Absolute algebra III-the saturated spectrum

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ABSOLUTE ALGEBRA
III–THE SATURATED SPECTRUM

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Abstract. We investigate the algebraic and topological preliminaries to a geometry in characteristic 1.

Date: March 15th, 2011.
2000 Mathematics Subject Classification. 06F05, 20M12, 08C99.
Key words and phrases. Characteristic one, spectra, Zariski topologies.
1. Introduction

This work is a sequel to [2] and [3], of which, except when explicitly stated otherwise, we shall keep the definitions and notations. In particular, we shall always equip the spectrum \( \text{Spec}(A) \) of a \( B_1 \)-algebra with the topology defined in [3], Proposition 3.15, and the prime spectrum \( Pr(A) \) of \( A \) with the topology defined in [3], Theorem 2.4.

We shall denote by \( SP \) the category whose objects are these \( B_1 \)-spectra and whose morphisms are the continuous maps between them.

For \( A \) a \( B_1 \)-algebra and \( S \) a subset of \( A \), let \( <S> \) denote the intersection of all the ideals of \( A \) containing \( S \) (there is always at least one such ideal: \( A \) itself). It is clear that \( <S> \) is an ideal of \( A \), and therefore is the smallest ideal of \( A \) containing \( S \). As in ring theory, one may see that

\[
< S > = \left\{ \sum_{j=1}^{n} a_j s_j | n \in \mathbb{N}, (a_1, ..., a_n) \in A^n, (s_1, ..., s_n) \in S^n \right\}.
\]
2. A NEW DESCRIPTION OF MAXIMAL CONGRUENCES

Let $A$ denote a $B_1$–algebra. For $P$ a saturated prime ideal (see [3], p.1786) of $A$, let us define a relation $S_P$ on $A$ by:

$$xS_Py \equiv (x \in P \text{ and } y \in P) \text{ or } (x \notin P \text{ and } y \notin P).$$

Then $S_P$ is a congruence on $A$: if $xS_Py$ and $x'S_Py'$, then one and only one of the following holds:

(i) $x \in P$, $y \in P$, $x' \in P$ and $y' \in P$,

(ii) $x \in P$, $y \in P$, $x' \notin P$ and $y' \notin P$,

(iii) $x \notin P$, $y \notin P$, $x' \in P$ and $y' \in P$,

(iv) $x \notin P$, $y \notin P$, $x' \notin P$ and $y' \notin P$.

In case (i), $x+x' \in P$ and $y+y' \in P$, whence $x+x'S_Py+y'$. In cases (ii) and (iv), $x+x' \notin P$ and $y+y' \notin P$ (as $P$ is saturated), whence $x+x'S_Py+y'$. Case (iii) is symmetrical relatively to case (ii), therefore, in all cases, $x+x'S_Py+y'$:

$xS_Py$ is compatible with addition.

In cases (i), (ii) and (iii), $xx' \in P$ and $yy' \in P$, whence $xx'S_Pyy'$; in case (iv) $xx' \notin P$ and $yy' \notin P$ (as $P$ is prime), whence also $xx'S_Pyy'$: $S_P$ is compatible with multiplication, hence is a congruence on $A$.

As $0 \in P$ and $1 \notin P$, $0 \notin S_P1$, therefore $S_P$ is nontrivial; but each $x \in A$ is either in $P$ (whence $xS_P0$) or not (whence $xS_P1$). It follows that

$$\frac{A}{S_P} = \{0, 1\} \simeq B_1,$$

in particular, $S_P$ is maximal: $S_P \in MaxSpec(A)$.

Obviously, $I(S_P) = P$.

Furthermore, let $(x, y) \in A^2$ be such that $xR_Py$; then there is $z \in P$ such that $x+z = y+z$. If $x \in P$ then $y+z = x+z \in P$, whence $y \in P$ (as $y + (y+z) = y+z$ and $P$ is saturated); symmetrically, $y \in P$ implies $x \in P$, whence the assertions $(x \in P)$ and $(y \in P)$ are equivalent, and $xS_Py$. We have shown that

$$R_P \leq S_P .$$

We shall denote by $\alpha_A$ the mapping

$$\alpha_A : Pr_s(A) \rightarrow MaxSpec(A)$$

$$P \mapsto S_P .$$

Let $R \in MaxSpec(A)$; then $R \in Spec(A)$, whence $I(R)$ is prime; by Theorem 3.8 of [3], it is saturated, i.e. $I(R) \in Pr_s(A)$. Let us set

