On Generalizations of Connes-Moscovici Characteristic Map

Mohammad HASSANZADEH

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Mars 2012

IHES/M/12/04

On Generalizations of Connes-Moscovici Characteristic Map

Mohammad Hassanzadeh

Institut des Hautes Études Scientifiques
IHES
mhassanz@ihes.fr

Abstract

In this paper we generalize the Connes-Moscovici characteristic map for cyclic cohomology of extended version of Hopf algebras called $\times$-Hopf algebras. To do this, we define a pairing for cyclic cohomology of module algebras and module coalgebras under the symmetry of a $\times$-Hopf algebra. We introduce more examples of similar generalized characteristic maps for quantum algebraic torus and enveloping algebras.

1 Introduction

The Connes-Moscovici characteristic map introduced in [CM98] has many applications in index theory, number theory and Hopf cyclic cohomology which is introduced in [CM01] and [CM00]. The characteristic map has been discovered in the study of computing the index of a transversally elliptic operator on a foliation. Connes and Moscovici introduced the Hopf algebras $\mathcal{H}_n$ which act on the algebra $\mathcal{A}_{FM} := C_c^\infty(FM \rtimes \Gamma)$ where M is a flat affine manifold, FM is the $GL^+(n,\mathbb{R})$-principal bundle of oriented frame on M and Γ is the pseudogroup of orientation preserving local diffeomorphisms of M. This action turns $\mathcal{A}_{FM}$ in to a $\mathcal{H}_n$-module algebra. Let (δ,σ) be a modular pair in involution for $\mathcal{H}_n$ and Tr be a δ-invariant σ-trace on $\mathcal{A}_{FM}$, then the following map

$$\gamma_{Tr} : \mathcal{H}_n^{\otimes n} \longrightarrow \operatorname{Hom}(\mathcal{A}_{FM}^{\otimes(n+1)}, \mathbb{C}), \tag{1.1}$$
$$\gamma_{Tr}(h_1 \otimes \cdots \otimes h_n)(a_0 \otimes \cdots \otimes a_n) = Tr(a_0 h_1(a_1) \cdots h_n(a_n)).$$

defines a map of cocyclic modules. Therefore they obtained the following characteristic map on the level of periodic cyclic cohomology.

$$\gamma_{Tr} : HP^*(\mathcal{H}_n) \longrightarrow HP^*(\mathcal{A}_{FM}),$$

where the left hand side stands for the Hopf cyclic cohomology of Connes-Moscovici Hopf algebra and on the right we have the cyclic cohomology of the algebra $\mathcal{A}_{FM}$. The great idea here is that although computing the cyclic cohomology of the algebras such as $\mathcal{A}_{FM}$ is a difficult task, one can compute the Hopf cyclic cohomology of the related symmetry as

$\mathcal{H}_n$ in this case and then use the characteristic map to transfer the cocycles and therefore information from the Hopf algebra of the symmetry to the algebra in question. In fact the terms of the form (1.1) appear in the computation of the local index formula of an elliptic operator D using the Chern-Connes character which is a finite sum of the expressions of the following form;

$$\int a_0[D, a_1]^{(k_1)} \cdots [D, a_m]^{(k_m)} |D|^{(-m+2k_1 \cdots +2k_m)}.$$

Here $a_i \in \mathcal{A}_{FM}$ and $[D, a]^{(k)}$ denotes the k-th iterated commutator of D^2 and $\int$ is the Diximier trace.

Later different extendings of the Connes-Moscovici characteristic map for Hopf algebras have been introduced in [KR3], [CR], [NS] and [Kay2]. Finally the author in [Kay3] has proved that all of these different setups produce isomorphic characteristic maps. Let us recall that the invariant trace Tr in fact is a 0-dimensional Hopf-cyclic cocycle and therefore the characteristic map can be viewed as the following pairing.

$$HP^0_{\mathcal{H}_n}(\mathcal{A}_{FM}, {}^1\mathbb{C}_\delta) \otimes HP^p_{\mathcal{H}_n}(\mathcal{H}_n, {}^1\mathbb{C}_\delta) \longrightarrow HP^p(\mathcal{A}_{FM}). \tag{1.2}$$

The authors in [KR2] have shown that if $\mathcal{H}$ is an arbitrary Hopf algebra, A a $\mathcal{H}$-module algebra and M a stable anti Yetter-Drinfeld module over $\mathcal{H}$ then there is a pairing of the following form;

$$HP^p_{\mathcal{H}}(A, M) \otimes HP^q_{\mathcal{H}}(\mathcal{H}, M) \longrightarrow HP^{p+q}(A). \tag{1.3}$$

Also there is a similar pairing on the level of cyclic and Hochschild cohomology. We refer the reader for more about Connes-Moscovici Characteristic map to [Kay1].

Connes and Moscovici's index theory of transversally elliptic operators lead beyond cyclic cohomology of Hopf algebras. Later in [CM01], Connes and Moscovici introduced a Hopf algebroid $\mathcal{H}_{FM}$ of transverse differential operators on $FM \overline{\rtimes}\, \Gamma_M$, the etale groupoid of germs of diffeomorphisms of M lifted to its frame bundle FM. It is now known that this is a $\times$-Hopf algebra. They have shown that $\mathcal{H}_{FM}$ acts on the algebra $\mathcal{A}_{FM} = C_c^\infty(FM \overline{\rtimes}\, \Gamma_M)$ and turns it in to a $\mathcal{H}_{FM}$-module algebra. They introduced an invariant faithful trace on $\mathcal{A}_{FM}$ and obtained a characteristic map as follows;

$$\gamma : HC^n(\mathcal{H}_{FM}) \longrightarrow HC^n(\mathcal{A}_{FM}). \tag{1.4}$$

In this paper, we will extend Connes-Moscovici characteristic map and define a version for $\times$-Hopf algebras.

This paper is organized as follows. In Section 2 we recall the basics of $\times$-Hopf algebras and study the $\times$-Hopf algebra structures of Connes-Moscovici Hopf algebroid, Kadisson bialgebroid, quantum algebraic torus and enveloping algebras. Also we review the module and comodule structures on $\times$-Hopf algebras. Specially we study the stable anti-Yetter-Drinfeld (SAYD) modules on the major examples of $\times$-Hopf algebras. In Section 3 we recall Hopf cyclic cohomology of $\times$-Hopf algebras and compute some examples. In Section 4 we introduce a pairing between cyclic cohomology of module algebras and module coalgebras under the symmetry of a $\times$-Hopf algebra. In this way we obtain a generalization of Connes-Moscovici characteristic map (1.4) for an extended version of Hopf algebras.

Acknowledgments: The author would like to thank the Institut des Hautes Études Scientifiques, IHES, for its hospitality and financial supports during his stay when the whole work was accomplished.

Contents

2 Preliminaries

In this section, we recall the definition and basic properties of $\times$-Hopf algebras. The notion of $\times$-Hopf algebras is introduced by Schauenburg in [Sch98]. It is a bialgebroid ($\times$-bialgebra) which is defined by Takeuchi [Tak] satisfying certain conditions. The notion of $\times$-Hopf algebras extends the Böhm-Szlachányi's Hopf algebroids [BSz] and many nice examples of J. H. Lu' Hopf algebroids [Lu]. One notes that a Böhm-Szlachányi's Hopf algebroid is not necessarily a Lu'Hopf algebroid and vice versa. Some nice quantum groupoids such as weak Hopf algebras with invertible antipodes and also Khalkhali-Rangipour's Para-Hopf algebroid [KR3] are examples of Böhm-Szlachányi's Hopf algebroids. Another interesting example is Connes-Moscovici Hopf algebroid which is originally understood as Lu'Hopf algebroid and also satisfies Böhm- Szlachányi's axioms. It is known that any Böhm-Szlachányi's Hopf algebroid (with invertible antipode) is also a $\times$-Hopf algebra.

