The twelve lectures in the (non)commutative geometry of differential equations

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THE TWELVE LECTURES IN THE (NON)COMMUTATIVE GEOMETRY OF DIFFERENTIAL EQUATIONS

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Abstract. These notes follow the twelve-lecture course in the geometry of nonlinear partial differential equations of mathematical physics. Briefly yet systematically, we outline the geometric and algebraic structures associated with such equations and study the properties of these structures and their inter-relations. The lectures cover the standard material about the infinite jet bundles, systems of differential equations (e.g., Lagrangian or Hamiltonian), their symmetries and conservation laws (together with the First and Second Noether Theorems), and the construction of the nonlocalities. Besides, in the lectures we introduce the calculus of variational multivectors — in terms of the Schouten bracket, or the antibracket — on the (non)commutative jet spaces and proceed with its applications to the variational Poisson formalism and the BRST- or BV-approach to the gauge systems.

The course differs from other texts on the subject by its greater emphasis on the physics that motivates the model geometries. Simultaneously, the course attests to the applicability of the algebraic techniques in the analysis of the geometry of fundamental interactions. These lectures could be a precursor to the study of the (quantum) field and string theory.

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Contents

Foreword

> Every block of stone has a statue inside it and it is the task of the sculptor to discover it.
> Michelangelo

This course is aimed to teach. Any other goals, such as: to bring more rigour into the treatment, to make the calculus more axiomatic, to reveal the possibilities of a further generalization, to reformulate the setup in coordinate-free algebraic terms, to broaden the horizons, and to bridge the gaps — or similar pontific activities — are secondary.

These notes do not contain any previously unknown, unpublished facts. The author rather faced the task of selection and systematization of the set of basic definitions, properties, and algorithms. The course communicates the bare minimum of the geometric theory in the amount which is practically useful for a mathematical physicist. The lectures indicate the applicability and limitations of this theory. Simultaneously, they prepare the curious scholar to a deeper study of the subject: the book [100], which has become the standard reference, and the book [79], which describes the application of the advanced algebraic techniques to differential equations, are highly recommended as the follow-up reading. The lectures also facilitate the understanding of the context and results in the current journal publications on this subject.

These notes are based on the lectures which were read by the author in the spring semesters in 2009 (at Utrecht University) and 2012 (at the University of Groningen). In the present form, the twelve chapters schedule the full-week course, lasting for six days with two standard lectures and one exercise class on each; on the seventh day, the audience experiences the shift of the paradigm.

The author declares his sole responsibility for the selection (resp., omission) of the material and its presentation. The arrangement of the topics reflects the author's personal view how the geometry of differential equations could be taught to the prepared novices. Each example or illustration, whenever quoted here for the educational purposes, is to be acknowledged as a token of gratitude to its author(s). Any discrepancies, instances of the incompleteness, factual errors (if any), or any mismatches in the notation are unintentional and should be promptly reported to the author.

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Introduction

These lectures in the geometry of nonlinear partial differential equations are devised for the students of mathematical and physical specializations who learn this discipline for the first time. The course acquaints the audience with the main notions and concepts; it neither fully covers the vast set of geometries nor reports all the modern methods and techniques. However, we believe that the familiarity with the language of jet bundles, which are ambient to differential equations, is indispensable to the specialists in classical mathematical physics (such as the General Relativity) and its quantisation (*e. g.*, QED, QCD, or string theory[38, 102]). It must be noted that the techniques and statements from the traditional paradigm of smooth manifolds, be they Poisson or be they the configuration spaces of the Euler–Lagrange systems, may no longer fully grasp the geometries of those models from QED, QCD, etc.

The course unveils the relations between the models of physical phenomena and the geometry behind such models. It outlines the limits of the applicability of the models. It then becomes possible to focus on the sets of initial axioms and sort out the *ad hoc* assumptions from the conceptual hypotheses such as the gauge invariance or the presence of infinitely many integrals of motion.

The course communicates the following **skills**:

- finding particular (classes of) solutions of nonlinear partial differential equations, and propagation of the already known solutions to families;
- finding the integrals of motion (more generally, finding the conserved currents);
- finding the variational Poisson structures and verification that a given (non)commutative bi-vector is Poisson;
- by using the same calculus of variational multivectors and involving the structures which are pertinent to the gauge systems, the introduction of the BRST- and BV-differentials and the construction of the extended BV-action that satisfies the classical variational master equation;
- inspecting whether the "spectral" parameter in a given zero-curvature representation is or is not removable by the gauge transformations.

In the narration, we balance between the coordinate-independent reasonings and the operation in local coordinates. To motivate and illustrate the main notions and constructions, we restrict ourself to the reasonable set of model equations. Nevertheless, these examples are multi-faceted so that they fit for all purposes. Such are the kinematic-integrable, bi-Hamiltonian Korteweg–de Vries equation (*i. e.*, the Drinfeld–Sokolov equation[22] associated with the root system A_1), the Darboux-integrable Liouville equation (*i. e.*, the nonperiodic 2D Toda chain associated with the root system A_1), and the Maxwell equation (*i. e.*, the Yang–Mills equation with the abelian gauge group $U(1)$).

At the same time, we balance between the maximal generality and the concrete realization of the abstract structures: for instance, we consider the homological equation $\boldsymbol{Q}^2 = 0$ on the jet super-spaces and from it we derive the (non)commutative variational Poisson formalism and the BRST- and BV-techniques.

The examples often indicate the branching point of theories and mark the exits from the main road to other concepts,[12] geometries,[23] and methods such as the supersymmetry, inverse scattering,[27] Hirota's tau-function,[96] (Poisson) vertex algebras,[46] the Yang–Baxter associativity equation,[104] or homotopy Lie structures[84] — to name only a few.

The superposition of mathematics and physics in this text makes it different fron either a traditional treatise on the formal geometry of differential equations or the reviews on gauge fields or Poisson mechanics. For a working mathematician, our approach may seem not entirely axiomatic. A theoretical physicist who waits to apply the new calculus in the own models will first face the great multitude of preliminary concepts which are declared to be the pre-requisites. We also note that we by-pass the detailed consideration of the *finite* jet spaces;[105] instead, we spend an extra time on the nonlocalities over the infinite jet spaces. Likewise, we study the "generic," higher-order Hamiltonian operators and thus, again, pass over the geometry of the hydrodynamic-type evolutionary systems.[24]

Summarizing, in these lectures we substantiate the key facts and mark the links to other domains of science.

Let us make several technical but important assumptions about the class of geometries which we are going to consider. The ground field will always be $\mathbb{R}$ and the real dimension $n \in \mathbb{N}$ of the base manifold M^n in the fibre bundles can be arbitrary positive integer (unlike $n = 1$ in CFT or in many surveys on the KdV-type systems). We study the non-graded setup of purely even manifolds and bundles. The odd *neighbours* will appear in due course: the parity reversion Π will be performed by hands, which will be stated explicitly. We do not develop on the plurality of the quantisation techniques (e. g., the deformation- or the BRST/BV-quantisation) which become available as soon as the jet bundle formalism is elaborated. Nor do we track the relation between the geometry of jets and the quantum groups, quantum integrable systems, and the classical Yang–Baxter equation. Although we introduce the calculus of variational multivectors in the noncommutative setup ab initio, the analysis stays at the non-quantum level (i. e., the Planck constant $\hbar$ almost never appears, with two important exceptions).

Unless it is stated otherwise, all maps (including those which define the smoothness classes of manifolds) are assumed to be infinitely smooth yet possibly not analytic. We emphasize that the power series "solutions" of the formally integrable differential equations can diverge outside the central point. This subtlety projects (1) onto our choice of the classes of sections of the bundles (so that we shall sometimes assume, for the vector bundles, that either the supports are finite, or the sections rapidly decay at the infinity, or are periodic but the period is not known in advance), and also projects (2) onto our hypotheses about the topology of the base manifolds (always assumed oriented) and of the total spaces of the bundles. Namely, the local triviality assumption can become insufficient whenever one attempts to integrate by parts or calculate the value of a functional at a given section by taking the integral over the entire base. Thus, the choice of the section classes versus the compactness of the base is correlated with the analytic integrability of the functionals. Usually, we shall discard the aspects of

topology. Likewise, we consider the spaces of jets of sections for the (vector) bundles but not the jets of submanifolds.

Lie groups and algebras are of course present, but their rôle is not emphatic (c. f. [22, 104]). On the same grounds, we exploit the geometric roots of the (non)commutative Poisson structures but avoid their study via the Fourier transform that converts them into the W-algebras and a suitable analog of the quantum BRST.[13]

The geometry of spaces of infinite jets for sections of vector bundles admits the immediate generalizations to the $\mathbb{Z}_2$-graded setup of supermanifolds,[77] to the jet spaces for maps of manifolds, and to noncommutative geometry. The calculus of (non)commutative variational multivectors has its renowned applications in the Hamiltonian formalism and in the geometry of the Euler–Lagrange gauge-invariant systems; both concepts are intimately related to the quantum string theory. The formalism also serves the following three model descriptions of the scattering: the Kontsevich deformation quantisation[68] (governed by the associativity equation[118] $|1\rangle \star (|2\rangle \star |3\rangle) = (|1\rangle \star |2\rangle) \star |3\rangle$ for the in-coming states $|i\rangle$), the pairwise-interaction factorization in Hirota's tau-function, and the inverse spectral transform. The correlation between the jet bundle formalism and Feynman's path integral approach to the quantum world is yet to be explored: there is more about it than meets the eye.

Part I. Lagrangian theory

While studying classical mechanics,[2, 85] we considered[51] the static, kinematics, and dynamics of material points: respectively, we analysed the structure of the configuration spaces, the non-relativistic motion along the true trajectories, and the interaction of massive particles in space (or in any suitable curved manifold). Special relativity theory allows us to approach the speed of light in the description of the motion of such points. Yet in both cases the objects were not stretched in space (e. g., they were like the electron — alternatively, the finite size of the objects could be neglected, such as in Kepler's problem problem for the attracting spheres). Besides, the objects did not contain any internal, hidden degrees of freedom which would determine the hypothetic fifth and, possiblly, higher dimensions that stay invisible for us.

However, the high-energy scattering experimentally confirms that the first not-yet-considered option is manifestly the case for the spatially-extended elementary particles such as the proton or the neutron, whereas the second option of a geometry with the hidden extra dimensions leads to the string theory (or the Kaluza–Klein formalism). The common feature of these geometries is the (implicit) presence of the maps $\varSigma^{D,1} \hookrightarrow M^{3,1}$ from the manifolds that carry some information about the interactions to the visible space-time where the measurements take place.

On the other hand, it is now generally accepted that the elementary particles of all sorts (whatever be their hidden structure) interact via the fields: primarily, the gauge fields, which are the connections in the principal fibre bundles over the Minkowski space-time with the structure groups such as $U(1)$ for Maxwell's electromagnetism, $SU(2)$ for the weak force, possibly $SU(3)$ for the strong force, and the pseudogroup of local diffeomorphisms for the gravity. This class of geometries is opposite to the former in the sense that the extra structures (the matter field $\psi(\boldsymbol{x})$, that is, the wave function, and the gauge field $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$, or the connection one-form) are sections of the appropriate bundles $\pi\colon E \to M^{3,1}$ over the space-time.

Our analysis covers the following set of model geometries:

- (graded-)commutative and noncommutative gauge fields over the space-time $M^{3,1}$;
- the configuration spaces for strings $\varSigma^{1,1} \hookrightarrow M^{3,1}$ or higher-dimensional discs $\mathrm{D}^{D,1} \hookrightarrow M^{3,1}$;
- closed, (non)commutative string-like quantum objects $\mathbb{S}^1_{\mathcal{A}}$ that carry the non-commutative fields $\mathcal{A}$ and propagate as sections of the bundles $\pi^{\mathrm{nC}}\colon E \xrightarrow{\mathbb{S}^1_{\mathcal{A}}} M^{3,1}$ over the space-time.

Whichever class of such geometries, or a combination of these classes be involved in a specific model, the motion (kinematics) and the interaction (dynamics) are described through partial differential equations. By construction, the concept of a differential equation reflects the idea that the unknowns satisfy a certain constraint (indeed, the equation itself) but we do not know in advance which particular solution we are having at our disposal. In other words, the formalism treats all the solutions (e.g., continuous paths) simultaneously. Let us study the geometric properties of such systems and then approach the techniques for their quantisation.[8, 10]

1. Jet space and structures on it

Let $\pi\colon E^{m+n} \to M^n$ be a vector bundle. Let $\boldsymbol{x} \in M^n$ be a point[2] and $\boldsymbol{u} \in \pi^{-1}(\boldsymbol{x})$ be a point in the fibre over $\boldsymbol{x}$.

We now iteratively construct the infinite jet space $J^\infty(\pi)$ which – locally over M^n – carries all the information about the local sections $\Gamma(\pi)$ at each point of the base M^n.

Remark 1.1. We treat the *admissible* sections as if they were infinitely smooth. However, let us bear in mind that such sections serve us to approximate those continuous sections which are possibly no more than (α, c)-Lipschitz with the coefficient $\alpha = 1$ and the constant c being the speed of light.

We notice that the equality of the values of two continuously differentiable sections at a point $\boldsymbol{x}_0 \in M^n$ is equivalent to the statement that their difference, also being a section, has a zero at $\boldsymbol{x}_0$:

$$\boldsymbol{s}_1 - \boldsymbol{s}_2 \in \mu_{\boldsymbol{x}_0}\Gamma(\pi) \stackrel{\text{def}}{=} \{\boldsymbol{s} \in \Gamma_{\text{loc}}(\pi) \mid \exists\, \boldsymbol{r} \in \Gamma_{\text{loc}}(\pi), \exists\, \mu \in C^\infty_{\text{loc}}(M^n) \\ \text{such that } \mu(\boldsymbol{x}_0) = 0, \boldsymbol{s} = \mu \cdot \boldsymbol{r}\}.$$

Indeed, this holds due to the following lemma.

Lemma 1.1 (Hadamard)**.** Let $f \in C^1(\mathbb{R})$ and $x_0 \in \mathbb{R}$, then there exists $g \in C^1(\mathbb{R})$ such that $f(x) = f(x_0) + (x - x_0) \cdot g(x)$ for every $x \in \mathbb{R}$.

Exercise 1.1. Write the Taylor–McLaurent expansion in the Hadamard form near $x_0 \in \mathbb{R}$ for a real-valued function $f \in C^{k+1}(\mathbb{R})$, $k \in \mathbb{N}$.

We say that two (sufficiently smooth) sections $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \Gamma_{\text{loc}}(\pi)$ are *tangent* at $\boldsymbol{x}_0 \in M^n$ with *tangency order* $k \geqslant 0$ if $(\boldsymbol{s}_1 - \boldsymbol{s}_2)(\boldsymbol{x}) \sim \overline{o}(|\boldsymbol{x} - \boldsymbol{x}_0|^k)$ for all $\boldsymbol{x}$ near $\boldsymbol{x}_0$ in M^n. By convention, the sections are tangent of order zero if only their values coincide at $\boldsymbol{x}_0$.

Remark 1.2. The Hadamard lemma implies that the partial derivatives of $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ at the point $\boldsymbol{x}_0$ coincide at all orders up to and including k whenever the sections are tangent of order k at that point. Because the tangency of order k is preserved under any C^∞-smooth reparametrizations $\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}(\boldsymbol{x})$ of the domain $V \subseteq M^n$ containing $\boldsymbol{x}_0$ (prove!), the analytic interpretation of the tangency is independent of the choice of local coordinates.

Consider the equivalence classes of (local) sections at a point,

$$[\boldsymbol{s}]^k_{\boldsymbol{x}_0} \stackrel{\text{def}}{=} \Gamma(\pi) / \left(\mu^{k+1}_{\boldsymbol{x}_0}\Gamma(\pi)\right), \quad k \geqslant 0.$$

Each element $\boldsymbol{s}$ of this class marks the set of sections $\boldsymbol{s}_1 \in \Gamma_{\text{loc}}(\pi)$ such that $\boldsymbol{s} - \boldsymbol{s}_1 = \overline{o}(|\boldsymbol{x} - \boldsymbol{x}_0|^k)$.

Definition 1.1. The space $J^k(\pi)$ of k-th jets of sections for the vector bundle π is the union

$$J^k(\pi) \stackrel{\text{def}}{=} \bigcup_{\substack{\boldsymbol{x} \in M^n \\ \boldsymbol{s} \in \Gamma_{\text{loc}}(\pi)}} [\boldsymbol{s}]^k_{\boldsymbol{x}}.$$

[2] In what follows we also let $\boldsymbol{x} = (x^1, \ldots, x^n)$ be the n-tuple of local coordinates on a simply connected domain $V \subseteq M^n$, while $\boldsymbol{u} = (u^1, \ldots, u^m)$ denotes the coordinates in $\pi^{-1}(V)$. We recall further that the vector spaces $\pi^{-1}(\boldsymbol{x})$ can themselves be the charts $U_\alpha \subseteq N^m$ in the target manifold.

For example, let $k=0$. Each point $\theta^0=[\boldsymbol{s}]^0_{\boldsymbol{x}}\in J^0(\pi)$ carries the information about the base point $\boldsymbol{x}\in M^n$ and the value $\boldsymbol{u}=\boldsymbol{s}(\boldsymbol{x})$ which is common for all the sections $\boldsymbol{s}$ which mark that equivalence class. Consequently, $J^0(\pi)\simeq E^{m+n}$, i. e., the total space of the bundle.

By Remark 1.2, the coordinates $\boldsymbol{x}_0$ of the base points and the values $\frac{\partial^{|\tau|}}{\partial\boldsymbol{x}^\tau}\Big|_{\boldsymbol{x}_0}(\boldsymbol{s}')$ of the derivatives for any representative $\boldsymbol{s}'\in[\boldsymbol{s}]^k_{\boldsymbol{x}_0}$ and all multi-indices τ such that $|\tau|\leqslant k$ comprise the entire set of parameters which uniquely determine a point $\theta^k=[\boldsymbol{s}]^k_{\boldsymbol{x}_0}\in J^k(\pi)$.

Exercise 1.2. Calculate the dimension $J^k(\pi)$ for $\pi\colon E^{m+n}\to M^n$ with m-dimensional fibers.

From now on, we denote by $\boldsymbol{u}$, $\boldsymbol{u}_{\boldsymbol{x}}$, ..., $\boldsymbol{u}_\sigma$ the jet fibre coordinates in $\pi^{-1}_{k,-\infty}(\boldsymbol{x})$; here $|\sigma|\leqslant k$ and $\boldsymbol{u}_\varnothing\equiv\boldsymbol{u}$ for the empty multi-index.

Exercise 1.3. Prove that $\pi_{k,-\infty}\colon J^k(\pi)\to M^n$ is a vector bundle if π was.

Exercise 1.4. Prove that the forgetful map $\pi_{k+1,k}\colon J^{k+1}(\pi)\to J^k(\pi)$ which discards the values of the topmost (the $(k+1)$-th order) derivatives at each point $\boldsymbol{x}\in M^n$ determines a vector bundle over $J^k(\pi)$ for all $k\geqslant 0$.

Unfortunately, the *finite*-order jet spaces $J^k(\pi)$ are very inconvenient for our practical purposes (with the only exceptions for the point geometry of $J^0(\pi)$ and the contact geometry of $J^1(\pi)$, which we do not study here in further detail). This is because most of the objects in differential calculus always reach the ceiling of the k-th derivatives at finitely many steps and then stop (e. g., consider an attempt of finding a formal power series solution for a given differential equation at a given point of the manifold M^n). The passage to the projective limit of $J^k(\pi)$ as $k\to+\infty$ radically improves the situation.

Definition 1.2. The *infinite jet space* $J^\infty(\pi)$ is the projective limit

$$J^\infty(\pi)\stackrel{\text{def}}{=}\varprojlim_{k\to+\infty}J^k(\pi),$$

that is, the minimal object such that there is the infinite chain of the epimorphisms $\pi_{\infty,k}\colon J^\infty(\pi)\to J^k(\pi)$ for every $k\geqslant 0$ and there is the vector bundle structure $\pi_{\infty,-\infty}\colon J^\infty(\pi)\to M^n$ such that the diagram with all admissible compositions of $\pi_{k+1,k}$, $\pi_{\infty,k}$ and $\pi_\infty\stackrel{\text{def}}{=}\pi_{\infty,-\infty}$ is commutative. A point $\theta^\infty\in J^\infty(\pi)$ is the infinite sequence $\theta^\infty=(\boldsymbol{x},\boldsymbol{u},\boldsymbol{u}_{\boldsymbol{x}},\ldots,\boldsymbol{u}_\sigma,\ldots)$ of (collections of) real numbers; here $|\sigma|\geqslant 0$ and $\boldsymbol{u}_\varnothing\equiv\boldsymbol{u}$.

The set $J^\infty(\pi)$ is an infinite-dimensional manifold with the projective-limit topology.

Our choice of the fibre coordinates $\boldsymbol{u}$, $\boldsymbol{u}_{\boldsymbol{x}}$, ..., $\boldsymbol{u}_\sigma$, $|\sigma|\geqslant k$, in the finite jet spaces $J^k(\pi)$ permits the identification of the points $\theta^\infty\in J^\infty(\pi)$ with the *formal* Taylor-McLaurent series on neighborhoofd of the points $\boldsymbol{x}$ of the base M^n. However, there is usually no hope that a given formal power series would converge to a local section of π at any point of M^n but the center of its expansion. For example, let $n=1$ and $m=1$ and set $\theta^\infty=(0,0!,1!,2!,3!,\ldots,k!,\ldots)\in J^\infty(\pi)$.

Therefore, the following result may seem very surprising.[105]

Lemma 1.2 (Borel)**.** Every point θ^∞ in the infinite jet space $J^\infty(\pi)$ does encode – up to the sections which are C^∞-flat at the central point $\boldsymbol{x} = \pi_\infty(\theta^\infty) \in M^n$ – a genuine infinitely smooth local section with the values of its derivatives at $\boldsymbol{x}$ prescribed by θ^∞ (although that section can be desperately non-analytic).

For the sake of brevity, let us consider the scalar case $n = 1$ and $m = 1$ with one independent variable x and the dependent variable u. We leave the extension of the proof to the setup $n \geqslant 1$ and $m \geqslant 1$ as an elementary exercise.

Proof. Let $(a_\sigma \mid \sigma \in \mathbb{N} \cup \{0\}) = (a_0, a_1, a_2, \ldots, a_\sigma, \ldots)$ be a sequence of real numbers. We now solve the problem of constructing an infinitely smooth (but not necessarily analytic) real-valued function $f \in C^\infty(\mathbb{R})$ such that the values of its derivatives $\frac{\partial^\sigma}{\partial x^\sigma}\Big|_{x=0} f$ taken at the origin are equal to the respective real constants a_σ for all $\sigma \geqslant 0$ (the derivative of zeroth order is of course the function itself).

Let $\eta \in C^\infty(\mathbb{R})$ be a smooth cut function such that $\eta(x) \equiv 1$ for all $|x| \leqslant \frac{1}{2}$ and $\eta(x) \equiv 0$ for all $|x| \geqslant 1$. By definition, set

$$t_\sigma(x) := x^\sigma \cdot \eta(x) \quad \text{so that} \quad t_\sigma^{(\tau)}(0) = t! \cdot \delta_\sigma^\tau$$

(here δ_σ^τ is the Kronecker delta) and put

$$\lambda_\sigma := \max(2^\sigma, |a_\sigma|, \|t_\sigma^{(\tau)}\|_{(\sigma)}) \quad \text{for all } \sigma \geqslant 0; \tag{1.1}$$

note that the support $|x| < 1$ of t_σ lies inside a compact in $\mathbb{R}$ and hence the supremum

$$\|t_\sigma^{(\tau)}\|_{(\sigma)} \stackrel{\text{def}}{=} \max_{\tau \leqslant \sigma} \sup_{x \in \mathbb{R}} |t_\sigma^{(\tau)}|$$

is finite[3] for each σ.

Consider the functions

$$f_\sigma(x) := \frac{a_\sigma}{\sigma!(\lambda_\sigma)^\sigma} \cdot t_\sigma(\lambda_\sigma x) \tag{1.2}$$

and compose them to the functional series $f(x) := \sum\limits_{\sigma=0}^{+\infty} f_\sigma(x)$, which obviously equals a_0 at $x = 0$.

We notice that our choice of the lower bound 2^σ for λ_σ (see (1.1)) produces the nested family of the cut functions $\eta(\lambda_\sigma x)$ whose supports $\operatorname{supp} \eta(\lambda_\sigma x) = \{|x| < 2^{-\sigma}\}$ shrink to an arbitrarily small neighbourhood of the origin after a sufficient number of steps. Therefore, at each $x \neq 0$ the series $f(x) = \sum\limits_{\sigma=0}^{+\infty} f_\sigma(x)$, as well as the functional series

$$f^{(\tau)}(x) := \sum_{\sigma=0}^{+\infty} f_\sigma^{(\tau)}(x), \quad 0 < \tau < \infty,$$

of the term-wise derivatives, contains only finitely many nonvanishing summands.[4] Because all of them are infinitely smooth on $\mathbb{R}$, we conclude that the series $f^{(\tau)}$ with $0 \leqslant \tau < \infty$ (here $f^{(\varnothing)} \equiv f$ by the usual convention) converge at all $x \neq 0$ and are infinitely smooth at all such points. This shows that the only point of possible discontinuity

[3] and attained: Note also that there may be no global bound upon the suprema of all the derivatives of t_σ on its support.

[4] The integer part of $1 - \log_2 |x|$ is the upper bound for such number at $|x| > 0$.

for either the sum itself or for the sums $f^{(\tau)}$ of the termwise derivatives is the origin $x = 0$. We claim that all the series $f^{(\tau)}$ converge uniformly on $x \leqslant 1$ for all $\tau < \infty$ and hence these sums, including also $f \equiv f^{(\varnothing)}$ which converges at $x = 0$, are continuous (in particular, at $x = 0$, which was the only point of our concern).

Indeed, let us apply the Weierstrass majorant test: if

$$\sup_{|x|\leqslant 1} |f_\sigma^{(\tau)}(x)| < M_\sigma^{(\tau)} = \text{const}(\sigma, \tau) \in \mathbb{R}$$

and the real number series $\sum\limits_{\sigma \geqslant 0} M_\sigma^{(\tau)}$ converges for each τ, then the functional series $f^{(\tau)}$ uniformly converges on $|x| \leqslant 1$ (so, to the respective continuous function in each case, $\tau \geqslant 0$).

To this end, for every $\tau > 0$ and $\sigma \geqslant 0$, let us estimate the norms

$$\|f_\sigma^{(\tau)}\|_{\sup} \stackrel{\text{def}}{=} \sup_{|x|\leqslant 1} |f_\sigma^{(\tau)}(x)|.$$

We have that

$$\sum_{\sigma=0}^{+\infty} \|f_\sigma^{(\tau)}\|_{\sup} \leqslant \sum_{\sigma=0}^{\tau+1} \frac{|a_\sigma|}{\sigma!} \cdot \frac{(\lambda_\sigma)^\tau}{(\lambda_\sigma)^\sigma} \cdot \|t_\sigma^{(\tau)}\|_{\sup} + \sum_{\sigma=\tau+2}^{+\infty} \frac{1}{\sigma!} \cdot \frac{1}{(\lambda_\sigma)^{\sigma-(\tau+2)}} \cdot \frac{|a_\sigma|}{\lambda_\sigma} \cdot \frac{\|t_\sigma^{(\tau)}\|_{\sup}}{\lambda_\sigma}. \quad (1.3)$$

The first summand in the right-hand side is a finite real number (depending on τ which is fixed); in the second term, the factors are nonnegative, and specifically,

- the power $\sigma - (\tau + 2) \geqslant 0$ for all σ so that $0 < (\lambda_\sigma)^{-(\sigma-(\tau+2))} \leqslant 1$ for $\lambda_\sigma \geqslant 2^\sigma$;
- the ratio $|a_\sigma|/\lambda_\sigma \leqslant 1$ holds by the definition of λ_σ;
- the norm
$$\|t_\sigma^{(\tau)}\|_{\sup} = \sup_{x\leqslant 1} |t_\sigma^{(\tau)}(x)| \leqslant \max_{\rho\leqslant\sigma} \sup_{x\in\mathbb{R}} |t_\sigma^{(\rho)}(x)| = \|t_\sigma\|_{(\sigma)}$$
because $\sigma > \tau$, which implies that $\|t_\sigma^{(\tau)}\|_{\sup}/\lambda_\sigma \leqslant \|t_\sigma\|_{(\sigma)}/\lambda_\sigma \leqslant 1$ by the definition of λ_σ;
- the series $\sum\limits_{\sigma=\tau+2}^{+\infty} 1/\sigma!$ converges (in fact, its sum does not exceed $e - 2$).

We conclude that the entire second summand in the right-hand side of inequality (1.3) does not exceed $e - 2$, whence the series $f^{(\tau)}(x) = \sum\limits_{\sigma\geqslant 0} f_\sigma^{(\tau)}(x)$ converges uniformly on $|x| \leqslant 1$. Therefore, its sum is a continuous function of x (in particular, at $x = 0$) for every $\tau > 0$, which implies also that the initial sum $f(x) = \sum\limits_{\sigma\geqslant 0} f_\sigma(x)$ is continuous on the entire disc $|x| \leqslant 1$ and continuously differentiable at all orders τ. We finally note that the –now legitimate– term-wise differentiation of f at $x = 0$ yields the prescribed values a_0, a_1, $\ldots$, a_σ, $\ldots$ of its derivatives. This completes the construction of an infinitely smooth local section for a given point $\theta^\infty \in J^\infty(\pi)$. $\square$

Exercise 1.5. Using the semi-logarithmic scale $(\pm \log_2 |x|,\ f(x))$, for all $x \in \mathbb{R}$ such that $|x| > 2^{-10}$ draw the left $(x < 0)$ and the right $(x > 0)$ components of the graph of an infinitely smooth section $f \in \Gamma(\pi\colon \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1)$ whose (divergent) Taylor expansion would be $\sum\limits_{k=0}^{+\infty} k!\, x^k$.

