The emerging p-adic Langlands programme

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Abstract. We give a brief overview of some aspects of the p-adic and modulo p Langlands programmes.

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1. Introduction

Fix p a prime number and $\overline{\mathbb{Q}_p}$ an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$. Let $\ell \neq p$ be another prime number and $\overline{\mathbb{Q}_\ell}$ an algebraic closure of $\mathbb{Q}_\ell$. If F is a field which is a finite extension of $\mathbb{Q}_p$ and n a positive integer, the celebrated local Langlands programme for GL_n ([48], [37], [38]) establishes a "natural" $1-1$ correspondence between certain $\overline{\mathbb{Q}_\ell}$-linear continuous representations ρ of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ on n-dimensional $\overline{\mathbb{Q}_\ell}$-vector spaces and certain $\overline{\mathbb{Q}_\ell}$-linear locally constant (or smooth) irreducible representations π of $\mathrm{GL}_n(F)$ on (usually infinite dimensional) $\overline{\mathbb{Q}_\ell}$-vector spaces. This local correspondence is moreover compatible with reduction modulo ℓ ([68]) and with cohomology ([49], [23], [37]). By "compatible with cohomology", we mean here that there exist towers of algebraic (Shimura) varieties $(S(K))_K$ over F of dimension d indexed by compact open subgroups K of $\mathrm{GL}_n(F)$ on which $\mathrm{GL}_n(F)$ acts on the right and such that the natural action of $\mathrm{GL}_n(F) \times \mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ on the inductive limit of ℓ-adic étale cohomology groups:

$$\varinjlim_{K} H^d_{\text{ét}}(S(K) \times_F \overline{\mathbb{Q}_p}, \overline{\mathbb{Q}_\ell}) \tag{1}$$

makes it a direct sum of representations $\pi \otimes \rho$ where ρ matches π by the previous local correspondence. (One can also take étale cohomology with values in certain locally constant sheaves of finite dimensional $\overline{\mathbb{Q}_\ell}$-vector spaces.)

Now $\mathrm{GL}_n(F)$ is a topological group (even a p-adic Lie group) and by [69] one can replace the above locally constant irreducible representations π of $\mathrm{GL}_n(F)$ on $\overline{\mathbb{Q}_\ell}$-vector spaces by continuous topologically irreducible representations $\widehat{\pi}$ of $\mathrm{GL}_n(F)$ on ℓ-adic Banach spaces (by a completion process which turns out to be

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reversible). This is quite natural as it gives now a $1-1$ correspondence between two kinds of continuous ℓ-adic representations. The original aim of the local p-adic Langlands programme is to look for a possible p-adic analogue of this ℓ-adic correspondence, that is:

Can one match certain linear continuous representations ρ of the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ on n-dimensional $\overline{\mathbb{Q}_p}$-vector spaces to certain linear continuous representations $\widehat{\pi}$ of $\mathrm{GL}_n(F)$ on p-adic Banach spaces, in a way that is compatible with reduction modulo p, with cohomology, and also with "p-adic families"?

It turns out that such a nice p-adic correspondence indeed exists between 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and certain continuous representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ on "unitary p-adic Banach spaces" (that is, with an invariant norm) which satisfies all of the above requirements. Based on the work of precursors ([50], [1], [2]) and on the papers [67], [59], [60], [61], the first cases were discovered and studied by the author in [6], [7], [8], [9], [15] and a partial programme was stated for $\mathrm{GL}_2(\mathbb{Q}_p)$ in [8]. The local p-adic correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$, together with its compatibility with "p-adic families" and with reduction modulo p, was then fully developed, essentially by Colmez, in the papers [19], [5], [3], [20], [21] after Colmez discovered that the theory of (φ,Γ)-modules was a fundamental intermediary between the representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and the representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ (see Berger's Bourbaki talk [4] and [12] for a historical account). These local results already have had important global applications by work of Kisin ([45]) and Emerton ([28]) as, combined with deformations techniques, they provide an almost complete proof of the Fontaine-Mazur conjecture ([31]). Finally, the important compatibility with cohomology is currently being written in [28]. Note that the relevant cohomology in that setting is not (1) but rather its p-adic completion, which is a much more intricate representation. Such p-adically completed cohomology spaces were introduced by Emerton in [24] (although some cases had been considered before, see, e.g., [51]). Their study as continuous representations of $\mathrm{GL}_n(F)\times\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ seems a mammoth task which is sometimes called the "global p-adic Langlands programme" (as these cohomology spaces are of a global nature). We sum up some of these results for $\mathrm{GL}_2(\mathbb{Q}_p)$ in §2.

At about the same time as the p-adic and modulo p theories for $\mathrm{GL}_2(\mathbb{Q}_p)$ were definitely flourishing, the theory modulo p for $\mathrm{GL}_2(F)$ and $F\neq\mathbb{Q}_p$ was discovered in [16], much to the surprise of everybody, to be much more involved. Although nothing really different happens on the Galois side when one goes from $\mathbb{Q}_p$ to F, the complications on the GL_2 side are roughly twofold: (i) there are infinitely many smooth irreducible (admissible) representations of $\mathrm{GL}_2(F)$ over any finite field containing the residue field of F (whereas when $F=\mathbb{Q}_p$ there is only a finite number of them) and (ii) the vast majority of them are much harder to study than for $F=\mathbb{Q}_p$. In particular (i) has the consequence that there is no possible naive $1-1$ correspondence as for the $F=\mathbb{Q}_p$ case and (ii) has the consequence that no one so far has been able to find an explicit construction of *one single* irreducible representation of $\mathrm{GL}_2(F)$ that isn't a subquotient of a principal series. The p-adic theory shouldn't be expected to be significantly simpler ([54]). And yet, cohomology spaces analogous to (1) are known to exist and to support interesting representations of $\mathrm{GL}_2(F)$ over $\overline{\mathbb{F}_p}$ (where $\overline{\mathbb{F}_p}$ is an algebraic closure of the finite field $\mathbb{F}_p$) as well as related representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$, but the representations of $\mathrm{GL}_2(F)$ occuring there seem to be of a very special type. We report on these phenomena in §3.

We then conclude this non-exhaustive survey in §4 more optimistically by mentioning, among other scattered statements, three theorems or conjectures available for $\mathrm{GL}_n(F)$ that give some kind of (p-adic or modulo p) relations between the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ side and the $\mathrm{GL}_n(F)$ side. Although they are quite far from any sort of correspondence, these statements are clearly part of the p-adic Langlands programme and will probably play a role in the future.

One word about the title. Strangely, the terminology "p-adic Langlands correspondence/programme" started to spread (at least in the author's memory) only shortly after preprints of [6], [7], [8], [9], [14], [19], [24], [53], [59], [60], [61], [70] were available (that is, around 2004), although of course p-adic considerations on automorphic forms (e.g., congruences modulo p between automorphic forms, p-adic families of automorphic forms) had begun years earlier with the fundamental work of Serre, Katz, Mazur, Hida, Coleman, etc. Maybe one of the reasons was that an important difference between the above more recent references and older ones was the focus on (i) topological group representation theory "à la Langlands" and (ii) purely p-adic aspects in relation with Fontaine's classifications of p-adic Galois representations.

The present status of the p-adic Langlands programme so far is thus the following: almost everything is known for $\mathrm{GL}_2(\mathbb{Q}_p)$ but most of the experts (including the author) are quite puzzled by the apparent complexity of whatever seems to happen for any other group. The only certainty one can have is that much remains to be discovered!

Let us introduce some notations. Recall that $\overline{\mathbb{Q}_p}$ (resp. $\overline{\mathbb{F}_p}$) is an algebraic closure of $\mathbb{Q}_p$ (resp. $\mathbb{F}_p$). If K is a finite extension of $\mathbb{Q}_p$, we denote by $\mathcal{O}_K$ its ring of integers, by ϖ_K a uniformizer in $\mathcal{O}_K$ and by $k_K := \mathcal{O}_K/(\varpi_K\mathcal{O}_K)$ its residue field.

Throughout the text, we denote by F a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$, by $q = p^f$ the cardinality of k_F and by $e = [F : \mathbb{Q}_p]/f$ the ramification index of F. For $x \in F^\times$, we let $|x| := q^{-\mathrm{val}_F(x)}$ where $\mathrm{val}_F(p) := e$. The Weil group of F is the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ of elements w mapping to an integral power $d(w)$ of the arithmetic Frobenius of $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$ (that is, $x \mapsto x^p$) via the map $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F) \to \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$.

Representations always take values either in E-vector spaces, in $\mathcal{O}_E$-modules or in k_E-vector spaces where E is always a "sufficiently big" finite extension of $\mathbb{Q}_p$. By "sufficiently big", we mean big enough so that we do not have to deal with rationality issues. For instance irreducible always means absolutely irreducible, we always assume $|\mathrm{Hom}(F,E)| = |\mathrm{Hom}(F,\overline{\mathbb{Q}_p})|$, $|\mathrm{Hom}(k_F,k_E)| = |\mathrm{Hom}(k_F,\overline{\mathbb{F}_p})|$, etc.

We normalize the reciprocity map $F^\times \hookrightarrow \mathrm{Gal}(\overline{\mathbb{Q}_p}/F)^{\mathrm{ab}}$ of local class field theory by sending inverses of uniformizers to arithmetic Frobeniuses. Via this map, we consider without comment Galois characters as characters of $F^\times$ by restriction. We denote by $\varepsilon : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character and by ω its reduction modulo p. Seen as a character of $\mathbb{Q}_p^\times$, ε is the identity on $\mathbb{Z}_p^\times$ and sends p to 1.

If A is any $\mathbb{Z}$-algebra, we denote by $B(A)$ (resp. $T(A)$) the upper triangular matrices (resp. the diagonal matrices) in $\mathrm{GL}_n(A)$. We denote by I (resp. I_1) the Iwahori subgroup (resp. the pro-p Iwahori subgroup) of $\mathrm{GL}_n(\mathcal{O}_F)$, that is, the matrices of $\mathrm{GL}_n(\mathcal{O}_F)$ that are upper triangular modulo ϖ_F (resp. upper unipotent modulo ϖ_F).