Let R be an algebra over the field of complex numbers $\mathbb{C}$. A left bialgebroid $\mathcal{K}$ over R consists of the data $(\mathcal{K}, \mathfrak{s}, \mathfrak{t})$. Here $\mathcal{K}$ is a $\mathbb{C}$-algebra, $\mathfrak{s} : R \longrightarrow \mathcal{K}$ and $\mathfrak{t} : R^{op} \longrightarrow \mathcal{K}$ are $\mathbb{C}$-algebra maps such that their range commute with one another. In terms of $\mathfrak{s}$ and $\mathfrak{t}$, $\mathcal{K}$ can be equipped with a R-bimodule structure as follows;

$$r_1.k.r_2 = \mathfrak{s}(r_1)\mathfrak{t}(r_2)k,$$

for all $r_1, r_2 \in R$ and $k \in \mathcal{K}$. Similarly, $\mathcal{K} \otimes_R \mathcal{K}$ is endowed with a natural R-bimodule structure. Also we assume that there is a R-bimodule maps $\Delta : \mathcal{K} \longrightarrow \mathcal{K} \otimes_R \mathcal{K}$ called coproduct and $\varepsilon : \mathcal{K} \longrightarrow R$ called counit via which $\mathcal{K}$ is a R-coring [BW]. For the coproduct we introduce the Sweedler summation notation $\Delta(k) = k^{(1)} \otimes k^{(2)}$, where implicit summation understood. The data $(\mathcal{K}, \mathfrak{s}, \mathfrak{t}, \Delta, \varepsilon)$ is called a left R-bialgebroid if the algebra and the coring structures have the following compatibility axioms for all $k, k' \in \mathcal{K}$ and $r \in R$;

i) $k^{(1)}\mathfrak{t}(r) \otimes_R k^{(2)} = k^{(1)} \otimes_R k^{(2)}\mathfrak{s}(r)$,

ii) $\Delta(1_{\mathcal{K}}) = 1_{\mathcal{K}} \otimes 1_{\mathcal{K}}$, and $\Delta(kk') = k^{(1)}k'^{(1)} \otimes k^{(2)}k'^{(2)}$,

iii) $\varepsilon(1_{\mathcal{K}}) = 1_R$ and $\varepsilon(kk') = \varepsilon(k\mathfrak{s}(\varepsilon(k')))$.

A left R-bialgebroid $(\mathcal{K}, \mathfrak{s}, \mathfrak{t}, \Delta, \varepsilon)$, is said to be a left $\times_R$-Hopf algebra if the following map

$$\nu : \mathcal{K} \otimes_{R^{op}} \mathcal{K} \longrightarrow \mathcal{K} \otimes_R \mathcal{K}, \qquad k \otimes_{R^{op}} k' \longmapsto k^{(1)} \otimes_R k^{(2)}k' \tag{2.1}$$

is bijective. In the domain of the map (2.1), R^{op}-module structures are given by right and left multiplication by $\mathfrak{t}(r)$ for $r \in R$. In the codomain of the map (2.1), R-module structures are given by right multiplication by $\mathfrak{s}$ and $\mathfrak{t}$. The maps ν and ν^{-1} are both right $\mathcal{K}$-linear. The image of ν^{-1} is denoted by

$$\nu^{-1}(h \otimes_{R^{op}} 1_{\mathcal{H}}) = h_- \otimes_R h_+.$$

The notation of a $\times$-Hopf algebra extends that of a Hopf algebra. In fact if $\mathcal{K}$ is a bialgebra, then injectivity of the map ν is equivalent to the fact that $\mathcal{K}$ is a Hopf algebra. In this case the inverse of the map ν is defined by

$$\nu^{-1}(h \otimes 1) = h_- \otimes_{R^{op}} h_+ = h^{(1)} \otimes_{R^{op}} S(h^{(2)}).$$

We note that in the bialgebroid structure we have the equal source and target maps $\mathfrak{s} = \mathfrak{t} : \mathbb{C} \longrightarrow \mathcal{K}$ given by $c \longmapsto c1_{\mathcal{K}}$.

Example 2.1. *Enveloping algebra* $R \otimes R^{op}$: Let R be an algebra over the field of complex numbers. The simplest example of a left $\times_R$-Hopf algebra which is not a Hopf algebra is $\mathcal{K} = R^e = R \otimes R^{op}$ with the source and target maps defined by

$$\mathfrak{s} : R \longrightarrow \mathcal{K}, \quad r \longmapsto r \otimes 1; \quad \mathfrak{t} : R^{op} \longrightarrow \mathcal{K}, \quad r \longmapsto 1 \otimes r,$$

comultiplication defined by

$$\Delta : \mathcal{K} \longrightarrow \mathcal{K} \otimes_R \mathcal{K}, \quad r_1 \otimes r_2 \longmapsto (r_1 \otimes 1) \otimes_R (1 \otimes r_2),$$

counit given by

$$\varepsilon : \mathcal{K} \longrightarrow R, \quad \varepsilon(r_1 \otimes r_2) = r_1 r_2,$$

and

$$\nu((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 \otimes 1 \otimes r_3 \otimes r_4 r_2,$$
$$\nu^{-1}((r_1 \otimes r_2) \otimes (r_3 \otimes r_4)) = r_1 \otimes 1 \otimes r_2 r_3 \otimes r_4,$$

where $r, r_1, r_2, r_3, r_4 \in R$.

Example 2.2. *Quantum algebraic torus A_θ*: The Laurent polynomials in two variables $\mathbb{C}[U, V, U^{-1}, V^{-1}]$ is a Hopf algebra which is a completion of the Hopf algebra of the group ring $\mathbb{Z} \times \mathbb{Z}$. We consider a well-known deformation of this Hopf algebra which is not a Hopf algebra anymore and it is called algebraic quantum torus denoted by A_θ. Let us recall that A_θ is an unital algebra over $\mathbb{C}$ generated by two invertible elements U and V satisfying $UV = qVU$, where $q = e^{2\pi i\theta}$ and θ is a real number. Let $R = \mathbb{C}[U, U^{-1}]$ be the algebra of Laurent polynomials. We define $\alpha = \beta : R \longrightarrow A_\theta$ be the natural embedding. One defines a coproduct $\Delta : A_\theta \longrightarrow A_\theta \otimes_R A_\theta$ given by

$$\Delta(U^n V^m) = U^n V^m \otimes_R V^m. \tag{2.2}$$

The counit map $\varepsilon : A_\theta \longrightarrow R$ is given by

$$\varepsilon(U^n V^m) = U^n. \tag{2.3}$$

Since the counit map is not an algebra map the quantum torus A_θ is not an bialgebra. Instead it is a left $\times_R$-bialgebroid by the coring structure defined above. Furthermore the following map

$$\begin{gathered}\nu : A_\theta \otimes_{R^op} A_\theta \longrightarrow A_\theta \otimes_R A_\theta \\ U^n V^m \otimes U^r V^s \longmapsto U^n V^m \otimes V^m U^r V^s = q^{-mr} U^n V^m \otimes U^r V^{s-m},\end{gathered}$$

is bijective where the inverse map is defined by

$$\nu^{-1} : U^n V^m \otimes U^r V^s \longmapsto U^n V^m \otimes V^{-m} U^r V^s = q^{mr} U^n V^m \otimes U^r V^{s-m} \tag{2.4}$$

This turns A_θ into a left $\times_R$-Hopf algebra.

Example 2.3. *Connes-Moscovici Hopf algebroid $\mathcal{H}_{FM}$* : In this example we show that Connes-Moscovici Hopf algebroid defined in [CM01] is a $\times$-Hopf algebra. Let M be a smooth manifold of dimension n with a finite atlas and FM be the frame bundle on M. Let Γ_M denotes the pseudogroup of all local diffeomorphims M where its elements are partial diffeomorphisms $\psi : Dom\psi \longrightarrow Ran\psi$, where the domain and the range of ψ are both open subsets of M. One can lift $\psi \in \Gamma_M$ to the frame FM. This prolongation is denoted by $\widetilde{\psi}$. We set

$$FM \rtimes \Gamma_M := \{(u, \widetilde{\varphi}), \quad \varphi \in \Gamma_M, \quad u \in Rang\widetilde{\varphi}\},$$

and

$$FM \,\overline{\rtimes}\, \Gamma_M := \{[u, \widetilde{\varphi}], \quad \varphi \in \Gamma_M, \quad u \in Rang\widetilde{\varphi}\},$$

where $[u, \widetilde{\varphi}]$ stands for the class of $(u, \widetilde{\varphi}) \in FM \rtimes \Gamma_M$ with respect to the following equivalence relation;

$$(u, \widetilde{\varphi}) \sim (v, \widetilde{\psi}), \quad \text{if} \quad u = v \quad \text{and} \quad \widetilde{\varphi}\,|_W = \widetilde{\psi}\,|_W .$$

Here W is an open neighborhood of u. Let

$$\mathcal{A}_{FM} := C_c^\infty(FM \overline{\rtimes} \Gamma_M), \tag{2.5}$$

be the smooth convolution algebra. Every element of this algebra is linearly spanned by monomials of the form fU_ψ^* where $f \in C_c^\infty(Dom\widetilde{\psi})$. One has

$$f_1U_{\psi_1}^* = f_2U_{\psi_2}^* \quad \text{iff} \quad f_1 = f_2 \quad \text{and} \quad \psi_1|_V = \psi_2|_V,$$

where V is a neighborhood of $Supp(f_1) = Supp(f_2)$. A multiplication is defined on $\mathcal{A}_{FM}$ by

$$f_1U_{\psi_1}^*.f_2U_{\psi_2}^* = f_1(f_2 \circ \widetilde{\psi_1})U_{\psi_2\psi_1}^*. \tag{2.6}$$

