Restrictions of generalized Verma modules to symmetric pairs

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Abstract

We initiate a new line of investigation on branching problems for generalized Verma modules with respect to reductive symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$. In general, Verma modules may not contain any simple module when restricted to a reductive subalgebra. In this article we give a necessary and sufficient condition on the triple $(\mathfrak{g}, \mathfrak{g}', \mathfrak{p})$ such that the restriction $X|_{\mathfrak{g}'}$ always contains simple $\mathfrak{g}'$-modules for any $\mathfrak{g}$-module X lying in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ attached to a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. Formulas are derived for the Gelfand–Kirillov dimension of any simple module occurring in a simple generalized Verma module. We then prove that the restriction $X|_{\mathfrak{g}'}$ is generically multiplicity-free for any $\mathfrak{p}$ and any $X \in \mathcal{O}^{\mathfrak{p}}$ if and only if $(\mathfrak{g}, \mathfrak{g}')$ is isomorphic to (A_n, A_{n-1}), (B_n, D_n), or (D_{n+1}, B_n). Explicit branching laws are also presented.

Keywords and phrases: branching law, symmetric pair, Verma module, highest weight module, flag variety, multiplicity-free representation

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Contents

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1 Program

Branching problems in representation theory ask how irreducible modules decompose when restricted to subalgebras. In the context of the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ of a semisimple Lie algebra $\mathfrak{g}$, branching problems are seemingly simple, however, it turns out that the restrictions behave wildly in general. For instance, the restrictions $X|_{\mathfrak{g}'_1}$ and $X|_{\mathfrak{g}'_2}$ of a $\mathfrak{g}$-module X lying in $\mathcal{O}$ may be completely different even when two reductive subalgebras $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ are conjugate to each other by an inner automorphism (see Examples 4.13, 4.14 for more details):

Example 1.1. *The restriction $X|_{\mathfrak{g}'_1}$ does not contain any simple $\mathfrak{g}'_1$-module, whereas $X|_{\mathfrak{g}'_2}$ decomposes into an algebraic direct sum of simple $\mathfrak{g}'_2$-modules.*

Example 1.2. *The Gelfand–Kirillov dimension of any simple $\mathfrak{g}'_1$-module occurring in $X|_{\mathfrak{g}'_1}$ is larger than that of any simple $\mathfrak{g}'_2$-module in $X|_{\mathfrak{g}'_2}$.*

The analysis of such phenomena brings us to the following problem to single out a good framework for the restriction $X|_{\mathfrak{g}'}$ where $\mathfrak{g}'$ is a (generalized) reductive subalgebra of $\mathfrak{g}$ and X Verma module of $\mathfrak{g}$.

Problem A. When does the restriction $X|_{\mathfrak{g}'}$ contain a simple $\mathfrak{g}'$-module?

Further, we raise the following problems when $X|_{\mathfrak{g}'}$ contains simple $\mathfrak{g}'$-modules.

Problem B. Find the 'size' of simple $\mathfrak{g}'$-modules occurring in $X|_{\mathfrak{g}'}$.

Problem C. Estimate multiplicities of simple $\mathfrak{g}'$-modules occurring in $X|_{\mathfrak{g}'}$.

Problem D. Find branching laws, in particular, for multiplicity-free cases.

Let us explain briefly our main results. We write $\mathfrak{B}$ for the full flag variety of $\mathfrak{g}$, and $\mathfrak{G}'$ for the set of conjugacy classes of $\mathfrak{g}'$ under the group $G := \mathrm{Int}(\mathfrak{g})$ of inner automorphisms. Then the 'framework' of the restriction $X|_{\mathfrak{h}}$ for $X \in \mathcal{O}$ and $\mathfrak{h} \in \mathfrak{G}'$ is classified by the quotient space $G\backslash(\mathfrak{B} \times \mathfrak{G}')$ under the diagonal action of G. More generally, we formulate such a statement in Theorem 2.1 in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ (see Subsection 2.1) for an arbitrary parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$.

In this article, we highlight the case where $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair. A special example of symmetric pairs is the diagonal case $(\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathrm{diag}(\mathfrak{g}_1))$, for which branching laws describe the decomposition of the tensor product of two representations (e.g. *fusion rules*).

For symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$, the cardinality of G-orbits on $\mathfrak{B} \times \mathfrak{G}'$ is finite, and our solution to Problem A in the category $\mathcal{O}^{\mathfrak{p}}$ is described in terms of the finite set $G\backslash(\mathfrak{P} \times \mathfrak{G}')$. Namely, we prove that the restriction $X|_{\mathfrak{g}'}$ contains simple $\mathfrak{g}'$-modules for any $X \in \mathcal{O}^{\mathfrak{p}}$ if and only if $(\mathfrak{p}, \mathfrak{g}')$ lies in a closed G-orbit on $\mathfrak{P} \times \mathfrak{G}'$ (Theorem 4.1).

In the study of Problem B, we use associated varieties (see e.g. [6, 15]) as a measure of the 'size' of $\mathfrak{g}'$-modules. We see that the associated variety $\mathcal{V}_{\mathfrak{g}'}(Y)$ of a simple $\mathfrak{g}'$-module Y occurring in the restriction $X|_{\mathfrak{g}'}$ is independent of Y if X is a simple $\mathfrak{g}$-module. The formulas of $\mathcal{V}_{\mathfrak{g}'}(Y)$ and its dimension (*Gelfand–Kirillov dimension*) are derived in Theorem 4.11.

It is notorious that multiplicities are often infinite in the branching laws for irreducible unitary representations when restricted to symmetric pairs, see [7]. In contrast, we prove in Theorem 4.15 that multiplicities are always finite in the branching laws with respect to symmetric pairs in the category $\mathcal{O}$, which gives an answer to Problem C.

Particularly interesting are 'multiplicity-free branching laws' where any simple $\mathfrak{g}'$-module occurs in the restriction $X|_{\mathfrak{g}'}$ at most once. We give two general multiplicity-free theorems with respect to symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$ in the parabolic category $\mathcal{O}^{\mathfrak{p}}$:

1) $\mathfrak{p}$ special, $(\mathfrak{g}, \mathfrak{g}')$ general (Theorem 5.1),

2) $\mathfrak{p}$ general, $(\mathfrak{g}, \mathfrak{g}')$ special (Theorem 5.4),

and then find branching laws corresponding to closed orbits in $G\backslash(\mathfrak{P} \times \mathfrak{G}')$.

Partial results of this article were presented at the conference in honor of Vinberg's 70th birthday at Bielefeld in Germany in 2007 and at the Winter School on Geometry and Physics in Cech Republic in 2010. The author is grateful to the organizers for their warm hospitality. In a subsequent paper [10], we shall apply the results here as a guiding principle to the construction of intertwining differential operators in parabolic geometry.

Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, 3, \cdots\}$.

2 Branching problem of Verma modules

In general, Verma modules may not contain any simple $\mathfrak{g}'$-module when restricted to a reductive subalgebra $\mathfrak{g}'$. In this section, we use the geometry of the double coset space $N_G(\mathfrak{g}')\backslash G/P$ and clarify the problem in Theorem 2.1, which will then serve as a foundational setting of branching problems for the category $\mathcal{O}^{\mathfrak{p}}$ in Theorem 4.1.

2.1 Generalized Verma modules

We begin with a quick review of the (parabolic) BGG category $\mathcal{O}^{\mathfrak{p}}$ and fix some notation. See [5] for a comprehensive introduction to this area.

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{j}$ a Cartan subalgebra. We write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{j})$ for the root system, $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ for the root space, and $\alpha^\vee$ for the coroot. We fix a positive system Δ^+, and define a Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ with nilradical $\mathfrak{n} := \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The BGG category $\mathcal{O}$ is defined to be the full subcategory of $\mathfrak{g}$-modules whose objects are finitely generated $\mathfrak{g}$-modules X such that X are $\mathfrak{j}$-semisimple and locally $\mathfrak{n}$-finite [2].

Let $\mathfrak{p}$ be a standard parabolic subalgebra, and $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$ its Levi decomposition with $\mathfrak{j} \subset \mathfrak{l}$. We set $\Delta^+(\mathfrak{l}) := \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{j})$, and define $\mathfrak{n}_-(\mathfrak{l}) := \oplus_{\alpha \in \Delta^+(\mathfrak{l})} \mathfrak{g}_{-\alpha}$. The parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ is defined to be the full subcategory of $\mathcal{O}$ whose objects X are locally $\mathfrak{n}_-(\mathfrak{l})$-finite. Then $\mathcal{O}^{\mathfrak{p}}$ is closed under submodules, quotients, and tensor products with finite dimensional representations.

The set of λ for which $\lambda|_{\mathfrak{j} \cap [\mathfrak{l}, \mathfrak{l}]}$ is dominant integral is denoted by

$$\Lambda^+(\mathfrak{l}) := \{\lambda \in \mathfrak{j}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \text{ for all } \alpha \in \Delta^+(\mathfrak{l})\}.$$

We write F_λ for the finite dimensional simple $\mathfrak{l}$-module with highest weight λ, inflate F_λ to a $\mathfrak{p}$-module via the projection $\mathfrak{p} \to \mathfrak{p}/\mathfrak{u} \simeq \mathfrak{l}$, and define the generalized Verma module by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{p}}^{\mathfrak{g}}(F_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda. \tag{2.1}$$

Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$, and any simple object in $\mathcal{O}^{\mathfrak{p}}$ is the quotient of some $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. We say $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of *scalar type* if F_λ is one-dimensional, or equivalently, if $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta(\mathfrak{l})$.

Let ρ be half the sum of positive roots. If $\lambda \in \Lambda^+(\mathfrak{l})$ satisfies

$$\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}_+ \text{ for all } \beta \in \Delta^+ \setminus \Delta(\mathfrak{l}), \tag{2.2}$$

then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple, see [3].

For $\mathfrak{p} = \mathfrak{b}$, we simply write $M^{\mathfrak{g}}(\lambda)$ for $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$. We note that $\mathcal{O}^{\mathfrak{b}} = \mathcal{O}$ by definition.

2.2 Framework of branching problems

Let $\mathfrak{g}'$ be a subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{G}'$ and $\mathfrak{P}$ the set of conjugacy classes of $\mathfrak{g}'$ and $\mathfrak{p}$, respectively. Let P be the parabolic subgroup of $G = \mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{p}$, and define the normalizer of $\mathfrak{g}'$ as

$$N_G(\mathfrak{g}') := \{g \in G : \mathrm{Ad}(g)\mathfrak{g}' = \mathfrak{g}'\}.$$

Then we have natural bijections: $G/P \simeq \mathfrak{P}$, $G/N_G(\mathfrak{g}') \simeq \mathfrak{G}'$, and hence

$$G\backslash(\mathfrak{P} \times \mathfrak{G}') \simeq N_G(\mathfrak{g}')\backslash\mathfrak{P} \simeq \mathfrak{G}'/P \simeq N_G(\mathfrak{g}')\backslash G/P. \tag{2.3}$$

Here, we let G act diagonally on $\mathfrak{P} \times \mathfrak{G}'$ in the left-hand side of (2.3).