$$\beta_A(R) := I(R) .$$

**Theorem 2.1.** The mappings

$$\alpha_A : Pr_s(A) \rightarrow MaxSpec(A)$$

and

$$\beta_A : MaxSpec(A) \rightarrow Pr_s(A)$$

are bijections, inverse of one another. They are continuous for the topologies on $Pr_s(A)$ and $MaxSpec(A)$ induced by the Zariski topologies on $Pr(A)$ and $Spec(A)$ defined in [3] (Theorem 2.4, resp. Proposition 3.15).
Proof. Let \( \mathcal{R} \in \text{MaxSpec}(A) \); then
\[
\alpha_A(\beta_A(\mathcal{R})) = \alpha_A(I(\mathcal{R})) = S_I(\mathcal{R}) .
\]

Let us assume \( x \mathcal{R} y \); then, if \( x \in I(\mathcal{R}) \) one has \( x \mathcal{R} 0 \), whence \( y \mathcal{R} 0 \) and \( y \in I(\mathcal{R}) \); by symmetry, \( y \in I(\mathcal{R}) \) implies \( x \in I(\mathcal{R}) \), thus \( (x \in I(\mathcal{R})) \) and \( (y \in I(\mathcal{R})) \) are equivalent, i.e. \( x S_I(\mathcal{R}) y \). We have proved that \( \mathcal{R} \leq S_I(\mathcal{R}) \). As \( \mathcal{R} \) is maximal, we have \( \mathcal{R} = S_I(\mathcal{R}) \), whence
\[
\alpha_A(\beta_A(\mathcal{R})) = S_I(\mathcal{R}) = \mathcal{R} ,
\]
and
\[
\alpha_A \circ \beta_A = \text{Id}_{\text{MaxSpec}(A)} .
\]

Let now \( \mathcal{P} \in \text{Pr}_s(A) \); then
\[
(\beta_A \circ \alpha_A)(\mathcal{P}) = \beta_A(\alpha_A(\mathcal{P}))
= \beta_A(S_{\mathcal{P}})
= I(S_{\mathcal{P}})
= \mathcal{P} ,
\]
whence
\[
\beta_A \circ \alpha_A = \text{Id}_{\text{Pr}_s(A)} ,
\]
and the first statement follows.

Let now \( F \) denote a closed subset of \( \text{Pr}_s(A) \); then \( F = G \cap \text{Pr}_s(A) \) for \( G \) a closed subset of \( \text{Pr}(A) \) and \( G = W(S) := \{ \mathcal{P} \in \text{Pr}(A) | S \subseteq \mathcal{P} \} \) for some subset \( S \) of \( A \). But then, for \( \mathcal{R} \in \text{MaxSpec}(A) \), \( \mathcal{R} \in \beta_A^{-1}(F) \) if and only if \( \beta_A(\mathcal{R}) \in F \), i.e. \( I(\mathcal{R}) \in G \cap \text{Pr}_s(A) \), that is \( I(\mathcal{R}) \in G \), or \( S \subseteq I(\mathcal{R}) \), which means \( \mathcal{R} \in V(S) \). Thus
\[
\beta_A^{-1}(F) = V(S) \cap \text{MaxSpec}(A)
\]
is closed in \( \text{MaxSpec}(A) \). We have shown the continuity of \( \beta_A \).

Let now \( H \subseteq \text{MaxSpec}(A) \) be closed ; then \( H = \text{MaxSpec}(A) \cap \mathcal{L} \) for some closed subset \( \mathcal{L} \) of \( \text{Spec}(A) \) and \( \mathcal{L} = V(T) \) for some subset \( T \) of \( A \). Then a saturated prime ideal \( \mathcal{P} \) of \( A \) belongs to \( \alpha_A^{-1}(H) \) if and only if \( \alpha_A(\mathcal{P}) \in H \), that is
\[
S_{\mathcal{P}} \in \text{MaxSpec}(A) \cap \mathcal{L} ,
\]
i.e.
\[
S_{\mathcal{P}} \in V(T)
\]
or \( T \subseteq I(S_{\mathcal{P}}) \). But \( I(S_{\mathcal{P}}) = \mathcal{P} \) whence \( \mathcal{P} \) belongs to \( \alpha_A^{-1}(H) \) if and only if \( T \subseteq \mathcal{P} \), that is
\[
\alpha_A^{-1}(H) = V(T) \cap \text{MaxSpec}(A) ,
\]
which is closed in \( \text{MaxSpec}(A) \). \( \square \)