We define the following algebra

$$\mathcal{R}_{FM} = C^\infty(FM), \tag{2.7}$$

which acts from left on $\mathcal{A}_{FM}$ by

$$r \triangleright fU^* = r.fU^*, \quad r \in \mathcal{R}_{FM}, \tag{2.8}$$

and from right by

$$fU^* \triangleleft r = (r \circ \widetilde{\psi}).fU^*, \quad r \in \mathcal{R}_{FM}. \tag{2.9}$$

In fact we obtain the source map $\alpha : \mathcal{R}_{FM} \longrightarrow \mathcal{A}_{FM}$ by the left action and the target map $\beta : \mathcal{R}_{FM}^{op} \longrightarrow \mathcal{A}_{FM}$ by the right action where both are algebra maps which their ranges commutate. Also we consider the action of an arbitrary vector field Z on FM which is given by

$$Z(fU^*) = Z(f)U^*, \quad fU^* \in \mathcal{A}_{FM}. \tag{2.10}$$

One notes that although a vector field acts by derivations on functions on the frame bundle, the action of the vector field on $\mathcal{A}_{FM}$ is not a derivation anymore. Now let

$$\mathcal{H}_{FM} \subset \mathcal{L}(\mathcal{A}_{FM}), \tag{2.11}$$

denotes those elements of the subalgebra of linear operators on $\mathcal{A}_{FM}$ which are generated by the three types of transformations; left multiplication, right multiplication and composition given in (2.8),(2.9) and (2.10). The elements of $\mathcal{H}_{FM}$ are called transverse differential operators on the groupoid $FM \overline{\rtimes} \Gamma_M$. One notes that $\alpha : \mathcal{R}_{FM} \longrightarrow \mathcal{H}_{FM}$ and $\beta : \mathcal{R}_{FM}^{op} \longrightarrow \mathcal{H}_{FM}$ endow $\mathcal{H}_{FM}$ with a $\mathcal{R}_{FM}$-bimodule structure. To define a $\mathcal{R}_{FM}$-coring structure on $\mathcal{H}_{FM}$, it is proved in [CM01] that $\mathcal{H}_{FM}$ has a Poincare-Birkhoff-Witt-type basis over $\mathcal{R}_{FM} \otimes \mathcal{R}_{FM}$ by fixing a torsion free connection on FM. To recall this basis, let $X_1, \cdots, X_n$ denote the standard vector fields corresponding to the standard basis of $\mathbb{R}^n$ and $\{Y_i^j\}$ be the fundamental vertical vector fields corresponding to the standard basis of $gl(n,\mathbb{R})$. These $n^2 + n$ vectors form a basis for the tangent space of FM at all points. Let $\delta_{jk}^i \in \mathcal{L}(\mathcal{A}_{FM})$ be the operators of multiplication defined in [CM01]. It is proven in

[CM01][Proposition 3] that transverse differential operators $\mathcal{Z}_I.\delta_\kappa$ form a basis for $\mathcal{H}_{FM}$ over $\mathcal{R}_{FM}\otimes\mathcal{R}_{FM}$, where

$$\mathcal{Z}_I = X_{i_1}\cdots X_{i_p}Y^{j_1}_{k_1}\cdots Y^{j_q}_{k_q}, \quad \text{and} \quad \delta_\kappa = \delta^{i_1}_{j_1k_1;\ell^1_1\cdots\ell^1_{p_1}}\cdots\delta^{i_r}_{j_rk_r;\ell^r_1\cdots\ell^r_{p_r}}, \tag{2.12}$$

and

$$\delta^i_{jk;\ell_1\cdots\ell_{p_1}} = [X_{\ell_r}\cdots[X_{\ell_1},\delta^i_{jk}],\cdots]. \tag{2.13}$$

We refer the reader to [CM01][Proposition 3] for definition of multi-indices I and κ. In order to define the coproduct, they have shown that the generators of $\mathcal{H}_{FM}$ acts on $\mathcal{A}_{FM}$ as a module algebra. This leads us to define a coproduct Δ_{FM} on $\mathcal{H}_{FM}$ which is not well-defined on $\mathcal{H}_{FM}\otimes\mathcal{H}_{FM}$, instead the ambiguity disappears in the tensor product over $\mathcal{R}_{FM}$. In the next step, the counit is defined in [CM01][Proposition 7] by

$$\varepsilon_{FM}:\mathcal{H}_{FM}\longrightarrow\mathcal{R}_{FM}, \quad \varepsilon(h)=h\triangleright 1. \tag{2.14}$$

Finally the authors in [CM01][proposition 8] defined a twisted antipode $\widetilde{S}_{FM}$ by defining a faithful trace $\tau:\mathcal{A}_{FM}\longrightarrow\mathbb{C}$. As a result, $\mathcal{H}_{FM}$ is a $\times_{\mathcal{R}_{FM}}$-Lu's Hopf algebroid by

$$(\Delta_{FM},\varepsilon_{FM},\widetilde{S}_{FM}). \tag{2.15}$$

The following canonical map

$$\mathcal{H}_{FM}\otimes_{\mathcal{R}^{op}_{FM}}\mathcal{H}_{FM}\longrightarrow\mathcal{H}_{FM}\otimes_{\mathcal{R}_{FM}}\mathcal{H}_{FM}, \tag{2.16}$$

which is given by

$$h\otimes h'\longmapsto h^{^{(1)}}\otimes h^{^{(2)}}h' \tag{2.17}$$

defines a $\times_{\mathcal{R}_{FM}}$-Hopf algebra structure on $\mathcal{H}_{FM}$. Here we mention a special case of Connes-Moscovici Hopf algebroid when $M=\mathbb{R}^n$ is the flat Euclidean space. It is proved in [CM2004] that

$$\mathcal{H}_{F\mathbb{R}^n}\cong\mathcal{R}_{F\mathbb{R}^n}\rtimes\mathcal{H}_n\ltimes\mathcal{R}_{F\mathbb{R}^n}, \tag{2.18}$$

is a Hopf algebroid where $\mathcal{H}_n$ is the Connes-Moscovici Hopf algebra [CM98]. In fact it can be shown that it is a left $\times$-Hopf algebra as follows. Generally speaking if H is a Hopf algebra and A is a H-module algebra then

$$\mathcal{H}_{CM}=A\rtimes H\ltimes A^{op}, \tag{2.19}$$

is a left $\times_A$-Hopf algebra, called Connes-Moscovici $\times$-Hopf algebra, by the following structure. The algebra structure is given by

$$(a\rtimes h\ltimes b)\cdot(a'\rtimes h'\ltimes b') = a(h^{^{(1)}}a')\rtimes h^{^{(2)}}h'\ltimes(h^{^{(3)}}b')b \tag{2.20}$$

The source and target maps $\alpha:A\longrightarrow\mathcal{H}$ and $\beta:A^{op}\longrightarrow\mathcal{H}$ are given by

$$\alpha(a)=a\rtimes 1\ltimes 1, \quad \beta(a)=1\rtimes 1\ltimes a. \tag{2.21}$$

The coring structure is given by the following coproduct;

$$\Delta(a \rtimes h \ltimes b) = (a \rtimes h^{(1)} \ltimes 1) \otimes_A (1 \rtimes h^{(2)} \ltimes b). \tag{2.22}$$

The counit $\varepsilon : \mathcal{H}_{CM} \longrightarrow A$ is defined by

$$\varepsilon(a \rtimes h \ltimes b) = a\varepsilon(h)b. \tag{2.23}$$

Furthermore we define $\nu : \mathcal{H}_{CM} \otimes_{A^{op}} \mathcal{H}_{CM} \longrightarrow \mathcal{H}_{CM} \otimes_A \mathcal{H}_{CM}$ is defined by

$$\nu((a \rtimes h \ltimes b) \otimes_{A^{op}} (a' \rtimes h' \ltimes b')) = (a \rtimes h^{(1)} \ltimes 1) \otimes_A \left(h^{(2)}a' \rtimes h^{(3)}h' \ltimes (h^{(4)}b')b\right), \tag{2.24}$$

with the inverse map given by

$$\nu^{-1}((a \rtimes h \ltimes b) \otimes_A (a' \rtimes h' \ltimes b')) = (a \rtimes h^{(1)} \ltimes 1) \otimes_{A^{op}} \left(S(h^{(4)}) \triangleright (ba) \rtimes S(h^{(4)})h' \ltimes S(h^{(2)}) \triangleright b'\right). \tag{2.25}$$

Example 2.4. *Kadison bialgebroid*: In this example we show that the Kadison bialgebroid $(A \otimes A^{op}) \bowtie H$ introduced in [Kad] is a $\times_A$-Hopf algebra. Here H is a Hopf algebra and A a left H-module algebra. First we recall the bialgebroid structure. The source map $\alpha : A \longrightarrow (A \otimes A^{op}) \bowtie H$ is given by

$$a \longmapsto (a \otimes 1) \otimes 1. \tag{2.26}$$

The target map $\beta : A \longrightarrow (A \otimes A^{op}) \bowtie H$ is given by

$$a \longmapsto (1 \otimes a) \otimes 1. \tag{2.27}$$

The algebra structure is given by the following multiplication rule;

$$(a \otimes b \otimes h) \cdot (a' \otimes b' \otimes h') = a(h^{(1)} \triangleright a') \otimes b'(S(h'^{(2)}) \triangleright b) \otimes h^{(2)}h'^{(1)}. \tag{2.28}$$