Let S be the set of complete representatives of the double coset $N_G(\mathfrak{g}')\backslash G/P$, and we write $\mathfrak{g}'_s := \mathrm{Ad}(s)^{-1}\mathfrak{g}'$ for $s \in S$. Then the branching problem for $\mathcal{O}^{\mathfrak{p}}$ with respect to a subalgebra belonging to $\mathfrak{G}'$ is 'classified' by the double coset $N_G(\mathfrak{g}')\backslash G/P$ in the following sense:

Theorem 2.1. *For any $X \in \mathcal{O}^{\mathfrak{p}}$ and any $\mathfrak{h} \in \mathfrak{G}'$, there exists $s \in S$ such that $X|_{\mathfrak{h}} \simeq \widetilde{X}|_{\mathfrak{g}'_s}$ for some $\widetilde{X} \in \mathcal{O}^{\mathfrak{p}}$ via a Lie algebra isomorphism between $\mathfrak{h}$ and $\mathfrak{g}'_s$.*

Proof of Theorem 2.1. Given $\mathfrak{h} \in \mathfrak{G}'$, we take $s \in S$ and $q \in P$ such that $\mathrm{Ad}((sq)^{-1})\mathfrak{g}' = \mathfrak{h}$. Clearly, we have a Lie algebra isomorphism $\mathrm{Ad}(q^{-1}) : \mathfrak{g}'_s \xrightarrow{\sim} \mathfrak{h}$.

For $X \in \mathcal{O}^{\mathfrak{p}}$, we define a new $\mathfrak{g}$-module structure on X by

$$Z \underset{q}{\cdot} v := (\mathrm{Ad}(q)^{-1}Z) \cdot v \quad \text{for } Z \in \mathfrak{g}, v \in X.$$

Since P normalizes $\mathfrak{p}$, this new module, to be denoted by $\widetilde{X}$, lies in $\mathcal{O}^{\mathfrak{p}}$. Then, for any Lie subalgebra $\mathfrak{v}$ of $\mathfrak{g}$, the restriction $\widetilde{X}|_{\mathfrak{v}}$ is isomorphic to the restriction $X|_{\mathrm{Ad}(q)^{-1}\mathfrak{v}}$ via the Lie algebra isomorphism $\mathfrak{v} \simeq \mathrm{Ad}(q)^{-1}\mathfrak{v}$. Applying this to $\mathfrak{v} := \mathfrak{g}'_s$, we get the following isomorphism:

$$X|_{\mathfrak{h}} = X|_{\mathrm{Ad}(q)^{-1}\,\mathrm{Ad}(s)^{-1}\mathfrak{g}'} \simeq \widetilde{X}|_{\mathfrak{g}'_s}$$

via the Lie algebra isomorphim $\mathrm{Ad}(q) : \mathfrak{h} \xrightarrow{\sim} \mathfrak{g}'_s$. Theorem 2.1 is thus proved. □

Remark 2.2. If $(\mathfrak{g}, \mathfrak{g}')$ is a semisimple symmetric pair (see Subsection 4.1), then S is a finite set (Matsuki [11]).

3 Discretely decomposable branching laws

In this section, we bring the concept of 'discretely decomposable restrictions' to the branching problem for the BGG category $\mathcal{O}^{\mathfrak{p}}$, and prove that the restriction $X|_{\mathfrak{g}'}$ contains simple $\mathfrak{g}'$-modules for $X \in \mathcal{O}^{\mathfrak{p}}$ if $\mathfrak{p}$ lies in a closed G'-orbit on the generalized flag variety $\mathfrak{P}$. In particular, it is the case if $\mathfrak{p}$ is $\mathfrak{g}'$-compatible (Definition 3.7). Under this assumption the character identities are derived for the restriction $X|_{\mathfrak{g}'}$ (Theorem 3.10).

3.1 Discretely decomposable modules $\mathcal{O}$

Suppose that $\mathfrak{g}$ is a reductive Lie algebra.

Definition 3.1. We say a $\mathfrak{g}$-module X is *discretely decomposable* if there is an increasing filtration $\{X_m\}$ of $\mathfrak{g}$-submodules of finite length such that $X = \bigcup_{m=0}^{\infty} X_m$. Further, we say X is *discretely decomposable in the category* $\mathcal{O}^{\mathfrak{p}}$ if all X_m can be taken from $\mathcal{O}^{\mathfrak{p}}$.

Here are obvious examples:

Example 3.2. 1) *Any $\mathfrak{g}$-module of finite length is discretely decomposable.*
2) (completely reducible case). *An algebraic direct sum of countably many simple $\mathfrak{g}$-modules is discretely decomposable.*

Remark 3.3. The concept of discretely decomposable $\mathfrak{g}$-modules was originally introduced in the context of $(\mathfrak{g}, K)$-modules in [8, Definition 1.1] as an algebraic analogue of unitary representations whose irreducible decompositions have no 'continuous spectrum'. Then the main issue of [7, 8] was to find a criterion for the discrete decomposability of the restriction of $(\mathfrak{g}, K)$-modules. We note that discrete decomposability in the generality of Definition 3.1 does not imply complete reducibility.

Suppose $\mathfrak{g}'$ is a reductive subalgebra, and $\mathfrak{p}'$ its parabolic subalgebra.

Lemma 3.4. *Let X be a simple $\mathfrak{g}$-module. Then the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable in the category $\mathcal{O}^{\mathfrak{p}'}$ if and only if there exists a $\mathfrak{g}'$-module $Y \in \mathcal{O}^{\mathfrak{p}'}$ such that $\operatorname{Hom}_{\mathfrak{g}'}(Y, X|_{\mathfrak{g}'}) \neq \{0\}$. In this case, any subquotient occurring in the $\mathfrak{g}'$-module $X|_{\mathfrak{g}'}$ lies in $\mathcal{O}^{\mathfrak{p}'}$.*

Proof. Suppose $\mathrm{Hom}_{\mathfrak{g}'}(Y, X|_{\mathfrak{g}'}) \neq 0$ for some $Y \in \mathcal{O}^{\mathfrak{p}'}$. Taking the subquotient of Y if necessary, we may assume Y is a simple $\mathfrak{g}'$-module. Let $\iota : Y \to X$ be an injective $\mathfrak{g}'$-homomorphism. For $m \in \mathbb{N}$, we denote by Y_m the image of the following $\mathfrak{g}'$-homomorphism:

$$\underbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}_{m} \otimes Y \to X, \quad (H_1 \otimes \cdots \otimes H_m) \otimes v \mapsto H_1 \cdots H_m \, \iota(v).$$

Then $X = \bigcup_{m=0}^{\infty} Y_m$ because X is simple. Moreover $Y_m \in \mathcal{O}^{\mathfrak{p}'}$ because $\mathcal{O}^{\mathfrak{p}'}$ is closed under quotients and tensor products with finite dimensional representations. Hence, the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable in $\mathcal{O}^{\mathfrak{p}'}$. Conversely, the 'only if' part is obvious because $\mathcal{O}^{\mathfrak{p}'}$ is closed under submodules. Finally, any subquotient of Y_m lies in $\mathcal{O}^{\mathfrak{p}'}$, whence the last statement. Thus Lemma 3.4 is proved. $\square$

3.2 Discretely decomposable restrictions for $\mathcal{O}^{\mathfrak{p}}$

Let $G = \mathrm{Int}(\mathfrak{g})$, P the parabolic subgroup of G with Lie algebra $\mathfrak{p}$ as before, and G' a reductive subgroup with Lie algebra $\mathfrak{g}'$. We ask when the restriction $X|_{\mathfrak{g}'}$ of $X \in \mathcal{O}^{\mathfrak{p}}$ is discretely decomposable in the sense of Definition 3.1.

Proposition 3.5. *If $G'P$ is closed in G then the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for any simple $\mathfrak{g}$-module X in $\mathcal{O}^{\mathfrak{p}}$.*

Proof. We set $P' := G' \cap P$. Suppose $G'P$ is closed in G. Then G'/P' is closed in the generalized flag variety G/P, and hence is compact. Therefore, the Lie algebra $\mathfrak{p}' := \mathfrak{g}' \cap \mathfrak{p}$ of P' must be a parabolic subalgebra of $\mathfrak{g}'$.

Let X be a simple object in $\mathcal{O}^{\mathfrak{p}}$. Then X is obtained as the quotient of some generalized Verma module, that is, there exists $\lambda \in \Lambda^+(\mathfrak{l})$ such that the composition map

$$F_\lambda \hookrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F_\lambda \twoheadrightarrow X$$

is non-trivial. Therefore, we get a non-zero $\mathfrak{g}'$-homomorphism

$$U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} (F_\lambda|_{\mathfrak{p}'}) \to X. \tag{3.1}$$

Since the $\mathfrak{g}'$-module $U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} (F_\lambda|_{\mathfrak{p}'})$ lies in $\mathcal{O}^{\mathfrak{p}'}$, the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable in the category $\mathcal{O}^{\mathfrak{p}'}$ owing to Lemma 3.4. $\square$

The converse statement of Proposition 3.5 will be proved in Theorem 4.1 under the assumption that $(\mathfrak{g}, \mathfrak{g}')$ is a semisimple symmetric pair.

The assumption of Proposition 3.5 fits well into the framework of Theorem 2.1. To see this, we make the following observation:

Lemma 3.6. *Retain the notation as in Subsection 2.2. Then the following conditions on the triple* $(\mathfrak{g}, \mathfrak{g}', \mathfrak{p})$ *are equivalent:*

(i) *The G'-orbit through $\mathfrak{p} \in \mathfrak{P}$ is closed.*

(ii) *$G'P$ is closed in G.*

Clearly these conditions are invariant under the conjugation of $\mathfrak{p}$ by an element of the group $N_G(\mathfrak{g}')$, and hence they are determined by the equivalence claseses in $N_G(\mathfrak{g}')\backslash G/P \simeq G\backslash(\mathfrak{P} \times \mathfrak{G}')$ (see (2.3)) containing $(\mathfrak{p}, \mathfrak{g}') \in \mathfrak{P} \times \mathfrak{G}'$.

3.3 $\mathfrak{g}'$-compatible parabolic subalgebra $\mathfrak{p}$

This subsection discusses a sufficient condition for the closedness of $G'P$ in G.

