

# Collapsing of Abelian Fibred Calabi-Yau Manifolds

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# COLLAPSING OF ABELIAN FIBRED CALABI-YAU MANIFOLDS

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ABSTRACT. We study the collapsing behaviour of Ricci-flat Kähler metrics on a projective Calabi-Yau manifold which admits an abelian fibration, when the volume of the fibers approaches zero. We show that away from the critical locus of the fibration the metrics collapse with locally bounded curvature, and along the fibers the rescaled metrics become flat in the limit. The limit metric on the base minus the critical locus is locally isometric to an open dense subset of any Gromov-Hausdorff limit space of the Ricci-flat metrics. We then apply these results to study metric degenerations of families of polarized hyperkähler manifolds in the large complex structure limit. In this setting we prove an analog of a result of Gross-Wilson for  $K3$  surfaces, which is motivated by the Strominger-Yau-Zaslow picture of mirror symmetry.

## 1. INTRODUCTION

A Calabi-Yau manifold  $M$  is a compact Kähler manifold with vanishing first Chern class  $c_1(M) = 0$  in  $H^2(M, \mathbb{R})$ . A fundamental theorem of Yau [45] says that on  $M$  there exists a unique Ricci-flat Kähler metric in each Kähler class. If we move the Kähler class towards a limit class on the boundary of the Kähler cone, we get a family of Ricci-flat Kähler metrics which degenerates in the limit. The general question of understanding the geometric behaviour of these metrics was raised by Yau [46, 47], Wilson [44] and others, and much work has been devoted to it, see for example [18, 29, 30, 34, 37, 38] and references therein. In this paper, we study metric degenerations of Ricci-flat Kähler metrics whose Kähler classes approach semi-ample non-big classes.

The first useful observation is that the diameters of a family of Ricci-flat Kähler metrics  $\tilde{\omega}_t$ ,  $t \in (0, 1]$ , on a Calabi-Yau manifold  $M$  are uniformly bounded if their Kähler classes  $[\tilde{\omega}_t]$  tend to a limit class  $\alpha$  on the boundary of the Kähler cone when  $t \rightarrow 0$  [37, 48]. Another special feature of the Kähler case is that the volume of the Ricci-flat metrics can be computed cohomologically, and to determine whether it will approach zero or stay bounded away from it, it is enough to calculate the self-intersection  $\alpha^n$  where

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$n = \dim_{\mathbb{C}} M$ . If  $\alpha^n$  is strictly positive, then it was proved by the second-named author [37] that the Ricci-flat metrics do not collapse, (i.e., there is a constant  $\nu > 0$  independent of  $t$  such that each  $\tilde{\omega}_t$  has a unit radius metric ball with volume bigger than  $\nu$ ), and in fact converge smoothly away from a subvariety. If  $\alpha^n$  is zero, then the total volume of the Ricci-flat metrics approaches zero, so one expects to have collapsing to a lower-dimensional space. This was shown to be the case for elliptically fibered  $K3$  surfaces by Gross-Wilson [18], and later the second-named author considered the higher dimensional case when the Calabi-Yau manifold  $M$  admits a holomorphic fibration to a lower-dimensional Kähler space, and the limit class is the pullback of a Kähler class [38]. The first goal of the present paper is to improve the convergence result in [38].

Let us now describe our first result in detail. Let  $(M, \omega_M)$  be a compact Calabi-Yau  $n$ -manifold which admits a holomorphic map  $f : M \rightarrow Z$  where  $(Z, \omega_Z)$  is a compact Kähler manifold. Thanks to Yau's theorem, we can assume that  $\omega_M$  is Ricci-flat. Denote by  $N = f(M)$  the image of  $f$ , and assume that  $N$  is an irreducible normal subvariety of  $Z$  with dimension  $m$ ,  $0 < m < n$ , and that the map  $f : M \rightarrow N$  has connected fibers. Denote by  $\omega_0 = f^*\omega_Z$ , which is a smooth nonnegative real  $(1, 1)$ -form on  $M$  whose cohomology class lies on the boundary of the Kähler cone of  $M$ , and denote also by  $\omega_N$  the restriction of  $\omega_Z$  to the regular part of  $N$ . For example, one can take either  $Z = N$  (if  $N$  is smooth), or  $Z = \mathbb{C}\mathbb{P}^N$  (if  $N$  is an algebraic variety). This second case arises whenever we have a line bundle  $L \rightarrow M$  which is semiample (some power is globally generated) and of Iitaka dimension  $m < n$ , so  $L$  is not big.

In general, given a map  $f : M \rightarrow N$  as above, there is a proper analytic subvariety  $S \subset M$  such that  $N \setminus f(S)$  is smooth and  $f : M \setminus S \rightarrow N \setminus f(S)$  is a smooth submersion (the set  $S$  is exactly where the differential  $df$  does not have full rank  $m$ ). For any  $y \in N \setminus f(S)$  the fiber  $M_y = f^{-1}(y)$  is a smooth Calabi-Yau manifold of dimension  $n - m$ , and it is equipped with the Kähler metric  $\omega_M|_{M_y}$ . The volume of the fibers  $\int_{M_y} (\omega_M|_{M_y})^{n-m}$  is a homological constant that does not depend on  $y$  in  $N \setminus f(S)$ , and we can assume that it equals 1. Consider the Kähler metrics on  $M$  given by  $\omega_t = \omega_0 + t\omega_M$ , with  $0 < t \leq 1$ , and call  $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t$  the unique Ricci-flat Kähler metric on  $M$  cohomologous to  $\omega_t$ , with potentials normalized by  $\sup_M \varphi_t = 0$ . They satisfy a family of complex Monge-Ampère equations

$$(1.1) \quad \tilde{\omega}_t^n = (\omega_t + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = c_t t^{n-m} \omega_M^n,$$

where  $c_t$  is a constant that has a positive limit as  $t \rightarrow 0$  (see (4.28)). A general  $C^0$  estimate  $\|\varphi_t\|_{C^0} \leq C$  (independent of  $t > 0$ ) for such equations was proved by Demailly-Pali [9] and Eyssidieux-Guedj-Zeriahi [10], generalizing work of Kołodziej [24]. In the case under consideration, much more is true: the second-named author's work [38] shows that there exists a smooth function  $\varphi$  on  $N \setminus f(S)$  so that as  $t$  goes to zero we have  $\varphi_t \rightarrow \varphi \circ f$  in

$C_{loc}^{1,\alpha}(M \setminus S, \omega_M)$  for any  $0 < \alpha < 1$ . Moreover  $\omega = \omega_N + \sqrt{-1}\partial\bar{\partial}\varphi$  is a Kähler metric on  $N \setminus f(S)$  with  $\text{Ric}(\omega) = \omega_{\text{WP}}$ . Here  $\omega_{\text{WP}}$  is the pullback of the Weil-Petersson metric from the moduli space of polarized Calabi-Yau fibers, which has appeared several times before in the literature [11, 18, 34, 38].

We now assume that the every fiber  $M_y$  with  $y \in N \setminus f(S)$  is biholomorphic to a complex torus (of course, it is enough to assume that just one smooth fiber is a complex torus). This is the case for example whenever  $M$  is hyperkähler. We also assume that  $M$  is projective, so we can take  $[\omega_M]$  to be the first Chern class of an ample line bundle. In this case we can improve the above result, thus answering Questions 4.1 and 4.2 of [39] in our setting:

**Theorem 1.1.** *If  $M$  is projective and if one (and hence all) of the fibers  $M_y$  with  $y \in N \setminus f(S)$  is a torus, then as  $t$  approaches zero the Ricci-flat metrics  $\tilde{\omega}_t$  converge in  $C_{loc}^\infty(M \setminus S, \omega_M)$  to  $f^*\omega$ , where  $\omega$  is a Kähler metric on  $N \setminus f(S)$  with  $\text{Ric}(\omega) = \omega_{\text{WP}}$ . Given any compact set  $K \subset M \setminus S$  there is a constant  $C_K$  such that the sectional curvature of  $\tilde{\omega}_t$  satisfies*

$$(1.2) \quad \sup_K |\text{Sec}(\tilde{\omega}_t)| \leq C_K,$$

for all small  $t > 0$ . Furthermore, on each torus fiber  $M_y$  with  $y \in N \setminus f(S)$  we have

$$(1.3) \quad \frac{\tilde{\omega}_t|_{M_y}}{t} \rightarrow \omega_{SF,y},$$

where  $\omega_{SF,y}$  is the unique flat metric on  $M_y$  cohomologous to  $\omega_M|_{M_y}$  and the convergence is smooth and uniform as  $y$  varies on a compact subset of  $N \setminus f(S)$ .

As remarked earlier, in the case of elliptically fibered  $K3$  surfaces ( $n = 2, m = 1$ ) this theorem follows from the work of Gross-Wilson [18]. In higher dimensions, in the very special case when  $S$  is empty, the theorem (except (1.2)) also follows from the work of Fine [11]. Both these works take a different approach from us, by constructing the Ricci-flat metrics  $\tilde{\omega}_t$  as small perturbations of semi-flat metrics (see section 3), which in [18] are glued to Ooguri-Vafa metrics near the singular fibers. By contrast, we work directly with the Ricci-flat metrics  $\tilde{\omega}_t$  and prove that they satisfy a priori estimates away from the singular fibers, which then implies the convergence results. This was also the approach taken by the second-named author in [38], where the convergence  $\tilde{\omega}_t \rightarrow f^*\omega$  was proved in a weaker topology (see also the work of Song-Tian [34] for the case of  $K3$  surfaces).

The curvature bound (1.2) in Theorem 1.1 does not hold if the generic fibers are not tori, as one can see for example by taking the product of two non-flat Calabi-Yau manifolds with the product Ricci-flat Kähler metric and then scaling one factor to zero. On the other hand, we believe that the assumption in Theorem 1.1 that  $M$  is projective is just technical and it should be possible to remove it.

We now describe the second main result of the paper, which concerns the Gromov-Hausdorff limit of our manifolds. The Gromov-Hausdorff distance  $d_{GH}$  was introduced by Gromov in the 1980's [15], and it defines a topology on the space of isometry classes of all compact metric spaces. For two compact metric spaces  $X$  and  $Y$ , the Gromov-Hausdorff distance of  $X$  and  $Y$  is

$$d_{GH}(X, Y) = \inf_Z \{d_H^Z(X, Y) \mid X, Y \hookrightarrow Z \text{ isometric embeddings}\},$$

where  $Z$  is a metric space and  $d_H^Z(X, Y)$  denotes the standard Hausdorff distance between  $X$  and  $Y$  regarded as subsets in  $Z$  by the isometric embeddings (see for example [15, 28] for more background). The Gromov-Hausdorff topology provides a framework to study families of compact metric spaces or Riemannian manifolds. We would like to understand the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_t)$  in Theorem 1.1. Since the volume of the whole manifold goes to zero, the manifolds  $(M, \tilde{\omega}_t)$  are collapsing. Furthermore, from Theorem 1.1 we know that on a Zariski open set of  $M$  the Ricci-flat metrics collapse with locally bounded curvature.

The collapsing of Einstein manifolds and Riemannian manifolds with definite curvature bounds in the Gromov-Hausdorff sense has been extensively studied from different viewpoints, see for example [2, 5, 6, 7, 8, 12, 18, 27, 28, 33] and the reference therein. These general theories provide us with results which are particularly strong in the case of Riemannian manifolds with bounded sectional curvature and Einstein manifolds of dimension 4. The first detailed analysis of the collapsing of geometrically interesting families of Einstein 4-manifolds was done by Anderson in [2]. More recently, a result of Cheeger-Tian [7] shows that on any sufficiently collapsed Ricci-flat Einstein 4-manifold with volume 1 there is a large open set  $U$  where the sectional curvature is bounded by a universal constant, and  $U$  admits an  $\mathcal{F}$ -structure, which is a generalization of torus fibration. Furthermore, by [27], the collapsed limits of Ricci-flat Einstein 4-manifolds with bounded Euler numbers are smooth Riemannian orbifolds away from a finite number of points. The metric structure of collapsed limits of higher-dimensional Einstein  $n$ -manifolds (and more generally manifolds with a uniform lower bound on the Ricci curvature) was extensively studied by Cheeger-Colding [5] and collaborators. Regarding the collapsed Gromov-Hausdorff limit of the Ricci-flat metrics in Theorem 1.1, we have the following result.

First of all, thanks to [37, 48] we know that the diameter of  $(M, \tilde{\omega}_t)$  satisfies

$$(1.4) \quad \text{diam}_{\tilde{\omega}_t}(M) \leq D,$$

for some constant  $D$  and for all  $t > 0$ . Furthermore, since  $\tilde{\omega}_t \rightarrow f^*\omega$  and the base  $N$  is not a point, we also have that  $\text{diam}_{\tilde{\omega}_t}(M) \geq D^{-1}$ . Given any sequence  $t_k \rightarrow 0$ , Gromov's precompactness theorem shows that a subsequence of  $(M, \tilde{\omega}_{t_k})$  converges to some compact path metric space  $(X, d_X)$  in the Gromov-Hausdorff topology. Note that because of the upper and lower

bounds for the diameter, if we rescaled the metrics  $\tilde{\omega}_{t_k}$  to have diameter equal to one, the Gromov-Hausdorff limit (modulo subsequences) would be isometric to  $(X, d_X)$  after a rescaling.

**Theorem 1.2.** *In the same setting as Theorem 1.1, for any such limit space  $(X, d_X)$  there is an open dense subset  $X_0 \subset X$  such that  $(X_0, d_X)$  is locally isometric to  $(N \setminus f(S), \omega)$ , i.e. there is a homeomorphism  $\phi : N \setminus f(S) \rightarrow X_0$  satisfying that, for any  $y \in N \setminus f(S)$ , there is a neighborhood  $B_y \subset N \setminus f(S)$  of  $y$  such that, for  $y_1$  and  $y_2 \in B_y$ ,*

$$d_\omega(y_1, y_2) = d_X(\phi(y_1), \phi(y_2)).$$

In fact we prove that  $X \setminus X_0$  has measure zero with respect to the renormalized limit measure of [5], which implies that  $X_0$  is dense in  $X$ . It would be interesting to prove that the metric completion of  $(N \setminus f(S), \omega)$  is isometric to  $(X, d_X)$ . In the case of  $K3$  surfaces this was proved by Gross-Wilson [18].

As an application of Theorem 1.1 and Theorem 1.2 we study the metric degenerations of families of polarized hyperkähler manifolds in the large complex structure limit. In [36], Stominger, Yau and Zaslow proposed a conjecture about constructing the mirror manifold of a given Calabi-Yau manifold via special Lagrangian fibrations. This became known as the SYZ conjecture, and has generated an immense amount of work, see for example [16, 17, 18, 25] and references therein. Later another version of the SYZ conjecture was proposed by Gross-Wilson [18], Kontsevich-Soibelman [25] and Todorov via degenerations of Ricci-flat Kähler-Einstein metrics. The conjecture says that if  $\{M_t\}$ ,  $t \in \Delta \setminus \{0\} \subset \mathbb{C}$ , is a family of polarized Calabi-Yau  $n$ -manifolds,  $\omega_t$  is the Ricci-flat Kähler-Einstein metric representing the polarization on  $M_t$ , and the complex structure of  $M_t$  tends to a large complex structure limit point in the deformation moduli space of  $M_t$  when  $t \rightarrow 0$ , then after rescaling  $(M_t, \omega_t)$  to have diameter 1, they collapse to a compact metric space  $(X, d_X)$  in the Gromov-Hausdorff sense. Furthermore, a dense open subset  $X_0 \subset X$  is a smooth manifold of real dimension  $n$ , and the codimension of  $X \setminus X_0$  is bigger or equal to 2. This conjecture holds trivially for tori, and was verified for  $K3$  surfaces by Gross-Wilson in [18].

