Noncommutative Toda Chains, Hankel quasideterminants and Painlevé II equation

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NONCOMMUTATIVE TODA CHAINS, HANKEL QUASIDETERMINANTS AND PAINLEVÉ II EQUATION

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Abstract. We construct solutions of an infinite Toda system and an analogue of the Painlevé II equation over noncommutative differential division rings in terms of quasideterminants of Hankel matrices.

Introduction

Let R be an associative algebra over a field with a derivation D. Set $Df = f'$ for any $f \in R$. Assume that R is a division ring. In this paper we construct solutions for the system of equations (0.1) over algebra R

$$(\theta_n'\theta_n^{-1})' = \theta_{n+1}\theta_n^{-1} - \theta_n\theta_{n-1}^{-1}, \quad n \geq 1 \tag{0.1-n}$$

assuming that $\theta_1 = \phi, \theta_0 = \psi^{-1}$, $\phi, \psi \in R$ and its "negative" counterpart (0.1')

$$(\eta_{-m}^{-1}\eta_{-m}')' = \eta_{-m}^{-1}\eta_{-m-1} - \eta_{-m+1}^{-1}\eta_{-m}, \quad m \geq 1 \tag{0.1'-m}$$

where $\eta_0 = \phi^{-1}, \eta_{-1} = \psi$.

Note that $\theta'\theta^{-1}$ and $\theta^{-1}\theta'$ are noncommutative analogues of the logarithmic derivative $(\log\theta)'$.

We use then the solutions of the Toda equations under a certain anzatz for constructing solutions of the *noncommutative Painlevé II equation*

$$P_{II}(u,\beta): \quad u'' = 2u^3 - 2xu - 2ux + 4(\beta + \frac{1}{2})$$

where $u, x \in R$, $x' = 1$ and β is a scalar parameter, $\beta' = 0$.

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Unlike papers [NGR] and [N] we consider here a "pure noncommutative" version of the Painlevé equation without any additional assumption for our algebra R.

Our motivation is the following. In the commutative case one can consider an infinite Toda system (see, for example [KMNOY, JKM]):

$$\tau_n''\tau_n - (\tau_n')^2 = \tau_{n+1}\tau_{n-1} - \phi\psi\tau_n^2 \tag{0.2-n}$$

with the conditions $\tau_1 = \phi, \tau_0 = 1, \tau_{-1} = \psi$.

Let $n \geq 1$. By setting $\theta_n = \tau_n/\tau_{n-1}$ the system can be written as

$$(\log \tau_n)'' = \theta_{n+1}\theta_n^{-1} - \phi\psi.$$

For $n = 1$ we have the equation (0.1-1)with $\theta_1 = \phi, \theta_0 = \psi^{-1}$. By subtracting equation (2.2-n) from (2.2-(n+1)) and replacing the difference $\log \tau_{n+1} - \log \tau_n$ by $\log \frac{\tau_{n+1}}{\tau_n}$ one can get (0.1-n).

Similarly, the system (0.2-m) for positive m implies the system $(0.1' - m)$ for $\theta_{-m} = \tau_{-m}/\tau_{-m+1}$.

By going from τ_n's to their consequtive relations we are cutting the system of equations parametrized by $-\infty < n < \infty$ to its "positive" and "negative" part.

A special case of the semi-infinite system (0.1) over noncommutative algebra with θ_0^{-1} formally equal to zero was treated in [GR2]. In this paper solutions of the Toda system (0.1) with $\theta_0^{-1} = 0$ were constructed as *quasideterminants* of certain Hankel matrices. It was the first application of quasideterminants introduced in [GR1] to noncommutative integrable systems. This line was continued by several reseachers, see, for example, [EGR1, EGR2], papers by Glasgow school [GN, GNO, GNS] and a recent [DFK].

In this paper we generalize the result of [GR2] for $\theta_0 = \psi^{-1}$ and extend it to the infinite Toda system. The solutions are also given in terms of quasideterminants of Hankel matrices but the computations are much harder. We follow here the commutative approach developed in [KMNOY, JKM] with some adjustments but our proofs are far from a straightforward generalization. In particular, for our proof we have to introduce and investigate *almost Hankel matrices* (see Section 2.2).

From solutions of the systems (0.1) and (0.1') under certain anzatz we deduce solutions for the noncommutative equation $P_{II}(u, \beta)$ for various parameters β (Theorem 3.2). This is a noncommutative development of an idea from [KM].

We start this paper by a reminder of basic properties of quasideterminants, then construct solutions of the systems (0.1) and (0.1'), then apply our results to noncommutative Painlevé II equations following the approach by [KM].

Our paper shows that a theory of "pure" noncommutative Painlevé equations and the related tau-functions can be rather rich and interesting. The Painlevé II type was chosen as a model and we are going to investigate other types of Painlevé equations.