A semisimple element $H \in \mathfrak{g}$ is said to be *hyperbolic* if the eigenvalues of $\mathrm{ad}(H)$ are all real. For a hyperbolic element H, we define the subalgebras

$$\mathfrak{u}_+ \equiv \mathfrak{u}_+(H), \quad \mathfrak{l} \equiv \mathfrak{l}(H), \text{ and } \mathfrak{u}_- \equiv \mathfrak{u}_-(H)$$

as the sum of the eigenspaces with positive, zero, and negative eigenvalues, respectively. Then

$$\mathfrak{p}(H) := \mathfrak{l}(H) + \mathfrak{u}_+(H) \tag{3.2}$$

is a Levi decomposition of a parabolic subalgera of $\mathfrak{g}$.

Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$.

Definition 3.7. We say $\mathfrak{p}$ is $\mathfrak{g}'$-*compatible* if there exists a hyperbolic element H in $\mathfrak{g}'$ such that $\mathfrak{p} = \mathfrak{p}(H)$.

If $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ is $\mathfrak{g}'$-compatible, then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ becomes a parabolic subalgebra of $\mathfrak{g}'$ with Levi decomposition

$$\mathfrak{p}' = \mathfrak{l}' + \mathfrak{u}'_+ := (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{u}_+ \cap \mathfrak{g}').$$

Then, using the notation of Subsection 3.2, we see that $G'/P' = G'/G' \cap P$ becomes a generalized flag variety, and therefore is closed in G/P. Hence, we get the following proposition from Proposition 3.5:

Proposition 3.8. *If $\mathfrak{p}$ is $\mathfrak{g}'$-compatible, then $G'P$ is closed in G and the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for any $X \in \mathcal{O}^{\mathfrak{p}}$.*

We note that the converse statement is not true (see also Theorem 4.1).

Example 3.9. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$, and $\mathfrak{g}' := \mathrm{diag}(\mathfrak{g}_1) \equiv \{(Z, Z) : Z \in \mathfrak{g}_1\}$. Then a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\mathfrak{g}'$-compatible if and only if $\mathfrak{p}$ is of the form $\mathfrak{p}_1 \oplus \mathfrak{p}_1$ for some parabolic subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}_1$.*

On the other hand, $G'P$ is closed in $G = G_1 \times G_1$ if and only if $\mathfrak{p}$ is of the form $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ for some parabolic subalgebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ containing a common Borel subalgebra.

3.4 Character identities

In this subsection, we prove the character identities of the restriction of generalized Verma modules to a reductive subalgebra $\mathfrak{g}'$ assuming that the parabolic subalgebras $\mathfrak{p}$ is $\mathfrak{g}'$-compatible.

Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ be a $\mathfrak{g}'$-compatible parabolic subalgebra of $\mathfrak{g}$ defined by a hyperbolic element $H \in \mathfrak{g}'$. We take a Cartan subalgebra $\mathfrak{j}'$ of $\mathfrak{g}'$ such that $H \in \mathfrak{j}'$, and extend it to a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$. Clearly, $\mathfrak{j} \subset \mathfrak{l}$ and $\mathfrak{j}' \subset \mathfrak{l}'$.

We recall that F_λ denotes the finite dimensional, simple module of $\mathfrak{l}$ with highest weight $\lambda \in \Lambda^+(\mathfrak{l})$. Likewise, let F'_δ denote that of $\mathfrak{l}'$ for $\delta \in \Lambda^+(\mathfrak{l}')$.

Given a vector space V we denote by $S(V) = \bigoplus_{k=0}^{\infty} S^k(V)$ the symmetric tensor algebra over V. We extend the adjoint action of $\mathfrak{l}'$ on $\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}'$ to $S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}')$. We set

$$m(\delta; \lambda) := \dim \mathrm{Hom}_{\mathfrak{l}'}(F'_\delta, F_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}')). \tag{3.3}$$

Theorem 3.10. *Suppose that $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ is a $\mathfrak{g}'$-compatible parabolic subalgebra of $\mathfrak{g}$, and $\lambda \in \Lambda^+(\mathfrak{l})$.*
1) *$m(\delta; \lambda) < \infty$ for all $\delta \in \Lambda^+(\mathfrak{l}')$.*
2) *In the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$, we have the following isomorphism:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\delta \in \Lambda^+(\mathfrak{l}')} m(\delta; \lambda) M_{\mathfrak{p}'}^{\mathfrak{g}'}(\delta). \tag{3.4}$$

Proof. Let $H \in \mathfrak{g}'$ be the hyperbolic element defining the parabolic subalgebra $\mathfrak{p}$. We denote by $\mathfrak{g}''$ the orthogonal complementary subspace of $\mathfrak{g}'$ in $\mathfrak{g}$

with respect to the Killing form. Since $\mathrm{ad}(H)$ preserves the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, the sum $\mathfrak{u}_-$ of negative eigenspaces of $\mathrm{ad}(H)$ decomposes as

$$\mathfrak{u}_- = \mathfrak{u}'_- \oplus \mathfrak{u}''_- := (\mathfrak{u}_- \cap \mathfrak{g}') \oplus (\mathfrak{u}_- \cap \mathfrak{g}''). \tag{3.5}$$

This is a decomposition of $\mathfrak{l}'$-modules, and hence, we have an $\mathfrak{l}'$-module isomorphism $S(\mathfrak{u}''_-) \simeq S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}')$.
1) Let $a(>0)$ be the minimum of the eigenvalues of $-\mathrm{ad}(H)$ on $\mathfrak{u}''_-$. Since $H \in \mathfrak{l}'$, we have

$$\mathrm{Hom}_{\mathfrak{l}'}(F'_\delta, F_\lambda \otimes S^k(\mathfrak{u}''_-)) = 0$$

for all k such that $k > \frac{1}{a}(\lambda(H) - \delta(H))$. In view of (3.3), we get $m(\delta;\lambda) < \infty$.
2) The formal character of the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is given by

$$\mathrm{ch}(M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)) = \mathrm{ch}(F_\lambda) \prod_{\alpha \in \Delta(\mathfrak{u}_-, \mathfrak{j})} (1 - e^\alpha)^{-1}. \tag{3.6}$$

Let us prove that its restriction to $\mathfrak{j}'$ equals the formal character of the right-hand side of (3.4). For this, we observe that $F_\lambda \otimes S(\mathfrak{u}''_-)$ is a semisimple $\mathfrak{l}'$-module, and therefore, it decomposes into the direct sum of simple $\mathfrak{l}'$-modules $\bigoplus_{\delta \in \Lambda^+(\mathfrak{l}')} m(\delta;\lambda) F'_\delta$, where $m(\delta, \lambda)$ is defined in (3.3). Turning to their formal characters, we get

$$\mathrm{ch}(F_\lambda)|_{\mathfrak{j}'} \prod_{\alpha \in \Delta(\mathfrak{u}''_-, \mathfrak{j}')} (1 - e^\alpha)^{-1} = \sum_{\delta \in \Lambda^+(\mathfrak{l}')} m(\delta;\lambda)\, \mathrm{ch}(F'_\delta). \tag{3.7}$$

Writing the multiset $\Delta(\mathfrak{u}_-, \mathfrak{j})|_{\mathfrak{j}'}$ as a disjoint union $\Delta(\mathfrak{u}''_-, \mathfrak{j}') \amalg \Delta(\mathfrak{u}'_-, \mathfrak{j}')$, we get from (3.6) and (3.7)

$$\begin{aligned}
\mathrm{ch}(M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))|_{\mathfrak{j}'} &= \mathrm{ch}(F_\lambda)|_{\mathfrak{j}'} \prod_{\alpha \in \Delta(\mathfrak{u}''_-, \mathfrak{j}')} (1 - e^\alpha)^{-1} \prod_{\alpha \in \Delta(\mathfrak{u}'_-, \mathfrak{j}')} (1 - e^\alpha)^{-1} \\
&= \sum_\delta m(\delta;\lambda)\, \mathrm{ch}(F'_\delta) \prod_{\alpha \in \Delta(\mathfrak{u}'_-, \mathfrak{j}')} (1 - e^\alpha)^{-1} \\
&= \sum_\delta m(\delta;\lambda)\, \mathrm{ch}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\delta)).
\end{aligned}$$

Hence (3.4) holds in the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$. $\square$

3.5 Multiplicity-free restriction

Retain the setting of the previous subsection. In particular, we suppose that $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ is a $\mathfrak{g}'$-compatible parabolic subalgebra of $\mathfrak{g}$. We will see in this subsection that the character identity in Theorem 3.10 leads us to multiplicity-free branching laws for generalized Verma modules when the $\mathfrak{l}'$-module $S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}')$ is multiplicity-free.

Definition 3.11. We say that a $\mathfrak{g}$-module V is a *multiplicity-free space* if the induced $\mathfrak{g}$-module on the symmetric algebra $S(V)$ is a multiplicity-free representation.

Multiplicity-free spaces for reductive Lie algebras were classified by V. Kac in the irreducible case, and by Benson–Ratcliff and Leahy independently in the reducible case (see [1]).

The following Corollary is an immediate consequence of Theorem 3.10:

Corollary 3.12. *Assume that $\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}'$ is an $\mathfrak{l}'$-multiplicity-free space. We denote by D the support of simple $\mathfrak{l}'$-modules occurring in $S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}')$, namely, $S(\mathfrak{u}_-/\mathfrak{u}_- \cap \mathfrak{g}') \simeq \bigoplus_{\delta \in D} F'_\delta$. Then any generalized Verma module $M_\mathfrak{p}^\mathfrak{g}(\lambda)$ of scalar type decomposes into a multiplicity-free sum of generalized Verma modules for $\mathfrak{g}'$ in the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$ as follows:*

$$M_\mathfrak{p}^\mathfrak{g}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\delta \in D} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda|_{\mathfrak{j}'} + \delta). \tag{3.8}$$

Remark 3.13. For a 'generic' λ, the formula (3.8) becomes a multiplicity-free direct sum of simple $\mathfrak{g}'$-modules. For instance, there is no extension among the modules $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda|_{\mathfrak{j}'} + \delta)$ ($\delta \in D$) if they have distinct $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal characters (e.g. Theorems 5.5, 5.6 and 5.7) or if $M_\mathfrak{p}^\mathfrak{g}(\lambda)$ has an invariant Hermitian inner product with respect to a certain real form of $\mathfrak{g}'$ (e.g. Theorem 5.1). See Section 5 for details.

4 Branching problems for symmetric pairs

The decomposition of the tensor product of two representations is an example of branching laws with respect to a special case of symmetric pairs, namely, the pair $\mathfrak{g}_1 \oplus \mathfrak{g}_1 \downarrow \mathrm{diag}(\mathfrak{g}_1)$. In this section, we discuss Problems A to D for semisimple symmetric pairs.

