Monodromy and Kawai-Lewellen-Tye Relations for Gravity Amplitudes

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MONODROMY AND KAWAI-LEWELLEN-TYE RELATIONS FOR GRAVITY AMPLITUDES*

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We are still learning intriguing new facets of the string theory motivated Kawai-Lewellen-Tye (KLT) relations linking products of amplitudes in Yang-Mills theories and amplitudes in gravity. This is very clearly displayed in computations of $\mathcal{N} = 8$ supergravity where the perturbative expansion show a vast number of similarities to that of $\mathcal{N} = 4$ super-Yang-Mills. We will here investigate how identities based on monodromy relations for Yang-Mills amplitudes can be very useful for organizing and further streamlining the KLT relations yielding even more compact results for gravity amplitudes.

Keywords: Amplitudes, Quantum gravity, String theory

1. Introduction

The search for a valid construction of quantum gravity has been on for most of the previous century initiated by Einstein's formulation of General Relativity in 1916 and the quantum mechanics revolution in the 1920ties. Physicists today are still hunting the answers to the ultimate questions, *e.g.* how was the universe formed and how does one comprehend the fabric of space and time? Quantum mechanical corrections to gravity are crucial for the exact answers but the fundamental concepts of such a quantum theory are unfortunately still very dim. In this paper we will investigate how we can learn about an ultimate theory of quantum gravity through studying symmetries in Yang-Mills theories and the links posed between Yang-Mills theories and gravity through string theory.

The combination of a traditional quantization and the extra symmetry introduced by a super-symmetrization of fundamental interactions appeared for a long

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while to be a way out of the troublesome ultraviolet divergences associated with a field theory for gravity. The most famous model is possibly is the one of maximal supersymmetry $\mathcal{N} = 8$ supergravity [1,2]. This theory arises as a low-energy effective description of string theory in four dimensions. Various later arguments based on supersymmetry point to a delay in the onset of ultraviolet divergences due to the extra symmetry [3–6] but it has long been the belief that only string theory should be completely free of UV divergences. However since no explicit ultraviolet divergences have been found so far in the four-dimensional four-graviton amplitude [7–12], the effective field-theory status of $\mathcal{N} = 8$ supergravity and its relation to string theory have been put into questions.

We know that the on-shell S-matrix elements in string theory depend on the scalars parameterizing the (classical) moduli space $E_{7(7)}(\mathbb{R})/(SU(8)/\mathbb{Z}_2)$ and that these are covariant under the discrete U-duality subgroup $E_{7(7)}(\mathbb{Z})$ [13,14]. However in supergravity the S-matrix elements are invariant under the continuous symmetry $E_{7(7)}(\mathbb{R})$ [15–18]. From the string theory viewpoint the relation between the four-dimensional Planck length ℓ_4 and the string scale $\ell_s = \sqrt{\alpha'}$ depend on the (four-dimensional) dilaton $\ell_4^2 = \alpha' y_4$ where $y_4 = g_s^2 \alpha'^3/(R_1 \cdots R_6)$ and R_i are the radii of compactification. The decoupling limit of string amplitudes goes as $\ell_s \to 0$, $1/R_i \to \infty$ and $R_i/\alpha' \to \infty$, keeping the four-dimensional Newton's constant $2\kappa_{(4)}^2 = 2\pi \ell_4^2$ fixed. This limit is singular since in this limit some non-perturbative states become massless and dominate the S-matrix [19,20]. These non-decoupling results do not imply that $\mathcal{N} = 8$ supergravity has perturbative ultraviolet problems however because of the lack of concrete data it has become urgent to clarify the status of the ultraviolet behavior of $\mathcal{N} = 8$ supergravity in four dimensions and its relation to string theory.

In recent years, by a combination of different inputs from string theory, supersymmetry, unitarity and due to remarkable progress in computational capacity, a huge number of amplitudes have been computed [21]. Surprisingly the ultraviolet behavior of $\mathcal{N} = 8$ supergravity occurs explicitly to be identical to the one of $\mathcal{N} = 4$ super-Yang-Mills at least through four loops [4,5,8,9,12,22–26]. These results have made it clear that $\mathcal{N} = 8$ supergravity has a much better perturbative expansion than power-counting naïvely suggests. It is still an open question if the perturbative expansions of the two theories are similar to all loop orders or what in given case will be the first loop order to have a dissimilarity. These and other aspects are discussed further in ref. [27].

