrm{ch}_2\big(\mathcal{O}_X(D_u)\big) = \tfrac{1}{2}\, u \cdot C^{-1} u \,. \tag{5.3}$$

General $Y_{p,q}$ spaces. With the parametrization of the toric fan given in appendix B, the principal divisors corresponding to the lattice vectors $(1, 0)$ and $(0, 1)$ are

$$\begin{aligned} \mathrm{div}\left(\chi^{(1,0)}\right) &= \sum_{i=0}^{n+1} x_i \, D_i \,, \\ \mathrm{div}\left(\chi^{(0,1)}\right) &= \sum_{i=0}^{n+1} y_i \, D_i \,, \end{aligned}$$

where we have introduced the notation $v_i = (x_i, y_i)$. Recalling that $v_0 = (1, 0)$, we arrive at the linear equivalences

$$\begin{aligned} [D_0] &= \frac{1}{y_{n+1}} \sum_{i=1}^{n} (x_{n+1}\, y_i - x_i \, y_{n+1}) \, [D_i] \,, \\ [D_{n+1}] &= -\frac{1}{y_{n+1}} \sum_{i=1}^{n} y_i \, [D_i] \,. \end{aligned}$$

By invoking eq. (B.3) from Appendix B and the relation

$$x_{i-1}\, y_{i+1} - x_{i+1}\, y_{i-1} = a_i \,,$$

which follows from eq. (B.3) and the fact that the resolved geometry is non-singular (i.e. the 2-cones have volume 1), we can easily verify that in $A_1(Y_{p,q}) \otimes \mathbb{Q}$ one has

$$[D_0] = -\sum_{i=1}^{n} \left(C^{-1}\right)^{1\,i} [D_i] \,, \tag{5.4}$$

$$[D_{n+1}] = -\sum_{i=1}^{n} \left(C^{-1}\right)^{n\,i} [D_i] \,. \tag{5.5}$$

We can now complete the set $\{[D_0], [D_{n+1}]\}$ to an integral generating set for $A_1(Y_{p,q})$ by setting $e^1 = [D_0]$, $e^n = [D_{n+1}]$ and defining e^i, $i = 2, \ldots, n-1$ recursively via

$$e^i = e^{i-1} - \sum_{j=i}^{n} c^i_j \, [D_i] - c^i_{n+1} \, e^n \,,$$

where

$$
\begin{aligned}
c^i_i &= 1\,, \\
c^i_j &= a_{j-1}\,c^i_{j-1} - 1\,, \quad j = i+1,\dots,n\,, \\
c^i_{n+1} &= -c^i_{n-1} + a_n\,c^i_n\,.
\end{aligned}
$$

The generators so defined satisfy

$$
e^i = -\sum_{j=1}^n \left(C^{-1}\right)^{ij}\,[D_j]\,.
$$

The computation now proceeds as above, yielding, for

$$
[D_u] = \sum_{i=1}^n u_i\,e^i\,,
$$

the second Chern character given by (5.3).

Combining the contributions from the line bundles and the ideal sheaves of points, we obtain the partition function

$$
Z_{gt}(Y_{p,q};q,v) = \frac{1}{\hat\eta(q)^{\chi(Y_{p,q})}}\,\sum_{u\in\mathbb{Z}^{d-1}} q^{-\frac12\,u\cdot C^{-1}u}\,w^u
$$

with $w^u := w_1^{u_1}\cdots w_n^{u_n}$. This coincides with the results obtained in ref. [8].