4.1 Criterion for discretely decomposable restriction

Let τ be an involutive automorphism of a semisimple Lie algebra $\mathfrak{g}$, and we denote the fixed point subalgebra by

$$\mathfrak{g}^\tau := \{Z \in \mathfrak{g} : \tau Z = Z\}.$$

The pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ is called a *semisimple symmetric pair*. Typical examples are the pairs $(\mathfrak{g}_1 \oplus \mathfrak{g}_1, \mathrm{diag}(\mathfrak{g}_1))$ ($\mathfrak{g}_1$: semisimple Lie algebra), $(\mathfrak{sl}_n, \mathfrak{so}_n)$, and $(\mathfrak{sl}_{p+q}, \mathfrak{s}(\mathfrak{gl}_p + \mathfrak{gl}_q))$.

We lift τ to an automorphism of the group $G = \mathrm{Int}(\mathfrak{g})$ of inner automorphisms, and set $G^\tau := \{g \in G : \tau g = g\}$. Then G^τ is a reductive subgroup of G with Lie algebra $\mathfrak{g}^\tau$.

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$, and X a $\mathfrak{g}$-module lying in $\mathcal{O}^{\mathfrak{p}}$. Problem A asks when the restriction $X|_{\mathfrak{g}^\tau}$ contains simple $\mathfrak{g}^\tau$-modules. We give its necessary and sufficient condition by the geometry of the generalized flag variety G/P associated to the parabolic subalgebra $\mathfrak{p}$:

Theorem 4.1. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, τ an involutive automorphism of $\mathfrak{g}$, and $\mathfrak{p}$ a parabolic subalgebra. Then the following three conditions on the triple $(\mathfrak{g}, \mathfrak{g}^\tau, \mathfrak{p})$ are equivalent:*

(i) *For any simple $\mathfrak{g}$-module X in $\mathcal{O}^{\mathfrak{p}}$, the restriction $X|_{\mathfrak{g}^\tau}$ contains at least one simple $\mathfrak{g}^\tau$-module.*

(ii) *For any simple $\mathfrak{g}$-module X in $\mathcal{O}^{\mathfrak{p}}$, the restriction $X|_{\mathfrak{g}^\tau}$ is discretely decomposable as a $\mathfrak{g}^\tau$-module in the sense of Definition 3.1.*

(iii) *$G^\tau P$ is closed in G.*

If one of (hence all of) the above three conditions is fulfilled then $\mathfrak{p}^\tau := \mathfrak{p} \cap \mathfrak{g}^\tau$ is a parabolic subalgebra of $\mathfrak{g}^\tau$, and any irreducible subquotient occurring in the restriction $X|_{\mathfrak{g}^\tau}$ belongs to the category $\mathcal{O}^{\mathfrak{p}^\tau}$.

In Proposition 4.6, the geometric condition (iii) in Theorem 4.1 will be reformalised as an algebraic condition.

Strategy of Proof of Theorem 4.1: We have already seen the equivalence (i) $\Longleftrightarrow$ (ii) in Lemma 3.4 and the implication (iii) $\Longrightarrow$ (ii) in Proposition 3.5 in a more general setting, i.e. without assuming that $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair. The non-trivial part is the implication (ii) $\Longrightarrow$ (iii), which will be proved in Subsection 4.4 after we establish some structual results on closed G^τ-orbit in G/P (Subsection 4.2).

We end this subsection with two very special cases of Theorem 4.1, namely, for $\mathfrak{p} = \mathfrak{b}$ (Borel) and for the pair $(\mathfrak{g} \oplus \mathfrak{g}, \operatorname{diag} \mathfrak{g})$:

Corollary 4.2. *Let $\mathcal{O}$ be the BGG category associated to a Borel subalgebra $\mathfrak{b}$, and τ an involutive automorphism of $\mathfrak{g}$. Then the following three conditions on $(\tau, \mathfrak{b})$ are equivalent:*

(i) *Any simple $\mathfrak{g}$-module in $\mathcal{O}$ contains at least one simple $\mathfrak{g}^\tau$-module when restricted to $\mathfrak{g}^\tau$.*

(ii) *Any simple $\mathfrak{g}$-module in $\mathcal{O}$ is discretely decomposable as a $\mathfrak{g}^\tau$-module in the sense of Definition 3.1.*

(iii) *$\tau\mathfrak{b} = \mathfrak{b}$.*

Proof. We shall see in Lemma 4.5 that $G^\tau B$ is closed in G if and only if $\tau\mathfrak{b} = \mathfrak{b}$. Hence, Corollary follows from Theorem 4.1. □

Corollary 4.3. *Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be two parabolic subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$. Then the following three conditions on $(\mathfrak{p}_1, \mathfrak{p}_2)$ are equivalent:*

(i) *For any simple $\mathfrak{g}$-module X_1 in $\mathcal{O}^{\mathfrak{p}_1}$ and X_2 in $\mathcal{O}^{\mathfrak{p}_2}$, the tensor product representation $X_1 \otimes X_2$ contains at least one simple $\mathfrak{g}$-module.*

(ii) *For any simple $\mathfrak{g}$-module X_1 in $\mathcal{O}^{\mathfrak{p}_1}$ and X_2 in $\mathcal{O}^{\mathfrak{p}_2}$, the tensor product representation $X_1 \otimes X_2$ is discretely decomposable as a $\mathfrak{g}$-module.*

(iii) *$\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic subalgebra.*

Proof. Let P_1 and P_2 be the parabolic subgroups of $G = \operatorname{Int}(\mathfrak{g})$ with Lie algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$, respectively. Then the diagonal G-orbit on $(G \times G)/(P_1 \times P_2)$ through the origin is given as $G/(P_1 \cap P_2)$, which is closed if and only if $\mathfrak{p}_1 \cap \mathfrak{p}_2$ is a parabolic algebra of $\mathfrak{g}$. Hence, Corollary is deduced from Theorem 4.1. □

4.2 Criterion for closed G^τ-orbit on G/P

As a preparation for the proof of Theorem 4.1, we establish some structural results for closed G^τ-orbits on the generalized flag variety G/P in this subsection. We note that the closedness condition for G^τ-orbits on G/P is much more complicated than that for the full flag variety G/B (cf. Lemma 4.5 below). The author is grateful to T. Matsuki for helpful discussions, in particular, for the proof of Proposition 4.6.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $G = \mathrm{Int}(\mathfrak{g})$, and τ an involutive automorphism of $\mathfrak{g}$ as before. We begin with:

Lemma 4.4.

1) *Let θ be a Cartan involution of $\mathfrak{g}$ commuting with τ. For any parabolic subalgebra $\mathfrak{p}$, there exist $h \in G^\tau$ and a Cartan subalgebra $\mathfrak{j}$ such that $\tau\mathfrak{j} = \theta\mathfrak{j} = \mathfrak{j}$ and $\mathfrak{j} \subset \mathrm{Ad}(h)\mathfrak{p}$. In particular, any parabolic subalgebra contains a τ-stable Cartan subalgebra.*

2) *A parabolic subalgebra is τ-stable if and only if it is $\mathfrak{g}^\tau$-compatible (see Definition 3.7).*

Proof. 1) This assertion holds for any Borel subalgebra of $\mathfrak{g}$ ([11, Theorem 1]). Hence, it holds also for any parabolic subalgebra.
2) Suppose $\mathfrak{p}$ is a τ-stable parabolic subalgebra. Take a τ-stable Cartan subalgebra $\mathfrak{j}$ contained in $\mathfrak{p}$. Then there exists $H \in \mathfrak{j}$ such that

$$\mathfrak{p} = \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g},\mathfrak{j}) \\ \alpha(H) \geq 0}} \mathfrak{g}_\alpha.$$

Since $\tau\mathfrak{p} = \mathfrak{p}$, $\alpha(H) \geq 0$ if and only if $\alpha(\tau H) \geq 0$, which is then equivalent to $\alpha(H + \tau H) \geq 0$. Therefore, the parabolic subalgebra $\mathfrak{p}$ equals $\mathfrak{p}(H + \tau H)$ with the notation (3.2), and thus it is $\mathfrak{g}^\tau$-compatible. Conversely, any $\mathfrak{g}^\tau$-compatible parabolic subalgebra is obviously τ-stable. □

We then deduce a simple characterization of closed G^τ-orbits on the full flag variety G/B from [11, Proposition 2] combined with Lemma 4.4 2):

Lemma 4.5. *The following three conditions on τ and a Borel subalgebra $\mathfrak{b}$ are equivalent:*

(i) *$G^\tau B$ is closed in G.*

(ii) $\tau\mathfrak{b} = \mathfrak{b}$.

(iii) $\mathfrak{b}$ *is* $\mathfrak{g}^\tau$*-compatible.*

Unfortunately, such a simple statement does not hold for a general parabolic subalgebra $\mathfrak{p}$. In fact, the condition $\tau\mathfrak{p} = \mathfrak{p}$ is stronger than the closedness of $G^\tau P$ (see Example 3.9). In order to give the right characterization for the closedness of $G^\tau P$, we let $\mathrm{pr}_\tau : \mathfrak{g} \to \mathfrak{g}^\tau$ be the projection defined by

$$\mathrm{pr}_\tau(Z) := \frac{1}{2}(Z + \tau Z). \tag{4.1}$$

For a subspace V in $\mathfrak{g}$, we define the ± 1 eigenspaces of τ by

$$V^{\pm\tau} := \{v \in V : \tau v = \pm v\}. \tag{4.2}$$

Note that $\mathrm{pr}_\tau(V) = V^\tau$ if V is τ-stable.

Proposition 4.6. *Suppose* $\mathfrak{p}$ *is a parabolic subalgebra with nilradical* $\mathfrak{u}$*, and* τ *is an involutive automorphism of* $\mathfrak{g}$*. Then, the following three conditions on the triple* $(\mathfrak{g}, \mathfrak{g}^\tau, \mathfrak{p})$ *are equivalent:*

(i) $G^\tau P$ *is closed in* G*.*

(ii) $\mathrm{pr}_\tau(\mathfrak{u})$ *is a nilpotent Lie algebra.*

(iii) $\mathrm{pr}_\tau(\mathfrak{u})$ *consists of nilpotent elements.*

We note that the parabolic subalgebra $\mathfrak{p}$ may not be τ-stable in Proposition 4.6. The idea of the following proof goes back to [12], which is to use a τ-stable Borel subalgebra contained in $\mathfrak{p}$ when $\mathfrak{p}$ itself is not τ-stable.

Proof. We take a Borel subalgebra $\mathfrak{b} \subset \mathfrak{p}$ such that $G^\tau B$ is relatively closed in $G^\tau P$. This is possible because $G^\tau\backslash G/B$ is a finite set.