Remark 1.3. By the Borel lemma, each point $\theta^\infty \in J^\infty(\pi)$ determines the class $[\boldsymbol{s}]^\infty_{\boldsymbol{x}_0}$ of local sections $\boldsymbol{s} \in \Gamma_{\mathrm{loc}}(\pi)$ of the initial bundle π such that for $\theta^\infty = (\boldsymbol{x}_0, \boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}}, \ldots, \boldsymbol{u}_\sigma, \ldots)$ and all multi-indices $|\sigma| \geqslant 0$, the derivatives $\frac{\partial^{|\sigma|}}{\partial \boldsymbol{x}^\sigma}\Big|_{\boldsymbol{x}_0}(\boldsymbol{s}) = \boldsymbol{u}_\sigma$ are equal to the given constants $\boldsymbol{u}_\sigma$ at the given point $\boldsymbol{x}_0 = \pi_\infty(\theta^\infty) \in M^n$ and remain continuous in a finite neighborhood $U_{\boldsymbol{x}_0} \subseteq M^n$ of $\boldsymbol{x}_0$. This yields the local section $j_\infty(\boldsymbol{s}) : U_{\boldsymbol{x}_0} \subseteq M^n \to J^\infty(\pi)$ of the bundle $\pi_\infty \colon J^\infty(\pi) \to M^n$ for all $\boldsymbol{x} \in U_{\boldsymbol{x}_0}$:

$$j_\infty(\boldsymbol{s})(U_{\boldsymbol{x}_0}) = \left\{ \boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x}), \boldsymbol{u}_{\boldsymbol{x}} = \big(\frac{\partial}{\partial \boldsymbol{x}}\boldsymbol{s}\big)(\boldsymbol{x}), \ldots, \boldsymbol{u}_\sigma = \big(\frac{\partial^{|\sigma|}}{\partial \boldsymbol{x}^\sigma}\boldsymbol{s}\big)(\boldsymbol{x}), \ldots \right\}.$$

Obviously, $(\pi_\infty \circ j_\infty(\boldsymbol{s}))(\boldsymbol{x}) \equiv \boldsymbol{x} \in U_{\boldsymbol{x}_0}$. The lift $j_\infty \colon \boldsymbol{s} \in \Gamma(\pi) \mapsto j_\infty(\boldsymbol{s}) \in \Gamma(\pi_\infty)$ is called the *infinite jet* of $\boldsymbol{s}$. In these terms, the Borel lemma substantiates the name of the infinite jet space $J^\infty(\pi)$ for indeed, it *is* the space of the infinite jets $[\boldsymbol{s}]^\infty_{\boldsymbol{x}} = j_\infty(\boldsymbol{s})(\boldsymbol{x})$ for sections $\boldsymbol{s}$ of π. The locus $j_\infty(\boldsymbol{s})(M^n) \subset J^\infty(\pi)$ is the *graph* $\Gamma_{\boldsymbol{s}}$ of the infinite jet of the section $\boldsymbol{s} \in \Gamma(\pi)$.

Let us continue the definition of basic algebraic structures on the infinite jet space $J^\infty(\pi)$.

Because the space $J^\infty(\pi)$ is infinite-dimensional and contains enough room to carry the information about the Taylor-McLaurent expansions of all sections of π, it would be incautious to define the ring of smooth functions on that space "as is." Namely, the right-hand side of the relation $s(x+\Delta x) = \sum\limits_{k=0}^{+\infty} \frac{(\Delta x)^k}{k!} s^{(k)}(x)$ depends smoothly (moreover, is linear) on the coordinates of the point $[s]^\infty_x$ that defines the class of sections containing s itself. But whenever the equality holds (i.e., s is locally analytic), its left-hand side breaks the locality with respect to the points of the base $M^n \ni x$, which can violate the causality principle in physical models.

Instead, we consider the infinite chain of the natural inclusion

$$\begin{aligned}\mathcal{F}_{-\infty} = C^\infty(M^n) \hookrightarrow C^\infty(E^{n+m}) = C^\infty(J^0(\pi)) \hookrightarrow C^\infty(J^1(\pi)) \hookrightarrow \ldots \\ \ldots \hookrightarrow \mathcal{F}_k(\pi) = C^\infty(J^k(\pi)) \hookrightarrow \mathcal{F}_{k+1}(\pi) \hookrightarrow \ldots,\end{aligned}$$

where the underlying manifolds $J^k(\pi)$ are finite-dimensional for all $k \geqslant 0$.

Definition 1.3. Let the ring $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ be the direct limit

$$\mathcal{F}(\pi) \stackrel{\text{def}}{=} \varinjlim_{k\to+\infty} \mathcal{F}_k(\pi),$$

i.e., $f \in \mathcal{F}(\pi)$ if and only if there exists $k = k(f) < \infty$ such that $f \in \mathcal{F}_k(\pi)$. The differential order $k(f)$ can be arbitrarily high but always finite. We denote $f(\boldsymbol{x}, [\boldsymbol{u}]) = f(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}}, \ldots, \boldsymbol{u}_\sigma)$ for $|\sigma| \leqslant k(f)$.

This agreement endows $\mathcal{F}(\pi)$ with the filtration by $k \in \mathbb{N} \cup \{0\} \cup \{-\infty\}$ which, we postulate, survives for all $\mathcal{F}(\pi)$-modules that appear in the sequel.

1.1. **Vector fields on $J^\infty(\pi)$.** Unlike it is on the finite jet spaces $J^k(\pi)$, there are two canonical classes of vector fields on $J^\infty(\pi)$: the total derivatives $\frac{\mathrm{d}}{\mathrm{d}x^i}$, which are horizontal with respect to the projection $(\pi_\infty)_*$ that maps $\frac{\mathrm{d}}{\mathrm{d}x^i} \mapsto \frac{\partial}{\partial x^i}$ without degeneration, and

the evolutionary derivations $\partial_{\varphi}^{(\boldsymbol{u})}$, which are π_∞-vertical. Both constructions stem from the intuitive idea of the chain rule and respect our choice of the jet variables $\boldsymbol{u}_\sigma$ such that the higher-order derivatives remain the descendants of the lower-order ones.

Definition 1.4. Let $1 \leqslant i \leqslant n = \dim M^n$. The total derivative $\frac{\mathrm{d}}{\mathrm{d}x^i} \colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ is the lift onto $J^\infty(\pi)$ of the partial derivative $\frac{\partial}{\partial x^i}$ on M^n and is defined by the formula

$$\left(\frac{\mathrm{d}}{\mathrm{d}x^i} f\right) \big(j_\infty(\boldsymbol{s})\big)(\boldsymbol{x}) = \frac{\partial}{\partial x^i}\Big(j_\infty(\boldsymbol{s})^*(f)\Big)(\boldsymbol{x}), \quad f \in \mathcal{F}, \quad \boldsymbol{s} \in \Gamma(\pi).$$

In other words, the derivation $\frac{\mathrm{d}}{\mathrm{d}x^i}$ is determined by its application to the infinite jets $j_\infty(s)(\mathcal{U}_\alpha)$ of local sections on $\mathcal{U}_\alpha \subseteq M^n$.

Exercise 1.6. Show that locally,

$$\frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} \sum_{|\sigma| \geqslant 0} u^j_{\sigma+1_i} \cdot \frac{\partial}{\partial u^j_\sigma},$$

and explain why this field admits no restrictions to $J^k(\pi)$ at finite $k \geqslant 0$.

Exercise 1.7. Show that the lift $\widehat{\ } \colon \frac{\partial}{\partial x^i} \mapsto \frac{\mathrm{d}}{\mathrm{d}x^i}$ is a flat connection.

The lift $\widehat{\ } : D_1(M^n) \to D_1(J^\infty(\pi))$ is the *Cartan connection* in the infinite jet bundle $\pi_\infty : J^\infty(\pi) \to M^n$. We denote by $\mathcal{C} = \operatorname{span}_{\mathcal{F}(\pi)} \left\langle \frac{\mathrm{d}}{\mathrm{d}x^i} \right\rangle \subset D_1(J^\infty(\pi))$ the distribution of the n-dimensional horizontal planes spanned by the total derivatives in the tangent spaces for $J^\infty(\pi)$ at all its points. This distribution is Frobenius: $[\mathcal{C}, \mathcal{C}] \subseteq \mathcal{C}$.

We are now interested in the construction of the vector fields X on $J^\infty(\pi)$ which would be symmetries of the Cartan distribution $[X, \mathcal{C}] \subseteq \mathcal{C}$, but which would not belong to it: $X \notin \mathcal{C}$. We note that it is always possible to remove the horizontal part of any such field by adding to it a suitably chosen linear combination of the total derivatives with appropriate coefficients from $\mathcal{F}(\pi)$. Consequently, without any loss of generality we assume that the sought-for field X is π_∞-vertical (i.e., its application to any $f \in C^\infty(M^n)$ gives zero). We thus reduce the condition for X to be a symmetry of $\mathcal{C}$ to the system of n equations

$$\left[X, \frac{\mathrm{d}}{\mathrm{d}x^i}\right] = 0, \quad 1 \leqslant i \leqslant n. \tag{1.4}$$

Exercise 1.8. Solve these equations for $X = \sum\limits_{j=1}^{m} \sum\limits_{|\sigma| \geqslant 0} a^j_\sigma \cdot \partial/\partial \boldsymbol{u}^j_\sigma$, where $a^j_\sigma \in \mathcal{F}(\pi)$ in a given coordinate chart, by reducing them to the recurrence relation(s) between the coefficients a^j_σ.

Solution, $m = 1$. We have that

$$\begin{aligned} &\left[\sum_\sigma a_\sigma \frac{\partial}{\partial u_\sigma}, \frac{\partial}{\partial x^i} + \sum_\tau u_{\tau+1_i} \cdot \frac{\partial}{\partial u_\tau}\right] \\ &\quad = \sum_\sigma \sum_\tau a_\sigma \cdot \delta^\sigma_{\tau+1_i} \cdot \frac{\partial}{\partial u_\tau} - \sum_\sigma \frac{\mathrm{d}}{\mathrm{d}x_i}(a_\sigma) \frac{\partial}{\partial u_\sigma} = \sum_{|\tau| \geqslant 0} \Big(a_{\tau+1_i} - \frac{\mathrm{d}}{\mathrm{d}x_i}(a_\tau)\Big) \cdot \frac{\partial}{\partial u_\tau} = 0, \end{aligned}$$

whence $a_{\sigma+1_i} = \frac{\mathrm{d}}{\mathrm{d}x_i}(a_\sigma)$ for each σ with $|\sigma| \geqslant 0$. □

Definition 1.5. The field
$$\partial_{\varphi}^{(\boldsymbol{u})}=\sum_{j=1}^{m}\sum_{|\sigma|\geqslant 0}\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^{\sigma}}(\varphi^{j})\frac{\partial}{\partial u_{\sigma}^{j}}$$
is the evolutionary derivation along the fibre of the infinite jet bundle $J^{\infty}(\pi)\to M^{n}$ over the vector bundle π with the fibre variable $\boldsymbol{u}$. The m-tuple $\varphi={}^{\mathrm{t}}(\varphi^{1},\ldots,\varphi^{m})\in\Gamma(\pi)\otimes_{C^{\infty}(M)}\mathcal{F}(\pi)$ is the generating section of $\partial_{\varphi}^{(\boldsymbol{u})}$.

Remark 1.4. By construction, the generating section of φ is a section of the induced vector bundle $\pi_{\infty}^{*}(\pi)\colon E\times_{M^{n}}J^{\infty}(\pi)\to J^{\infty}(\pi)$; here we implicitly use the fact that $\pi\colon E^{m+n}\to M^{n}$ is a vector bundle and hence the tangent spaces at the points of its fibres are the fibres themselves (otherwise, the construction would be $\varphi\in\Gamma(\pi_{\infty}^{*}(T\pi))$ for a fibre bundle π).

We denote $\varkappa(\pi)\equiv\Gamma(\pi_{\infty}^{*}(\pi))$ for brevity. Also, we shall often identify the evolutionary vector fields $\partial_{\varphi}^{(\boldsymbol{u})}\in\Gamma(TJ^{\infty}(\pi))$ with their generating sections $\varphi\in\varkappa(\pi)$, and denote by the usual commutator $[\,,\,]$ the Lie algebra structure which is induced on $\varkappa(\pi)$ by the bracket of such vector fields.

Exercise 1.9. Show that the commutator of any two evolutionary vector fields $\partial_{\varphi_1}^{(\boldsymbol{u})}$ and $\partial_{\varphi_2}^{(\boldsymbol{u})}$ is again an evolutionary vector field:
$$\bigl[\partial_{\varphi_1}^{(\boldsymbol{u})},\partial_{\varphi_2}^{(\boldsymbol{u})}\bigr]=\partial_{[\varphi_1,\varphi_2]}^{(\boldsymbol{u})},\quad\text{where}\quad[\varphi_1,\varphi_2]=\partial_{\varphi_1}^{(\boldsymbol{u})}(\varphi_2)-\partial_{\varphi_2}^{(\boldsymbol{u})}(\varphi_1)$$
with the component-wise application of the evolutionary vector fields to the elements of $\varkappa(\pi)$.

Remark 1.5. In applications, it is very convenient to identify the generating section $\varphi\in\varkappa(\pi)$ of evolutionary derivations with the right-hand sides of the (now introduced) autonomous evolutionary differential equations $\dot{\boldsymbol{u}}=\varphi(\boldsymbol{x},[\boldsymbol{u}])$. The structure of the evolutionary field $\partial_{\varphi}^{(\boldsymbol{u})}$ essentially states that $\dot{\boldsymbol{u}}_{\sigma}=\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^{\sigma}}(\dot{\boldsymbol{u}})$. The only drawback of this –very intuitive– identification is that it is illegal because the Cauchy–Kovalevskaya theorem for the existence of the integral trajectories of such fields is not available. So, there are no integral trajectories and hence there is no "time," the derivative with respect to which we would be eager to denote by that dot.

1.2. **Differential forms on** $J^{\infty}(\pi)$**.** By using the Cartan connection $\partial/\partial x^{i}\mapsto\mathrm{d}/\mathrm{d}x^{i}$, we lift the de Rham differential $\mathrm{d}_{\mathrm{dR}(M^{n})}$ on the base M^{n} of the vector bundle $\pi\colon E^{m+n}\to M^{n}$ to the *horizontal differential*
$$\bar{\mathrm{d}}=\sum_{i=1}^{n}\mathrm{d}x^{i}\cdot\frac{\mathrm{d}}{\mathrm{d}x^{i}}$$
on the infinite jet space $J^{\infty}(\pi)$. By convention, the horizontal differential (specifically, the vector fields $\mathrm{d}/\mathrm{d}x^{i}$ contained in it) acts on its arguments by the Lie derivative.

On the infinite jet space $J^{\infty}(\pi)$, the de Rham differential $\mathrm{d}_{\mathrm{dR}(J^{\infty}(\pi))}$, which we understand as the derivation which respects the filtered $\mathcal{F}(\pi)$-module structure for the direct limit of the infinite chain of algebras of differential forms on M^{n} and on the finite jet

spaces $J^k(\pi)$, splits into the horizontal component $\bar{\mathrm{d}}$ and the vertical part $\mathrm{d}_{\mathcal{C}}$, which is the Cartan differential. We have that $\mathrm{d}_{\mathrm{dR}(J^\infty(\pi))} = \bar{\mathrm{d}} + \mathrm{d}_{\mathcal{C}}$ so that $\bar{\mathrm{d}}^2 = 0$, $\mathrm{d}_{\mathcal{C}}^2 = 0$, and $\mathrm{d}_{\mathcal{C}} \circ \bar{\mathrm{d}} + \bar{\mathrm{d}} \circ \mathrm{d}_{\mathcal{C}} = 0$.

The space $\Lambda^1(\pi)$ of differential one-forms inherits from $\bigcup_{k=0}^{+\infty} \Lambda^1(J^k(\pi)) \cup \Lambda^1(M^n)$ the filtration by the orders k of the jet bundle variables that occur in the differentials, and is also endowed by the filtered $\mathcal{F}(\pi)$-module structure under the left multiplication of forms by smooth functions on $J^\infty(\pi)$. Moreover, the space $\Lambda^1(\pi)$ splits as the direct sum of two $\mathcal{F}(\pi)$-modules,

$$\Lambda^1(\pi) = \bar{\Lambda}^1(\pi) \oplus \mathcal{C}^1\Lambda(\pi),$$

where $\bar{\Lambda}^1(\pi)$ is the space of horizontal forms with the differentials $\mathrm{d}x^i$ along the base M^n, and we choose the other summand as follows: $\mathcal{C}^1\Lambda(\pi) = \mathrm{Ann}(\mathcal{C})$ so that $\omega(X) = 0$ for any horizontal vector field $X \in \mathcal{C}$ and a one-form $\omega \in \mathcal{C}^1\Lambda(\pi)$.

The Cartan differential $\mathrm{d}_{\mathcal{C}}$ produces a basis in $\mathcal{C}^1\Lambda$: to each fibre variable u^j_σ it assigns the Cartan form $\omega^j_\sigma = \mathrm{d}u^j_\sigma - \sum_{i=1}^{n} u^j_{\sigma+1_i}\,\mathrm{d}x^i$. The convention that the fields $\mathrm{d}/\mathrm{d}x^k$ act via the Lie derivative implies that

$$\frac{\mathrm{d}}{\mathrm{d}x^k}(\omega^j_\sigma) = \omega^j_{\sigma+1_k} \qquad \text{for all } |\sigma| \geqslant 0,\ 1 \leqslant k \leqslant n, \text{ and } 1 \leqslant j \leqslant m.$$

The filtered $\mathcal{F}(\pi)$-modules of differential r-forms on $J^\infty(\pi)$ are also split for all $r \geqslant 1$:

$$\Lambda^r(\pi) = \bigoplus_{p+q=r} \bar{\Lambda}^p(\pi) \otimes \mathcal{C}^q\Lambda(\pi),$$

where $\bar{\Lambda}^p(\pi)$ consists of the purely horizontal p-forms (note that $\bar{\Lambda}^p(\pi) = 0$ whenever $p > n$), and $\mathcal{C}^q\Lambda(\pi)$ is the ideal of q-forms which vanish on their q arguments $X_1, \ldots, X_q$ whenever at least one of the vector fields $X_j \in D_1(J^\infty(\pi))$ is horizontal: $X_j \in \mathcal{C}$.

The horizontal differential $\overline{\mathrm{d}}$ and the Cartan differential $\mathrm{d}_{\mathcal{C}}$ generate the bi-complex:[5]

$$
\begin{tikzcd}
 & 0 & 0 & \\
\Lambda^n(M^n) \arrow[r] & \overline{\Lambda}^n(\pi) \arrow[u] \arrow[r, "\mathrm{d}_{\mathcal{C}}"] & \overline{\Lambda}^n(\pi) \otimes \mathcal{C}^1\Lambda(\pi) \arrow[u] \arrow[r] & \cdots \\
\Lambda^{n-1}(M^n) \arrow[u, "\mathrm{d}_{\mathrm{dR}(M^n)}"] \arrow[r] & \overline{\Lambda}^{n-1}(\pi) \arrow[u, "\overline{\mathrm{d}}"] \arrow[r, "\mathrm{d}_{\mathcal{C}}"] & \overline{\Lambda}^{n-1}(\pi) \otimes \mathcal{C}^1\Lambda(\pi) \arrow[u, "\overline{\mathrm{d}}"] \arrow[r] & \cdots \\
\vdots \arrow[u] & \vdots \arrow[u, "\overline{\mathrm{d}}"] & \vdots \arrow[u, "\overline{\mathrm{d}}"] & \\
\Lambda^1(M^n) \arrow[u] \arrow[r] & \overline{\Lambda}^1(\pi) \arrow[u] \arrow[r, "\mathrm{d}_{\mathcal{C}}"] & \overline{\Lambda}^1(\pi) \otimes \mathcal{C}^1\Lambda(\pi) \arrow[u] \arrow[r] & \cdots \\
C^\infty(M^n) \arrow[u, "\mathrm{d}_{\mathrm{dR}(M^n)}"] \arrow[r] & \mathcal{F}(\pi) \arrow[u, "\overline{\mathrm{d}}"] \arrow[r, "\mathrm{d}_{\mathcal{C}}"] & \mathcal{C}^1\Lambda(\pi) \arrow[u, "\overline{\mathrm{d}}"] \arrow[r] & \cdots . \\
 & \mathbb{R} \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
\tag{1.5}
$$

The bi-complex is obviously finite (it consists of $n+1$ lines) with respect to the application of the horizontal differential $\overline{\mathrm{d}}$ that adds to its argument one $\mathrm{d}x^i$ along the base. At the same time, the complex extends infinitely to the right because there are infinitely many fibre coordinates u^j_σ, $|\sigma| \geqslant 0$, on the infinite jet space $J^\infty(\pi)$.

Let us now suppose that both the topology of the total space E for the bundle π and the choice of the class of the sections are such that the integration by parts makes sense. We then pass to the *horizontal cohomology*

$$
\overline{H}^p(\pi) = \frac{\ker \overline{\mathrm{d}} \colon \overline{\Lambda}^p(\pi) \to \overline{\Lambda}^{p+1}(\pi)}{\operatorname{im} \overline{\mathrm{d}} \colon \overline{\Lambda}^{p-1}(\pi) \to \overline{\Lambda}^p(\pi)}
$$

with respect to the differential $\overline{d}$, in all the entries of the bi-complex above. This produces the elements

$$
E_1^{p,q} = \frac{\ker \overline{\mathrm{d}} \colon \overline{\Lambda}^p \otimes \mathcal{C}^q\Lambda}{\operatorname{im} \overline{\mathrm{d}} \colon \overline{\Lambda}^{p-1} \otimes \mathcal{C}^q\Lambda}
$$

[5]Traditionally and mainly due to the typographical reasons, the horizontal differential $\overline{\mathrm{d}}$ is drawn as the vertical arrow pointing upward, while the vertical differential $\mathrm{d}_{\mathcal{C}}$ is –of course– drawn horizontally, in the direction from left to right.

of the diagram

$$
\begin{tikzcd}
0 & 0 & 0 & \\
\overline{H}^n(\pi) \arrow[u] \arrow[r, "\delta"] & E_1^{n,1} \arrow[u] \arrow[r] & E_1^{n,2} \arrow[u] \arrow[r] & \cdots \\
\overline{H}^{n-1}(\pi) \arrow[u, "\bar{\mathrm{d}}"'] \arrow[r] & E_1^{n-1,1} \arrow[u, "\bar{\mathrm{d}}"'] \arrow[r] & \cdots \arrow[u, "\bar{\mathrm{d}}"'] & \\
\vdots \arrow[u, "\bar{\mathrm{d}}"'] & \vdots \arrow[u] & & \\
\mathcal{F}(\pi) \arrow[u, "\bar{\mathrm{d}}"'] \arrow[r] & \mathcal{C}^1\Lambda(\pi)/\sim \arrow[u] \arrow[r] & \cdots . &
\end{tikzcd}
\tag{1.6}
$$

This diagram will be one of the main objects of our further study. The elements of its upper-left corner $\overline{H}^n(\pi)$ are called the Lagrangians or the Hamiltonians. Each Lagrangian $\mathcal{L} = \int L\,\mathrm{d}\boldsymbol{x}$ or, equivalently, a Hamiltonian $\mathcal{H}$ is the equivalence class of n-th horizontal forms. Such classes determine the nonlinear differential operators, also called functionals, on the space $\Gamma(\pi)$ of sections of the bundle π, i.e.,

$$\mathcal{H}\colon \Gamma(\pi) \ni \boldsymbol{s} \mapsto \int_{M^n} j_\infty(\boldsymbol{s})^*(L)(\boldsymbol{x})\ \mathrm{dvol}(M^n).$$

Exercise 1.10. Show that the arrow $\delta(\cdot) \stackrel{\text{def}}{=} {:}\mathrm{d}_{\mathcal{C}}(\cdot){:}$, which is the restriction of the Cartan differential onto the horizontal cohomology classes such that the normalization throws all the derivatives off the Cartan forms by the multiple integration by parts, determines the usual *variational derivative* $\delta/\delta\boldsymbol{u}$ (the Euler operator).

Remark 1.6. The geometry which we have outlined in this lecture admits the immediate generalizations to

- the $\mathbb{Z}_2$-graded setup of vector superbundles $(\pi^0|\pi^1)\colon E^{(m_0+n_0|m_1+n_1)} \longrightarrow M^{(n_0|n_1)}$ over supermanifolds,
- the infinite jet spaces for smooth maps of smooth (super)manifolds, and
- the spaces of infinite jets of maps from manifolds to noncommutative associative algebras with m generators (see Lecture 8).

Problem 1.1. Prove this formulation of the Hadamard lemma: For a function f which is piecewise continuously differentiable on an open set $U \subseteq \mathbb{R}$ and for any $x_0, x \in U$ there is a function g defined on U such that

$$f(x) = f(x_0) + (x - x_0) \cdot g(x).$$

Problem 1.2. Establish the Leibniz rule for the total derivative $\frac{\mathrm{d}}{\mathrm{d}x^i}(f_1 \circ f_2)$ of the product of any two differential functions f_1, $f_2 \in \mathcal{F}(\pi)$.

Problem 1.3. Express $\frac{\partial}{\partial u_k}(\frac{d}{dx}f)$ in terms of $\partial f/\partial u_\ell$ and total derivatives of such partial derivatives.

Problem 1.4. Explain why the Leibniz rule fails for $\partial^{(\boldsymbol{u})}_{f\cdot\varphi}$, where $f \in \mathcal{F}(\pi)$, $\varphi \in \varkappa(\pi)$.

Problem 1.5. Show that generally, $f \cdot \partial^{(\boldsymbol{u})}_{\varphi}$ is **not** an evolutionary vector field.

Problem 1.6. For an object f, define (whenever that formula makes sense)

$$\ell^{(\boldsymbol{u})}_f(\delta\boldsymbol{u}) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} f(\boldsymbol{u} + \varepsilon\delta\boldsymbol{u}).$$

Prove that $\partial^{(\boldsymbol{u})}_{\varphi}(f) = \ell^{(\boldsymbol{u})}_f(\varphi)$.

- Show that

$$\ell^{(\boldsymbol{u})}_{\frac{\mathrm{d}}{\mathrm{d}x^i}(f)} = \frac{\mathrm{d}}{\mathrm{d}x^i} \circ \ell^{(\boldsymbol{u})}_f.$$

Problem 1.7. Construct a queer example of a "function" on $J^\infty(\pi)$ such that at each $\boldsymbol{x} \in M^n$ its differential order is $< \infty$ but globally over the base M^n it is **not**.

Problem 1.8. Lift the vector field $X = A(\boldsymbol{x}, \boldsymbol{u})\,\partial/\partial\boldsymbol{x} + B(\boldsymbol{x}, \boldsymbol{u})\,\partial/\partial\boldsymbol{u}$ from $J^0(\pi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2)$ to $J^\infty(\pi)$.

Problem 1.9. Derive the identity $\ell^{(\boldsymbol{u})}_w = \ell^{(\mathfrak{m})}_w \circ \ell^{(\boldsymbol{u})}_{\mathfrak{m}}$ for $w = w\big[\mathfrak{m}[\boldsymbol{u}]\big]$.

Problem 1.10. Prove that $\frac{\delta}{\delta\boldsymbol{u}} \circ \frac{\mathrm{d}}{\mathrm{d}x^i} \equiv 0$ for a trivial vector bundle π over $\mathbb{R}^n$ and hence deduce that $\delta \circ \bar{\mathrm{d}} \equiv 0$, where δ is the restriction of the Cartan differential d_C onto the upper-left corner $\overline{H}^n(\pi)$ in Diagram (1.6).

Problem 1.11 ([54])**.** Prove the identity

$$W\left(1, \frac{x}{1!}, \frac{x^2}{2!}, \ldots, \widehat{\frac{x^k}{k!}}, \ldots, \frac{x^N}{N!}\right) = \frac{x^{N-k}}{(N-k)!}, \qquad 0 \leqslant k \leqslant N,$$

for the Wronskian determinant W.

2. Differential equations $\mathcal{E}^\infty \subseteq J^\infty(\pi)$

Let us accept that systems of differential equations are specified by the collections $\mathcal{E} = \{\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0\}$ of r differential constraints $\boldsymbol{F} = {}^{\mathrm{t}}(F^1, \ldots, F^r)$ upon the unknowns $\boldsymbol{u}$ and their derivatives $\boldsymbol{u}_\sigma$ up to arbitrarily high but always finite order $|\sigma| \leqslant k(j)$ for each F^j.

Remark 2.1. Neither the number of equations not their differential orders are well-defined invariants for a given system $\mathcal{E}$ (except for the case $m = r = k = 1$). The number r of equations and m of unknowns are in general not correlated in any way. Moreover, it can be that infinitely many equations are imposed on one function $\boldsymbol{u}$ in infinitely many variables x^i: e.g., consider the Kadomtsev–Petviashvili hierarchy[96] (see also [99]).

Exercise 2.1. Rewrite the scalar hyperbolic Liouville equation $u_{xy} = \exp(2u)$ as an evolutionary system.

Remark 2.2. The equivalence problem for partial differential equations, i. e., the task of checking whether two given systems can be transformed one into another by a differential substitution or the like –and if they can be, finding these transformations–seems unsolvable without extra assumptions. A certain progress in this direction has been recently achieved after the introduction of the fundamental Lie algebras,[42] which are the jet-bundle realizations of the concept of fundamental groups (and those allow us to distinguish between the topological spaces). The fundamental Lie algebras tend to be infinite-dimensional and by now they are calculated for relatively few nonlinear systems (primarily, for several important evolution equations).

The r-component left-hand side $\boldsymbol{F}$ of the given system $\mathcal{E}$ is a section of the bundle $\pi_\infty^*(\xi)\colon N^{r+n} \times_{M^n} J^\infty(\pi) \to J^\infty(\pi)$ induced by the map $\pi_\infty\colon J^\infty(\pi) \to M^n$ from a suitable vector bundle $\xi\colon N^{n+r} \to M^n$ with the r-dimensional fibres (along which $F^1, \ldots, F^r$ will be coordinates). We denote by P the left $\mathcal{F}(\pi)$-module of sections of the induced bundle:6

$$P \stackrel{\text{def}}{=} \Gamma(\pi_\infty, \xi) = \Gamma(\xi) \otimes_{C^\infty(M)} \mathcal{F}(\pi)$$

Differential equations impose the constraints $\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0$ upon the sections of the bunde π. However, these constraints may not all be differentialy independent. For example, consider the (overdetermined) system $\mathcal{E} = \{F^1 = u_x = 0,\ F^2 = u \cdot u_{xx} = 0\}$ such that $F^2 - u \cdot \frac{d}{dx}(F^1) \equiv 0$ irrespective of the choice of a section $u = s(x)$.