In the third main result of this paper we consider this conjecture for higher dimensional hyperkähler manifolds. We will describe it briefly here, and give a more complete description in Section 2. Let  $(M, I)$  be a compact complex manifold of complex dimension  $2n$  with a Ricci-flat Kähler metric  $\omega_I$  with holonomy the full group  $Sp(n)$ . In particular  $M$  is Calabi-Yau (in our definition), and furthermore it has a hyperkähler structure. We assume that there is an ample line bundle over  $M$  with the first Chern class  $[\omega_I]$ , that we have a holomorphic fibration  $f : M \rightarrow N$  as before with  $N$  a projective variety, and that there is a holomorphic section  $s : N \rightarrow M$ . Under these assumptions, it is known that  $N = \mathbb{C}\mathbb{P}^n$  [22], and that the smooth fibers of  $f$  are complex Lagrangian tori [26]. If we perform a hyperkähler rotation

of the complex structure, the fibers become special Lagrangian, and we are exactly in the setup of Strominger, Yau and Zaslow [36]. We furthermore assume that the polarization induced on the torus fibers is principal. In this case, the SYZ mirror symmetry picture predicts that  $M$  is mirror to itself, and that a large complex structure limit is mirror to a large Kähler structure limit. We use this as our definition of large complex structure limit, so we have a family of polarized hyperkähler structures  $(M, \check{\Omega}_s)$  with Ricci-flat Kähler metric  $\check{\omega}$  which approach a large complex structure limit as  $s \rightarrow \infty$ . By assuming the validity of a standard conjecture on hyperkähler manifolds (Conjecture 2.3), we can perform a hyperkähler rotation and a normalization to reduce exactly to the setup covered by Theorems 1.1 and 1.2, and we can prove:

**Theorem 1.3.** *In the above situation, denote  $\check{M}_s$  the hyperkähler manifold with period  $\check{\Omega}_s$ , and  $d_s = \text{diam}_{\check{\omega}}(\check{M}_s)$ . Then, for any sequence  $s_k \rightarrow \infty$ , a subsequence of  $(\check{M}_{s_k}, d_{s_k}^{-2}\check{\omega})$  converges in the Gromov-Hausdorff sense to a compact metric space  $(X, d_X)$ . Furthermore, there is an open dense subset  $X_0 \subset X$  such that  $(X_0, d_X)$  is local isometric to an open non-complete smooth Riemannian manifold  $(N_0, g)$  with  $\dim_{\mathbb{R}} N_0 = \frac{1}{2} \dim_{\mathbb{R}} M$ .*

This proves the conjecture of Gross-Wilson [18], Kontsevich-Soibelman [25] and Todorov in our situation, modulo these assumptions, except for the statement that  $\text{codim}_{\mathbb{R}}(X \setminus X_0) \geq 2$  where more arguments are needed. Again, this was proved by Gross-Wilson [18] in the case of K3 surfaces.

This paper is organized as follows. In Section 2 we study SYZ mirrors of some hyperkähler manifolds, and derive Theorem 1.3 as a consequence of Theorems 1.1 and 1.2. In Section 3 we construct semi-flat background metrics on the total space of a holomorphic torus fibration. Theorem 1.1 is proved in Section 4 while Theorem 1.2 is proved in Section 5.

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## 2. HYPERKÄHLER MIRROR SYMMETRY

In this section we discuss a version of mirror symmetry for hyperkähler manifolds analogous to the one used for K3 surfaces in [18].

The situation for general hyperkähler manifolds is considerably less developed, however, and we shall have to make many assumptions in this discussion. The goal is to show, modulo these assumptions, that one obtains the expected Gromov-Hausdorff collapse at a large complex structure limit of hyperkähler manifolds, and that the limit can be identified using the main results of this paper. This is completely analogous to [18], and this discussion represents a summary of known results.

First we review known facts about periods of hyperkähler manifolds. Fix  $M$  a manifold of real dimension  $4n$  which supports a hyperkähler manifold structure with holonomy being the full group  $Sp(n)$ . (When a hyperkähler manifold has this full group as holonomy, it is said to be *irreducible*.) Set  $L = H^2(M, \mathbb{Z})$ ,  $L_{\mathbb{R}} := L \otimes_{\mathbb{Z}} \mathbb{R}$ ,  $L_{\mathbb{C}} := L \otimes_{\mathbb{Z}} \mathbb{C}$ . Then there is a real-valued non-degenerate quadratic form  $q_M : L \rightarrow \mathbb{R}$ , called the *Beauville-Bogomolov form*, with the property that there is a constant  $c$  such that

$$q_M(\alpha)^n = c \int_M \alpha^{2n}$$

for  $\alpha \in L$ , of signature  $(+, +, +, -, \dots, -)$ . We write  $q_M(\cdot, \cdot)$  for the induced pairing, with  $q_M(\alpha, \alpha) = q_M(\alpha)$ .

We can define the *period domain* of  $M$  to be

$$\mathcal{P}_M := \{[\Omega] \in \mathbb{P}(L_{\mathbb{C}}) \mid q_M(\Omega) = 0, \quad q_M(\Omega, \bar{\Omega}) > 0\}.$$

The Teichmüller space of  $M$ ,  $\mathbf{Teich}_M$ , is the set of hyperkähler complex structures on  $M$  modulo elements of  $\text{Diff}_0(M)$ , the diffeomorphisms of  $M$  isotopic to the identity. By the Bogomolov-Tian-Todorov theorem, this is a (non-Hausdorff) manifold. There is a period map

$$\text{Per} : \mathbf{Teich}_M \rightarrow \mathcal{P}_M$$

taking a complex structure on  $M$  to the class of the line  $H^{2,0}(M)$ . Then  $\text{Per}$  is étale, and was proved to be surjective by Huybrechts in [21]. Although we shall not make use of this here, we note that recently Verbitsky [41] proved a suitably formulated global Torelli theorem. However, one must keep in mind that  $\text{Per}$  is not, in general, a diffeomorphism.

Next consider a complex structure on  $M$  and Ricci-flat Kähler metric  $\omega_I$  making  $M$  hyperkähler. Then a choice of a holomorphic symplectic two-form  $\Omega_I$ , along with  $\omega_I$ , completely determines this structure. In particular, if we write  $\Omega_I = \omega_J + \sqrt{-1}\omega_K$ , we can normalize  $\Omega_I$  so that  $q_M(\omega_I) = q_M(\omega_J) = q_M(\omega_K)$ . Furthermore, necessarily  $q_M(\omega_I, \omega_J) = q_M(\omega_I, \omega_K) = q_M(\omega_J, \omega_K) = 0$ . The triple  $\omega_I, \omega_J, \omega_K$  is called a *hyperkähler triple*. It gives rise to an  $S^2$  worth of complex structures compatible with the same hyperkähler metric: in particular, one has the  $J$  complex structure with holomorphic symplectic form  $\Omega_J := \omega_K + \sqrt{-1}\omega_I$  and Kähler form  $\omega_J$ , and the  $K$  complex structure with holomorphic symplectic form  $\Omega_K := \omega_I + \sqrt{-1}\omega_J$  and Kähler form  $\omega_K$ .



We use these facts to speculate on mirror symmetry for hyperkähler manifolds, starting with the Strominger-Yau-Zaslow point of view. Suppose that we are given a complex structure on  $M$  such that there is a fibration  $f : M \rightarrow N$ , with fibers being holomorphic Lagrangian subvarieties of  $M$ . Suppose furthermore that  $N$  is a Kähler manifold. Then by results of Matsushita (see [26] and [16], Proposition 24.8 for these results) the smooth fibers of  $f$  are complex tori and  $N$  is a Fano manifold with  $b_2(N) = 1$ . Furthermore, if  $M$  is projective of complex dimension  $2n$  then  $N = \mathbb{C}\mathbb{P}^n$  by a result of Hwang [22]. Let  $\omega_I$  be a Ricci-flat Kähler form on  $M$ . Write the holomorphic symplectic form  $\Omega_I$  on  $M$  as  $\omega_J + \sqrt{-1}\omega_K$ . Then after hyperkähler rotation, there is a complex structure with holomorphic symplectic form  $\Omega_K = \omega_I + \sqrt{-1}\omega_J$  and Kähler form  $\omega_K$ . If  $M_y$  is a fiber of  $f$ , then  $\omega_J|_{M_y} = \omega_K|_{M_y} = 0$ , from which it follows that  $\text{Im}(\Omega_K^n)|_{M_y} = 0$ , so the fibers of  $f$  are special Lagrangian.

The Strominger-Yau-Zaslow conjecture [36] predicts that mirror symmetry can be explained via dualizing such a special Lagrangian torus fibration. In a general situation, it can be hard to dualize torus fibrations, because of singular fibres. The case that  $M$  is a K3 surface, treated in detail in [18], is rather special because Poincaré duality gives a canonical isomorphism between a two-torus and its dual.

With some additional assumptions, a similar situation holds in the hyperkähler case. Suppose that the Kähler form  $\omega_I$  is integral, so that there is an ample line bundle  $\mathcal{L}$  on  $X$  whose first Chern class is represented by  $\omega_I$ . The restriction of this line bundle to a non-singular fiber  $M_y$  then induces a polarization of some type  $(d_1, \dots, d_n)$ . In particular there is a canonical map  $M_y \rightarrow M_y^\vee$  given by

$$M_y \ni x \mapsto \mathcal{L}|_{M_y} \otimes t_x^* \mathcal{L}^{-1}|_{M_y} \in M_y^\vee.$$

Here  $M_y^\vee$  is the dual abelian variety to  $M_y$ , classifying degree zero line bundles on  $M_y$ , and  $t_x : M_y \rightarrow M_y$  is given by translation by  $x$ , which makes sense once one chooses an origin in  $M_y$ . The kernel of this map is  $(\mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z})^{\oplus 2}$ . In particular, if  $f$  possesses a section  $s : N \rightarrow M$ , and  $N_0 := N \setminus f(S)$  where  $S$  is the critical locus of  $f$ , then the dual of  $f^{-1}(N_0) \rightarrow N_0$  can be described as a quotient map, given by dividing out by the kernel of the polarization on each fiber. One can then hope that this dual fibration can be compactified to a hyperkähler manifold.

In general, if  $M_y$  carries a polarization of type  $(d_1, \dots, d_n)$ , it is not difficult to check that the dual abelian variety  $M_y^\vee$  carries a polarization of type  $(d_n/d_n, d_n/d_{n-1}, \dots, d_n/d_1)$ . Thus it is possible that the SYZ dual hyperkähler manifold need not be the same as  $M$ . There do indeed exist examples of abelian variety fibrations on hyperkähler manifolds which are not principally polarized; these were discovered by Justin Sawon, see Example 3.8 and Remark 3.9 of [31]. It is quite possible these fibrations do

not have duals which are hyperkähler manifolds, as a natural compactification might be a holomorphic symplectic variety without a holomorphic symplectic resolution of singularities.

On the other hand, if  $\omega_I$  induces a *principal* polarization on each fiber  $M_y$ , i.e., the map  $M_y \rightarrow M_y^\vee$  is an isomorphism, then the SYZ dual of the fibration  $f^{-1}(N_0) \rightarrow N_0$ , assuming again the existence of a section, can be canonically identified with  $f^{-1}(N_0) \rightarrow N_0$ , and thus it is natural to consider  $f : M \rightarrow N$  to be a self-dual fibration, at least at the purely topological level. In this case, and only in this case, SYZ mirror symmetry predicts that hyperkähler manifolds are self-mirror. The idea that hyperkähler manifolds should be self-mirror was first suggested and explored by Verbitsky in [40].

In this case only, we can be more explicit about mirror symmetry. We summarize our assumptions so far:

**Assumptions 2.1.** *Let  $M_I$  be a hyperkähler manifold with  $f : M_I \rightarrow N$  a complex torus fibration, along with a section  $s : N \rightarrow M_I$  and an ample line bundle  $\mathcal{L}$  with first Chern class represented by a hyperkähler metric  $\omega_I$ . We assume further the induced polarization on the smooth fibers of  $f$  is principal and that  $N$  is projective.*

Thus, with these assumptions, it is natural to assume that mirror symmetry exchanges complex and Kähler moduli for the fixed underlying space  $M$ . This can be described at the level of period domains as follows.

Let  $\sigma \in L_{\mathbb{R}}$  be the class represented by  $\omega_I$ . Fix an integral Kähler class  $\omega_N$  on  $N$ , and let  $E \in L$  be represented by  $f^*\omega_N$ , so that  $q_M(E) = 0$ .

**Lemma 2.2.** *In the above situation, we have  $q_M(E, \sigma) \neq 0$ .*

*Proof.* By [16], Exercise 23.2, we have

$$q_M(E, \sigma) \int_M \sigma^{2n} = 2q_M(\sigma) \int_M \sigma^{2n-1} \wedge f^*\omega_N \neq 0,$$

so  $q_M(E, \sigma) \neq 0$ . □

Denote by  $E^\perp \subseteq L_{\mathbb{R}}$  the orthogonal complement of  $E$  under  $q_M$ , and denote by  $E^\perp/E$  the quotient space  $E^\perp/\mathbb{R}E$ . Then  $q_M$  induces a quadratic form on  $E^\perp/E$ . Let

$$\mathcal{C}(M) := \{x \in E^\perp/E \mid q_M(x) > 0\},$$

and define the *complexified Kähler moduli space* of  $M$  to be

$$\mathcal{K}(M) := E^\perp/E \oplus i\mathcal{C}(M) \subseteq (E^\perp/E) \otimes \mathbb{C}.$$

We then have an isomorphism

$$m_{E, \sigma} : \mathcal{K}(M) \rightarrow \mathcal{P}_M \setminus E^\perp$$

via, representing an element of  $(E^\perp/E) \otimes \mathbb{C}$  by  $\alpha \in E^\perp \otimes \mathbb{C}$ ,

$$\alpha \mapsto \left[ \frac{1}{q_M(E, \sigma)} \sigma + \alpha - \frac{1}{2} \left( \frac{q_M(\sigma)}{q_M(E, \sigma)^2} + q_M(\alpha) + 2 \frac{q_M(\alpha, \sigma)}{q_M(E, \sigma)} \right) E \right].$$

Indeed, one first checks that this is independent of which representative  $\alpha$  is chosen. Then one notes that the coefficient of  $E$  is chosen so that  $q_M(m_{E, \sigma}(\alpha)) = 0$ , and  $q_M(m_{E, \sigma}(\alpha), m_{E, \sigma}(\bar{\alpha})) = 2q_M(\text{Im } \alpha) > 0$  by assumption that  $\alpha \in \mathcal{K}(M)$ . Further,  $m_{E, \sigma}$  is clearly injective, since  $\alpha = m_{E, \sigma}(\alpha) - \sigma/q_M(E, \sigma) \pmod{E}$ . It is surjective, since given  $[\Omega] \in \mathcal{P}_M \setminus E^\perp$ , we can rescale  $\Omega$  so that  $q_M(\Omega, E) = 1$ , and then  $[\Omega] = m_{E, \sigma}(\Omega - \sigma/q_M(E, \sigma) \pmod{E})$ .