Let us consider the special case in which \( A \) is in the image of \( \mathcal{F} : A = \mathcal{F}(M) \), for \( M \) a commutative monoid. Let \( \mathcal{P} \) be a prime ideal of \( M \); as seen in [3], Theorem 4.2, \( \tilde{\mathcal{P}} \) is a saturated prime ideal in \( A \), and one obtains in this way a bijection between \( \text{Spec}_{\mathcal{D}}(M) \) and \( \text{Pr}_s(A) \). The following is now obvious:
Theorem 2.2. The mapping

$$\psi_M : \text{Spec}_D(M) \to \text{MaxSpec}(\mathcal{F}(M))$$

$$P \mapsto \alpha_{\mathcal{F}(M)}(\tilde{P})$$

is a bijection.

Two particular cases are of special interest:

1. $M$ is a group; then $\text{Spec}_D(M) = \{ \emptyset \}$, whence $\text{MaxSpec}(\mathcal{F}(G))$ has exactly one element.

2. $M = C_n := \langle x_1, \ldots, x_n \rangle$ is the free monoid on $n$ variables $x_1, \ldots, x_n$.

Then the elements of $\text{Spec}_D(M)$ are the $(P_J)_{J \subseteq \{1, \ldots, n\}}$, where

$$P_J := \bigcup_{j \in J} x_j C_n$$

(a fact that was already used in [3], Example 4.4). Then

$$\psi_M(P_J) = \alpha_{\mathcal{F}(M)}(\tilde{P}_J) = \tilde{S}_J,$$

whence $x \psi_M(P_J)y$ if and only if either $(x \in \tilde{P}_J$ and $y \in \tilde{P}_J$) or $(x \notin \tilde{P}_J$ and $y \notin \tilde{P}_J$). But we have seen in [3], Theorem 4.5, that $\mathcal{F}(M) = B_1[x_1, \ldots, x_n]$ could be identified with the set of finite formal sums of elements of $M$. Obviously, an element $x$ of $\mathcal{F}(M)$ belongs to $\tilde{P}_J$ if and only if at least one of its components involves at least one factor $x_j (j \in J)$. It is now clear that, using the notation of [3], Definition 4.6 and Theorem 4.7,

$$\psi_M(P_J) = \tilde{J}.$$

We hereby recover the description of $\text{MaxSpec}(B_1[x_1, \ldots, x_n])$ given in [1] (Theorems 4.7, 4.8 and 4.10).

The following result will be useful

Theorem 2.3. Any proper saturated ideal of a $B_1$–algebra $A$ is contained in a saturated prime ideal of $A$.

Proof. Let $J$ be a proper saturated ideal of $A$; as $I(\mathcal{R}_J) = \overline{J} = J \neq A$, $\mathcal{R}_J \neq \mathcal{C}_0(A)$. By Zorn’s Lemma, one has $\mathcal{R}_J \leq \mathcal{R}$ for some $\mathcal{R} \in \text{MaxSpec}(A)$. According to Theorem 1.1, $\mathcal{R} = \alpha(\mathcal{P}) = S_\mathcal{P}$ for a saturated prime ideal $\mathcal{P}$ of $A$, therefore $\mathcal{R}_J \leq S_\mathcal{P}$ and

$$J = \overline{J} = I(\mathcal{R}_J) \subseteq I(\mathcal{S}_\mathcal{P}) = \mathcal{P}. \quad \Box$$
3. Functorial properties of spectra

Let \( \varphi : A \to C \) denote a morphism of \( B_1 \)-algebras, and let \( R \in \text{Spec}(C) \). We define a binary relation \( \tilde{\varphi}(R) \) on \( A \) by:

\[
\forall(a, a') \in A^2 \quad a \varphi(R) a' \equiv \varphi(a) \varphi(R) \varphi(a').
\]

It is clear that \( \tilde{\varphi}(R) \) is a congruence on \( A \), and that

\[
I(\tilde{\varphi}(R)) = \varphi^{-1}(I(R)).
\]