Let us consider the *horizontal infinite jet bundle* $\overline{J^\infty}(\pi_\infty^*(\xi)) \equiv \overline{J^\infty}(\xi_\pi) \stackrel{\text{def}}{=} J^\infty(\xi) \times_{M^n} J^\infty(\pi) \to J^\infty(\pi)$ and introduce the ring $\mathcal{F}_\infty(\pi, \xi)$ of the smooth functions in $[\boldsymbol{u}]$ and $[\boldsymbol{F}]$ on the total space $\overline{J^\infty}(\xi_\pi)$. Suppose that the system $\mathcal{E}$ admits nonlinear differential constraints $\boldsymbol{\Phi}(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}]) \equiv 0$ that hold for all $\boldsymbol{s} \in \Gamma(\pi)$. (We emphasize that $\boldsymbol{\Phi}$ does depend on the equation $\boldsymbol{F}$ and still can depend on the variables $\boldsymbol{u}_\sigma$ with $|\sigma| \leqslant k(j)$, that is, on points $\theta^\infty \in J^\infty(\pi)$.) Then $\boldsymbol{\Phi}$ itself is a section of yet another bundle induced by the projection $\overline{J^\infty}(\xi_\pi) \to J^\infty(\pi) \to M^n$ over the horizontal jet space $\overline{J^\infty}(\xi_\pi)$ from a suitable vector bundle ξ^1 with fibre cordinates $\Phi^1, \ldots, \Phi^q$ over M^n. We denote by P_1

6The construction of P makes the componentwise application of the total derivative $\mathrm{d}/\mathrm{d}x^i$ to the sections of $\boldsymbol{F} \in P$ legitimate.

the $\mathcal{F}_\infty(\pi,\xi)$-module $P_1 = \Gamma(\xi^1)\otimes_{C^\infty(M)}\mathcal{F}_\infty(\pi,\xi)$ of sections of the new induced bundle so that $\boldsymbol{\Phi}\in P_1$.

Arguing as above, we inductively obtain the collection $\xi,\xi^1,\xi^2,\ldots,\xi^i,\ldots$ of the auxiliary vector bundles over M^n. Likewise we get the chain of the horizontal jet spaces

$$M^n \leftarrow J^\infty(\pi) \leftarrow \overline{J^\infty}(\xi_\pi) \leftarrow \overline{J^\infty}(\xi^1_{\xi_\pi}) \leftarrow \ldots,$$

and the sequence of the *horizontal modules* of sections of the induced bundles, $P\ni \boldsymbol{F}$, $P_1\ni\boldsymbol{\Phi}$, $\ldots$, $P_i\ni\boldsymbol{\Phi}_i$ such that $\boldsymbol{\Phi}_i(\boldsymbol{x},[\boldsymbol{u}],[\boldsymbol{\Phi}],\ldots,[\boldsymbol{\Phi}_{i-1}])\equiv 0$ for all $\theta^\infty\in J^\infty(\pi)$, $[\boldsymbol{F}]^\infty_{\boldsymbol{x}}\in J^\infty(\xi)$, …, $[\boldsymbol{\Phi}]^\infty_{\boldsymbol{x}}\in J^\infty(\xi^{i-1})$.

Example 2.1. Consider the system of three differential equations upon three unknowns,

$$\begin{cases} F^1 = u^3_y - u^2_z + u^2u^3_t = 0,\\ F^2 = u^1_z - u^3_x + u^3u^1_t = 0,\\ F^3 = u^2_x - u^1_y + u^1u^2_t = 0.\end{cases}$$

This system is the Frobenius integrability condition $\mathrm{d}\omega\big|_{\omega=0} = 0$ for the three-dimensional distribution determined in the four-space $\mathbb{R}^4$ with the coordinates (x,y,z,t) by the one-form $\omega = \mathrm{d}t - u^1\,\mathrm{d}x - u^2\,\mathrm{d}y - u^3\,\mathrm{d}z$ (see [79, §V.2.6]). The constraint

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + u^1\frac{\mathrm{d}}{\mathrm{d}t} - u^1_t\right)(F^1) + \left(\frac{\mathrm{d}}{\mathrm{d}y} + u^2\frac{\mathrm{d}}{\mathrm{d}t} - u^2_t\right)(F^2) + \left(\frac{\mathrm{d}}{\mathrm{d}z} + u^3\frac{\mathrm{d}}{\mathrm{d}t} - u^3_t\right)(F^3) \equiv 0 \tag{2.1}$$

holds for all sections $\boldsymbol{u} = \boldsymbol{s}(x,y,z,t)$ of the trivial bundle $\pi\colon\mathbb{R}^3\times\mathbb{R}^4\to\mathbb{R}^4$; note that the coefficients of the linear differential relation $\Phi([\boldsymbol{u}],[\boldsymbol{F}])\equiv 0$ between the equations $\boldsymbol{F}=0$ do depend on the jet variables for $J^\infty(\pi)$.

Exercise 2.2. Verify formula (2.1).

Definition 2.1. The [(non)linear differential constraints between the, recursively] equations $\boldsymbol{F}=0$ are called the *Noether identities* (e. g., in the mathematical physics liturature on gauge fields), *Bianchi identities* (in differential geometry) or *syzygies* (in the formal geometry of differential equations).

The Yang–Mills equations (e. g., Maxwell's equations for the electromagnetic field) or the Einstein equations for gravity are the best-known examples of the (non)linear systems that admit one generation of Noether's identities. We adress the geometry of such models in more detail in Lecture 6, where we relate gauge symmetries to the Noether identities for the systems at hand.

Exercise 2.3. Prove that the *evolutionary* systems $\boldsymbol{u}_t = (\mathbf{t},\boldsymbol{x},[\boldsymbol{u}])$ never possess any Noether identities.

Remark 2.3. The labelling by the index $i=1,\ldots,r$ for the equations $F^i=0$ in a system $\mathcal{E}=\{\boldsymbol{F}=0\}$ originates from the enumeration by i for the r components of the fibres of the auxiliary bundle ξ such that $\boldsymbol{F}\in\Gamma(\pi^*_\infty(\xi)) = P$. This shows that any (re)parametrizations of the unknowns $\boldsymbol{u}$ and the equations $\boldsymbol{F}$ are in general not correlated at all. They can be performed entirely independently: e. g., we can swap two equations in a system and this makes no effect — neither on the unknowns nor on the

class of solutions. Still there are two convenient identifications which are brought in per force.

- For determined ($r \equiv m$) systems of autonomous evolution equations
$$u_t^1 = f^1(\boldsymbol{x}, [\boldsymbol{u}]), \quad \ldots, \quad u_t^m = f^m(\boldsymbol{x}, [\boldsymbol{u}]),$$
it is possible to identify $\xi \simeq \pi'$ so that $P = \varkappa(\pi')$ and the equation $\mathcal{E} = \{F^i = u_t^i - f^i = 0\}$ determines the evolutionary vector field $\partial^{(\boldsymbol{u})}$ on the *smaller* bundle $\mathbb{R} \times J^\infty(\pi') \to \mathbb{R} \times (M')^{n-1} \ni (t, \boldsymbol{x})$ in which the time variable t is decoupled;
- By the definition of Euler's variatonal derivative,
$$\vec{\delta}\mathcal{L} = \frac{\overrightarrow{\delta}\,\mathcal{L}}{\delta \boldsymbol{u}} \cdot \delta \boldsymbol{u} \cong \mathrm{d}_{\mathcal{C}}\mathcal{L}, \qquad \mathcal{L} \in \overline{H}^n(\pi), \tag{2.2}$$
the module P containing the Euler–Lagrange equations $\mathcal{E}_{\mathrm{EL}} = \{F_i = \delta\mathcal{L}/\delta u^i = 0\}$ has the structure of the $\mathcal{F}(\pi)$-module $\widehat{\varkappa(\pi)} = \mathrm{Hom}_{\mathcal{F}(\pi)}(\varkappa(\pi), \overline{\Lambda}^n(\pi))$.

In the above notation, we have that the bundle $\xi = \widehat{\pi}\colon \widehat{E} \to M^n$ is the dual bundle with the dual vector spaces $\widehat{\pi}^{-1}(\boldsymbol{x}) = (\pi^{-1}(\boldsymbol{x}))^*$ chosen as the m-dimensional fibres over the points $\boldsymbol{x} \in M^n$, so that $P = \Gamma(\pi_\infty^*(\widehat{\pi}))$.

Remark 2.4. However, by default a reparametrization of cordinates on $J^\infty(\pi)$, — e. g., a reciprocal transformation, a hodograph transformation, or an invertible change $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}[\boldsymbol{u}]$ of the unknowns — produces no echo on the equations: $\boldsymbol{F} = 0 \mapsto \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}[\widetilde{\boldsymbol{u}}]]) = 0$. Any transformation $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{F}}[\boldsymbol{F}]$ of the equations (e. g., transcribing them again in the evolutionary notation $\widetilde{u}_t^i = \widetilde{f}^i(t, \boldsymbol{x}, [\widetilde{\boldsymbol{u}}])$ if the original symstem $\mathcal{E}$ was evolutionary) is then the act of our will.

Suppose that a partial differential equation $\boldsymbol{F} = 0$ is given and that it defines a nonempty locus $\mathcal{E} = \{\boldsymbol{F} = 0\} \subseteq J^k(\pi)$. We assume further that $\mathcal{E}$ is a submanifold in that jet space.

Exercise 2.4. Find an example of the differential equation of order $k > 0$ that does not contain even a single point of $J^k(\pi)$. What could be the set of solutions of such an equation?

Definition 2.2. A section $\boldsymbol{s} \in \Gamma(\pi)$ is a *solution* of an equation $\mathcal{E}$ if the graph of its k-th jet $j_k(\boldsymbol{s})$ is entirely contained in the equation manifold $\mathcal{E} \subseteq J^k(\pi)$:
$$j_k(\boldsymbol{s})(M^n) = \bigcup_{\boldsymbol{x} \in M^n} \{\boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x}), \boldsymbol{u}_{x^i} = \frac{\partial \boldsymbol{s}}{\partial x^i}(\boldsymbol{x}), \ldots, \boldsymbol{u}_\sigma = \frac{\partial^{|\sigma|}\boldsymbol{s}}{\partial \boldsymbol{x}^\sigma}(\boldsymbol{x}),\ |\sigma| \leqslant k\} \subseteq \mathcal{E}.$$

In the remaining part of this lecture, we motivate and state the assumptions that characterize the class of *formally integrable* differential equations. These are the systems which admit the formal power series solutions, regardless of any convergency, at all points $\boldsymbol{x} \in M^n$. However, our present reasoning is not aimed at the construction of true solutions for such systems; we postpone this till the next lecture.

So, we now attempt the iterative construction of the better and more better approximations to a solution of $\mathcal{E} \in J^k(\pi)$ at a point $\boldsymbol{x}_0 \in M^n$. Namely, let us take the Taylor–McLaurent polynomials $\boldsymbol{s}_{k+\ell} = \sum\limits_{0 \leqslant |\sigma| \leqslant k+\ell} \boldsymbol{u}_\sigma \cdot (\boldsymbol{x} - \boldsymbol{x}_0)^\sigma$ of higher and higher

degrees as $\ell \to +\infty$. We are going to construct these approximations by using the sequence of points $[\boldsymbol{s}]_{\boldsymbol{x}_0}^{k+\ell} \in J^{k+\ell}(\pi)$ in such a way that the discrepancy between the penalty function $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{s}_{k+\ell})$ and zero has the order $\overline{o}(|\boldsymbol{x}-\boldsymbol{x}_0|^\ell)$ near $\boldsymbol{x}_0$.

First, we suppose that the manifold $\mathcal{E} \subseteq J^k(\pi)$ is projected onto the total space $E = J^0(\pi)$ of the bundle π under the map $\pi_{k,0} = \pi_{1,0} \circ \cdots \circ \pi_{k,k-1}$. In other words, let the equation $\mathcal{E}$ contain no sub-equations $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{u}) = 0$ of differential order zero.

Example 2.2. The system $u_x+u = 0$, $u_x+\sin u = f(x)$ does contain the non-differential equation $\sin u = u + f(x)$.

Such non-differential equations should be eliminated in advance by shrinking the geometry of the initial bundle π (during this procedure the system $\mathcal{E}$ can turn into the tautology $0 = 0$ or become incompatible).

Pick any $\boldsymbol{x}_0 \in M^n$ and consider a point $\theta^k \in \mathcal{E} \subseteq J^k(\pi)$ such that $\pi_{k,-\infty}(\theta^k) = \boldsymbol{x}_0$; such $\theta^k \in \mathcal{E}$ exists because we assumed that $\mathcal{E} \to J^0(\pi)$ is epi and $\pi\colon E \to M^n$ is also an epimorphism.

The relation $\theta^k = (\boldsymbol{x}_0, \boldsymbol{u}, \boldsymbol{u}_x, \ldots, \boldsymbol{u}_\sigma \mid |\sigma| = k) \in \mathcal{E}$ for $\theta^k = [\boldsymbol{s}]_{\boldsymbol{x}_0}^k$ essentially states that there is the Talyor–McLaurent polynomial $\boldsymbol{s}_k = \sum\limits_{0\leqslant|\sigma|\leqslant k} \boldsymbol{u}_\sigma \cdot (\boldsymbol{x}-\boldsymbol{x}_0)^\sigma \in [\boldsymbol{s}]_{\boldsymbol{x}_0}^k$ of degree k whose coefficients $\boldsymbol{u}, \boldsymbol{u}_x, \ldots, \boldsymbol{u}_\sigma$ satisfy the *algebraic* equation $\boldsymbol{F}(\boldsymbol{x}_0, [\boldsymbol{s}]_{\boldsymbol{x}_0}^k) = 0$ at the point $\boldsymbol{x}_0$ (but not necessarily satisfy the equation $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{s}_k(\boldsymbol{x})) = 0$ at any $\boldsymbol{x} \neq \boldsymbol{x}_0$ from any finite neighbourhood of $\boldsymbol{x}_0 \in M^n$.

Thus, a reformulation of our first assumption about the class of "admissible" equations is the postulated existence of a solution θ^k for $F(\boldsymbol{x}_0, \theta^k) = 0$ at every point $\boldsymbol{x}_0 \in M^n$ for all *Cauchy data* $\theta^{k-1} = \pi_{k,k-1}(\theta^k) = 0$, i. e., for any values of the unknown function(s) and the derivatives of all orders strictly less than the differential order k of $\mathcal{E}$. To establish the equivalence between our earlier hypothesis $\mathcal{E} \xrightarrow{\text{epi}} E = J^0(\pi)$ and the epimorphism $\mathcal{E} \xrightarrow{\text{epi}} J^{k-1}(\pi)$ it suffices to note that every differential equation can be written as a system of first-order equations so that $k-1 \equiv 0$. In turn, the present reformulation is a particular case of a more general statement which will be our next restriction upon the class of $\mathcal{E}$, see below.

The possibility to match the coordinates $\boldsymbol{u}_\sigma$ of θ^k over one point $\boldsymbol{x}_0 \in M^n$ recalls us that the set $\mathcal{E}$ is not empty but does not imply that the Taylor–McLaurent polynomial $\boldsymbol{s}_k(\boldsymbol{x})$ is a solution of $\mathcal{E} = \{\boldsymbol{F} = 0\}$ at all $\boldsymbol{x}_0 \in M^n$ near $\boldsymbol{x}_0$. Usually, there is no reason why it should be so, for indeed, we obtained the local section $\boldsymbol{s}_k \in \Gamma_{\mathrm{loc}}(\pi)$ by matching its derivatives at one point ($\boldsymbol{x}_0 \in M^n$) only instead of doing so at **all** points $\boldsymbol{x} \in M^n$. Consequently, $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{s}_k(\boldsymbol{x})) = \overline{o}(1)$ that need not be zero except only at $\boldsymbol{x}_0$.

We now use the freedom in the choice of the still unfixed values of the $(k+1)$-th and higher-order derivatives at $\boldsymbol{x}_0$, aiming to suppress the deviation $\boldsymbol{F}(\boldsymbol{x}_0, [\boldsymbol{s}]_{\boldsymbol{x}_0}^k)$ entirely by making it $\overline{o}(|\boldsymbol{x}-\boldsymbol{x}_0|^\ell)$ at the ℓ-th step in the ascent along the tower

$$\begin{aligned}\boldsymbol{x}_0 \leftarrow \text{any } \theta^0 = (\boldsymbol{x}_0, \boldsymbol{u}_0) \leftarrow \text{any Cauchy data } \theta^{k-1} \leftarrow \theta^k = [\boldsymbol{s}]_{\boldsymbol{x}_0}^k \leftarrow \\ \leftarrow \theta^{k+1} = [\boldsymbol{s}]_{\boldsymbol{x}_0}^{k+1} \leftarrow \ldots \leftarrow \theta^{k+\ell} = [\boldsymbol{s}]_{\boldsymbol{x}_0}^{k+\ell} \leftarrow \ldots \leftarrow \theta^\infty = [\boldsymbol{s}]_{\boldsymbol{x}_0}^\infty \in J^\infty(\pi).\end{aligned}$$

Namely, consider a point $\theta^{k+1} \in J^{k+1}(\pi)$ such that $\pi_{k+1,k}(\theta^{k+1}) = \theta^k$ from our previous reasoning; the new point θ^{k+1} carries all the information from θ^k *and* the values of the

$(k+1)$-th order derivatives at $\boldsymbol{x}_0$ for the class $[\boldsymbol{s}]^{k+1}_{\boldsymbol{x}_0}$ of local sections. We plug the point θ^{k+1} in the equation: the values of the lower-order (i. e., not exceeding k and already containied in θ^k) derivatives were enough to verify $\boldsymbol{F}(\boldsymbol{x}_0, [\boldsymbol{s}]^{k+1}_{\boldsymbol{x}_0}) = 0$ but, taken alone, could produce the discrepancy $\overline{o}(1)$ at $\boldsymbol{x} \neq \boldsymbol{x}_0$. We now use the $(k+1)$-th order "top" of $[\boldsymbol{s}]^{k+1}_{\boldsymbol{x}_0}$ in order to lessen the deviation:

$$\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{s}]^{k}_{\boldsymbol{x}_0}) = \overline{o}(1) \quad \to \quad \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{s}]^{k+1}_{\boldsymbol{x}_0}) = \overline{o}(|\boldsymbol{x} - \boldsymbol{x}_0|), \qquad \boldsymbol{x} \in M^n.$$

To this end, we solve together the algebraic equations

$$F(\boldsymbol{x}, \boldsymbol{s}_{k+1}(\boldsymbol{x}))\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = 0 \quad \text{and} \quad \frac{\partial}{\partial x^i}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} F(\boldsymbol{x}, \boldsymbol{s}_{k+1}(\boldsymbol{x})) = 0, \qquad 1 \leqslant i \leqslant n.$$

We suppose that there exists a solution to this system (which is not automatic and which again restricts the class of "admissible" differential equations, see Example 2.4 below).

Remark 2.5. We assume further that the equation $\mathcal{E}$ is such that the attempt to correlate the values $\boldsymbol{u}_\sigma$ at $|\sigma| = k+1$ between themselves and the lower-order coefficients does not alter the previously found values u_τ at $|\tau| \leqslant k$. Geometrically, this means that the differential consequences of the equation in the system $\mathcal{E}$ never combine to any lower-order constraints that would overdetermine the initial system.

Example 2.3. The system $\mathcal{X}(u) = 0$, $\mathcal{Y}(u) = 0$, which is given by two vector fields $\mathcal{X} = \sum\limits_{i=1}^{n} \mathcal{X}^i \cdot \frac{\mathrm{d}}{\mathrm{d}x^i}$ and $\mathcal{Y} = \sum\limits_{i=1}^{n} \mathcal{Y}^j \cdot \frac{\mathrm{d}}{\mathrm{d}x^j}$, has order $k = 1$ but its solutions must necessarily satisfy the extra compatibility condition $[\mathcal{X}, \mathcal{Y}](u) = 0$. Being of differential order one, this consequence $\mathcal{X}(\mathcal{Y}(u)) - \mathcal{Y}(\mathcal{X}(u)) = 0$ of the initial two equations is in general linearly independent from them and hence restricts the system.

We proceed by induction over $\ell \in \mathbb{N}$ and keep on solving the algebraic systems

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{s}_{k+\ell}(\boldsymbol{x}))\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} = 0, \quad \ldots, \quad \frac{\partial^{|\tau|}}{\partial \boldsymbol{x}^\tau}\Big|_{\boldsymbol{x}=\boldsymbol{x}_0} \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{s}_{k+\ell}(\boldsymbol{x})) = 0,$$

where $|\tau| \leqslant \ell$ and $\boldsymbol{s}_{k+\ell} = \sum\limits_{0 \leqslant |\sigma| \leqslant k+\ell} \boldsymbol{u}_\sigma \cdot (\boldsymbol{x} - \boldsymbol{x}_0)^\sigma$, with respect to the values $u^i_\sigma \in \mathbb{R}$ of the fibre coordinates of the points $\theta^{k+\ell} = [\boldsymbol{s}]^{k+\ell}_{\boldsymbol{x}_0}$. We thus construct the sequence $\{\theta^{k+\ell} = [\boldsymbol{s}]^{k+\ell}_{\boldsymbol{x}_0} \in J^{k+\ell}(\pi)\}$ of the points that raise –to ℓ at the least– the *order of tangency* between the manifold $\mathcal{E} = \{\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0\} \subseteq J^k(\pi)$ and the graph $\Gamma^k_{\boldsymbol{s}_{k+\ell}} = \{j_k(\boldsymbol{s}_{k+\ell})(\boldsymbol{x}),\ \boldsymbol{x} \in U_{\boldsymbol{x}_0} \subseteq M^n\} \subseteq J^k(\pi)$ over the neighbourhood $U_{\boldsymbol{x}_0} \ni \boldsymbol{x}_0$. (The discrepancy $\overline{o}(1)$ near $\boldsymbol{x}_0$ meant just the intersection, which is the tangency of order zero by convention.)

Definition 2.3. The *ℓ-th prolongation* $\mathcal{E}^{(\ell)} \subseteq J^{k+\ell}(\pi)$ of a differential equation $\mathcal{E} \subseteq J^k(\pi)$ is the locus $\mathcal{E}^{(\ell)} \stackrel{\text{def}}{=} \{\theta^{k+\ell} = [\boldsymbol{s}_{k+\ell}]^{k+\ell}_{\boldsymbol{x}_0} \mid \boldsymbol{x}_0 \in M^n,\ \boldsymbol{s}_{k+\ell} \in \Gamma_{\mathrm{loc}}(\pi)$, and $j_k(\boldsymbol{s}_{k+\ell})(M^n)$ is tangent[7] to $\mathcal{E}$ at $\theta^\ell = [\boldsymbol{s}_{k+\ell}]^k_{\boldsymbol{x}_0}$ with order $\leqslant \ell\}$. In the sequel, we require that the set $\mathcal{E}^{(\ell)}$ is nonempty and it is a submanifold in $J^{k+\ell}(\pi)$.

[7]By Lecture 1 and the implicit function theorem, the property of two submanifolds to be tangent of order ℓ at their touch point $\boldsymbol{x}_0$ is independent of the choice of local coordinate $\boldsymbol{x}$ near $\boldsymbol{x}_0$ on either of the manifolds.

Example 2.4. The first prolongation $\mathcal{E}^{(1)}$ for the system $\mathcal{E} = \{u_{xz} = y, u_{yz} = -x\}$ is empty because the third derivative of u_{xyz} of any solution of $\mathcal{E}$ must be equal simultaneously to two distinct real numbers at all values of the arguments x, y, and z. Indeed, the first equation implies that $u_{xyz} = (u_{xz})_y = +1$, whereas the other equation yields $u_{xyz} = (u_{yz})_x = -1$. This contradiction shows that $\mathcal{E}^{(1)} = \varnothing$.

Besides, we extend the assumption in Remark 2.5 onto the tower

$$M^n \xleftarrow{\text{epi}} J^0(\pi) \xleftarrow{\text{epi}} J^{k-1}(\pi) \xleftarrow{\text{epi}} \mathcal{E} \xleftarrow{\text{epi}} \mathcal{E}^{(1)} \xleftarrow{\text{epi}} \ldots \xleftarrow{\text{epi}} \mathcal{E}^{(\ell)} \xleftarrow{\text{epi}} \mathcal{E}^{(\ell+1)} \xleftarrow{\text{epi}} \ldots$$

of (hence, presumed existing) prolongations $\mathcal{E}^{(\ell)}$ for $\mathcal{E}$ at all orders $\ell \in \mathbb{N}$: let it be that no differential consequences of the equation $\mathcal{E}$ may retro-act and shrink the set of the previously found approximations $[\boldsymbol{s}]_{\boldsymbol{x}_0}^{\ell'}$ at all $\ell' < \ell$.

We finally pass to the nonempty projective limit as $\ell \to +\infty$ and reach the *infinite prolongation*

$$\mathcal{E}^\infty = \varprojlim_{\ell \to +\infty} \mathcal{E}^{(\ell)} \subseteq J^\infty(\pi)$$

for the equation $\mathcal{E}$. Reading backwards the definition of the total derivative, we can see that the object $\mathcal{E}^\infty$ is described by the infinite system of differential equations

$$\mathcal{E}^\infty \simeq \left\{ \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0,\ \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0,\ \ldots,\ \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma} \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0, \ldots \mid |\sigma| \geqslant 0 \right\}.$$

Each point $\theta^\infty = [\boldsymbol{s}]_{\boldsymbol{x}_0}^\infty \in \mathcal{E}^\infty$ determines the class of sections $\boldsymbol{s}$ which are infinitely smooth near $\boldsymbol{x}_0$ and satisfy the equation $\mathcal{E}$ exactly at that point $\boldsymbol{x}_0 \in M^n$. The existence of such sections is guaranteed by the entire set of assumptions which we imposed on the class of **formally integrable** systems $\mathcal{E}$; those do possess the nonempty infinite prolongation. However, the situation with our ability to *find* a global solution of $\mathcal{E}$ remains exactly where it was when we started. This is because our reasoning was attached to a point $\boldsymbol{x}_0$ and does not suggest any technique for glueing such pointwise-defined sections by a proper choice of the representatives from the infinite classes $[\boldsymbol{s}]_{\boldsymbol{x}_0}^\infty$ (see Borel's lemma). The great convenience of dealing with the infinite prolongations $\mathcal{E}^\infty$ rather than with the systems $\mathcal{E} \subseteq J^k(\pi)$ will reveal itself in the next lecture, permitting the effective search for the genuine exact solutions of $\mathcal{E}$ by using the symmetries of $\mathcal{E}^\infty$.

Remark 2.6. Suppose that a system $\mathcal{E} = \{\boldsymbol{F} = 0\}$ is not normal, whence there exists a differential relation between the equations. Moreover, assume that this constrain is linear: $\Delta(\boldsymbol{F}) = 0$ on $J^\infty(\pi)$ for some $\Delta \in \mathcal{C}\mathrm{Diff}(P_0, P_1)$. It then follows in a standard way that, whatever be $\varphi \in \varkappa(\pi)$ in the Leibniz formula

$$\partial_\varphi^{(\boldsymbol{u})}\big(\Delta(\boldsymbol{F})\big) = \partial_\varphi^{(\boldsymbol{u})}(\Delta)(\boldsymbol{F}) + \Delta\big(\partial_\varphi^{(\boldsymbol{u})}(\boldsymbol{F})\big),$$

the equality $0 = \partial_\varphi^{(\boldsymbol{u})}\big(\Delta(\boldsymbol{F})\big) \doteq \Delta\big(\partial_\varphi^{(\boldsymbol{u})}(\boldsymbol{F})\big)$ holds on $\mathcal{E}^\infty$. This yields the on-shell equality $\Delta \circ \ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\Big|_{\mathcal{E}^\infty} \doteq 0$ on $\mathcal{E}^\infty$.

Definition 2.4. The equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ such that

$$\Delta \circ \ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\Big|_{\mathcal{E}^\infty} \doteq 0 \text{ on } \mathcal{E}^\infty \quad \text{implies} \quad \Delta \doteq 0 \text{ on } \mathcal{E}^\infty$$

is called *ℓ-normal*.

We conclude that the presence of a Noether identity for $\mathcal{E}$ breaks the ℓ-normality of this system.

Problem 2.1. Find the profile of the travelling wave solution $u = f(x - ct)$ for the Korteweg-de Vries equation $u_t + u_{xxx} + 6uu_x = 0$.

Problem 2.2. Find the value of the constant α such that the Cole–Hopf substitution $u = \frac{\alpha v_x}{v}$ transforms the Burgers equation $u_t = u_{xx} + uu_x$ to the heat equation $v_t = v_{xx}$.

Problem 2.3. Show that the Legendre transform (discovered in 1787) $\mathcal{L} = \{\Phi = xu_x + yu_y - u, p = u_x, q = u_y\}$ brings the nonlinear equation $\mathcal{E}_{\min\Sigma} = \{(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} = 0\}$ for minimal area surfaces $\{x, y, z = u(x, y)\} \subset \mathbb{E}^3$ to the linear form $\mathcal{L}(\mathcal{E}_{\min\Sigma}) = \{(1 + p^2)\Phi_{pp} + 2pq\Phi_{pq} + (1 + q^2)\Phi_{qq} = 0\}$.
- Find the parametric form of a general solution to $\mathcal{E}_{\min\Sigma}$.

Problem 2.4. Prove that the quantities[112] $w = u_x^2 - u_{xx}$ and $\overline{w} = u_y^2 - u_{yy}$ are such that $\frac{\mathrm{d}}{\mathrm{d}y}\Big|_{\mathcal{E}_{\mathrm{Liou}}}(w) \doteq 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{\mathcal{E}_{\mathrm{Liou}}}(\overline{w}) \doteq 0$ on $\mathcal{E}_{\mathrm{Liou}} = \{u_{xy} = \exp(2u)\}$.
- Let $w = f(x)$ and $\overline{w} = g(y)$ and (as Liouville did in 1854) derive the general solution(s) $u = \frac{1}{2} \ln \frac{\mathcal{X}'(x)\mathcal{Y}'(y)}{Q^2(\mathcal{X}(x)+\mathcal{Y}(y))}$ of the Liouville equation by a straightforward integration. What are the admissible mappings Q? (Hint: There exists a map $f(x) \mapsto \mathcal{X}(x)$ and $g(y) \mapsto \mathcal{Y}(y)$.)

Problem 2.5. Write down explicitly the Noether relations between the components of the Yang–Mills equations with the structure Lie group $U(1)$, $SU(2)$, or $SU(3)$; show that there is no second generation of the identities between the known ones.

Problem 2.6 ([96, 58])**.** Let $k_\ell \in \mathbb{R}$ be the wave number, $\alpha_\ell > 0$ be an arbitrary phase shift; set $A \equiv 2$ and $A_{\ell m} = \frac{(k_\ell - k_m)^2}{(k_\ell + k_m)^2}$ for the future pairwise interactions for any two waves. By definition, put $\omega_\ell = -k_\ell^3$ (which is the frequency) and $\eta_\ell = k_\ell x + \omega_\ell t$ (the phase). Consider the tau-function

$$\tau_{k_1,\ldots,k_N}(\eta_\ell) = \sum_{\substack{\mu_\ell=0,1\\1\leqslant\ell\leqslant N}} \exp\left(\sum_{\ell=1}^{N} \mu_\ell \cdot (\eta_\ell + \log(\alpha_\ell)) + \sum_{1\leqslant\ell\leqslant m\leqslant N} \mu_\ell\mu_m \log A_{\ell m}\right).$$

in other words,

$$\begin{aligned}\tau_{k_1,\ldots,k_N}(\eta_\ell) = 1 &+ \alpha_1 \exp\eta_1 + \ldots + \alpha_N \exp\eta_N + \\ &+ \alpha_1\alpha_2 A_{12}\exp(\eta_1 + \eta_2) + \ldots + \alpha_{N-1}\alpha_N A_{N-1,N}\exp(\eta_{N-1} + \eta_N) + \ldots.\end{aligned}$$

Show that

$$u_{k_1,k_2,\ldots,k_N}(x, t) = A\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\tau_{k_1,\ldots,k_N}(\eta_\ell) \qquad (2.3)$$

is the N-soliton solution of the Korteweg–de Vries equation $u_t + u_{xxx} + 6uu_x = 0$.