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1. Quasideterminants

The notion of quasideterminants was introduced in [GR1], see also [GR2-3, GGRW].

Let $A = ||a_{ij}||$, $i, j = 1, 2, \dots, n$ be a matrix over an associative unital ring. Denote by A^{pq} the $(n-1)\times(n-1)$ submatrix of A obtained by deleting the p-th row and q-th column. Let r_i be the row matrix $(a_{i1}, a_{i2}, \dots \hat{a}_{ij}, \dots, a_{in})$ and c_j be the column matrix with entries $(a_{1j}, a_{2j}, \dots \hat{a}_{ij}, \dots, a_{nj})$.

For $n = 1$, $|A|_{11} = a_{11}$. For $n > 1$ the *quasideterminant* $|A|_{ij}$ is defined if the matrix A^{ij} is invertible. In this case

$$|A|_{ij} = a_{ij} - r_i A^{ij} c_j.$$

If the inverse matrix $A^{-1} = ||b_{pq}||$ exists then $b_{pq} = |A|_{qp}^{-1}$ provided that the quasideterminant is invertible.

If R is commutative then $|A|_{ij} = \det A / \det A^{ij}$ for any i and j.

Examples. (a) For the generic 2×2-matrix $A = (a_{ij})$, $i, j = 1, 2$, there are four quasideterminants:

$$\begin{aligned} |A|_{11} &= a_{11} - a_{12}a_{22}^{-1}a_{21}, & |A|_{12} &= a_{12} - a_{11}a_{21}^{-1}a_{22}, \\ |A|_{21} &= a_{21} - a_{22}a_{12}^{-1}a_{11}, & |A|_{22} &= a_{22} - a_{21}a_{11}^{-1}a_{12}. \end{aligned}$$

(b) For the generic 3×3-matrix $A = (a_{ij})$, $i, j = 1, 2, 3$, there are 9 quasideterminants. One of them is

$$\begin{aligned} |A|_{11} = a_{11} &- a_{12}(a_{22} - a_{23}a_{33}^{-1}a_{32})^{-1}a_{21} - a_{12}(a_{32} - a_{33}a_{23}^{-1}a_{22})^{-1}a_{31} \\ &- a_{13}(a_{23} - a_{22}a_{32}^{-1}a_{33})^{-1}a_{21} - a_{13}(a_{33} - a_{32}\cdot a_{22}^{-1}a_{23})^{-1}a_{31}. \end{aligned}$$

Here are the transformation properties of quasideterminants. Let $A = ||a_{ij}||$ be a square matrix of order n over a ring R.

(i) The quasideterminant $|A|_{pq}$ does not depend on permutations of rows and columns in the matrix A that do not involve the p-th row and the q-th column.

(ii) *The multiplication of rows and columns.* Let the matrix $B = ||b_{ij}||$ be obtained from the matrix A by multiplying the i-th row by $\lambda \in R$ from the left, i.e., $b_{ij} = \lambda a_{ij}$ and $b_{kj} = a_{kj}$ for $k \neq i$. Then

$$|B|_{kj} = \begin{cases} \lambda |A|_{ij} & \text{if } k = i, \\ |A|_{kj} & \text{if } k \neq i \text{ and } \lambda \text{ is invertible.} \end{cases}$$

Let the matrix $C = ||c_{ij}||$ be obtained from the matrix A by multiplying the j-th column by $\mu \in R$ from the right, i.e. $c_{ij} = a_{ij}\mu$ and $c_{il} = a_{il}$ for all i and $l \neq j$. Then

$$|C|_{i\ell} = \begin{cases} |A|_{ij}\mu & \text{if } l = j, \\ |A|_{i\ell} & \text{if } l \neq j \text{ and } \mu \text{ is invertible.} \end{cases}$$

(iii) *The addition of rows and columns.* Let the matrix B be obtained from A by replacing the k-th row of A with the sum of the k-th and l-th rows, i.e., $b_{kj} = a_{kj} + a_{lj}$, $b_{ij} = a_{ij}$ for $i \neq k$. Then

$$|A|_{ij} = |B|_{ij}, \qquad i = 1, \dots k-1, k+1, \dots n, \quad j = 1, \dots, n.$$

We will need the following property of quasideterminants sometimes called the *noncommutative Lewis Carroll identity.* It is a special case of the *noncommutative Sylvester identity* from [GR1-2] or *heredity principle* formulated in [GR3].