The comultiplication is given by

$$(a \otimes b) \otimes h \longmapsto ((a \otimes 1) \otimes h^{(1)}) \otimes_A ((1 \otimes b) \otimes h^{(2)}) \tag{2.29}$$

Furthermore the counit $\varepsilon : (A \otimes A^{op}) \bowtie H \longrightarrow A$ is given by

$$(a \otimes b) \otimes h \longmapsto a(h \triangleright b). \tag{2.30}$$

Let $A^e = A \otimes A^{op}$. We define $\nu : A^e \otimes H \otimes_{A^{op}} A^e \otimes H \longrightarrow A^e \otimes H \otimes_A A^e \otimes H$ by

$$(a \otimes b \otimes h) \otimes_{A^{op}} (a' \otimes b' \otimes h') \longmapsto (a \otimes 1 \otimes h^{(1)}) \otimes_A \left((h^{(2)} \triangleright a') \otimes b'(S(h'^{(2)}) \triangleright b) \otimes h^{(3)}h'^{(1)}\right), \tag{2.31}$$

with the following inverse map;

$$(a \otimes b \otimes h) \otimes_A (a' \otimes b' \otimes h') \longrightarrow (a \otimes 1 \otimes h^{(1)}) \otimes_{A^{op}} \left((b \otimes 1 \otimes S(h^{(2)}) \cdot (a' \otimes b' \otimes h')\right) \tag{2.32}$$

It is proved by Panaite and Van Oystaeyen in [PVO] that the Connes-Moscovici bialgebroid is isomorphic to the Kadison bialgebroid;

$$A \rtimes H \ltimes A^{op} \cong (A \otimes A^{op}) \bowtie H. \tag{2.33}$$

This isomorphism is given by

$$\chi : (A \otimes A^{op}) \bowtie H \longrightarrow A \otimes H \otimes A^{op}, \quad a \otimes b \otimes h \longmapsto a \otimes h^{^{(1)}} \otimes h^{^{(2)}} b, \tag{2.34}$$

and

$$\chi^{-1} : a \otimes h \otimes b \longmapsto a \otimes S(h^{^{(2)}}) \triangleright b \otimes h^{^{(1)}}. \tag{2.35}$$

It is mentioned in [PVO] that if $S^2 = \mathrm{Id}$ then the map χ induces an isomorphism on the level of Böhm-Szlachányi Hopf algebroid. We recall that two $\times$-Hopf algebras $\mathcal{K}$ and $\mathcal{H}$ are isomorphic if there exists a map $\zeta : \mathcal{K} \longrightarrow \mathcal{H}$ which commutes with all bialgebroid structures and furthermore $\zeta\nu = \nu\zeta$. Similarly one has the following statement.

Lemma 2.5. *Connes-Moscovici and Kadison $\times$-Hopf algebras $A \rtimes H \ltimes A^{op}$ and $(A \otimes A^{op}) \bowtie H$ are isomorphic.*

Here we briefly recall the definitions of modules, comodules and stable anti-Yetter-Drienfeld (SAYD) modules for a left $\times$-Hopf algebra. A right module of a left $\times_R$-Hopf algebra $\mathcal{K}$ is a right $\mathcal{K}$-module M. A right $\mathcal{K}$-module M can be equipped with a R-bimodule structure as follows:

$$r \cdot m = s(r) \cdot m, \quad \text{and} \quad m \cdot r = t(r) \cdot m$$

A left comodule of a left $\times_R$-Hopf algebra $\mathcal{K}$ is defined to be a left comodule of the underlying R-coring $(\mathcal{K}, \Delta, \varepsilon)$, that is, a left R-module M, together with a left R-module map $M \longrightarrow \mathcal{K} \otimes_R M$, $m \longmapsto m_{(-1)} \otimes_R m_{(0)}$, satisfying coassociativity and counitality axioms. One notes that a left $\mathcal{K}$-comodule M can be equipped with a R-bimodule structure by introducing a right R-action as follows,

$$m \cdot r := \varepsilon(m_{(-1)} s(r)) \cdot m_{(0)},$$

for $r \in R$ and $m \in M$. With respect to the resulting bimodule structure, $\mathcal{K}$-comodule maps are R-bimodule maps. In the special case, the left $\mathcal{K}$-coaction on M is an R-bimodule map in the sense that for all $r, r' \in R$ and $m \in M$, we have;

$$(r \cdot m \cdot r')_{(-1)} \otimes_R (r \cdot m \cdot r')_{(0)} = s(r) m_{(-1)} s(r') \otimes_R m_0. \tag{2.36}$$

Furthermore, for all $m \in M$ and $r \in R$ we have;

$$m_{(-1)} \otimes_R m_{(0)} \cdot r = m_{(-1)} t(r) \otimes_R m_{(0)}. \tag{2.37}$$

Let M be a right $\mathcal{K}$-module and a left $\mathcal{K}$-comodule. We say M is an anti Yetter-Drinfeld, AYD, module provided that the following two conditions hold.

i) The R-bimodule structures on M, underlying its module and comodule structures, coincide. That is, for $m \in M$ and $r \in R$,

$$m \cdot r = m \triangleleft \mathfrak{s}(r), \qquad \text{and} \qquad r \cdot m = m \triangleleft \mathfrak{t}(r),$$

where $r \cdot m$ denotes the left R-action on the left $\mathcal{K}$-comodule M and $r \cdot m$ is the canonical right action.

ii) For $k \in \mathcal{K}$ and $m \in M$ we have;

$$(m \triangleleft k)_{(-1)} \otimes (m \triangleleft k)_{(0)} = k_{+}^{(2)} m_{(-1)} k^{(1)} \otimes_R m_{(0)} \triangleright k_{-}^{(2)}. \tag{2.38}$$

The anti Yetter-Drinfeld module M is said to be stable if in addition for any $m \in M$ we have $m_{(0)} m_{(-1)} = m$.

Example 2.6. A map δ is called a right character [BSz, Lemma 2.5], for the $\times_R$-Hopf algebra $\mathcal{K}$ if it satisfies the following conditions:

$$\delta(k\mathfrak{s}(r)) = \delta(k)r, \quad \text{for} \quad k \in \mathcal{K} \quad \text{and} \quad r \in R, \tag{2.39}$$

$$\delta(k_1 k_2) = \delta(\mathfrak{s}(\delta(k_1))k_2), \quad \text{for} \quad k_1, k_2 \in \mathcal{K}, \tag{2.40}$$

$$\delta(1_{\mathcal{K}}) = 1_R. \tag{2.41}$$

As an example, for any right $\times_R$-Hopf algebra, the counit ε is a right character. We recall from [BS, Example 2.18] and [HR1, Example 2.5], let $\sigma \in \mathcal{K}$ be a group-like element and the map $\delta : \mathcal{K} \longrightarrow R$ be a right character. The following action and coaction,

$$r \triangleleft k = \delta(\mathfrak{s}(r)k), \quad \text{and} \quad r \longmapsto \mathfrak{s}(r)\sigma \otimes 1 \tag{2.42}$$

define a right $\mathcal{K}$-module and left $\mathcal{K}$-comodule structure on R, respectively. These action and coaction amount to a right-left anti Yetter-Drinfeld module on R if and only if, for all $r \in R$ and $k \in \mathcal{K}$ we have;

$$\mathfrak{s}(\delta(k))\sigma = \mathfrak{t}(\delta(k^{(2)-}))k^{(2)+}\sigma k^{(1)}, \quad \text{and} \quad \varepsilon(\sigma\mathfrak{s}(r)) = \delta(\mathfrak{s}(r)). \tag{2.43}$$

The anti Yetter-Drinfeld module R is stable if in addition $\delta(\mathfrak{s}(r)\sigma) = r$, for all $r \in R$. We denote this SAYD module over $\mathcal{K}$ by ${}^{\sigma}R_{\delta}$.

Example 2.7. *SAYD modules on the enveloping algebras*:
As a special case of the previous example when $\mathcal{K} = R \otimes R^{op}$, it is shown in [HR1] that a homogenous element $x \otimes y \in \mathcal{K}$ is a group-like element if and only if $xy = yx = 1$. Furthermore if δ be a character then $\delta(r \otimes r') = rr'$ for all $r \otimes r' \in \mathcal{K}$. As a result ${}^{x\otimes x^{-1}}R_{\delta}$ is a right-left SAYD module over the $\times_R$-Hopf algebra $\mathcal{K} = R \otimes R^{op}$ by the following action and coaction;

$$r_2 r r_1 = r \triangleleft (r_1 \otimes r_2), \qquad r \longmapsto (rx \otimes x^{-1}) \otimes 1, \tag{2.44}$$

where $r, r_1, r_2, \in R$ and x is an element of the center of R.