In particular \( I(\tilde{\varphi}(R)) \) is a prime ideal of \( A \), hence \( \tilde{\varphi}(R) \in \text{Spec}(A) \) if \( \tilde{\varphi} \) maps \( \text{Spec}(C) \) into \( \text{Spec}(A) \). Let \( F := V(S) \) be a closed subset of \( \text{Spec}(A) \), and let \( R \in \text{Spec}(C) \); then \( R \in \tilde{\varphi}^{-1}(F) \) if and only if \( \tilde{\varphi}(R) \in F \), that is \( S \subseteq I(\tilde{\varphi}(R)) \), or \( S \subseteq \varphi^{-1}(I(R)) \), i.e. \( \varphi(S) \subseteq I(R) \), or \( R \in V(\varphi(S)) \). Therefore \( \tilde{\varphi}^{-1}(F) = V(\varphi(S)) \) is closed in \( \text{Spec}(C) \); \( \tilde{\varphi} \) is continuous.

Furthermore, for \( \varphi : A \to C \) and \( \psi : C \to D \) one has

\[
\tilde{\varphi} \circ \varphi = \tilde{\psi} \circ \psi : \text{Spec}(D) \to \text{Spec}(A).
\]

It follows that the equations \( H(A) = \text{Spec}(A) \) and \( H(\varphi) = \tilde{\varphi} \) define a contravariant functor \( H \) from \( \mathcal{Z}_n \) to \( \mathcal{S} \).

Let \( J \) denote an ideal in \( C \), and let us assume \( aR_{\varphi^{-1}(J)}a' \); then there is \( x \in \varphi^{-1}(J) \) with \( a + x = a' + x \). Then \( \varphi(x) \in J \) and

\[
\varphi(a) + \varphi(x) = \varphi(a + x) = \varphi(a') + \varphi(x)
\]

whence \( \varphi(a)R_J\varphi(a') \) and \( a\tilde{\varphi}(R_J)a' \). We have established

**Proposition 3.1.** Let \( A \) and \( C \) denote \( B_1 \)-algebras, \( \varphi : A \to C \) a morphism and \( J \) an ideal of \( C \) : then

\[
R_{\varphi^{-1}(J)} \leq \tilde{\varphi}(R_J).
\]

**Theorem 3.2.** Let \( A \) and \( C \) denote two \( B_1 \)-algebras, and \( \varphi : A \to C \) a morphism. Then \( \tilde{\varphi} : \text{Spec}(C) \to \text{Spec}(A) \) maps \( \text{MaxSpec}(C) \) into \( \text{MaxSpec}(A) \), and the diagram

\[
\begin{array}{ccc}
\text{MaxSpec}(C) & \xrightarrow{\tilde{\varphi}} & \text{MaxSpec}(A) \\
\downarrow_{\alpha_C} & & \downarrow_{\alpha_A} \\
\text{Pr}_s(C) & \xrightarrow{\varphi^{-1}} & \text{Pr}_s(A)
\end{array}
\]

commutes.

**Proof.** Let \( \mathcal{P} \in \text{Pr}_s(C) \), then, for all \((a, a') \in A^2 \)

\[
a\tilde{\varphi}(\mathcal{P})a' \iff \varphi(a)\tilde{\varphi}\varphi(a')
\]

\[
\iff (\varphi(a) \in \mathcal{P} \text{ and } \varphi(a') \in \mathcal{P}) \text{or}(\varphi(a) \notin \mathcal{P} \text{ and } \varphi(a') \notin \mathcal{P})
\]

\[
\iff (a \in \varphi^{-1}(\mathcal{P}) \text{ and } a' \in \varphi^{-1}(\mathcal{P})) \text{or}(a \notin \varphi^{-1}(\mathcal{P}) \text{ and } a' \notin \varphi^{-1}(\mathcal{P}))
\]

\[
\iff a\mathcal{S}_{\varphi^{-1}(\mathcal{P})}a'.
\]
Therefore

\[
(\tilde{\varphi} \circ \alpha_C)(\mathcal{P}) = \tilde{\varphi}(\alpha_C(\mathcal{P})) \\
= \tilde{\varphi}(S_P) \\
= S_{\varphi^{-1}(P)} \\
= \alpha_A(\varphi^{-1}(\mathcal{P})) \\
= (\alpha_A \circ \varphi^{-1})(\mathcal{P})
\]

whence \(\tilde{\varphi} \circ \alpha_C = \alpha_A \circ \varphi^{-1}\).