A smooth representation of a topological group is a representation such that any vector is fixed by a non-empty open subgroup. A smooth representation of $\mathrm{GL}_n(\mathcal{O}_F)$ over a field is admissible if its subspace of invariant elements under any open (compact) subgroup of $\mathrm{GL}_n(\mathcal{O}_F)$ is finite dimensional. We recall that the socle of a smooth representation of a topological group over a field is the (direct) sum of all its irreducible subrepresentations.

We call a Serre weight for $\mathrm{GL}_n(\mathcal{O}_F)F^\times$ any smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_F)F^\times$ over k_E. In particular, a Serre weight is finite dimensional, $F^\times$ acts on it by a character and its restriction to $\mathrm{GL}_n(\mathcal{O}_F)$ is irreducible. In other references (e.g., [18] or [39]), a Serre weight is just a smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_F)$ over k_E; however, in all representations we consider, $F^\times$ acts by a character, and it is very convenient to extend the action to $\mathrm{GL}_n(\mathcal{O}_F)F^\times$.

2. The group $\mathrm{GL}_2(\mathbb{Q}_p)$

We assume here $F = \mathbb{Q}_p$. The p-adic Langlands programme for $\mathrm{GL}_2(\mathbb{Q}_p)$ and 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is close to being finished. We sum up below some of the local and global aspects of the theory.

2.1. The modulo p local correspondence. We first describe the modulo p Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (at least in the "generic" case), which is much easier than the p-adic one and which was historically found before.

Let σ be a Serre weight for $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ and denote by:

$$\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma$$

the k_E-vector space of functions $f:\mathrm{GL}_2(\mathbb{Q}_p)\to\sigma$ which have compact support modulo $\mathbb{Q}_p^\times$ and such that $f(kg)=\sigma(k)(f(g))$ for $(k,g)\in\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times\times\mathrm{GL}_2(\mathbb{Q}_p)$. We endow this space with the left and smooth action of $\mathrm{GL}_2(\mathbb{Q}_p)$ defined by $(gf)(g'):=f(g'g)$. By a standard result, one has ([2]):

$$\mathrm{End}_{\mathrm{GL}_2(\mathbb{Q}_p)}\Big(\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma\Big)=k_E[T]$$

for a certain Hecke operator T. One then defines:

$$\pi(\sigma,0):=\Big(\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times}^{\mathrm{GL}_2(\mathbb{Q}_p)}\sigma\Big)/(T).$$

One can prove that the representations $\pi(\sigma,0)$ are irreducible and admissible ([6]). The representations $\pi(\sigma,0)$ form the so-called *supersingular* representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\chi_i:\mathbb{Q}_p^\times\to k_E^\times$, $i\in\{1,2\}$ be smooth multiplicative characters and define:

$$\begin{array}{cccccc}\chi_1\otimes\chi_2 & : & B(\mathbb{Q}_p) & \to & k_E^\times \\ & & \begin{pmatrix}a & b\\ 0 & d\end{pmatrix} & \mapsto & \chi_1(a)\chi_2(d).\end{array}$$

Denote by:

$$\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2$$

the k_E-vector space of locally constant functions $f:\mathrm{GL}_2(\mathbb{Q}_p)\to k_E$ such that $f(hg)=(\chi_1\otimes\chi_2)(h)f(g)$ for $(h,g)\in B(\mathbb{Q}_p)\times\mathrm{GL}_2(\mathbb{Q}_p)$. We endow this space with the same left and smooth action of $\mathrm{GL}_2(\mathbb{Q}_p)$ as previously. The representations $\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2$ are admissible. They are irreducible if $\chi_1\neq\chi_2$ and have length 2 otherwise ([1], [2]). They form the so-called *principal series*. The supersingular representations together with the Jordan-Hölder factors of the principal series exhaust the smooth irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E with a central character ([2], [6]).

Theorem 2.1. *For $\chi_1\neq\chi_2$ and $\chi_1\neq\chi_2\omega^{\pm1}$ the k_E-vector space:*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1\otimes\chi_2\omega^{-1},\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_2\otimes\chi_1\omega^{-1}\big)$$

has dimension 1.

Proof. This follows for instance from [16, Cor.8.6] but other (and earlier) proofs can be found in [27] and [21, §VII]. □

Note that the assumptions on χ_i imply that both principal series in Theorem 2.1 are irreducible distinct (and hence that any extension between them has their central character) and that $\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}(\chi_2,\chi_1)$ also has dimension 1.

For g in the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, let:

$$\omega_2(g) := \frac{g(\sqrt[p^2-1]{-p})}{\sqrt[p^2-1]{-p}} \in \mu_{p^2-1}(\overline{\mathbb{Q}_p}) \xrightarrow{\sim} \mathbb{F}_{p^2}^\times \hookrightarrow k_E^\times$$

be Serre's level 2 fundamental character (where the first map is reduction modulo p and where we choose an arbitrary field embedding $\mathbb{F}_{p^2} \hookrightarrow k_E$). For $0 \le r \le p-1$, we denote by σ_r the unique Serre weight for $\mathrm{GL}_2(\mathbb{Z}_p)\mathbb{Q}_p^\times$ such that $\sigma_r(p) = 1$ and σ_r has dimension $r+1$ (in fact $\sigma_r|_{\mathrm{GL}_2(\mathbb{Z}_p)} \simeq \mathrm{Sym}^r(k_E^2)$). For $0 \le r \le p-1$, we denote by ρ_r the unique continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over k_E such that its determinant is ω^{r+1} and its restriction to inertia is $\omega_2^{r+1} \oplus \omega_2^{p(r+1)}$.

The modulo p local correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ can be defined as follows.

Definition 2.2. (i) For $0 \le r \le p-1$ and $\chi : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to k_E^\times$, the representation $\pi(\sigma_r, 0) \otimes (\chi \circ \det)$ corresponds to $\rho_r \otimes \chi$.
(ii) For $\chi_1 \notin \{\chi_2, \chi_2\omega^{\pm 1}\}$ the representation associated to the unique non-split (resp. split) extension in:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(\mathbb{Q}_p)}\big(\mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_{B(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}\chi_2 \otimes \chi_1\omega^{-1}\big)$$

corresponds to the representation associated to the unique non-split (resp. split) extension in $\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}(\chi_2,\chi_1)$.

For more general 2-dimensional reducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, the corresponding representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are a bit more subtle to define and we refer the reader to [27] or [21, §VII]. When representations (on both side) are semi-simple, the above correspondence was first defined in [6]. Note that Definition 2.2 requires one to check that whenever there is an isomorphism between the $\mathrm{GL}_2(\mathbb{Q}_p)$-representations involved, the corresponding Galois representations are also isomorphic. The correspondence of Definition 2.2 (without restrictions on the χ_i) can now be realized using the theory of (φ,Γ)-modules (see §2.3), which makes it much more natural.

2.2. Over E: first properties.

We now switch to continuous representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E and explain the first properties of the p-adic local correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

We fix a p-adic absolute value $|\cdot|$ on E extending the one on $F = \mathbb{Q}_p$ and recall that a (p-adic) norm on an E-vector space V is a function $\|\cdot\| : V \to \mathbb{R}_{\ge 0}$ such that $\|v\| = 0$ if and only if $v = 0$, $\|\lambda v\| = |\lambda|\|v\|$ ($\lambda \in E$, $v \in V$) and $\|v+w\| \le \mathrm{Max}(\|v\|, \|w\|)$ ($v, w \in V$). Any norm on V defines a metric $\|v - w\|$

which in turns defines a topology on V by the usual recipe. A (p-adic) Banach space over E is an E-vector space endowed with a topology coming from a norm and such that the underlying metric space is complete. All norms on a Banach space over E defining its topology are equivalent.

Definition 2.3. (i) A Banach space representation of a topological group G over E is a Banach space B over E together with a linear action of G by continuous automorphisms such that the natural map $G \times B \to B$ is continuous.
(ii) A Banach space representation B of G over E is unitary if there exists a norm $\| \cdot \|$ on B defining its topology such that $\|gv\| = \|v\|$ for all $g \in G$ and $v \in B$.

If G is compact, any Banach space representation of G is unitary but this is not true if G is not compact, e.g., $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Let B be a unitary Banach space representation of G and $B^0 := \{v \in B,\ \|v\| \leq 1\}$ the unit ball with respect to an invariant norm on B (giving its topology); then $B^0 \otimes_{\mathcal{O}_E} k_E$ is a smooth representation of G over k_E. A unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is said to be admissible if $B^0 \otimes_{\mathcal{O}_E} k_E$ is admissible. This does not depend on the choice of B^0 ([60, §3], [8, §4.6]). The category of unitary admissible Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E is abelian ([60]).

To any Banach space representation B of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, one can associate two subspaces $B^{\mathrm{alg}} \subset B^{\mathrm{an}}$ which are stable under $\mathrm{GL}_2(\mathbb{Q}_p)$. We define $B^{\mathrm{an}} \subset B$ (the locally analytic vectors) to be the subspace of vectors $v \in B$ such that the function $\mathrm{GL}_2(\mathbb{Q}_p) \to B$, $g \mapsto gv$ is locally analytic in the sense of [61]. We define $B^{\mathrm{alg}} \subset B^{\mathrm{an}}$ (the locally algebraic vectors) to be the subspace of vectors $v \in B$ for which there exists a compact open subgroup $H \subset \mathrm{GL}_2(\mathbb{Q}_p)$ such that the H-representation $\langle H \cdot v \rangle \subset B|_H$ is isomorphic to a direct sum of finite dimensional (irreducible) algebraic representations of H. In general one has $B^{\mathrm{alg}} = 0$, but if B is admissible as a representation of the compact group $\mathrm{GL}_2(\mathbb{Z}_p)$ it is a major result due to Schneider and Teitelbaum (which holds in much greater generality) that the subspace B^{an} is never 0 and is even dense in B ([62]).

Inspired by the modulo p correspondence of Definition 2.2 and by lots of computations on locally algebraic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ ([7], [15]), the author suggested in [8, §1.3] (see also [25, §3.3]) the following partial “programme”.