Example 2.8. *SAYD modules on the quantum algebraic torus*:
Let A_θ be the algebraic torus by the left $\times$-Hopf algebra structure given in the Example 2.2. It is obvious that every element of the form V^m is a group-like element. Furthermore, the map : $\delta : A_\theta \longrightarrow \mathbb{C}[U, U^{-1}]$ which is given by

$$\delta(U^n V^m) = q^{nm} U^n, \qquad U^n V^m \in A_\theta, \tag{2.45}$$

is a right character. The only group-like element which satisfies the stability condition with respect to this right character is the unit element. As a result of (2.42), the following action and coaction endow ${}^1R_\delta = \mathbb{C}[U, U^{-1}]$ with a SAYD structure on A_θ as follows;

$$U^k \triangleleft U^n V^m = q^{(k+n)m} U^{k+n}, \qquad U^n \longmapsto U^n \otimes 1. \tag{2.46}$$

Example 2.9. *SAYD modules on the Connes-Moscovici $\times$-Hopf algebra*:
Suppose the homogenous element $a \otimes h \otimes b \in A \otimes H \otimes A^{op}$ is a group-like element. Using the A-coring structure defined in the Example 2.3 we obtain;

$$(a \otimes h^{^{(1)}} \otimes 1) \otimes (1 \otimes h^{^{(2)}} \otimes b) = (a \otimes h \otimes b) \otimes (a \otimes h \otimes b). \tag{2.47}$$

This implies that $a = b = 1$ and therefore h is a group-like element of H. Therefore the group-like elements of Connes-Moscovici $\times$-Hopf algebra $H_{CM} = A \otimes H \otimes A^{op}$ are of the form $1 \otimes \sigma \otimes 1$ where σ is a group-like element of the Hopf algebra H. We define $\delta : H_{CM} \longrightarrow A$ given by

$$\delta(a \otimes h \otimes b) = \varepsilon(h) f(ba), \tag{2.48}$$

where $f : A \longrightarrow A$ is an unital algebra map satisfying $f^2 = f$ and $f(h \triangleright a) = \varepsilon(h) f(a)$. One can check that δ is a right character.

3 Cyclic cohomology of $\times$-Hopf algebras

Cyclic cohomology of Hopf algebras is discovered by Connes-Moscovici in their ground breaking work on local index theory [CM98]. Their work is followed by important calculations of Hopf cyclic cohomology of quantum groups by Khalkhali-Rangipour in [KR1], Kustermans-Rognes-Tuset in [KRT] and aslo Hadfield-Krähmer in [HK1] and [HK2]. The cyclic cohomology of Lu's Hopf algebroid (which is defined in [CM01] and [Ko]) and of Khalkhali-Rangipour Para- Hopf algebroids [KR3] are defined with *trivial* coefficients, the underlying algebra of the coring structure. The generalized SAYD coefficients for extended versions of Hopf algebras first is defined in [BS] for $\times$-Hopf algebras and later has been generalized in [HR1], [HR2] and [KK]. In this section, we review the cyclic cohomology of algebras and coalgebras under the symmetry of a left $\times_R$-Hopf algebra with coefficients in a SAYD module.

Let R be an algebra over $\mathbb{C}$ and $\mathcal{K}$ be a left $\times_R$-Hopf algebra. A left $\mathcal{K}$-module coring C is a R-coring and left $\mathcal{K}$-module with one and the same underlying R-bimodule structure, such that counit ε and comultiplication Δ both are left $\mathcal{K}$-linear. We consider the left

$\mathcal{K}$-module structure of R by $k \triangleright r := \varepsilon(ks(r))$ and left $\mathcal{K}$-module structure of $C \otimes_R C$ is by diagonal action. This means;

$$\varepsilon(k \triangleright c) = k \triangleright \varepsilon(c) = \varepsilon(ks(\varepsilon(c))), \tag{3.1}$$

$$\Delta(k \triangleright c) = k^{^{(1)}} \triangleright c^{^{(1)}} \otimes_R k^{^{(2)}} \triangleright c^{^{(2)}}. \tag{3.2}$$

For any left $\mathcal{K}$-module coring C and a right-left SAYD module M over $\mathcal{K}$, one defines a cocyclic module as follows. Let

$${}_{\mathcal{K}}C^n(C, M) = M \otimes_{\mathcal{K}} C^{\otimes_R (n+1)}.$$

We abbreviate $\widetilde{c} = c_0 \otimes_R \cdots \otimes_R c_n$ and define the following cofaces, codegeneracies and cocyclic maps.

$$\begin{aligned}
d_i(m \otimes_{\mathcal{K}} \widetilde{c}) &= m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R \Delta(c_i) \otimes_R \cdots \otimes_R c_n,\\
d_{n+1}(m \otimes_{\mathcal{K}} \widetilde{c}) &= m_{(0)} \otimes_{\mathcal{K}} c_0^{(2)} \otimes_R c_1 \otimes_R \cdots \otimes_R c_n \otimes_R m_{(-1)} c_0^{(1)},\\
s_i(m \otimes_{\mathcal{K}} \widetilde{c}) &= m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R \varepsilon(c_i) \otimes_R \cdots \otimes_R c_n,\\
t_n(m \otimes_{\mathcal{K}} \widetilde{c}) &= m_{(0)} \otimes_{\mathcal{K}} c_1 \otimes_R \cdots \otimes_R m_{(-1)} c_0.
\end{aligned}$$

One verifies that $({}_{\mathcal{K}}C^n(C, M), d, s, t)$ is a cocyclic module. We denote the cyclic cohomology of this cocyclic module by ${}_{\mathcal{K}}HC^n(C, M)$.
Now we describe the cyclic cohomology of a module algebra under the symmetry of a $\times$-Hopf algebra. A left $\mathcal{K}$-module algebra A is a $\mathbb{C}$-algebra and a left $\mathcal{K}$-module satisfying the following conditions for all $k \in \mathcal{K}$, $a, a' \in A$ and $r \in R$;

i) $k \triangleright 1_A = s(\varepsilon(k)) \triangleright 1_A$,

ii) $h \triangleright (aa') = (k^{(1)} \triangleright a)(k^{(2)} \triangleright a')$,

iii) $(\mathfrak{t}(r) \triangleright a)a' = a(s(r) \triangleright a')$, (multiplication is R-balanced).

For any left $\mathcal{K}$-module algebra A and a right-left SAYD module M over $\mathcal{K}$, we set

$${}_{\mathcal{K}}C^n(A, M) = \mathrm{Hom}_{\mathcal{K}}(M \otimes_R A^{\otimes_R (n+1)}, R),$$

to be the set of $\mathcal{K}$-linear maps from $M \otimes A^{\otimes_R (n+1)}$ to R. The following cofaces, codegeneracies and cocyclic maps will define a cocyclic module;

$$\begin{aligned}
(\delta_i f)(m \otimes_R \otimes_R a_0 \otimes_R \cdots \otimes_R a_n) &= f(m \otimes_R \otimes a_0 \otimes_R \cdots \otimes_R a_i a_{i+1} \otimes_R \cdots \otimes_R a_n),\\
(\delta_n f)(m \otimes_R \otimes_R a_0 \otimes_R \cdots \otimes_R a_n) &= f(m_{<0>} \otimes_R a_n(m_{(-n)} a_0) \otimes_R \ldots \otimes_R m_{(-1)} a_{n-1}),\\
(\sigma_i f)(m \otimes_R \otimes_R a_0 \otimes_R \cdots \otimes_R a_n) &= f(m \otimes_R a_0 \otimes_R \cdots \otimes_R 1 \otimes_R \cdots \otimes_R a_n),\\
(\tau_n f)(m \otimes_R \otimes_R a_0 \otimes_R \cdots \otimes_R a_n) &= f(m_0 \otimes_R a_n \otimes_R m_{(-n)} a_0 \otimes \ldots \otimes_R m_{(-1)} a_{n-1}).
\end{aligned}$$

The cyclic cohomology of this cocyclic module is denoted by ${}_{\mathcal{K}}\widetilde{HC}^n(A, M)$ which generalizes the dual cyclic cohomology defined in [KR1].