Let $A = ||a_{ij}||$, $i, j = 1, 2, \dots, n$. Consider the followng $(n-1) \times (n-1)$-submatrices $X = ||x_{pq}||$, $p, q = 1, 2, \dots, n-1$ of A: matrix $A_0 = ||a_{pq}||$ obtained from A by deleting its n-th row and n-th column; matrix $B = ||b_{pq}||$ obtained from A by deleting its $(n-1)$-th row and n-th column; matrix $C = ||c_{pq}||$ obtained from A by deleting its n-th row and $(n-1)$-th column; matrix $D = ||d_{pq}||$ obtained from A by deleting its $(n-1)$-th row and $(n-1)$-th column. Then

$$|A|_{nn} = |D|_{n-1,n-1} - |B|_{n-1,n-1} |A_0|^{-1}_{n-1,n-1} |C|_{n-1,n-1} \tag{1.1.}$$

2. Quasideterminant solutions of noncommutative Toda equations

2.1. Noncommutative Toda equations in bilinear form. Let F be a commutative field and R be an associative ring containing F-algebra. Let $D : R \to R$ be a derivation over F, i.e. an F-linear map satisfying the Leibniz rule $D(ab) = D(a) \cdot b + a \cdot D(b)$ for any $a, b \in R$. Also, $D(\alpha) = 0$ for any $\alpha \in F$. As usual, we set $u' = D(u), u'' = D(D(u)), \dots$. Recall that $D(v^{-1}) = -v^{-1}v'v^{-1}$ for any invertible $v \in R$.

Let $\phi, \psi \in R$ and R be a division ring. We construct now solutions for the noncommutative Toda equations (0.1) and (0.1') assuming that $\theta_0 = \psi^{-1}$, $\theta_1 = \phi$ and $\eta_0 = \phi^{-1}$, $\eta_{-1} = \psi$.

Set (cf. [KMNOY, JKM] for the commutative case) $a_0 = \phi, b_0 = \psi$ and

$$(2.1) \quad a_n = a'_{n-1} + \sum_{i+j=n-2, i,j\geq 0} a_i \psi a_j, \quad b_n = b'_{n-1} + \sum_{i+j=n-2, i,j\geq 0} b_i \phi b_j, \quad n \geq 1.$$

Construct Hankel matrices $A_n = ||a_{i+j}||$, $B_n = ||b_{i+j}||, i, j = 0, 1, 2 \dots, n$.

Theorem 2.1. *Set $\theta_{p+1} = |A_p|_{p,p}$, $\eta_{-q-1} = |B_q|_{q,q}$. The elements θ_n for $n \geq 1$ satisfy the system (0.1) and the elements $\eta_{-m}, m \geq 1$ satisfy the system (0.1').*

This theorem can be viewed as a noncommutative generalization of Theorem 2.1 from [KMNOY]. In [KMNOY] it was proved that in the commutative case the Hankel determinants $\tau_{n+1} = \det A_n, n \geq 0, \tau_0 = 1, \tau_{-n-1} = \det B_n$, $n \leq 0$ satisfy the system (0.2).

Example. The (noncommutative) logarithmic derivative $\theta_1'\theta_1^{-1}$ satisfies the noncommutative Toda equation (0.1-1):

$$(\theta_1'\theta_1^{-1})' = \theta_2\theta_1^{-1} - \phi\psi.$$

In fact,

$$(\theta_1'\theta_1^{-1})' = (a_1 a_0^{-1})' = (a_2 - a_0\psi a_0)a_0^{-1} - (a_1 a_0^{-1})^2$$
$$= (a_2 - a_1 a_0^{-1} a_1)a_0^{-1} - a_0\psi = \theta_2\theta_1^{-1} - \phi\psi.$$

Our proof of Theorem 2.1. in the general case is based on properties of quasideterminants of almost Hankel matrices.

2.2. Almost Hankel matrices and their quasideterminants. We define *almost Hankel* matrices $H_n(i,j) = ||a_{st}||$, $s, t = 0, 1, \dots, n$, $i, j \geq 0$ for a sequence $a_0, a_1, a_2, \dots$ as follows. Set $a_{nn} = a_{i+j}$ and for $s, t < n$

$$a_{s,t} = a_{s+t}, \quad a_{n,t} = a_{i+t}, \quad a_{s,n} = a_{s+n}.$$

and $a_{nn} = a_{i+j}$.

Note that $H_n(n,n)$ is a Hankel matrix.

Denote by $h_n(i,j)$ the quasideterminant $|H_n(i,j)|_{nn}$. Then $h_n(i,j) = 0$ if at least one of the inequalities $i < n$, $j < n$ holds.

Lemma 2.2.

$$h_n(i,j)' = \kappa_n(i,j) - \sum_{p=1}^{i} a_{p-1}\psi h_n(i-p,j) - \sum_{q=1}^{j} h_n(i,j-q)\psi a_{q-1} \tag{2.2}$$

where

$$\kappa_n(i,j) = h_n(i+1,j) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j). \tag{2.3a}$$

Also,

$$= h_n(i,j+1) - h_n(i,n)h_{n-1}^{-1}(n-1,n-1)h_{n-1}(n-1,j). \tag{2.3bb}$$

Note that some summands $h_n(i-p,j)$, $h_n(i,j-q)$ in formula (2.2) can be equal to zero.