We can then view the mirror map  $m_{E, \sigma}$  described above as realising mirror symmetry on the level of period domains as follows, defining an exchange of data

$$(M, \Omega, \mathbf{B} + \sqrt{-1}\omega) \leftrightarrow (M, \check{\Omega}, \check{\mathbf{B}} + \sqrt{-1}\check{\omega}).$$

Here  $[\Omega], [\check{\Omega}] \in \mathcal{P}_M$ , with  $q_M(E, \Omega), q_M(E, \check{\Omega}) \neq 0$ , so that we can assume  $\Omega$  and  $\check{\Omega}$  are normalized with  $q_M(E, \Omega) = q_M(E, \check{\Omega}) = 1$ . Furthermore,  $\mathbf{B}, \check{\mathbf{B}} \in E^\perp/E$  and  $\omega, \check{\omega} \in E^\perp$  satisfy  $q_M(\omega, \Omega) = q_M(\check{\omega}, \check{\Omega}) = 0$  and  $q_M(\omega), q_M(\check{\omega}) > 0$ . The relationship between the two triples is that  $\check{\Omega} = m_{E, \sigma}(\mathbf{B} + \sqrt{-1}\omega)$  and  $\check{\mathbf{B}}, \check{\omega}$  are the unique cohomology classes satisfying the above conditions and  $\Omega = m_{E, \sigma}(\check{\mathbf{B}} + \sqrt{-1}\check{\omega})$ . Indeed,  $\check{\mathbf{B}}$  and  $\check{\omega}$  exist, since as  $q_M(E, \Omega) = 1$ , we can write  $\Omega = \frac{1}{q_M(E, \sigma)}\sigma + \check{\mathbf{B}} + \sqrt{-1}\check{\omega} \pmod{E}$ , and replacing a chosen representative  $\check{\omega}$  with  $\check{\omega} - (q_M(\check{\omega}, \sigma)/q_M(E, \sigma) - q_M(\check{\omega}, \mathbf{B}))E$ , one guarantees that  $q_M(\check{\Omega}, \check{\omega}) = 0$ .

This mirror symmetry on the level of period domains doesn't quite give an exact mirror symmetry on the level of moduli spaces, since global Torelli does not in general hold for hyperkähler manifolds, so there might be a number of choices of complex structure on  $M$  with period  $[\Omega]$ . In addition,  $\omega$  or  $\check{\omega}$  need not represent a Kähler form except for very general choices of complex structure.

Nevertheless, this allows us to identify a large complex structure limit as being mirror to a large Kähler limit. The family

$$\left( M, \Omega = \frac{1}{q_M(E, \sigma)}\sigma + \check{\mathbf{B}} + \sqrt{-1}\check{\omega} \pmod{E}, s\omega \right),$$

represents a large Kähler limit, with the Kähler class moving off to infinity while the complex structure is fixed, and this is mirror to the triple

$$\left( M, \check{\Omega}_s = \frac{1}{q_M(E, \sigma)}\sigma + \sqrt{-1}s\omega \pmod{E}, \check{\mathbf{B}} + \sqrt{-1}\check{\omega} \right).$$

If for each  $s$ , we have an actual hyperkähler manifold with period  $\check{\Omega}_s$  and Kähler form  $\check{\omega}$ , we would like to understand the limiting metric behaviour.

To do so, we use hyperkähler rotation, and to do this we need to normalize the holomorphic symplectic form, defining

$$\check{\Omega}_s^{\text{nor}} = s^{-1} \sqrt{\frac{q_M(\check{\omega})}{q_M(\omega)}} \check{\Omega}_s.$$

Then we have  $q_M(\text{Re } \check{\Omega}_s^{\text{nor}}) = q_M(\text{Im } \check{\Omega}_s^{\text{nor}}) = q_M(\check{\omega})$ . So  $\text{Re } \check{\Omega}_s^{\text{nor}}$ ,  $\text{Im } \check{\Omega}_s^{\text{nor}}$  and  $\check{\omega}$  form a hyperkähler triple, and hence we can hyperkähler rotate to obtain a hyperkähler manifold with holomorphic two-form

$$\check{\Omega}_{s,J} := \text{Im } \check{\Omega}_s^{\text{nor}} + \sqrt{-1}\check{\omega} = \sqrt{\frac{q_M(\check{\omega})}{q_M(\omega)}} \left( \omega - \frac{q_M(\omega, \sigma)}{q_M(E, \sigma)} E \right) + \sqrt{-1}\check{\omega}$$

and Kähler form

$$\check{\omega}_{s,J} = \text{Re } \check{\Omega}_s^{\text{nor}} = \sqrt{\frac{q_M(\check{\omega})}{q_M(\omega)}} \left[ \frac{1}{s} \left( \frac{1}{q_M(E, \sigma)} \sigma - \frac{1}{2} \frac{q_M(\sigma)}{q_M(E, \sigma)^2} E \right) + \frac{s}{2} q_M(\omega) E \right].$$

We note that the period  $\check{\Omega}_{s,J}$  is in fact independent of  $s$ , so we can fix the complex structure on  $M$  independent of  $s$ . Assume that  $E$  is the first Chern class of a nef line bundle on  $M$  with respect to a complex structure with period  $\check{\Omega}_{s,J}$ , and  $\check{\omega}_{s,J}$  is a Kähler class with respect to this complex structure if  $s \geq s_0$ , for some  $s_0 \gg 0$ . We now take  $s = s_0 \sqrt{\frac{t+1}{t}}$ , so that as  $t$  goes to zero,  $s$  goes to infinity and we define the rescaled metrics

$$\check{\omega}_{t,J}^{\text{nor}} = \sqrt{t(t+1)} \check{\omega}_{s(t),J} = t \check{\omega}_{s_0,J} + \frac{s_0}{2} \sqrt{q_M(\check{\omega}) q_M(\omega)} E.$$

So as  $t \rightarrow 0$ ,  $\check{\omega}_{t,J}^{\text{nor}}$  moves on a straight line towards  $\frac{s_0}{2} \sqrt{q_M(\check{\omega}) q_M(\omega)} E$ , and  $\check{\omega}_{s_0,J}$  is Kähler.

To relate this to the results of this paper, we have the following conjecture, stated in [19, 42]:

**Conjecture 2.3.** *Let  $M$  be an irreducible hyperkähler manifold and  $\mathcal{L}$  a non-trivial nef bundle on  $M$ , with  $q_M(c_1(\mathcal{L})) = 0$ . Then  $\mathcal{L}$  induces a holomorphic map  $f' : M \rightarrow N'$  to a projective variety  $N'$  with  $\mathcal{L}^m \cong f'^*(\mathcal{O}(1))$  for some  $m > 0$ .*

If such a map exists, it is necessarily a holomorphic Lagrangian fibration. If furthermore  $M$  is projective then  $N' = \mathbb{C}\mathbb{P}^n$  by [22]. This conjecture follows from the log abundance conjecture if some multiple of  $\mathcal{L}$  is effective, and has been studied for example in [1, 4, 19, 42].

Let us suppose this conjecture holds. By choosing  $s_0$  properly, we assume that  $\frac{s_0}{2} \sqrt{q_M(\check{\omega}) q_M(\omega)}$  is an integer, and thus  $\frac{s_0}{2} \sqrt{q_M(\check{\omega}) q_M(\omega)} E = f'^* \alpha$  for an ample class  $\alpha$  on  $N'$ , where  $f'$  and  $N'$  are obtained by Conjecture 2.3. Because of the hyperkähler rotation, the Riemannian metrics defined by  $(\check{\Omega}_s^{\text{nor}}, \check{\omega})$  and by  $(\check{\Omega}_{s,J}, \check{\omega}_{s,J})$  are the same. Therefore, to understand the Gromov-Hausdorff limit of the large complex structure limit  $(M, \check{\Omega}_s^{\text{nor}}, \check{\omega})$  (this is the same that appears in the statement of Theorem

1.3) we can instead consider  $(M, \check{\Omega}_{s,J}, \check{\omega}_{s,J})$ . Now  $\check{\Omega}_{s,J}$  is independent of  $s$ , so we are simply changing the Kähler class, and the rescaled metrics  $\check{\omega}_{t,J} = \sqrt{t(t+1)}\check{\omega}_{s(t),J} = t\check{\omega}_{s_0,J} + f'^*\alpha$  move towards  $f'^*\alpha$  along a straight line. Therefore we are exactly in the setting of Theorem 1.1 and Theorem 1.2, which describe the Gromov-Hausdorff limit of  $(M, \check{\omega}_{t,J})$  as  $t$  goes to zero. But as remarked in the Introduction, we also have that the diameter of  $\check{\omega}_{t,J}$  is bounded uniformly away from zero and infinity, so if we further rescale the metrics  $\check{\omega}_{t,J}$  to have diameter 1, then up to a subsequence the Gromov-Hausdorff limit only changes by a rescaling, and Theorem 1.3 follows.

### 3. SEMI-FLAT METRICS

In this section we discuss semi-flat forms and metrics, extending some results in [18, 20] to our setting.

In general a closed real  $(1, 1)$ -form  $\omega_{SF}$  on an open set  $U \subset M \setminus S$  will be called semi-flat if its restriction to each torus fiber  $M_y \cap U$  with  $y \in f(U)$  is a flat metric, which we will always assume to be cohomologous to  $\omega_M|_{M_y}$ . If  $\omega_{SF}$  is also Kähler then we will call it a semi-flat metric. Semi-flat forms can also be defined when the fibers  $M_y$  are not tori but general Calabi-Yau manifolds, by requiring that the restriction to each fiber be Ricci-flat (see [34, 38]). They were first introduced by Greene-Shapere-Vafa-Yau in [14].

Fix now a small ball  $B \subset N \setminus f(S)$  with coordinates  $y = (y_1, \dots, y_m)$ , and consider the preimage  $f : U = f^{-1}(B) \rightarrow B$ . This is a holomorphic family of complex tori, and if  $B$  is small enough it has a holomorphic section  $\sigma_0$ , which we also fix. We can then define a complex Lie group structure on each fiber  $M_y = f^{-1}(y)$  with unit  $\sigma_0(y)$ . We claim that this family is locally isomorphic to a family of the form  $f' : (B \times \mathbb{C}^{n-m})/\Lambda \rightarrow B$ , where  $h : \Lambda \rightarrow B$  is a lattice bundle with fiber  $h^{-1}(y) = \Lambda_y \cong \mathbb{Z}^{2n-2m}$ , so that  $M_y \cong \mathbb{C}^{n-m}/\Lambda_y$ . To see this, note that each fiber  $M_y = f^{-1}(y)$  is a torus biholomorphic to  $\mathbb{C}^{n-m}/\Lambda_y$  for some lattice  $\Lambda_y$  that varies holomorphically in  $y$ . We choose a basis  $v_1(y), \dots, v_{2n-2m}(y)$  of this lattice, which varies holomorphically in  $y$ . Given these lattices we can construct the family  $f'$  by taking the quotient of  $B \times \mathbb{C}^{n-m}$  by the  $\mathbb{Z}^{2n-2m}$ -action given by  $(n_1, \dots, n_{2n-2m}) \cdot (y, z) = (y, z + \sum_i n_i v_i(y))$ , where  $z = (z_1, \dots, z_{n-m}) \in \mathbb{C}^{n-m}$ . Note that different choices of generators give isomorphic quotients. By construction the fiber  $f'^{-1}(y)$  is biholomorphic to  $f^{-1}(y)$  for all  $y \in B$ . A theorem of Kodaira-Spencer [23] (see also [43, Satz 3.6]) then implies that the families  $f$  and  $f'$  are locally isomorphic, so up to shrinking  $B$  there exists a biholomorphism  $(B \times \mathbb{C}^{n-m})/\Lambda \rightarrow U$  compatible with the projections to  $B$ , proving our claim. With this identification, the section  $\sigma_0 : B \rightarrow U$  is induced by the map  $B \rightarrow B \times \mathbb{C}^{n-m}$  given by  $y \mapsto (y, 0)$ .

Composing this biholomorphism with the quotient map  $B \times \mathbb{C}^{n-m} \rightarrow (B \times \mathbb{C}^{n-m})/\Lambda$  by the  $\mathbb{Z}^{2n-2m}$ -action we get a holomorphic map  $p : B \times \mathbb{C}^{n-m} \rightarrow U$  such that  $f \circ p(y, z) = y$  for all  $(y, z)$ , and  $p$  is a local isomorphism (the map  $p$  is also the universal covering map of  $U$ ).

We now assume that  $M$  is projective and  $[\omega_M]$  is an integral class, so each complex torus fiber  $M_y$ ,  $y \in B$ , can be polarized by  $[\omega_M]$ , which gives an ample polarization of type  $(d_1, \dots, d_{n-m})$  for some sequence of integers  $d_1 | d_2 | \dots | d_{n-m}$ . By [3], Proposition 8.1.1, one can then assume that  $\Lambda$  is generated by  $d_1 e_1, \dots, d_{n-m} e_{n-m}, Z_1, \dots, Z_{n-m} \in \mathbb{C}^{n-m}$ , where  $e_1, \dots, e_{n-m}$  is the standard basis for  $\mathbb{C}^{n-m}$ . Furthermore, the matrix  $Z$  with columns  $Z_1, \dots, Z_{n-m}$  must satisfy  $Z = Z^t$  and  $\text{Im} Z$  positive definite. Also, on the fibre  $M_y$ , the Kähler form  $\sum_{i,j} \sqrt{-1} (\text{Im} Z)_{ij} dz^i \wedge d\bar{z}^j$  is cohomologous to  $\omega_M|_{M_y}$ . Let

$$g_{ij} = (\text{Im} Z)_{ij}^{-1}.$$

Note that  $Z$  depends on  $y \in B$ , as does  $g_{ij}$ . Recall that we have the fiber coordinates  $z_1, \dots, z_{n-m}$ . Consider the function

$$\eta(y, z) = \sum_{i,j} -\frac{g_{ij}(y)}{2} ((z_i - \bar{z}_i)(z_j - \bar{z}_j)).$$

We would first like to show that  $\sqrt{-1} \partial \bar{\partial} \eta$  is invariant under translation by flat sections of the Gauss-Manin connection on  $B \times \mathbb{C}^{n-m}$  (this is the connection on this bundle such that sections of  $\Lambda$  are flat sections of the bundle). It is enough to check invariance under translation by  $\lambda s$  for  $s$  one of the generators of  $\Lambda$ ,  $\lambda \in \mathbb{R}$ . First, consider the composition of  $\eta$  with a general translation  $z_i \mapsto z_i + \tau_i(y)$ :

$$\begin{aligned} & \sum_{i,j} -\frac{g_{ij}}{2} ((z_i + \tau_i - \bar{z}_i - \bar{\tau}_i)(z_j + \tau_j - \bar{z}_j - \bar{\tau}_j)) \\ &= \eta - \sum_{i,j} \frac{g_{ij}}{2} ((\tau_i - \bar{\tau}_i)(z_j - \bar{z}_j) + (\tau_j - \bar{\tau}_j)(z_i - \bar{z}_i) + (\tau_i - \bar{\tau}_i)(\tau_j - \bar{\tau}_j)) \\ &= \eta - \sum_{i,j} g_{ij} \left( (\tau_i - \bar{\tau}_i)(z_j - \bar{z}_j) + \frac{1}{2} (\tau_i - \bar{\tau}_i)(\tau_j - \bar{\tau}_j) \right), \end{aligned}$$

the last equality by the symmetry  $g_{ij} = g_{ji}$ . We now consider two cases. If  $\tau_i = \lambda \delta_{ik}$  for some  $k$ , so that  $\tau_i$  is real, then in fact the above formula reduces to  $\eta$ , so  $\eta$  is itself invariant under this translation. Secondly, if we take  $\tau_i = \lambda Z_{ik}$  for some  $k$ ,  $\lambda \in \mathbb{R}$ , we obtain

$$\begin{aligned} & \eta - \sum_{i,j} (\text{Im} Z)_{ij}^{-1} (2\lambda \sqrt{-1} (\text{Im} Z)_{ik} (z_j - \bar{z}_j) - 2\lambda^2 (\text{Im} Z)_{ik} (\text{Im} Z)_{jk}) \\ &= \eta - \sum_j 2\delta_{jk} \lambda \sqrt{-1} (z_j - \bar{z}_j) - 2\lambda^2 \delta_{jk} (\text{Im} Z)_{jk}. \end{aligned}$$

Applying  $\partial \bar{\partial}$  kills the correction term, so  $\sqrt{-1} \partial \bar{\partial} \eta$  is invariant under this action. This means that  $\sqrt{-1} \partial \bar{\partial} \eta$  is the pullback under  $p$  of a two-form  $\omega_{SF}$

on  $U$

$$(3.1) \quad p^* \omega_{SF} = \sqrt{-1} \partial \bar{\partial} \eta,$$

and  $\omega_{SF}$  is semi-flat since its restriction to a fiber is  $\sqrt{-1} \sum_{i,j} g_{ij}(y) dz^i \wedge d\bar{z}^j$ , a flat metric on  $M_y$  cohomologous to  $\omega_M|_{M_y}$ . Note that the function  $\eta$  on  $B \times \mathbb{C}^{n-m}$  has the scaling property

$$(3.2) \quad \eta(y, \lambda z) = \lambda^2 \eta(y, z),$$

for all  $\lambda \in \mathbb{R}$ .