(i) $\Longrightarrow$ (ii) Suppose $G^\tau P$ is closed in G. Then $G^\tau B$ is also closed in G. Owing to Lemma 4.5, $\mathfrak{b}$ is τ-stable, and therefore, so is the nilradical $\mathfrak{n}$ of $\mathfrak{b}$. Thus, $\mathrm{pr}_\tau(\mathfrak{n}) = \mathfrak{n}^\tau$. Since $\mathfrak{u} \subset \mathfrak{n}$, we get $\mathrm{pr}_\tau(\mathfrak{u}) \subset \mathrm{pr}_\tau(\mathfrak{n}) = \mathfrak{n}^\tau$.

For $X, Y \in \mathfrak{g}$, a simple computation shows

$$2[\mathrm{pr}_\tau(X), \mathrm{pr}_\tau(Y)] = \mathrm{pr}_\tau([X, Y]) + \mathrm{pr}_\tau([X, \tau Y]).$$

If $X, Y \in \mathfrak{u}$, then $[X, Y] \in \mathfrak{u}$ and $[X, \tau Y] \in [\mathfrak{u}, \mathfrak{n}] \subset \mathfrak{u}$. Hence $\mathrm{pr}_\tau(\mathfrak{u})$ is a Lie subalgebra. Since $\mathrm{pr}_\tau(\mathfrak{u})$ is contained in $\mathfrak{n}^\tau$, we conclude that $\mathrm{pr}_\tau(\mathfrak{u})$ is a nilpotent Lie algebra. Thus, (i) $\Longrightarrow$ (ii) is proved.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (i). Since the conditions (i) and (iii) remain the same if we replace $\mathfrak{p}$ by $\mathrm{Ad}(h)(\mathfrak{p})$ for some $h \in G^\tau$, we may and do assume that $\mathfrak{p}$ contains a Cartan subalgebra $\mathfrak{j}$ such that $\tau\mathfrak{j} = \theta\mathfrak{j} = \mathfrak{j}$ by Lemma 4.4. Then $\theta\alpha = -\alpha$ for any $\alpha \in \Delta(\mathfrak{g}, \mathfrak{j})$.

Suppose $G^\tau P$ is not closed in G. By the Matsuki duality [11], we see that $G^{\tau\theta}P$ is not open in G. Therefore, there exists $\alpha \in \Delta(\mathfrak{u}, \mathfrak{j})$ such that $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{g}^{\tau\theta} + \mathfrak{p}$. Take a non-zero $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$. In view that

$$X_{-\alpha} = (X_{-\alpha} + \tau\theta X_{-\alpha}) - \tau\theta X_{-\alpha} \in \mathfrak{g}^{\tau\theta} + \mathfrak{g}_{\tau\alpha},$$

we see $\mathfrak{g}_{\tau\alpha} \not\subset \mathfrak{p}$ because otherwise $X_{-\alpha}$ would be contained in $\mathfrak{g}^{\tau\theta} + \mathfrak{p}$. Hence, $\mathfrak{g}_{-\tau\alpha} \subset \mathfrak{u}$ and $\tau\alpha \neq \alpha$.

Take a non-zero $X_\alpha \in \mathfrak{g}_\alpha$ and we set $X := X_\alpha + \tau\theta X_\alpha \in \mathfrak{g}_\alpha + \mathfrak{g}_{-\tau\alpha} \subset \mathfrak{u}$.

Case 1. Suppose $X \neq 0$. Let $Y := \mathrm{pr}_\tau(X)$. Clearly, $\theta Y = Y$. Moreover, $Y \neq 0$ because $\tau\alpha \neq \alpha$. This means that $\mathrm{pr}_\tau(\mathfrak{u})$ contains a non-zero semisimple element.

Case 2. Suppose $X = 0$. Let $Y := X_\alpha + \tau X_\alpha = X_\alpha - \theta X_\alpha$. Then $Y \neq 0$ and $\theta Y = -Y$. Again, this means that $\mathrm{pr}_\tau(\mathfrak{u})$ contains a non-zero semisimple element.

Thus we have proved the contraposition, "not (i) $\Longrightarrow$ not (iii)". Hence the proof of Proposition has been completed. □

The nilradical of the Lie algbra $\mathfrak{p}^\tau$ is given explicitly as follows:

Proposition 4.7. *Under the equivalent conditions (i)-(iii) in Proposition 4.6, $\mathfrak{p}^\tau$ is a parabolic subalgebra of $\mathfrak{g}^\tau$ having the following Levi decomposition:*

$$\mathfrak{p}^\tau = \mathfrak{l}^\tau + \mathrm{pr}_\tau(\mathfrak{u}).$$

Proof of Proposition 4.7. We take a Borel subalgebra $\mathfrak{b} \subset \mathfrak{p}$ such that $G^\tau B$ is closed, and a τ-stable Cartan subalgebra $\mathfrak{j}$ contained in $\mathfrak{b}$ as in the proof of Proposition 4.6.

Given a $\mathfrak{j}$-stable subspace $V = \bigoplus_{\alpha \in \Delta(V)} \mathfrak{g}_\alpha$ in $\mathfrak{g}$, we denote by $\Delta(V)$ the multiset of $\mathfrak{j}$-weights. (Here we note that the multiplicity of the zero weight in V may be larger than one.) We divide $\Delta(V)$ into the disjoint union

$$\Delta(V) = \Delta(V)_{\mathrm{I}} \amalg \Delta(V)_{\mathrm{II}} \amalg \Delta(V)_{\mathrm{III}},$$

subject to the condition (I) $\tau\alpha = \alpha$ and $\tau|_{\mathfrak{g}_\alpha} = \mathrm{id}$, (II) $\tau\alpha = \alpha$ and $\tau|_{\mathfrak{g}_\alpha} = -\mathrm{id}$, and (III) $\tau\alpha \neq \alpha$. Accordingly, we have a direct sum as vector spaces:

$$V^\tau = \bigoplus_{\alpha\in\Delta(V)_{\mathrm{I}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha,\tau\alpha\in\Delta(V)_{\mathrm{III}}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\tau\alpha})^\tau,$$
$$\mathrm{pr}_\tau(V) = \bigoplus_{\alpha\in\Delta(V)_{\mathrm{I}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in\Delta(V)_{\mathrm{III}}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\tau\alpha})^\tau.$$

In particular, we get

$$\begin{aligned}\mathfrak{p}^\tau &= \bigoplus_{\alpha\in\Delta(\mathfrak{p})_{\mathrm{I}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha,\tau\alpha\in\Delta(\mathfrak{p})_{\mathrm{III}}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\tau\alpha})^\tau \\ &= \bigoplus_{\alpha\in\Delta(\mathfrak{l})_{\mathrm{I}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha,\tau\alpha\in\Delta(\mathfrak{l})_{\mathrm{III}}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\tau\alpha})^\tau \oplus \bigoplus_{\alpha\in\Delta(\mathfrak{u})_{\mathrm{I}}} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha,\tau\alpha\in\Delta(\mathfrak{u})_{\mathrm{III}}} (\mathfrak{g}_\alpha + \mathfrak{g}_{\tau\alpha})^\tau \\ &= \mathfrak{l}^\tau \oplus \mathrm{pr}_\tau(\mathfrak{u}).\end{aligned}$$

Here we have used $\tau\mathfrak{u} \subset \mathfrak{p}$ in the second equality. Thus Proposition 4.7 is proved. □

4.3 Application of associated varieties to restrictions

In this subsection, we apply associated varieties of $\mathfrak{g}$-models to the study of branching problems.

Suppose X is a finitely generated $\mathfrak{g}$-module. We take a finite dimensional subspace X_0 which generates X as a $\mathfrak{g}$-module. Let $U(\mathfrak{g}) = \cup_{k\geq 0} U_k(\mathfrak{g})$ be a natural filtration of the enveloping algebra of $\mathfrak{g}$. Then, $X_k := U_k(\mathfrak{g})X_0$ ($k \in \mathbb{N}$) gives a filtration $\{X_k\}_k$ satisfying

$$X = \bigcup_{k=0}^{\infty} X_k, \qquad U_i(\mathfrak{g})X_j = X_{i+j} \ (i,j \geq 0).$$

Then, $\mathrm{gr}\, X := \bigoplus_{k=0}^{\infty} X_k/X_{k-1}$ is a finitely generated module of the commutative algebra $\mathrm{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g})$. The *associated variety* of the $\mathfrak{g}$-module X is a closed subset $\mathcal{V}_\mathfrak{g}(X)$ of $\mathfrak{g}^*$ defined by

$$\mathcal{V}_\mathfrak{g}(X) := \mathrm{Supp}_{S(\mathfrak{g})}(\mathrm{gr}\, X).$$

Then $\mathcal{V}_\mathfrak{g}(X)$ is independent of the choice of the generating subspace X_0. We recall the following basic properties:

Lemma 4.8 ([6, Chapter 17])**.** 1) *If* $0 \longrightarrow X_1 \longrightarrow X \longrightarrow X_2 \longrightarrow 0$ *is an exact sequence of* $\mathfrak{g}$*-modules, we have* $\mathcal{V}_{\mathfrak{g}}(X) = \mathcal{V}_{\mathfrak{g}}(X_1) \cup \mathcal{V}_{\mathfrak{g}}(X_2)$.
2) *For any finite dimensional* $\mathfrak{p}$*-module* F, $\mathcal{V}_{\mathfrak{g}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F) = \mathfrak{p}^{\perp}$.

Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$, and $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'} : \mathfrak{g}^* \to \mathfrak{g}'^*$ the restriction map. We set $\mathfrak{p}' := \mathfrak{g}' \cap \mathfrak{p}$ and $\mathfrak{p}'^{\perp} := \{\lambda \in (\mathfrak{g}')^* : \lambda|_{\mathfrak{p}'} \equiv 0\}$.

Lemma 4.9. *Let* X *be a simple* $\mathfrak{g}$*-module lying in* $\mathcal{O}^{\mathfrak{p}}$.
1) *If* Y *is a simple* $\mathfrak{g}'$*-module such that* $\mathrm{Hom}_{\mathfrak{g}'}(Y, X|_{\mathfrak{g}'}) \neq \{0\}$ *then*

$$\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{V}_{\mathfrak{g}'}(Y) \subset (\mathfrak{p}')^{\perp}. \tag{4.3}$$

2) *If* Y_i *are simple* $\mathfrak{g}'$*-modules such that* $\mathrm{Hom}_{\mathfrak{g}'}(Y_i, X|_{\mathfrak{g}'}) \neq \{0\}$ $(i = 1, 2)$*, then* $\mathcal{V}_{\mathfrak{g}'}(Y_1) = \mathcal{V}_{\mathfrak{g}'}(Y_2)$.