Motivated by string theory [28] where the massless spectrum of $\mathcal{N} = 8$ supergravity can be factorized as the tensorial product of two copies of $\mathcal{N} = 4$ super-Yang-Mills theories, one can organize $\mathcal{N} = 8$ supergravity tree-level amplitudes according to the KLT relations [8,21,28–32] which we will write schematically in the following way

$$\mathcal{M}_{\text{Gravity}}^{\text{tree}} \sim \sum_{ij} K_{ij} \mathcal{A}_{\text{Yang−Mills}}^{i\,L} \times \mathcal{A}_{\text{Yang−Mills}}^{j\,R}. \tag{1}$$

Here $\mathcal{M}_{\text{Gravity}}$, $\mathcal{A}^{i\,L}_{\text{Yang-Mills}}$, $\mathcal{A}^{j\,R}_{\text{Yang-Mills}}$ are gravity and color ordered Yang-Mills amplitudes and K_{ij} is a specific function of kinematic invariants needed to ensure that the tree-level gravity amplitude has the correct analytic structure.

The simple KLT relations between theories of gravity and two gauge theories are observed directly in on-shell S-matrix elements but have no motivation at the Lagrangian level (This is true even if part of the Lagrangian is rearranged as a product of Yang-Mills types of interactions at the two-derivative level [32–34] or for higher derivative corrections [35]. In the case of pure gravity one needs to take into account the contribution from the dilaton in employing the KLT relations.)

Because of their high degree of supersymmetry both $\mathcal{N} = 4$ super-Yang-Mills and $\mathcal{N} = 8$ supergravity loop amplitudes are cut constructible in $D = 4 - 2\epsilon$ dimensions and surprisingly the knowledge of the tree-level amplitudes is enough for reconstructing the full higher-loop amplitudes [9,10,12,15,36].

We will here discuss tree amplitudes from the point of view of the classical $\mathcal{N} = 8$ theory, which can be constructed from the $\mathcal{N} = 4$ super-Yang-Mills tree-level amplitudes using the KLT relation in (1). (For effective theories of gravity [30] one can also employ KLT relations in a slightly modified fashion taking into account higher derivative operators introduced through counterterms to ultraviolet divergences.)

We will next discuss the construction of tree-level amplitudes in Yang-Mills and gravity from a minimal basis of amplitudes following [37].

2. Minimal basis for Yang-Mills and Gravity tree-level amplitudes

The n-point amplitude in open string theory with $U(N)$ gauge group reads

$$\mathcal{A}_n = i g_{\text{YM}}^{n-2} (2\pi)^D \, \delta^D(k_1 + \cdots + k_n) \sum_{(a_1,\ldots,a_n) \in S_n/\mathbb{Z}_n} \text{tr}(T^{a_1} \cdots T^{a_n}) \, \mathcal{A}_n(a_1, \cdots, a_n), \quad (2)$$

where D is any number of dimensions obtained by dimensional reduction from 26 dimensions if we consider the bosonic string, or 10 dimensions in the supersymmetric case. The field theory amplitudes are obtained by taking the limit $\alpha' \to 0$. A new series of amplitude identities between different color-ordered amplitudes based on monodromy for integrations in string theory was derived in [37] (see [38,39] for related discussions). The real part of these relations relates the n-point amplitude with different orderings as

$$\mathcal{A}_n(\beta_1, \ldots, \beta_r, 1, \alpha_1, \ldots, \alpha_s, n) = (-1)^r$$

$$\times \Re\mathfrak{e}\Big[\prod_{1 \leq i < j \leq r} e^{2i\pi\alpha'(k_{\beta_i} \cdot k_{\beta_j})} \sum_{\sigma \subset \text{OP}\{\alpha\} \cup \{\beta^T\}} \prod_{i=0}^{s} \prod_{j=1}^{r} e^{(\alpha_i, \beta_j)} \mathcal{A}_n(1, \sigma, n) \Big]. \quad (3)$$

Here $e^{(\alpha,\beta)} \equiv e^{2i\pi\alpha'(k_\alpha \cdot k_\beta)}$ if $x_\beta > x_\alpha$ and 1 otherwise, α_0 denotes the leg 1 at point 0. The imaginary part give the following amplitude relation

$$0 = \Im\mathfrak{m}\Big[\prod_{1 \leq i < j \leq r} e^{2i\pi\alpha'(k_{\beta_i} \cdot k_{\beta_j})} \sum_{\sigma \subset \text{OP}\{\alpha\} \cup \{\beta^T\}} \prod_{i=0}^{s} \prod_{j=1}^{r} e^{(\alpha_i, \beta_j)} \mathcal{A}_n(1, \sigma, n) \Big]. \quad (4)$$

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We define the $(n-3)!$ color ordered amplitudes $\mathcal{B}_\sigma = \mathcal{A}_n(1, \sigma(2), \cdots, \sigma(n-2), n-1, n)$ with $\sigma \in \mathfrak{S}_{n-3}$ denoting a permutation of the legs $(2, \ldots, n-2)$. As a consequence of (3) and (4) any color ordered amplitudes associated with the permutation σ' of the external legs can be expanded [37]

$$\mathcal{A}_n(\sigma'(1), \cdots, \sigma'(n)) = \sum_{\sigma \in \mathfrak{S}_n} c_{\sigma'}^\sigma \, \mathcal{B}_\sigma \,, \tag{5}$$

where $c_{\sigma'}^\sigma$ are functions of the $\mathcal{S}_{p,q} = \sin(2\pi\alpha' \, p \cdot q)$ and p and q are sums of the external momenta. This implies that $\{\mathcal{B}_\sigma; \sigma \in \mathfrak{S}_{n-3}\}$ provides a minimal basis in which all other color ordered amplitudes can be expanded.