Since $h_n(i,j) = 0$ when $i < n$ or $j < n$ we have the following corollary.

Corollary 2.3.

$$h_n(n,n) = \kappa_n(n,n),$$

$$h_n(i,n)' = \kappa_n(i,n) - \sum_{s=1}^{i} a_{s-1}\psi h_n(i-s,n),$$

$$h_n(n,j)' = \kappa_n(n,j) - \sum_{v=1}^{j} h_n(n,j-v)\psi a_{v-1}.$$

Proof of Lemma 2.2. We prove Lemma 2.2 by induction. By definition,

$$h_1(i,j)' = a_{i+j+1} - \sum_{k=0}^{i+j-1} a_k\psi a_{i+j-1-k} - (a_{i+1} - \sum_{s=0}^{i-1} a_s\psi a_{i-1-2})a_0^{-1}a_j$$

$$+a_ia_0^{-1}a_1a_0^{-1} - a_ia_0^{-1}(a_{j+1} - \sum_{t=0}^{j-1} a_{j-1-t}\psi a_t).$$

Set

$$\kappa_1(i,j) = a_{i+j+1} - a_{i+1}a_0^{-1}a_j + a_ia_0^{-1}a_1a_0^{-1}a_j - a_ia_0^{-1}a_{j+1},$$

we can check formulas (2.3a) and (2.3b). The rest of the proof for $n = 1$ is easy.

Assume now that formula (2.2) is true for $n \geq 1$ and prove it for $n+1$. By the noncommutative Sylvester identity (1.1)

$$h_{n+1}(i,j) = h_n(i,j) - h_n(i,n-1)h_n^{-1}(n,n)h_n(n,j). \tag{2.4}$$

Set $h_{n+1}(i,j)' = \kappa_{n+1} + r_{n+1}(i,j)$ where κ_{n+1} contains all terms without ψ. Then

$$\kappa_{n+1}(i,j) = \kappa_n(i,j) - \kappa_n(i,n)h_n^{-1}(n,n)h_n(n,j)$$
$$+h_n(i,n)h_n^{-1}(n,n)\kappa_n(n,n)h_n^{-1}(n,n)h_n(n,j) - h_n(i,n)h_n^{-1}(n,n)\kappa_n(n,j).$$

By induction, the first two terms can be written as

$$h_n(i+1,j) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j)$$

$$+[h_n(i+1,n) - h_{n-1}(i,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,n)]h_n^{-1}(n,n)h_n(n,j)$$
$$= h_n(i+1,j) - h_n(i+1,n)h_n^{-1}(n,n)h_n(n,j).$$

This expression equals to $h_{n+1}(i+1,j)$ by the Sylvester identity.

The last two terms in $\kappa_{n+1}(i,j)$ can be written as

$$h_n(i,n)h_n^{-1}(n,n)[h_n(n+1,n)-h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,n)]h_n^{-1}(n,n)h_n(n,j)$$

$$-h_n(i,n)h_n^{-1}(n,n)[h_n(n+1,j) - h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1)h_n(n,j)]$$
$$= h_n(i,n)h_n^{-1}(n,n)[h_n(n+1,n) - h_n(i,n)h_n^{-1}(n,n)h_n(n+1,j)]$$
$$= h_n(i,n)h_n^{-1}(n,n)h_{n+1}(n+1,j)$$

also by the Sylvester identity.

Therefore, $\kappa_{n+1}(i,j)$ satisfies formula (2.3a). Formula (2.3b) can be obtained in a similar way.

Let us look at the terms containing ψ. According to the inductive assumption

$$h_n(i,j)' = \kappa_n(i,j) - \sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,j) - \sum_{\ell=1}^{j} h_n(i,j-\ell)\psi a_{\ell-1}.$$

Using the Corollary 2.3 and formula (2.2) for n one can write $r_{n+1}(i,j)$ as

$$-\sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,j) - \sum_{\ell=1}^{j} h_n(i,j-\ell)\psi a_{\ell-1}$$

$$+\sum_{k=1}^{i} a_{k-1}\psi h_n(i-k,n)]h_n^{-1}(n,n)h_n(n,j)$$

$$+h_n(i,n)h_n^{-1}(n,n)[\sum_{\ell=1}^{n} a_{u-1}\psi h_n(n,j-\ell)\psi a_{\ell-1}$$

$$= -\sum_{k=1}^{i} a_{k-1}\psi[h_n(i-k,j) - h_n(i-k,n)h_n^{-1}(n,n)h_n(n,j)]$$

$$-\sum_{\ell=1}^{j}[h_n(i,j-\ell) + h_n(i,n)h_n^{-1}(n,n)h_n(n,j-\ell)]\psi a_{\ell-1}.$$

Our lemma follows now from the Sylvester identity applied to each expression in square brackets.