Proof. 1) Since $\mathcal{O}^{\mathfrak{p}}$ is closed under tensor products with finite dimensional representations, the proof for the first inclusion in (4.3) parallels to the proof of [8, Theorem 3.1] by using the double filtration of X.

For the second inclusion in (4.3), we use the notation of the proof of Proposition 3.5 and let Y be the image of (3.1). Then it follows from Lemma 4.8 that $\mathcal{V}_{\mathfrak{g}'}(Y) \subset \mathcal{V}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} (F_\lambda|_{\mathfrak{p}'})) = \mathfrak{p}'^{\perp}$.
2) The proof is the same as that of [8, Theorem 3.7] in the category of $(\mathfrak{g}, K)$-modules. □

Remark 4.10. An analogous result to Lemma 4.9 2) was proved in [4] in the special case where X is the oscillator representation of $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{R})$ in the context of compact dual pair correspondence. In this case, $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(X))$ coincides with the associated variety $\mathcal{V}_{\mathfrak{g}'}(Y)$. It is plausible that $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}'}(\mathcal{V}_{\mathfrak{g}}(X)) = \mathcal{V}_{\mathfrak{g}'}(Y)$ in the generality of the setting in Lemma 4.9. We shall prove this assertion in Theorem 4.11 below for symmetric pairs $(\mathfrak{g}, \mathfrak{g}^{\tau})$.

4.4 Proof of Theorem 4.1

The equivalence of Theorem 4.1 has been already proved in Section 4.1 except for the implication (ii) ⇒ (iii). We are ready to complete the proof.

Proof of Theorem 4.1, (ii) ⇒ (iii). By the Killing form, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Then the projection $\mathrm{pr}_{\mathfrak{g}\to\mathfrak{g}^{\tau}} : \mathfrak{g}^* \to (\mathfrak{g}^{\tau})^*$ is given as the map $\mathrm{pr}_{\tau} : \mathfrak{g} \to \mathfrak{g}^{\tau}$ (see (4.1)). Further, $\mathfrak{p}^{\perp} = \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{p}} \equiv 0\}$ is isomorphic to the nilpotent radical $\mathfrak{u}$ of the parabolic subalgebra $\mathfrak{p}$.

We take a generalized Verma module $X := M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ with generic parameter $\lambda \in \Lambda^+(\mathfrak{l})$ (cf. (2.2)). Then it follows from Lemma 4.8 that $\mathcal{V}_{\mathfrak{g}}(X) = \mathfrak{u}$. Therefore, if the restriction $X|_{\mathfrak{g}^\tau}$ is discretely decomposable, then $\mathrm{pr}_\tau(\mathfrak{u})$ consists of nilpotent elements by Lemma 4.9. In turn, $G^\tau P$ is closed in G owing to Proposition 4.6. Thus, the proof of Theorem 4.1 is completed. □

4.5 Associated varieties of irreducible summands

We retain the previous notation: $\mathfrak{p}$ is a parabolic subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$, and τ an involutive automorphism of $\mathfrak{g}$. In this subsection, we give an explicit formula for the associated variety $\mathcal{V}_{\mathfrak{g}^\tau}(Y)$ and the Gelfand–Kirillov dimension $\mathrm{DIM}(Y)$ of irreducible summands Y.

Theorem 4.11. *Suppose* $(\mathfrak{g}, \mathfrak{g}^\tau, \mathfrak{p})$ *satisfies one of* (*hence, all of*) *the equivalent conditions in Theorem 4.1. Let* $X = M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *be a simple generalized Verma module, and* Y *a simple* $\mathfrak{g}'$*-module such that* $\mathrm{Hom}_{\mathfrak{g}'}(Y, X|_{\mathfrak{g}'}) \neq \{0\}$. *Then,*

$$\mathcal{V}_{\mathfrak{g}^\tau}(Y) = \mathrm{pr}_\tau(\mathfrak{u}) \quad \textit{and} \quad \mathrm{DIM}(Y) = \dim \mathfrak{g}^\tau/\mathfrak{p}^\tau.$$

Proof of Theorem 4.11. The nilradical of the parabolic subalgebra $\mathfrak{p}^\tau$ is given by $\mathrm{pr}_\tau(\mathfrak{u})$ in Proposition 4.7. Hence, via the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$, the inclusive relation (4.3) is written as

$$\mathrm{pr}_\tau(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{V}_{\mathfrak{g}^\tau}(Y) \subset \mathrm{pr}_\tau(\mathfrak{u}). \tag{4.4}$$

Since $\mathcal{V}_{\mathfrak{g}}(X) = \mathfrak{u}$, the three terms in (4.4) must be the same, and therefore $\mathcal{V}_{\mathfrak{g}^\tau}(Y) = \mathrm{pr}_\tau(\mathfrak{u})$.

The Gelfand–Kirillov dimension $\mathrm{DIM}(Y)$ is given by the dimension of the associated variety $\mathcal{V}_{\mathfrak{g}^\tau}(Y)$, and thus we have $\mathrm{DIM}(Y) = \dim \mathrm{pr}_\tau(\mathfrak{u})$, which equals $\dim \mathfrak{p}^\tau - \dim \mathfrak{l}^\tau = \dim \mathfrak{g}^\tau - \dim \mathfrak{p}^\tau$ by Proposition 4.7. □

Remark 4.12. There are finitely many G^τ-orbits on the generalized flag variety G/P by [11]. Among them, suppose $G^\tau y_j P$ $(j = 1, 2, \cdots, k)$ are closed in G. Correspondingly we realize $\mathfrak{g}^\tau$ as a subalgebra of $\mathfrak{g}$ by

$$\iota_j : \mathfrak{g}^\tau \hookrightarrow \mathfrak{g}, \quad Z \mapsto \mathrm{Ad}(y_j)^{-1}(Z).$$

Then $(\mathfrak{g}, \iota_j(\mathfrak{g}^\tau))$ form symmetric pairs defined by the involutions $\tau_j := \mathrm{Ad}(y_j^{-1}) \circ \tau \circ \mathrm{Ad}(y_j) \in \mathrm{Aut}(\mathfrak{g})$. Theorem 4.1 implies that the restrictions $X|_{\iota_j(\mathfrak{g}^\tau)}$ are discretely decomposable for any $X \in \mathcal{O}^{\mathfrak{p}}$ and for any j ($j = 1, \dots, k$). Obviously, the Lie algebras $\iota_j(\mathfrak{g}^\tau)$ are isomorphic to each other, but $\dim(\mathfrak{p} \cap \iota_j(\mathfrak{g}^\tau))$ may differ. Accordingly, the Gelfand–Kirillov dimension of simple summands in the restrictions $X|_{\iota_j(\mathfrak{g}^\tau)}$ depends on j. See Examples 4.13 and 4.14 below.

Example 4.13 ($A_{p+q-1} \downarrow A_{p-1} \times A_{q-1}$)**.** *Let $p, q \geq 2$, $\mathfrak{g} = \mathfrak{sl}_{p+q}(\mathbb{C})$, $\mathfrak{p}$ its parabolic subalgebra whose nilradical is the Heisenberg Lie algebra of dimension $2(p+q)-3$, and $\mathfrak{g}' = \mathfrak{s}(\mathfrak{gl}_p(\mathbb{C}) \oplus \mathfrak{gl}_q(\mathbb{C}))$. Then, there are four injective homomorphisms $\iota_j : \mathfrak{g}' \to \mathfrak{g}$ $(1 \leq j \leq 4)$ such that each ι_j induces closed G'-orbits on G/P. Let $\tau_j \in \mathrm{Aut}(\mathfrak{g})$ be defined as in Remark 4.12. It turns out that $\mathfrak{p}$ is τ_j-compatible for all j. Further, the Gelfand–Kirillov dimension is given by*

$$\mathrm{DIM}(Y) = \begin{cases} p+q-2 & (j = 1, 2) \\ 2p-3 & (j = 3) \\ 2q-3 & (j = 4) \end{cases}$$

for any simple $\mathfrak{g}'$-module Y and for any simple generalized Verma module $X = M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ such that $\mathrm{Hom}_{\mathfrak{g}'}(Y, X|_{\iota_j(\mathfrak{g}')}) \neq 0$.

Example 4.14 ($C_n \downarrow A_n$)**.** *Let $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C})$ (complex symplectic Lie algebra of rank n), $\mathfrak{p}$ the Siegel parabolic subalgebra, and $\mathfrak{g}' = \mathfrak{gl}_n(\mathbb{C})$. Then there are $(n+1)$ injective homomorphisms $\iota_j : \mathfrak{g}' \to \mathfrak{g}$ $(0 \leq j \leq n)$ such that each ι_j induces closed $GL_n(\mathbb{C})$-orbits on $Sp(n, \mathbb{C})/P$ and that*

$$\mathrm{DIM}(Y) = j(n-j)$$

for any simple $\mathfrak{g}'$-module if Y occurs in the restriction $X|_{\iota_j(\mathfrak{g}')}$ where X is any simple generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$.

Sketch of the proof. We take ι_j so that $\mathfrak{g}_{\mathbb{R}} \simeq \mathfrak{sp}(n, \mathbb{R})$ and $\mathfrak{g}_{\mathbb{R}} \cap \iota_j(\mathfrak{g}') \simeq \mathfrak{u}(j, n-j)$ with the notation as will be explained in Subsection 5.1. □

4.6 Finite multiplicity theorem

The multiplicities in branching laws behave much simpler in the category $\mathcal{O}$ than those in the context of unitary representations (see Example 4.17 below).

Here is a finite multiplicity theorem in the category $\mathcal{O}$.

Theorem 4.15 (finite multiplicity theorem)**.** *Let τ be an involutive automorphism of a complex semisimple Lie algebra $\mathfrak{g}$. Then*

$$\dim \mathrm{Hom}_{\mathfrak{g}^\tau}(Y, X|_{\mathfrak{g}^\tau}) < \infty \tag{4.5}$$

for any simple $\mathfrak{g}$-module X in the category $\mathcal{O}$ and any simple $\mathfrak{g}^\tau$-module Y.

Proof of Theorem 4.15. Suppose $\mathrm{Hom}_{\mathfrak{g}^\tau}(Y, X|_{\mathfrak{g}^\tau}) \neq 0$ for some X and Y. Denote $\mathfrak{b}$ by the Borel subalgebra that defines the category $\mathcal{O}$. Then it follows from Theorem 4.1 and Lemma 4.5 that $\mathfrak{b}$ is $\mathfrak{g}^\tau$-compatible.

We now apply Theorem 3.10 to the $\mathfrak{g}^\tau$-compatible Borel subalgebra $\mathfrak{b}$, and conclude that $\mathrm{Hom}_{\mathfrak{g}^\tau}(Y, M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)) < \infty$ for any Verma module $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$.