Fix V a linear continuous potentially semi-stable 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E with distinct Hodge-Tate weights $w_1 < w_2$. As in §4.1 below, following Fontaine ([30]) one can associate to V a Weil-Deligne representation to which (after semi-simplifying its underlying Weil representation) one can in turn attach a smooth admissible infinite dimensional representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E by the classical local Langlands correspondence (slightly modified as in §2.4 or §4.1 below). We denote by $\overline{V}^{\mathrm{ss}}$ the semi-simplification of $V^0 \otimes_{\mathcal{O}_E} k_E$ where V^0 is any Galois $\mathcal{O}_E$-lattice in V. To V one should be able to attach an admissible unitary Banach space representation $B(V)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E satisfying the following properties:

(i) $V \simeq V'$ if and only if $B(V) \simeq B(V')$ if and only if $B(V)^{\mathrm{an}} \simeq B(V')^{\mathrm{an}}$;

(ii) if V is irreducible then $B(V)$ is topologically irreducible; if V is reducible and indecomposable (resp. semi-simple) then $B(V)$ is reducible and indecomposable (resp. semi-simple);

(iii) for any unit ball $B^0 \subset B(V)$ preserved by $\mathrm{GL}_2(\mathbb{Q}_p)$, the semi-simplification of $B^0 \otimes_{\mathcal{O}_E} k_E$ corresponds to $\overline{V}^{\mathrm{ss}}$ under the modulo p correspondence of Definition 2.2;

(iv) the $\mathrm{GL}_2(\mathbb{Q}_p)$-subrepresentation $B(V)^{\mathrm{alg}}$ is isomorphic to:

$$\det{}^{w_1} \otimes_E \mathrm{Sym}^{w_2-w_1-1}(E^2) \otimes_E \pi.$$

When V is irreducible, (ii) and (iv) imply that $B(V)$ is a suitable completion of the locally algebraic representation $B(V)^{\mathrm{alg}} = \det{}^{w_1} \otimes_E \mathrm{Sym}^{w_2-w_1}(E^2) \otimes_E \pi$ with respect to an invariant norm. This property is the basic idea which initially motivated the above programme: what is missing to recover V from w_1, w_2 and its associated Weil-Deligne representation, or equivalently from $B(V)^{\mathrm{alg}}$, is a certain weakly admissible Hodge filtration ([22]). This missing data should precisely correspond to an invariant norm on $B(V)^{\mathrm{alg}}$. For instance, when V is irreducible and becomes crystalline over an abelian extension of $\mathbb{Q}_p$, such a filtration turns out to be unique (see, e.g., [32, §3.2]). Correspondingly one finds that there is a unique class of invariant norms on $B(V)^{\mathrm{alg}}$ in that case ([5, §5.3], [55]).

The first instances of $B(V)$ were constructed "by hand" for V semi-stable and small values of $w_2 - w_1$ in [7], [8] and [9]. Shortly after these examples were worked out, Colmez discovered that there was a way to define $B(V)$ directly out of Fontaine's (φ, Γ)-module of V ([19], [5]), thus explaining the above basic idea and also the compatibility (iii) with Definition 2.2 (the latter was checked in detail by Berger [3]). Using the (φ, Γ)-module machinery, Colmez was ultimately able to fulfil the above programme and even to associate a $B(V)$ to *any* linear continuous 2-dimensional representation V of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E. It was then recently proved by Paškūnas that these $B(V)$ and their Jordan-Hölder constituents essentially exhaust all topologically irreducible admissible unitary Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E.

2.3. (φ, Γ)-modules and the theorems of Colmez and of Paškūnas. We first briefly recall what a (φ, Γ)-module is ([30]) and then state the main results on the Banach space representations $B(V)$.

Let $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\sqrt[p^\infty]{1})/\mathbb{Q}_p)$ and note that the p-adic cyclotomic character ε canonically identifies Γ with $\mathbb{Z}_p^\times$. If $a \in \mathbb{Z}_p^\times$, let $\gamma_a \in \Gamma$ be the unique element such that $\varepsilon(\gamma_a) = a$. Let $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ be the p-adic completion of $\mathcal{O}_E[[X]][\frac{1}{X}]$ equipped with the unique ring topology such that a basis of neighbourhoods of 0 is:

$$\Big(p^n \mathcal{O}_E[[X]][\frac{1}{X}]^\wedge + X^m \mathcal{O}_E[[X]]\Big)_{n\geq 0, m\geq 0}.$$

We endow $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ with the unique $\mathcal{O}_E$-linear continuous Frobenius φ such that $\varphi(X^j) := ((1+X)^p - 1)^j$ ($j \in \mathbb{Z}$) and with the unique $\mathcal{O}_E$-linear continuous action of Γ such that ($a \in \mathbb{Z}_p^\times$):

$$\gamma_a(X^j) := ((1+X)^a - 1)^j = \left(\sum_{i=1}^{+\infty} \frac{a(a-1)\cdots(a-i+1)}{i!} X^i\right)^j.$$

We extend φ and Γ by E-linearity to the field $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$. Note that the actions of φ and Γ commute and preserve the subring $\mathcal{O}_E[[X]]$.

A (φ, Γ)-module over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ (resp. $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$) is an $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$-module of finite type (resp. an $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$-vector space of finite dimension) D equipped with the topology coming from that on $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ together with a homomorphism $\varphi : D \to D$ such that $\varphi(sd) = \varphi(s)\varphi(d)$ and with a continuous action of Γ such that $\gamma(sd) = \gamma(s)\gamma(d)$ and $\gamma \circ \varphi = \varphi \circ \gamma$ ($s \in \mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ or $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$, $d \in D$, $\gamma \in \Gamma$). A (φ, Γ)-module over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ or $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$ is said to be *étale* if moreover the image of φ generates D, in which case φ is automatically injective. There is a third important $\mathcal{O}_E$-linear map $\psi : D \to D$ on any étale (φ, Γ)-module D defined by $\psi(d) := d_0$ if $d = \sum_{i=0}^{p-1}(1+X)^i\varphi(d_i) \in D$ (any d determines uniquely such $d_i \in D$ as D is étale). The map ψ is surjective, commutes with Γ and satisfies by definition $(\psi\circ\varphi)(d) = d$. The main theorem is the following equivalence of categories due to Fontaine (we won't need more details here, see [29]).

Theorem 2.4. *There is an equivalence of categories between the category of $\mathcal{O}_E$-linear continuous representations of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *on finite type $\mathcal{O}_E$-modules (resp. on finite dimensional E-vector spaces) and étale* (φ, Γ)*-modules over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ (resp. over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$).*

If T (resp. V) is an $\mathcal{O}_E$-linear continuous representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on a finite type $\mathcal{O}_E$-module (resp. on a finite dimensional E-vector space), we denote by $D(T)$ (resp. $D(V)$) the corresponding (φ, Γ)-module over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ (resp. over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$).

Let V be any linear continuous 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E and $\chi : \mathbb{Q}_p^\times \to \mathcal{O}_E^\times$ any continuous character. For $d \in D(V)^{\psi=0} := \{d \in D(V),\ \psi(d) = 0\}$, one can prove that the formula:

$$w_\chi(d) := \lim_{n\to+\infty} \sum_{i\in\mathbb{Z}_p^\times \bmod p^n} \chi(i)^{-1}(1+X)^i\gamma_{-i^2}\big(\varphi^n\psi^n\big((1+X)^{-i^{-1}}d\big)\big)$$

converges in $D(V)^{\psi=0}$ and that $w_\chi^2(d) = d$ ([21, §II]). One defines the following E-vector space (recalling that $(1-\varphi\psi)(D(V)) \subseteq D(V)^{\psi=0}$):

$$D(V) \boxtimes_\chi \mathbb{P}^1 := \{(d_1, d_2) \in D(V) \times D(V), (1-\varphi\psi)(d_1) = w_\chi\big((1-\varphi\psi)(d_2)\big)\}.$$

Note that $(d_1, d_2) \in D(V) \boxtimes_\chi \mathbb{P}^1$ is determined by $\varphi\psi(d_1)$ and d_2, or by d_1 and $\varphi\psi(d_2)$. One can show that the following formulas define an action of the group $\mathrm{GL}_2(\mathbb{Q}_p)$ on $D(V) \boxtimes_\chi \mathbb{P}^1$ (even if V has dimension ≥ 2):

(i) if $a \in \mathbb{Q}_p^\times$, $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)(d_1, d_2) := (\chi(a)d_1, \chi(a)d_2)$;

(ii) if $a \in \mathbb{Z}_p^\times$, $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)(d_1, d_2) := (\gamma_a(d_1), \chi(a)\gamma_{a^{-1}}(d_2))$;

(iii) $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)(d_1, d_2) := (d_2, d_1)$;

(iv) $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)(d_1, d_2)$ is the unique element (d_1', d_2') of $D(V)\boxtimes_\chi\mathbb{P}^1$ such that $\varphi\psi(d_1') := \varphi(d_1)$ and $d_2' := \chi(p)\psi(d_2)$;

(v) if $b \in p\mathbb{Z}_p$, $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)(d_1, d_2)$ is the unique element (d_1', d_2') of $D(V) \boxtimes_\chi \mathbb{P}^1$ such that $d_1' := (1 + X)^b d_1$ and:

$$\varphi\psi(d_2') := \chi(1+b)^{-1}(1+X)^{-1}w_\chi\Big(\gamma_{1+b}\Big((1+X)^b w_\chi\big((1+X)^{(1+b)^{-1}}\varphi\psi(d_2)\big)\Big)\Big).$$

All of the above mysterious formulas were first discovered in the case V crystalline, where everything can be made very explicit ([5], [19]), and then extended more or less *verbatim* to any V.