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A Euler characteristic of torus invariant subschemes

In this appendix, we will calculate the Euler characteristic $\chi(\mathcal{O}_Y)$ of torically invariant subschemes Y using Čech cohomology. We will compute the cohomology with respect to the canonical torically invariant open cover $\{U_i\}$ of X, where each U_i corresponds to a maximal cone. We choose the index i to enumerate consecutive maximal cones in anti-clockwise order. In the case that X is compact, we will identify the first and last open set of our cover. The collection of sets thus defined has the properties

$$U_i \cap U_j \begin{cases} = (\mathbb{C}^*)^2 \,, & j \neq i \pm 1 \,, \\ \supset (\mathbb{C}^*)^2 \,, & j = i \pm 1 \,, \end{cases} \tag{A.1}$$

and

$$U_i \cap U_j \cap U_k = (\mathbb{C}^*)^2$$

for any i, j, k. Our strategy to compute $\chi(\mathcal{O}_Y)$ is as follows. Given

$$V = \bigcup_{i=a}^{b} U_i \,, \tag{A.2}$$

let

$$A_V = \mathcal{O}_Y(U_{a-1} \cup V \cup U_{b+1})\big|_V$$

be the space of global sections of $\mathcal{O}_Y(V)$ that lift to $\mathcal{O}_Y(U_{a-1})$ and $\mathcal{O}_Y(U_{b+1})$, and define

$$\chi_V = \dim_{\mathbb{C}}(A_V) - \check{h}^1\big(\mathcal{O}_Y\big|_{U_{a-1}\cup V \cup U_{b+1}}\big) \,.$$

Note that we avoid the use of $\check{h}^0(\mathcal{O}_Y|_{U_{a-1}\cup V\cup U_{b+1}})$ in place of $\dim_{\mathbb{C}}(A_V)$, as the former could be infinite. We will compute χ_V for $V = U_i$, and given two adjacent such sets, such as V in eq. (A.2) and

$$W = \bigcup_{i=b+1}^{c} U_i \,,$$

together with their respective integers χ_V and χ_W, we will determine $\chi_{V\cup W}$. Applying this gluing operation a finite number of times will yield $\chi(\mathcal{O}_Y)$.

The monomial generators of $\mathcal{O}_Y(U_i)$ are in one-to-one correspondence with the boxes of a possibly infinite Young tableau π, such that π_k and π_k^T (with π_k/π_k^T the number of boxes in the k-th row/column of π) stabilize at large k, to λ_i and λ_{i+1} say. The boxes with coordinates (k, l), $k \geq \lambda_i$, $l \geq \lambda_{i+1}$ correspond to sections that restrict to 0 outside of U_i. Each such box contributes one to $\chi(\mathcal{O}_Y)$. In the following, we can hence restrict attention to subschemes Y without embedded points.

We turn to the calculation of χ_{U_i}. Without loss of generality, we can assume

$$\begin{aligned} \mathcal{O}_Y(U_{i-1}) &= \mathbb{C}\big[\tfrac{1}{x}\,,\, x^{a_{i-1}}\, y\big] \,/\, \big((\tfrac{1}{x})^{\lambda_{i-1}}\, (x^{a_{i-1}} y)^{\lambda_i}\big) \,, \\ \mathcal{O}_Y(U_i) &= \mathbb{C}[x, y] \,/\, \big(x^{\lambda_{i+1}}\, y^{\lambda_i}\big) \,, \\ \mathcal{O}_Y(U_{i+1}) &= \mathbb{C}\big[x\, y^{a_i}\,,\, \tfrac{1}{y}\big] \,/\, \big((x\, y^{a_i})^{\lambda_{i+1}}\, (\tfrac{1}{y})^{\lambda_{i+2}}\big) \,, \end{aligned}$$

together with

$$\begin{aligned} \mathcal{O}_Y(U_{i-1,i}) &= \mathbb{C}\big[x\,,\, \tfrac{1}{x}\,,\, y\big] \,/\, \big(y^{\lambda_i}\big) \,, \\ \mathcal{O}_Y(U_{i,i+1}) &= \mathbb{C}\big[x\,,\, y\,,\, \tfrac{1}{y}\big] \,/\, \big(x^{\lambda_{i+1}}\big) \,, \end{aligned}$$

and

$$\mathcal{O}_Y(U_{i-1,i,i+1}) = 0\,.$$

We have introduced the notation $U_{i,\ldots,j} = U_i \cap \ldots \cap U_j$. A moment's thought yields