We now claim that on  $U$  the semi-flat form  $\omega_{SF}$  is nonnegative definite. To check this, it is enough to check at one point on each fiber, because of the invariance of this form. We check at the point  $z_1 = \cdots = z_{n-m} = 0$ , where the form is  $\sqrt{-1} \sum_{i,j} g_{ij} dz^i \wedge d\bar{z}^j$ , which is clearly nonnegative definite. It follows that  $\omega_{SF} \geq 0$ , and moreover that given any Kähler metric  $\omega'$  on  $B$  the form  $\omega_{SF} + f^* \omega'$  is a semi-flat Kähler metric on  $U$ .

Suppose now that we have a holomorphic section  $\sigma : B \rightarrow U$  of the map  $f$ . We will denote by  $T_\sigma : U \rightarrow U$  the fiberwise translation by  $\sigma$  (with respect to the section  $\sigma_0$ ). If we choose any local lift of  $\sigma$  to  $B \times \mathbb{C}^{n-m}$ , given by  $y \mapsto (y, \tilde{\sigma}(y))$ , then the translation  $T_\sigma$  is induced by the map  $B \times \mathbb{C}^{n-m} \rightarrow B \times \mathbb{C}^{n-m}$  given by  $(y, z) \mapsto (y, z + \tilde{\sigma}(y))$  (the choice of lift  $\tilde{\sigma}$  is irrelevant). We also have a map  $T_{-\sigma} : U \rightarrow U$  given by fiberwise translation by  $-\sigma$  (with respect to  $\sigma_0$ ), which is induced by  $(y, z) \mapsto (y, z - \tilde{\sigma}(y))$ . The two translation are biholomorphisms of  $U$  and are inverses to each other. For later purposes, we will need the following version of the  $\partial\bar{\partial}$ -Lemma, which is analogous to [18, Lemma 4.3] (see also [20, Proposition 4.6]), except that we work away from the singular fibers.

**Proposition 3.1.** *Let  $\omega$  be any Kähler metric on  $U$  cohomologous to  $\omega_{SF}$  in  $H^2(U, \mathbb{R})$ . Then there exist a holomorphic section  $\sigma : B \rightarrow U$  of  $f$  and a smooth real function  $\xi$  on  $U$  such that*

$$(3.3) \quad T_\sigma^* \omega_{SF} - \omega = \sqrt{-1} \partial \bar{\partial} \xi$$

on  $U$ .

*If in addition  $\omega$  is also semi-flat, then  $\xi$  is constant on each fiber  $M_y$  and is therefore the pullback of a function from  $B$ .*

*Proof.* By assumption there is a 1-form  $\zeta$  on  $U$  such that

$$\omega_{SF} - \omega = d\zeta = \partial \zeta^{0,1} + \bar{\partial} \zeta^{1,0}, \quad \bar{\partial} \zeta^{0,1} = 0,$$

where  $\zeta = \zeta^{0,1} + \zeta^{1,0}$  and  $\zeta^{0,1} = \bar{\zeta}^{1,0}$ .

We claim that  $(0, 1)$ -forms

$$(3.4) \quad \theta_j = \sqrt{-1} \bar{\partial} \left( \sum_{i=1}^{n-m} g_{ij}(y) (z_i - \bar{z}_i) \right), \quad j = 1, \dots, n-m,$$

are invariant under translations by flat sections of the Gauss-Manin connection on  $B \times \mathbb{C}^{n-m}$ , and thus descend to  $(0, 1)$ -forms on  $U$ . It is enough to

check invariance under translation by  $\lambda s$  where  $s$  is a generator of  $\Lambda$  and  $\lambda \in \mathbb{R}$ . First, consider a general translation  $z_i \mapsto z_i + \tau_i(y)$ . If  $\tau_i = \lambda \delta_{ik}$  for some  $k$ , so that  $\tau_i$  is real, then  $\theta_j$  are invariant. If  $\tau_i = \lambda Z_{ik}$  for some  $k$ ,  $\lambda \in \mathbb{R}$ , we obtain

$$\begin{aligned} \sum_{i=1}^{n-m} g_{ij}(z_i + \lambda Z_{ik} - \bar{z}_i - \lambda \bar{Z}_{ik}) &= \sum_{i=1}^{n-m} g_{ij}(z_i - \bar{z}_i) \\ &\quad + 2\sqrt{-1} \sum_{i=1}^{n-m} (\operatorname{Im} Z)_{ij}^{-1} \lambda (\operatorname{Im} Z)_{ik} \\ &= \sum_{i=1}^{n-m} g_{ij}(z_i - \bar{z}_i) + 2\lambda \sqrt{-1} \delta_{jk}. \end{aligned}$$

Applying  $\bar{\partial}$  kills the correction term, so  $\theta_j$  are invariant, and therefore they define  $(0, 1)$ -forms on  $U$ . Since, for any  $y \in B$ ,

$$(3.5) \quad p^*(\theta_j|_{M_y}) = -\sqrt{-1} \sum_{i=1}^{n-m} g_{ij}(y) d\bar{z}_i,$$

is fiberwise constant and  $g_{ij}$  is non-degenerate, we have that  $[\theta_i|_{M_y}]$ ,  $i = 1, \dots, n-m$  is a basis of  $H^{0,1}(M_y)$ .

We claim that there are holomorphic functions  $\sigma_i : B \rightarrow \mathbb{C}$  such that

$$(3.6) \quad \zeta^{0,1} = \sum_{i=1}^{n-m} \sigma_i \theta_i + \bar{\partial} h,$$

for a complex-valued function  $h$  on  $U$ . To prove this, note that  $H^{0,1}(U) = H^1(U, \mathcal{O}_U)$  which by the Leray spectral sequence for  $f$  is isomorphic to  $H^0(B, R^1 f_* \mathcal{O}_U)$  since  $H^k(B, f_* \mathcal{O}_U) = H^k(B, \mathcal{O}_B) = 0$  for  $k \geq 1$ . It follows that a  $\bar{\partial}$ -closed  $(0, 1)$ -form on  $U$  represents the zero class if and only if its restriction to  $M_y$  represents the zero class in  $H^{0,1}(M_y)$  for all  $y \in B$ . Consider now the  $(0, 1)$ -forms  $d\bar{y}^i$ ,  $1 \leq i \leq m$ , on  $B$  and denote their pullbacks to  $U$  by the same symbol. Then at each point of  $U$  the forms  $\{\theta_j\}$ ,  $1 \leq j \leq n-m$  together with  $\{d\bar{y}^i\}$ ,  $1 \leq i \leq m$ , form a basis of  $(0, 1)$ -forms. We can then write

$$\zeta^{0,1} = \sum_{j=1}^{n-m} w_j \theta_j + \sum_{i=1}^m h_i d\bar{y}^i,$$

where  $w_j, h_i$  are smooth complex functions on  $U$ . If we now restrict to a fiber  $M_y$  we get  $\zeta^{0,1}|_{M_y} = \sum_{j=1}^{n-m} w_j \theta_j|_{M_y}$ , and the functions  $w_j$  restricted to  $M_y$  can be thought of as functions on  $\mathbb{C}^{n-m}$  which are periodic with period  $\Lambda_y$ . There is a holomorphic  $T^{2n-2m}$ -action on  $U$  which is induced by the action of  $\mathbb{R}^{2n-2m}$  on  $B \times \mathbb{C}^{n-m}$  given by  $x \cdot (y, z) = (y, z + \sum_j x_j \tau_j(y))$ , where  $\tau_j(y)$  is a basis for the lattice  $\Lambda_y$  (the choice of which is irrelevant). If  $\alpha$  is a function or differential form on  $U$  or  $M_y$ , we will denote by  $\tilde{\alpha}$  its



average with respect to the  $T^{2n-2m}$ -action. In particular, if  $\alpha$  is a function on  $U$  then  $\tilde{\alpha}$  is the pullback of a function from  $B$ . We now call  $\sigma_j = \tilde{w}_j$ ,  $1 \leq j \leq n-m$ , which are functions of  $y \in B$  only. We clearly have that  $\tilde{\theta}_j = \theta_j$  and  $\widetilde{d\bar{y}^i} = d\bar{y}^i$ , so

$$\widetilde{\zeta^{0,1}|_{M_y}} = \sum_{j=1}^{n-m} \sigma_j(y) \theta_j|_{M_y}.$$

Now the  $T^{2n-2m}$ -action on  $M_y$  is generated by holomorphic vector fields and therefore acts trivially on the Dolbeault cohomology  $H^{0,1}(M_y)$ , which implies that

$$[\zeta^{0,1}|_{M_y}] = [\widetilde{\zeta^{0,1}|_{M_y}}] = \sum_{j=1}^{n-m} \sigma_j(y) [\theta_j|_{M_y}],$$

in  $H^{0,1}(M_y)$  for all  $y \in B$ . If we show that the  $\sigma_j(y)$  are holomorphic, then the  $(0,1)$ -form  $\zeta^{0,1} - \sum_j \sigma_j(y) \theta_j$  on  $U$  would be  $\bar{\partial}$ -closed and cohomologous to zero in  $H^{0,1}(U)$ , thus proving (3.6).

Call now  $V_j$ ,  $1 \leq j \leq n-m$  and  $W_i$ ,  $1 \leq i \leq m$  the  $T^{2n-2m}$ -invariant  $(0,1)$ -type vector fields on  $U$  which are the dual basis to  $\theta_j, d\bar{y}^i$ . We have that  $V_j = \sqrt{-1} \sum_{k=1}^{n-m} g^{jk} \frac{\partial}{\partial \bar{z}_k}$ , where  $g^{jk}$  is the inverse matrix of  $g_{jk}$ , and the vector fields  $\frac{\partial}{\partial \bar{z}_k}$  are well-defined on  $U$ . We will not need the explicit formula for  $W_i$ , but just the fact that if a function  $f$  on  $U$  is the pullback of a function on  $B$  then  $W_i(f) = \frac{\partial f}{\partial \bar{y}_i}$ .

To see why  $\sigma_j(y)$  is holomorphic, compute

$$\begin{aligned} 0 = \bar{\partial} \zeta^{0,1} &= \sum_{i,j} W_i(w_j) d\bar{y}^i \wedge \theta_j + \sum_{i,j} V_i(w_j) \theta_i \wedge \theta_j \\ &\quad + \sum_{i,j} W_j(h_i) d\bar{y}^j \wedge d\bar{y}^i + \sum_{i,j} V_j(h_i) \theta_j \wedge d\bar{y}^i. \end{aligned}$$

Since each  $V_j$  is a linear combination of  $\frac{\partial}{\partial \bar{z}_k}$ , we have that the functions  $V_i(w_j)$  and  $V_j(h_i)$  have average zero on each fiber. Taking the average then gives

$$0 = \bar{\partial} \widetilde{\zeta^{0,1}} = \sum_{i,j} \frac{\partial \sigma_j}{\partial \bar{y}_i} d\bar{y}^i \wedge \theta_j + \sum_{i,j} \frac{\partial \tilde{h}_i}{\partial \bar{y}_j} d\bar{y}^j \wedge d\bar{y}^i.$$

Since the forms  $d\bar{y}^i \wedge \theta_j$  and  $d\bar{y}^j \wedge d\bar{y}^i$  are linearly independent at every point, this implies that  $\sigma_j(y)$  are indeed holomorphic.

Let now  $T_\sigma : U \rightarrow U$  be the translation induced by the section  $\sigma = (p \circ \sigma_1, \dots, p \circ \sigma_{n-m})$ , where  $p : B \times \mathbb{C}^{n-m} \rightarrow U$  is the quotient map. Since

$$\begin{aligned} & \sum_{i,j} -\frac{g_{ij}}{2} ((z_i + \sigma_i - \bar{z}_i - \bar{\sigma}_i)(z_j + \sigma_j - \bar{z}_j - \bar{\sigma}_j)) \\ &= \eta - \sum_{i,j} \frac{g_{ij}}{2} ((\sigma_i - \bar{\sigma}_i)(z_j - \bar{z}_j) + (\sigma_j - \bar{\sigma}_j)(z_i - \bar{z}_i) + (\sigma_i - \bar{\sigma}_i)(\sigma_j - \bar{\sigma}_j)) \\ &= \eta - \sum_{i,j} g_{ij} \left( (\sigma_i - \bar{\sigma}_i)(z_j - \bar{z}_j) + \frac{1}{2}(\sigma_i - \bar{\sigma}_i)(\sigma_j - \bar{\sigma}_j) \right), \end{aligned}$$

we have

$$\begin{aligned} p^*T_\sigma^*\omega_{SF} - p^*\omega_{SF} &= -\sqrt{-1}\partial\bar{\partial} \sum_{i,j} g_{ij}(\sigma_i - \bar{\sigma}_i)(z_j - \bar{z}_j) + \sqrt{-1}\partial\bar{\partial}\Phi(y) \\ &= p^* \left( -\partial \sum_i \sigma_i \theta_i - \bar{\partial} \sum_i \bar{\sigma}_i \bar{\theta}_i \right) + \sqrt{-1}\partial\bar{\partial}\Phi(y), \end{aligned}$$

where  $\Phi(y) = -\sum_{i,j} \frac{g_{ij}}{2} (\sigma_i - \bar{\sigma}_i)(\sigma_j - \bar{\sigma}_j)$  is a real function of  $y$  only. We have just proved that

$$\omega_{SF} - \omega = \partial\zeta^{0,1} + \bar{\partial}\bar{\zeta}^{0,1} = \partial \sum_i \sigma_i \theta_i + \partial\bar{\partial}h + \bar{\partial} \sum_i \bar{\sigma}_i \bar{\theta}_i + \bar{\partial}\bar{\partial}h.$$

Thus

$$p^*T_\sigma^*\omega_{SF} - p^*\omega = p^*\sqrt{-1}\partial\bar{\partial}(2\text{Im}h + \Phi),$$

which proves (3.3) with  $\xi = 2\text{Im}h + \Phi$ .  $\square$

#### 4. ESTIMATES AND SMOOTH CONVERGENCE

In this section we prove a priori estimates of all orders for the Ricci-flat metrics  $\tilde{\omega}_t$  which are uniform on compact sets of  $M \setminus S$ , and then use these to prove Theorem 1.1. These estimates improve the results in [38], and use crucially the assumptions that  $M$  is projective and that the smooth fibers  $M_y$  are tori.

**Lemma 4.1.** *There is a constant  $C$  such that on  $U$  the Ricci-flat metrics  $\tilde{\omega}_t$  satisfy*

$$(4.1) \quad C^{-1}(\omega_0 + t\omega_M) \leq \tilde{\omega}_t \leq C(\omega_0 + t\omega_M),$$

for all small  $t > 0$ .