For any étale (φ, Γ)-module D over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$, let $D^\natural \subset D$ be the smallest compact $\mathcal{O}_E[[X]]$-submodule which generates D over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ and which is preserved by ψ (one can prove that such a module exists). If D is an étale (φ, Γ)-module over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge[\frac{1}{p}]$, choose any lattice $D_0 \subset D$, that is any étale (φ, Γ)-module D_0 which is free over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ and generates D, and let $D^\natural := D_0^\natural[\frac{1}{p}]$. Going back to our 2-dimensional V, for $(d_1, d_2) \in D(V) \boxtimes_\chi \mathbb{P}^1$ and $n \in \mathbb{Z}_{\geq 0}$, let $(d_1^{(n)}, d_2^{(n)}) := \left(\begin{smallmatrix} p^n & 0 \\ 0 & 1 \end{smallmatrix}\right)(d_1, d_2) \in D(V) \boxtimes_\chi \mathbb{P}^1$. Note that from the iteration of (iv) above and from $\psi \circ \varphi = \mathrm{Id}$, one gets $\psi(d_1^{(n+1)}) = d_1^{(n)}$. One then defines the following subspace of $D(V) \boxtimes_\chi \mathbb{P}^1$:

$$D(V)^\natural \boxtimes_\chi \mathbb{P}^1 := \{(d_1, d_2) \in D(V) \boxtimes_\chi \mathbb{P}^1, d_1^{(n)} \in D(V)^\natural \text{ for all } n \in \mathbb{Z}_{\geq 0}\}.$$

Now let $\chi(x) = \chi_V(x) := (x|x|)^{-1}\det(V)(x)$ $(x \in \mathbb{Q}_p^\times)$. It turns out that, for such a χ, $D(V)^\natural \boxtimes_\chi \mathbb{P}^1$ is preserved by $\mathrm{GL}_2(\mathbb{Q}_p)$ inside $D(V) \boxtimes_\chi \mathbb{P}^1$. The stability of the subspace $D(V)^\natural \boxtimes_{\chi_V} \mathbb{P}^1$ by $\mathrm{GL}_2(\mathbb{Q}_p)$ is the most subtle part of the theory and, so far, the only existing proof (following a suggestion of Kisin) is by analytic continuation from the crystalline case (see [21, §II.3]).

We can now state the main theorem giving the local p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ in the case V is irreducible ([21]).

Theorem 2.5. *Assume V is irreducible. Then the quotient:*

$$B(V) := D(V) \boxtimes_{\chi_V} \mathbb{P}^1 / D(V)^\natural \boxtimes_{\chi_V} \mathbb{P}^1$$

together with the induced action of $\mathrm{GL}_2(\mathbb{Q}_p)$ *above is naturally an admissible unitary topologically irreducible Banach space representation of* $\mathrm{GL}_2(\mathbb{Q}_p)$ *over* E *satisfying properties (i) to (iii) of §2.2. Moreover,* $B(V)^{\mathrm{alg}} \neq 0$ *if and only if* V *is potentially semi-stable with distinct Hodge-Tate weights, and* $B(V)$ *then satisfies property (iv)*[1] *of §2.2.*

A unit ball of $B(V)$ is $B(T) := D(T) \boxtimes_{\chi_V} \mathbb{P}^1 / D(T)^\natural \boxtimes_{\chi_V} \mathbb{P}^1$ where $T \subset V$ is any Galois $\mathcal{O}_E$-lattice (one can extend all the previous constructions with $D(T)$ instead of $D(V)$). For the second part of property (i) of §2.2, one has to use that the subspace $B(V)^{\mathrm{an}} \subset B(V)$ of locally analytic vectors admits an analogous construction in terms of the (φ, Γ)-module of V over the Robba ring ([21, §V.2]). When V is reducible, a reducible $B(V)$ can also be constructed as an extension between two continuous principal series in a way analogous to (ii) of Definition 2.2 (see [19] or [27] or [47], see also [46]).

There is a nice functorial way to recover in all cases $D(T)$ from $B(T)$ (and hence $D(V)$ from $B(V)$) as follows. Let $n \in \mathbb{Z}_{>0}$, $T^\vee := \mathrm{Hom}_{\mathcal{O}_E}(T, \mathcal{O}_E)$ and let $\sigma \subset B(T^\vee)/p^n B(T^\vee)$ be any $\mathcal{O}_E$-submodule of finite type that generates $B(T^\vee)/p^n B(T^\vee)$ as a $\mathrm{GL}(\mathbb{Q}_p)$-representation (such a σ exists as a consequence of property (iii) of §2.2). Consider the $\mathcal{O}_E/p^n\mathcal{O}_E$-module:

$$\mathrm{Hom}_{\mathcal{O}_E/p^n\mathcal{O}_E}\left(\sum_{m\geq 0}\begin{pmatrix} p^m & \mathbb{Z}_p \\ 0 & 1\end{pmatrix}\sigma, \mathcal{O}_E/p^n\mathcal{O}_E\right) \tag{2}$$

where the left entry is the $\mathcal{O}_E/p^n\mathcal{O}_E$-submodule of $B(T^\vee)/p^n B(T^\vee)$ generated by σ under the matrices $\left(\begin{smallmatrix} p^m & a \\ 0 & 1\end{smallmatrix}\right)$, $a \in \mathbb{Z}_p$, $m \in \mathbb{Z}_{\geq 0}$. The natural action of $\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1\end{smallmatrix}\right)$ (resp. of $\left(\begin{smallmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1\end{smallmatrix}\right)$) on $\sum_{m\geq 0}\left(\begin{smallmatrix} p^m & \mathbb{Z}_p \\ 0 & 1\end{smallmatrix}\right)\sigma$ makes (2) a module over the Iwasawa algebra $\mathcal{O}_E[[\left(\begin{smallmatrix} 1 & \mathbb{Z}_p \\ 0 & 1\end{smallmatrix}\right)]] = \mathcal{O}_E[[\mathbb{Z}_p]] = \mathcal{O}_E[[X]]$ where $X := [\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)] - 1$ (resp. endows (2) with an action of $\Gamma \simeq \mathbb{Z}_p^\times$). After tensoring (2) by $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ over $\mathcal{O}_E[[X]]$, one can moreover define a natural Frobenius φ coming from the action of $\left(\begin{smallmatrix} p^{-1} & 0 \\ 0 & 1\end{smallmatrix}\right)$. The final result turns out to be an étale (φ, Γ)-module over $\mathcal{O}_E[[X]][\frac{1}{X}]^\wedge$ (killed by p^n) which is independent of the choice of σ and isomorphic to $D(T)/p^n D(T)$ ([21, §IV]). One then recovers $D(T)$ by taking the projective limit over n.

This last functor $B(T) \mapsto D(T)$ has revealed itself to be of great importance. For instance it allowed Kisin to give in many cases another construction of $B(V)$ more amenable to deformation theory ([46]) and it was a key ingredient in Kisin's or Emerton's proof of almost all cases of the Fontaine-Mazur conjecture ([45], [28]). Together with Kisin's construction, it was also used by Paškūnas to recently prove the following nice theorem ([56]):

[1] Some of the arguments of [21] here rely on the global results of [28], in particular on Theorem 2.7 below. Hence property (iv) might not yet be completely proven in a few cases like $p = 2$ or $\overline{V}^{\mathrm{ss}} \cong \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ up to twist, etc.

Theorem 2.6. *Assume $p \geq 5$, then the above functor $B(T) \mapsto D(T)$ induces (after tensoring by E) a bijection between isomorphism classes of:*

(i) admissible unitary topologically irreducible Banach space representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E which are not subquotients of continuous parabolic inductions of unitary characters;

(ii) irreducible 2-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over E.

Finally, let us mention that this functor has been extended to a more general setting in [64].

2.4. Local-global compatibility.

The local correspondence of §2.3 turns out to be realized on suitable cohomology spaces of (towers of) modular curves. This aspect, usually called "local-global compatibility" (as the cohomology spaces have a global origin), is the deepest and most important part of the theory.

Denote by $\mathbb{A}$ the adèles of $\mathbb{Q}$, $\mathbb{A}_f \subset \mathbb{A}$ the finite adèles and $\mathbb{A}_f^p \subset \mathbb{A}_f$ the finite adèles outside p. For any compact open subgroup K_f of $\mathrm{GL}_2(\mathbb{A}_f)$, consider the following complex curve:

$$Y(K_f)(\mathbb{C}) := \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/K_f\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}).$$

For varying K_f, $(Y(K_f)(\mathbb{C}))_{K_f}$ forms a projective system on which $\mathrm{GL}_2(\mathbb{A}_f)$ naturally acts on the right ($g \in \mathrm{GL}(\mathbb{A}_f)$ maps $Y(K_f)(\mathbb{C})$ to $Y(g^{-1}K_fg)(\mathbb{C})$). Likewise, for each fixed compact open subgroup $K_f^p \subset \mathrm{GL}_2(\mathbb{A}_f^p)$ and varying compact open subgroups $K_{f,p}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, $(Y(K_f^pK_{f,p})(\mathbb{C}))_{K_{f,p}}$ forms a projective system on which $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on the right. One considers the following "completed cohomology spaces":

$$\begin{aligned} \widehat{H}^1(K_f^p) &:= \Big(\varprojlim_n \varinjlim_{K_{f,p}} H^1\big(Y(K_f^pK_{f,p})(\mathbb{C}), \mathcal{O}_E/p^n\mathcal{O}_E\big)\Big) \otimes_{\mathcal{O}_E} E \\ \widehat{H}^1 &:= \varinjlim_{K_f^p} \widehat{H}^1(K_f^p) \end{aligned}$$

where H^1 is usual Betti cohomology and where $K_{f,p}$ (resp. K_f^p) runs over the compact open subgroups of $\mathrm{GL}(\mathbb{Q}_p)$ (resp. of $\mathrm{GL}(\mathbb{A}_f^p)$). The group $\mathrm{GL}_2(\mathbb{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbb{A}_f)$) acts on $\widehat{H}^1(K_f^p)$ (resp. on $\widehat{H}^1$) and one can prove that each $\widehat{H}^1(K_f^p)$ is an admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over E, an open unit ball being given by $\varprojlim \varinjlim H^1\big(Y(K_f^pK_{f,p})(\mathbb{C}), \mathcal{O}_E/p^n\mathcal{O}_E\big)$ (this result, due to Emerton, actually holds in much greater generality, see [24, §2]). Moreover, all the Betti cohomology spaces $H^1\big(Y(K_f^pK_{f,p})(\mathbb{C}), \mathcal{O}_E/p^n\mathcal{O}_E\big)$ can be identified with étale cohomology spaces, in particular they carry a natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We thus also have a (commuting) action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\widehat{H}^1(K_f^p)$ and $\widehat{H}^1$.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ be a linear continuous representation (where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$) and for each prime number ℓ let ρ_ℓ be the restriction of ρ to a decomposition group at ℓ. By the classical local Langlands correspondence as in [38], if $\ell \neq p$ one can associate to ρ_ℓ (after maybe semi-simplifying the action of Frobenius) a smooth irreducible representation π'_ℓ of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ over E. We slightly modify π'_ℓ as follows: if π'_ℓ is infinite dimensional, we let $\pi_\ell(\rho_\ell) := \pi'_\ell \otimes |\det|^{-\frac{1}{2}}$. If π'_ℓ is finite dimensional (that is, 1-dimensional), we let $\pi_\ell(\rho_\ell)$ be the unique principal series which has $\pi'_\ell \otimes |\det|^{-\frac{1}{2}}$ as unique irreducible quotient ($\pi_\ell(\rho_\ell)$ is a non-split extension of $\pi'_\ell \otimes |\det|^{-\frac{1}{2}}$ by a suitable twist of the Steinberg representation). For $\ell = p$, recall we have the unitary admissible Banach space representation $B(\rho_p)$ of §2.3. The following theorem is currently being proven by Emerton ([28]).