Example 3.1. *Cyclic cohomology of a $\times_R$-Hopf algebra with coefficients in R*:
If R is a SAYD module for the $\times$-Hopf $\mathcal{K}$ as explained in Example 2.6, then the related cocyclic module is;

$$
{}_{\mathcal{K}}C^n(\mathcal{K},R) = R\otimes_{\mathcal{K}} \underbrace{\mathcal{K}\otimes_R\cdots\otimes_R\mathcal{K}}_{n+1\ times} \cong \underbrace{\mathcal{K}\otimes_R\cdots\otimes_R\mathcal{K}}_{n\ times}. \tag{3.3}
$$

The isomorphism is given by

$$
\rho(r\otimes_{\mathcal{K}} k_0\otimes_R\cdots\otimes_R k_n) = \mathfrak{s}(r\triangleleft k_0)k_1\otimes_R\cdots\otimes_R k_n, \tag{3.4}
$$

which is in fact;

$$
\rho(r\otimes_{\mathcal{K}} k_0\otimes_R\cdots\otimes_R k_n) = \mathfrak{s}(\delta(\mathfrak{s}(r)k_0))k_1\otimes_R\cdots\otimes_R k_n, \tag{3.5}
$$

and the inverse map is given by

$$
\rho^{-1}(k_1\otimes_R\cdots\otimes_R k_n) = 1_R\otimes_{\mathcal{K}} 1_{\mathcal{K}}\otimes_R k_1\otimes_R\cdots\otimes_R k_n. \tag{3.6}
$$

Then the cocyclic module (3.3) is simplified to the following one.

$$
\begin{aligned}
&d_0(k_1\otimes_R\cdots\otimes_R k_n) = 1_{\mathcal{K}}\otimes_R k_1\otimes_R \otimes_R k_n,\\
&d_i(k_1\otimes_R\cdots\otimes_R k_n) = k_1\otimes_R\cdots\otimes_R \Delta(k_i)\otimes_R\cdots\otimes_R k_n,\\
&d_{n+1}(k_1\otimes_R\cdots\otimes_R k_n) = k_1\otimes_R\cdots\otimes_R k_n\otimes_R \sigma,\\
&s_0(k_1\otimes_R\cdots\otimes_R k_n) = \mathfrak{s}(\delta(k_1))k_2\otimes_R k_3\otimes_R\cdots\otimes_R k_n,\\
&s_i(k_1\otimes_R\cdots\otimes_R k_n) = k_1\otimes_R\cdots\otimes_R \varepsilon(k_i)\otimes_R\cdots\otimes_R k_n,\\
&t_n(k_1\otimes_R\cdots\otimes_R k_n) = \mathfrak{s}(\delta(k_1))k_2\otimes_R k_3\otimes_R\cdots\otimes_R k_n\otimes_R \sigma.
\end{aligned}
$$

Example 3.2. *Cyclic cohomology of the universal algebra R^e*:
For any unital algebra R on the field of complex numbers $\mathbb{C}$, we have

$$
C^n_{R^e}(R^e,R) = R\otimes_{R^e}\otimes \underbrace{R^e\otimes_R\cdots\otimes_R R^e}_{n+1\ times} \cong \underbrace{R\otimes\cdots\otimes R}_{n\ times}\otimes R^{op}. \tag{3.7}
$$

Similar to [CM01] we use the unit map to define a map of cocyclic modules

$$
C^n(\mathbb{C}\otimes\mathbb{C}^{op}) \longrightarrow C^n(R\otimes R^{op}).
$$

Now we fix an unital linear functional $\varphi\in R^*$ to define a homotopy map $\mathbf{s}: C^n(R^e)\longrightarrow C^{n-1}(R^e)$ which is given by

$$
\mathbf{s}(r_1\otimes\cdots\otimes r_n) = \varphi(r_1)r_2\otimes\cdots\otimes r_n, \tag{3.8}
$$

where $r_1,\cdots,r_n\in R$. One easily checks that $\mathbf{s}$ commutes with face maps and therefore we obtain an isomorphism on the level of Hochschild and consequently cyclic cohomology for the cocyclic modules $C^n(\mathbb{C}\otimes\mathbb{C}^{op})$ and $C^n(R\otimes R^{op})$. Therefore we obtain;

$$
HC^*(R\otimes R^{op}) \cong HC^*(\mathbb{C}\otimes\mathbb{C}^{op}) \cong HC^*(\mathbb{C}). \tag{3.9}
$$

Example 3.3. *Cyclic cohomology of Connes-Moscovici and Kadison $\times$-Hopf algebras*:
For the Connes-Moscovici and Kadison $\times$-Hopf algebras we have;

$$HC^*(A \otimes H \otimes A^{op}) \cong HC^*(A^e \otimes H) \cong HC^*(H). \tag{3.10}$$

Here the first isomorphism is a result of the Lemma 2.5. In fact as it is shown in [CM01], there is a R-bimodule isomorphism for Connes-Moscovici Hopf algebra as follows;

$$H_{CM} = R \otimes H \otimes R^{op} \cong \alpha(R) \otimes \beta(R) \otimes H = R \otimes R^{op} \otimes H. \tag{3.11}$$

Therefore one can transfer the $\times$-Hopf algebra structure to $R \otimes R^{op} \otimes H$. It is easily observed that

$$\Delta_{H_{CM}} = \Delta_{R^e} \otimes \Delta_H, \quad \text{and} \quad \varepsilon_{H_{CM}} = \varepsilon_{R^e} \otimes \varepsilon_H. \tag{3.12}$$

This shows that H_{CM} is actually isomorphic to the external tensor product between the bicoalgebroid R^e and the coalgebra H over the complex numbers. In fact we have;

$$\delta_{H_{CM}} = \delta_{R^e} \otimes \delta_H, \quad \text{and} \quad \sigma_{H_{CM}} = \sigma_{R^e} \otimes \sigma_H. \tag{3.13}$$

By applying Eilenberg-Zilber theorem we obtain

$$HH^*(H_{CM}) \cong HH^*(R^e) \otimes HH^*(H) \cong HH^*(H), \tag{3.14}$$

where the second isomorphism is obtained by the Example 3.2. In fact the composition of the two isomorphism is given by the canonical inclusion homomorphism $f : H \longrightarrow H_{CM}$ which is also a morphism of $\times$-Hopf algebras. This enables us to obtain a map of cocyclic modules ${}_HC^*(H, \mathbb{C}) \longrightarrow_{H_{CM}} C^*(H_{CM}, R)$. Therefore the Connes-Long exact sequence of cocyclic modules relating Hochschild and cyclic cohomology implies the isomorphism on the level of cyclic cohomology.

Example 3.4. *Cyclic cohomology of the quantum algebraic torus*:
A normal Harr system $\varrho : A_\theta \longrightarrow \mathbb{C}[U, U^{-1}]$ is introduced for the bialgebroid structure of the quantum algebraic torus in [KR3] as follows;

$$\varrho(U^n V^m) = \delta_{m,0} U^n. \tag{3.15}$$

This leads to having a contracting homotopy as defined in [KR3] and therefore we obtain;

$$HC^{2i+1}(A_\theta) = 0, \quad HC^{2i}(A_\theta) = \mathbb{C}[U, U^{-1}], \quad \text{for all} \quad i \geq 0 \tag{3.16}$$

4 Module algebras paired with module coalgebras

Let $\mathcal{K}$ be a left $\times_R$-Hopf algebra, M be a right-left SAYD module over $\mathcal{K}$, A be a left $\mathcal{K}$-module algebra and C be a left $\mathcal{K}$-module coring. Let C acts on A from left satisfying the following conditions;

$$(hc)a = h(ca), \tag{4.1}$$
$$c(ab) = (c^{(1)}a)(c^{(2)}b), \tag{4.2}$$
$$c \rhd 1_A = \varepsilon(c) \rhd 1_A. \tag{4.3}$$

Let $B = \mathrm{Hom}_{\mathcal{K}}(C, A)$ be the set of maps from the R-coring C to A which are both $\mathcal{K}$-linear and R-linear. The space B is an algebra over $\mathbb{C}$ by the multiplication $*$ which is given by;

$$(f * g)(c) = f(c^{^{(1)}})f(c^{^{(2)}}). \tag{4.4}$$

We denote that B is a R-bimodule by $(r \triangleright f)(c) = f(r \triangleright c)$ and similarly for the right R-action. There exists an unital algebra map given by $\eta_A : R \longrightarrow A, \quad \eta_A(r) = r \triangleright 1_A = s(r) \triangleright 1_A$. Therefore B has the unit element $\eta_B = \eta_A \circ \varepsilon_C$. We remind that the cyclic cohomology of the algebra B is computed by the cohomology of the following cocyclic module.

$$\delta_i^a(\varphi)(b_0 \otimes \cdots b_{n+1}) = \varphi(b_0 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots b_{n+1}),$$
$$\sigma_i^a(\varphi)(b_0 \otimes \cdots b_n) = \varphi(b_0 \otimes \cdots \otimes \varepsilon(b_i) \otimes \cdots b_n),$$
$$\tau_i^a(\varphi)(b_0 \otimes \cdots b_n) = \varphi(b_n \otimes b_0 \otimes \cdots \otimes b_{n-1}).$$

Let

$$C_{a,c}^{n,n} =_{\mathcal{K}} C^n(A, M) \otimes \ _{\mathcal{K}}C^n(C, M),$$

be the diagonal complex which is a cocyclic module by $(\delta_n \otimes d_n, \sigma_n \otimes s_n \otimes \tau_n \otimes t_n)$. We define the following map;

$$\Psi_c : C_{a,c}^{n,n} \longrightarrow \mathrm{Hom}(B^{\otimes_R (n+1)}, \mathbb{C}),$$
$$\Psi_c(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n)(f_0 \otimes_R \cdots \otimes_R f_n) = \phi(m \otimes_R f_0(c_0) \otimes_R \cdots \otimes_R f_n(c_n)).$$

Here $f_i \in B$, for all $0 \le i \le n$ and $\phi \in_{\mathcal{K}} C^n(A, M)$. The map Ψ_c is well-defined because f_i's are $\mathcal{K}$ and R linear and ϕ is equivariant.