Since any simple $\mathfrak{g}$-module $X \in \mathcal{O}$ is obtained as the quotient of a Verma module, (4.5) follows. □

Remark 4.16. We recall that Theorem 3.10 counts the multiplicities in the subquotients. Therefore, the multiplicities of Y occurring in the restriction $X|_{\mathfrak{g}^\tau}$ as *subquotients* are also finite.

Theorem 4.15 should be compared with the fact that the multiplicities are often infinite in the branching laws of the restriction of an irreducible unitary representation with respect to a semisimple symmetric pair (see [9]):

Example 4.17. *There exists an irreducible unitary representation π of $G = SO(5, \mathbb{C})$ and two irreducible unitary representations Y_1 and Y_2 of the subgroup $G' = SO(3, 2)$ satisfying the following three conditions:*

(1) $0 < \dim \mathrm{Hom}_{G'}(Y_1, \pi|_{G'}) < \infty$.

(2) $\dim \mathrm{Hom}_{G'}(Y_2, \pi|_{G'}) = \infty$.

(3) $\mathrm{DIM}(Y_1) = 3$, $\mathrm{DIM}(Y_2) = 4$.

Here, $\mathrm{Hom}_{G'}(\cdot, \cdot)$ *denotes the space of continuous G'-intertwining operators, and* $\mathrm{DIM}(Y)$ *stands for the Gelfand–Kirillov dimension of the underlying $(\mathfrak{g}', K')$-module of the unitary representation Y of G'.*

5 Multiplicity-free branching laws

In this section we prove two multiplicity-free theorems for the restriction of generalized Verma modules with respect to symmetric pairs $(\mathfrak{g}, \mathfrak{g}')$:

- $\mathfrak{p}$ is special and $(\mathfrak{g}, \mathfrak{g}')$ is general (Theorem 5.1),
- $\mathfrak{p}$ is general and $(\mathfrak{g}, \mathfrak{g}')$ is special (Theorem 5.4).

Correspondingly, explicit branching laws are also derived (Theorems 5.2, 5.5, 5.6, and 5.7).

5.1 Parabolic subalgebra with abelian nilradical

We begin with multiplicity-free branching laws of the restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ with respect to symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ in the case where $\mathfrak{p}$ is a certain maximal parabolic subalgebra.

An abstract feature of the results here boils down to the following:

Theorem 5.1. *Suppose $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_+$ is a parabolic subalgebra such that the nilradical $\mathfrak{u}_+$ is abelian. Then for any involutive automorphism τ such that $\tau\mathfrak{p} = \mathfrak{p}$, the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ of scalar type is decomposed into a multiplicity-free direct sum of simple $\mathfrak{g}^\tau$-modules if $\lambda \in \Lambda^+(\mathfrak{l})$ is sufficiently negative, i.e. $\langle \lambda, \alpha \rangle \ll 0$ for all $\alpha \in \Delta(\mathfrak{u}_+)$.*

Theorem 5.1 is deduced from an explicit formula of the irreducible decomposition. To give its description, we write $\mathfrak{g} = \mathfrak{u}_- + \mathfrak{l} + \mathfrak{u}_+$ for the Gelfand–Naimark decomposition, and take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{l}$ such that $\mathfrak{l}^\tau$ contains $\mathfrak{j}^\tau$ as a maximal abelian subspace (see (4.2) for notation). Let $\Delta(\mathfrak{u}_-^{-\tau}, \mathfrak{j}^\tau)$ be the set of weights of $\mathfrak{u}_-^{-\tau}$ with respect to $\mathfrak{j}^\tau$. The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu_1, \cdots, \nu_k\}$ in $\Delta(\mathfrak{u}_-^{-\tau}, \mathfrak{j}^\tau)$ inductively as follows: ν_j is the highest root among the elements in $\Delta(\mathfrak{u}_-^{-\tau}, \mathfrak{j}^\tau)$ that are strongly orthogonal to $\nu_1, \cdots, \nu_{j-1}$ $(1 \leq j \leq k-1)$. The cardinality k coincides with the split rank of the semisimple symmetric space $G_{\mathbb{R}}/G_{\mathbb{R}}^\tau$.

Then we have

Theorem 5.2. *Suppose that $\mathfrak{p}$ and τ are as in Theorem 5.1. Then, for any sufficiently negative λ, the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ decomposes into a multiplicity-free direct sum of generalized Verma modules of $\mathfrak{g}^\tau$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau} \simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_l \geq 0 \\ a_1, \cdots, a_l \in \mathbb{N}}} M_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(\lambda|_{\mathfrak{j}^\tau} + \sum_{j=1}^{l} a_j \nu_j). \tag{5.1}$$

Proof of Theorem 5.2. Suppose that $\mathfrak{p}$ is a parabolic subalgebra such that its nilradical $\mathfrak{u}_+$ is abelian. Then $\mathfrak{p}$ is automatically a maximal parabolic subalgebra. Further, it follows from [13] that there exists a real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$ such that $G_{\mathbb{R}}/(G_{\mathbb{R}} \cap P)$ is a Hermitian symmetric space of non-compact type, where $G_{\mathbb{R}}$ is the connected real form of $G = \mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}_{\mathbb{R}}$. The group $K_{\mathbb{R}} := G_{\mathbb{R}} \cap P$ is a maximal compact subgroup of $G_{\mathbb{R}}$, and the complexification of its Lie algebra gives a Levi part, denoted by $\mathfrak{l}$, of $\mathfrak{p}$.

Let θ be the involution of $\mathfrak{g}$ defined by

$$\theta|_{\mathfrak{l}} = \mathrm{id}, \quad \theta|_{\mathfrak{u}_- + \mathfrak{u}_+} = -\mathrm{id}\,.$$

Then, θ stabilizes $\mathfrak{g}_{\mathbb{R}}$ and $\mathfrak{p}$, and the restriction $\theta|_{\mathfrak{g}_{\mathbb{R}}}$ is a Cartan involution of the real semisimple Lie algebra $\mathfrak{g}_{\mathbb{R}}$. Since θ commutes with τ, $\tau\theta$ defines another involution of $\mathfrak{g}$. We use the same symbol to denote its lift to the group G. Then $K_{\mathbb{R}}^{\tau\theta} = G_{\mathbb{R}}^{\tau\theta} \cap P$ is a maximal compact subgroup of $G_{\mathbb{R}}^{\tau\theta}$, and has a complexified Lie algebra $\mathfrak{l}^\tau$. Further, $G_{\mathbb{R}}^{\tau\theta}/(G_{\mathbb{R}}^{\tau\theta} \cap P) = G_{\mathbb{R}}^{\tau\theta}/K_{\mathbb{R}}^{\tau\theta}$ becomes also a Hermitian symmetric space whose holomorphic tangent space at the origin is identified with $\mathfrak{u}_-^{-\tau}$. It then follows from W. Schmid [14] that the symmetric algebra $S(\mathfrak{u}_-^{-\tau})$ decomposes into the multiplicity-free sum of simple $\mathfrak{l}^\tau$-modules as

$$S(\mathfrak{u}_-^{-\tau}) \simeq \bigoplus_{\delta \in D} F'_\delta,$$

where δ is the highest weight of F'_δ and

$$D := \{\sum_{j=1}^{k} a_j \nu_j : a_1 \geq \cdots \geq a_k \geq 0,\ a_1, \ldots, a_k \in \mathbb{N}\}.$$

Applying Corollary 3.12, we see that the identity (5.1) holds in the Grothendieck group of $\mathfrak{g}^\tau$-modules. Finally, let us show that the restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ decomposes as a direct sum of $\mathfrak{g}^\tau$-modules as given in (5.1) if λ is sufficiently negative.

For this, let $\widetilde{G_{\mathbb{R}}}$ be the universal covering group of $G_{\mathbb{R}}$, and $\widetilde{K_{\mathbb{R}}}$ that of $K_{\mathbb{R}}$. Then the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is isomorphic to the underlying $(\mathfrak{g}, \widetilde{K_{\mathbb{R}}})$-module of a highest weight representation of $\widetilde{G_{\mathbb{R}}}$ which is unitarizable if $\langle \lambda, \alpha \rangle \ll 0$ for any $\alpha \in \Delta(\mathfrak{u}_+)$. Hence, the identity (5.1) in the Grothendieck group holds as $\mathfrak{g}^\tau$-modules. □

Remark 5.3. As we have seen in the above proof, Theorems 5.1 and 5.2 are equivalent to the theorems on branching laws of unitary highest weight representations of a real semisimple Lie group $\widetilde{G_{\mathbb{R}}}$. In the latter formulation, the corresponding results were previously proved in [9, Theorem B] by a geometric method based on reproducing kernels and 'visible actions' on complex manifolds. See also [9, Theorem 5.3].

5.2 Multiplicity-free pairs

Next, we consider multiplicity-free branching laws of the restriction $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ in the case where $\mathfrak{p} = \mathfrak{b}$ (Borel subalgebra). In general, the 'smaller' the parabolic subalgebra $\mathfrak{p}$ is, the 'larger' the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ becomes. Hence, we expect that the multiplicity-free property of the restriction $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ in the extreme case $\mathfrak{p} = \mathfrak{b}$ should give the strongest constraints on the pair $(\mathfrak{g}, \mathfrak{g}^\tau)$. In this subsection, we determine for which symmetric pair $(\mathfrak{g}, \mathfrak{g}^\tau)$ the restriction $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ is still multiplicity-free.

Before stating a theorem, we recall from Corollary 4.2 that any simple $\mathfrak{g}$-module in $\mathcal{O}$ contains at least one simple $\mathfrak{g}^\tau$-module if and only if $G^\tau B$ is closed in G, or equivalently, $\mathfrak{b}$ is τ-stable.

Theorem 5.4. *Let $\mathfrak{g}$ be a complex simple Lie algebra, and $(\mathfrak{g}, \mathfrak{g}^\tau)$ a complex symmetric pair. Then the following three conditions are equivalent:*

(i) *$(\mathfrak{g}, \mathfrak{g}^\tau)$ is isomorphic to $(\mathfrak{sl}_{n+1}(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C}))$ or $(\mathfrak{so}_{n+1}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$.*

(ii) *For any τ-stable Borel subalgebra $\mathfrak{b}$, the restriction $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ is multiplicity-free as $\mathfrak{g}^\tau$-modules for any generic λ.*

(iii) *The restriction $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ is multiplicity-free as $\mathfrak{g}^\tau$-modules for some λ and some τ-stable Borel subalgebra $\mathfrak{b}$.*

Proof of Theorem 5.4. (i) $\Longrightarrow$ (ii). We shall give an explicit branching law of the restriction $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)$ with respect to the symmetric $(\mathfrak{g}, \mathfrak{g}^\tau)$ which is isomorphic to $(\mathfrak{sl}_{n+1}(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C}))$ or $(\mathfrak{so}_{n+1}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$ in Subsections 5.3–5.5.