Theorem 2.7. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(E)$ be a linear continuous representation which is unramified outside a finite set of primes and such that the determinant of one (or equivalently any) complex conjugation is -1. Let $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(k_E)$ be the semi-simplification modulo p of ρ. Assume $p > 2$, $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[p]{1}))}$ irreducible and $\overline{\rho}|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)} \not\cong \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ or $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ up to twist. Then the $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1)$ decomposes as a restricted tensor product:*[2]

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1) \simeq B(\rho_p) \otimes_E \left(\otimes'_{\ell \neq p} \pi_\ell(\rho_\ell) \right).$$

Note that this theorem in particular states that $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1)$ is always non-zero. It is thus at the same time a local-global compatibility result *and* a modularity result! When ρ comes from a modular form (so that one already knows $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1) \neq 0$) and when moreover ρ_p is semi-stable, it was proven in [8], [5] and [13] that, for a suitable K_f^p, one has $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1(K_f^p)) \simeq B(\rho_p)$. These results were the first cohomological incarnations of the representations $B(V)$ of §2.3. Note that the case where ρ_p is crystalline and irreducible is easy here. Indeed, as ρ is modular, one knows that the locally algebraic representation $\det^{w_1} \otimes_E \mathrm{Sym}^{w_2-w_1-1}(E^2) \otimes_E \pi_p$ (see property (iv) of §2.2) embeds into $\widehat{H}^1(K_f^p)$ and its closure has to be $B(\rho_p)$ since this Banach space is its only unitary completion (see the end of §2.2).

The proof of Theorem 2.7 uses many ingredients, such as the aforementioned local-global compatibility in the crystalline case, the density in the space of all ρ of those ρ such that ρ_p is crystalline, Serre's modularity conjecture ([44]), Colmez's last functor at the end of §2.3, Mazur's deformation theory, Kisin's construction of $D(V)$ ([46]), etc. In fact, Theorem 2.7 is a consequence of an even stronger result giving a *full* description of the $\mathrm{GL}_2(\mathbb{A}_f)$-representation $\widehat{H}^1_{\overline{\rho}}$ (where $\widehat{H}^1_{\overline{\rho}}$ is the localization of $\widehat{H}^1$ at the maximal Hecke ideal defined by $\overline{\rho}$) and not just of $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1) = \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(\rho, \widehat{H}^1_{\overline{\rho}})$ ([28]).

[2] Depending on normalizations, one may have to replace ρ_p and the ρ_ℓ here by their duals or their Cartier duals.

3. The group $\mathrm{GL}_2(F)$

After the group $\mathrm{GL}_2(\mathbb{Q}_p)$, it is natural to look at the group $\mathrm{GL}_2(F)$, where many new phenomena appear and where the theory is thus still in its infancy. We describe below some of these new aspects, starting with the modulo p theory.

3.1. Why the $\mathrm{GL}_2(\mathbb{Q}_p)$ theory cannot extend directly. Let us start with reducible 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E. One of the first naive hopes in order to extend the modulo p Langlands correspondence from $\mathrm{GL}_2(\mathbb{Q}_p)$ to $\mathrm{GL}_2(F)$ in that case (see (ii) of Definition 2.2) was the following: since, if $F = \mathbb{Q}_p$, the unique non-split (resp. split) $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-extension:

$$\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$$

corresponds to the unique non-split (resp. split) $\mathrm{GL}_2(\mathbb{Q}_p)$-extension:

$$0 \longrightarrow \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1} \longrightarrow * \longrightarrow \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1} \longrightarrow 0$$

(at least in "generic" cases) then for general F the space of extensions:

$$\mathrm{Ext}^1_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)}(\chi_2, \chi_1)$$

(which has generic dimension $[F : \mathbb{Q}_p]$) would hopefully be (canonically) isomorphic to the space of extensions:

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F)}(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1})$$

thus yielding a nice correspondence.

Unfortunately, this turned out to be completely wrong.

Theorem 3.1. *Assume $F \neq \mathbb{Q}_p$. For $\chi_1 \neq \chi_2$ one has:*

$$\mathrm{Ext}^1_{\mathrm{GL}_2(F)}\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}\big) = 0.$$

Proof. This follows from [16, Thm.8.1] together with [16, Thm.7.16(i)] and [16, Cor.6.6]. □

Remark 3.2. In fact, at least for $\chi_1 \neq \chi_2$ and $\chi_1 \neq \chi_2\omega^{\pm 1}$, one can prove that $\mathrm{Ext}^i_{\mathrm{GL}_2(F)}\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}\big) = 0$ for $0 \leq i \leq [F : \mathbb{Q}_p]-1$ and that $\mathrm{Ext}^{[F:\mathbb{Q}_p]}_{\mathrm{GL}_2(F)}\big(\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}, \mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}\big)$ has dimension 1.

Let us now consider irreducible 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E. Just as for $F = \mathbb{Q}_p$, we define the following smooth representations of $\mathrm{GL}_2(F)$:

$$\pi(\sigma, 0) := \big(\mathrm{c} - \mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}^{\mathrm{GL}_2(F)} \sigma\big)/(T)$$

where σ is a Serre weight for $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ (the definition of T holds for any F). Recall that for $F=\mathbb{Q}_p$ the representations $\pi(\sigma,0)$ are all irreducible admissible.

Again, this turns out to be wrong for $F\neq\mathbb{Q}_p$.

Theorem 3.3. *Assume $F\neq\mathbb{Q}_p$. For any Serre weight σ the representation $\pi(\sigma,0)$ is of infinite length and is not admissible.*

Proof. When k_F is strictly bigger than $\mathbb{F}_p$, this can be derived from the results of [16], in particular Theorem 3.4 below. When $k_F=\mathbb{F}_p$, one can prove (by an explicit calculation) that $\pi(\sigma,0)$ contains $\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}^{\mathrm{GL}_2(F)}\sigma'$ for some Serre weight σ', which implies both statements as this representation is neither of finite length nor admissible. $\square$

3.2. So many representations of $\mathbf{GL_2}(\boldsymbol{F})$.

We survey most of the results so far on smooth admissible representations of $\mathrm{GL}_2(F)$ over k_E.

It is not known how to define an irreducible quotient of $\pi(\sigma,0)$ by *explicit* equations, although we know such quotients exist by an abstract argument using Zorn's lemma ([2]). The classification of all irreducible representations of $\mathrm{GL}_2(F)$ over k_E with a central character remains thus unsettled. But one can prove that there exist many irreducible admissible quotients of $\pi(\sigma,0)$ with, for instance, a given $\mathrm{GL}_2(\mathcal{O}_F)F^\times$-socle (containing σ). This is enough to show that irreducible representations of $\mathrm{GL}_2(F)$ over k_E are far more "numerous" than irreducible representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E. This also turns out to be useful as the representations of $\mathrm{GL}_2(F)$ appearing in étale cohomology groups over k_E analogous to (1) are expected to have specific $\mathrm{GL}_2(\mathcal{O}_F)F^\times$-socles (see §3.3 below).

Denote by $\mathcal{N}(F)$ the normalizer of the Iwahori subgroup I inside $\mathrm{GL}_2(F)$, that is, $\mathcal{N}(F)$ is the subgroup of $\mathrm{GL}_2(F)$ generated by I, the scalars $F^\times$ and the matrix $\left(\begin{smallmatrix}0&1\\ \varpi_F&0\end{smallmatrix}\right)$. The following theorem was proved in [16, §9] using and generalizing constructions of Paškūnas based on the existence and properties of injective envelopes of Serre weights for $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ ([53]).

Theorem 3.4. *Assume $p>2$. Let D_0 be a finite dimensional smooth representation of $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ over k_E with a central character and $D_1\subseteq D_0|_{IF^\times}$ a non-zero subrepresentation of $IF^\times$. For each k_E-linear action of $\mathcal{N}(F)$ on D_1 that induces the $IF^\times$-action, there exists a smooth admissible representation π of $\mathrm{GL}_2(F)$ over k_E with a central character such that the following diagram commutes:*

$$\begin{tikzcd} D_0 \arrow[r, hook] & \pi \\ D_1 \arrow[u, hook] \arrow[r, hook] & \pi \arrow[u, equal] \end{tikzcd}$$

(where the two horizontal injections are respectively $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ and $\mathcal{N}(F)$-equi-

variant), such that π is generated by D_0 under $\mathrm{GL}_2(F)$ *and such that :*

$$\mathrm{socle}(\pi|_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}) = \mathrm{socle}(D_0).$$

In general, it is not straightforward to construct explicitly such pairs (D_0, D_1) with a compatible action of $\mathcal{N}(F)$ on D_1, but there is one case where it is: the case where the pro-p subgroup I_1 of I acts trivially on D_1, for instance if $D_1 = D_0^{I_1}$ (which is never 0 as I_1 is pro-p). Indeed, in that case, D_1 is just a direct sum of characters of $IF^\times$ (as I/I_1 has order prime to p) and an action of $\left(\begin{smallmatrix} 0 & 1 \\ \varpi_F & 0 \end{smallmatrix}\right)$ is then essentially a certain permutation of order 2 on these characters. Moreover for such pairs (D_0, D_1) the assumption $p > 2$ in Theorem 3.4 is unnecessary. These examples are enough to show that there are infinitely many irreducible admissible non-isomorphic quotients of the representations $\pi(\sigma, 0)$, for instance because there are infinitely many D_0 containing σ for which there exist many non-isomorphic compatible actions of $\mathcal{N}(F)$ on $D_0^{I_1}$ such that any π as in Theorem 3.4 is irreducible and is not a subquotient of a principal series (see [16] when k_F is not $\mathbb{F}_p$).