$$\begin{aligned} \dim_{\mathbb{C}}(A_{U_i}) &= \sum_{s=0}^{\lambda_i - 1} \sum_{r=0}^{a_{i-1}\,s} 1 + \sum_{s=0}^{\lambda_{i+1}-1} \sum_{r=0}^{a_i\,s} 1 - \lambda_{i+1}\,\lambda_i \\ &= a_{i-1}\,\frac{(\lambda_i - 1)\,\lambda_i}{2} + \lambda_i + a_i\,\frac{(\lambda_{i+1}-1)\,\lambda_{i+1}}{2} + \lambda_{i+1} - \lambda_{i+1}\,\lambda_i\,, \end{aligned}$$

and

$$\check{h}^1\big(\mathcal{O}_Y\big|_{U_{i-1}\cup U_i \cup U_{i+1}}\big) = 0\,,$$

hence

$$\chi_{U_i} = \dim_{\mathbb{C}}(A_{U_i})\,.$$

Next, we turn to the compution of $\chi_{V\cup W}$ given χ_V and χ_W, using the notation for V and W introduced above. Note that by eq. (A.1), $V \cap W = U_b \cap U_{b+1}$. If we consider the sum $\chi_V + \chi_W$ as an approximation to $\chi_{V\cup W}$, then we make the following mistakes:

- We count generators of A_V and A_W that have the same restriction to $V\cap W$ twice.
- We count generators of $A_V|_{V\cap W}$ that do not lift to A_W, and likewise elements of $A_W|_{V\cap W}$ that do not lift to A_V.
- We do not subtract new contributions to $\check{h}^1$, i.e. generators in $\mathcal{O}_Y(V\cap W)$ that are not exact.

Now consider the space $B_{V,W} = \mathcal{O}_Y(U_b \cup U_{b+1})|_{U_b\cap U_{b+1}}$. Generators of this space either lift to both A_V and A_W, or to either A_V or A_W but not both, or to neither. By the following lemma, the generators of $B_{V,W}$ are hence in one-to-one correspondence with the elements over-counted above.

Lemma A.1. *Elements in $B_{V,W}$ that lift neither to A_V nor to A_W lie in $\check{H}^1(V\cup W)$.*

Proof. A preimage r under the Čech differential δ of an element in $\check{C}^1(V\cup W)$ of the form

$$s_{ij} = \begin{cases} a & \text{if } \{i,j\} = \{b, b+1\} \\ 0 & \text{otherwise} \end{cases}\,,$$

must satisfy

$$r_i - r_j\big|_{U_{i,j}} = \begin{cases} \pm\, a & \text{if } \{i,j\} = \{b,b+1\} \\ 0 & \text{otherwise}\ . \end{cases}\,,$$

Such an element exists if and only if a lifts to A_V or A_W. □

Finally, the number of generators of $B_{V,W}$ already entered into our computation of the dimension of A_{U_i} above,

$$\dim_{\mathbb{C}}(B_{V,W}) = \sum_{s=0}^{\lambda_{b+1}-1} \sum_{r=0}^{a_b\,s} 1\,.$$

Combining all of these observations, we arrive at the desired formula for the holomorphic Euler characteristic of Y,

$$\chi(\mathcal{O}_Y) = \sum_{i=1}^{n} \left(a_i \, \frac{\lambda_i\,(\lambda_i - 1)}{2} + \lambda_i - \lambda_i\,\lambda_{i+1} \right) .$$

B Toric surfaces

B.1 General non-singular toric surfaces

A non-singular toric surface is determined by a sequence of integral vectors v_i in $\mathbb{Z}^2$, taken in counter-clockwise order, that generate the 1-cones of the toric fan. For compact surfaces, we will enumerate these v_1 through v_n. Any two adjacent vectors span a 2-cone in this case, and by non-singularity, generate the lattice. For non-compact surfaces, two of the 2-cones have only one neighbor. For this case, we denote the outer-most vectors by v_0 and v_{n+1}, giving rise to a total of $n+2$ 1-cones.