3. Symmetries of differential equations

In this lecture we study the geometric algorithm for finding the infinitesimal symmetries of the infinite prolongations $\mathcal{E}^\infty$ for systems $\mathcal{E}$ of differential equations. This technique makes it possible to classify partial differential equations with respect to their symmetry properties. Also, it allows us to find exact solutions of the systems $\mathcal{E}$ and then propagate them to families. For example, the multi-soliton solutions of the KdV-like equations can be obtained as the stationary points of the linear combinations of higher flows in the hierarchies for such equations. The classification of the models by their symmetries can go both ways: we either arrange a given set of systems into classes or postulate the invariance group for the equations of motion (possibly, for their Lagrangian functional). In the latter case, the further symmetry reductions produce new model equations which are also interesting (e. g., the 2D Toda chains emerge via the cylindric symmetry reduction[88] of the Yang–Mills equations).

Although we discard the presence of the boundary from our reasoning, we recall that the construction of the infinitesimal symmetries, which satisfy the linear determining relations irrespective of the nonlinearity degree in the equations $\mathcal{E}$, also arises in the homotopy approach to the solution of boundary value problems for differential equations. The symmetries are also interesting per se, as the geometric objects. In particular, together with the generating sections of conservation laws (see next lecture) they appear in the nonlocal terms of the *weakly nonlocal* recursion and Hamiltonian or symplectic operators (see Lecture 7). In other words, the knowledge of such structures hints us the ansatz for the yet unknown operators. Furthermore, the search for the recursion and Hamiltonian operators (more generally, Noether operators, see Lectures 7 and 9, respectively) for differential equations can also be reformulated in terms of the technique which we outline here.

We finally remark that some symmetries stand apart from all the other types of invariance, e. g., the *supersymmetry* that couples the even and odd fields in the $\mathbb{Z}_2$-graded models.

The main advantage in the transition from the equations $\mathcal{E} \subseteq J^k(\pi)$ to their infinite prolongations $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ is the existence of the total derivatives $\mathrm{d}/\mathrm{d}x^i$ and evolutionary derivations $\partial_\varphi^{(\boldsymbol{u})}$ on $J^\infty(\pi)$ and on $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ in particular; these two fields do not exist in general on $J^{k+\ell}(\pi)$ at any $\ell < \infty$. The use of these structures now allows us to convert the problem of finding the symmetries of a given formally integrable equation $\mathcal{E}$ to a purely algorithmic procedure.

From the very beginning we pass to the infinitesimal standpoints and consider the vector fields instead of the diffeomorphisms. (Note that by doing this on $J^\infty(\pi)$ we may lose the control over the existence of those diffeomorphisms of the infinite-dimensional manifolds, see Remark 1.5 on p. 15.)

Secondly, we are interested only in those transformations of $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ which respect the correspondence $\frac{\mathrm{d}^{|\tau|}}{\mathrm{d}x^\tau}(u_\sigma) = u_{\sigma+\tau}$ between the jet variables. That is, the sought-for vector fields must map sections $j_\infty(\boldsymbol{s})$ of $\pi_\infty\colon J^\infty(\pi) \to M^n$ to sections again, and for that purpose, preserve the distribution $\mathcal{C}$ of the Cartan planes spanned by the total derivatives, see Eq. (1.4). In Lecture 1 we learned that such fields are

$$X = \sum\nolimits_{i=1}^n a^i \cdot \mathrm{d}/\mathrm{d}x^i + \partial_\varphi^{(\boldsymbol{u})} \tag{3.1}$$

with arbitrary $a^i \in \mathcal{F}(\pi)$, $\varphi \in \varkappa(\pi)$.

Exercise 3.1. Prove that the equality

$$\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} + \sum_{j=1}^{m} b^j \frac{\partial}{\partial u^j} + \ldots = \sum_{i=1}^{n} a^i \frac{\mathrm{d}}{\mathrm{d}x^i} + \partial_{\varphi}^{(\boldsymbol{u})}$$

where a^i, $b^j \in \mathcal{F}(\pi)$ and $\varphi^j = b^j - \sum_{i=1}^{n} a^i \cdot u^j_{x^i}$, determines the proper rule for the reconstruction of the missing terms in the left-hand side such that the entire vector field is a symmetry of the Cartan distribution on $J^\infty(\pi)$.

We claim that the fields $X = a^i \cdot \mathrm{d}/\mathrm{d}x^i \in \mathcal{C}$ induce the trivial transformations on the space $\Gamma(\pi)$ of sections: namely, each graph $\mathit{\Gamma}_{\boldsymbol{s}} = j_\infty(\boldsymbol{s})(M^n)$ of $\boldsymbol{s} \in \Gamma(\pi)$ glides along itself. Indeed, under the usual hypothesis that all the sections at hand are continuously differentiable, the restriction of the horizontal field $X = a^i\, \partial/\partial x^i + a^i\, u^j_{x^i}\, \partial/\partial u^j + \ldots$ onto the graph $\mathit{\Gamma}_{\boldsymbol{s}} = \{\boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x}),\ \boldsymbol{u}_{x^i} = \frac{\partial \boldsymbol{s}}{\partial x^i}(\boldsymbol{x}), \ldots \mid \boldsymbol{x} \in M^n\}$ yields the velocities $\dot{x}^i = a^i(\boldsymbol{x}, [\boldsymbol{s}]^\infty_{\boldsymbol{x}}) \cdot 1$ and $\dot{u}^j = a^i \cdot \dfrac{\partial \boldsymbol{s}^j}{\partial x^i}(\boldsymbol{x})$, which amounts to the definition of a derivative and of a vector field and therefore holds tautologically on $\mathit{\Gamma}_{\boldsymbol{s}}$. The integral form of such transformations is $\boldsymbol{s}(\boldsymbol{x}) \mapsto \boldsymbol{s}(\boldsymbol{x} + \Delta\boldsymbol{x}(\boldsymbol{x}, \boldsymbol{s}))$ for some nonlinear operators $\Delta\boldsymbol{x}$; this simply means that the values of ear section $\boldsymbol{s} \in \Gamma(\pi)$ are recalculated at the new points after the application of a section-dependent local diffeomorphism to the base $M^n \ni \boldsymbol{x}$, while the section $\boldsymbol{s}$ itself remains unchanged.

Conversely, the π_∞-vertical evolutionary vector fields $\partial_{\varphi}^{(\boldsymbol{u})}$ on $J^\infty(\pi)$ leave the points of the base M^n intact, but transform the sections $\boldsymbol{s}$. Therefore, it is logical to consider the classes (3.1) of infinitesimal transformations of the jets of sections and mark these classes by the π_∞-vertical representatives $\partial_{\varphi}^{(\boldsymbol{u})}$. (We notice that in practical situations this convention may result in not the shortest possible formulas, see section 3.1 below.)

We note further that we are interested in the construction of the distribution-preserving transformations which are defined on the space of all sections at once. Indeed, we operate with the jet variables $\boldsymbol{u}_\sigma$ instead of the derivatives of the sections, which brings the flavour of operads into the jet bundle paradigm. In what follows, we do not consider the transformations which (even locally) are defined for only a subset of the set of all sections for π. Nevertheless, the maximal generality nourishes the ever-present risk of obtaining the zero denominators when the expressions are evaluated at a given local section at the end of the day.

We now consider the space of evolutionary vector fields which preserve the Cartan distribution on $J^\infty(\pi)$ and, in the sense which is explained immediately below, preserve the infinite prolongation $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ of a given system $\mathcal{E}$. Such fields transform into itself the a priori unknown set of solutions of $\mathcal{E}$ and propagate them to the families $\boldsymbol{s}^\varepsilon$ whenever the Cauchy problems

$$\dot{\boldsymbol{s}}^\varepsilon = \varphi(\boldsymbol{x}, [\boldsymbol{s}^\varepsilon(\boldsymbol{x})]), \qquad \boldsymbol{s}^\varepsilon|_{\varepsilon=0} = \boldsymbol{s}_0 \in \operatorname{Sol}\mathcal{E} = \{\boldsymbol{s} \in \Gamma(\pi) \mid j_k(\boldsymbol{s})(M^n) \subseteq \mathcal{E}\}$$

can be integrated.

We recall that the infinite prolongation $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ of an equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ is described by the infinite system of differential relations

$$\boldsymbol{F} = 0, \quad \tfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\boldsymbol{F} = 0, \quad \ldots, \quad \tfrac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}(\boldsymbol{F}) = 0, \ \ldots, \qquad |\sigma| \geqslant 0. \tag{3.2}$$

Consider the ring $\mathcal{F}(\pi)$ of smooth functions on $J^\infty(\pi)$ and take the ideal $I(\mathcal{E}^\infty) \subseteq \mathcal{F}(\pi)$ generated by the functions $\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}F^j$, $1 \leqslant j \leqslant r$ or, equivalently, differentially generated in $\mathcal{F}(\pi)$ by the components F^1, …, F^r of the left-hand sides $\boldsymbol{F} \in P = \Gamma(\xi) \otimes_{C^\infty(M)} \mathcal{F}(\pi)$ for $\mathcal{E}$. We have that $\mathcal{F}(\pi) \cdot I(\mathcal{E}^\infty) \subseteq I(\mathcal{E}^\infty)$. Note that the proper part (in fact, a subring in $\mathcal{F}(\pi)$) of the ideal $I(\mathcal{E}^\infty) \subseteq \mathcal{F}(\pi)$ is the pull-back of the ring $\mathcal{F}(\xi)$ on $J^\infty(\xi)$ under the evaluation $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])$.

Definition 3.1. An *infinitesimal symmetry* of a formally integrable differential equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ is the evolutionary vector field $\partial^{(\boldsymbol{u})}_\varphi$ that preserves the ideal $I(\mathcal{E}^\infty) \subseteq \mathcal{F}(\pi)$ in the ring of smooth functions on $J^\infty(\pi)$.

Being a derivation, the field $\partial^{(\boldsymbol{u})}_\varphi$ acts by the Leibniz rule; being an *evolutionary* derivation, this field dives under the total derivatives. Therefore, for $\partial^{(\boldsymbol{u})}_\varphi$ to be a symmetry of the equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ it is not only necessary, but also sufficient that

$$\partial^{(\boldsymbol{u})}_\varphi(F^j) \in I(\mathcal{E}^\infty), \qquad 1 \leqslant j \leqslant r.$$

However, we note that there still remain (constituting yet another ideal — prove!) the symmetries whose generating sections $\varphi \in \varkappa(\pi) \equiv \Gamma(\pi) \otimes_{C^\infty(M^n)} \mathcal{F}(\pi)$ are composed by arbitrary elements of the ideal $I(\mathcal{E}^\infty) \subseteq \mathcal{F}(\pi)$. Such evolutionary vector fields induce the nontrivial transformations of the Cartan distribution on $J^\infty(\pi)$ and preserve the ideal $I(\mathcal{E}^\infty)$ but they identically vanish at all points of $\mathcal{E}^\infty$.

Exercise 3.2. State the converse: the components of the generating section φ for a symmetry which vanishes on $\mathcal{E}^\infty$ belong to the equation's ideal $I(\mathcal{E}^\infty)$.

We remark that the symmetries which vanish on-shell are abundant in the commutation tables for the nontrivial symmetries of the gauge systems. We say that the symmetries $\partial^{(\boldsymbol{u})}_\varphi\big|_{\mathcal{E}^\infty} = 0$ are *improper*. Let us introduce an important concept that will allow us to get rid of the improper symmetries.

Definition 3.2. The set of *internal coordinates* on the infinite prolongation $\mathcal{E}^\infty$ of a system $\mathcal{E}$ is the maximal subset of the set of variables $\boldsymbol{x}$, $\boldsymbol{u}$, $\boldsymbol{u}_{\boldsymbol{x}}$, …, $\boldsymbol{u}_\sigma$, …, $|\sigma| \geqslant 0$, possessing the following two properties:
- at all points of $\mathcal{E}^\infty$ and for each $i = 1$, …, n, the derivative $\frac{\mathrm{d}}{\mathrm{d}x^i}(u^j_\sigma)$ of an internal coordinate u^j_σ can be expressed as a differential function in the internal coordinates;
- there are no differential-functional relations $\Psi = 0$ between the internal coordinates.

Note that the equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ imposes at least one such constraint upon the set of *all* variables on $J^\infty(\pi)$, so that the maximal set of the independent ones is "strictly less than everything." We also note that the internal coordinates *exist* because we required that the prolongations $\mathcal{E}^{(\ell)}$ are submanifolds in $J^{k+\ell}(\pi)$ at all $\ell \in \mathbb{N} \cup \{0\}$; for the same reason, the set of internal coordinates is filtered by ℓ.

Example 3.1. The list $(t, x, u, u_x, u_{xx}, \ldots, u_k, \ldots)$ is a convenient set of internal coordinates on the infinite prolongation of the Burgers equation $u_t = u_{xx} + uu_x$. The set

$(t,\, x,\, u,\, u_x,\, u_t,\, u_{tx},\, u_{txx},\, \ldots,\, u_{tx\cdots x},\, \ldots)$ is an inconvenient set of internal coordinates for the same equation. The set $(t,\, x,\, u,\, u_x,\, u_t,\, u_{xx},\, u_{tx},\, u_{tt})$ is not a set of the internal coordinates for the Burgers equation.

Exercise 3.3. Show that the sections $\varphi^x = u_x$ and $\varphi^t = u_{xx} + uu_x$ yield the proper symmetries of the Burgers equation. Next, show that $\varphi^F = u_t - u_{xx} - uu_x$ determines a symmetry of the Burgers equation but it is improper; can φ^F be expressed only via the internal coordinates and remain nonzero, whatever be your choice of that set ?

Exercise 3.4. Prove that the substitution of the functions which calculate the derivatives of the internal coordinates by virtue of $\mathcal{E}^\infty$ for such derivatives in any function from the ideal $I(\mathcal{E}^\infty)$ yields the zero function.

Together with Exercise 3.2, the statement above helps us to quotient out the improper symmetries of any formally integrable equation. From now on, we assume that at all points of $\mathcal{E}^\infty$ the generating section φ of a symmetry $\partial_\varphi^{(\boldsymbol{u})}$ for $\mathcal{E}$ is written by using the internal coordinates. So, we take the following theorem for the definition of proper infinitesimal symmetries of $\mathcal{E}$.

Theorem 3.1. *Under all the above assumptions, the restriction of the evolutionary vector field $\partial_\varphi^{(\boldsymbol{u})}$ onto $\mathcal{E}^\infty$ is a* proper infinitesimal symmetry *of the equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ if and only if the determining equation*

$$\partial_\varphi^{(\boldsymbol{u})}\big|_{\mathcal{E}^\infty}(\boldsymbol{F}) \doteq 0 \tag{3.3}$$

holds by virtue of system (3.2); *we denote by the symbol $\doteq$ the equality which is valid on-shell.*

- *Besides, there is a linear total differential operator $\nabla = \nabla_\varphi$ (moreover, linear also in φ) such that*
$$\partial_\varphi^{(\boldsymbol{u})}(\boldsymbol{F}) = \nabla_\varphi(\boldsymbol{F}) \quad \text{on } J^\infty(\pi).$$
- *The space of proper infinitesimal symmetries $\partial_\varphi^{(\boldsymbol{u})}$ retains the Lie algebra structure $[\,,\,]$ from the Lie algebra of vector fields*; *we denote by* $\operatorname{sym}\mathcal{E}$ *the Lie algebra of (proper) infinitesimal symmetries for $\mathcal{E}$.*

Proof. Only the second statement needs a brief comment.[8] When the restriction onto $\mathcal{E}^\infty$ of the evolutionary derivation $\partial_\varphi^{(\boldsymbol{u})}$, the components of whose section are already expressed through the internal coordinates, acts on the function $F^j \in \mathcal{F}(\pi)$ by the Leibniz rule, it encounters either the internal coordinates or the first total derivatives of the internal coordinates; this can always be achieved by rewriting the entire initial system $\mathcal{E}$ in a larger bundle. In the latter case (i. e., when a total derivative of an internal coordinate is met), the evolutionary field dives under the total derivatives and there it produces the functions which depend on the internal coordinates only. In turn, each of those total derivatives then itself acts by the Leibniz rule onto the argument.[9]

We now notice that the external (i. e., not internal) variables may appear during that second action linearly. This implies that the equation $\boldsymbol{F} = 0$ or its differential

[8] At this point, an ox-eye daisy was drawn on the blackboard, its yellow centre indicating the internal coordinates and the white petals depicting those derivatives of the internal coordinates which are themselves not the internal coordinates.

[9] The reasoning that follows also works in the equality (4.1).

consequences $\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma}\boldsymbol{F}=0$ are used at this point only once. But by definition, an expression which vanishes on-shell and which is linear in $\boldsymbol{F}$ or, possibly, its higher-order differential consequences is the value of a linear differential operator in total derivatives applied to $\boldsymbol{F}$: we have that $\nabla=\nabla_\varphi\in\mathcal{C}\mathrm{Diff}(P,P)$. The linearity of ∇_φ in φ is obvious. $\square$

Remark 3.1. The determining equation $\partial^{(\boldsymbol{u})}_\varphi(\boldsymbol{F})\doteq 0$ on $\mathcal{E}^\infty$ for $\varphi\in\operatorname{sym}\mathcal{E}$ is linear and homogeneous: Its right-hand side vanishes by virtue of the equation which does not change, it is only the sections of π_∞ which experience the infinitesimal transformation. Nevertheless, the inhomogeneous generalization of the determining equation appear naturally in the construction of Gardner's deformations $\mathfrak{m}_\varepsilon\colon\mathcal{E}_\varepsilon\to\mathcal{E}\big|_{\varepsilon=0}$ for completely integrable systems (see Lecture 12). The section φ then describes the infinitesimal variation of the contraction map $\mathfrak{m}_\varepsilon$, whereas the infinitesimal modification of the system $\mathcal{E}_\varepsilon=\{\boldsymbol{F}(\varepsilon)=0\}$ contributes to the nonhomogeneity.

How can any solution $\varphi\in\operatorname{sym}\mathcal{E}$ of the determining equation (3.3) be found or, in broader terms, where can we take the symmetries from? Let us note in passing that if the equation $\mathcal{E}$ is linear (or can be transformed to a linear system, see Problem 3.5), then the store of its symmetries is immense: any shift –infinitesimal or finite– by a solution of that equation is its symmetry. However, for true nonlinear models this trick is impossible, so the straightforward solution of the determining equation (3.3) must be performed (see below). Retrospectively, when sufficient experimental evidence allows us to detect the existence of symmetries and study their distribution along their differential orders or any other weights or degree (see Exercise 3.9 on p. 36), we check whether the system admits a *master-symmetry* $\varphi_M\colon\varphi_i\mapsto\varphi_{i+1}=[\varphi_M,\varphi_i]$ that yields the (in)finite sequence (see Problem 3.5). Likewise, we could try to find a recursion, e. g., a linear differential operator $R\colon\varphi_i\mapsto\varphi_{i+1}$ and guess its seed symmetry φ_0 (see Lecture 7). Finally, field theory models admit gauge symmetries. These are the most powerful: to find them, we do not even need to solve any determining equations (see Lecture 6).

The practical algorithm for the search of proper infinitesimal symmetries of a formally integrable differential equation $\mathcal{E}=\{\boldsymbol{F}=0\}$ is as follows.

(1) Introduce the internal coordinates on the infinite prolongation $\mathcal{E}^\infty$.
(2) Restrict the differential order of the sections φ which are expressed via the internal coordinates. If this order equals one,[10] the symmetry –whenever it exists and is found– is called *classical*. If the section φ of a classical symmetry is linear with respect to the first-order derivatives of the unknowns (c. f. Exercise 3.1 on p. 28), then the symmetry at hand is a *point* symmetry. Otherwise, this classical symmetry is *contact*. However, if the section φ essentially depends on the jet variables $\boldsymbol{u}_\sigma$ of orders $|\sigma|>1$ for all choices of the internal coordinates on $\mathcal{E}^\infty$, then this is a higher symmetry of $\mathcal{E}$.
(3) Write the determining equation
$$\partial^{(\boldsymbol{u})}_\varphi(\boldsymbol{F})\doteq 0\text{ on }\mathcal{E}^\infty.$$

[10]Consider the Burgers equation $u_t=u_{xx}+uu_x$. The generating section $\varphi(t,x,u,u_x,u_t)$ has order one and is expressed using the set of internal coordinates $(t,x,u,u_x,u_t,u_{tx},u_{txx},\ldots)$. The same section would in general depend on the variables t,x,u,u_x, and u_{xx} when expressed via the internal coordinates $(t,x,u,u_x,u_{xx},u_{xxx},\ldots)$ on the infinite prolongation.

and solve it for φ (e. g., see section 3.1 below). Usually, the solutions are obtained by inspecting how their dependence on the highest-order derivatives $\boldsymbol{u}_\sigma$ shows up in the determining equation in the coefficients of the jet variables $u_{\sigma+\sigma'}$ which were not present initially, neither in the section φ nor in the differential functions F^j of positive differential order $|\sigma'|$.

(4) Calculate the table of commutators $[\varphi_i, \varphi_j]$ for the already known solutions of (3.3) or estimate the dependence on the top-order derivatives for the entries of this table by using the intermediate estimates for the would-be solutions (see above); the second option sometimes helps us to demonstrate the non-existence of a partially "found" solution with a given top-order dependence.

(5) Inspect whether a master-symmetry $\varphi_M \in \operatorname{sym}\mathcal{E}$ is available so that by taking the commutator with φ_M one propagates symmetries to sequences.

(6) Inspect whether a (non)local recursion differential operator $R\colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ is available for the symmetry algebra $\operatorname{sym}\mathcal{E}$, and prove that the sequences of symmetry generators which it produces from the "seed" sections φ_0 remain local.

(7) Find φ-invariant solutions $\boldsymbol{s}_\varphi(\boldsymbol{x})$ of $\mathcal{E} = \{\boldsymbol{F} = 0\}$ by solving[11] the systems $\{\boldsymbol{F} = 0,\, \varphi = 0\}$.

(8) Integrate the classical symmetries of $\mathcal{E}$ to the finite transformations of the solution set by solving the systems

$$\dot{\boldsymbol{x}} = \boldsymbol{a}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}}), \quad \dot{\boldsymbol{u}} = \boldsymbol{b}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}}), \quad \dot{\boldsymbol{u}}_{\boldsymbol{x}} = (\boldsymbol{c}(\boldsymbol{a}, \boldsymbol{b}))(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{\boldsymbol{x}})$$

of ordinary differential equations on $J^1(\pi)$. By the implicit function theorem, this yields the families $\boldsymbol{s}^\varepsilon(\boldsymbol{x})$ of solutions for the Cauchy data $u\big|_{\varepsilon=0} = \boldsymbol{s}(\boldsymbol{x}) \in \operatorname{Sol}(\mathcal{E})$ and sufficiently small $|\varepsilon| \in \mathbb{R}$.

(9) If necessary, solve the given boundary value problem for $\mathcal{E}$ either by matching the already known classes of exact solutions of $\mathcal{E}$ to the boundary data or by performing the homotopy from known solutions of $\mathcal{E}$ to a solution satisfying the given conditions. Note that the π-vertical velocity $\dot{\boldsymbol{s}}^\varepsilon = \varphi(\boldsymbol{x}, [\boldsymbol{s}^\varepsilon(\boldsymbol{x})])$ satisfies the equation (3.3) at all $\varepsilon \in (0, 1)$ during the homotopy. However, one has to substantiate it separately that the auxiliary linear boundary value problems do have the solutions for all the intermediate configurations, c. f. [57].

(10) If applicable, analyse the obstructions to the existence of the symmetries for $\mathcal{E}$ or recursion differential operators for $\operatorname{sym}\mathcal{E}$ and resolve those obstructions by introducing the nonlocalities (see Lecture 7).

(11) If applicable, track the correspondence between the symmetries of $\mathcal{E}$ and the conservation laws for it (e. g., using the First Noether Theorem, see Lecture 5), and write down the conserved charges.

(12) If applicable, derive the gauge symmetries of $\mathcal{E}$ from the Noether identities between the equations in the system $\mathcal{E}$ by using the the Second Noether Theorem (see Lecture 6).

This is enough. We do not advise the immediate reconstruction of the conserved currents that correspond to the Noether gauge symmetries.

[11] It is nontrivial that for $\varphi \in \operatorname{sym}\mathcal{E}$ the new system which contains the extra constraint $\varphi = 0$ remains solvable (see [44]). We note further that there are *other* techniques to consistently overdetermine a given differential equation, see [82].

3.1. **Classical symmetries of the Burgers equation.** The Burgers equation
$$u_t = u_{xx} + uu_x \tag{3.4}$$
is a model equation that describes weakly-nonlinear dissipative phenomena such as the behaviour of rarified interstellar dust or plasma. The quadratic term in the nonlinear transfer equation $u_t = uu_x$ suggests that the velocity is proportional to the density $u(x,t)$, which results in the shock waves; the dissipative term in the right-hand side of (3.4) resolves the gradient catastrophe by taking the collisions into account.

Let us choose the standard set of internal coordinates on the infinite prolongation $\mathcal{E}^\infty$ of (3.4):
$$t, x, u, u_x, u_{xx}, u_{xxx}, u_{4x}, \ldots,$$
and let φ be the generating section of an evolutionary vector field $\partial^{(u)}_\varphi$ on $\mathcal{E}^\infty$; we assume that φ is expressed in terms of the internal coordinates. For φ to be symmetry of (3.4), it must satisfy the determining equation
$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi) \doteq \frac{\mathrm{d}^2}{\mathrm{d}x^2}(\varphi) + \varphi \cdot u_x + u \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\varphi) \quad \text{on } \mathcal{E}^\infty. \tag{3.5}$$
We now aim at finding the classical symmetries so we restrict the set of arguments of φ to $\varphi(t,x,u,u_x,u_{xx})$. Under this assumption, the linearized equation upon φ becomes
$$\begin{aligned}
&\frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial u}(u_{xx} + uu_x) + \frac{\partial\varphi}{\partial u_x}(u_{xxx} + u_x^2 + uu_{xx}) + \frac{\partial\varphi}{\partial u_{xx}}(u_{4x} + 3u_xu_{xx} + uu_{xxx}) \doteq \\
&\doteq \varphi\cdot u_x + u\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial u}u_x + \frac{\partial\varphi}{\partial u_x}u_{xx} + \frac{\partial\varphi}{\partial u_{xx}}u_{xxx}\right) + \left(\frac{\partial^2\varphi}{\partial x^2} + 2\frac{\partial^2\varphi}{\partial x\partial u}u_x + \right.\\
&\left. +2\frac{\partial^2\varphi}{\partial x\partial u_x}u_{xx} + 2\frac{\partial^2\varphi}{\partial x\partial u_{xx}}u_{xxx}\right) + \left(\frac{\partial^2\varphi}{\partial u^2}u_x^2 + 2\frac{\partial^2\varphi}{\partial u\partial u_x}u_xu_{xx} + \right.\\
&\left. +2\frac{\partial^2\varphi}{\partial u\partial u_{xx}}u_xu_{xxx} + \frac{\partial\varphi}{\partial u}u_{xx}\right) + \left(\frac{\partial^2\varphi}{\partial u_x^2}u_{xx}^2 + 2\frac{\partial^2\varphi}{\partial u_x\partial u_{xx}}u_{xx}u_{xxx} + \frac{\partial\varphi}{\partial u_x}u_{xxx}\right) + \\
&+\left(\frac{\partial^2\varphi}{\partial u_{xx}^2}u_{xxx}^2 + \frac{\partial\varphi}{\partial u_{xx}}u_{4x}\right) \quad \text{on } \mathcal{E}^\infty.
\end{aligned} \tag{3.6}$$

Exercise 3.5. Express $\frac{\mathrm{d}}{\mathrm{d}t}(u_x)$ and $\frac{\mathrm{d}}{\mathrm{d}t}(u_{xx})$ by virtue of the Burgers equation and recognize the corresponding terms in (3.6).

We notice that equation (3.6) contains the (powers of) internal coordinates u_{xxx} and u_{4x} which were not initially present — neither in the Burgers equation nor in the list of arguments of φ. Therefore, condition (3.6) determines the *system* of equations: each coefficient of that polynomial in u_{xxx} and u_{4x}, including the free term which can depend only on (t,x,u,u_x,u_{xx}), vanishes separately. Let us inspect the coefficients of the respective monomials:

u_{4x}: $\frac{\partial\varphi}{\partial u_{xx}} \equiv \frac{\partial\varphi}{\partial u_{xx}}$ is satisfied identically (c. f. Problem 4.3 on page 52);

u_{xxx}^2: $\partial^2\varphi/\partial u_{xx}^2 = 0$, which implies that
$$\varphi = A(t,x,u,u_x)\cdot u_{xx} + B(t,x,u,u_x);$$

u_{xxx}: $\frac{\partial\varphi}{\partial u_x} + A\cdot u = u\cdot A + 2\frac{\partial A}{\partial x} + 2u_x\cdot\frac{\partial A}{\partial u} + 2u_{xx}\cdot\frac{\partial A}{\partial u_x} + \frac{\partial\varphi}{\partial u_x}$. We easily recognize the total derivative $\frac{\mathrm{d}}{\mathrm{d}x}(A)$ and then from $2\cdot\frac{\mathrm{d}}{\mathrm{d}x}(A) = 0$ we deduce that $A = A(t)$.