*Proof.* This estimate is contained in the second-named author's work [38], although it is not explicitly stated there. To see this, start from [38, (3.24)], which gives a constant  $C$  so that on  $U$  we have

$$C^{-1}(t\omega_M) \leq \tilde{\omega}_t.$$

Then use [38, Lemma 3.1] to get

$$C^{-1}\omega_0 \leq \tilde{\omega}_t,$$

and so adding these two inequalities we get

$$C^{-1}(\omega_0 + t\omega_M) \leq \tilde{\omega}_t,$$

or in other words  $\text{tr}_{\tilde{\omega}_t}\omega_t \leq C$  on  $U$ , where  $\omega_t = \omega_0 + t\omega_M$  as before. To get the reverse inequality, we note that on  $U$  we have

$$\text{tr}_{\omega_t}\tilde{\omega}_t \leq (\text{tr}_{\tilde{\omega}_t}\omega_t)^{n-1} \frac{\tilde{\omega}_t^n}{\omega_t^n} \leq C \frac{\tilde{\omega}_t^n}{\omega_t^n} \leq C,$$

where the last inequality follows from [38, (3.23)]. We thus get the reverse inequality

$$\tilde{\omega}_t \leq C(\omega_0 + t\omega_M),$$

thus proving (4.1).  $\square$

From now on we fix a small ball  $B \subset N \setminus f(S)$ , and as before we call  $U = f^{-1}(B)$  and we have the holomorphic covering map  $p : B \times \mathbb{C}^{n-m} \rightarrow U$ , with  $f \circ p(y, z) = y$  where  $(y, z) = (y_1, \dots, y_m, z_1, \dots, z_{n-m})$  the standard coordinates on  $B \times \mathbb{C}^{n-m}$ . We let  $\lambda_t : B \times \mathbb{C}^{n-m} \rightarrow B \times \mathbb{C}^{n-m}$  be the dilation

$$\lambda_t(y, z) = \left( y, \frac{z}{\sqrt{t}} \right),$$

which takes the lattice  $\sqrt{t}\Lambda_y$  to  $\Lambda_y$ . If we pull back the Kähler potential  $\varphi_t$  on  $U$  via  $p$  we get a function  $\varphi_t \circ p$  on  $B \times \mathbb{C}^{n-m}$  which is periodic in  $z$  with period  $\Lambda_y$ , i.e.  $\varphi_t \circ p(y, z + \ell) = \varphi_t \circ p(y, z)$  for all  $\ell \in \Lambda_y$ . The function  $\varphi_t \circ p \circ \lambda_t$  is then periodic in  $z$  with period  $\sqrt{t}\Lambda_y$ . Note that since  $\omega_0$  is the pullback of a metric from  $N \setminus f(S)$ , we have  $\lambda_t^* p^* \omega_0 = p^* \omega_0$ .

Recall now that we have a nonnegative definite semi-flat form  $\omega_{SF}$  on  $U$ , and that  $\omega_0 + \omega_{SF}$  is then a semi-flat Kähler metric on  $U$ . Since  $U$  is diffeomorphic to a product  $B \times M_y$ , it follows that  $\omega_{SF}$  and  $\omega_M$  are cohomologous on  $U$ . We now apply Proposition 3.1 and get a holomorphic section  $\sigma : B \rightarrow U$  and a real function  $\xi$  on  $U$  such that

$$(4.2) \quad T_\sigma^* \omega_{SF} - \omega_M = \sqrt{-1} \partial \bar{\partial} \xi$$

on  $U$ , where  $T_\sigma$  is the fiberwise translation by  $\sigma$ .

**Lemma 4.2.** *There is a constant  $C$  such that on the whole of  $B \times \mathbb{C}^{n-m}$  we have*

$$(4.3) \quad C^{-1} p^*(\omega_0 + \omega_{SF}) \leq \lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t \leq C p^*(\omega_0 + \omega_{SF}),$$

for all small  $t > 0$ .

*Proof.* First of all notice that after replacing  $U$  with a slightly smaller open set, the semi-flat metric  $\omega_0 + \omega_{SF}$  is uniformly equivalent to  $\omega_M$ , which implies that

$$(4.4) \quad C^{-1}(\omega_0 + t\omega_{SF}) \leq \omega_0 + t\omega_M \leq C(\omega_0 + t\omega_{SF}),$$

for all small  $t > 0$ . Thanks to Lemma 4.1 on  $U$  we have that

$$C^{-1}(\omega_0 + tT_{-\sigma}^*\omega_M) \leq T_{-\sigma}^*\tilde{\omega}_t \leq C(\omega_0 + tT_{-\sigma}^*\omega_M),$$

and since  $T_{-\sigma}^*\omega_M$  is uniformly equivalent to  $\omega_M$  we also have that

$$(4.5) \quad C^{-1}(\omega_0 + t\omega_M) \leq T_{-\sigma}^*\tilde{\omega}_t \leq C(\omega_0 + t\omega_M),$$

and combining (4.4) and (4.5) we get

$$(4.6) \quad C^{-1}(\omega_0 + t\omega_{SF}) \leq T_{-\sigma}^*\tilde{\omega}_t \leq C(\omega_0 + t\omega_{SF}),$$

on  $U$ . If we pull back (4.6) by  $p \circ \lambda_t$  we get

$$(4.7) \quad C^{-1}(p^*\omega_0 + t\lambda_t^*p^*\omega_{SF}) \leq \lambda_t^*p^*T_{-\sigma}^*\tilde{\omega}_t \leq C(p^*\omega_0 + t\lambda_t^*p^*\omega_{SF}),$$

on all of  $B \times \mathbb{C}^{n-m}$ . We claim that on the whole of  $B \times \mathbb{C}^{n-m}$  we have that

$$(4.8) \quad t\lambda_t^*p^*\omega_{SF} = p^*\omega_{SF}.$$

In fact, the construction of  $\omega_{SF}$  in section 3 gives that  $p^*\omega_{SF} = \sqrt{-1}\partial\bar{\partial}\eta$ , for a function  $\eta$  on  $B \times \mathbb{C}^{n-m}$  that satisfies

$$(4.9) \quad \eta \circ \lambda_t(y, z) = \eta\left(y, \frac{z}{\sqrt{t}}\right) = \frac{1}{t}\eta(y, z),$$

for all  $(y, z)$  in  $B \times \mathbb{C}^{n-m}$  and any  $t > 0$ . It follows then that

$$(4.10) \quad t\lambda_t^*p^*\omega_{SF} = t\lambda_t^*\sqrt{-1}\partial\bar{\partial}\eta = t\sqrt{-1}\partial\bar{\partial}(\eta \circ \lambda_t) = \sqrt{-1}\partial\bar{\partial}\eta = p^*\omega_{SF},$$

as claimed. Combining (4.7) and (4.8) we get the bound (4.3).  $\square$

**Proposition 4.3.** *Given any compact set  $K$  in  $B \times \mathbb{C}^{n-m}$  and any  $k \geq 0$  there exists a constant  $C$  independent of  $t > 0$  such that*

$$(4.11) \quad \|\lambda_t^*p^*T_{-\sigma}^*\tilde{\omega}_t\|_{C^k(K, \delta)} \leq C,$$

where  $\delta$  is the Euclidean metric on  $B \times \mathbb{C}^{n-m}$ .

*Proof.* We pull back (1.1) via  $T_{-\sigma} \circ p \circ \lambda_t$  and get

$$\begin{aligned} (\lambda_t^*p^*T_{-\sigma}^*\tilde{\omega}_t)^n(y, z) &= c_t t^{n-m} (\lambda_t^*p^*T_{-\sigma}^*\omega_M)^n(y, z) \\ &= c_t (p^*T_{-\sigma}^*\omega_M)^n\left(y, \frac{z}{\sqrt{t}}\right), \end{aligned}$$

since the pullback under  $\lambda_t$  of any volume form  $f(y, z)dy^1 \wedge \dots \wedge d\bar{z}^{n-m}$  on  $B \times \mathbb{C}^{n-m}$  equals  $t^{m-n}f(y, \frac{z}{\sqrt{t}})dy^1 \wedge \dots \wedge d\bar{z}^{n-m}$ . We now claim that in fact we have

$$(p^*T_{-\sigma}^*\omega_M)^n\left(y, \frac{z}{\sqrt{t}}\right) = (p^*T_{-\sigma}^*\omega_M)^n(y, z).$$

To see this, consider the  $(n, 0)$ -form

$$dy^1 \wedge \dots \wedge dy^m \wedge dz^1 \wedge \dots \wedge dz^{n-m}$$

on  $B \times \mathbb{C}^{n-m}$ . This form is invariant under the  $\mathbb{Z}^{2n-2m}$ -action described above

$$(n_1, \dots, n_{2n-2m}) \cdot (y, z) = (y, z + \sum_i n_i v_i(y)),$$

where  $(y, z) = (y_1, \dots, y_m, z_1, \dots, z_{n-m})$ , and so it descends to a holomorphic  $(n, 0)$ -form to the quotient  $(B \times \mathbb{C}^{n-m})/\Lambda$  and using the biholomorphism with  $U$  we get a holomorphic  $(n, 0)$ -form  $\Omega$  on  $U$ . We can then consider the volume form  $(\sqrt{-1})^{n^2} \Omega \wedge \bar{\Omega}$ , and we have

$$T_{-\sigma}^* \omega_M^n = h \cdot (\sqrt{-1})^{n^2} \Omega \wedge \bar{\Omega},$$

where  $h$  is a smooth positive function on  $U$ . Taking  $\sqrt{-1} \partial \bar{\partial} \log$  of both sides we get

$$\sqrt{-1} \partial \bar{\partial} \log h = \sqrt{-1} \partial \bar{\partial} \log \frac{T_{-\sigma}^* \omega_M^n}{(\sqrt{-1})^{n^2} \Omega \wedge \bar{\Omega}} = 0,$$

since  $T_{-\sigma}^* \omega_M$  is Ricci-flat and  $\Omega$  is a holomorphic  $(n, 0)$ -form. So  $\log h$  is pluriharmonic on  $U$ , and this implies that its restriction to any fiber  $M_y$  with  $y \in B$  is constant. Pulling back via  $p$  we get

$$(p^* T_{-\sigma}^* \omega_M^n)(y, z) = (h \circ p)(y, z) (\sqrt{-1})^{n^2} dy^1 \wedge \dots \wedge d\bar{z}^{n-m},$$

but since  $h$  is constant along the fibers of  $f$  and  $p$  is compatible with the projection to  $B$  we get that the function  $(h \circ p)(y, z)$  on  $B \times \mathbb{C}^{n-m}$  is independent of  $z$ . In particular we have

$$(p^* T_{-\sigma}^* \omega_M)^n \left( y, \frac{z}{\sqrt{t}} \right) = (p^* T_{-\sigma}^* \omega_M)^n(y, z),$$

and so the rescaled metrics  $\lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t$  satisfy the nondegenerate complex Monge-Ampère equation

$$(\lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t)^n = (p^* \omega_0 + t \lambda_t^* p^* T_{-\sigma}^* \omega_M + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t)^n = c_t (p^* T_{-\sigma}^* \omega_M)^n$$

on  $B \times \mathbb{C}^{n-m}$ , where we have set

$$\tilde{\varphi}_t = \varphi_t \circ T_{-\sigma} \circ p \circ \lambda_t.$$

We claim that the estimates (4.11) hold. To see this, we use (4.2) and get

$$(4.12) \quad p^* \omega_{SF} = p^* T_{-\sigma}^* \omega_M + p^* T_{-\sigma}^* \sqrt{-1} \partial \bar{\partial} \xi,$$

for a function  $\xi$  on  $U$ . On  $B \times \mathbb{C}^{n-m}$  we can then use (4.10) and (4.12) and write

$$(4.13) \quad \begin{aligned} \lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t &= p^* \omega_0 + t \lambda_t^* p^* T_{-\sigma}^* \omega_M + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t \\ &= p^* \omega_0 + t \lambda_t^* p^* (\omega_{SF} - T_{-\sigma}^* \sqrt{-1} \partial \bar{\partial} \xi) + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t \\ &= p^* \omega_0 + p^* \omega_{SF} - t \lambda_t^* p^* T_{-\sigma}^* \sqrt{-1} \partial \bar{\partial} \xi + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_t \\ &= p^* (\omega_0 + \omega_{SF}) + \sqrt{-1} \partial \bar{\partial} u_t, \end{aligned}$$

where for simplicity we write  $u_t = \tilde{\varphi}_t - t(\xi \circ T_{-\sigma} \circ p \circ \lambda_t)$ . The functions  $u_t$  are uniformly bounded in  $C^0(B \times \mathbb{C}^{n-m})$  because of the  $L^\infty$  bound for  $\varphi_t$

from [9, 10] and because  $\xi$  is a fixed function on  $U$ . The functions  $u_t$  satisfy the complex Monge-Ampère equations

$$(4.14) \quad (p^*\omega_0 + p^*\omega_{SF} + \sqrt{-1}\partial\bar{\partial}u_t)^n = c_t(p^*T_{-\sigma}^*\omega_M)^n$$

on  $B \times \mathbb{C}^{n-m}$ , and on any compact subset  $K$  of  $B \times \mathbb{C}^{n-m}$  the Kähler metric  $p^*(\omega_0 + \omega_{SF})$  is  $C^\infty$  equivalent to the Euclidean metric  $\delta$  (with constants that depend only on  $K$ ). The bounds (4.3) imply that

$$C^{-1}\delta \leq p^*(\omega_0 + \omega_{SF}) + \sqrt{-1}\partial\bar{\partial}u_t \leq C\delta,$$

on  $K$  for all small  $t > 0$ , where  $C$  depends on  $K$ . The constants  $c_t$  are bounded uniformly and away from zero. After shrinking  $K$  slightly we can then apply the Evans-Krylov theory (as explained for example in [13, 32]) and Schauder estimates to get higher order estimates  $\|u_t\|_{C^k(K,\delta)} \leq C(k)$  for all  $k \geq 0$ , thus proving (4.11).  $\square$

**Lemma 4.4.** *Given any compact set  $K \subset M \setminus S$  there is a constant  $C_K$  such that the sectional curvature of  $\tilde{\omega}_t$  satisfies*

$$(4.15) \quad \sup_K |\text{Sec}(\tilde{\omega}_t)| \leq C_K,$$

for all small  $t > 0$ .

*Proof.* We can assume that  $K$  is sufficiently small so that  $f(K) \subset B$  for a ball  $B$  as before, and that there is a compact set  $K' \subset B \times \mathbb{C}^{n-m}$  so that  $p : K' \rightarrow T_\sigma(K)$  is a biholomorphism. We then have

$$\begin{aligned} \sup_K |\text{Sec}(\tilde{\omega}_t)| &= \sup_{T_\sigma(K)} |\text{Sec}(T_{-\sigma}^*\tilde{\omega}_t)| = \sup_{K'} |\text{Sec}(p^*T_{-\sigma}^*\tilde{\omega}_t)| \\ &= \sup_{\lambda_t^{-1}(K')} |\text{Sec}(\lambda_t^*p^*T_{-\sigma}^*\tilde{\omega}_t)|. \end{aligned}$$

For  $t > 0$  small enough, the sets  $\lambda_t^{-1}(K)$  are all contained in a fixed compact set  $K'' \subset B \times \mathbb{C}^{n-m}$ . From (4.3) and (4.11) we then get a uniform bound for the sectional curvatures of  $\lambda_t^*p^*T_{-\sigma}^*\tilde{\omega}_t$  on  $K''$ , and this proves (4.15).  $\square$

**Lemma 4.5.** *Given any compact set  $K$  in  $B \times \mathbb{C}^{n-m}$  and any  $k \geq 0$  there exists a constant  $C$  independent of  $t > 0$  such that*

$$(4.16) \quad \|p^*T_{-\sigma}^*\tilde{\omega}_t\|_{C^k(K,\delta)} \leq C,$$

where  $\delta$  is the Euclidean metric on  $B \times \mathbb{C}^{n-m}$ .

*Proof.* Given  $K$ , for all  $t > 0$  small enough the sets  $\lambda_t^{-1}(K)$  are all contained in a fixed compact set  $K' \subset B \times \mathbb{C}^{n-m}$ . We wish to deduce (4.16) from (4.11).