The coordinate transformation between neighboring 2-cones is of the form

$$(x, y) \;\longmapsto\; \left(\tfrac{1}{x}\,,\, x^a\,y\right) .$$

For each 2-cone, generated by integral vectors v_i and v_{i+1}, one determines the integer a by considering the generator v_{i+1} of the neighboring cone, counter-clockwise, which satisfies

$$v_{i+1} = -v_{i-1} + a_i\,v_i\,. \tag{B.1}$$

The torically invariant prime divisors of a toric manifold are in one-to-one correspondence with the integral generators of the 1-cones. In the conventions introduced above, the number of compactly supported divisors is n, both in the compact and the non-compact case. The intersection matrix of the compactly supported divisors is determined as follows. Two divisors whose associated 1-cones span a 2-cone of the fan intersect transversally, while all others are disjoint. The self-intersection number $D_i^2 = -a_i$ is determined by the constant a_i in eq. (B.1) (recall that we are excluding $i = 0$ and $i = n+1$; indeed, due to the non-compact support of the associated divisors, the naive intersection number here is not defined). The intersection matrix $-C$ for the compactly supported toric divisors therefore has the form

$$C = \begin{pmatrix} a_1 & -1 & 0 & \dots & 0 \\ -1 & a_2 & -1 & \dots & 0 \\ 0 & -1 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & -1 \\ 0 & 0 & \dots & -1 & a_n \end{pmatrix} . \tag{B.2}$$

B.2 Hirzebruch-Jung surfaces

Hirzebruch-Jung spaces $X = Y_{p,q}$ are non-compact toric surfaces, parametrized by two positive coprime integers p and q with $p > q$. They are defined as the resolutions of $A_{p,q}$

quotients, i.e. the quotients of $\mathbb{C}^2$ by the action of the cyclic group $\mathbb{Z}_p$ generated by

$$\Gamma = \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix} ,$$

where $\xi = \mathrm{e}^{2\pi \mathrm{i}/p}$. The toric fan for the singular space is given by the two 1-cones generated by the integral vectors $v_0 = (1,0)$ and $v_{n+1} = (-q,p)$ respectively, and the 2-cone generated by the pair. The resolution is obtained via subdivision with 1-cones generated by integral vectors $v_1, \ldots, v_n$ such that

$$v_{i-1} + v_{i+1} = a_i \, v_i \tag{B.3}$$

for $i = 1, \ldots, n$. The integers a_i can be read off from the continued fraction expansion of p/q,

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots a_{n-1} - \frac{1}{a_n}}} .$$

With these entries, C of eq. (B.2) is positive definite, the intersection matrix hence negative definite.

Topologically, ALE spaces are resolutions of A_n singularities. These are the Hirzebruch-Jung spaces $Y_{n+1,n}$. For these surfaces, $a_i = 2$ for $i = 1, \ldots, n$, and a choice of integral vectors generating the 1-cones of the toric fan is given by

$$v_0 = (1,0) , \quad v_1 = (0,1) , \quad \ldots , \quad v_{n+1} = (-n, n+1) .$$

We have depicted the fan for the surface A_2 in Figure 7.

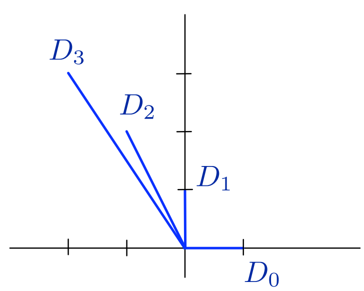

Figure 7: The toric fan for A_2, with the torically invariant divisors indicated.