(ii) $\Longrightarrow$ (iii). Obvious.

(iii) $\Longrightarrow$ (i). We take a τ-stable Levi decomposition $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$. Then, it follows from Theorem 3.10 that $M_{\mathfrak{b}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau}$ is multiplicity-free only if $S(\mathfrak{n}^{-\tau})$ is multiplicity-free as a $\mathfrak{j}^\tau$-module. In turn, this happens only if the weights of $\mathfrak{n}^{-\tau}$ are linearly independent over $\mathbb{Q}$, which leads us to the following inequality

$$\dim \mathfrak{n}^{-\tau} \leq \dim \mathfrak{j}^\tau,$$

or equivalently,

$$\dim \mathfrak{g} - \dim \mathfrak{g}^\tau \leq \operatorname{rank} \mathfrak{g} + \operatorname{rank} \mathfrak{g}^\tau. \tag{5.2}$$

In view of the classification of complex symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ with $\mathfrak{g}$ simple, the inequality (5.2) holds only if $(\mathfrak{g}, \mathfrak{g}^\tau)$ is isomorphic to $(\mathfrak{sl}_{n+1}(\mathbb{C}), \mathfrak{gl}_n(\mathbb{C}))$ or $(\mathfrak{so}_{n+1}(\mathbb{C}), \mathfrak{so}_n(\mathbb{C}))$. □

In Subsections 5.3–5.5, we shall fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and consider B-conjugacy classes of involutions τ instead of considering G^τ-conjugacy classes of Borel subalgebras by fixing τ. With this convention, we shall use the abbreviation $M^{\mathfrak{g}}(\lambda)$ for $M^{\mathfrak{g}}_{\mathfrak{b}}(\lambda)$.

5.3 Branching laws for $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_n$

Let $\mathfrak{g} := \mathfrak{gl}_{n+1}(\mathbb{C})$ and $\mathfrak{g}' := \mathfrak{gl}_1(\mathbb{C}) \oplus \mathfrak{gl}_n(\mathbb{C})$. We observe that there are $(n+1)$ closed $GL_n(\mathbb{C})$-orbits on the full flag variety of $GL_{n+1}(\mathbb{C})$. Correspondingly, there are essentially $n+1$ different settings for discretely decomposable restrictions of the Verma module $M^{\mathfrak{g}}(\lambda)$ to $\mathfrak{g}'$ by Theorems 2.1 and 4.1.

In order to fix notation, let $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}_+$ be the standard Borel subalgebra of consisting of upper triangular matrices in $\mathfrak{g}$, and $\mathfrak{j}$ the Cartan subalgebra consisting of diagonal matrices. For $1 \leq l \leq n+1$, we realize $\mathfrak{g}'$ as a subalgebra of $\mathfrak{g}$ by letting $\iota_l(\mathfrak{g}')$ be the centralizer of the matrix unit E_{ll}. For $k = (k_1, \cdots, \widehat{k_l}, \cdots, k_{n+1}) \in \mathbb{N}^n$, we set

$$\operatorname{ind}_l k := k_1 + \cdots + k_{l-1} - k_{l+1} - \cdots - k_{n+1}.$$

In what follows, $\boxtimes$ denotes the outer tensor product representation of the direct product of Lie algebras.

Theorem 5.5 ($A_n \downarrow A_{n-1}$). *Suppose $\lambda_i - \lambda_j \notin \mathbb{Z}$ for any distinct i, j in $\{1, \cdots, \widehat{l}, \cdots, n+1\}$. Then the restriction of the Verma module of $\mathfrak{g}$ decomposes into a multiplicity-free direct sum of simple Verma modules of $\mathfrak{g}'$.*

$$\begin{aligned} &M^{\mathfrak{gl}_{n+1}}(\lambda)|_{\iota_l(\mathfrak{gl}_1 \oplus \mathfrak{gl}_n)} \\ &\simeq \bigoplus_{k \in \mathbb{N}^n} \mathbb{C}_{\lambda_l + \operatorname{ind}_l k} \boxtimes M^{\mathfrak{gl}_n}(\lambda_1 - k_1, \cdots, \lambda_{l-1} - k_{l-1}, \lambda_{l+1} + k_{l+1}, \cdots, \lambda_{n+1} + k_{n+1}). \end{aligned} \tag{5.3}$$

Proof. We fix l $(1 \leq l \leq n+1)$ once and for all. Let $\tau \equiv \tau_l$ be the involution of $\mathfrak{g}$ such that $\mathfrak{g}^\tau = \iota_l(\mathfrak{g}')$. With our choice of $\mathfrak{j}$, we have $\mathfrak{j}^\tau = \mathfrak{j} \simeq \mathbb{C}^{n+1}$, and the set of characters of $\mathfrak{j}^\tau$ is identified with $\mathbb{C}^{n+1}$. We apply Corollary 3.12 to the $\mathfrak{j}^\tau$-module $\mathfrak{n}_-^{-\tau}$:

$$\mathfrak{n}_-^{-\tau} = \bigoplus_{i=1}^{l-1} \mathfrak{g}_{-e_i+e_l} \oplus \bigoplus_{i=l+1}^{n+1} \mathfrak{g}_{-e_l+e_j}.$$

Extending this to the symmetric algebra $S(\mathfrak{n}_-^{-\tau})$, we have $\mathfrak{j}^\tau$-isomorphism:

$$S(\mathfrak{n}_-^{-\tau}) \simeq \bigoplus_{k \in \mathbb{N}^n} (-k_1, \ldots, -k_{l-1}, \operatorname{ind}_l k, k_{l+1}, \cdots, k_{n+1}).$$

Therefore, the identity (5.3) holds in the Grothendieck group by Corollary 3.12.

Since $\lambda_i - \lambda_j \notin \mathbb{Z}$ for any i, j, the Verma modules appearing in the right-hand side of (5.3) have distinct infinitesimal characters. Therefore, there is no extension among these representations. Hence (5.3) is a direct sum decomposition. □

5.4 Branching laws for $\mathfrak{so}(2n+1) \downarrow \mathfrak{so}(2n)$

Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{g}' = \mathfrak{so}_{2n}(\mathbb{C})$ and G' be the connected subgroup of $G = \operatorname{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{g}'$. Then there are two closed G'-orbits on the full flag variety G/B, which are conjugate to each other by an element of the normalizer $N_G(\mathfrak{g}')$. Thus it follows from Theorem 2.1 that there is essentially the unique triple $(\mathfrak{g}, \mathfrak{g}', \mathfrak{b})$ satisfying the equivalent conditions of Theorem 4.1.

To fix notation, we may and do assume that $\mathfrak{g}' \cap \mathfrak{b}$ contains a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ and that

$$\begin{aligned}\Delta^+(\mathfrak{g}, \mathfrak{j}) =& \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\},\\ \Delta^+(\mathfrak{g}', \mathfrak{j}) =& \{e_i \pm e_j : 1 \leq i < j \leq n\}.\end{aligned}$$

Theorem 5.6 $(B_n \downarrow D_n)$**.** *Suppose $\lambda_i \pm \lambda_j \notin \mathbb{Z}$ for any $1 \leq i < j \leq n$.*

$$M^{\mathfrak{so}_{2n+1}}(\lambda)|_{\mathfrak{so}_{2n}} = \bigoplus_{k \in \mathbb{N}^n} M^{\mathfrak{so}_{2n}}(\lambda - k). \tag{5.4}$$

Proof of Theorem 5.6. Let τ be the involution of $\mathfrak{g}$ such that $\mathfrak{g}' = \mathfrak{g}^\tau$. Applying Corollary 3.12 to the $\mathfrak{j}$-module:

$$S(\mathfrak{n}_-^{-\tau}) = S(\bigoplus_{i=1}^{n} \mathfrak{g}_{-e_i}) \simeq \bigoplus_{k\in\mathbb{N}^n} (-k_1, \cdots, -k_n),$$

we get (5.4) in the Grothendieck group. The assumption $\lambda_i \pm \lambda_j \notin \mathbb{Z}$ assures that every summand in (5.4) is simple. Further, there is no extension among $M^{\mathfrak{so}_{2n}}(\lambda - k)$ because they have a distinct $\mathfrak{Z}(\mathfrak{g}')$-infinitesimal characters. □

5.5 Branching laws for $\mathfrak{so}(2n+2) \downarrow \mathfrak{so}(2n+1)$

Let $\mathfrak{g} = \mathfrak{so}_{2n+2}(\mathbb{C})$ and $\mathfrak{g}' = \mathfrak{so}_{2n+1}(\mathbb{C})$. Then there exists a unique closed G'-orbit on the full flag variety G/B. To fix notation, we suppose that our Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ is defined by the positive system

$$\Delta^+(\mathfrak{g}, \mathfrak{j}) = \{e_i \pm e_j : 1 \leq i < j \leq n+1\},$$

and that $\mathfrak{j}' := \mathfrak{j} \cap \mathfrak{g}'$ is given by $\{H \in \mathfrak{j} : e_{n+1}(H) = 0\}$. Then $\mathfrak{b}' := \mathfrak{b} \cap \mathfrak{g}'$ is a Borel subalgebra of $\mathfrak{g}'$ given by a positive system

$$\Delta^+(\mathfrak{g}', \mathfrak{j}') = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\}.$$

Theorem 5.7 ($D_{n+1} \downarrow B_n$)**.** *Suppose $\lambda_i \pm \lambda_j \notin \mathbb{Z}$ for any $1 \leq i < j \leq n$. We set $\lambda := (\lambda_1, \cdots, \lambda_n)$. Then*

$$M^{\mathfrak{so}_{2n+2}}(\lambda, \lambda_{n+1})|_{\mathfrak{so}_{2n+1}} \simeq \bigoplus_{k\in\mathbb{N}^n} M^{\mathfrak{so}_{2n+1}}(\lambda - k). \tag{5.5}$$

Proof. Let τ be the defining involution of $\mathfrak{g}' = \mathfrak{so}_{2n+1}(\mathbb{C})$. Then

$$\mathfrak{n}_-^{-\tau} = \bigoplus_{i=1}^{n} (\mathfrak{g}_{-e_i+e_{n+1}} + \mathfrak{g}_{-e_i-e_{n+1}})^{-\tau},$$

and hence we have an isomorphism

$$S(\mathfrak{n}_-^{-\tau}) \simeq \bigoplus_{k\in\mathbb{N}^n} (-k_1, \cdots, -k_n)$$

as $\mathfrak{j}'$-modules. Therefore, (5.5) follows from Corollary 3.12. □

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