To see this, write on  $B \times \mathbb{C}^{n-m}$

$$\begin{aligned} \lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t = & \sqrt{-1} \left( \sum_{i,j} A_{i\bar{j}}(t, y, z) dz^i \wedge d\bar{z}^j + \sum_{i,j} B_{i\bar{j}}(t, y, z) dy^i \wedge d\bar{y}^j \right. \\ & \left. + \sum_{i,j} C_{i\bar{j}}(t, y, z) dy^i \wedge d\bar{z}^j + \sum_{i,j} D_{i\bar{j}}(t, y, z) dz^i \wedge d\bar{y}^j \right). \end{aligned}$$

Thanks to (4.11), on  $K'$  the coefficients  $A, B, C, D$  satisfy uniform  $C^k$  estimates in the variables  $(y, z)$  independent of  $t$ . We then pull back this equation via the map  $\lambda_{1/t}$  (the inverse of  $\lambda_t$ ) and get

$$\begin{aligned} p^* T_{-\sigma}^* \tilde{\omega}_t = & \sqrt{-1} \left( t \sum_{i,j} A_{i\bar{j}}(t, y, z\sqrt{t}) dz^i \wedge d\bar{z}^j + \sum_{i,j} B_{i\bar{j}}(t, y, z\sqrt{t}) dy^i \wedge d\bar{y}^j \right. \\ & \left. + \sqrt{t} \sum_{i,j} C_{i\bar{j}}(t, y, z\sqrt{t}) dy^i \wedge d\bar{z}^j + \sqrt{t} \sum_{i,j} D_{i\bar{j}}(t, y, z\sqrt{t}) dz^i \wedge d\bar{y}^j \right), \end{aligned}$$

and the new coefficients are uniformly bounded in  $C^k$  on  $K$ , thus proving (4.16).  $\square$

**Proposition 4.6.** *As  $t$  goes to zero we have*

$$\tilde{\omega}_t \rightarrow f^* \omega$$

in  $C_{loc}^\infty(M \setminus S, \omega_M)$ , where  $\omega = \omega_N + \sqrt{-1} \partial \bar{\partial} \varphi$  is a Kähler metric on  $N \setminus f(S)$  with  $\text{Ric}(\omega) = \omega_{\text{WP}}$  as in Theorem 1.1.

*Proof.* Recall that  $\tilde{\omega}_t = \omega_0 + t\omega_M + \sqrt{-1} \partial \bar{\partial} \varphi_t$ , so that

$$p^* T_{-\sigma}^* \tilde{\omega}_t = p^* \omega_0 + t p^* T_{-\sigma}^* \omega_M + \sqrt{-1} \partial \bar{\partial} (\varphi_t \circ T_{-\sigma} \circ p).$$

We now fix a compact set  $K \subset M \setminus S$ , which we can assume is sufficiently small so that  $f(K) \subset B$  for a ball  $B$  as before, and that there is a compact set  $K' \subset B \times \mathbb{C}^{n-m}$  such that  $p : K' \rightarrow T_{-\sigma}(K)$  is a biholomorphism. From (4.16) (together with the  $L^\infty$  bound for  $\varphi_t$  from [9, 10]) we see that

$$\|\varphi_t \circ T_{-\sigma} \circ p\|_{C^k(K', \delta)} \leq C(k),$$

and therefore also

$$(4.17) \quad \|\varphi_t\|_{C^k(K, \omega_M)} \leq C(k),$$

since  $T_{-\sigma} \circ p : K' \rightarrow K$  is a fixed biholomorphism. From [38] we know that  $\varphi_t \rightarrow f^* \varphi$  in  $C_{loc}^{1,\alpha}(M \setminus S, \omega_M)$ , and so (4.17) implies that  $\varphi_t \rightarrow f^* \varphi$  in  $C_{loc}^\infty(M \setminus S, \omega_M)$ , and therefore that  $\tilde{\omega}_t \rightarrow f^* \omega$  in  $C_{loc}^\infty(M \setminus S, \omega_M)$ .  $\square$

As a corollary of this, for any compact subset  $K \subset M \setminus S$ , there is a positive function  $\varepsilon(t)$  which goes to zero as  $t \rightarrow 0$ , such that

$$(4.18) \quad f^* \omega - \varepsilon(t) \omega_M \leq \tilde{\omega}_t \leq f^* \omega + \varepsilon(t) \omega_M$$

on  $K$ , as well as

$$(4.19) \quad e^{-\varepsilon(t)} f^* \omega \leq \tilde{\omega}_t.$$

We now finish the proof of Theorem 1.1. We have already proved the first two statements in Proposition 4.6 and Lemma 4.4, and it remains to prove (1.3). We will present two proofs of (1.3), one which uses the fact that the fibers are tori, and another one which only uses the convergence result in Proposition 4.6.

For the first proof, we need the following lemma

**Lemma 4.7.** *As  $t$  goes to zero we have*

$$(4.20) \quad \lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t \rightarrow p^*(\omega_{SF} + f^* \omega)$$

in  $C_{loc}^\infty(B \times \mathbb{C}^{n-m}, \delta)$ , where  $\delta$  is the Euclidean metric.

*Proof.* Recall that from (4.13) we see that on  $B \times \mathbb{C}^{n-m}$

$$\lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t = p^*(\omega_0 + \omega_{SF}) + \sqrt{-1} \partial \bar{\partial} u_t,$$

where the functions  $u_t = \tilde{\varphi}_t - t \lambda_t^* p^* T_{-\sigma}^* \xi$  have uniform  $C^\infty$  bounds on compact sets. We need to show that as  $t$  goes to zero we have  $u_t \rightarrow (f \circ p)^* \varphi$  in  $C_{loc}^\infty(B \times \mathbb{C}^{n-m}, \delta)$ , where  $f^* \varphi$  is the  $C^{1,\alpha}$  limit of  $\varphi_t$  from [38]. To prove this we need another estimate from the second-named author's work [38, (3.9)], which implies that there is a constant  $C$  (that depends on the initial choice of  $B$ ) so that for all  $0 < t \leq 1$  we have

$$(4.21) \quad \sup_{y \in B} \text{osc}_{M_y} \varphi_t \leq Ct.$$

We now use this together with the fact that  $\varphi_t \rightarrow f^* \varphi$  in  $C^0$  to get that for any  $(y, z)$  in  $B \times \mathbb{C}^{n-m}$  we have

$$\begin{aligned} |\tilde{\varphi}_t(y, z) - (f \circ p)^* \varphi(y, z)| &= \left| \varphi_t \circ T_{-\sigma} \circ p \left( y, \frac{z}{\sqrt{t}} \right) - \varphi(y) \right| \\ &\leq \left| \varphi_t \circ p \left( y, \frac{z}{\sqrt{t}} - \tilde{\sigma}(y) \right) - \varphi_t \circ p(y, z) \right| \\ &\quad + |\varphi_t \circ p(y, z) - ((f^* \varphi) \circ p)(y)| \\ &\leq Ct + \sup_U |\varphi_t - f^* \varphi|, \end{aligned}$$

where in the last line we used (4.21) because the points  $p(y, \frac{z}{\sqrt{t}} - \tilde{\sigma}(y))$  and  $p(y, z)$  lie in the same fiber  $M_y$ . Letting  $t$  go to zero we see that  $\tilde{\varphi}_t \rightarrow (f \circ p)^* \varphi$  in  $C^0(B \times \mathbb{C}^{n-m})$ . On the other hand we have that  $t \lambda_t^* p^* \xi \rightarrow 0$  in  $C^0(B \times \mathbb{C}^{n-m})$ , and so  $u_t \rightarrow (f \circ p)^* \varphi$  in  $C^0(B \times \mathbb{C}^{n-m})$ . Thanks to the higher order estimates for  $u_t$ , we also have that  $u_t \rightarrow (f \circ p)^* \varphi$  in  $C_{loc}^\infty(B \times \mathbb{C}^{n-m}, \delta)$ , up to shrinking  $B$  slightly.  $\square$

We can now complete the proof of Theorem 1.1.



*Proof.* Recall that thanks to Lemma 4.7, on  $B \times \mathbb{C}^{n-m}$  we can write

$$\lambda_t^* p^* T_{-\sigma}^* \tilde{\omega}_t - p^*(\omega_{SF} + f^* \omega) = E_t,$$

where the error term  $E_t$  is a  $(1,1)$ -form that goes to zero smoothly on compact sets. From (4.8) we also have that

$$E_t = \lambda_t^* p^*(T_{-\sigma}^* \tilde{\omega}_t - f^* \omega - t\omega_{SF}).$$

If we restrict the form  $T_{-\sigma}^* \tilde{\omega}_t - f^* \omega - t\omega_{SF}$  to a fiber  $M_y$  and divide by  $t$  we get

$$\frac{E_t}{t} \Big|_{\{y\} \times \mathbb{C}^{n-m}} = \lambda_t^* p^* \left( \frac{T_{-\sigma}^* \tilde{\omega}_t|_{M_y}}{t} - \omega_{SF,y} \right)$$

Pulling back this via the map  $\lambda_{1/t}$  (the inverse of  $\lambda_t$ ) we get

$$\frac{\lambda_{1/t}^* E_t}{t} \Big|_{\{y\} \times \mathbb{C}^{n-m}} = p^* \left( \frac{T_{-\sigma}^* \tilde{\omega}_t|_{M_y}}{t} - \omega_{SF,y} \right).$$

Explicitly we have  $\lambda_{1/t}(y, z) = (y, z\sqrt{t})$ , which implies that  $\lambda_{1/t}^* dz^i = \sqrt{t} dz^i$ , and so

$$\frac{\lambda_{1/t}^* E_t}{t} \Big|_{\{y\} \times \mathbb{C}^{n-m}}(y, z) = E_t \Big|_{\{y\} \times \mathbb{C}^{n-m}}(y, z\sqrt{t}),$$

which goes to zero smoothly as  $t$  approaches zero, uniformly in  $y$ . It follows that  $\frac{T_{-\sigma}^* \tilde{\omega}_t|_{M_y}}{t}$  converges smoothly to  $\omega_{SF,y}$ , and the convergence is uniform as  $y$  varies on compact sets of  $N \setminus f(S)$ . Pulling back via  $T_\sigma$ , and using the fact that  $T_\sigma^* \omega_{SF,y} = \omega_{SF,y}$ , we see that also  $\frac{\tilde{\omega}_t|_{M_y}}{t}$  converges smoothly to  $\omega_{SF,y}$ , as desired.  $\square$

**Remark 4.8.** Note that in particular we get the estimate

$$\sup_{M_y} |\nabla(\tilde{\omega}|_{M_y})|_{\omega_M}^2 \leq Ct^2,$$

which improves [38, (2.11)].

We now give a second proof of (1.3). In fact we show that in general (1.3) follows from Proposition 4.6, without assuming that  $M$  is projective or that the fibers  $M_y$  are tori (in general  $M_y$  is a Calabi-Yau manifold). This will finish the proof of Theorem 1.1.

**Proposition 4.9.** *Assume the same setting as in the Introduction, except that  $M$  need not be projective and  $M_y$  need not be a torus. If we have that*

$$(4.22) \quad \tilde{\omega}_t \rightarrow f^* \omega$$

*in  $C_{loc}^\infty(M \setminus S, \omega_M)$ , where  $\omega$  is as before, then on each fiber  $M_y$  with  $y \in N \setminus f(S)$  we have*

$$(4.23) \quad \frac{\tilde{\omega}_t|_{M_y}}{t} \rightarrow \omega_{SF,y},$$

where  $\omega_{SF,y}$  is the unique Ricci-flat metric on  $M_y$  cohomologous to  $\omega_M|_{M_y}$  and the convergence is smooth and uniform as  $y$  varies on a compact subset of  $N \setminus f(S)$ .

*Proof.* For simplicity of notation call  $\omega_y = \omega_M|_{M_y}$  and  $\tilde{\omega}_y = \tilde{\omega}_t|_{M_y}$ . On each fiber  $M_y$  we have that  $\text{Ric}(\omega_y) = \sqrt{-1}\partial\bar{\partial}F_y$  for some smooth function  $F_y$  normalized by  $\int_{M_y}(e^{F_y} - 1)\omega_y^{n-m} = 0$ . The functions  $F_y$  vary smoothly in  $y \in N \setminus f(S)$ , because so do the Kähler metrics  $\omega_y$ . The unique Ricci-flat metric on  $M_y$  cohomologous to  $\omega_y$  is given by  $\omega_{SF,y} = \omega_y + \sqrt{-1}\partial\bar{\partial}\zeta_y$  and solves the complex Monge-Ampère equation on  $M_y$

$$\omega_{SF,y}^{n-m} = (\omega_y + \sqrt{-1}\partial\bar{\partial}\zeta_y)^{n-m} = e^{F_y}\omega_y^{n-m}.$$

Recall from [38, Section 2] that we have

$$\omega_0^m \wedge \omega_M^{n-m} = H\omega_M^n,$$

where  $H \geq 0$  is a smooth function on  $M$  that vanishes precisely on  $S$ . A simple calculation [38, (3.5)] shows that on  $M_y$  we have

$$\text{Ric}(\omega_y) = -\sqrt{-1}\partial\bar{\partial}\log H + (\text{Ric}(\omega_M))|_{M_y} = -\sqrt{-1}\partial\bar{\partial}\log H,$$

since we picked  $\omega_M$  to be Ricci-flat. It follows that on  $M_y$  the functions  $F_y$  and  $-\log H$  differ by a constant, which we can identify as follows: thanks to Yau's estimates, the functions  $\zeta_y$  vary smoothly in  $y$  and so they define a smooth function  $\zeta$  on  $M \setminus S$ . We then defined  $\omega_{SF} = \omega_M + \sqrt{-1}\partial\bar{\partial}\zeta$ , which is a semi-flat form on  $M \setminus S$  (here semi-flat means that its restriction to each fiber  $M_y$  is Ricci-flat). This semi-flat form is in general different from the one constructed locally in section 3, although they are equal when restricted to each fiber  $M_y$ . Even though  $\omega_{SF}$  is not necessarily nonnegative, on  $M \setminus S$  the  $(n, n)$ -form  $\omega_{SF}^{n-m} \wedge \omega_0^m$  is strictly positive, and so we can define a smooth positive function  $G$  on  $M \setminus S$  by

$$(4.24) \quad G = \frac{\omega_M^n}{\omega_0^m \wedge \omega_{SF}^{n-m}}.$$

It is shown in [35, Lemma 3.3], [38, p.445] that  $G$  is a positive constant on each fiber  $M_y$ , and we claim we have

$$(4.25) \quad e^{F_y} = \frac{1}{GH}.$$

This is because on  $M_y$  we have

$$\frac{1}{H} = \frac{\omega_M^n}{\omega_0^m \wedge \omega_M^{n-m}} = \frac{\omega_M^n}{\omega_0^m \wedge \omega_{SF}^{n-m}} \cdot \frac{\omega_{SF,y}^{n-m}}{\omega_y^{n-m}} = Ge^{F_y}.$$

On  $M_y$  we can then write, using (1.1), (4.25)

$$\begin{aligned}
\left(\frac{\tilde{\omega}_y}{t}\right)^{n-m} &= t^{m-n} \frac{\tilde{\omega}_y^{n-m}}{\omega_y^{n-m}} \omega_y^{n-m} = t^{m-n} \frac{\tilde{\omega}_y^{n-m} \wedge \omega_0^m}{\omega_M^{n-m} \wedge \omega_0^m} \omega_y^{n-m} \\
(4.26) \qquad &= \frac{\tilde{\omega}_y^{n-m} \wedge \omega_0^m}{\tilde{\omega}_t^n} \cdot \frac{c_t}{H} \omega_y^{n-m} \\
&= \frac{\tilde{\omega}_y^{n-m} \wedge \omega_0^m}{\tilde{\omega}_t^n} (c_t G) e^{F_y} \omega_y^{n-m}.
\end{aligned}$$

We also have a pointwise identity on  $M_y$

$$\frac{\tilde{\omega}_y^{n-m} \wedge \omega_0^m}{\tilde{\omega}_t^n} = \frac{\tilde{\omega}_y^{n-m} \wedge \omega_0^m}{\binom{n}{m} \tilde{\omega}_y^{n-m} \wedge \tilde{\omega}_t^m} = \frac{\omega_y^{n-m} \wedge \omega_0^m}{\binom{n}{m} \omega_y^{n-m} \wedge \tilde{\omega}_t^m},$$

and we will write

$$f_t = c_t G \frac{\omega_y^{n-m} \wedge \omega_0^m}{\binom{n}{m} \omega_y^{n-m} \wedge \tilde{\omega}_t^m},$$

so that we can recast (4.26) as

$$(4.27) \qquad \left(\frac{\tilde{\omega}_y}{t}\right)^{n-m} = f_t \omega_{SF,y}^{n-m}.$$