C Characteristic classes of coherent sheaves

In physics, we are most familiar with characteristic classes assigned to vector bundles on a manifold X, taking values in $H^*(X,\mathbb{Z}) \otimes \mathbb{Q}$ (the sheaf cohomology of the locally constant sheaf); we often deal with the image in de Rham cohomology. Chern classes form basic building blocks for all characteristic classes. The total Chern class satisfies the multiplicative property

$$c_t(\mathcal{E}) = c_t(\mathcal{E}') \cdot c_t(\mathcal{E}'') \tag{C.1}$$

whenever the bundles $\mathcal{E}$, $\mathcal{E}'$, and $\mathcal{E}''$ fit into an exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0\,.$$

The product in eq. (C.1) is the cup product or the wedge product, respectively.

A refined version of characteristic classes takes values in the Chow ring $A_*(X) \otimes \mathbb{Q}$ of X. One path between the two definitions is via the splitting principle and the relation of line bundles to divisors. We take the image of an irreducible closed subscheme Y in the Chow ring to be the underlying closed subset endowed with the reduced induced structure, Y_{red}, with multiplicity given by the length of the local ring $\mathcal{O}_{y,Y}$ at the generic point y of Y_{red} (we will unravel this perhaps unfamiliar sounding definition below in the simple case of a zero dimensional subscheme).

The Grothendieck group $K(X)$ of a scheme X is the free abelian group generated by the coherent sheaves on X, modulo the relation $\mathcal{F} - \mathcal{F}' - \mathcal{F}''$ whenever these sheaves fit into an exact sequence. As coherent sheaves on well-behaved schemes allow for a finite locally free resolution, the property (C.1) allows for the extension of the definition of characteristic classes to all of $K(X)$.

For a complete scheme X of dimension n, there is a degree map $A_n(X) \to \mathbb{Z}$. In the cohomological description, this corresponds to taking the integral over the underlying compact manifold of X. This map can be defined on non-compact X for cycles of $A_n(X)$ that have a representative with compact support.

With these definitions, we can derive the Euler characteristic (3.1) of the structure sheaf of a subscheme Y as follows. The ideal sheaf $\mathcal{I}_Y$ associated to Y is defined via the exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0\,.$$

By additivity of the Euler characteristic, one has

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - \chi(\mathcal{I}_Y)\,. \tag{C.2}$$

If we consider $\mathcal{I}_Y$ as an abstract sheaf, forgetting about its embedding into $\mathcal{O}_X$, then it is isomorphic to an invertible sheaf on X, i.e. a line bundle. This is simply the familiar correspondence between divisors $D = [Y]$ and line bundles $\mathcal{O}_X(D)$,

$$\mathcal{O}_X(-D) = \mathcal{I}_Y\,.$$

To calculate $\chi(\mathcal{O}_Y)$ using eq. (C.2), we invoke the Hirzebruch-Riemann-Roch theorem to compute $\chi(\mathcal{O}_X)$ and $\chi(\mathcal{O}_X(-D))$. It states that *the Euler characteristic of a locally free sheaf $\mathcal{E}$ of rank r on a nonsingular projective variety X of dimension n is given by*

$$\chi(\mathcal{E}) = \deg \left(\mathrm{ch}(\mathcal{E}) \cdot \mathrm{td}\,(X)\right)_n\,,$$

where $(\,-\,)_n$ denotes the component of degree n in the Chow ring $A_*(X) \otimes \mathbb{Q}$ and a dot denotes intersection product.

When X is a surface, the Todd class is given by

$$\mathrm{td}\,(X) = 1 - \tfrac{1}{2}\,K_X + \tfrac{1}{12}\left(K_X^2 + c_2(X)\right)\,,$$

where $K_X = -c_1(X)$ is the canonical divisor. Since $\mathrm{ch}(\mathcal{O}_X) = 1$ one finds

$$\chi(\mathcal{O}_X) = \tfrac{1}{12}\left(K_X^2 + c_2(X)\right).$$

For a line bundle $\mathcal{O}_X(-D)$, one has $\mathrm{ch}(\mathcal{O}_X(-D)) = \exp(-D)$. Therefore,

$$\chi\big(\mathcal{O}_X(-D)\big) = \tfrac{1}{2}\, D\cdot(D+K_X) + \tfrac{1}{12}\left(K_X^2 + c_2(X)\right).$$

Collecting these results, we arrive at formula (3.1).