Incidentally we have proved that \(\tilde{\varphi}\) maps \(\text{MaxSpec}(C) = \alpha_C(\text{Pr}_s(C))\) into \(\alpha_A(\text{Pr}_s(A)) = \text{MaxSpec}(A)\), i.e. the first assertion. \(\square\)
4. Nilpotent radicals and prime ideals

The usual theory generalizes without major problem to $B_1$–algebras.

**Theorem 4.1.** In the $B_1$–algebra $A$, let us define

$$\text{Nil}(A) := \{x \in A | (\exists n \geq 1) x^n = 0\}.$$ 

Then $\text{Nil}(A)$ is a saturated ideal of $A$, and one has

$$\bigcap_{P \in Pr(A)} P = \bigcap_{P \in Pr_s(A)} P = \text{Nil}(A).$$

**Proof.** Let $M := \bigcap_{P \in Pr(A)} P$ and $N = \bigcap_{P \in Pr_s(A)} P$. If $x \in \text{Nil}(A)$ and $P \in Pr(A)$, then, for some $n \geq 1$, $x^n = 0 \in P$, whence (as $P$ is prime) $x \in P : \text{Nil}(A) \subseteq M$.

As $Pr_s(A) \subseteq Pr(A)$, we have $M \subseteq N$.

Let now $y \notin \text{Nil}(A)$; then $(\forall n \in N) y^n \neq 0$.

Define

$$E := \{J \in Id_s(A) | (\forall n \geq 0) x^n \notin J\}.$$

This set is nonempty ($\{0\} \in E$) and inductive for $\subseteq$, therefore, by Zorn’s Lemma, there exists a maximal element $P$ of $E$. As $1 = x^0 \notin P$, $P \neq A$.

Let us assume $ab \in P$, $a \notin P$ and $b \notin P$; then $P + Ab$ and $P + Ab$ are saturated ideals of $A$ strictly containing $P$, whence there exists two integers $m$ and $n$ with $x^m \in P + Ab$ and $x^n \in P + Ab$. By definition of the closure of an ideal, there are $u = p_1 + \lambda a \in P + Aa$ and $v = p_2 + \mu b \in P + Ab$ such that $x^m + u = u$ and $x^n + v = v$. Then

$$ub = p_1 b + \lambda(ab) \in P$$

and

$$x^m b + ub = (x^m + u)b = ub,$$

whence, as $P$ is saturated, $x^m b \in P$.

Then

$$x^m v = x^m p_2 + \mu x^m b \in P;$$

as

$$x^{m+n} + x^m v = x^m(x^n + v) = x^m v,$$

we obtain $x^{m+n} \in P$, a contradiction.

Therefore $P$ is prime and saturated and $x = x^1 \notin P$, whence $x \notin N$. We have proved that $N \subseteq \text{Nil}(A)$, whence $M = N = \text{Nil}(A)$. $\square$

**Corollary 4.2.**

$$\text{Nil}(A) = \bigcap_{P \in Pr(A)} P.$$
Proof.

$$\text{Nil}(A) = \bigcap_{\mathcal{P} \in \mathcal{P}(A)} \mathcal{P} \quad \text{(by Theorem 3.1)}$$

$$\subseteq \bigcap_{\mathcal{P} \in \mathcal{P}(A)} \mathcal{P}$$

$$\subseteq \bigcap_{\mathcal{P} \in \mathcal{P}(A)} \mathcal{P}$$

$$= \bigcap_{\mathcal{P} \in \mathcal{P}(A)} \mathcal{P}$$

$$= \text{Nil}(A) \quad \text{(also by Theorem 3.1)}.$$

Definition 4.3. For $I$ an ideal of $A$, we define the root $r(I)$ of $I$ by

$$r(I) := \{ x \in A | (\exists n \geq 1) x^n \in I \}.$$ 

Lemma 4.4. (i) $r(I)$ is an ideal of $A$.

(ii) $r(I) \subseteq r(I)$; in particular, if $I$ is saturated then so is $r(I)$.

(iii) $r(\{0\}) = \text{Nil}(A)$.

Proof. (i) Obviously, $0 \in r(I)$.