Because the monodromy relations hold for all polarization configurations and any smaller number of dimensions by a trivial dimensional reduction, it follows immediately that they hold for any choice of external legs corresponding to the full $\mathcal{N} = 1$, $D = 10$ supermultiplet and in dimensional reductions thereof [40].

In the case of the four-gluon amplitude one have

$$\mathcal{A}_4(1,2,3,4) = \frac{\Gamma(1-\alpha's)\Gamma(1-\alpha't)}{\Gamma(1-\alpha'u)} \left(\frac{n_s}{s} + \frac{n_t}{t}\right), \tag{6}$$

$$\mathcal{A}_4(1,3,2,4) = \frac{\Gamma(1-\alpha'u)\Gamma(1-\alpha't)}{\Gamma(1-\alpha's)} \left(-\frac{n_u}{u} - \frac{n_t}{t}\right), \tag{7}$$

$$\mathcal{A}_4(2,1,3,4) = \frac{\Gamma(1-\alpha's)\Gamma(1-\alpha'u)}{\Gamma(1-\alpha't)} \left(\frac{n_s}{s} + \frac{n_u}{u}\right), \tag{8}$$

where n_s, n_t and n_u depends on the polarizations and the external momenta.

The monodromy relations (3) and (4)

$$\mathcal{A}_4(1,3,2,4) = \frac{\sin(2\pi\alpha' \, s)}{\sin(2\pi\alpha' \, u)} \mathcal{A}_4(1,2,3,4), \quad \mathcal{A}_4(2,1,3,4) = \frac{\sin(2\pi\alpha' \, t)}{\sin(2\pi\alpha' \, u)} \mathcal{A}_4(1,2,3,4), \tag{9}$$

imply that the numerator factors satisfy the Jacobi like relation $n_s = n_t + n_u$. The generalization to higher points gives the new amplitude relations recently conjectured by Bern et al. in ref. [31]. The string theory monodromy identities for the Kawai-Lewellen-Tye relationship between closed and open string amplitudes give highly symmetric forms for tree-level amplitudes where the tree-level gravity amplitudes are expanded in a basis obtained by the left/right tensorial product of gauge color ordered amplitudes

$$\mathcal{M}_n = \sum_{\sigma, \sigma' \in \mathfrak{S}_{n-3}} \mathcal{G}^{\sigma, \sigma'}(k_i \cdot k_j) \, \mathcal{B}_\sigma^L \mathcal{B}_{\sigma'}^R \,. \tag{10}$$

As a direct application of our procedure, we can rewrite the Kawai-Lewellen-Tye relations at four-point level as

$$\mathcal{M}_4 = \frac{\kappa_{(4)}^2}{\alpha'} \frac{\mathcal{S}_{k_1, k_2} \mathcal{S}_{k_1, k_4}}{\mathcal{S}_{k_1, k_3}} \mathcal{A}_4^L(1,2,3,4) \mathcal{A}_4^R(1,2,3,4). \tag{11}$$

The field theory limit of the string amplitude (11), $\alpha' \to 0$ gives the symmetric form of the gravity amplitudes of [31]

$$\mathcal{M}_4^{FT} = \kappa_{(4)}^2 \frac{st}{u} \left(\frac{n_s}{s} + \frac{n_t}{t}\right) \left(\frac{\tilde{n}_s}{s} + \frac{\tilde{n}_t}{t}\right) = -\kappa_{(4)}^2 \left(\frac{n_s \tilde{n}_s}{s} + \frac{n_t \tilde{n}_t}{t} + \frac{n_u \tilde{n}_u}{u}\right) . \quad (12)$$

Here we have made use of the on-shell relation $s + t + u = 0$ and the four-point Jacobi relation $n_u = n_s - n_t$. Similarly considerations at higher-point order will be detailed in [41].

3. Conclusions

We have discussed the interesting link posed by the Kawai-Lewellen-Tye (KLT) string theory relations between products of amplitudes in Yang-Mills theories and amplitudes in gravity. We here observed how identities based on monodromy relations for Yang-Mills amplitudes and the KLT relations can be employed to yield very compact results for gravity amplitudes. It would be interesting to analyze the role of the monodromies at loop order since this would allow us to further understand the similarities of the perturbative expansion of $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super Yang-Mills.

4. Acknowledgments

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