Finally, we present a simple application of the definition of degree presented above to the case of zero dimensional subschemes.

Lemma C.1. *The degree of an irreducible zero dimensional subscheme Y of a scheme X of finite type over an algebraically closed field k is given by* $\dim_k(H^0(Y, \mathcal{O}_Y))$.

Proof. The question is local, so we can assume X affine with coordinate ring A, and $Y = \mathrm{Spec}\, A/I$. The generic point y of Y_{red} is $\sqrt{I}$ (this is prime by the irreducibility assumption). The local ring at the generic point is $\mathcal{O}_{y,Y} = (A/I)_{\sqrt{I}} = A/I$. The second equality follows from $\dim A/I = 0$. As the length of A/I as a module over k is equal to its dimension as a vector space, the lemma follows. □

D The Hilbert scheme of $\mathbb{CP}^2$

The Hilbert scheme of hypersurfaces in projective space (zero sets of homogeneous polynomials in $\mathbb{P}^r$) is easily obtained. For a hypersurface Y of degree d, whose Hilbert polynomial is given by

$$P_Y^{\mathbb{P}^r}(t) = \binom{r+t}{r} - \binom{r+t-d}{r},$$

it is given by projective space $\mathbb{P}^N$, with $N = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-d\,H)) - 1$ and H the hyperplane divisor of $\mathbb{P}^r$. Consider in particular the case $r = 2$. As defined above, the hypersurfaces can be non-reduced and reducible, but they cannot include embedded points (as principal ideals do not possess embedded components). Including such points increases the Euler characteristic of the subscheme. Hence the constant term of $P_Y^{\mathbb{P}^2}(t)$,

$$P_Y^{\mathbb{P}^2}(0) = \frac{3d - d^2}{2},$$

is a lower bound for the Euler characteristic of a degree d subscheme of $\mathbb{P}^2$. In fact, this follows from a corollary of Hartshorne [35].

Theorem D.1. *Let k be a field, $r > 0$ an integer and $p \in \mathbb{Q}[z]$ a numerical polynomial. Then a necessary and sufficient condition that p be the Hilbert polynomial of a proper closed subscheme of $\mathbb{P}_k^r$ is that when p is written in the form*

$$p(z) = \sum_{t=0}^{\infty}\left[\binom{z+t}{t+1} - \binom{z+t-m_t}{t+1}\right],$$

one has $m_0 \geq m_1 \geq \ldots \geq m_{r-1} \geq 0$ and $m_r = m_{r+1} = \ldots = 0$.

In the case of interest,

$$p(z) = m_1\, z + \frac{2m_0 + m_1 - m_1^2}{2}\,,$$

from which our claim follows.

This observation suggests a factorization of the Hilbert scheme of curves on $\mathbb{P}^2$ as

$$\mathrm{Hilb}^{\mathbb{P}^2}_{d\,t+n} = \mathrm{Hilb}^{\mathbb{P}^2}_{d\,t+n_d} \times \left(\mathbb{P}^2\right)^{[n-n_d]}\,,$$

for $n \geq n_d = \frac{3d-d^2}{2}$. It follows from Theorem A.1 that the left-hand side is empty for $n < n_d$.

At the level of the underlying topological spaces, this decomposition follows easily. With $S = \mathbb{C}[x_0, x_1, x_2]$ the homogeneous coordinate ring of $\mathbb{P}^2$, the subschemes of $\mathbb{P}^2$ are in one-to-one correspondence with homogeneous ideals I of S, via the map $I \mapsto \mathrm{Proj}\, S/I$. Since S is a unique factorization domain, we can decompose I uniquely into irreducible ideals

$$I = \Big(\prod_i I_1^i\Big)\,\Big(\prod_j I_0^j\Big)\,,$$

where the I_1^i are generated by one element and correspond to subschemes of dimension 1, and the I_0^j are generated by more than one element and correspond to subschemes of dimension 0.

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