Notice that the functions  $f_t$  are the restriction to  $M_y$  of smooth functions on  $M \setminus S$ . We claim that as  $t$  approaches zero the functions  $f_t$  converge to 1 in  $C_{loc}^\infty(M \setminus S, \omega_M)$ . To see this, first of all note that by definition we have

$$(4.28) \qquad \lim_{t \rightarrow 0} c_t = \binom{n}{m} \frac{\int_M \omega_0^m \wedge \omega_M^{n-m}}{\int_M \omega_M^n} > 0,$$

see also [10], [38, (2.6)]. We now use the assumption (4.22), and so the functions  $f_t$  converge smoothly to

$$(4.29) \qquad G \binom{n}{m} \frac{\int_M \omega_0^m \wedge \omega_M^{n-m}}{\int_M \omega_M^n} \cdot \frac{\omega_M^{n-m} \wedge \omega_0^m}{\binom{n}{m} \omega_M^{n-m} \wedge (f^* \omega)^m}.$$

To see why this equals one, recall from [38, (4.3)] that the limit metric  $\omega$  on  $N \setminus f(S)$  satisfies

$$(4.30) \qquad \omega^m = G \frac{\int_M \omega_0^m \wedge \omega_M^{n-m}}{\int_M \omega_M^n} \omega_N^m,$$

where our function  $G$  is defined so that it differs from the function  $F$  in [38, (4.3)] by the constant factor  $\int_M (\omega_0 + \omega_M)^n / \int_M \omega_M^n$ . Substituting (4.30) into (4.29) we see that the limit of  $f_t$  equals

$$\frac{\omega_M^{n-m} \wedge \omega_0^m}{\omega_M^{n-m} \wedge (f^* \omega_N)^m} = 1.$$

Note now that from the main result of [38] we have that on each fiber  $M_y$

$$(4.31) \qquad C^{-1} \omega_y \leq \frac{\tilde{\omega}_y}{t} \leq C \omega_y,$$

where  $C$  is uniform as  $y$  varies in a compact set of  $N \setminus f(S)$ . From the definition on  $M_y$  we have

$$\frac{\tilde{\omega}_y}{t} = \omega_y + \sqrt{-1} \partial \bar{\partial} \left( \frac{\varphi_t}{t} \right),$$

where  $\frac{\varphi_t}{t}$  satisfies the  $C^0$  estimate (4.21). The metrics  $\frac{\tilde{\omega}_y}{t}$  satisfy the complex Monge-Ampère equations on  $M_y$

$$(4.32) \quad \left( \frac{\tilde{\omega}_y}{t} \right)^{n-m} = \left( \omega_y + \sqrt{-1} \partial \bar{\partial} \left( \frac{\varphi_t}{t} \right) \right)^{n-m} = f_t e^{F_y} \omega_y^{n-m},$$

and we have just shown that the functions  $f_t e^{F_y}$  are bounded in  $C^\infty(M_y, \omega_y)$  and away from zero, so we can apply the theory of Evans-Krylov and Schauder estimates on  $M_y$  to (4.32) (using (4.21) and (4.31)) to get bounds

$$\left\| \frac{\tilde{\omega}_y}{t} \right\|_{C^k(M_y, \omega_y)} \leq C(k),$$

independent of  $t$ . It follows that given any sequence  $t_i \rightarrow 0$  we can find a subsequence (still denoted by  $t_i$ ) and a smooth Kähler metric  $\alpha_y$  on  $M_y$  so that  $\frac{\tilde{\omega}_y}{t_i} \rightarrow \alpha_y$  in  $C^\infty(\omega_y)$ . Equation (4.27) in the limit becomes

$$\alpha_y^{n-m} = \omega_{SF,y}^{n-m},$$

and so by the uniqueness of Ricci-flat metrics in a given cohomology class we must have  $\alpha_y = \omega_{SF,y}$ . Therefore the whole sequence  $\frac{\tilde{\omega}_y}{t}$  converges smoothly to  $\omega_{SF,y}$  as desired, and the convergence is uniform as  $y$  varies on compact sets of  $N \setminus f(S)$ .  $\square$

**Remark 4.10.** In fact the proof of Proposition 4.9 shows that if we just have that  $\tilde{\omega}_t \rightarrow f^* \omega$  in  $C_{loc}^0(M \setminus S)$  (or in the  $C^2$  topology of Kähler potentials) then (1.3) holds in the  $C^{1,\alpha}$  topology of Kähler potentials. It seems that just having  $\tilde{\omega}_t \rightarrow f^* \omega$  in the  $C^{1,\alpha}$  topology of Kähler potentials (which is proved in [38] in general) is not quite enough to deduce (1.3).

## 5. GROMOV-HAUSDORFF CONVERGENCE

In this section we study the collapsed Gromov-Hausdorff limits of the Ricci-flat metrics  $\tilde{\omega}_t$  and prove Theorem 1.2.

**Lemma 5.1.** *There is an open subset  $X_0 \subset X$  such that  $(X_0, d_X)$  is locally isometric to  $(N_0, \omega)$  where  $N_0 = N \setminus f(S)$ , i.e. there is a homeomorphism  $\phi : N_0 \rightarrow X_0$  such that, for any  $y \in N_0$ , there is a neighborhood  $B_y \subset N_0$  of  $y$  satisfying that, if  $y_1$  and  $y_2 \in B_y$ ,*

$$d_\omega(y_1, y_2) = d_X(\phi(y_1), \phi(y_2)).$$

Furthermore, for any  $y \in N_0$ , there is a compact neighborhood  $B \subset N_0$  and a holomorphic section  $s : B \rightarrow f^{-1}(B)$ , i.e.,  $f \circ s = \text{id}$ , such that  $s(y) \rightarrow \phi(y)$  under the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ .

*Proof.* Let  $A$  be a countable dense subset of  $N_0$ , and  $K \subset N_0$  be a compact subset with the interior  $\text{int } K$  non-empty. Let  $\{B_i\}$  be a finite covering of  $K$  with small Euclidean balls such that each the concentric balls  $B'_i$  of half radius still cover  $K$ . Let  $s_i : B_i \rightarrow f^{-1}(B_i)$  be sections on  $B_i$ , i.e., holomorphic maps with  $f \circ s_i = \text{id}$ .

Now, we define a map  $\phi$  from  $A \cap K = \{a_1, a_2, \dots\}$  to  $X$ . Suppose that the point  $a_1$  lies inside the ball  $B'_i$ , and consider the points  $s_i(a_1)$  inside  $M$ . Under the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ , a subsequence of these points converges to a point  $b_1$  in  $X$ , because the diameter of  $(M, \tilde{\omega}_{t_k})$  is uniformly bounded. If  $a_1$  also lies inside another ball  $B'_j$ , then (1.3) (or also [38, (2.10)]) shows that  $d_{\tilde{\omega}_{t_k}}(s_i(a_1), s_j(a_1)) \rightarrow 0$  when  $t_k \rightarrow 0$ . Thus, by passing to subsequences, both  $s_i(a_1)$  and  $s_j(a_1)$  converge to the same point  $b_1 \in X$  under the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ . We then define  $\phi(a_1) = b_1$ . For  $a_2$ , by repeating the above procedure, we obtain that a subsequence  $s_{i_j}(a_j)$ ,  $j = 1, 2$ , converges to  $b_j \in X$ ,  $j = 1, 2$ , respectively. Define  $\phi(a_2) = b_2$ . By repeating this procedure and with a diagonal argument, we can find a subsequence of  $(M, \tilde{\omega}_{t_k})$ , denoted by  $(M, \tilde{\omega}_{t_k})$  also, such that  $s_{i_j}(a_j)$  converges to  $b_j \in X$  along the Gromov-Hausdorff convergence. For any  $a_j \in A \cap K$ , define  $\phi(a_j) = b_j$ .

Now, we prove that  $\phi : A \cap \text{int } K \rightarrow X$  is injective. If it is not true, there are  $y_1, y_2 \in A \cap \text{int } K$  such that  $y_1 \neq y_2$ , and  $\phi(y_1) = \phi(y_2)$ , which implies  $d_{\tilde{\omega}_{t_k}}(s_{i_1}(y_1), s_{i_2}(y_2)) \rightarrow 0$ . If  $\gamma_k$  is a minimal geodesic in  $(M, \tilde{\omega}_{t_k})$  connecting  $s_{i_1}(y_1)$  and  $s_{i_2}(y_2)$ , then

$$C^{-1} \text{length}_{\omega_N}(f(\gamma_k) \cap K) \leq \text{length}_{\tilde{\omega}_{t_k}}(\gamma_k \cap f^{-1}(K)) \leq d_{\tilde{\omega}_{t_k}}(s_{i_1}(y_1), s_{i_2}(y_2)),$$

by (4.1) for a constant  $C > 0$  independent of  $k$ . Thus, if  $f(\gamma_k) \subset K$  for  $t_k \ll 1$ ,

$$d_{\omega_N}(y_1, y_2) \leq C \text{length}_{\omega_N}(f(\gamma_k)) \rightarrow 0,$$

or, if  $f(\gamma_k) \cap N \setminus K$  are not empty by passing to a subsequence,

$$d_{\omega_N}(y_1, \partial K) + d_{\omega_N}(\partial K, y_2) \leq C \text{length}_{\omega_N}(f(\gamma_k) \cap K) \rightarrow 0.$$

In both cases, we obtain contradictions. Thus  $\phi : A \cap \text{int } K \rightarrow X$  is injective.

Note that there is a  $r > 0$  such that, for any  $y \in \text{int } K$ , the metric ball  $B_\omega(y, r)$  is a geodesically convex set, i.e. for any  $y_1$  and  $y_2 \in B_\omega(y, r)$ , there is a minimal geodesic  $\gamma \subset B_\omega(y, r)$  connecting  $y_1$  and  $y_2$ , which implies

$$d_\omega(y_1, y_2) = \text{length}_\omega(\gamma) \leq 2r.$$

We take  $r \ll 1$  such that there is a  $B'_i$  with  $B_\omega(y, 2r) \subset B'_i$ . If  $y_1, y_2 \in A$ , by Proposition 4.6,

$$\begin{aligned} d_X(\phi(y_1), \phi(y_2)) &= \lim_{t_k \rightarrow 0} d_{\tilde{\omega}_{t_k}}(s_i(y_1), s_i(y_2)) \\ &\leq \lim_{t_k \rightarrow 0} \text{length}_{\tilde{\omega}_{t_k}}(s_i(\gamma)) \\ &= \text{length}_\omega(\gamma) \\ &= d_\omega(y_1, y_2). \end{aligned}$$

If  $\gamma_k$  is a minimal geodesic in  $(M, \tilde{\omega}_{t_k})$  connecting  $s_i(y_1)$  and  $s_i(y_2)$ , then (4.19) implies that

$$e^{-\frac{\varepsilon(t_k)}{2}} \text{length}_\omega(f(\gamma_k) \cap B_\omega(y, 2r)) \leq \text{length}_{\tilde{\omega}_{t_k}}(\gamma_k) \longrightarrow d_X(\phi(y_1), \phi(y_2)),$$

for some function  $\varepsilon(t) \rightarrow 0$  as  $t \rightarrow 0$ . If  $f(\gamma_k) \subset B_\omega(y, 2r)$  for  $t_k \ll 1$  by passing to a subsequence,

$$\text{length}_\omega(f(\gamma_k)) \geq \text{length}_\omega(\gamma),$$

since  $\gamma$  is a minimal geodesic in  $(N_0, \omega)$ . If  $f(\gamma_k) \cap N_0 \setminus B_\omega(y, 2r)$  is not empty for  $t_k \ll 1$ , then there is a  $\bar{y} \in f(\gamma_k) \cap N_0 \setminus B_\omega(y, 2r)$ . Since  $y_1, y_2 \in B_\omega(y, r)$  and  $f(\gamma_k)$  connects  $y_1$  and  $y_2$ ,

$$\text{length}_\omega(f(\gamma_k) \cap B_\omega(y, 2r)) \geq d_\omega(y_1, \bar{y}) + d_\omega(y_2, \bar{y}) \geq 2r \geq \text{length}_\omega(\gamma).$$

In both cases,

$$\begin{aligned} d_\omega(y_1, y_2) &= \text{length}_\omega(\gamma) \\ &\leq \lim_{t_k \rightarrow 0} \text{length}_\omega(f(\gamma_k) \cap B_\omega(y, 2r)) \\ &\leq d_X(\phi(y_1), \phi(y_2)). \end{aligned}$$

Thus

$$d_\omega(y_1, y_2) = d_X(\phi(y_1), \phi(y_2)),$$

i.e.  $\phi : (A \cap \text{int } K, d_\omega) \longrightarrow (X, d_X)$  is a local isometric embedding. If  $\{y_{1,j}\}$  and  $\{y_{2,j}\}$  are two sequences in  $A \cap \text{int } K$  such that  $\lim_{j \rightarrow \infty} d_\omega(y_{i,j}, y) = 0$  for  $i = 1, 2$ , then  $\lim_{j \rightarrow \infty} d_\omega(y_{1,j}, y_{2,j}) = 0$  and  $\{y_{1,j}, y_{2,j}\} \subset B_\omega(y, r)$  for  $j \gg 1$ . Hence  $d_\omega(y_{1,j}, y_{2,j}) = d_X(\phi(y_{1,j}), \phi(y_{2,j}))$  and  $d_\omega(y_{i,j}, y_{i,j+\ell}) = d_X(\phi(y_{i,j}), \phi(y_{i,j+\ell}))$  for  $j \gg 1$  and any  $\ell \geq 0$ , which implies that  $\{\phi(y_{1,j})\}$  and  $\{\phi(y_{2,j})\}$  are two Cauchy sequences, and converge to a unique point  $x \in X$ . By defining  $\phi(y) = x$ ,  $\phi$  extends to a unique map, denoted still by  $\phi$ , from  $\text{int } K$  to  $X$  which is also a local isometric embedding.

Now we prove that  $\phi(\text{int } K)$  is an open subset of  $X$ . Let  $x \in \phi(\text{int } K)$ , i.e. there is a  $y \in \text{int } K$  such that  $\phi(y) = x$ , and let  $x' \in X$  with  $d_X(x, x') < \rho$  for a constant  $\rho < \frac{1}{8}d_\omega(y, \partial K)$ . From the above construction,  $y \in B'_i$  for a  $B'_i$ , and  $s_i(y) \rightarrow x$  under Gromov-Hausdorff convergence. There is a sequence of points  $p_k \in (M, \tilde{\omega}_{t_k})$  such that  $p_k \rightarrow x'$  under the Gromov-Hausdorff convergence. If  $\gamma'_k$  is a minimal geodesic connecting  $s_i(y)$  and  $p_k$

in  $(M, \tilde{\omega}_{t_k})$ , then

$$d_{\tilde{\omega}_{t_k}}(s_i(y), p_k) = \text{length}_{\tilde{\omega}_{t_k}}(\gamma'_k) \longrightarrow d_X(x, x').$$

Equation (4.19) implies that, for  $k \gg 1$ ,

$$\begin{aligned} \frac{1}{2} \text{length}_{\omega}(f(\gamma'_k) \cap K) &\leq e^{-\frac{\varepsilon(t_k)}{2}} \text{length}_{\omega}(f(\gamma'_k) \cap K) \\ &\leq \text{length}_{\tilde{\omega}_{t_k}}(\gamma'_k) \\ &< 2\rho < \frac{1}{4} d_{\omega}(y, \partial K). \end{aligned}$$

Thus  $f(p_k) \in K' \subset \text{int } K$  where  $K'$  is a compact subset of  $\text{int } K$ . By passing to a subsequence,  $f(p_k) \rightarrow y'$  in  $(K', \omega)$ . By Proposition 4.6,  $d_{\tilde{\omega}_{t_k}}(p_k, s_{i_k}(f(p_k))) \rightarrow 0$  when  $t_k \rightarrow 0$ , and, thus,  $s_{i_k}(f(p_k)) \rightarrow x'$  under the Gromov-Hausdorff convergence. The above construction shows that  $\phi(y') = x'$ , which implies that  $\{x' | d_X(x, x') < \rho\} \subset \phi(\text{int } K)$ . Hence  $\phi(\text{int } K)$  is open, and  $\phi : \text{int } K \rightarrow \phi(\text{int } K)$  is a homeomorphism.