If $x \in r(I)$ and $y \in r(I)$, then $x^m \in I$ for some $m \geq 1$ and $y^n \in I$ for some $n \geq 1$, whence

$$x^m + y^n \in I,$$

as $x^j \in I$ for $j \geq m$ and $y^{m+n-1-j} \in I$ for $j \leq m-1$ (as, then, $m+n-1-j \geq n$). Therefore $x + y \in r(I)$.

For $a \in A$, $(ax)^m = a^m x^m \in I$, whence $ax \in r(I)$. Therefore $r(I)$ is an ideal of $A$.

(ii) Let $x \in r(I)$ then there is $u \in r(I)$ such that $x + u = u$, and there is $n \geq 1$ such that $u^n \in I$. Let us show by induction on $j \in \{0, \ldots, n-1\}$ that $u^{n-j}x^j \in I$. This is clear for $j = 0$. Let then $j \in \{0, \ldots, n-1\}$, and assume that $u^{n-j-1}x^j \in I$; then

$$u^{n-j-1}x^j + u^{n-j}x^j = u^{n-j-1}x^j(x + u) = u^{n-j-1}x^j u = u^{n-j}x^j,$$

whence $u^{n-j-1}x^j+1 \in I$. Thus, for $j = n$, we obtain

$$x^n = u^{n-n}x^n \in I,$$

whence $x \in r(I)$. 

If $I$ is saturated, then
\[ r(I) \subseteq \overline{r(I)} \subseteq r(I) \text{ (by the above)} \]
whence $r(I) = \overline{r(I)}$ is saturated.

(iii) That assertion is obvious.

\[ \square \]

**Proposition 4.5.** For each saturated ideal $I$ of the $B_1$-algebra $A$
\[ r(I) = \bigcap_{P \in Pr_s(A) ; I \subseteq P} P . \]

**Remark 4.6.** For $I = \{0\}$, this is part of Theorem 4.1.

**Proof.** Let $x \in r(I)$, and let $P \in Pr_s(A)$ with $I \subseteq P$; then, for some $n \geq 1$ $x^n \in I$, whence $x^n \in P$ and $x \in P$:
\[ r(I) \subseteq \bigcap_{P \in Pr_s(A) ; I \subseteq P} P . \]

Let now $y \in A$, $y \notin r(I)$, and denote by $\pi$ the canonical projection
\[ \pi : A \twoheadrightarrow \tilde{A} := \frac{A}{\mathcal{R}_I} . \]
As $I$ is saturated, one has
\[ \forall n \geq 1 \ y^n \notin \overline{I} , \]
whence
\[ \forall n \geq 1 \ y^n \notin \mathcal{R}_I 0 , \]
or
\[ \forall n \geq 1 \ \pi(y)^n = \pi(y^n) \neq \overline{0} . \]
Therefore $\pi(y) \notin \text{Nil}(\tilde{A})$, whence, according to Theorem 3.1, there exists a saturated prime ideal $\tilde{P}$ of $\tilde{A}$ such that $\pi(y) \notin \tilde{P}$. But then $P := \pi^{-1}(\tilde{P})$ is a saturated prime ideal of $A$ containing $I$ with $y \notin P$, whence
\[ y \notin \bigcap_{P \in Pr_s(A) ; I \subseteq P} P . \]

\[ \square \]
5. Topology of spectra

We can now establish the basic topological properties of the spectra \( Pr_s(A) \) (analogous, in our setting, to Corollary 1.1.8 and Proposition 1.1.10(ii) of [1]).

**Theorem 5.1.** \( Pr_s(A) \) and \( \text{MaxSpec}(A) \) are \( T_0 \) and quasi-compact.

**Proof.** According to Theorem 1.1, \( Pr_s(A) \) and \( \text{MaxSpec}(A) \) are homeomorphic, therefore it is enough to establish the result for \( Pr_s(A) \).

Let \( \mathcal{P} \) and \( \mathcal{Q} \) denote two different points of \( Pr_s(A) \); then either \( \mathcal{P} \not\subseteq \mathcal{Q} \) or \( \mathcal{Q} \not\subseteq \mathcal{P} \). Let us for instance assume that \( \mathcal{P} \not\subseteq \mathcal{Q} \); then \( \mathcal{Q} \not\in W(\mathcal{P}) \); set
\[
O := Pr_s(A) \cap (Pr(A) \setminus W(\mathcal{P})) .
\]
Then \( O \) is an open set in \( Pr_s(A) \), \( \mathcal{Q} \in O \) and, obviously, \( \mathcal{P} \not\in O \). Therefore \( Pr_s(A) \) is \( T_0 \).