To eliminate the risk of error in calculations, we substitute the ansatz

$$\varphi = A(t)\cdot u_{xx} + B(t,x,u,u_x) \tag{3.7}$$

back in (3.5) instead of inspecting the free term of (3.6) "as is." We thus obtain a more manageable expression,

$$\begin{aligned}
&Au_{xx} + A(u_{4x} + 3u_xu_{xx} + uu_{xxx}) + \\
&\quad + \left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial u}(u_{xx}+uu_x) + \frac{\partial B}{\partial u_x}\left(u_{xxx}+u_x^2+uu_{xx}\right)\right) \doteq \\
&\quad \doteq Au_xu_{xx} + Bu_x + Auu_{xxx} + u\left(\frac{\partial B}{\partial x} + \frac{\partial B}{\partial u}u_x + \frac{\partial B}{\partial u_x}u_{xx}\right) + \\
&\quad + Au_{4x} + \left(\frac{\partial^2 B}{\partial x^2} + 2\frac{\partial^2 B}{\partial x\partial u}u_x + 2\frac{\partial^2 B}{\partial x\partial u_x}u_{xx} + \frac{\partial^2 B}{\partial u^2}u_x^2 + 2\frac{\partial^2 B}{\partial u\partial u_x}u_xu_{xx} + \right.\\
&\quad \left. + \frac{\partial B}{\partial u}u_{xx} + \frac{\partial^2 B}{\partial u_x^2}u_{xx}^2 + \frac{\partial B}{\partial u_x}u_{xxx}\right).
\end{aligned} \tag{3.8}$$

Again, let us look at the top-order derivatives in (3.8):

u_{xx}^2**:** $\partial^2 B/\partial u_x^2 = 0$ implies that

$$B = C(t,x,u)\cdot u_x + D(t,x,u). \tag{3.9}$$

u_{xx}**:** $\dot A + 2Au_x = 2\frac{\partial^2 B}{\partial x\partial u_x} + 2\frac{\partial^2 B}{\partial u\partial u_x}u_x = 2\frac{\partial C}{\partial x} + 2\frac{\partial C}{\partial u}u_x$, where we use formula (3.9).

The equality which appears at u_{xx} splits further because it is polynomial in u_x:

u_x**:** $A(t) = C_u$,

1**:** $\dot A = 2C_x$; we denote by the subscripts the partial derivatives with respect to x and u, respectively, and by the dot the time-derivative.

Resolving the equations for C, we conclude that

$$C = A(t)u + \tfrac12 \dot A(t)x + Z(t),$$

and therefore,

$$B = A(t)uu_x + \tfrac12\dot A(t)xu_x + Z(t)u_x + D(t,x,u).$$

One more time we substitute everything back in (3.8). Because we have already processed the coefficients of all positive powers of u_{xx} and higher-order derivatives and made those coefficients identically equal to zero, we now omit them while rewriting (3.8). There only remains

$$\begin{aligned}
&(\dot A uu_x + \tfrac12\ddot A xu_x + \dot D) + (Auu_x^2 + \tfrac12\dot A xu_x^2) + (\dot Z u_x + Zu_x^2) \doteq \\
&(Auu_x^2 + \tfrac12\dot A xu_x^2 + Du_x + Zu_x^2) + (\tfrac12\dot A uu_x + uD_x) + D_{xx} + 2D_{xu}u_x + D_{uu}u_x^2.
\end{aligned} \tag{3.10}$$

As before, we inspect the coefficients of the polynomial in the top-order derivatives:

u_x^2**:** $D_{uu} = 0$, so that

$$D = E(t,x)\cdot u + F(t,x);$$

u_x**:** $\tfrac12\dot A u + \tfrac12\ddot A x + \dot Z = D + 2D_{xu} = Eu + F + 2E_x.$

This equality itself splits with respect to the first and zeroth powers of u, yielding the coefficients of

u**:** $\frac{1}{2}\dot{A} = E$, whence
$$E = \tfrac{1}{2}\dot{A}(t);$$

1**:** $\frac{1}{2}\ddot{A}x + \dot{Z} = F$.

This combines to
$$D = \tfrac{1}{2}\dot{A}u + \tfrac{1}{2}\ddot{A}x + \dot{Z}.$$
There remains the free term in (3.10):

1**:** $\dot{D} = uD_x + D_{xx}$, which is
$$\tfrac{1}{2}\ddot{A}u + \tfrac{1}{2}\dddot{A}x + \ddot{Z} = u \cdot \frac{1}{2}\ddot{A}.$$
The equality splits in the powers of x:

x**:** $\dddot{A} = 0$,

1**:** $\ddot{Z} = 0$.

We conclude that
$$A = \alpha t^2 + \beta t + \gamma,$$
$$Z = \delta \cdot t + \varepsilon.$$
Having expressed the functions A, B, C, D, E, F, and Z via arbitrary constants α, β, γ, δ, and ε, we obtain the solution
$$\varphi = \alpha \cdot (t^2u_t + txu_x + tu + x) + \beta \cdot (2tu_t + xu_x + u) + \gamma \cdot u_t + \delta \cdot (tu_x + 1) + \varepsilon \cdot u_x.$$
It determines the five-dimensional vector space of classical symmetries of the Burgers equation. We claim that their physical sense is as follows:

- the β-term is the scaling-invariance under the correlated dilation of all variables;
- the γ-term and the ε-term are the translations;
- the δ-term is the Galilean symmetry which induces a change of variables such that the new coordinate system moves uniformly and rectilinearly along the old one.

Exercise 3.6. Calculate the commutation table for these five symmetries.

Remark 3.2. The Burgers equation $u_t = u_{xx} + uu_x$ admits not only the five classical but also infinitely many *higher* symmetries whose generating sections are of the form $\varphi = t^\ell \cdot u_k + \ldots$ for all integers $k \geqslant 1$ and $0 \leqslant \ell \leqslant k$.

Exercise 3.7. Find all third-order symmetries $\varphi(t, x, u, u_x, u_{xx}, u_{xxx})$ with $\frac{\partial \varphi}{\partial u_{xxx}} \not\equiv 0$ for the Burgers equation (3.4).

Finally, let us support our claim about the physical sense of the classical symmetries for (3.4) by integrating them to finite transformations of the variables. We recall that the evolutionary vector fields $\partial_\varphi^{(\boldsymbol{u})}$ mark the equivalence classes (3.1) of infinitesimal transformations of the Cartan distribution, which means that any linear combination of the total derivatives with arbitrary functional coefficients can be freely added to $\partial_\varphi^{(\boldsymbol{u})}$.

Moreover, if $\partial_{\varphi}^{(\boldsymbol{u})}$ is a symmetry of a given differential equation, then the resulting vector fields (3.1) are also its symmetries.

Consider first the generator $\varphi_5 = u_x$ of a symmetry of the Burgers equation. By adding $-\mathrm{d}/\mathrm{d}x$ to this field, we infer that $\partial_{\varphi}^{(u)} \sim -\partial/\partial x$, which is the translation:

$$\dot{x} = -1, \quad \dot{t} = 0, \quad \dot{u} = 0.$$

Indeed, the elementary integration yields the mapping

$$x \mapsto x - \tau, \quad t \mapsto t, \quad u \mapsto u, \qquad \tau \in \mathbb{R},$$

of the variables so that each solution $u = s(x,t)$ is shifted along x:

$$s(x,t) \mapsto s(x - \tau, t).$$

Exercise 3.8. Show that the infinitesimal symmetry $\varphi_3 = u_t$ of (3.4) produces the time-shift $s(x,t) \mapsto s(x, t - \tau)$ in the solutions.

The dilation $\varphi_2 = 2tu_t + xu_x + u$ belongs to the same class (3.1) as the vector field

$$-2t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} + \ldots,$$

which is the scaling of the variables. Indeed, the system

$$\dot{x} = -x, \quad \dot{t} = -2t, \quad \dot{u} = u$$

has the solution

$$x(\tau) = x \cdot \exp(-\tau) = x \cdot \lambda^{-1}, \quad t(\tau) = t \cdot \exp(-2\tau) = t \cdot \lambda^{-2}, \quad u(\tau) = u \cdot \exp(\tau) = u \cdot \lambda.$$

Exercise 3.9. Derive the scaling-invariance of the Burgers equation $u_t = u_{xx} + uu_x$ from the fact that it is homogeneous with respect to the *weights* if one sets $[x] = -1 \Leftrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}x}\right] = 1$, $[t] = -2 \Leftrightarrow \left[\frac{\mathrm{d}}{\mathrm{d}t}\right] = 2$, and $[u] = 1$ (the weights sum up under the multiplication of variables and/or the action upon them by the total derivatives, c. f. [65]).

Exercise 3.10. Is the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ homogeneous with respect to any set of weights ascribed to x, t, and w? If yes, find the scaling symmetry for the KdV equation without solving the determining equation (3.3).

Next, consider the symmetry $\varphi_4 = tu_x + 1$ of the Burgers equation (3.4). By subtracting $t \cdot \frac{\partial}{\partial x}$ from the evolutionary vector field $\partial_{\varphi_4}^{(u)}$, we obtain the vector field $-t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} + \ldots$. Consider the system

$$\dot{x} = -t, \quad \dot{t} = 0, \quad \dot{u} = 1.$$

Its solution,

$$x(\tau) = x - \tau t, \quad t(\tau) = t, \quad u(\tau) = u + \tau,$$

determines the Galilean transformation of the variables at each time t.

Exercise 3.11. Integrate the infinitesimal symmetry $\varphi_1 = t^2 u_t + \ldots$ of the Burgers equation to a finite transformation of the variables x, t, and u.

3.2. **Symmetry-invariant solutions of the Liouville equation.** The hyperbolic Liouville equation is

$$u_{xy} = \exp(2u). \tag{3.11}$$

Let us first find its finite symmetries and then, taking their infinitesimal counterparts, construct a class of exact solutions for (3.11). To match the notation used in the literature on conformal field theory (CFT) where the Liouville equation plays a prominent rôle, we complexify the independent variables and let $x = z$ and $y = \overline{z}$ be the respective (anti)holomorphic coordinates on $\mathbb{C}$.

Exercise 3.12. Show that under an arbitrary chiral transformation $z \mapsto \zeta(z)$, $\overline{z} \mapsto \overline{\zeta}(\overline{z})$ the field $\mathcal{U}(\zeta, \overline{\zeta})$ is a solution of the Liouville equation $\mathcal{U}_{z,\overline{z}} = \exp(2\mathcal{U})$ if and only if

$$u(z, \overline{z}) = \mathcal{U}(\zeta(z), \overline{\zeta}(\overline{z})) + \log\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{\frac{1}{2}}\left(\frac{\mathrm{d}\overline{\zeta}}{\mathrm{d}\overline{z}}\right)^{\frac{1}{2}} \tag{3.12}$$

solves the same equation $u_{z\overline{z}} = \exp(2u)$.

Remark 3.3. The exponents $\frac{1}{2}$ of the (anti)holomorphic derivatives in (3.12) are the *conformal weights* (or *dimension*[102]) $(\Delta, \overline{\Delta}) = \left(\frac{1}{2}, \frac{1}{2}\right)$ of the Liouville field u (or $\exp(u)$).

Now let us take a close-to-identity chiral transformation

$$\begin{cases} z \mapsto \zeta(z) = z - \varepsilon f(z) + \overline{o}(\varepsilon), \\ \overline{z} \mapsto \overline{\zeta}(\overline{z}) = \overline{z} - \varepsilon g(\overline{z}) + \overline{o}(\varepsilon), \end{cases}$$

which induces the mapping of the field

$$\begin{aligned} u \mapsto U = u - \tfrac{1}{2}\log\left|\frac{\mathrm{d}\zeta}{\mathrm{d}z}\cdot\frac{\mathrm{d}\overline{\zeta}}{\mathrm{d}\overline{z}}\right|\Bigg|_{\substack{z=z(\zeta),\\ \overline{z}=\overline{z}(\overline{\zeta})}} + \overline{o}(\varepsilon) = \\ = u - \tfrac{1}{2}\log|1 - \varepsilon f'(z) + \overline{o}(\varepsilon)| - \tfrac{1}{2}\log|1 - \varepsilon g'(\overline{\zeta}) + \overline{o}(\varepsilon)| + \overline{o}(\varepsilon) = \\ = u + \frac{\varepsilon}{2}f'(z) + \frac{\varepsilon}{2}g'(z) + \overline{o}(\varepsilon). \end{aligned}$$

The infinitesimal version of such reparametrization (as $\varepsilon \to 0$) is

$$\dot{z} = -f(z), \quad \dot{\overline{z}} = -g(\overline{z}), \quad \dot{u} = \tfrac{1}{2}f'(z) + \tfrac{1}{2}g'(\overline{z}).$$

By adding $f(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z} + g(\overline{z})\cdot\frac{\mathrm{d}}{\mathrm{d}\overline{z}}$ to the vector field at hand, we obtain the evolutionary vector field $\partial_{\varphi}^{(u)}$ with the generating section

$$\varphi = u_z \cdot f(z) + u_{\overline{z}} \cdot g(\overline{z}) + \tfrac{1}{2}f'(z) + \tfrac{1}{2}g'(\overline{z}) = \square\big(f(z)\big) + \overline{\square}\big(g(\overline{z})\big),$$

here f and g are arbitrary functions of their arguments. This yields the *classical* infinitesimal symmetries of the Liouville equation whose full Lie algebra of higher symmetries is immense (c.f. Problem 3.1).

Let us construct the φ-invariant solutions of the Liouville equation $u_{xy} = \exp(2u)$ by overdetermining it with the constraint $\varphi = 0$; from now on, we return to the notation x, y for the independent variables. We first solve the linear first-order partial differential equation

$$u_x \cdot f(x) + u_y \cdot g(y) + \tfrac{1}{2}\left(f'(x) + g'(y)\right) = 0$$

upon $u(x, y)$ by using the method of characteristics. We have that

$$\frac{\mathrm{d}x}{f(x)} = \frac{\mathrm{d}y}{g(y)} = -\frac{\mathrm{d}u}{\frac{1}{2}\left(f'(x) + g'(y)\right)} = \frac{f' \cdot \frac{\mathrm{d}x}{f} + g' \cdot \frac{\mathrm{d}y}{g}}{f'(x) + g'(y)}.$$

The first equality gives us the integral

$$\int \frac{\mathrm{d}x}{f(x)} - \int \frac{\mathrm{d}y}{g(y)} = C_1 = \text{const};$$

using the last equality, we deduce that

$$u + \tfrac{1}{2} \ln |f(x) \cdot g(y)| = C_2 = \text{const},$$

whence $\Pi(C_1, C_2) = 0$ with arbitrary Π implicitly determines the solutions $u(x, y)$. Because Π can be resolved with respect to u, we conclude that the general solution of the equation $\varphi = 0$ is

$$u(x, y) = \Phi\bigl(\mathcal{X}(x) + \mathcal{Y}(y)\bigr) + \tfrac{1}{2} \ln |\mathcal{X}'(x) \cdot \mathcal{Y}'(y)|,$$

where Φ is an arbitrary function of its argument $v = \mathcal{X}(x) + \mathcal{Y}(y)$ and we put $\mathcal{X}(x) = \int \mathrm{d}x/f(x)$ and $\mathcal{Y}(y) = -\int \mathrm{d}y/g(y)$ so that $f(x) = 1/\mathcal{X}'(x)$ and $g(y) = -1/\mathcal{Y}'(y)$.

Second, we plug this ansatz for u in the Liouville equation $u_{xy} = \exp(2u)$ and obtain the *ordinary* differential equation

$$\Phi'' = \exp(2\Phi).$$

Noting that it does not contain the argument v of Φ explicitly, we perform the standard order reduction $\Phi'(v) := h(\Phi)$ and integrate the equation $h' \cdot h = \exp(2\Phi)$, which finally yields the first-order ordinary differential equation

$$(\Phi')^2 = C + \exp(2\Phi). \tag{3.13}$$

The analytic form of its solutions depends on the choice of the sign for the integration constant C. If $C = 0$, we have that $\Phi = -\ln |v - v_0|$, but $v_0 \in \mathbb{R}$ is absorbed by a redefinition of $\mathcal{X}(x)$ or $\mathcal{Y}(y)$.

Exercise 3.13. Integrate Eq. (3.13) whenever $C > 0$ or $C < 0$.

We thus obtain the general solution(s) of the Liouville equation $u_{xy} = \exp(2u)$:

$$\begin{aligned} u_{C=0}(x, y) &= \tfrac{1}{2} \ln \frac{|\mathcal{X}'(x)\mathcal{Y}'(y)|}{(\mathcal{X}(x) + \mathcal{Y}(y))^2}, \\ u_{C>0}(x, y) &= \tfrac{1}{2} \ln \frac{|\mathcal{X}'(x)\mathcal{Y}'(y)|}{\sinh^2(\mathcal{X}(x) + \mathcal{Y}(y))}, \\ u_{C<0}(x, y) &= \tfrac{1}{2} \ln \frac{|\mathcal{X}'(x)\mathcal{Y}'(y)|}{\sin^2(\mathcal{X}(x) + \mathcal{Y}(y))}. \end{aligned}$$

Note that $v(x, y) = \mathcal{X}(x) + \mathcal{Y}(y)$ is itself a general solution of the wave equation $v_{xy} = 0$.

Exercise 3.14. Find the substitutions between the functions $v = \mathcal{X}(x) + \mathcal{Y}(y)$ which transform each of the three formulas (with $C = 0$, $C < 0$, and $C > 0$) into any of the other two (so that it suffices to remember only the first formula at $C = 0$).

Problem 3.1. Prove that $\varphi = (u_x + \frac{1}{2}\frac{d}{dx})f(x,[w]) + (u_y + \frac{1}{2}\frac{d}{dy})g(y,[w])$ is a (higher) symmetry of $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$ for all smooth f and g, here $w = u_x^2 - u_{xx}$ and $\overline{w} = uy^2 - u_{yy}$.

- Find the commutator relations between such symmetries.
- Set $f := w$ and $g \equiv 0$. Calculate the induced velocity $\partial_\varphi^{(u)}(w)$ and compare it with $-\frac{1}{2}\frac{d^3}{dx^3}(w) + 3w\frac{d}{dx}(w)$, see also Problems 9.5–9.6 on p. 95.
- Show that the evolution

$$\begin{cases} \dot{\mathcal{X}} = \mathcal{X}' \cdot \{\mathcal{X}(x), x\} \equiv \mathcal{X}''' - \dfrac{3}{2}\dfrac{(\mathcal{X}'')^2}{\mathcal{X}'}, \\ \dot{\mathcal{Y}} = 0 \end{cases}$$

yields the evolution $\dot{u} = \square(w)$ by virtue of the already known general solution $u(x,y) = \frac{1}{2}\ln\frac{|\mathcal{X}'(x)\cdot\mathcal{Y}'(y)|}{(\mathcal{X}(x)+\mathcal{Y}(y))^2}$ of $\mathcal{E}_{\text{Liou}}$ (the evolution $\dot{\mathcal{X}} = \mathcal{X}' \cdot \{\mathcal{X}(x), x\}$ is the Krichever–Novikov equation, or Schwarz-KdV; $\{\mathcal{X}(x), x\}$ is the Schwarzian derivative).
- Is the space of general solutions of $\mathcal{E}_{\text{Liou}}$ arcwise connected ?

Problem 3.2. Find the classical symmetries of the Korteweg-de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ and integrate them to finite transformations of the variables.

Problem 3.3. Find five classes of solutions for the Burgers equation $u_t = u_{xx} + uu_x$ which are invariant with respect to its five classical symmetries.

Problem 3.4. Find three solutions of the boundary-value problem

$$\begin{cases} \Delta u = \exp(2u) \\ u|_{r=1} = 0 \\ 0 \leqslant r < 1;\ 0 \leqslant \alpha \leqslant 2\pi. \end{cases}$$

and show that these solutions exhibit very different behaviour as $r \to 0$.

Problem 3.5. Using the generator of exact solutions in parametric form for the minimal surface equation (see [52]), plot two graphs of such surfaces.

Problem 3.6 ([54])**.** Calculate the conformal dimension of the Wronskian determinant

$$W(f_1, \ldots, f_N)$$

of N functions $f_j\colon \mathbb{R} \to \mathbb{R}$ (by definition, a *function* is a scalar field of conformal dimension zero). Is the Wronskian of N functions itself a function ? a vector ? a tensor ?

4. Conservation laws

We now formalize the language that will allow us to find, classify, and use the conserved currents associated with systems of differential equations. The currents reflect the continuity properties and manifest the presence of conserved integral values, or *charges*, for all solutions of the system at hand. In this lecture we explain in precisely what sense the equivalence classes of conserved currents are dual to the notion of nontrivial proper infinitesimal symmetries for differential equations; also, we outline the practical technique for the search of the equivalence classes of the currents, or *conservation laws*. The two Noether theorems, which we prove in Lectures 5 and 6, correlate the symmetries and conservation laws for the Euler–Lagrange models.

The origins of various types of conserved currents for partial differential equations are slightly more diverse than those for the symmetries (which necessarily preserve the Cartan distribution). Firstly, the currents stem from a nontrivial topology of the bundle π and so reveal the presence and the structure of the bundle's (co)homology groups regardless of any differential equations imposed on the sections. Second, if the equation at hand admits gauge symmetries, then there appears the class of non-topological *improper* conserved currents, which on one hand, carry the information about the gauge group but, nevertheless, vanish on-shell. Thirdly, we study the nontrivial proper currents that are conserved by virtue of the given system $\mathcal{E}$ and its differential consequences. The presence of sufficiently many non-topological, non-trivial, proper conservation laws for the system at hand is the most serious argument in favour of its integrability (e. g., via the inverse scattering). In turn, the kinematic integrability technique reveals the direction for the non-abelian generalization of the notion of conservation laws (see Lecture 12). Retrospectively, conservation laws produce the abelian contributions to the non-abelian zero-curvature representations for nonlinear systems (see Example 12.5 on p. 131).

Let us also note that the knowledge of conserved currents for a system $\mathcal{E}$ under study is highly practical in the numerical modelling: it allows one to control the stability and precision of the algorithms or, alternatively, raise the accuracy by inserting the continuity constraints into the scheme and thus preventing the leakage of energy or other conserved values. Finally, the generating sections ψ_η of conservation laws $\int\eta$ for equations $\mathcal{E}$ are interesting per se. In particular, these sections appear in the nonlocal terms of the recursion and symplectic operators (see Lecture 7), which hints us a part of the ansatz for such structures.

In the previous lecture we first introduced the infinitesimal symmetries; likewise, we now pass to the local setup and study the continuity equations of form $\operatorname{div}\eta \doteq 0$ on $\mathcal{E}^\infty$. The transition to the global picture will require some assumptions about the topology, which we discuss below.

Definition 4.1. A *conserved current* η for a system $\mathcal{E}$ is the continuity equation

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x^i}\Big|_{\mathcal{E}^\infty} (\eta_i) \doteq 0 \text{ on } \mathcal{E}^\infty,$$

where $\eta_i(\boldsymbol{x}, [\boldsymbol{u}])$ are the coefficients of the horizontal $(n-1)$-form

$$\eta = \sum_{i=1}^{n} (-1)^{i+1} \eta_i \cdot \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^i} \wedge \ldots \wedge \mathrm{d}x^n \in \overline{\Lambda}^{n-1}(\pi).$$

The conservation of η is the equality

$$\bar{\mathrm{d}}\big|_{\mathcal{E}^\infty} \eta \doteq 0 \text{ on } \mathcal{E}^\infty,$$

i. e., the form η is $\bar{\mathrm{d}}$-closed on-shell.

Example 4.1 ($n = 1$)**.** Consider a system of ordinary differential equations $\mathcal{E}$ and a function $\mathcal{C}(t, \boldsymbol{u}, \boldsymbol{u}', \ldots, \boldsymbol{u}^{(k)}) \in \mathcal{F}_k(\pi) \hookrightarrow \overline{\Lambda}^0(\pi)$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}(t, [\boldsymbol{u}]) \doteq 0$ by virtue of $\mathcal{E}$ and its consequences. This means that $\mathcal{C}(t, [\boldsymbol{s}]) = \mathrm{const}(\boldsymbol{s})$ for every solution $\boldsymbol{s}$ of $\mathcal{E}$; in other words the conserved quantity $\mathcal{C}$ is the *first integral*[12] of the equation $\mathcal{E}$. The value of the constant is determined by the Cauchy data that uniquely specify the solution $\boldsymbol{s}$. The conservation of energy, momentum, angular momentum, and the Runge–Lentz vector in Kepler's problem for the orbital motion is an illustration of this concept in mechanical models.

Though it is immediate to check whether a given horizontal form η is $\bar{\mathrm{d}}$-closed on a given equation, there are several independent ways for the form η to be exact or to emerge from the topology of the bundle π. We set the intermediate goal to describe the $\bar{\mathrm{d}}$-cohomology group of the non-trivially conserved non-topological currents that do no vanish identically on-shell. Therefore, let us examine the nature of those excessive elements. First, we recall that each globally defined $\bar{\mathrm{d}}$-exact q-form on $J^\infty(\pi)$ is $\bar{\mathrm{d}}$-closed off-shell. For instance, the equality $\frac{\mathrm{d}}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$ holds regardless of the Burgers equation $u_t = u_{xx} + uu_x$. Let us pass to the horizontal cohomology on $J^\infty(\pi)$:

$$\overline{H}^q(\pi) = \frac{\{\eta \in \overline{\Lambda}^q \mid \bar{\mathrm{d}}\eta = 0\}}{\{\eta \in \overline{\Lambda}^q \mid \eta = \bar{\mathrm{d}}\gamma\}}.$$

Definition 4.2. A conservation law $\int\eta \in \overline{H}^{n-1}(\mathcal{E})$ for an equation $\mathcal{E}$ is the equivalence class of conserved currents,

$$\bar{\mathrm{d}}\big|_{\mathcal{E}^\infty} (\eta) \doteq 0 \text{ on } \mathcal{E}^\infty,$$

modulo the globally defined exact currents $\bar{\mathrm{d}}\,\xi \in \int 0$.

Still if the topology of the bundle π is nontrivial for some q, $0 \leqslant q \leqslant n$, there are the q-cocycles which are not coboundaries, that is, which are off-shell closed but are not equal to the differential of a globally defined $(q-1)$-form. Let us consider the model situation: the form $\mathrm{d}\varphi$ on the circle $\mathbb{S}^1$ with the *local* coordinate φ is closed but is not the differential of any *globally* defined (zero-degree) form because the angular coordinate φ is not uniquely defined at all points of the circle at once. Likewise, the total space E of the bundle π –or the equation $\mathcal{E}$ in $J^k(\pi)$, which is topologically indistinguishable from $J^0(\pi) = E$ because $\pi_{k,0}\colon J^k(\pi) \to J^0(\pi)$ is a vector bundle

[12]This example reveals an interesting approach to Riemannian surfaces: Consider a two-component system of first-order ordinary differential equations upon the unknowns $x(t)$ and $y(t)$, and let $\mathcal{C}(t, x, y)$ be a first integral. Whenever $\mathcal{C}$ is a cubic, the equality $\mathcal{C} = \mathrm{const}$ determines the elliptic curve for each value of the constant, c.f. [66].

and the projection $\mathcal{E} \to J^0(\pi)$ is an epimorphism– can contain such cocycles. They constitute the group $H^q(\mathcal{E})$ at $0 \leqslant q \leqslant n$; in particular the space $H^{n-1}(\mathcal{E})$ consists of the topological conservation laws. The corresponding continuity equations are independent of the solutions of $\mathcal{E}$. Let us quotient out such topological contribution and consider the group of *non-topological* conservation laws

$$c.l.(\mathcal{E}) = \overline{H}^{n-1}(\mathcal{E})/H^{n-1}(\mathcal{E}).$$

For the sake of pedagogical transparency, we postulate that the topology of the bundle π is trivial: $H^q(\pi) = 0$ for $0 < q < n$. This convention, which we accept for the rest of the course, allows us to continue denoting by $\overline{H}^{n-1}(\mathcal{E})$ the space of non-topological conservation laws for $\mathcal{E}$.

Example 4.2. The continuity relation $\frac{\mathrm{d}}{\mathrm{d}t}(u) \doteq \frac{\mathrm{d}}{\mathrm{d}x}(u_x + \frac{1}{2}u^2)$ determines the nontrivial non-topological conserved current for the Burgers equation $u_t = u_{xx} + uu_x$.

4.1. **Generating sections of conservation laws.** We now describe the two-step method for the systematic search of conservation laws for systems $\mathcal{E} = \{\boldsymbol{F} = 0\}$ of differential equations. Within this approach, to each conserved current η we assign its generating section ψ_η that satisfies the linear determining equation $(\ell_{\boldsymbol{F}}^{(\boldsymbol{u})})^\dagger(\psi_\eta) \doteq 0$ on $\mathcal{E}^\infty$. Solving the latter for ψ_η (here we use the techniques which are also applicable in the search of symmetries, see the previous lecture) and filtering out the *irrelevant solutions* –whose possible existence in a new phenomenon in the contrast with the one-to-one correspondence between the solutions of (3.3) and symmetries of $\mathcal{E}$,– we then reconstruct the currents η by using the homotopy[13] which is based on the constructive proof of the Poincaré lemma. Let us repeat that from now on we discard the topology of the bundle π and, on top of this, operate with the classes of sections that respect the Green formula for the integration by parts.

Suppose that the current $\eta \in \overline{\Lambda}^{n-1}(\pi)$ is conserved on the equation $\mathcal{E} = \{\boldsymbol{F} = 0 \mid \boldsymbol{F} \in P = \Gamma\bigl(\pi_\infty^*(\xi)\bigr)\}$: we have that $\bar{\mathrm{d}}\big|_{\mathcal{E}^\infty}\eta \doteq 0$ on $\mathcal{E}^\infty$. Let us express the coefficients $\eta_i(\boldsymbol{x}, [\boldsymbol{u}])$ of η via the internal coordinates on $\mathcal{E}^\infty$; this quotients out the improper terms that are "invisible" on-shell. Now repeating for the derivation $\bar{\mathrm{d}}$ the reasoning which was used in the proof of Theorem 3.1 (see footnote 9 on p. 30), we conclude that for each η there exists the linear total differential operator $\square\colon P \to \overline{\Lambda}^n(\pi)$ such that

$$\bar{\mathrm{d}}\eta = \square(\boldsymbol{F}) \text{ on } J^\infty(\pi). \tag{4.1}$$

Let us introduce the notation $\mathrm{Ber}(\pi) \stackrel{\text{def}}{=} \overline{\Lambda}^n(\pi)$ for the $\mathcal{F}(\pi)$-module of the highest, n-th degree "horizontal volume" forms on $J^\infty(\pi)$ and consider the $\mathcal{F}(\pi)$-module $\widehat{\mathrm{Ber}(\pi)} = \mathrm{Hom}_{\mathcal{F}(\pi)}\bigl(\mathrm{Ber}(\pi), \overline{\Lambda}^n(\pi)\bigr)$ dual to $\mathrm{Ber}(\pi)$ with respect to the coupling $\langle\,,\rangle\colon \widehat{\mathrm{Ber}(\pi)} \times \mathrm{Ber}(\pi) \to \overline{\Lambda}^n(\pi)$; by construction, $\widehat{\mathrm{Ber}(\pi)} \simeq \mathcal{F}(\pi)$. We take the equality

$$\bar{\mathrm{d}}\eta = \langle 1, \square(\boldsymbol{F})\rangle, \qquad \text{here } 1 \in \widehat{\mathrm{Ber}(\pi)},$$

[13]Likewise, the homotopy formula for the upper line of Diagram 1.6 permits the reconstruction of the Lagrangian functionals $\mathcal{L} \in \overline{H}^n(\pi)$ for the systems $\mathcal{E} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$ of Euler–Lagrange equations.

and project it to the horizontal cohomology group $\overline{H}^n(\pi)$; we continue denoting by $\langle\,,\rangle$ the coupling that now takes its values in $\overline{H}^n(\pi)$. Integrating by parts, we establish the equivalence

$$\bar{\mathrm{d}}\eta \cong \langle \square^\dagger(1), \boldsymbol{F}\rangle,$$

where $\square^\dagger\colon \widehat{\mathrm{Ber}(\pi)} \to \widehat{P} = \mathrm{Hom}_{\mathcal{F}(\pi)}\bigl(P, \overline{\Lambda}^n(\pi)\bigr)$ is the *adjoint operator*.