We now give two series of examples of such pairs (D_0, D_1).

The first examples are very explicit and arise from the generalization of Serre's modularity conjecture in [18] (see also [58]). For these examples we assume F unramified over $\mathbb{Q}_p$. To any linear continuous 2-dimensional representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E is associated in [18] a finite set $\mathcal{W}(\rho)$ of Serre weights which generically has $2^{[F:\mathbb{Q}_p]}$ elements. Let $D_0(\rho)$ be a linear representation of $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ over k_E such that:

(i) $\mathrm{soc}_{\mathrm{GL}_2(\mathcal{O}_F)F^\times} D_0(\rho) = \oplus_{\sigma\in\mathcal{W}(\rho)}\sigma$

(ii) the action of $\mathrm{GL}_2(\mathcal{O}_F)$ on $D_0(\rho)$ factors through $\mathrm{GL}_2(\mathcal{O}_F) \twoheadrightarrow \mathrm{GL}_2(k_F)$

(iii) $D_0(\rho)$ is maximal for inclusion with respect to (i) and (ii).

If ρ is sufficiently generic (in a sense that can be made precise, see [16, §11]), one can prove that such a $D_0(\rho)$ exists, is unique, and that $D_1(\rho) := D_0(\rho)^{I_1}$ can be endowed with (many) compatible actions of $\mathcal{N}(F)$. For each such action of $\mathcal{N}(F)$, Theorem 3.4 applied to $(D_0, D_1) := (D_0(\rho), D_1(\rho))$ gives a smooth admissible representation π of $\mathrm{GL}_2(F)$. In fact, based on explicit computations in special cases ([43]), it is expected that the number of isomorphism classes of π as in Theorem 3.4 will be *strictly bigger* than one for *each* action of $\mathcal{N}(F)$ on $D_1(\rho)$ as soon as $F \neq \mathbb{Q}_p$. Denote by $\Pi(\rho)$ the set of isomorphism classes of all π given by Theorem 3.4 for all compatible actions of $\mathcal{N}(F)$ on $D_1(\rho)$. The following result is proved in [16].

Theorem 3.5. *If ρ is (sufficiently generic and) irreducible, then any π in $\Pi(\rho)$ is irreducible. If ρ is (sufficiently generic and) reducible, then any π in $\Pi(\rho)$ is reducible.*

Remark 3.6. When ρ is irreducible, one could replace $D_0(\rho)$ by its subrepresentation $\langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\rho)\rangle$ as one can prove that any π as in Theorem 3.4 for $(\langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\rho)\rangle, D_1(\rho))$ contains $D_0(\rho)$ in that case, i.e., is in $\Pi(\rho)$.

In the case ρ is reducible, $\rho \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, any π in $\Pi(\rho)$ is reducible because it strictly contains the representation $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}$. By Theorem 3.1 (which can be applied as the genericity of ρ entails in particular $\chi_1 \neq \chi_2$), it cannot be an extension of $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}$ by $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}$. So what could π look like in this case? Consider the following two propositions, the first one being in [16, §19] and the second one being elementary.

Proposition 3.7. *If ρ is (sufficiently generic and) reducible split, then some of the π in $\Pi(\rho)$ are semi-simple with $[F : \mathbb{Q}_p] + 1$ non-isomorphic Jordan-Hölder factors, two of them being the above two principal series (which are irreducible for ρ sufficiently generic) and the others being irreducible admissible quotients of representations $\pi(\sigma, 0)$.*

Proposition 3.8. *If ρ is (sufficiently generic and) reducible, then the tensor induction of ρ from* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ *to* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *is a successive extension of $[F : \mathbb{Q}_p] + 1$ non-isomorphic semi-simple representations, two of them being 1-dimensional.*

If ρ is reducible non-split, then any extension between two consecutive semi-simple representations as in Proposition 3.8 is non-split and the two 1-dimensional representations are the unique irreducible subobject and the unique irreducible quotient of the tensor induction of ρ. If ρ is split, the link between the $[F : \mathbb{Q}_p] + 1$ Jordan-Hölder factors of π in Proposition 3.7 and the $[F : \mathbb{Q}_p] + 1$ semi-simple representations of the tensor induction of ρ in Proposition 3.8 can be made much more convincing by using (φ, Γ)-modules (see [10]). In particular, the above two propositions suggest that, among all the π in $\Pi(\rho)$ for ρ reducible and sufficiently generic, some of them (the "good" ones) should have exactly $[F : \mathbb{Q}_p] + 1$ distinct Jordan-Hölder factors as in Proposition 3.7 and should be:

(i) semi-simple if ρ is reducible split, or

(ii) uniserial with $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_2 \otimes \chi_1\omega^{-1}$ as unique irreducible subobject and $\mathrm{Ind}_{B(F)}^{\mathrm{GL}_2(F)} \chi_1 \otimes \chi_2\omega^{-1}$ as unique irreducible quotient if ρ is reducible non-split.

This gives a possible explanation for Theorem 3.1: there is no extension for $F \neq \mathbb{Q}_p$ because we are missing the "middle" Jordan-Hölder factors!

The second examples of pairs (D_0, D_1) are constructed in [42] (no assumption on F here). Let π be an irreducible *not necessarily admissible* representation of $\mathrm{GL}_2(F)$ over k_E with a central character and $\sigma \subset \pi|_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}$ a Serre weight for $\mathrm{GL}_2(\mathcal{O}_F)F^\times$ (which always exists). One first defines an $\mathcal{N}(F)$-subrepresentation $D_1(\pi)$ of π as follows (with notations analogous to (2)):

$$D_1(\pi) := \left(\sum_{m \geq 0} \begin{pmatrix} \varpi_F^m & \mathcal{O}_F \\ 0 & 1 \end{pmatrix} \sigma \right) \bigcap \begin{pmatrix} 0 & 1 \\ \varpi_F & 0 \end{pmatrix} \left(\sum_{m \geq 0} \begin{pmatrix} \varpi_F^m & \mathcal{O}_F \\ 0 & 1 \end{pmatrix} \sigma \right)$$

(which is checked to be preserved by $\mathcal{N}(F)$ inside π). One can prove that $D_1(\pi)$ does not depend on the choice of the Serre weight σ in π and that it always contains π^{I_1}. One then considers the pair $(D_0(\pi), D_1(\pi))$ with $D_0(\pi) := \langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\pi)\rangle \subset \pi$.

Theorem 3.9. *If $D_1(\pi)$ is finite dimensional, then there is a unique representation of $\mathrm{GL}_2(F)$ as in Theorem 3.4 with $(D_0, D_1) := (D_0(\pi), D_1(\pi))$ (even if $p = 2$) and it is the representation π. In particular π is then admissible.*

This theorem is proved in [42]. In fact, [42] proves more: (i) without any assumption on $D_1(\pi)$ the pair $(D_0(\pi), D_1(\pi))$ *always* uniquely determines π and (ii) $D_1(\pi)$ is finite dimensional if and only if π is of finite presentation (i.e., is a quotient of some $\mathrm{c}-\mathrm{Ind}_{\mathrm{GL}_2(\mathcal{O}_F)F^\times}^{\mathrm{GL}_2(F)}\sigma$ by an invariant subspace which is finitely generated under $\mathrm{GL}_2(F)$). However, if $F \neq \mathbb{Q}_p$ it is not known in general whether $D_1(\pi)$ is or isn't finite dimensional, and it seems quite hard to determine $D_1(\pi)$ explicitly if π is not a subquotient of a principal series. For those π in $\Pi(\rho)$, note that one has the inclusions $D_1(\rho) \subseteq \pi^{I_1} \subseteq D_1(\pi)$ hence also $\langle \mathrm{GL}_2(\mathcal{O}_F) \cdot D_1(\rho)\rangle \subseteq D_0(\pi)$ with equalities if $F = \mathbb{Q}_p$.

3.3. Questions on local-global compatibility. We conclude our discussion of the modulo p theory for $\mathrm{GL}_2(F)$ with questions on local-global compatibility.

Let L be a totally real finite extension of $\mathbb{Q}$ with ring of integers $\mathcal{O}_L$. Assume for simplicity that p is inert in L (i.e., $p\mathcal{O}_L$ is a prime ideal) and let L_p denote the completion of L at p and $\mathbb{A}_{L,f}^p$ the finite adèles of L outside p. To any quaternion algebra D over L which splits at only one of the infinite places and which splits at p and to any compact open subgroup $K_f^p \subset (D \otimes_L \mathbb{A}_{L,f}^p)^\times$, one can associate a tower of Shimura algebraic curves $(S(K_f^p K_{f,p}))_{K_{f,p}}$ over L where $K_{f,p}$ runs over the compact open subgroups of $(D \otimes_L L_p)^\times \simeq \mathrm{GL}_2(L_p)$. Analogously to the case $L = \mathbb{Q}$ and $D = \mathrm{GL}_2$ of §2.4, one would like to understand:

$$\varinjlim_{K_{f,p}} H^1_{\text{ét}}\big(S(K_f^p K_{f,p}) \times_L \overline{\mathbb{Q}}, k_E\big)$$

as a representation of $\mathrm{GL}_2(L_p) \times \mathrm{Gal}(\overline{\mathbb{Q}}/L)$ over k_E. Fix a linear continuous totally odd (i.e., any complex conjugation has determinant -1) irreducible representation:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/L) \to \mathrm{GL}_2(k_E).$$

One can at least state the following conjecture which generalizes one of the main conjectures of [18].