Let \( (U_i)_{i \in I} \) denote an open cover of \( Pr_s(A) \):
\[
Pr_s(A) = \bigcup_{i \in I} U_i ;
\]
each \( Pr_s(A) \setminus U_i \) is closed, whence \( Pr_s(A) \setminus U_i = Pr_s(A) \cap W(S_i) \) for some subset \( S_i \) of \( A \). Therefore \( Pr_s(A) \cap (\bigcap_{i \in I} W(S_i)) = \emptyset \), i.e. \( Pr_s(A) \cap W(\bigcup_{i \in I} S_i) = \emptyset \). Therefore \( Pr_s(A) \cap W(< \bigcup_{i \in I} S_i >) = \emptyset \), whence, according to Theorem 2.3, \( < \bigcup_{i \in I} S_i > = A \). Let \( J =< \bigcup_{i \in I} S_i > \); then \( 1 \in J \), hence there is \( x \in J \) such that \( 1 + x = x \). Furthermore, there exist \( n \in \mathbb{N} \), \( (i_1, ..., i_n) \in I^n \), \( x_{i_k} \in S_{i_k} \) and \( (a_1, ..., a_n) \in A^n \) such that \( x = a_1 x_{i_1} + ... + a_n x_{i_n} \). But then
\[
1 + a_1 x_{i_1} + ... + a_n x_{i_n} = a_1 x_{i_1} + ... + a_n x_{i_n}
\]
whence
\[
1 \in \{x_{i_1}, ..., x_{i_n}\} \subseteq \bigcup_{j=1}^n S_{i_j}
\]
and
\[
\bigcup_{j=1}^n S_{i_j} = A .
\]
It follows that
\[
Pr_s(A) \cap W(\bigcup_{j=1}^n S_{i_j}) = \emptyset ,
\]
that is
\[
Pr_s(A) \cap \bigcap_{j=1}^n W(S_{i_j}) = \emptyset ,
\]
or
\[
Pr_s(A) = \bigcup_{j=1}^n U_{i_{i_j}} .
\]
\( Pr_s(A) \) is quasi-compact. \( \square \)

For \( f \in A \), let
\[
D(f) := Pr_s(A) \setminus (Pr_s(A) \cap W(\{f\})
\]
\[
= \{\mathcal{P} \in Pr_s(A) : f \not\in \mathcal{P}\}.
\]
Proposition 5.2. (1) Each $D(f)(f \in A)$ is open and quasi-compact in $Pr_s(A)$.

(2) The family $(D(f))_{f \in A}$ is an open basis for $Pr_s(A)$.

Proof. (1) The openness of $D(f)$ is obvious. Let us assume $D(f) = \bigcup_{i \in I} U_i$, where the $U_i$'s are open sets in $D(f)$. Each $U_i$ can be written as

$$U_i = D(f) \cap V_i,$$

for $V_i$ an open set in $Pr_s(A)$, i.e. $Pr_s(A) \setminus V_i = W(S_i)$ for $S_i$ a subset of $A$. Then

$$D(f) \subseteq \bigcup_{i \in I} V_i = Pr_s(A) \setminus \left(\bigcap_{i \in I} W(S_i)\right),$$

whence

$$Pr_s(A) \cap W\left(\bigcup_{i \in I} S_i\right) \subseteq W(\{f\}),$$

that is, setting

$$S := \bigcup_{i \in I} S_i,$$

$$f \in \bigcap_{P \in W(S) \cap Pr_s(A)} P = \bigcap_{P \in Pr_s(A); S \subseteq P} P.$$

Therefore, by Proposition 2.4, $f \in r(\langle S \rangle)$: there is $n \geq 1$ such that $f^n \in \langle S \rangle$. Thus, there is $g \in \langle S \rangle$ such that $f^n + g = g$; one has $g = \sum_{j=1}^{m} a_j s_j$ for $a_j \in A$, $s_j \in S$; for each $j \in \{1, ..., m\}$, $s_j \in S_{i_j}$ for some $i_j \in I$. Let $S_0 = \{s_1, ..., s_m\}$; then $g \in \langle \bigcup_{j=1}^{m} S_{i_j} \rangle$, whence $f^n \in \langle \bigcup_{j=1}^{m} S_{i_j} \rangle$, and reading the above argument in reverse order with $S$ replaced by $\bigcup_{j=1}^{m} S_{i_j}$ yields that

$$D(f) = \bigcup_{j=1}^{m} U_{i_j},$$

whence the quasi-compactness of $D(f)$.