Corollary 2.3 and formula (2.3a) immediately imply

Corollary 2.4. *For $n > 1$*

$$h_n(n,n)'h_n^{-1}(n,n) = h_n(n+1,n)h_n^{-1}(n,n) - h_{n-1}(n,n-1)h_{n-1}^{-1}(n-1,n-1).$$

Note in the right hand side we have a difference of left quasi-Plücker coordinates (see [GR3]).

2.3. Proof of Theorem 2.1.

Our solution of the Toda system (0.1) follows from Corollary 2.4 and the following lemma.

Lemma 2.5. *For $k > 0$*

$$[h_k(k+1,k)'h_k^{-1}(k,k)]' = h_{k+1}(k+1,k+1)h_k^{-1}(k,k) - a_0\psi.$$

Proof. Corollary 2.3 and formula (2.3b) imply

$$h_k(k+1,k)' = h_k(k+1,k+1) - h_k(k+1,k)h_{k-1}^{-1}(k-1,k-1)h_{k-1,k} - a_0\psi h_k(k,k)$$

because $h_k(k+1-s,k) = 0$ for $s > 1$.

Then, using again formula (2.3b) one has

$$[h_k(k+1,k)'h_k^{-1}(k,k)]' =$$

$$[h_k(k+1,k+1) - h_k(k+1,k)h_{k-1}^{-1}(k-1,k-1)h_{k-1}(k-1,k) - a_0\psi h_k(k,k)]h_k^{-1}(k,k)$$

$$-h_k(k+1,k)h_k^{-1}(k,k)][h_k(k,k+1) - h_k(k,k)h_{k-1}^{-1}(k-1,k-1)h_{k-1}(k-1,k)]h_k^{-1}(k,k)$$

$$= [h_k(k+1,k+1) - h_k(k+1,k)h_k^{-1}(k,k)h_k(k,k+1)]h_k^{-1}(k,k) - a_0\psi =$$

$$h_{k+1}(k+1,k+1)h_k^{-1} - a_0\psi$$

by the Sylvester formula.

Theorem 2.1. now follows from Corollary 2.4 and Lemma 2.5. The statement for η_{-m}, $m \geq 1$ can be proved in a similar way.

3. Noncommutative Painlevè II

3.1 Commutative Painlevè II and Hankel determinants: motivation. The Painlevè II (P_{II}) equation (with commutative variables)

$$u'' = 2u^3 - 4xu + 4(\beta + \frac{1}{2})$$

admits unique rational solution for a half-integer value of the parameter α. These solutions can be expressed in terms of logarithmic derivatives of ratios of Hankel-type determinants. Namely, if $\beta = N + \frac{1}{2}$ then

$$u = \frac{d}{dx} \log \frac{\det A_{N+1}(x)}{\det A_N(x)},$$

where $A_N(x) = ||a_{i+j}||$ where $i, j = 0, 1, \ldots, n-1$. The entries of the matrix are polynomials $a_n(x)$ subjected to the recurrence relations:

$$a_0 = x, \quad a_1 = 1, a_n = a'_{n-1} + \sum_{i=0}^{n-1} a_i a_{n-1-i}.$$

(see [JKM])

3.2 Noncommutative and "quantum" Painlevè II. We will consider here a *noncommutative* version of P_{II} which we will denote $\mathrm{nc} - P_{II}(x, \beta)$:

$$u'' = 2u^3 - 2xu - 2ux + 4(\beta + \frac{1}{2}),$$

where $x, u \in R, \quad x' = 1$ and β is a central scalar parameter ($\beta \in F, \beta' = 0$).

This equation is a specialization of a general noncommutative Painlevé II system with respect to three dependent noncommutative variables u_0, u_1, u_2:

$$\begin{aligned} u_0' &= u_0 u_2 + u_2 u_0 + \alpha_0 \\ u_1' &= -u_1 u_2 - u_2 u_1 + \alpha_1 \\ u_2' &= u_1 - u_0. \end{aligned}$$

Indeed, taking the derivative of the third and using the first and second, we get

$$u_2'' = -(u_0 + u_1) u_2 - u_2 (u_0 + u_1) + \alpha_1 - \alpha_0.$$

Then we have:

$$(u_0 + u_1)' = -u_2' u_2 - u_2 u_2' + \alpha_0 + \alpha_1$$

and, immediately

$$-(u_0 + u_1) = u_2^2 - (\alpha_0 + \alpha_1) x - \gamma, \, \gamma \in F.$$

Compare with u_2'' we obtain the following $\mathrm{nc} - P_{II}$:

$$u_2'' = 2u_2^3 - (\alpha_0 + \alpha_1) x u_2 - (\alpha_0 + \alpha_1) u_2 x - 2\gamma u_2 + \alpha_1 - \alpha_0.$$

Our equation corresponds the choice $\gamma = 0, \alpha_1 = 2(\beta + 1), \alpha_0 = -2\beta$.