Proposition 4.1. *The map Ψ_c defines a cyclic map between the diagonal cocyclic modules $C_{a,c}^{n,n}$ and the cocyclic module of the algebra B, i.e. $C^n(B)$.*

Proof. First we show that Ψ_c commutes with cofaces.

$$\Psi_c(\delta_i \otimes d_i(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n+1})$$
$$\Psi_c(\delta_i(\phi) \otimes d_i(m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n+1})$$
$$\Psi_c(\delta_i(\phi) \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_i^{^{(1)}} \otimes c_i^{^{(2)}} \otimes_R c_n)(f_0 \otimes_R \cdots \otimes_R f_{n+1})$$
$$\delta_i(\phi)(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R f_i(c_i^{^{(1)}}) \otimes_R f_{i+1}(c_i^{^{(2)}}) \otimes_R \cdots \otimes_R f_{n+1}(c_n))$$
$$\phi(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R f_i(c_i^{^{(1)}}) f_{i+1}(c_i^{^{(2)}}) \otimes_R \cdots \otimes_R f_{n+1}(c_n))$$
$$\phi(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R (f_i * f_{i+1})(c_i) \otimes_R \cdots \otimes_R f_{n+1}(c_n))$$
$$(\delta_i^a \Psi_c(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n+1}).$$

Here we show that Ψ_c commutes with codegeneracies.

$$\begin{aligned}
&\Psi_c(\sigma_i \otimes s_i(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n-1})\\
&\Psi_c(\sigma_i(\phi) \otimes s_i(m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n-1})\\
&\Psi_c(\sigma_i(\phi) \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R \varepsilon(c_i) \otimes_R \cdots \otimes_R c_n)(f_0 \otimes_R \cdots \otimes_R f_n)\\
&\sigma_i(\phi)\,(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R \varepsilon(c_i) \otimes_R \cdots \otimes_R f_{n-1}(c_n))\\
&\phi\,(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R \varepsilon(c_i)1 \otimes_R \cdots \otimes_R f_{n-1}(c_n))\\
&\phi\,(m \otimes_{\mathcal{K}} f_0(c_0) \otimes_R \cdots \otimes_R \eta_B(c_i) \otimes_R \cdots \otimes_R f_{n-1}(c_n))\\
&(\sigma_i^a \Psi_c(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_{n-1}).
\end{aligned}$$

The following computation shows that Ψ_c commutes with cyclic maps.

$$\begin{aligned}
&\Psi_c(\tau_n \otimes t_n(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_n)\\
&\Psi_c(\tan(\phi) \otimes t_n(m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n))(f_0 \otimes_R \cdots \otimes_R f_n)\\
&\tau\phi(m_{<0>} \otimes f_0(c_1) \otimes_R \cdots \otimes_R m_{<-1>} f_n(c_0))\\
&\phi(m_{<0><0>} \otimes_R m_{<-1>} f_n(c_0) \otimes_R m_{<0><-n>} f_0(c_1) \otimes_R \cdots \otimes_R m_{<0><-1>} f_{n-1}(c_n))\\
&\phi(m_{<0>} \otimes_R m_{<-1>}{}^{(1)} f_n(c_0) \otimes_R m_{<0>}{}^{(2)} f_0(c_1) \otimes_R \cdots \otimes_R m_{<0>}{}^{(n+1)} f_{n-1}(c_n))\\
&\phi(m_{<0>} \otimes_R m_{<-1>} \triangleright (f_n(c_0) \otimes_R f_0(c_1) \cdots \otimes_R f_{n-1}(c_n))\\
&\phi(m \otimes_R f_n(c_0) \otimes_R f_0(c_1) \cdots \otimes_R f_{n-1}(c_n))\\
&\Psi_c(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n)(f_n \otimes_R f_0 \otimes_R \cdots \otimes_R f_{n-1})\\
&(\tau_n^a \Psi_c(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n)(f_o \otimes_R \cdots \otimes_R f_n).
\end{aligned}$$

□

We define an unital algebra map $\lambda : A \longrightarrow B = \mathrm{Hom}_{\mathcal{K}}(A, C)$ given by $\lambda(a)(c) = ca$. The condition (4.1) implies the $\mathcal{K}$-linearity of the map $\lambda(a) \in B$ and therefore λ is well-defined. The condition (4.2) shows that the map λ is multiplicative and finally (4.3) proves that λ is unital. Therefore we obtain a map of cocyclic modules $\lambda : C^*(B, \mathbb{C}) \longrightarrow C^*(A, \mathbb{C})$. We set

$$\Psi := \lambda \circ \Psi_c : C^{n,n}_{a,c} \longrightarrow C^*(A, \mathbb{C}). \tag{4.5}$$

where

$$\Psi(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_n)(a_0 \otimes \cdots \otimes a_n) = \varphi(m \otimes_R c_0 a_0 \otimes_R \cdots \otimes_R c_n a_n).$$

Theorem 4.2. *Let R be a an unital $\mathbb{C}$-algebra, $\mathcal{K}$ be a left $\times_R$-Hopf algebra, M be a right-left SAYD module over $\mathcal{K}$, A be a left $\mathcal{K}$-module algebra and C be a left $\mathcal{K}$-module coalgebra. Let C acts A satisfying* (4.1)*,* (4.2) *and* (4.3)*. We have the following pairing on the level of cyclic cohomology,*

$$\sqcup : \widetilde{HC}^p_{\mathcal{K}}(A, M) \otimes HC^q_{\mathcal{K}}(C, M) \longrightarrow HC^{p+q}(A),$$

given by

$$\sqcup = \Psi AW,$$

where AW is the Alexander-Whitney map. There are similar pairings for Hochschild and periodic cyclic cohomology.

Remark 4.3. Similar to [R1] one shows that the coproduct $\sqcup$ introduced in Theorem 4.2 is given by the following explicit formula on the level of Hochschild cohomology.

$$\sqcup_H : C^p_{\mathcal{K}}(A, M) \otimes C^q_{\mathcal{K}}(C, M) \longrightarrow C^{p+q}(A), \tag{4.6}$$

$$(\phi \otimes m \otimes_{\mathcal{K}} c_0 \otimes_R \cdots \otimes_R c_q)(a_0 \otimes \cdots \otimes a_{p+q}) =$$
$$\phi(m_{<0>} \otimes (c_0^{(p+1)} a_0)(c_1 a_1) \cdots (c_q a_q) \otimes_R m_{<-p>}(c_0^{(1)} a^{q+1}) \otimes_R \cdots \otimes_R m_{<-1>}(c_0^{(p)} a^{p+q})).$$

Example 4.4. In the theorem 4.2, let $M = R$. As a result we have

$$C^0_{\mathcal{K}}(A, R) = R \otimes_R A \cong A,$$

where the isomorphism is given by $r \otimes a \longmapsto \mathfrak{s}(r) \triangleright a$ and $1_R \otimes a \longleftarrow a$. Therefore a 0-Hopf cocycle is an $\mathcal{K}$-linear map $Tr : A \longrightarrow R$ where

$$Tr(\mathfrak{s}(r) \triangleright (a_1 a_2)) = Tr(a_2(\mathfrak{s}(r)\sigma \triangleright a_1)), \quad r \in R, a_1, a_2 \in A. \tag{4.7}$$

If R is unital we obtain

$$Tr(a_1 a_2) = Tr(a_2(\sigma \triangleright a_1)), \quad a_1, a_2 \in A. \tag{4.8}$$

Such a trace is called a σ-trace. Also Tr is a right $\mathcal{K}$-linear map. Therefore since R is unital we have $Tr(k \triangleright a) = Tr(a) \triangleleft k$. Using the definition of the action defined in (2.42) we have

$$Tr(k \triangleright a) = \delta(\mathfrak{s}(Tr(a))k). \tag{4.9}$$

Such a trace is called a δ-trace. One notes when $\mathcal{K}$ is a Hopf algebra, this condition is equivalent to $Tr(ha) = \delta(h)Tr(a)$. In fact a σ-trace which is δ-invariant is a 0-cocycle. Therefore for $p = 0$ in the Theorem 4.2 we obtain;

$$HC^n_{\mathcal{K}}(\mathcal{K}, R) \longrightarrow HC^n(A) \tag{4.10}$$
$$Tr(k_0 \otimes_R \cdots \otimes_R k_n)(a_0 \otimes_R \otimes \cdots \otimes_R) = Tr(a_0 k_0(a_1) \cdots k_n(a_n)).$$