Let  $K_0 \subset \cdots \subset K_j \subset K_{j+1} \subset \cdots \subset N_0$  be a family of compact subsets with  $N_0 = \bigcup_j \text{int } K_j$ . Given each  $K_j$ , the above argument con-

structs a local isometric embedding  $\phi_j : (\text{int } K_j, \omega) \rightarrow (X, d_X)$ , which is a homeomorphism onto the image  $\phi_j(\text{int } K_j)$ . By the same argument as above,  $\phi_j$  extends to a local isometric embedding  $\phi_{j+1} : (\text{int } K_{j+1}, \omega) \rightarrow (X, d_X)$ , i.e.  $\phi_{j+1}|_{\text{int } K_j} = \phi_j$ , which is a homeomorphism onto the image  $\phi_{j+1}(\text{int } K_{j+1})$ . By a diagonal argument, we obtain a local isometry  $\phi : (N_0, \omega) \rightarrow (\phi(N_0), d_X) \subset (X, d_X)$ .  $\square$

The above lemma proves the existence of  $\phi$  in Theorem 1.2, and is an analog of Lemma 4.1 in [30] for the collapsing case. In the rest of this section, we prove that  $X_0 = \phi(N_0)$  is dense in  $X$ .

Let  $\bar{x} \in X_0$  and  $\bar{p}_k \in M$  such that  $\bar{p}_k \rightarrow \bar{x}$  under the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ , and let

$$\underline{V}_k(p, r) = \frac{\text{Vol}_{\tilde{\omega}_{t_k}}(B_{\tilde{\omega}_{t_k}}(p, r))}{\text{Vol}_{\tilde{\omega}_{t_k}}(B_{\tilde{\omega}_{t_k}}(\bar{p}_k, 1))},$$

for any  $p \in M$  and  $r > 0$ . By Theorem 1.6 in [5], there is a continuous function  $\underline{V}_0 : X \times [0, \infty) \rightarrow [0, \infty)$  such that, if  $p_k \rightarrow x$  under the convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ , then

$$(5.1) \quad \underline{V}_k(p_k, r) \longrightarrow \underline{V}_0(x, r).$$

By Theorem 1.10 in [5],  $\underline{V}_0$  induces a unique Radon measure  $\nu$  on  $X$  such that

$$(5.2) \quad \nu(B_{d_X}(x, r)) = \underline{V}_0(x, r), \quad \text{and} \quad \frac{\nu(B_{d_X}(x, r_1))}{\nu(B_{d_X}(x, r_2))} \geq \mu(r_1, r_2) > 0,$$

for any  $x \in X$ ,  $r_1 \leq r_2$ , where  $\mu(r_1, r_2)$  is a function of  $r_1$  and  $r_2$ . For any compact subset  $K \subset X$ ,

$$\nu(K) = \lim_{\delta \rightarrow 0} \nu_\delta(K) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i V_0(x_i, r_i) \mid r_i < \delta \right\},$$

where  $\bigcup_i B_{d_X}(x_i, r_i) \supset K$ . By scaling  $\tilde{\omega}_t$  and  $\omega$  by one positive number, we assume that  $B_\omega(\phi^{-1}(\bar{x}), 2) \subset N_0$  and is a geodesically convex set.

**Lemma 5.2.** *There is a constant  $v > 0$  such that*

$$\nu(X) = v \int_M \omega_M^n, \quad V_0(x, r) = v \int_{f^{-1}(B_\omega(\phi^{-1}(x), r))} \omega_M^n,$$

whenever  $x \in X_0$  and  $r \leq 1$  is such that  $B_\omega(\phi^{-1}(x), 2r)$  is a geodesically convex subset of  $(N_0, \omega)$ .

*Proof.* If  $p_k \rightarrow x$  under the convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ , we claim that a subsequence of  $p_k$  converges to a point  $p' \in f^{-1}(\phi^{-1}(x))$  under the metric  $\omega_M$  on  $M$ . By Lemma 5.1, there is a compact neighborhood  $B \subset N_0$  of  $\phi^{-1}(x)$  and a section  $s : B \rightarrow f^{-1}(B)$  such that  $s(\phi^{-1}(x)) \rightarrow x$  under the Gromov-Hausdorff convergence of  $(M, \tilde{\omega}_{t_k})$  to  $(X, d_X)$ . Thus  $d_{\tilde{\omega}_{t_k}}(p_k, s(\phi^{-1}(x))) \rightarrow 0$  when  $t_k \rightarrow 0$ . By Lemma 4.1, there are curves  $\gamma_k$  connecting  $p_k$  and  $s(\phi^{-1}(x))$  such that  $\text{length}_{\tilde{\omega}_{t_k}}(\gamma_k) = d_{\tilde{\omega}_{t_k}}(p_k, s(\phi^{-1}(x)))$ , and

$$\text{length}_{\omega_0}(f(\gamma_k) \cap B) = \text{length}_{f^*\omega_0}(\gamma_k \cap f^{-1}(B)) \leq C^{\frac{1}{2}} \text{length}_{\tilde{\omega}_{t_k}}(\gamma_k) \rightarrow 0.$$

For a  $k \gg 1$ , if there is a  $y_k \in f(\gamma_k) \setminus B$ , then

$$\text{length}_{\omega_0}(f(\gamma_k) \cap B) \geq d_{\omega_0}(y_k, \phi^{-1}(x)) \geq \rho,$$

where  $\rho > 0$  such that  $B_{\omega_0}(\phi^{-1}(x), \rho) \subset B$ , which is a contradiction. Thus  $f(\gamma_k) \subset B$  for  $k \gg 1$ ,  $\text{length}_{\omega_0}(f(\gamma_k)) \rightarrow 0$  and  $f(p_k)$  converges to  $\phi^{-1}(x)$  under the metric  $\omega_0$ . By passing to a subsequence,  $p_k$  converges to a point  $p'$  under the metric  $\omega_M$ . Since  $f^*\omega_0 \leq C'\omega_M$  for a constant  $C' > 0$ ,  $d_{\omega_0}(f(p_k), f(p')) \leq C'^{\frac{1}{2}} d_{\omega_M}(p_k, p') \rightarrow 0$ . Hence  $f(p') = \phi^{-1}(x)$  and  $p' \in f^{-1}(\phi^{-1}(x))$ .

Let  $r$  satisfy  $r \leq 1$ , and  $B_\omega(\phi^{-1}(x), 2r)$  is a geodesically convex subset of  $(N_0, \omega)$ . If  $q \in f^{-1}(B_\omega(\phi^{-1}(x), 2r))$ , there is a curve  $\bar{\gamma}$  connecting  $p'$  and  $q$  such that  $f(\bar{\gamma})$  is the unique minimal geodesic connecting  $\phi^{-1}(x)$  and  $f(q)$ . Thanks to (4.18) we have

$$f^*\omega - \varepsilon(t_k)\omega_M \leq \tilde{\omega}_{t_k} \leq f^*\omega + \varepsilon(t_k)\omega_M$$

where  $\varepsilon(t_k) \rightarrow 0$  when  $t_k \rightarrow 0$ , on  $f^{-1}(B_\omega(\phi^{-1}(x), 2r))$ . We obtain that

$$\begin{aligned} d_{\tilde{\omega}_{t_k}}(p', q) &\leq \text{length}_{\tilde{\omega}_{t_k}}(\bar{\gamma}) \\ &\leq \text{length}_\omega(f(\bar{\gamma})) + C\varepsilon(t_k)^{\frac{1}{2}} \\ &= d_\omega(\phi^{-1}(x), f(q)) + C\varepsilon(t_k)^{\frac{1}{2}}. \end{aligned}$$



If  $\bar{\gamma}_k$  is a minimal geodesic of  $\tilde{\omega}_{t_k}$  connecting  $p'$  and  $q$ , then (4.19) gives

$$d_{\tilde{\omega}_{t_k}}(p', q) = \text{length}_{\tilde{\omega}_{t_k}}(\bar{\gamma}_k) \geq e^{-\frac{\varepsilon(t_k)}{2}} \text{length}_{\omega}(f(\bar{\gamma}_k) \cap B_{\omega}(\phi^{-1}(x), 2r)).$$

If  $f(\bar{\gamma}_k) \subset B_{\omega}(\phi^{-1}(x), 2r)$ , then

$$\text{length}_{\omega}(f(\bar{\gamma}_k) \cap B_{\omega}(\phi^{-1}(x), 2r)) \geq \text{length}_{\omega}(f(\bar{\gamma})) = d_{\omega}(\phi^{-1}(x), f(q)),$$

and, otherwise,

$$\text{length}_{\omega}(f(\bar{\gamma}_k) \cap B_{\omega}(\phi^{-1}(x), 2r)) \geq 2r \geq \text{length}_{\omega}(f(\bar{\gamma})) = d_{\omega}(\phi^{-1}(x), f(q)),$$

by the same argument as in the proof of Lemma 5.1. Thus

$$e^{-\frac{\varepsilon(t_k)}{2}} d_{\omega}(\phi^{-1}(x), f(q)) \leq d_{\tilde{\omega}_{t_k}}(p', q) \leq d_{\omega}(\phi^{-1}(x), f(q)) + C\varepsilon(t_k)^{\frac{1}{2}}$$

where  $C$  is a constant independent of  $t_k$ ,  $p'$  and  $q$ . Of course if  $k$  is large we will have that

$$d_{\omega}(\phi^{-1}(x), f(q)) - C\varepsilon(t_k)^{\frac{1}{2}} \leq e^{-\frac{\varepsilon(t_k)}{2}} d_{\omega}(\phi^{-1}(x), f(q)).$$

Thanks to (4.1), there is constant  $C > 0$  independent of  $t_k$  such that  $\tilde{\omega}_{t_k} \leq C\omega_M$  on  $f^{-1}(B_{\omega}(\phi^{-1}(x), 2r))$ . Let  $\gamma'_k$  be minimal geodesics of  $\omega_M$  connecting  $p_k$  and  $p'$ , which satisfy  $\gamma'_k \subset f^{-1}(B_{\omega}(\phi^{-1}(x), 2r))$  for  $k \gg 1$ . Thus

$$d_{\tilde{\omega}_{t_k}}(p', p_k) \leq \text{length}_{\tilde{\omega}_{t_k}}(\gamma'_k) \leq C^{\frac{1}{2}} \text{length}_{\omega_M}(\gamma'_k) = C^{\frac{1}{2}} d_{\omega_M}(p', p_k) \rightarrow 0.$$

The triangle inequality shows that

$$|d_{\tilde{\omega}_{t_k}}(p_k, q) - d_{\omega}(\phi^{-1}(x), f(q))| \leq C\varepsilon(t_k)^{\frac{1}{2}} + C^{\frac{1}{2}} d_{\omega_M}(p', p_k).$$

Hence there is a function  $\rho(t_k)$  of  $t_k$  such that  $\rho(t_k) \rightarrow 0$  when  $t_k \rightarrow 0$ , and

$$f^{-1}(B_{\omega}(\phi^{-1}(x), r - \rho(t_k))) \subset B_{\tilde{\omega}_{t_k}}(p_k, r) \subset f^{-1}(B_{\omega}(\phi^{-1}(x), r + \rho(t_k))).$$

We obtain that

$$\lim_{t_k \rightarrow 0} \int_{B_{\tilde{\omega}_{t_k}}(p_k, r)} \omega_M^n = \int_{f^{-1}(B_{\omega}(\phi^{-1}(x), r))} \omega_M^n.$$

Note that

$$\tilde{\omega}_{t_k}^n = c_{t_k} t_k^{n-m} \omega_M^n.$$

Hence

$$\begin{aligned} \underline{V}_k(p_k, r) &= \frac{\text{Vol}_{\tilde{\omega}_{t_k}}(B_{\tilde{\omega}_{t_k}}(p_k, r))}{\text{Vol}_{\tilde{\omega}_{t_k}}(B_{\tilde{\omega}_{t_k}}(\bar{p}_k, 1))} \\ &= \frac{\int_{B_{\tilde{\omega}_{t_k}}(p_k, r)} c_{t_k} t_k^{n-m} \omega_M^n}{\int_{B_{\tilde{\omega}_{t_k}}(\bar{p}_k, 1)} c_{t_k} t_k^{n-m} \omega_M^n} \rightarrow \frac{\int_{f^{-1}(B_{\omega}(\phi^{-1}(x), r))} \omega_M^n}{\int_{f^{-1}(B_{\omega}(\phi^{-1}(\bar{x}), 1))} \omega_M^n}, \end{aligned}$$

when  $t_k \rightarrow 0$ . By (5.1),

$$\underline{V}_0(x, r) = v \int_{f^{-1}(B_{\omega}(\phi^{-1}(x), r))} \omega_M^n, \quad \text{where } v = \left( \int_{f^{-1}(B_{\omega}(\phi^{-1}(\bar{x}), 1))} \omega_M^n \right)^{-1}.$$

Recall the diameter bound (1.4)

$$\text{diam}_{\tilde{\omega}_{t_k}}(M) \leq D$$

for a constant  $D > 0$ . Using (5.1), we have

$$\nu(X) = \underline{V}_0(x, D) = \lim_{t_k \rightarrow 0} \underline{V}_k(p_k, D) = v \int_M \omega_M^n.$$

□

*Proof of Theorem 1.2.* We prove that  $X_0 \subset X$  is dense. If this is not true, there is a metric ball  $B_{d_X}(x', \rho) \subset X \setminus X_0$ . Note that

$$\text{diam}_{d_X}(X) = \lim_{t_k \rightarrow 0} \text{diam}_{\tilde{\omega}_{t_k}}(M) \leq D.$$

Because of (5.2), we have

$$\nu(B_{d_X}(x', \rho)) \geq \mu(\rho, D)\nu(X) = \varpi > 0.$$

For any compact subset  $K \subset X_0$ ,

$$\nu(K) \leq \nu(X) - \varpi = v \int_M \omega_M^n - \varpi$$

by Lemma 5.2. If  $B_{d_X}(x_i, r_i)$  is a family of metric balls in  $(X, d_X)$  such that  $r_i < \delta \ll 1$ ,  $B_{d_X}(x_i, 2r_i)$  is a geodesically convex subset of  $X_0$ , and  $\bigcup_i B_{d_X}(x_i, r_i) \supset K$ , then

$$\sum_i \underline{V}_0(x_i, r_i) = \sum_i v \int_{f^{-1}(\phi^{-1}(B_{d_X}(x_i, r_i)))} \omega_M^n \geq v \int_{f^{-1}(\phi^{-1}(K))} \omega_M^n$$

by Lemma 5.2. Thus

$$v \int_{f^{-1}(\phi^{-1}(K))} \omega_M^n \leq \lim_{\delta \rightarrow 0} \nu_\delta(K) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_i \underline{V}_0(x_i, r_i) \mid r_i < \delta \right\} = \nu(K).$$

By taking  $K$  large enough such that

$$\nu(K) \geq v \int_{f^{-1}(N_0)} \omega_M^n - \frac{\varpi}{2} = v \int_M \omega_M^n - \frac{\varpi}{2},$$

we obtain a contradiction. □

**Remark 5.3.** In fact, the same proof shows that  $\nu(X \setminus X_0) = 0$ .

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