Remark. *The noncommutative Painlevé II system above is the straightforward generalization of the analogues system in* [NGR] *when the variables $u_i, i = 0, 1, 2$ are subordinated to some commutation relations. Here we don't assume that the "independent" variable x commutes with u_i.*

Going further with this analogy we will write a "fully non-commutative" Hamiltonian of the system

$$H = \frac{1}{2}(u_0u_1 + u_1u_0) + \alpha_1 u_2$$

and introduce the "canonical" variables

$$p := u_2,\ q := u_1,\ x := \frac{1}{2}(u_0 + u_1 + u_2^2).$$

Proposition 3.1. *Let a triple (x, p, q) be a "solution" of the "Hamiltonian system" with the Hamiltonian H and $\alpha_1 = 2(\beta + 1)$.*

$$p_x = -H_q$$
$$q_x = H_p.$$

Then p satisfies the $nc - P_{II}$:

$$p_{xx} = 2p^3 - 2px - 2xp + 4(\beta + \frac{1}{2}).$$

Proof. Straightforward computation gives that:

$$p_x = p^2 + 2q - 2x$$
$$q_x = \alpha_1 - (qp + pq).$$

Taking $p_{xx} = p_xp + pp_x + 2q_x - 2$ and substituting p_x and q_x we obtain the result.

We give (for the sake of completeness) the explicit expression of the Painlevé Hamiltonian H in the "canonical" coordinates:

$$H(x, p, q) = qx + xq - q^2 - \frac{1}{2}(qp^2 + p^2q) + 2(\beta + 1)p.$$

3.3 Solutions of the noncommutative Painlevé and of the Toda system.

Theorem 3.2. *Let ϕ and ψ satisfy the following identities:*

$$\psi^{-1}\psi'' = \phi''\phi^{-1} = 2x - 2\phi\psi, \tag{3.1}$$

$$\psi\phi' - \psi'\phi = 2\beta. \tag{3.2}$$

Then
1) $u_n = \theta_n'\theta_n^{-1}$ satisfies $nc - P_{II}(x, \beta + n - 1)$;
2) $u_{-n} = \theta_{-n}'\theta_{-n}^{-1}$ satisfies $nc - P_{II}(x, \beta - n)$.

Let us start with the following useful (though slightly technical) lemma

Lemma 3.3. *Under the conditions of the Theorem 3.1 we have the chain of identities ($n \geq 0$):*
1) $\theta'_n\theta_n^{-1} + \theta'_{n-1}\theta_{n-1}^{-1} = 2(\beta + n - 1)\theta_{n-1}\theta_n^{-1}$.
2) $\theta''_n\theta_n^{-1} = 2(x - \theta_n\theta_{n-1}^{-1})$.

Proof. Remark that the first step in the chain ($n = 1$) directly follows from our assumption: $\theta_1 = \phi, \theta_0 = \psi^{-1}$:

$$\phi'\phi^{-1} + (\psi^{-1})'\psi = 2\beta\psi^{-1}\phi^{-1}.$$

Indeed, we have

$$\phi'\phi^{-1} - \psi^{-1}\psi' = 2\beta\psi^{-1}\phi^{-1}$$

where the result:

$$\psi\phi' - \psi'\phi = 2\beta.$$

The second step ($n = 2$) is a little bit tricky.

We consider the Toda equation $(\phi'\phi^{-1})' = \theta_2\phi^{-1} - \phi\psi$ and find easily θ_2 (using $\phi''\phi^{-1} = 2x - 2\phi\psi$):

$$\theta_2 = 2x\phi - \phi\psi\phi - (\phi'\phi^{-1})2\phi.$$

Taking the derivation and using the same Toda and the first step identity, we get

$$\theta'_2 = 2\phi(\beta + 1) - \phi'\phi^{-1}\theta_2.$$

The second ($n = 2$) identity is rather straightforward:

$$\theta''_2 + (\theta'_1\theta_1^{-1})'\theta_2 + (\theta'_1\theta_1^{-1})\theta'_2 = 2(\beta + 1)\theta'_1.$$

Again using the Toda and the first identity we obtain finally:

$$\theta''_2\theta_2^{-1} + \theta_2\phi^{-1} - \phi\psi - (\phi'\phi^{-1})^2 = 0$$

and then

$$\theta''_2\theta_2^{-1} + 2x - 2(\phi\psi + (\phi'\phi^{-1})^2) = \theta''_2\theta_2^{-1} - 2(x - \theta_2\phi^{-1}) = 0.$$

We will discuss one more step, namely the passage from $n = 2$ to $n = 3$ (then the recurrence will be clear). We want to show that:
1) $\theta'_3\theta_3^{-1} + \theta'_2\theta_2^{-1} = 2(\beta + 2)\theta_2\theta_3^{-1}$;
2) $\theta''_3\theta_3^{-1} = 2(x - \theta_3\theta_2^{-1})$.