Definition 4.3. The section

$$\psi_\eta \stackrel{\mathrm{def}}{=} \square^\dagger(1) \in \widehat{P} = \Gamma(\pi_\infty^*(\widehat{\xi}\,)) = \Gamma(\widehat{\xi}\,) \otimes_{C^\infty(M^n)} \mathcal{F}(\pi)$$

is the *generating section* of the conservation law $\int\eta$ for the equation

$$\mathcal{E} = \{\boldsymbol{F} = 0 \mid \boldsymbol{F} \in P = \Gamma(\pi_\infty^*(\xi)) = \Gamma(\xi) \otimes_{C^\infty(M^n)} \mathcal{F}(\pi)\}.$$

Exercise 4.1. Prove that neither the operator $\square$ nor the generating section ψ_η, with their coefficients expressed in the internal coordinates on $\mathcal{E}^\infty$, depend on the choice of a current η from the equivalence class $\int\eta$.

Remark 4.1. By their definition, the generating sections ψ_η belong to the $\mathcal{F}(\pi)$-module $\widehat{P}$ which is $\langle\,,\rangle$-dual to the module P of equations. Therefore, there is in general no well-defined coupling between the modules of such sections ψ_η and generating sections $\varphi \in \varkappa(\pi)$ of evolutionary vector fields.

Exercise 4.2. Let $\mathcal{L} \in \overline{\Lambda}^n(\pi)$. Prove that $\frac{\delta}{\delta\boldsymbol{u}}(\mathcal{L}) = (\ell_{\mathcal{L}}^{(\boldsymbol{u})})^\dagger(1)$, where $1 \in \widehat{\mathrm{Ber}(\pi)}$ and $\ell_{\mathcal{L}}^{(\boldsymbol{u})\dagger}\colon \widehat{\mathrm{Ber}(\pi)} \to \widehat{\varkappa(\pi)} = \mathrm{Hom}_{\mathcal{F}(\pi)}\bigl(\varkappa(\pi), \overline{\Lambda}^n(\pi)\bigr)$.

From Problem 1.10 we know that under a suitable assumption about the topology of the bundle π we have

$$\frac{\delta}{\delta\boldsymbol{u}}(\bar{\mathrm{d}}\eta) \equiv 0 \in \widehat{\varkappa(\pi)}.$$

Using the equivalence $\bar{\mathrm{d}}\eta \cong \langle\psi_\eta, F\rangle$ in $\overline{H}^n(\pi)$, we continue the equality and combine Exercise 4.2 with the Leibniz rule:

$$0 = \ell^{(\boldsymbol{u})\,\dagger}_{\langle\psi_\eta,\boldsymbol{F}\rangle} = {\ell^{(\boldsymbol{u})}_{\psi_\eta}}^\dagger(\boldsymbol{F}) + {\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}}^\dagger(\psi_\eta), \quad \text{where } \begin{cases} \ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\colon \varkappa(\pi) \to P; & {\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}}^\dagger\colon \widehat{P} \to \widehat{\varkappa(\pi)}, \\ \ell^{(\boldsymbol{u})}_{\psi_\eta}\colon \varkappa(\pi) \to \widehat{P}; & \ell^{(\boldsymbol{u})\,\dagger}_{\psi_\eta}\colon P \to \widehat{\varkappa(\pi)}. \end{cases}$$

Restricting this equality onto $\mathcal{E}^\infty$, we obtain

Theorem 4.1. *The generating section $\psi_\eta \in \widehat{P}$ of a conservation law $\int\eta$ for a system $\mathcal{E} = \{\boldsymbol{F} = 0 \mid \boldsymbol{F} \in P\}$ satisfies the determining equation*

$$(\ell^{(\boldsymbol{u})}_{\boldsymbol{F}})^\dagger(\psi_\eta) \doteq 0 \text{ on } \mathcal{E}^\infty. \tag{4.2}$$

Its structure explains why its solutions are sometimes called cosymmetries.

Remark 4.2. There is an abundant store of special-type solution for the determining equation (4.2) if the system $\mathcal{E} = \{\boldsymbol{F} = 0 \mid \boldsymbol{F} \in P = \Gamma(\pi_\infty^*(\xi))\}$ admits a Noether relation $\Phi \in P_1$. Indeed, suppose that

$$\Phi(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])]) \equiv 0 \text{ on } J^\infty(\pi), \tag{4.3}$$

in other words, the identical-zero value in the right-hand side does not alter under any (e. g., infinitesimal) shifts of sections of π_∞. This means that for any $\varphi \in \varkappa(\pi)$, the induced velocity of the left-hand side along $\partial_\varphi^{(\boldsymbol{u})}$ vanishes:

$$\partial_\varphi^{(\boldsymbol{u})}\left(\Phi(\boldsymbol{x},[\boldsymbol{u}],[\boldsymbol{F}])\right)+\bigl(\ell_\Phi^{(\boldsymbol{F})}\circ\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\bigr)(\varphi)=0,$$

where we have passed to the infinite horizontal jet bundle $\overline{J^\infty}(\pi_\xi)\to J^\infty(\pi)$ so that $\boldsymbol{u}$ and $\boldsymbol{F}$ become independent jet variables, see Lecture 2. Let us couple this identity with any element $\boldsymbol{p}$ of the dual module $\widehat{P_1}$ and integrate by parts. We obtain the equality

$$\langle\ell_\Phi^{(\boldsymbol{u})\dagger}(\boldsymbol{p})+\bigl(\ell_F^{(\boldsymbol{u})\dagger}\circ\ell_\Phi^{(\boldsymbol{F})\dagger}\bigr)(\boldsymbol{p}),\varphi\rangle\cong 0,$$

which holds for all $\varphi\in\varkappa(\pi)$ simultaneously. Therefore,

$$\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger}\bigl(\ell_\Phi^{(\boldsymbol{F})\dagger}(\boldsymbol{p})\bigr)+\ell_\Phi^{(\boldsymbol{u})\dagger}(\boldsymbol{p})=0.$$

Now let us note that the system $\mathcal{E}=\{\boldsymbol{F}=0\}$ *exhausts* the set of relations between the unknowns $\boldsymbol{u}$ so that there can be no extra equations upon these variables. On the other hand, we notice that the Noether identity (4.3) holds, in particular, at all points of $\mathcal{E}^\infty\subseteq J^\infty(\pi)$, at which we replace the generators (3.2) of the ideal $I(\mathcal{E}^\infty)$ by the zeroes. Consequently, the constraint $\Phi(\boldsymbol{x},[\boldsymbol{u}],[0])$ may not contain any terms which would *not* depend on $\boldsymbol{F}$, whence we conclude that the linearization $\ell_\Phi^{(\boldsymbol{u})}$ vanishes on-shell. We thus establish the on-shell equality

$$\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger}\left(A(\boldsymbol{p})\right)\doteq 0 \text{ on } \mathcal{E}^\infty,\qquad \boldsymbol{p}\in\widehat{P_1},$$

where we put $A=\ell_\Phi^{(\boldsymbol{F})\dagger}$. This shows that the on-shell defined section $\psi=A(\boldsymbol{p})$ is a solution of (4.2) for each $\boldsymbol{p}\in\widehat{P_1}$. However, there is no guarantee that a nontrivial conservation law will be available on $\mathcal{E}^\infty$ for such cosymmetry $\psi=A(\boldsymbol{p})$ or (another option) that even if existing, the current will not vanish everywhere on $\mathcal{E}^\infty$. The latter is typical for the gauge-invariant models; we study this in further detail in Lectures 5 and 6.

Remark 4.3. The determining equation (4.2) can admit *irrelevant* solutions ψ that do not correspond to any non-trivial conserved current for the system $\mathcal{E}$. Let us establish a convenient verification procedure for ψ to be the genuine generating section of a conservation law for an *evolutionary* system $\mathcal{E}$; we refer to [79, §V.2.6-7] for the general case.

In contrast with the gauge models, the evolutionary systems never admit any Noether identities between the equations. Let us examine the correspondence between the conservation laws and their generating sections that appears in the evolutionary case.[14]

Proposition 4.2. Let $\mathcal{E}=\{\boldsymbol{u}_t=\boldsymbol{f}(t,\boldsymbol{x},[\boldsymbol{u}])\}$ be an evolutionary system and let the current $\eta=\rho\,\mathrm{d}\boldsymbol{x}+\sum\limits_{i=1}^{n-1}(-1)^i\,\eta_i\,\mathrm{d}t\wedge\mathrm{d}x^1\wedge\ldots\wedge\widehat{\mathrm{d}x^i}\wedge\ldots\wedge\mathrm{d}x^{n-1}$, where $\rho,\eta_i\in C^\infty(\mathcal{E}^\infty)$, be conserved:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{\mathcal{E}^\infty}(\rho)+\sum_i\tfrac{\mathrm{d}}{\mathrm{d}x^i}\big|_{\mathcal{E}^\infty}(\eta_i)\doteq 0 \text{ on } \mathcal{E}^\infty.$$

[14] The proof of Proposition 4.2 states the assertion in any fixed system of local coordinates.

Then its generating section is equal to

$$\psi_\eta = \ell_\rho^{(\boldsymbol{u})\dagger}(1) = \tfrac{\delta}{\delta \boldsymbol{u}}(\rho);$$

because of this, the section ψ_η is sometimes called the (*variational*) *gradient* of η for the evolutionary system $\mathcal{E}$.

Proof. Let us represent the time-derivative on-shell in two different ways. First we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathcal{E}^\infty} \doteq \frac{\partial}{\partial t} + \partial_{\boldsymbol{f}}^{(\boldsymbol{u})}\bigg|_{\mathcal{E}^\infty}, \qquad \text{whence} \quad \frac{\partial \rho}{\partial t} + \partial_{\boldsymbol{f}}^{(\boldsymbol{u})}(\rho) + \sum_{i=1}^{n-1} \frac{\mathrm{d}}{\mathrm{d}x^i}(\eta_i) \doteq 0 \text{ on } \mathcal{E}^\infty.$$

At the same time,

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \partial_{\boldsymbol{u}_t}^{(\boldsymbol{u})} \text{ on } J^\infty(\pi).$$

Expressing the flux from the previous formula, we deduce that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \sum_{i=1}^{n-1} \frac{\mathrm{d}}{\mathrm{d}x^i}(\eta_i) \doteq \frac{\partial \rho}{\partial t} + \partial_{\boldsymbol{u}_t}^{(\boldsymbol{u})}(\rho) - \left(\frac{\partial \rho}{\partial t} + \partial_{\boldsymbol{f}}^{(\boldsymbol{u})}(\rho)\right) = \partial_{\boldsymbol{u}_t - \boldsymbol{f}}^{(\boldsymbol{u})}(\rho) = \ell_\rho^{(\boldsymbol{u})}(\boldsymbol{u}_t - \boldsymbol{f}).$$

This yields the operator $\square = \ell_\rho^{(\boldsymbol{u})}$ that ensures the on-shell conservation of the current η, so that its generating section ψ_η equals $\square^\dagger(1) = (\ell_\rho^{(\boldsymbol{u})})^\dagger(1) = \delta\rho/\delta\boldsymbol{u}$ by Exercise 4.2. $\square$

Remark 4.4. For evolutionary systems, Proposition 4.2 suggests us to discard those solutions ψ of the determining equation (4.2) which do not belong to the image of the variational derivative; the latter is checked by using Theorem 5.1 on p. 55. Still, for a generic non-evolutionary system there is no analog of Proposition 4.2 that would prescribe some "canonical" shape of the generating sections for conservation laws.

Finally, let us parameterize the velocities of the conservation laws under the infinitesimal symmetries of differential equations.

Proposition 4.3. Let $\varphi \in \varkappa(\pi)$ be the generating section of a symmetry of an equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ so that

$$\partial_\varphi^{(\boldsymbol{u})}(\boldsymbol{F}) = \nabla_\varphi(\boldsymbol{F}).$$

Suppose that a current $\eta \in \overline{\Lambda}^{n-1}(\pi)$ is conserved on $\mathcal{E}^\infty$, which means that $\overline{\mathrm{d}} = \square(\boldsymbol{F})$, and we set $\psi_\eta = \square^\dagger(1)$. Then

- the current $\dot{\eta} = \partial_\varphi^{(\boldsymbol{u})}(\eta) \in \overline{\Lambda}^{n-1}(\pi)$ is also conserved on $\mathcal{E}^\infty$;
- its generating section equals $\psi_{\dot{\eta}} = \partial_\varphi^{(\boldsymbol{u})}(\psi_\eta) + \nabla_\varphi^\dagger(\psi_\eta)$.

Proof. By construction, the evolutionary fields dive under the total derivatives, whence the on-shell conservation of η implies the on-shell conservation of $\dot{\eta}$ because the derivation of $\partial_\varphi^{(\boldsymbol{u})}$ preserves the ideal $I(\mathcal{E}^\infty)$ (again, see the proof of Theorem 3.1 on p. 30). We see that

$$\overline{\mathrm{d}}\big(\partial_\varphi^{(\boldsymbol{u})}(\eta)\big) = \partial_\varphi^{(\boldsymbol{u})}(\overline{\mathrm{d}}\eta) = \partial_\varphi^{(\boldsymbol{u})}(\square)(\boldsymbol{F}) + \square\big(\partial_\varphi^{(\boldsymbol{u})}(\boldsymbol{F})\big) = \big(\partial_\varphi^{(\boldsymbol{u})}(\square)\big)(\boldsymbol{F}) + \square\big(\nabla_\varphi(\boldsymbol{F})\big).$$

Let us multiply this equality in $\overline{\Lambda}^n(\pi)$ by $1 \in \widehat{\mathrm{Ber}(\pi)}$ and integrate by parts:

$$\psi_{\dot\eta} = \bigl(\partial_\varphi^{(\boldsymbol{u})}(\Box)\bigr)^\dagger(1) + \nabla_\varphi^\dagger\bigl(\Box^\dagger(1)\bigr).$$

Finally, instead of letting the evolutionary derivation dive under the total derivatives in the adjoint operator $\Box^\dagger$, we take it out and then extend its application to the argument $1 \in \widehat{\mathrm{Ber}(\pi)}$ of $\Box^\dagger$ by the Leibniz rule, which of course alters nothing. We thus obtain

$$\psi_{\dot\eta} = \partial_\varphi^{(\boldsymbol{u})}\bigl(\Box^\dagger(1)\bigr) + \nabla_\varphi^\dagger\bigl(\Box^\dagger(1)\bigr),$$

whence the assertion follows. $\Box$

4.2. **The homotopy formula.** Having introduced the machinery of generating functions ψ_η for conservation laws $\int\eta$, we now consider the second step in the construction of conserved currents η for differential equations. Namely, we shall derive the homotopy formula that yields the inverse mapping $\psi_\eta \mapsto \eta$. We emphasize that this reconstruction is itself iterative: we first reduce the problem on the jet space $J^\infty(\pi)$ to a smaller problem on its base M^n by contracting the fibres of π_∞, and then the Poincaré lemma works by further contraction of star-shaped domains in M^n to one point. Let us therefore study this geometry “bottom-up”; a formula for finding the Lagrangian $\mathcal{L}$ of a given Euler–Lagrange equation $\delta\mathcal{L}/\delta\boldsymbol{u} = 0$, also applicable to the reconstruction of the conserved densities for evolutionary systems (see Proposition 4.2), will be the intermediate output of this reasoning.

4.2.1. *Poincaré's lemma.* The principle which we are going to study sounds as follows, provided that we forget about the devil who hides in the details. Let N^m be a manifold (which could be the fibre over M^n in the bundle π) and d be the de Rham differential on it. Suppose that there is a *homotopy operator* s such that $\omega = \mathrm{d}\,(\mathsf{s}(\omega)) + \mathsf{s}\,(\mathrm{d}(\omega))$ for any differential form ω (the construction of such s will be somewhat special for zero-forms $\omega \in \Lambda^0(N^m) = C^\infty(N^m)$, i. e., functions). Then the de Rham complex

$$0 \to \mathbb{R} \hookrightarrow C^\infty(N^m) \xrightarrow{\mathrm{d}} \Lambda^1(N^m) \xrightarrow{\mathrm{d}} \ \ldots\ \xrightarrow{\mathrm{d}} \Lambda^m(N^m) \to 0 \tag{4.4}$$

is locally exact: $\mathrm{im}\,\mathrm{d}_i = \ker \mathrm{d}_{i+1}$ for $i < m$. Indeed, if $\mathrm{d}\omega = 0$, then the homotopy amounts to $\omega = \mathrm{d}\,(\mathsf{s}(\omega)) \in \mathrm{im}\,\mathrm{d}$. The *locality* of this statement means that from now on we operate only on star-shaped domains $V_\alpha \ni \boldsymbol{u}_0$ in the manifold N^m where $\boldsymbol{u}$ is a local coordinate system; by definition of a manifold, it is legitimate to think that $V_\alpha \subseteq \mathbb{R}^m$ with $\boldsymbol{u}_0 = 0$ and V_α is such that for each point $\boldsymbol{u} \in V_\alpha$ the entire segment $\{\lambda\boldsymbol{u} \mid 0 \leqslant \lambda \leqslant 1\}$ is contained in V_α.

To construct the homotopy operator s on $V_\alpha \subseteq N^m$ explicitly, let us recall some basic properties of the Lie derivative.

Exercise 4.3. Show that the Lie derivative $\mathrm{L}_X(\omega)$ of a differential form ω along a vector field X is

$$\mathrm{L}_X(\omega) = \mathrm{d}(\mathrm{i}_X(\omega)) + \mathrm{i}_X(\mathrm{d}\omega), \tag{4.5}$$

where $\mathrm{i}_X \equiv X\lrcorner$ is the insertion of the vector field in the differential form (by convention, the insertion lands on the first, leftmost slot whenever the form is regarded as a multilinear mapping of several arguments).

Exercise 4.4. Show that the de Rham differential d commutes with mappings f of manifolds: $f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$ for any differential form ω on the target space of f.

Let X be a vector field on $V_\alpha \subseteq N^m$ such that its integral trajectories $\{\boldsymbol{u} \mapsto \exp(\mu X)(\boldsymbol{u})\}$ are defined for all $\mu \in [0, \varepsilon)$ by virtue of the Cauchy–Kovalevskaya theorem for some $\varepsilon \in \mathbb{R}$ and suppose that a given differential form is regular along those pieces of the integral trajectories (see Remark 4.5 below). Let us integrate (4.5) along the trajectories, always transporting the velocities to the initial point $\boldsymbol{u} \in V_\alpha$ by the contravariant mapping of differential forms. We obtain that

$$\exp(\varepsilon X)^* \left(\omega|_{\exp(\varepsilon X)(\boldsymbol{u})}\right) - \omega|_{\boldsymbol{u}} = \int_0^\varepsilon \exp(\mu X)^* \left(\mathrm{L}_X(\omega)|_{\exp(\mu X)(\boldsymbol{u})}\right) \mathrm{d}\mu,$$

which is the Newton–Leibniz formula. Using Exercises 4.3 and 4.4, we continue the equality:

$$= \int_0^\varepsilon \left(\mathrm{d}\left(\exp(\mu X)^* \left(\mathrm{i}_X(\omega)|_{\exp(\mu X)(\boldsymbol{u})}\right)\right) + \exp(\mu X)^* \left(\mathrm{i}_X(\mathrm{d}\omega)|_{\exp(\mu X)(\boldsymbol{u})}\right)\right) \mathrm{d}\mu.$$

It is important that *first*, the field X is inserted into the forms ω or $\mathrm{d}\omega$, respectively, and strictly *after* that, the resulting forms of smaller degree are evaluated at the point $\exp(\mu X)(\boldsymbol{u})$ of the integral trajectory containing $\boldsymbol{u}$ at $\mu = 0$. By definition, we put

$$\mathsf{s}_X^\varepsilon(\omega)|_{\boldsymbol{u}} \stackrel{\text{def}}{=} \int_0^\varepsilon \exp(\mu X)^* \left((X \lrcorner\, \omega)|_{\exp(\mu X)(\boldsymbol{u})}\right) \mathrm{d}\mu.$$

Using this notation, we conclude that

$$\exp(\varepsilon X)^* \left(\omega|_{\exp(\varepsilon X)(\boldsymbol{u})}\right) - \omega|_{\boldsymbol{u}} = \mathrm{d} \circ \mathsf{s}_X^\varepsilon(\omega)|_{\boldsymbol{u}} + \mathsf{s}_X^\varepsilon(\mathrm{d}\omega)|_{\boldsymbol{u}}\,. \tag{4.6}$$

Now it is almost obvious what vector field on $V_\alpha \ni \boldsymbol{u}$ one should take in order to connect by a straight line the given point $\boldsymbol{u}$ with the centre of the star-shaped domain, and what will be the limits for variation of ε.

Lemma 4.4 (Poincaré)**.** Let $V_\alpha \subseteq \mathbb{R}^m$ be a star-shaped domain centered at the origin. Then the de Rham complex (4.4) on V_α is exact.

Remark 4.5. The proof[107] (see below) reveals that the assertion of Poincaré's lemma is extremely sensitive to the analytic class of forms which we deal with. Specifically, the proof is unable to detect (without inversion) that $-\mathrm{d}u/u^2 = \mathrm{d}(1/u)$ for all $u > 0$ or even that $\ln u\, \mathrm{d}u = \mathrm{d}\,(u \cdot (-1 + \ln u))$, although the primitive of $\ln u\, \mathrm{d}u$ is bounded and continuous at $0 \leqslant u \leqslant u_{\max} < \infty$ and is continuously differentiable at $u > 0$.

Proof. Choose $X_D = \sum_i u^i \cdot \partial/\partial u^i$ so that the flow of X_D is the dilation centered at the origin: it is $\boldsymbol{u} \mapsto \exp(\varepsilon X_D)(\boldsymbol{u}) = e^\varepsilon \cdot \boldsymbol{u} \in V_\alpha$ for all $\varepsilon \in (-\infty, 0]$; note that $\exp(\varepsilon X_D)(\boldsymbol{u}) \to 0$ as $\varepsilon \to -\infty$. Let $\omega = \sum\limits_{|I|=k} \alpha_I(\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{u}^I$ be a k-form (we first let $k > 0$), whence for all $\varepsilon \leqslant 0$ we have that

$$\exp(\varepsilon X_D)^* \left(\omega|_{\exp(\varepsilon X_D)(\boldsymbol{u})}\right) = \exp(\varepsilon X_D)^* \Big(\sum_I \alpha_I(e^\varepsilon \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{u}^I\Big) = \sum_{|I|=k} \alpha_I(e^\varepsilon \boldsymbol{u}) \cdot e^{k\varepsilon} \mathrm{d}\boldsymbol{u}^I.$$

Let us introduce the convenient notation $\omega[e^\varepsilon \boldsymbol{u}] = \exp(\varepsilon X_D)^* \omega[\boldsymbol{u}]$. Then from (4.6) we infer that

$$\omega[e^\varepsilon \boldsymbol{u}] - \omega[u] = \mathrm{d} \circ \mathsf{s}_{X_D}^\varepsilon(\omega) + \mathsf{s}_{X_D}^\varepsilon(\mathrm{d}\omega),$$

where

$$\mathsf{s}^{\varepsilon}_{X_D}(\omega) = \int_0^{\varepsilon} (X_D \lrcorner\, \omega)[e^{\mu}\boldsymbol{u}]\,\mathrm{d}\mu.$$

Let us change the variables via $\mu = \ln\lambda$; the limits $|_0^{-\infty}$ for ε imply that $(e^{\mu} = \lambda)|_1^0$. Therefore,

$$\mathsf{s}^{\varepsilon}_{X_D}(\omega) = -\int_{e^2}^{1} (X_D \lrcorner\, \omega)[\lambda\boldsymbol{u}]\frac{\mathrm{d}\lambda}{\lambda}.$$

In order to regularize the behaviour of the integrand as $\lambda \to 0$ whenever $\varepsilon \to -\infty$, we appropriately restrict the class of admissible differential forms ω. (We also emphasize that the coefficients of ω may not vanish at the origin but attain some finite nonzero values, c. f. Problem 4.7.) Finally, let us pass to the limit $\varepsilon \to -\infty$ and put

$$\mathsf{s}(\omega) = -\int_0^1 (X_D \lrcorner\, \omega)[\lambda\boldsymbol{u}]\frac{\mathrm{d}\lambda}{\lambda},$$

which determines the required homotopy for that restricted sub-class of forms.

The case $k = 0$ of zero-forms (i. e., functions $\omega|_{\boldsymbol{u}} = f(\boldsymbol{u})$, $f \in C^{\infty}(N^m)$), which contain no differentials that would regularize the integrand, leads to the equality

$$f(0) - f(\boldsymbol{u}) = \mathrm{d}\mathsf{s}^{-\infty}_{X_D}(f)(\boldsymbol{u}) + \mathsf{s}^{-\infty}_{X_D}(\mathrm{d}f)(\boldsymbol{u}) = 0 + \mathsf{s}^{-\infty}_{X_D}(\mathrm{d}f)(\boldsymbol{u})$$

(the first term vanishes because $X_D \lrcorner\, f \equiv 0$). Consequently, if $\mathrm{d}f = 0$, then $f(0) = f(\boldsymbol{u})$ for any $\boldsymbol{u} \in V_{\alpha}$, that is, the function f is a constant. This confirms the exactness of the de Rham complex $0 \to \mathbb{R} \hookrightarrow C^{\infty}(N^m) \xrightarrow{\mathrm{d}} \ldots$ in the term $C^{\infty}(N^m)$: the inclusion corresponds to the constant functions. The proof is complete. $\square$

Our goal is to extend the homotopy formula from the geometry of usual manifolds N^m to the spaces $J^{\infty}(\pi)$ of infinite jets of sections for the vector bundles π with m-dimensional fibres over M^n. Under such generalization, de Rham complex (4.4) splits as it is shown in Diagram (1.5) on p. 46.

Let us first regard the coordinates $\boldsymbol{x}$ along the base M^n as formal parameters; simultaneously, we pay no attention to the presence of the differentials $\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}$, $0 \leqslant p \leqslant n$, but we focus on the sub-complexes which are generated by the Cartan differential $\mathrm{d}_{\mathcal{C}}$.

Under all suitable assumptions about the topology of the spaces at hand (note that at the moment the star-shaped domains stand vertically along the fibres) and about the admissible smoothness classes of sections, the Poincaré lemma works for the never-ending complex

$$0 \to C^{\infty}(M^n) \hookrightarrow \mathcal{F}(\pi) \xrightarrow{\mathrm{d}_{\mathcal{C}}} \mathcal{C}^1\Lambda(\pi) \xrightarrow{\mathrm{d}_{\mathcal{C}}} \mathcal{C}^2\Lambda(\pi) \xrightarrow{\mathrm{d}_{\mathcal{C}}} \ldots \tag{4.7}$$

and also for the remaining n complexes which are obtained via the tensor multiplication over $\mathcal{F}(\pi)$ by $\overline{\Lambda}^p(\pi)$, here $0 \leqslant p \leqslant n$. In particular, the upper line in (1.5) is

$$0 \to \Lambda^n(M^n) \hookrightarrow \underline{\overline{\Lambda}^n(\pi)} \xrightarrow{\mathrm{d}_{\mathcal{C}}} \overline{\Lambda}^n(\pi) \otimes \mathcal{C}^1\Lambda(\pi) \xrightarrow{\mathrm{d}_{\mathcal{C}}} \overline{\Lambda}^n(\pi) \otimes \mathcal{C}^2\Lambda(\pi) \xrightarrow{\mathrm{d}_{\mathcal{C}}} \ldots \quad .$$

We recall that the underlined term contains the Lagrangians and the horizontal cohomology classes for elements in its image under $\mathrm{d}_{\mathcal{C}}$ encode the Euler–Lagrange equations.

By repeating the constructive proof of Poincaré's lemma and now using the dilation $X_D = \sum\limits_{|\sigma|\geqslant 0} \boldsymbol{u}_\sigma \cdot \partial/\partial \boldsymbol{u}_\sigma = \partial_{\boldsymbol{u}}^{(\boldsymbol{u})}$ in the fibre over each point $\boldsymbol{x} \in M^n$ of the base, we obtain the homotopy

$$\mathsf{s}(\omega) = \int_0^1 \left(\partial_{\boldsymbol{u}}^{(\boldsymbol{u})} \lrcorner\, \omega\right)(\boldsymbol{x}, [\lambda \boldsymbol{u}]) \frac{\mathrm{d}\lambda}{\lambda} \tag{4.8}$$

such that (notice the signs in these two formulas!)

$$\omega(\boldsymbol{x}, [\boldsymbol{u}]) - \omega(\boldsymbol{x}, [0 \cdot \boldsymbol{u}]) = \mathrm{d}_{\mathcal{C}} \circ \mathsf{s}(\omega) + \mathsf{s}(\mathrm{d}_{\mathcal{C}}(\omega))$$

for any Cartan q-form $\omega \in \overline{\Lambda}^p(\pi) \otimes \mathcal{C}^q\Lambda(\pi)$ with $q > 0$. (The special case $\Lambda^p(M^n) \hookrightarrow \overline{\Lambda}^p(\pi)$ at $q = 0$ is covered by the pointwise reasoning over M^n: it corresponds to the inclusion $\mathbb{R} \hookrightarrow C^\infty(N^m)$ for the fibres.)

Corollary 4.5. The Lagrangian $\mathcal{L}$ of the Euler–Lagrange equation $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$ is obtained via the homotopy formula

$$\delta\mathcal{L}(\boldsymbol{x}, [\boldsymbol{u}]) = \delta \int \mathrm{d}\boldsymbol{x} \int_0^1 F_i(\boldsymbol{x}, [\lambda\boldsymbol{u}]) \cdot u^i \,\mathrm{d}\lambda = \boldsymbol{F}\,\delta\boldsymbol{u}, \tag{4.9}$$

which we deduce from (4.8) for the corner $p = n$, $q = 0$.