Conjecture 3.10. *If $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)}$ is sufficiently generic (in the sense of [16, §11]) then for each compact open subgroup $K_f^p \subset (D \otimes_L \mathbb{A}_{L,f}^p)^\times$ one has:*

$$\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/L)}\Big(\rho, \varinjlim_{K_{f,p}} H^1_{\text{ét}}\big(S(K_f^p K_{f,p}) \times_L \overline{\mathbb{Q}}, k_E\big)\Big) \simeq \pi_p^n$$

for some integer $n \geq 0$ and some π_p in the set[3] $\Pi(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)})$ *(see §3.2).*

Note that Conjecture 3.10 does not state that the above space of homomorphisms is non-zero, but that, *if* it is non-zero, then it is a number of copies of some π_p in $\Pi(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)})$. Conjecture 3.10 is known for $L = \mathbb{Q}$ ([28]). For $L \neq \mathbb{Q}$, some non-trivial evidence for this conjecture and for a variant with 0-dimensional Shimura varieties and H^0 (instead of Shimura curves and H^1) can be found in [18], [57], [33] and [11] (see also [58] and [35]). If Conjecture 3.10 holds, the main crucial questions are then (recalling from §3.2 that $\Pi(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)})$ is a huge set if $L_p \neq \mathbb{Q}_p$):

Question 3.11. *Does π_p in Conjecture 3.10 only depend on $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)}$? How can one "distinguish" the π_p of Conjecture 3.10 in the purely local set $\Pi(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)})$?*

If the answer to the first question is yes, then this will enable one to define a genuine modulo p local Langlands correspondence for $\mathrm{GL}_2(F)$ that is compatible with cohomology. Again, the answer is of course yes if $L = \mathbb{Q}$.

3.4. Over E. The modulo p theory being so involved, it is not surprising that very little is known in characteristic 0. We just state here the main theorem of [54], which shows that one also has too many admissible unitary topologically irreducible Banach space representations of $\mathrm{GL}_2(F)$ over E when $F \neq \mathbb{Q}_p$.

Theorem 3.12. *Let π be a smooth irreducible admissible representation of $\mathrm{GL}_2(F)$ over k_E. Then there exists an admissible unitary topologically irreducible Banach space representation B of $\mathrm{GL}_2(F)$ over E and a unit ball $B^0 \subset B$ preserved by $\mathrm{GL}_2(F)$ such that:*

$$\mathrm{Hom}_{\mathrm{GL}_2(F)}(\pi, B^0 \otimes_{\mathcal{O}_E} k_E) \neq 0.$$

In particular, because of the results of §3.2, one should not expect a naive extension of Theorem 2.6 to hold for $F \neq \mathbb{Q}_p$. The question whether one can always choose B above such that $\pi \simeq B^0 \otimes_{\mathcal{O}_E} k_E$ is open (except for $F = \mathbb{Q}_p$ where the answer is yes and is already essentially in [7]). If such a B does not always exist, maybe one should only consider those π for which it does, i.e., those π which lift to characteristic 0.

All the other results concerning $\mathrm{GL}_2(F)$ over E are very partial so far. In some cases, one can for instance associate to a 2-dimensional semi-stable p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over E a locally $\mathbb{Q}_p$-analytic strongly admissible (in the sense of [61]) representation of $\mathrm{GL}_2(F)$ over E which generalizes the representation from the $F = \mathbb{Q}_p$ case and that one would wish to find inside completed cohomology spaces analogous to the $\widehat{H}^1(K_f^p)$ of §2.4 (see, e.g., [65] for the non-crystalline case). However, if this holds, it is likely that for $F \neq \mathbb{Q}_p$ this locally $\mathbb{Q}_p$-analytic representation is only a strict subrepresentation of the "correct" (unknown) locally $\mathbb{Q}_p$-analytic representation(s) of $\mathrm{GL}_2(F)$.

[3] As in Theorem 2.7, depending on normalizations, one may have to replace $\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/L_p)}$ here by its dual or its Cartier dual.

4. Other groups

If not much is known for $\mathrm{GL}_2(F)$, almost nothing is known for groups other than $\mathrm{GL}_2(F)$, even conjecturally, although some non-trivial results start to appear in various cases like $\mathrm{GL}_3(\mathbb{Q}_p)$ ([66]) or quaternion algebras ([36]). We content ourselves here to mention briefly a few results and conjectures that have been stated for $\mathrm{GL}_n(F)$ and that give some kind of "relations" between the $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ side and the $\mathrm{GL}_n(F)$ side. Although these relations are very far from any kind of correspondence, it is plausible that they will play some role in the future.

4.1. Invariant lattices and admissible filtrations. Locally algebraic representations of $\mathrm{GL}_n(F)$ over E (such as the representations $B(V)^{\mathrm{alg}}$ of §2.2) which are "related" to continuous n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over E (e.g., that appear as subrepresentations in completed cohomology spaces) should have invariant $\mathcal{O}_E$-lattices, as is clear from the $\mathrm{GL}_2(\mathbb{Q}_p)$ case (§2.2). It turns out that a necessary condition for a locally algebraic representation of $\mathrm{GL}_n(F)$ to have invariant lattices is essentially a well-known condition in Fontaine's theory called "weakly admissible".

Let us fix (r, N, D) a Weil-Deligne representation on an n-dimensional E-vector space D where r is the underlying representation of the Weil group of F (which has open kernel) and N the nilpotent endomorphism on D satisfying the usual relation $r(w)\circ N\circ r(w)^{-1} = p^{d(w)}N$ (for w in the Weil group of F, see introduction for $d(w)$).

To (r, N, D), one can associate a smooth irreducible representation π' of $\mathrm{GL}_n(F)$ over E by the classical local Langlands correspondence as in [38] (after semi-simplifying r). We then slightly modify it as in §2.4: if π' is generic, we let $\pi := \pi' \otimes |\det|^{\frac{(1-n)}{2}}$. If π' is not generic, we replace π' by a certain parabolic induction π'' which has π' as unique irreducible quotient (see [17, §4]) and let $\pi := \pi'' \otimes |\det|^{\frac{(1-n)}{2}}$.

For each embedding $\tau : F \hookrightarrow E$, let us fix n integers $i_{1,\tau} < i_{2,\tau} < \cdots < i_{n,\tau}$. We denote by σ_τ the algebraic representation of GL_n over E of highest weight $-i_{1,\tau} - (n-1) \geq -i_{2,\tau} - (n-2) \geq \cdots \geq -i_{n,\tau}$ that we see as a representation of $\mathrm{GL}_n(F)$ via the embedding $\tau : F \hookrightarrow E$. We then set $\sigma := \otimes_\tau \sigma_\tau$. This is a finite dimensional representation of $\mathrm{GL}_n(F)$ over E.

Any p-adic potentially semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ on an n-dimensional E-vector space V gives rise to some (r, N, D) and some $(i_{j,\tau})_{j,\tau}$ as follows ([30]). Let F' be a finite Galois extension of F such that $V|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F')}$ becomes semi-stable and set:

$$D := (\mathrm{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}_p}/F')} \otimes_{F'_0 \otimes E} E$$

where B_{st} is Fontaine's semi-stable period ring, F'_0 is the maximal unramified subfield in F' and $F'_0 \hookrightarrow E$ is any embedding. It is an n-dimensional E-vector space endowed with a nilpotent endomorphism N coming from the one on B_{st}. We define $r(w)$ on D by $r(w) := \varphi^{-d(w)} \circ \overline{w}$ where w is any element in the Weil group of F, $\overline{w}$ its image in $\mathrm{Gal}(F'/F)$ and φ the semi-linear endomorphism coming from the action of the Frobenius on B_{st} (as $\varphi^{-d(w)} \circ \overline{w}$ is $F'_0 \otimes E$-linear, $r(w)$ goes down to D). Finally, the $i_{j,\tau}$ are just the opposite of the various Hodge-Tate weights of V (n weights for each embedding $\tau : F \hookrightarrow E$).

The following conjecture was stated in [17, §4].

Conjecture 4.1. *Fix (r, N, W) and $(i_{j,\tau})_{j,\tau}$ as above. There exists an invariant $\mathcal{O}_E$-lattice on the locally algebraic $\mathrm{GL}_n(F)$-representation $\sigma \otimes_E \pi$ if and only if the data $\big((r, N, W), (i_{j,\tau})_{j,\tau}\big)$ comes from a p-adic n-dimensional potentially semi-stable representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$.

The following theorem gives one complete direction in the above conjecture. After many cases were proved in [63] and [17], its full proof was given in [41].

Theorem 4.2. *If there exists an invariant $\mathcal{O}_E$-lattice on $\sigma \otimes_E \pi$ then the data $\big((r, N, W), (i_{j,\tau})_{j,\tau}\big)$ comes from a p-adic n-dimensional potentially semi-stable representation of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$.

The proof is divided into four steps. (i) It is essentially trivial if π is supercuspidal. Hence one can restrict to the non-supercuspidal cases. (ii) Using a result of Emerton ([26, Lem.4.4.2]), one deduces from the existence of an invariant lattice on $\sigma \otimes_E \pi$ a finite number of inequalities relating the numbers $i_{j,\tau}$ to the "powers of p" in the action of r. (iii) These inequalities are just what is needed so that there exists a weakly admissible filtration on a certain (φ, N)-module naturally associated to (r, N, W). (iv) Such a filtration gives an n-dimensional potentially semi-stable representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ by the main result of [22].

The other direction in Conjecture 4.1 is much harder. Apart from trivial or scattered partial results, the only case which is completely known is again that of $\mathrm{GL}_2(\mathbb{Q}_p)$ ([5]).

4.2. Supersingular modules and irreducible Galois representations. We now state a theorem on Hecke-Iwahori modules for $\mathrm{GL}_n(F)$ over k_E in relation with irreducible n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E.