Example 4.5. In the Theorem 4.2, let $K = R^e$, $M = R$ and $p = 0$. we obtain a characteristic map as follows;

$$HC^n_{R \otimes R^{op}}(R \otimes R^{op}, R) \longrightarrow HC^n(A). \tag{4.11}$$

Using Example 3.2, we have;

$$HC^n(\mathbb{C}) \longrightarrow HC^n(A). \tag{4.12}$$

Example 4.6. We apply the Theorem 4.2 to Connes-Moscovici $\times$-Hopf algebra. In fact let $C = \mathcal{K} = A \otimes H \otimes A^{op}$, $M = A$ and $p = 0$. Using Example 3.3 we obtain the following map;

$$HC^*(H) \longrightarrow HC^*(A). \tag{4.13}$$

Example 4.7. In the theorem 4.2, let $\mathcal{K} = A_\theta$, $M = R = \mathbb{C}[U, U^{-1}]$ and $p = 0$. Using Example 3.4 for any $\mathcal{K}$-module algebra A we obtain the following map;

$$\mathbb{C}[U, U^{-1}] \longrightarrow HC^{2n}(A). \tag{4.14}$$

Example 4.8. In the theorem 4.2, let $B \subseteq A$ be an algebra extensions, $R = B$, $\mathcal{K} = C = B^e$, and $M = B$. Then we obtain the following pairing;

$$\widetilde{HC}^p_{B^e}(A, B) \otimes HC^q_{B^e}(B^e, B) \longrightarrow HC^{p+q}(A). \tag{4.15}$$

Example 4.9. Let $\mathcal{B}$ be a right $\times$-Hopf algebra, A be a right comodule algebra [HR1], $B = T^{\mathcal{B}}$ be the space of invariant coactions, and $\mathcal{K}$ a left $\times_B$-Hopf algebra. If ${}_{\mathcal{K}}A(B)^{\mathcal{B}}$ is an equivaraint Hopf Galois extension as defined in [HR1] then it is shown that

$$\widetilde{HC}^p_{\mathcal{K}}(A, M) \cong HC^p_{\mathcal{B}}(\mathcal{B}, \widetilde{M}),$$

where $\widetilde{M} = B \otimes_{\mathcal{K}} M$. Therefore using the theorem 4.2 we obtain the following pairing;

$$\widetilde{HC}^p_{\mathcal{B}}(\mathcal{B}, \widetilde{M}) \otimes HC^q_{\mathcal{K}}(\mathcal{K}, M) \longrightarrow HC^{p+q}(A). \tag{4.16}$$

References

[BSz] G. Böhm, and K. Szlachányi, *Hopf algebroids with bijective antipodes: axioms, integrals, and duals*, J. Algebra 274, no. 2, pp. 708–750, (2004).

[BS] G. Böhm, and D. Stefan, *(co)cyclic (co)homology of bialgebroids: An approach via (co)monads*, Commun. Math. Phys **282**, pp. 239–286, (2008).

[BW] T. Brzezinski, and R. Wisbauer, *Corings and Comodules*, London Mathematical Society Lecture Note Series **309** Cambridge University Press, Cambridge, (2003).

[C-Book] A. Connes, *Noncommutative geometry*, Academic Press, (1994).

[CM98] A. Connes, and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys. **198**, pp. 199–246, (1998).

[CM00] A. Connes, and Moscovici, *Cyclic cohomology and Hopf algebra symmetry*, Lett. Phys. 52 (1), pp. 1–28, (2000).

[CM01] A. Connes, and H. Moscovici, *Differential cyclic cohomology and Hopf algebraic structures in transverse geometry*, in: Essays on geometry and related topics. Vol 1-2, Monogr. Enseign. Math. **38**, Enseignement Math, Geneva, pp. 217–255, (2001).

[CM2004] A. Connes, and H. Moscovici, *Rankin-Cohen brackets and the Hopf algebra of transverse geometry*, Moscow Mathematical Journal, V. 4. no. 1, pp. 111–130, (2004).

[HK1] T. Hadfield, and U. Krähmer, *Twisted homology of quantum SL(2)*, K-Theory 34, no. 4, pp. 327–360, (2005).

[HK2] T. Hadfield, and U. Krähmer, *Twisted homology of quantum SL(2)-part II*, Journal of K-Theory 6, no. 1, pp. 69–98, (2010).

[CR] M. Crainic, *Cyclic cohomology of Hopf algebras*, J. Pure Appl. Algebra 166, no. 1-2, pp. 29–66, (2002).

[HKRS1] P.M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, *Stable anti-Yetter-Drinfeld modules*, C. R. Math. Acad. Sci. Paris **338**, pp. 587–590, (2004).

[HKRS2] P.M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, *Hopf-cyclic homology and cohomology with coefficients*, C. R. Math. Acad. Sci. Paris **338**, pp. 667–672, (2004).

[HR1] M. Hassanzadeh, B. Rangipour, *Equivariant Hopf Galois extensions and Hopf cyclic cohomology*, To apear in Journal of noncommutative Geometry, 33 pages, (2010).

[HR2] M. Hassanzadeh, B. Rangipour, *Quantum Groupoids and their Hopf Cyclic Cohomology*, http://arxiv.org/abs/1108.0030, 48 pages, (2011).

[Kad] L. Kadison, *Pseudo-Galois extensions and Hopf algebroids*, in: Modules and Comodules, (Porto conf. Sept. (2006) for Robert Wisbauer) eds. Brzezinski et al Birkhauser Trends in Math. XII, pp. 247–264, (2008).

[Kay1] A. Kaygun, *A survey on Hopf-cyclic cohomology and Connes-Moscovici characteristic map*, Contemporary Mathematics, Vol. 546 pp. 171–179, (Proceedings of Henrifest), (2011)

[Kay2] A. Kaygun, *Products in Hopf-cyclic cohomology, Homology, Homotopy and Applications*, 10, no. 2, pp. 115–133, (2008).

[Kay3] A. Kaygun, *The universal Hopf cyclic theory*, Journal of Noncommutative Geometry Vol.**2**, No.2, pp. 333-351, (2008).

[Ko] N. Kowalzig, *Hopf Algebroids and Their Cyclic Theory*, Ph. D. Thesis, (2009).

[KK] N. Kowalzig, and U. Krähmer, *Cyclic structures in algebraic (co)homology theories*, Homology, Homotopy and Applications, Vol. 13, No. 1, pp. 297–318, (2011).

[KR1] M. Khalkhali, and B. Rangipour, *A new cyclic module for Hopf algebras*, K-Theory 27, pp. 111–131, (2002).

[KR2] M. Khalkhali, and B. Rangipour,*Cup products in Hopf-cyclic cohomology*, C. R. Math. Acad. Sci. Paris 340, no. 1, pp. 914, (2005).

[KR3] M. Khalkhali, and B. Rangipour, *Para-Hopf algebroids and their cyclic cohomology*, Lett. Math. Phys. 70, no. 3, pp. 259–272, (2004).

[KRT] J. Kustermans, J. Rognes, and L. Tuset, *The Connes-Moscovici approach to cyclic cohomology for compact quantum groups*, K-Theory 26, no. 2, pp. 101–137, (2002).

[Lu] J.H. Lu, *Hopf algebroids and quantum groupoids*, Int. J. Math. 7, pp. 4770, (1996).

[NS] I. M. Nikonov, and G. I. Sharygin, *On the Hopf-type cyclic cohomology with coefficients*, C.-algebras and elliptic theory, Trends Math., Birkhauser, Basel, pp. 203–212, (2006).

[PVO] F. Panaite, and F. Van Oystaeyen, *Some bialgebroids constructed by Kadison and Connes-Moscovici are isomorphic*, Applied Categorical Structures, Vol. 14, No. 5-6, pp. 627-632, (2006).

[R1] B. Rangipour, *Cup products in Hopf cyclic cohomology via cyclic modules I*, Homology Homotopy Appl. Volume 10, Number 2, 273-286, (2008).

[Sch98] P. Schauenburg, *Bialgebras over noncommutative rings and a structure theorem for Hopf bimodules*, Appl. Categorical Str. **6**, pp. 193–222, (1998).

[Sch] P. Schauenburg, *Duals and doubles of quantum groupoids ($\times_R$-Hopf algebras)*, in: N. Andruskiewitsch, W.R. Ferrer-Santos and H.-J. Schneider (eds.) Contemp. Math. **267**, Providence, pp. 273–293, (2000).

[Tak] M. Takeuchi, *Group of algebras over $A \otimes \overline{A}$*, J. Math. Soc. Japan 90, pp. 459–492, (1977).