(2) Let $U$ be an open set in $Pr_s(A)$, and $P \in U$. We have $Pr_s(A) \setminus U = Pr_s(A) \cap W(S)$ for some subset $S$ of $A$. As $P \notin W(S)$, $S \notin P$, whence there is an $s \in S$ with $s \notin P$. It is now clear that $P \in D(s)$ and

$$D(s) \subseteq Pr_s(A) \setminus W(S) = U.\□$$
6. Remarks on the One–generator Case

Let us now consider the case of a nontrivial monogenic $B_1$–algebra containing strictly $B_1$, i.e. $A = \frac{B_1[x]}{x}$ is a quotient of the free algebra $B_1[x]$ with $x \not\sim 0$, $x \sim 1$. Denote by $\alpha$ the image of $x$ in $A$; then $\alpha \not\in \{0, 1\}$, and $\alpha$ generates $A$ as a $B_1$–algebra.

Let us suppose that, for some $(u,v) \in A^2$, $\alpha u = 1 + \alpha v$; then $\alpha$ is not nilpotent, as from $\alpha^n = 0$ would follow $0 = \alpha^n v = \alpha^{n-1}(\alpha v) = \alpha^{n-1}(1+\alpha u) = \alpha^{n-1} + \alpha^n u = \alpha^{n-1}$, whence $\alpha^{n-1} = 0$ and, by induction, $1 = \alpha^0 = 0$, a contradiction.

Therefore three cases may appear

(i) $\alpha$ is nilpotent.

(ii) $\alpha$ is not nilpotent and there does not exist $(u,v) \in A^2$ such that $\alpha u = 1 + \alpha v$.

(iii) $(\alpha$ is not nilpotent) and there exist $(u,v) \in A^2$ such that $\alpha u = 1 + \alpha v$.

In case (i), any prime ideal of $A$ must contain $\alpha$, hence contain $\alpha A$; the ideal $\alpha A$ is, according to the above remark, saturated, and is not contained in a strictly bigger saturated ideal other than $A$ itself (in both cases, as any element of $A$ not in $\alpha A$ is of the shape $1 + \alpha x$). Therefore $Pr_s(A) = \alpha A$, whence $Nil(A) = \alpha A$. In this case we see that

$$\frac{A}{R_{Nil(A)}} \approx B_1.$$ 

In cases (ii) and (iii), no power of $\alpha$ belongs to $Nil(A)$; as $Nil(A)$ is saturated, it follows that $Nil(A) = \{0\}$. In fact, $A$ is integral, whence $\{0\} \in Pr_s(A)$. If $\mathcal{P} \in Pr_s(A)$ and $\mathcal{P} \neq \{0\}$, then $\mathcal{P}$ contains some power of $\alpha$, hence contains $\alpha$, hence contains $\alpha A$. As above we see that $\mathcal{P} = \alpha A$; but, in case (iii), $\alpha A$ is not saturated. In case (ii) it is easy to see that $\alpha A$ is prime and saturated. Therefore:

in case (ii), $Pr_s(A) = \{\{0\}, \alpha A \}$; $\{0\}$ is a generic point ($\{\{0\}\} = Pr_s(A)$), and $\alpha A$ a “closed point” ($\{\alpha A\}$ is closed);

in case (iii), $Pr_s(A) = \{\{0\}\}$.

One may remark that $B_1[x]$ itself falls into case (ii).

In [3], pp. 75–79, we have enumerated (up to isomorphism) monogenic $B_1$–algebras of cardinality $\leq 5$. It is easy to see where these algebras fall in the above classification ; we keep the numbering used in [3]. Let then $3 \leq |A| \leq 5$. We have the following repartition

Case (i) : (6),(8),(12),(15),(18),(24)

Case (ii): (7),(10),(11),(16),(19),(25),(26)

Case (iii): (5),(9),(13),(14),(17),(20),(21),(22),(23),(27),(28)
Acknowledgment. This paper was written during a stay at I.H.E.S. (December 2010-March 2011). I am grateful to the staff and the colleagues who managed to make this stay pleasant and stimulating. I am also deeply indebted to Professor Alain Connes for his constant moral support.

References


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