From the second Toda and second identity we get

$$\theta_3 = 2x\theta_2 - \theta_2\theta_1^{-1}\theta_2 - \theta'_2\theta_2^{-1}\theta'_2.$$

It implies

$$\theta_3' = 2\theta_2 + 2x\theta_2' - \theta_2'\theta_1^{-1}\theta_2 + \theta_2\theta_1^{-1}\theta_1'\theta_1^{-1}\theta_2 - \theta_2\theta_1^{-1}\theta_2' -$$

$$-2(x - \theta_2\theta_1^{-1})\theta_2' + (\theta_2'\theta_2^{-1})^2\theta_2' - \theta_2'\theta_2^{-1}(2x - 2\theta_2\theta_1^{-1})\theta_2.$$

We simplify and obtain from this

$$\theta_3' = 2\theta_2 + \theta_2\theta_1^{-1}(\theta_2' + \theta_1'\theta_1^{-1}\theta_2) + \theta_2'\theta_1^{-1}\theta_2 + (\theta_2'\theta_2^{-1})^2\theta_2' - 2\theta_2'\theta_2^{-1}x\theta_2.$$

By the identity for θ_2' we have

$$\theta_3' = 2\theta_2 + \theta_2\theta_1^{-1}\cdot 2(1+\alpha)\theta_1 + \theta_2'\theta_1^{-1}\theta_2 + \theta_2'\theta_2^{-1}(-\theta_3 - \theta_2\theta_1^{-1}\theta_2).$$

which assure the first identity for $n = 3$.

Now we prove the second.

Set $a = 2(\beta + 2)$. We have

$$\theta_3' = a\theta_2 - (\theta_2'\theta_2^{-1})\theta_3.$$

Take the second derivation:

$$\theta_3'' = a\theta_2' - (\theta_2'\theta_2^{-1})'\theta_3 - \theta_2'\theta_2^{-1}\theta_3'.$$

By using the formula for θ_3' we have

$$\theta_3'' = a\theta_2' - (\theta_2'\theta_2^{-1})'\theta_3 - \theta_2'\theta_2^{-1}(a\theta_2 - \theta_2'\theta_2^{-1}\theta_3).$$

The terms with a are cancelled and we have

$$\theta_3'' = -(\theta_2'\theta_2^{-1})'\theta_3 + (\theta_2'\theta_2^{-1})^2\theta_3.$$

Note that

$$-(\theta_2'\theta_2^{-1})' + (\theta_2'\theta_2^{-1})^2 = \theta_2''\theta_2^{-1} - 2(\theta_2'\theta_2^{-1})'.$$

We already know that the first summand in the right hand side equals $2(x - \theta_2\theta_1^{-1})$ and by our Toda system

$$(\theta_2'\theta_2^{-1})' = \theta_3\theta_2^{-1} - \theta_2\theta_1^{-1}.$$

The $n-$th step of the recurrence goes as follows: from $n-$th Toda and recurrence conjecture we have

$$\theta_{n+1} = 2x\theta_n - \theta_n\theta_{n-1}^{-1}\theta_n - \theta_n'\theta_n^{-1}\theta_n'.$$

It implies

$$\theta'_{n+1} = 2\theta_n + 2x\theta'_n - \theta'_n\theta_{n-1}^{-1}\theta_n + \theta_n\theta_{n-1}^{-1}\theta'_{n-1}\theta_{n-1}^{-1}\theta_n - \theta_n\theta_{n-1}^{-1}\theta'_n -$$

$$-2(x - \theta_n\theta_{n-1}^{-1})\theta'_n + (\theta'_n\theta_n^{-1})^2\theta'_n - \theta'_n\theta_n^{-1}(2x - 2\theta_n\theta_{n-1}^{-1})\theta_n.$$

Then, after some simplifications we get

$$\theta'_{n+1} = 2\theta_n + \theta_n\theta_{n-1}^{-1}(\theta'_n + \theta'_{n-1}\theta_{n-1}^{-1}\theta_n) + \theta'_n\theta_{n-1}^{-1}\theta_n + (\theta'_n\theta_n^{-1})^2\theta'_n - 2\theta'_n\theta_n^{-1}x\theta_n.$$