Remark 4.6. However, it is a priori not obvious whether the Poincaré lemma on $J^\infty(\pi)$ states that each $\overline{\mathrm{d}}$-closed q-form is locally $\overline{\mathrm{d}}$-exact. Indeed, the definition of the total derivatives, the condition $j_\infty(s)^*(\overline{\mathrm{d}}\eta) \equiv 0$, and the usual Poincaré lemma on the star-shaped domains in the base M^n of the bundle π imply that for each $\boldsymbol{s} \in \Gamma_{\mathrm{loc}}(\pi)$ and $\mathrm{d}_{\mathrm{dR}(M^n)}(j_\infty(\boldsymbol{s})^*(\eta)) = 0$ there exists a $(q-1)$-form $\xi(\boldsymbol{s}) \in \Lambda^{q-1}_{\mathrm{loc}}(M^n)$ such that $j_\infty(\boldsymbol{s})^*(\eta) = \mathrm{d}_{\mathrm{dR}(M^n)}\xi(\boldsymbol{s})$. Yet at this moment it is entirely unclear why and under which extra assumptions should such section-dependent forms $\xi(\boldsymbol{s})$ on M^n glue to the section-independent horizontal form $\xi \in \overline{\Lambda}^{q-1}(\pi)$ on $J^\infty(\pi)$. We now address this subtlety.

4.2.2. *Reconstruction of conserved currents:* $\psi_\eta \mapsto \eta$. The nontriviality of this reconstruction problem is clear because now the horizontal differential forms (or the horizontal differential $\overline{\mathrm{d}}$) and the differential-functional coefficients which depend on $\boldsymbol{u}$ or the derivatives co-exist along two different directions along the jet space $J^\infty(\pi)$ (respectively, the horizontal and π_∞-vertical directions specified by the Cartan distribution). Therefore, we have two simultaneous tasks which are

1. to find a way for returning from the divergences $\overline{\mathrm{d}}\eta \in \overline{\Lambda}^n(\pi)$ to the currents $\eta(\boldsymbol{x}, [\boldsymbol{u}]) \in \overline{\Lambda}^{n-1}(\pi)$;
2. to find a way for planting the currents $\eta(\boldsymbol{x}, [\boldsymbol{u}]) \in \overline{\Lambda}^{n-1}(\pi)$ from $\eta_0(\boldsymbol{x}) \in \Lambda^{n-1}(M^n) \hookrightarrow \overline{\Lambda}^{n-1}(\pi)$.

For indeed, if we obtain a p-form $h(\boldsymbol{x})\,\mathrm{d}^p\boldsymbol{x}$ with $p < n$ at some stage of our reasoning, then it is $\overline{\mathrm{d}}$-exact by the usual Poincaré lemma. We fulfill the first task by re-introducing the Koszul differential s, and then we perform the homotopy which reinstates the dependence on $[\boldsymbol{u}]$, thus accomplishing the other task.

We now aim at the construction of the operator s that yields the homotopy $\mathsf{s} \circ \overline{\mathrm{d}} + \overline{\mathrm{d}} \circ \mathsf{s} = \mathbf{1}$ between the last two terms in the zeroth column $0 \to \mathbb{R} \hookrightarrow \mathcal{F}(\pi) \to$

$\ldots \to \overline{\Lambda}^n(\pi) \to 0$ of Diagram (1.5). The following reasoning is not canonical, its result is non-unique and depends on the choice of local coordinates (and the volume form) so that at the end of the day one glues the objects by the partition of unity. Again, we assume that the topology and the choice of admissible classes of sections are such that the integration by parts makes sense.

First let us describe an auxiliary algebraic structure. Instead of the spaces $\mathcal{C}\mathrm{Diff}\bigl(\mathcal{F}(\pi)\text{-module}, \overline{\Lambda}^p(\pi)\bigr)$ of form-valued total differential operators[15]

$$\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\cdot\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma},$$

we consider an isomorphic (as vector spaces) algebra $\mathfrak{A}$ generated by

- n even symbols $D_1, \ldots, D_n$ which identically correspond to the n total derivatives $\mathrm{d}/\mathrm{d}x^1, \ldots, \mathrm{d}/\mathrm{d}x^n$ on $J^\infty(\pi)$ under the isomorphism, and also by
- n odd symbols $\xi_1, \ldots, \xi_n$ which anticommute between themselves and which are the *placeholders* for the n one-forms $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$, that is, the odd symbols ξ_i indicate the *absence* of the respective differentials $\mathrm{d}x^i$:

$$\xi_{j_1}\cdot\ldots\cdot\xi_{j_{n-p}}\cdot D_\sigma \simeq \pm\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p}\cdot\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}, \tag{4.10}$$

where

$$\{j_1,\ldots,j_{n-p}\}\cup\{i_1,\ldots,i_p\} = \{1,2,\ldots,n\}.$$

(In fact, we have introduced the Hodge structure $*$; the proper choice of the signs in the isomorphism (4.10) is prescribed by the requirement that the operator $\overline{\mathrm{d}}$ is a derivation and a differential; this will be discussed later, c.f. Problem 4.6.)

The interpretation of the horizontal differential $\overline{\mathrm{d}}$ on $\mathcal{C}\mathrm{Diff}\bigl(\cdot, \overline{\Lambda}^p(\pi)\bigr)$ in the new language of the algebra $\mathfrak{A}$ is as follows. On one side of the isomorphism, we have

$$\mathcal{C}\mathrm{Diff}\bigl(\cdot,\overline{\Lambda}^p(\pi)\bigr) \xrightarrow{\overline{\mathrm{d}}} \mathcal{C}\mathrm{Diff}(\cdot,\overline{\Lambda}^{p+1}(\pi)),$$

which is given by the formula

$$\overline{\mathrm{d}}\left(\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\cdot\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}\right) = \sum_{i=1}^n \mathrm{d}x^i\wedge\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_p}\cdot\frac{\mathrm{d}^{|\sigma|+1}}{\mathrm{d}\boldsymbol{x}^{\sigma+1_i}}.$$

At the same time, in the algebra $\mathfrak{A}$ the operator $\overline{\mathrm{d}}$ remains a derivation (i. e., it works by the Leibniz rule) and a differential (for this, let us bear in mind that $\overline{\mathrm{d}}(\mathrm{d}x^\alpha)\equiv 0$). The understanding of ξ_i's as placeholders for the absent $\mathrm{d}x^i$'s implies that $\overline{\mathrm{d}}$ on $\mathfrak{A}$ is the odd derivation whose action on the generators is

$$\overline{\mathrm{d}}(\xi_i) = D_i, \qquad \overline{\mathrm{d}}(D_i) = 0.$$

Note that the disappearance of the placeholder ξ_i for $\mathrm{d}x^i$ in the image of ξ_i under $\overline{\mathrm{d}}$ means that the differentials $\mathrm{d}x^i$ and the total derivatives $\mathrm{d}/\mathrm{d}x^i$ emerge in pairs, which must be expected for $\overline{\mathrm{d}} = \sum_{i=1}^n \mathrm{d}x^i\cdot\mathrm{d}/\mathrm{d}x^i$ on $J^\infty(\pi)$. In other words, the placeholder

[15]The module which we shall actually use is $\varkappa(\pi)$; such operators will be provided by taking the linearizations $\ell^{(\boldsymbol{u})}_{\langle F,\cdot\rangle}$.

withdraws when the owner of that place returns, but also *vice versa*. Namely, let us define another odd operator on $\mathfrak{A}$,

$$\mathsf{s}(D_i) = \xi_i, \qquad \mathsf{s}(\xi_i) = 0,$$

for which we postulate the Leibniz rule. This odd derivation is not quite, but almost what we need to build the homotopy.

Exercise 4.5. Show that

$$(\mathsf{s} \circ \overline{\mathrm{d}})(\xi_\tau \cdot D_\sigma) = (|\tau| + |\sigma|)\, \xi_\tau \cdot D_\sigma,$$

which means that the derivation $[\mathsf{s}, \overline{\mathrm{d}}]_+$ is the *weight counting operator* (instead of the desired identity mapping).

Finally, we put

$$\bar{\mathsf{s}}(\xi_\tau \cdot D_\sigma) \stackrel{\text{def}}{=} \frac{\pm}{|\tau| + |\sigma|}\, \mathsf{s}(\xi_\tau \cdot D_\sigma).$$

Because we shall apply this formula only to the upper-left corner of bi-complex (1.5), where $|\tau| = 0$, we again postpone the discussion about the proper choice of the signs. In the meantime, we constatate that $\bar{\mathsf{s}}$ generally stops being a derivation after the division by $|\tau| + |\sigma|$.

We now write the homotopy formula $\psi_\eta \mapsto \eta$ for conserved currents on a given differential equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$. Let ψ_η be a relevant generating section (i. e., suppose η exists for a solution of (4.2)), and set

$$\omega = \langle \psi_\eta, \boldsymbol{F} \rangle = \psi_\eta \boldsymbol{F}\, \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^n \in \overline{\Lambda}^n(\pi).$$

By definition, we have that $\overline{\mathrm{d}}(\eta) = \nabla(\boldsymbol{F})$ and $\psi_\eta = \nabla^\dagger(1)$. We now project to the highest horizontal cohomology $\overline{H}^n(\pi)$ — that is, the integration by parts is declared admissible — and take *any* $\varphi \in \varkappa(\pi)$, c. f. section 4.2.1. We have that

$$\partial_\varphi^{(\boldsymbol{u})}(\overline{\mathrm{d}}\eta) \cong \partial_\varphi^{(\boldsymbol{u})}(\langle \psi_\eta, \boldsymbol{F} \rangle) = \langle 1, \ell_\omega^{(\boldsymbol{u})}(\varphi) \rangle = \langle \underline{\ell_\omega^{(\boldsymbol{u})\dagger}(1)}, \varphi \rangle + \overline{\mathrm{d}}\big(G(\ell_\omega^{(\boldsymbol{u})})(\varphi)\big),$$

where the underlined term vanishes for $\omega \cong \overline{\mathrm{d}}\eta$ in view of Exercise 4.2 and Problem 1.10, and where the avatar G of the Koszul differential (here the notation G stands for Green's formula for integration by parts) is

$$G\colon \mathcal{C}\mathrm{Diff}\big(\varkappa(\pi), \overline{\Lambda}^n(\pi)\big) \to \mathrm{Hom}\big(\varkappa(\pi), \overline{\Lambda}^{n-1}(\pi)\big),$$
$$G\Big(\sum_{|\sigma| \geqslant 0} a_\sigma^i \mathrm{d}\boldsymbol{x} \cdot \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}\Big) = \sum_{|\sigma|>0} \sum_{j \in \sigma} \frac{(-)^{|\sigma|-1}}{|\sigma|} \frac{\mathrm{d}^{|\sigma|-1}}{\mathrm{d}\boldsymbol{x}^{\sigma - 1_j}}(a_\sigma^i) \cdot \omega_{(-j)},$$

where

$$\omega_{(-j)} = (-)^{j+1} \mathrm{d}x^1 \wedge \ldots \wedge \widehat{\mathrm{d}x^j} \wedge \ldots \wedge \mathrm{d}x^n$$

and the exclusion $\sigma - 1_j$ is the subtraction of unit from the strictly positive σ_j in the multi-index $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Now it is almost obvious what evolutionary vector field on $J^\infty(\pi)$ one should take in order to contract the fibres of π_∞. Namely, take $\varphi = \boldsymbol{u}$ so that $\partial_{\boldsymbol{u}}^{(\boldsymbol{u})}$ is the dilation whose flow is

$$A_\varepsilon\colon (\boldsymbol{x}, \boldsymbol{u}_\sigma) \longmapsto (\boldsymbol{x}, e^\varepsilon \boldsymbol{u}_\sigma), \qquad |\sigma| \geqslant 0,$$

and let $\varepsilon \in (-\infty, 0]$. Note that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} A^*_\varepsilon(\omega) = A^*_\varepsilon\left(\partial^{(\boldsymbol{u})}_\varphi(\omega)\right) = A^*_\varepsilon\left(\ell^{(\boldsymbol{u})}_\omega(\varphi)\right)$$

and recall from the previous paragraph that $\ell^{(\boldsymbol{u})}_\omega(\varphi) = \overline{\mathrm{d}}\left(G(\ell^{(\boldsymbol{u})}_\omega)(\varphi)\right)$. We finally obtain

$$\langle \psi_\eta, \boldsymbol{F}\rangle - \underline{(\psi_\eta \boldsymbol{F})(\boldsymbol{x}, [0\cdot \boldsymbol{u}])\mathrm{d}\boldsymbol{x}} = A^*_0(\omega) - A^*_{-\infty}(\omega) = \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(A^*_\varepsilon(\omega)\right)\mathrm{d}\varepsilon =$$
$$= \overline{\mathrm{d}} \int_0^1 G\left(\ell^{(\boldsymbol{u})}_{\langle\psi_\eta,\boldsymbol{F}\rangle}\right)(\boldsymbol{u})\Big|_{(\boldsymbol{x},[\lambda\boldsymbol{u}])} \frac{\mathrm{d}\lambda}{\lambda} = \overline{\mathrm{d}}(\eta + \ldots). \quad (4.11)$$

The underlined top-form is locally exact by the usual Poincaré lemma for $\Lambda^n(M^n)$ and hence it contributes with trivial terms to the current η, but this is of course inessential. In conclusion, the last line of (4.11) establishes the mapping $\psi_\eta \mapsto \eta$ of generating sections to conserved currents, which solves the reconstruction problem.

Problem 4.1. In a complete parallel with Lecture 3, introduce the definition of the *improper* conserved currents which vanish on-shell and analyse the properties of their generating sections.

The globally-defined non-trivial but improper conserved currents may still be interesting,[109] e. g., if they correspond to the Noether gauge symmetries of the Euler–Lagrange models (see Lectures 5 and 6).

Problem 4.2. Express the (i,j)-th entry of the adjoint operator $A^\dagger$ for a matrix, linear total differential operator $A = \| \sum\limits_\sigma a^{ij}_\sigma \cdot \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma} \|^{j=1,\ldots,q}_{i=1,\ldots,p}$, where $a^{ij}_\sigma \in \mathcal{F}(\pi)$.

Problem 4.3. Prove that a scalar evolution equation $u_t = u_{2\lambda} + \ldots$ of even differential order 2λ upon $u(t,x)$ can not admit arbitrarily high order ($k \gg 2\lambda$) solutions $\psi = \psi(t,x,u,\ldots,u_k)$ of the determining equation (4.2) and hence such evolution equation can not have any nontrivial conserved currents of differential orders $\gg \lambda$.
- Find the estimate upon $\max k$ such that $\partial\psi/\partial u_k \not\equiv 0$.

Problem 4.4. Find the generating function of the conservation law $\int \eta = \int w\,\mathrm{d}x$ for $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$; here $w = u_x^2 - u_{xx}$.

Problem 4.5. Show that the linearization $\ell^{(u)}_F$ for $\mathcal{E}_{\text{Liou}} = \{F = u_{xy} - \exp(2u) = 0\}$ is self-adjoint.
- Prove that $\psi = (u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x})\, f(x) + (u_y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y})\, g(y)$ is a cosymmetry for $\mathcal{E}_{\text{Liou}}$ for all smooth functions f and g.
- Reconstruct a conserved current η for $\mathcal{E}_{\text{Liou}}$ if f is given but $g \equiv 0$ and if g is given and $f \equiv 0$. (Have you encountered the resulting quantities before?)

Problem 4.6. Establish the proper choice of the signs in the correspondence

$$\xi_{j_1}\cdot \ldots \cdot \xi_{j_{n-q}} \cdot D_\sigma \simeq \pm dx^{i_1} \wedge \ldots \wedge dx^{i_q} \cdot D_\sigma$$

(so that the differential $\overline{\mathrm{d}}$ is indeed the one).

Problem 4.7. Reconstruct the Lagrangian of the Liouville equation.

Problem 4.8. Find at least five inequivalent conservation laws for the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$.

Problem 4.9 ([55]). Find the classical symmetries of the equation $u_{xy} = \exp(-u_{zz})$.
- Find the (classes of) solutions which stay invariant with respect to these symmetries.
- Find five conservation laws for the equation $u_{xy} = \exp(-u_{zz})$.

Problem 4.10. Show that the *heavenly equation*[106] $u_{xyzz} = (\exp(-u_{zz}))_{zz}$ is Euler–Lagrange.
- Which of its classical symmetries preserve its Lagrangian?

5. Euler–Lagrange equations

The hour has come when we bring together the concepts of symmetry and conservation: this fundamental relation of Nature is expressed in the First and the Second Noether Theorem, both of which we will have proven by the end of the next lecture.

Let $\mathcal{L} \in \overline{H}^n(\pi)$ be a Lagrangian functional. We recall that the map

$$\begin{array}{rcl}\mathcal{L}\colon \Gamma(\pi) & \to & \Bbbk, \\ \boldsymbol{s} & \mapsto & \int_{M^n} j_\infty(\boldsymbol{s})^* \mathcal{L}\end{array}$$

takes each section of the bundle π to the ground field $\Bbbk$ (or to a noncommutative associative algebra, see Lectures 8–9). The value $\mathcal{L}(\boldsymbol{s})$ can be used further, for instance to determine the contribution $\exp\left(\frac{i}{\hbar}\mathcal{L}(\boldsymbol{s})\right)$ of the section $\boldsymbol{s}$ to the path integral over the space of sections (possibly satisfying given boundary conditions).

In the meantime, we are interested in finding the stationary points of the functional $\mathcal{L}$. Because the derivation of the Euler–Lagrange equations is standard, let us focus on the geometric origin of this procedure. Consider diagram (1.6); we notice[16] that it remains in our power to pick the representatives of the cohomology classes in the image of the restriction $:\!\mathrm{d}_\mathcal{C}\!:$ of the Cartan differential $\mathrm{d}_\mathcal{C}$ to the highest horizontal cohomology $\overline{H}^n(\pi)$. Let us establish the following convention: we integrate by parts in the image of $\mathrm{d}_\mathcal{C}$ and, by using another convention —the total derivatives (hence $\overline{\mathrm{d}}$) act on the Cartan forms via the Lie derivative so that $\mathrm{d}_\mathcal{C}\boldsymbol{u}_\sigma = \mathrm{d}_\mathcal{C}\left(\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}(\boldsymbol{u})\right) = \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}(\mathrm{d}_\mathcal{C}\boldsymbol{u}_\varnothing)$— we transform the Cartan differentials $\mathrm{d}_\mathcal{C}\boldsymbol{u}_\sigma$ of the jet variables $\boldsymbol{u}_\sigma$ to the differentials $\mathrm{d}_\mathcal{C}\boldsymbol{u}_\varnothing$ of the m fibre variables $u^1, \ldots, u^m$ in the bundle π. This convention yields the *variation* $\delta = :\!\mathrm{d}_\mathcal{C}\!:$,

$$\overrightarrow{\delta}\mathcal{L} = \frac{\overrightarrow{\delta\mathcal{L}}}{\delta\boldsymbol{u}} \cdot \delta\boldsymbol{u},$$

where the arrow indicates the direction along which the differentials $\delta\boldsymbol{u}$ are transported. (This indication matters in the $\mathbb{Z}_2$-graded and non-commutative theories, see Part II of this course.)

Exercise 5.1. Show that

$$\frac{\overrightarrow{\delta\mathcal{L}}}{\delta\boldsymbol{u}} = \sum_\sigma (-)^\sigma \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}\left(\frac{\overrightarrow{\delta L}}{\delta\boldsymbol{u}_\sigma}\right), \qquad \text{where } \mathcal{L} = \int L(\boldsymbol{x}, [\boldsymbol{u}])\, \mathrm{d}^n\boldsymbol{x}.$$

Exercise 5.2. Establish the isomorphism

$$E_0^{n,1} = \mathcal{C}^1\Lambda(\pi) \otimes_{\mathcal{F}(\pi)} \overline{\Lambda}^n(\pi) \simeq \mathcal{C}\mathrm{Diff}\left(\varkappa(\pi), \overline{\Lambda}^n(\pi)\right).$$

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[16] Variational bi-complex (1.5) is the zeroth term $\{E_0^{p,q} | 0 \leqslant p \leqslant n,\ q \in \mathbb{N}\cup\{0, -\infty\}\}$ of Vinogradov's $\mathcal{C}$-spectral sequence; Diagram (1.6) with $\{E_1^{p,q} | 0 \leqslant p \leqslant n,\ q \geqslant 0\}$ its first term so that, in particular, $\overline{H}^n(\pi) = E_1^{n,0}$ and $\widehat{\varkappa(\pi)} \simeq E_1^{n,1}$. The modules

$$P_0 = P \ni \boldsymbol{F}, \quad P_1 \ni \boldsymbol{\Phi}, \quad P_2, \quad \ldots, \quad P_\lambda,$$

containing the equations, the Noether identities, and higher generations of the syzygies determine how fast the spectral sequence's cohomology groups $E_{r+1}^{p,q}$ with respect to differentials $\mathrm{d}_r^{p,q}$ at the points (p, q) converge to the limit $E_\infty^{p,q}$ (here $\mathrm{d}_0^{p,q} = \overline{\mathrm{d}}$ and $\mathrm{d}_1^{p,q} = :\!\mathrm{d}_\mathcal{C}\!:$).

Exercise 5.3. Verify the horizontal cohomology class equivalence

$$\left\langle \psi\, \mathrm{d}_{\mathcal{C}} \boldsymbol{u}_\sigma, \partial_\varphi^{(\boldsymbol{u})} \right\rangle \cong \left\langle :\psi\, \mathrm{d}_{\mathcal{C}} \boldsymbol{u}_\sigma:, \partial_\varphi^{(\boldsymbol{u})} \right\rangle = \Big\langle \sum_\sigma (-)^\sigma \tfrac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}(\psi), \varphi \Big\rangle. \tag{5.1}$$

Remark 5.1. As soon as the integration by parts is allowed, the term $E_1^{n,1}$ acquires the structure of the $\mathcal{F}(\pi)$-module $\widehat{\varkappa(\pi)} = \mathrm{Hom}_{\mathcal{F}(\pi)}\big(\varkappa(\pi), \overline{\Lambda}^n(\pi)\big)$ which is $\langle\,,\,\rangle$-dual to $\varkappa(\pi)$.

Omitting the differentials $\mathrm{d}_{\mathcal{C}}\boldsymbol{u}$ and the derivatives $\partial/\partial\boldsymbol{u}$ in their coupling (5.1), we conclude that the stationary-point condition $\delta\mathcal{L}(\boldsymbol{s}) = 0$ upon the sections $\boldsymbol{s} \in \Gamma(\pi)$ amounts to the system of the Euler–Lagrange equations

$$\mathcal{E}_{\mathrm{EL}} = \big\{ \boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0 \mid \mathcal{L} \in \overline{H}^n(\pi) \big\} \subseteq J^k(\pi). \tag{5.2}$$

This also reveals the important isomorphism $P \simeq \widehat{\varkappa(\pi)}$ for the module of left-hand sides of Euler–Lagrange equations $\boldsymbol{F} \in P$ (c. f. Remark 2.3 on p. 21).

Remark 5.2. The isomorphism $P \simeq \widehat{\varkappa(\pi)}$ remains, although it stops being canonical, under those reparametrisations $\widetilde{\boldsymbol{F}} = \widetilde{\boldsymbol{F}}[\boldsymbol{F}]$ of the equations $\boldsymbol{F} = 0$ which are *not* induced by a change $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}[\boldsymbol{u}]$ of the unknowns. In principle, such uncorrelated procedures are possible, but they are *not geometric*.

Likewise, homotopy formula (4.9) on p. 49,

$$\boldsymbol{F}\,\delta\boldsymbol{u} = F_i(\boldsymbol{x}, [\boldsymbol{u}])\, \mathrm{d}_{\mathcal{C}} u^i - F_i(\boldsymbol{x}, [0\cdot\boldsymbol{u}])\, \mathrm{d}_{\mathcal{C}}(0\cdot u^i) = \overrightarrow{\delta} \int \mathrm{d}^n\boldsymbol{x} \int_0^1 F_i(\boldsymbol{x}, [\lambda\boldsymbol{u}]) u^i\, \mathrm{d}\lambda,$$

is fragile and fails to reconstruct the Lagrangian $\mathcal{L} = \int \mathrm{d}^n\boldsymbol{x} \int_0^1 F_i(\boldsymbol{x}, [\lambda\boldsymbol{u}]) u^i\, \mathrm{d}\lambda$ if the canonical transcript $\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u}$ of the Euler–Lagrange equations is lost.

Therefore, we need a working criterion that allows us to check whether a given system $\mathcal{E} = \{\boldsymbol{F} = 0\}$ has the form $\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u}$ for some $\mathcal{L} \in \overline{H}^n(\pi)$.

Exercise 5.4. Suppose that all arguments θ^∞ are restricted to a star-shaped domain in $J^\infty(\pi)$. From the identity $\mathrm{d}_{\mathcal{C}}^2 = 0$ deduce the (very convenient) condition $\mathrm{d}_{\mathcal{C}}(\nabla(\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_\varnothing)) \overset{?}{=} 0$ which verifies whether a given total differential operator ∇ is in fact the linearisation of some element f in an $\mathcal{F}(\pi)$-module: $\nabla = \ell_f^{(\boldsymbol{u})}$.

Theorem 5.1 (Helmholtz)**.** *The following two statements are equivalent*:

- *The cohomology group $E_2^{n,1}$ vanishes with respect to the restriction* $:\mathrm{d}_{\mathcal{C}}:$ *of the Cartan differential* $\mathrm{d}_{\mathcal{C}}$ *to the horizontal cohomology groups in the upper line of Diagram* (1.6);
- *The left-hand side $\boldsymbol{F} \in \widehat{\varkappa(\pi)}$ of the equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ has the form $\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u}$ for some $\mathcal{L} \in \overline{H}^n(\pi)$ if and only if the linearisation of $\boldsymbol{F}$ with respect to the unknowns $\boldsymbol{u}$ is self-adjoint*:

$$\boldsymbol{F} = \frac{\delta\mathcal{L}}{\delta\boldsymbol{u}} \qquad \Longleftrightarrow \qquad \ell_{\boldsymbol{F}}^{(\boldsymbol{u})} = \ell_{\boldsymbol{F}}^{(\boldsymbol{u})\,\dagger}.$$

(Note that $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})} \colon \varkappa(\pi) \to P \simeq \widehat{\varkappa(\pi)}$ and $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger} \colon \widehat{P} = \mathrm{Hom}_{\mathcal{F}(\pi)}\big(P, \overline{\Lambda}^n(\pi)\big) \simeq \varkappa(\pi) \to \widehat{\varkappa(\pi)} \simeq P$ so that the operator equality makes sense.) In fact, the second clause of the Helmholtz criterion is a reformulation of the empty-cohomology first statement.

Proof. The Cartan differential $\mathrm{d}_{\mathcal{C}}$ maps

$$\int \boldsymbol{F}\,\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\,\mathrm{d}^n\boldsymbol{x} \longmapsto \int \mathrm{d}_{\mathcal{C}}\boldsymbol{F}\wedge \mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\,\mathrm{d}^n\boldsymbol{x} = \int \ell^{(\boldsymbol{u})}_{\boldsymbol{F}}(\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing})\wedge \mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\,\mathrm{d}^n\boldsymbol{x},$$

where we define the application of the linearization to the Cartan one-forms (instead of the application to $\varkappa(\pi)$, which was defined earlier) by the rule

$$\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}(\mathrm{d}_{\mathcal{C}}\Delta\boldsymbol{u}) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \mathrm{d}_{\mathcal{C}}\boldsymbol{F}(\boldsymbol{x},[\boldsymbol{u}+\varepsilon\Delta\boldsymbol{u}]).$$

Let us split the first wedge factor into two halves and integrate by parts:

$$\begin{aligned}\mathrm{d}_{\mathcal{C}}\int \boldsymbol{F}\,\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\,\mathrm{d}^n\boldsymbol{x} &\cong \frac{1}{2}\int \Big(\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}(\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing})\wedge \mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing} + \mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\wedge (\ell^{(\boldsymbol{u})}_{\boldsymbol{F}})^{\dagger}(\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing})\Big)\,\mathrm{d}^n\boldsymbol{x}\\ &= \frac{1}{2}\int \Big(\ell^{(\boldsymbol{u})}_{\boldsymbol{F}} - (\ell^{(\boldsymbol{u})}_{\boldsymbol{F}})^{\dagger}\Big)(\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing})\wedge \mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\cdot \mathrm{d}^n\boldsymbol{x}.\end{aligned}$$

If the term $\int \boldsymbol{F}\,\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\mathrm{d}^n\boldsymbol{x} \in E_1^{n,1}$ is exact (that is, if it equals $\delta\mathcal{L}$ for $\mathcal{L}\in \overline{H}^n(\pi)$), then $\mathrm{d}_{\mathcal{C}}(\delta\mathcal{L}) = 0$ because $\mathrm{d}_{\mathcal{C}}^2 = 0$, and the linearization is self-adjoint. Conversely, the vanishing of the cohomology group $E_2^{n,1}$ means the existence of the element $\mathcal{L}\in\overline{H}^n(\pi)$ for each closed element $\int F\,\mathrm{d}_{\mathcal{C}}\boldsymbol{u}_{\varnothing}\cdot \mathrm{d}^n\boldsymbol{x}\in E_1^{n,1}$. □

Remark 5.3. The generating sections φ of symmetries and the generating sections ψ of conservation laws for the Euler–Lagrange equations, written in canonical form (5.2), satisfy the same determining equation on-shell: compare (3.3) and (4.2) in view of Theorem 5.1. Hence each section ψ is a symmetry of the system $\mathcal{E}_{\mathrm{EL}}$, c. f. Remark 4.2.

5.1. **First Noether Theorem.**

Definition 5.1. The evolutionary vector field $\partial^{(\boldsymbol{u})}_{\varphi}$ is a *Noether symmetry* of the Lagrangian $\mathcal{L}\in\overline{H}^n(\pi)$ if the derivation preserves the horizontal cohomology class of the functional:

$$\partial^{(\boldsymbol{u})}_{\varphi}(\mathcal{L}) = \int \bar{\mathrm{d}}\xi \text{ on } J^{\infty}(\pi), \qquad \xi\in\overline{\Lambda}^{n-1}(\pi).$$

Exercise 5.5. Prove that the commutator of two Noether symmetries is again a Noether symmetry, so that they constitute the Lie algebra $\mathrm{sym}\,\mathcal{L}$.

Lemma 5.2. Let $\mathcal{L}\in\overline{H}^n(\pi)$ and $\varphi\in\varkappa(\pi)$. Then

$$\frac{\delta}{\delta\boldsymbol{u}}\left(\partial^{(\boldsymbol{u})}_{\varphi}(\mathcal{L})\right) = \partial^{(\boldsymbol{u})}_{\varphi}\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{u}}\right) + \ell^{(\boldsymbol{u})\dagger}_{\varphi}\left(\frac{\delta\mathcal{L}}{\delta\boldsymbol{u}}\right) \text{ on } J^{\infty}(\pi). \tag{5.3}$$

In particular, if $E_2^{n,0} = 0$, that is, if $\delta\omega = 0$ implies $\omega = \bar{\mathrm{d}}\xi$, $\xi\in\overline{\Lambda}^{n-1}(\pi)$, then the search for the Noether symmetries of a Lagrangian $\mathcal{L}\in\overline{H}^n(\pi)$ can be performed by solving the equation $\frac{\delta}{\delta\boldsymbol{u}}(\partial^{(\boldsymbol{u})}_{\varphi}(\mathcal{L})) = 0$ on $J^{\infty}(\pi)$ for $\varphi\in\varkappa(\pi)$.