Let $\mathcal{H}_1$ be the Hecke algebra of I_1 over k_E, that is, $\mathcal{H}_1 := k_E[I_1 \backslash \mathrm{GL}_n(F)/I_1]$. The usual product of double cosets makes $\mathcal{H}_1$ a non-commutative k_E-algebra of finite type. An $\mathcal{H}_1$-module M over k_E is a k_E-vector space endowed with a linear right action of $\mathcal{H}_1$. By Schur's lemma, the center $\mathcal{Z}_1$ of $\mathcal{H}_1$ acts on a simple (and thus finite dimensional) $\mathcal{H}_1$-module M by a character with values in k_E called the central character of M. The commutative k_E-subalgebra $\mathcal{Z}_1$ is generated by (cosets of) scalars, by certain elements of $k_E[I_1 \backslash I/I_1] = k_E[I/I_1]$ and by $n - 1$

cosets $Z_1, \cdots, Z_{n-1}$. A finite dimensional simple $\mathcal{H}_1$-module is said to be supersingular if its central character sends all these Z_i to 0 ([71]).

The following nice numerical coincidence was conjectured in [71] and completely proved in [52].

Theorem 4.3. *The number of simple n-dimensional supersingular $\mathcal{H}_1$-modules over k_E is equal to the number of linear continuous n-dimensional irreducible representations of* $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ *over k_E.*

Let us briefly give the case $n = 3$ as an example. The number of (isomorphism classes of) continuous 3-dimensional irreducible representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E with determinant mapping a fixed choice of Frobenius to $1 \in k_E$ is easily checked to be $\frac{q^3-q}{3}$. The number of 3-dimensional simple supersingular $\mathcal{H}_1$-modules over k_E with central character mapping a fixed choice of uniformizer to $1 \in k_E$ turns out to be:

$$2\Big(q - 1 + (q-1)(q-2) + \frac{(q-1)(q-2)(q-3)}{6}\Big).$$

The reader can check that these two numbers are just the same (whence the theorem for $n = 3$ by varying the central character/determinant).

For $(n, F) = (2, \mathbb{Q}_p)$, the functor $\pi \mapsto \pi^{I_1}$ induces a bijection between smooth irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over k_E and 2-dimensional simple supersingular $\mathcal{H}_1$-modules over k_E ([70]), but this already completely breaks down when $n = 2$ and $F \neq \mathbb{Q}_p$ (see §3). The meaning of Theorem 4.3 in terms of smooth representations of $\mathrm{GL}_n(F)$ over k_E (if any) thus remains mysterious for $(n, F) \neq (2, \mathbb{Q}_p)$.

4.3. Serre weights and Galois representations. We have seen in §3 that the set of Serre weights $\mathcal{W}(\rho)$ associated in [18] and [58] to a linear continuous 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/F)$ over k_E is expected to be the set of simple summands (forgetting possible multiplicities) of the $\mathrm{GL}_2(\mathcal{O}_F)F^\times$-socle of some smooth admissible representation of $\mathrm{GL}_2(F)$ over k_E. (Without restrictions on ρ, one may indeed have multiplicities in this socle.) This yields a non-trivial link between the weights in Serre-type conjectures and the modulo p Langlands programme for $\mathrm{GL}_2(F)$.

For $\mathrm{GL}_n(\mathbb{Q}_p)$ when $n > 2$ the modulo p Langlands programme is essentially open (although there is recent progress in the classification of "non-supersingular" smooth irreducible admissible representations of $\mathrm{GL}_n(F)$ over k_E, see [40]). But the set of Serre weights $\mathcal{W}(\rho)$ has been generalized by Herzig and Gee in [39] and [34] to linear continuous n-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ over k_E.

For integers $a_1 \geq a_2 \geq \cdots \geq a_n$ such that $a_i - a_{i+1} \leq p - 1$ for all i we let $F(a_1, \cdots, a_n)$ denote the restriction to $\mathrm{GL}_n(\mathbb{F}_p)$ of the GL_n-socle of the algebraic dual Weyl module for GL_n of highest weight $(t_1, \cdots, t_n) \mapsto t_1^{a_1} t_2^{a_2} \cdots t_n^{a_n}$ (see

[39, §3.1]). The $F(a_1, \cdots, a_n)$ exhaust the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_p)$ (equivalently of $\mathrm{GL}_n(\mathbb{Z}_p)$) over k_E.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_n(k_E)$ be any linear continuous representation. Its determinant has the form $\omega^m \mathrm{unr}$ where unr is an unramified character of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ and m an integer. We can see unr as a character of $\mathrm{GL}_n(\mathbb{Z}_p)\mathbb{Q}_p^\times$ which is trivial on $\mathrm{GL}_n(\mathbb{Z}_p)$.

Definition 4.4. The set $\mathcal{W}(\rho)$ of Serre weights for $\mathrm{GL}_n(\mathbb{Z}_p)\mathbb{Q}_p^\times$ associated to ρ is the set of $F(a_1, \cdots, a_n) \otimes \mathrm{unr}$ such that ρ has a crystalline lift with Hodge-Tate weights $a_1 + n - 1, a_2 + n - 2, \cdots, a_n$.

Definition 4.4 is quite general but not at all explicit. When ρ is sufficiently generic and semi-simple, a conjectural but much more explicit description of the weights of $\mathcal{W}(\rho)$ has been given in [39] (which was actually written before [34]). The method of [39] is first to associate to ρ (in fact to its restriction to the inertia subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$) a finite dimensional "Deligne-Lusztig" representation $\sigma(\rho)$ of $\mathrm{GL}_n(\mathbb{F}_p)$ over E. For instance, if $\rho = \oplus_{i=1}^n \chi_i$ is a direct sum of characters of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, then:

$$\sigma(\rho) := \mathrm{Ind}_{B(\mathbb{F}_p)}^{\mathrm{GL}_n(\mathbb{F}_p)} \chi$$

where $\chi : B(\mathbb{F}_p) \twoheadrightarrow T(\mathbb{F}_p) \to E^\times$, $(t_1, \cdots, t_n) \mapsto [\chi_1(t_1)\chi_2(t_2) \cdots \chi_n(t_n)]$ (note that $\chi_i|_{1+p\mathbb{Z}_p} = 1$ and $[\cdot]$ is here the multiplicative representative). If ρ is sufficiently generic, all Jordan-Hölder factors $F(a_1, \cdots, a_n)$ of the modulo p semi-simplification $\overline{\sigma}(\rho)^{\mathrm{ss}}$ of $\sigma(\rho)$ (that is, of the semi-simplification of the reduction modulo ϖ_E of any invariant $\mathcal{O}_E$-lattice in $\sigma(\rho)$) are such that $a_i - a_{i+1} \leq p - 2$ for all i. If ρ is moreover semi-simple, the set $\mathcal{W}(\rho)$ of Definition 4.4 is then expected to be the set of Serre weights:

$$F\big(a_n + (n-1)(p-2), a_{n-1} + (n-2)(p-2), \cdots, a_2 + p - 2, a_1\big) \otimes \mathrm{unr}$$

for $F(a_1, \cdots, a_n)$ a Jordan-Hölder factor of $\overline{\sigma}(\rho)^{\mathrm{ss}}$.

Changing notations, let:

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(k_E)$$

be a linear continuous irreducible odd representation, that is, either $p = 2$ or the eigenvalues of the image of a complex conjugation are:

$$\underbrace{1, \cdots, 1}_{n_+ \text{ times}}, \underbrace{-1, \cdots, -1}_{n_- \text{ times}}$$

with $-1 \leq n_+ - n_- \leq 1$. Let N be the Artin conductor of ρ measuring its ramification at primes other than p and let unr_p be as above the unramified part of $\det(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)})$. Then the "Serre conjecture" of [39] and [33] states that the Serre weights of $\mathcal{W}(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)})$ should be exactly those Serre weights

$F(a_1, \cdots, a_n) \otimes \mathrm{unr}_p$ for $\mathrm{GL}_n(\mathbb{Z}_p)\mathbb{Q}_p^\times$ such that ρ "arises" from a non-zero Hecke eigenclass in some group cohomology $H^*(\Gamma_1(N), F(a_1, \cdots, a_n))$. Here $\Gamma_1(N) \subset \mathrm{SL}_n(\mathbb{Z})$ is the subgroup of matrices with last row congruent to $(0, \cdots, 0, 1)$ modulo N (see [39, §6] for details).

The results of §3 suggest that the Serre weights of $\mathcal{W}(\rho|_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)})$ may form (up to multiplicities) the $\mathrm{GL}_n(\mathbb{Z}_p)\mathbb{Q}_p^\times$-socle of interesting smooth admissible representations of $\mathrm{GL}_n(\mathbb{Q}_p)$ over k_E (that remain to be discovered if $n > 2$). But one should keep in mind the following numbers. Assuming ρ is semi-simple, for $n = 2$ one has generically $|\mathcal{W}(\rho)| = 2$, and for $n = 3$ one should have $|\mathcal{W}(\rho)| = 9$, but then $\mathcal{W}(\rho)$ rapidly grows: $n = 4$ should give $|\mathcal{W}(\rho)| = 88$ and $n = 5$ should give $|\mathcal{W}(\rho)| = 1640$! Also, consider for instance the case $n = 3$ and $\rho = \oplus_{i=1}^3 \chi_i$ with ρ sufficiently generic. Then 6 of the 9 weights of $\mathcal{W}(\rho)$ are easily checked to be the $\mathrm{GL}_3(\mathbb{Z}_p)\mathbb{Q}_p^\times$-socle of 6 natural principal series representations of $\mathrm{GL}_3(\mathbb{Q}_p)$ analogous to the 2 principal series in (ii) of Definition 2.2. But there are 3 remaining Serre weights and their combinatorics suggests that they might form the $\mathrm{GL}_3(\mathbb{Z}_p)\mathbb{Q}_p^\times$-socle of an irreducible admissible representation of $\mathrm{GL}_3(\mathbb{Q}_p)$ that does not occur in any (strict) parabolic induction, i.e., of a supersingular representation of $\mathrm{GL}_3(\mathbb{Q}_p)$. We thus may have a phenomenon analogous to what happens with $\mathrm{GL}_2(F)$ (see Proposition 3.7 and the discussion that follows) except that the possible appearance here of this "extra" supersingular constituent seems now quite mysterious.

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