By the recurrent formula for θ'_n we have

$$\theta'_{n+1} =$$

$$2\theta_n + \theta_n\theta_{n-1}^{-1}(2(\beta+n-1)\theta_{n-1} - \theta'_{n-1}\theta_{n-1}^{-1}\theta_n) + \theta'_n\theta_{n-1}^{-1}\theta_n + (\theta'_n\theta_n^{-1})^2\theta'_n - 2\theta'_n\theta_n^{-1}x\theta_n =$$

$$= 2(\beta + n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n + \theta'_n\theta_n^{-1}(\theta'_n\theta_{n-1}^{-1}\theta'_n - 2x\theta_n) =$$

$$= 2(\beta + n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n + \theta'_n\theta_n^{-1}(-\theta_{n+1} - \theta_n\theta_{n-1}^{-1}\theta_n) =$$

$$2(\beta + n)\theta_n + \theta'_n\theta_{n-1}^{-1}\theta_n - \theta'_n\theta_n^{-1}\theta_{n+1} - \theta'_n\theta_{n-1}^{-1}\theta_n.$$

which assure the first identity for $n + 1$.

We leave the proof of the second identity for any n as an easy (though a bit lengthy) exercise similar to the case $n = 3$ above.

Lemma 3.4. *For $n = 1$ the left logarithmic derivative $\phi'\phi^{-1} =: u_1$ satisfies to $nc - P_{II}(x, \beta)$.*

Proof. From the previous lemma we have from the first Toda equation:

$$(\phi'\phi^{-1})' = \theta_2\phi^{-1} - \phi\psi = \phi''\phi^{-1} - (\phi'\phi^{-1})^2 = 2(x - \phi\psi) - u_1^2$$

and hence

$$\theta_2\phi^{-1} = 2x - \phi\psi - u_1^2.$$

In other hand, taking the derivative of the first Toda, we get

$$u_1'' = (\theta_2\phi^{-1} - \phi\psi)' = \theta_2'\phi^{-1} - \theta_2\phi^{-1}u_1 - (\phi'\psi + \phi\psi').$$

We replace $\theta_2'\phi^{-1}$ by

$$2(\beta + 1) - u_1\theta_2\phi^{-1} = 2(\beta + 1) - u_1(2x - \phi\psi - u_1^2).$$

Finally we obtain

$$u_1'' = 2u_1^3 - 2u_1x - 2xu_1 + 2(\beta + 1) + u_1\phi\psi + \phi\psi u_1 - (\phi'\psi + \phi\psi'),$$

but

$$u_1\phi\psi + \phi\psi u_1 - (\phi'\psi + \phi\psi') = \phi\psi\phi'\phi^{-1} - \phi\psi' = 2\beta$$

which gives the desired result.

Our proof of Theorem 3.2 in the general case almost *verbatim* repeats the proof of the Lemma 3.4.

Proof. Let $u_n := \theta_n'\theta_n^{-1}$. Now the same arguments, from the Lemma 3.2, show that:
a) $\theta_{n+1}\theta_n^{-1} = 2x - \theta_n\theta_{n-1}^{-1} - u_n^2$;
b) $\theta_{n+1}'\theta_n^{-1} = 2(\beta+n) - \theta_n'\theta_n^{-1}\theta_{n+1}\theta_n^{-1}$;
c) $u_n'' = 2u_n^3 - 2xu_n - 2u_nx + 2(\beta+n) + \theta_n\theta_{n-1}^{-1}(\theta_n'\theta_n^{-1} + \theta_{n-1}'\theta_{n-1}^{-1})$. This implies that $u_n'' = 2u_n^3 - 2xu_n - 2u_nx + 4(\beta + n - \frac{1}{2})$.

Remark. *Starting with the "negative" Toda hierarchy (0-1'-m) we can formulate the similar identities for η_m. Using these identities we can prove the second statement of the Theorem 3.1.*

4. Discussion and perspectives

We have developed an approach to integrability of a fully noncommutative analog of the Painlevé equation. We construct solutions of this equation related to the "fully noncommutative" Toda chain, generalizing the results of [GR2, EGR1]. This solutions admit an explicit description in terms of Hankel quasideterminants.

We consider here only the noncommutative generalization of Painlevé II but it is not difficult to write down some noncommutative analogs of other Painlevé transcendants. It is interesting to study their solutions, noncommutative $\tau-$ functions, etc. We hope that our equation (like its "commutative" prototype) is a part of a whole noncommutative Painlevé hierarchy which relates (via a noncommutative Miura transform) to the noncommutative m-KdV and m-KP hierarchies (see i.e. [EGR1-2],[GN],[GNS]). Another interesting problem is to study a noncommutative version of isomonodromic transformations problem for our Painlevé equation. The natural approach to this problem is a noncommutative generalization of generating functions, constructed in [JKM]. The noncommutative "non-autonomous" Hamiltonian should be studied more extensively. It would be interesting to find noncommutative analogs of Okamoto differential equations [OK] and to generalize the description of Darboux-Bäcklund transformations for their solutions.

We shall address these and other open questions in the forthcoming papers.

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