Proof. Let $\Delta\in\mathcal{C}\mathrm{Diff}\left(\varkappa(\pi),\overline{\Lambda}^n(\pi)\right)$. Integrating by parts, we obtain that $\Delta(\varphi)\cong \Delta_0(\varphi) + \bar{\mathrm{d}}(\Delta'(\varphi))$, where the differential order of Δ_0 is zero and $\Delta'\in\mathcal{C}\mathrm{Diff}\big(\varkappa(\pi),\overline{\Lambda}^{n-1}(\pi)\big)$. Then the linearization of $\Delta(\varphi)$ is $\ell^{(\boldsymbol{u})}_{\Delta(\varphi)} = \ell^{(\boldsymbol{u})}_{\Delta_0(\varphi)} + \bar{\mathrm{d}}\circ\ell^{(\boldsymbol{u})}_{\Delta'(\varphi)}$. From Exercise 4.2 we have that

$$\tfrac{\delta}{\delta\boldsymbol{u}}(\Delta(\varphi)) = \ell^{(\boldsymbol{u})\dagger}_{\Delta(\varphi)}(1) = \ell^{(\boldsymbol{u})\dagger}_{\Delta_0(\varphi)}(1) - \ell^{(\boldsymbol{u})}_{\Delta'(\varphi)}(\bar{\mathrm{d}}(1)),$$

where $\overline{\mathrm{d}}^{\dagger} = -\overline{\mathrm{d}}$ and $1 \in \widehat{\mathrm{Ber}(\pi)}$. The subtrahend vanishes because $\overline{\mathrm{d}}(1) \equiv 0$, whence by using the Leibniz rule we conclude that

$$\tfrac{\delta}{\delta \boldsymbol{u}}(\Delta(\varphi)) = \ell_{\Delta_0}^{(\boldsymbol{u})\dagger}(\varphi) + \ell_{\varphi}^{(\boldsymbol{u})\,\dagger}(\Delta_0).$$

Finally, let us take $\Delta := \ell_{\mathcal{L}}^{(\boldsymbol{u})}$, which implies that $\Delta_0(\varphi) = \langle \delta\mathcal{L}/\delta\boldsymbol{u}, \varphi\rangle$. Taking into account the Helmholtz criterion

$$\ell_{\delta\mathcal{L}/\delta\boldsymbol{u}}^{(\boldsymbol{u})\dagger} = \ell_{\delta\mathcal{L}/\delta\boldsymbol{u}}^{(\boldsymbol{u})}$$

(see Theorem 5.1), we obtain

$$\tfrac{\delta}{\delta \boldsymbol{u}}(\partial_{\varphi}^{(\boldsymbol{u})}(\mathcal{L})) = \ell_{\delta\mathcal{L}/\delta\boldsymbol{u}}^{(\boldsymbol{u})}(\varphi) + \ell_{\varphi}^{(\boldsymbol{u})\dagger}\left(\delta\mathcal{L}/\delta\boldsymbol{u}\right),$$

whence the assertion follows. □

Corollary 5.3. Each Noether symmetry $\varphi_{\mathcal{L}} \in \operatorname{sym}\mathcal{L}$ of a Lagrangian $\mathcal{L} \in \overline{H}^n(\pi)$ is a symmetry of the corresponding Euler–Lagrange equation $\mathcal{E}_{\mathrm{EL}} = \{\delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$:

$$\operatorname{sym}\mathcal{L} \subseteq \operatorname{sym}\mathcal{E}_{\mathrm{EL}}. \tag{5.4}$$

Indeed, the left hand side in (5.3) is identically zero on $J^{\infty}(\pi)$ because $\varphi_{\mathcal{L}}$ is a Noether symmetry, whereas the last term in the right hand side of the same formula vanishes on-shell because $\ell_{\varphi}^{(\boldsymbol{u})\,\dagger}$ is a total differential operator applied to $\boldsymbol{F}$.

By permitting ourself a minimal abuse of language, we shall say that such a section $\varphi_{\mathcal{L}}$ is a Noether symmetry of the Euler–Lagrange *equation* $\mathcal{E}_{\mathrm{EL}} = \{\delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$ instead of the rigorous attribution to the Lagrangian *functional* $\mathcal{L}$.

Remark 5.4. Lemma 5.2 explains us why the inverse statement to (5.4) is in general false: If $\partial_{\varphi}^{(\boldsymbol{u})}(\boldsymbol{F}) = \nabla_{\varphi}(\boldsymbol{F})$ on $J^{\infty}(\pi)$ for $\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u}$, the operator ∇_{φ} may not be equal to $\ell_{\varphi}^{(\boldsymbol{u})\,\dagger}$. For example, consider the scaling symmetry of Newton's equations in Kepler's problem of orbital motion.

Remark 5.5. To find the Noether symmetries of an Euler–Lagrange equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ it is not obligatory to know the Lagrangian. It suffices to know only of the existence of that functional. Moreover, the system $\mathcal{E}$ need not be written in the canonical way $\delta\mathcal{L}/\delta\boldsymbol{u} = 0$: We may have that $\mathcal{E}$ is equivalent to it under some reparametrisation of the equations. This justifies a great flexibility in our dealings with the Euler–Lagrange equations. For instance, it becomes legitimate to diagonalise their symbols (if possible).

Theorem 5.4 (First Noether Theorem)**.** *Let* $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0 \in P \simeq \widehat{\varkappa(\pi)} \mid \mathcal{L} \in \overline{H}^n(\pi)\}$ *be a system of Euler–Langrange equations.*

- *A section* $\varphi \in \varkappa(\pi)$ *is a Noether symmetry of the Lagrangian* $\mathcal{L}$ *if and only if* $\varphi \in \varkappa(\pi) \simeq \widehat{P}$ *is the generating section of a conserved current* $\eta \in \overline{\Lambda}^{n-1}(\pi)$ *for the Euler–Lagrange equation* $\mathcal{E}_{\mathrm{EL}}$:

$$\varphi \in \operatorname{sym}\mathcal{L} \quad\Longleftrightarrow\quad \exists\,\eta \in \overline{\Lambda}^{n-1}(\pi) : \overline{\mathrm{d}}\eta = \langle 1, \nabla(\boldsymbol{F})\rangle, \quad \varphi = \nabla^{\dagger}(1),$$

where $\nabla \in \mathcal{C}\mathrm{Diff}\left(P, \overline{\Lambda}^n(\pi)\right)$ *and* $1 \in \widehat{\mathrm{Ber}(\pi)}$.

- *If, moreover, the system $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = 0\}$ is* normal *(that is, it admits no nontrivial syzygies $\boldsymbol{\Phi} \in P_1$: the identity $\boldsymbol{\Phi}(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])]) = 0$ on $J^\infty(\pi)$ forces $\boldsymbol{\Phi} \equiv 0$ and $P_1 = \{0\}$), then the current η is nontrivial and* proper *($\eta\big|_{\mathcal{E}^\infty_{\mathrm{EL}}} \not\equiv 0$) for every $\varphi(\boldsymbol{x}, [\boldsymbol{u}]) \not\equiv 0$.*

Reciprocally, the kernel of the map $\varphi_{\mathcal{L}} \to \eta\big|_{\mathcal{E}^\infty_{\mathrm{EL}}}$ from the space of Noether symmetries $\varphi_{\mathcal{L}}$ of $\mathcal{L}$ to the space of currents on $\mathcal{E}^\infty_{\mathrm{EL}}$ that are conserved by virtue of the equation $\mathcal{E}_{\mathrm{EL}} = \{\delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$ consists of the Noether gauge symmetries *$\varphi_{\mathcal{L}} = A(\boldsymbol{p})$, where $\boldsymbol{p} \in \widehat{P_1}$ and $A\colon \widehat{P_1} \to \varkappa(\pi)$. Stemming from the nontrivial linear Noether identities $\boldsymbol{\Phi} = A^\dagger(\boldsymbol{F}) = 0$ on $J^\infty(\pi)$ between the equations $\boldsymbol{F} = 0$, such symmetries yield the* improper *currents η which vanish on-shell: $\eta\big|_{\mathcal{E}^\infty_{\mathrm{EL}}} \doteq 0$.*

Proof. Let $\eta \in \overline{\Lambda}^{n-1}(\pi)$ be a conserved current (here we do not exclude the possibility that η is trivial or vanishes on-shell):

$$\overline{\mathrm{d}}\eta = \langle 1, \nabla(\boldsymbol{F})\rangle = \langle \nabla^\dagger(1), \boldsymbol{F}\rangle + \overline{\mathrm{d}}\gamma \text{ on } J^\infty(\pi), \qquad \gamma \in \overline{\Lambda}^{n-1}(\pi).$$

Denote by φ the generating section $\nabla^\dagger(1) \in \widehat{P}$ of the current η. Recalling that $P \simeq \widehat{\varkappa(\pi)}$ and $\widehat{P} \simeq \varkappa(\pi)$ for the Euler–Lagrange equation $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$, we swap the terms, $\langle\varphi, \boldsymbol{F}\rangle = \langle\boldsymbol{F}, \varphi\rangle$ so that now the variational covector $\langle \int \boldsymbol{F}\, \mathrm{d}_{\mathcal{C}}\boldsymbol{u} \cdot \mathrm{d}^n\boldsymbol{x}|$ stands on the left and the vector $|\partial^{(\boldsymbol{u})}_\varphi\rangle$ is on the right. From Exercise 4.2 we deduce that

$$\langle \boldsymbol{F}, \varphi\rangle = \Big\langle \ell^{(\boldsymbol{u})\dagger}_{\mathcal{L}}(1), \varphi\Big\rangle \cong \Big\langle 1, \ell^{(\boldsymbol{u})}_{\mathcal{L}}(\varphi)\Big\rangle + \overline{\mathrm{d}}\beta,$$

where $1 \in \widehat{\mathrm{Ber}(\pi)}$ and $\beta \in \overline{\Lambda}^{n-1}(\pi)$. Inspecting the equalities back to the starting point, we conclude that $\partial^{(\boldsymbol{u})}_\varphi(\mathcal{L}) = \overline{\mathrm{d}}(\eta - \gamma - \beta)$ on $J^\infty(\pi)$, that is, φ is a Noether symmetry of $\mathcal{L}$. By reading the formulas backwards, we establish for $\varphi \in \operatorname{sym}\mathcal{L}$ the existence of the current $(\eta - \gamma) \in \overline{\Lambda}^{n-1}(\pi)$ which is conserved (at least) on-shell so that $\overline{\mathrm{d}}(\eta - \gamma) = \langle\boldsymbol{F}, \varphi\rangle$.

The second statement of the First Noether Theorem heralds the Second Noether Theorem and reveals the following property of the current $\eta - \gamma$: it can be *improper*, that is, vanish identically[17] at all points of the infinite prolongation $\mathcal{E}^\infty_{\mathrm{EL}}$.

Namely, suppose that $A^\dagger(\boldsymbol{F}) = 0 \in P_1$ on $J^\infty(\pi)$ is a *linear* Noether identity between the equations $\boldsymbol{F} = 0$ (this is indeed the case whenever $\varphi = A(\boldsymbol{p}) \in \varkappa(\pi)$ is a Noether symmetry of the Lagrangian $\mathcal{L}$ for every $\boldsymbol{p} \in \widehat{P_1}$; we show this in the next lecture). Then

$$\overline{\mathrm{d}}(\eta - \gamma) = \langle\boldsymbol{F}, A(\boldsymbol{p})\rangle \cong \langle\boldsymbol{p}, A^\dagger(\boldsymbol{F})\rangle + \overline{\mathrm{d}}\lambda = \overline{\mathrm{d}}\lambda \text{ on } J^\infty(\pi)$$

because $A^\dagger(\boldsymbol{F}) = 0$ on $J^\infty(\pi)$. Therefore, $\eta - \gamma = \lambda + \overline{\mathrm{d}}\xi$ for some $\xi \in \overline{\Lambda}^{n-2}(\pi)$. But let us notice that the current λ, which emerged during the integration by parts, $\langle\boldsymbol{F}, A(\boldsymbol{p})\rangle \cong \langle\boldsymbol{p}, A^\dagger(\boldsymbol{F})\rangle$, does contain $\boldsymbol{F}$ or its descendants (3.2) inside all the terms of all its components. In other words, the current λ, which can only differ from the sought-for current $\eta - \gamma$ by a trivial element $\overline{\mathrm{d}}\xi$, identically vanishes on $\mathcal{E}^\infty$. The restriction

[17] We recall that we had to quotient out the improper *symmetries* of equations by passing to the internal coordinates.

$j_\infty(\boldsymbol{s})^*\lambda_i$ of each component λ_i in $\lambda = \sum\limits_{i=1}^{n} \lambda_i \, \mathrm{d}x^i$ equals zero for every solution $\boldsymbol{s}$ of the system $\mathcal{E}_{EL}$. □

Remark 5.6. This paradoxical 'invisibility' of the improper currents did in fact lead to the understanding of gauge symmetries.[109] We shall consider them in the next lecture.

Problem 5.1. Show that the operator $\nabla = xu_x^2 + 2xuu_x\frac{d}{dx}$ is the linearization of some f, $\nabla = \ell_f^{(u)}$, and find that f.

Problem 5.2. Show that the equations

$$m \cdot \tfrac{\mathrm{d}}{\mathrm{d}t}(r^2 \cdot \dot{\phi}) = 0, \qquad m\ddot{r} - mr\dot{\phi}^2 + \alpha/r^2 = 0$$

upon $\phi(t)$ and $r(t)$ are Euler–Lagrange, and reconstruct the Lagrangian.

Problem 5.3. Prove that the two-component Toda systems

$$\mathcal{E}_{\text{Toda}} = \{u^i_{xy} = \exp(K^i_j u^j) \mid 1 \leqslant i, j \leqslant 2\}$$

are Euler–Lagrange if K is the Cartan matrix of one of the rank two semi-simple Lie algebras A_2, $B_2 \simeq C_2$, $D_2 \simeq A_1 \oplus A_1$, or G_2. Find the Lagrangians of those systems. Are the linearisations $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}$ of the left-hand sides of these equations $\mathcal{E} = \{\boldsymbol{F} = 0\}$ always self-adjoint ?

Problem 5.4 ([62])**.** Prove that $\varphi_{\mathcal{L}} = \bigl(u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\bigr)\bigl(\delta\mathcal{H}(x,[w])/\delta w\bigr)$, where $w = u_x^2 - u_{xx}$, is a Noether symmetry of the Liouville equation $u_{xy} = \exp(2u)$ for each $\mathcal{H}$, and identify the bundles which are involved in this construction.

Problem 5.5. Establish the conservation of energy, angular momentum, and the Runge–Lentz vector for Newton's equations of motion in the central potential $V(r) = -\alpha/r$, $\alpha = \text{const}$.

- Find the respective contact Noether symmetries; is the scaling-invariance a Noether symmetry ?

Problem 5.6. Find five classical Noether symmetries of the heavenly equation[106]

$$u_{xyzz} = \tfrac{\mathrm{d}^2}{\mathrm{d}z^2}\bigl(\exp(-u_{zz})\bigr).$$

Problem 5.7. Prove Ibragimov's identity:

$$\partial_\varphi^{(\boldsymbol{u})} = \varphi \cdot \frac{\delta}{\delta \boldsymbol{u}} + \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}x^i} \circ Q_{\varphi,i},$$

where

$$Q_{\varphi,i} = \sum_{\tau} \sum_{\rho+\sigma=\tau} (-1)^\sigma \frac{\mathrm{d}^{|\rho|}}{\mathrm{d}\boldsymbol{x}^\rho}(\varphi) \cdot \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma} \circ \frac{\partial}{\partial \boldsymbol{u}_{\tau+1_i}}.$$

6. Gauge symmetries

We recall that every Euler–Lagrange system $\mathcal{E}_{\mathrm{EL}} = \{F_i := \delta\mathcal{L}/\delta q^i = 0 \mid \mathcal{L} \in \overline{H}^n(\pi)\}$ contains as many equations as there are unknowns q^i in it. In Remark 2.3 on p. 21 we pointed out that *usually* the equations in Euler–Lagrange systems are conveniently enumerated (more precisely, labelled) by the respective unknowns q^i which explicitly occur in the left-hand sides; let us remember that the horizontal module $P_0 = \Gamma(\pi_\infty^*(\xi))$ of the sections F is then $P_0 \simeq \widehat{\varkappa(\pi)}$. Because of this, the generating sections $\psi \in \widehat{P_0}$ of conservation laws for $\mathcal{E}_{\mathrm{EL}}$ acquire the nature of Noether symmetries $\varphi_{\mathcal{L}} \in \operatorname{sym}\mathcal{E}_{\mathrm{EL}}$, which is indeed the case by virtue of the First Noether Theorem (up to, possibly, the non-identical Noether maps $\widehat{P_0} \to \varkappa(\pi)$, see [56, 62]).

However, we emphasize that the admissible reparametrizations of the fields $\boldsymbol{q}$ and of the equations $\boldsymbol{F} = 0$ are entirely unrelated. This produces its due effect on the transcription of the infinitesimal symmetries $\dot{\boldsymbol{q}} = \varphi\bigl(\boldsymbol{x}, [\boldsymbol{q}]\bigr)$ in the former case and, in the latter, on the objects that lie in or are dual to the horizontal module $P_0 \ni \boldsymbol{F}$ of the equations, e. g., on the generating sections ψ of conservation laws or the objects such as the antifields $\boldsymbol{q}^\dagger$ (thus, more appropriately denoted by $\boldsymbol{F}^\dagger$) or the ghosts $\boldsymbol{\gamma}$ and the antighosts $\boldsymbol{\gamma}^\dagger$ (we shall address them in Lecture 11). The 2D Toda chains (see Problem 5.3) offer us an example of such discorrelation between $\varphi_{\mathcal{L}}$ and ψ, which is brought in by force due to a purely æsthetic tradition of writing hyperbolic systems with their symbols cast into diagonal shape.

Although the Euler–Lagrange systems $\mathcal{E}_{\mathrm{EL}}$ are never overdetermined, there may appear the differential constraints (also called *syzygies* or the *Noether identities*) between the equations of motion,

$$\boldsymbol{\Phi}\bigl(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])]\bigr) \equiv 0 \text{ on } J^\infty(\pi), \qquad \boldsymbol{\Phi} \in P_1 = \Gamma\bigl(\pi_\infty^*(\xi_1)\bigr).$$

The relations $\boldsymbol{\Phi}_{i+1}\bigl(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}], \ldots, [\boldsymbol{\Phi}_i]\bigr) \equiv 0$ between the relations, valid identically for all $\boldsymbol{x}$, $\boldsymbol{u}$, ..., $\boldsymbol{\Phi}_{i-1}$ (here $\boldsymbol{\Phi}_0 = \boldsymbol{F} \in P_0$) give rise to possibly several but finitely many generations of the horizontal modules $P_i \ni \boldsymbol{\Phi}_i$ for $i > 0$. In this course, let us assume that the given action functional $\mathcal{L} \in \overline{H}^n(\pi)$ determines the system $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = 0\}$ of equations of motion with one generation of the constraints $\boldsymbol{\Phi}\bigl(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}]\bigr) = 0$ and that there are no further relations between the already known ones. For example, such is the case of the Maxwell equations or, more generally, Yang–Mills equations for which only the modules P_0 and P_1 are nontrivial (see below).

Theorem 6.1 (Second Noether Theorem). *Let $\mathcal{E}_{\mathrm{EL}} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0 \mid \mathcal{L} \in \overline{H}^n(\pi)\}$ be an Euler–Lagrange system that admits the Noether identities $\boldsymbol{\Phi} = {}^{\mathrm{t}}\bigl(\Phi^1, \ldots, \Phi^\lambda\bigr) \in P_1$ such that*

$$\Phi^i\bigl(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])]\bigr) = 0 \quad \text{on } J^\infty(\pi), \qquad 1 \leqslant i \leqslant \lambda. \tag{6.1}$$

Then relations (6.1) *between the equations $F_j = 0$, $1 \leqslant j \leqslant m$, yield the linear total differential operator $A\colon \widehat{P_1} \to \operatorname{sym}\mathcal{E}_{\mathrm{EL}}$ that generates the symmetries of the system $\mathcal{E}_{\mathrm{EL}}$.*

> By definition, an Euler–Lagrange system of differential equations which admits a nontrivial Noether identity is called a *gauge system*. Its symmetries of the form $\varphi = A_i(\boldsymbol{p}_i)$ are *gauge symmetries*. The module $\widehat{P_1} \ni \boldsymbol{p}_i$

is then constituted by the field-dependent *gauge parameters* $\boldsymbol{p}_i\bigl(\boldsymbol{x},[\boldsymbol{u}]\bigr)$, which are the even neighbours of the odd *ghosts* $\boldsymbol{b}_i$ (see Lecture 11).

Suppose further that the gauge symmetry generator $A \in \mathcal{C}\mathrm{Diff}(\widehat{P_1}, \mathrm{sym}\,\mathcal{E}_{\mathrm{EL}})$ *can be extended by using the* improper *symmetries to an operator* (*which we continue denoting by* A) *that takes values in the Lie algebra* $\mathfrak{g}(\pi) = \bigl(\varkappa(\pi), [\,,\,]\bigr)$ *of evolutionary vector fields on the entire jet space* $J^\infty(\pi) \supseteq \mathcal{E}^\infty_{\mathrm{EL}}$. *Then the following two conditions are equivalent*:

- *For each* $\boldsymbol{p} \in \widehat{P_1}$, *the evolutionary vector field* $\partial^{(\boldsymbol{u})}_{A(\boldsymbol{p})}$ *is a Noether symmetry of the Lagrangian* $\mathcal{L} \in \overline{H}^n(\pi)$,
$$\partial^{(\boldsymbol{u})}_{A(\boldsymbol{p})}(\mathcal{L}) \cong 0 \in \overline{H}^n(\pi);$$
- *The operator* $A\colon \widehat{P_1} \to \varkappa(\pi)$ *determines the* linear *Noether relations*
$$\boldsymbol{\Phi}\bigl(\boldsymbol{x},[\boldsymbol{u}],[\boldsymbol{F}(\boldsymbol{x},[\boldsymbol{u}])]\bigr) = A^\dagger(\boldsymbol{F}) = 0 \text{ on } J^\infty(\pi)$$
between the equations of motion.

Remark 6.1. Generally, the constraints Φ^i need not be linear. Furthermore, in the cases of a very specific geometry of $\mathcal{E}$ there may appear the symmetry-producing operators which do not issue from any differential relations between the equations of motion. The approach which we have formulated so far treats all such structures in a uniform way.

Proof. Suppose that the identity $\boldsymbol{\Phi}\bigl(\boldsymbol{x},[\boldsymbol{u}],[\boldsymbol{F}[\boldsymbol{x},[\boldsymbol{u}])]\bigr) \equiv 0$ holds on $J^\infty(\pi)$ irrespective of a section $\boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x})$. Consequently, this identity is indifferent to the arbitrary infinitesimal shifts, which are given by the π_∞-vertical evolutionary vector fields $\partial^{(\boldsymbol{u})}_\varphi$ on $J^\infty(\pi)$. The chain rule implies that[18]
$$\partial^{(\boldsymbol{u})}_\varphi\left(\boldsymbol{\Phi}(\boldsymbol{x},[\boldsymbol{u}],[\boldsymbol{F}])\right) + \bigl(\ell^{(\boldsymbol{F})}_{\boldsymbol{\Phi}} \circ \ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\bigr)(\varphi) = 0 \in P_1,$$
where the composition $\circ$ yields the linear map from $\varkappa(\pi)$ to P_1. Take any $\boldsymbol{p} \in \widehat{P_1}$ and couple it with both sides of the above equality in P_1; this gives us
$$\bigl\langle \boldsymbol{p}, \ell^{(\boldsymbol{u})}_{\boldsymbol{\Phi}}(\varphi) + \bigl(\ell^{(\boldsymbol{F})}_{\boldsymbol{\Phi}} \circ \ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\bigr)(\varphi)\bigr\rangle = 0 \quad \text{on } \overline{J^\infty}\bigl(\xi_\pi\bigr).$$
Next, let us integrate by parts, staying in the equivalence class of the zero in the horizontal cohomology. We obtain the equality
$$\bigl\langle \ell^{(\boldsymbol{u})\dagger}_{\boldsymbol{\Phi}}(\boldsymbol{p}) + \bigl(\ell^{(\boldsymbol{u})\dagger}_{\boldsymbol{F}} \circ \ell^{(\boldsymbol{F})\dagger}_{\boldsymbol{\Phi}}\bigr)(\boldsymbol{p}), \varphi\bigr\rangle \cong 0 \qquad \text{for any } \varphi \in \varkappa(\pi).$$
Therefore, the $\langle\,,\,\rangle$-dual factor from $\widehat{\varkappa(\pi)}$ itself vanishes off-shell, that is,
$$\ell^{(\boldsymbol{u})\dagger}_{\boldsymbol{\Phi}}(\boldsymbol{p}) + \bigl(\ell^{(\boldsymbol{u})\dagger}_{\boldsymbol{F}} \circ \ell^{(\boldsymbol{F})\dagger}_{\boldsymbol{\Phi}}\bigr)(\boldsymbol{p}) = 0 \quad \text{on } \overline{J^\infty}(\xi_\pi).$$
Let us recall that the linearization $\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\colon \varkappa(\pi) \to P_0 \simeq \widehat{\varkappa(\pi)}$ and the adjoint linearization $\ell^{(\boldsymbol{u})\dagger}_{\boldsymbol{F}}\colon \widehat{P_0} \simeq \varkappa(\pi) \to \widehat{\varkappa(\pi)} \simeq P_0$ *coincide* for the Euler–Lagrange system $\mathcal{E}_{\mathrm{EL}} = \bigl\{\boldsymbol{F} = 0\bigr\}$ due to the Helmholtz criterion (see Theorem 5.1 on p. 55; we assume that the topology of the bundle π fits):
$$\ell^{(\boldsymbol{u})}_{\boldsymbol{F}} = \bigl(\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\bigr)^\dagger \quad \Longleftrightarrow \quad \exists\ \mathcal{L} \in \overline{H}^n(\pi) \mid \boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u}.$$

[18]In the beginning of the proof of the first claim we repeat our earlier reasoning from Remark 4.2 on p. 43.

This implies that

$$\ell_{\boldsymbol{\Phi}}^{(\boldsymbol{u})\dagger}(\boldsymbol{p}) + \ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\bigl(\ell_{\boldsymbol{\Phi}}^{(\boldsymbol{F})\dagger}(\boldsymbol{p})\bigr) = 0 \quad \text{on } \overline{J^\infty}(\xi_\pi). \tag{6.2}$$

As before, the system $\mathcal{E}_{\text{EL}} = \{\boldsymbol{F} = 0\}$ exhausts the set of equations which are imposed on the sections of the bundle π, whence the linearization with respect to $\boldsymbol{u}$ of the Noether identities $\boldsymbol{\Phi}$ vanishes on-shell: indeed, all its coefficients contain $[\boldsymbol{F}]$ in all the terms and thus equal zero whenever we restrict the syzygies $\boldsymbol{\Phi}\bigl(\boldsymbol{x}, [\boldsymbol{u}], [0]\bigr) \doteq 0$ on $\mathcal{E}_{\text{EL}}^\infty$ and pass to the internal coordinates. We finally deduce the equality

$$\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\left(\ell_{\boldsymbol{\Phi}}^{(\boldsymbol{F})^\dagger}(\boldsymbol{p})\right) \doteq 0 \quad \text{on } \mathcal{E}_{\text{EL}}^\infty,$$

which states that the on-shell defined section $\varphi = \bigl(\ell_{\boldsymbol{\Phi}}^{(\boldsymbol{F})}\bigr)^\dagger(\boldsymbol{p})$ is a symmetry of the Euler–Lagrange system for each $\boldsymbol{p} \in \widehat{P_1}$. By definition, put

$$A := \bigl(\ell_{\boldsymbol{\Phi}}^{(\boldsymbol{F})}\bigr)^\dagger \colon \widehat{P_1} \to \operatorname{sym} \mathcal{E}_{\text{EL}}^\infty.$$

This matrix differential operator is the sought-for generator of the infinitesimal gauge symmetries of the equations of motion.

Second, under the additional assumption that the symmetries obtained in the images of the operators are Noether, the existence of the respective sections $\boldsymbol{\Phi}$ is justified easily (besides, these constraints appear to be *linear*). Actually, suppose that the image of the extended operator $A\colon \widehat{P_1} \to \varkappa(\pi)$ preserves the cohomology class $\mathcal{L} \in \overline{H}^n(\pi)$ of the action, that is, $\partial_{A(\boldsymbol{p})}^{(\boldsymbol{u})}(\mathcal{L}) \cong 0 \in \overline{H}^n(\pi)$ on $J^\infty(\pi)$. The construction of the Euler–Lagrange equation $\mathcal{E}_{\text{EL}} = \{\boldsymbol{F} = \delta\mathcal{L}/\delta\boldsymbol{u} = 0\}$ from the action $\mathcal{L}$ implies that $\partial_{A(\boldsymbol{p})}^{(\boldsymbol{u})}(\mathcal{L}) \cong \bigl\langle \delta\mathcal{L}/\delta\boldsymbol{u}, A(\boldsymbol{p})\bigr\rangle \cong \bigl\langle \boldsymbol{p}, A^\dagger(\boldsymbol{F})\bigr\rangle \cong 0 \in \overline{H}^n(\pi)$ for all sections $\boldsymbol{p} \in \widehat{P_1}$. Therefore, $A^\dagger(\boldsymbol{F}) = 0$ on $J^\infty(\pi)$; in other words, the Noether identity $\boldsymbol{\Phi}\bigl(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}])]\bigr) = 0$ amounts to the linear differential relation between the equations of motion so that the linearization of the syzygies $\boldsymbol{\Phi}$ with respect to $\boldsymbol{F}$ coincides with the adjoint to the generator of the Noether symmetries. Obviously, the converse is also true: whenever the equality $A^\dagger(\boldsymbol{F}) = 0$ holds on $J^\infty(\pi)$, the evolutionary vector field $\partial_{A(\boldsymbol{p})}^{(\boldsymbol{u})}$ preserves the action $\mathcal{L}$ for all $\boldsymbol{p} \in \widehat{P_1}$. $\square$

6.1. **Maxwell's equation.** Let us illustrate how the Second Noether Theorem works: we now study the geometry of Maxwell's equation for the electromagnetic field.[25, 97]

Denote by $M^{3,1}$ the space-time with the Minkowski metric $g_{\mu\nu}$, which one can always bring locally to the form $\operatorname{diag}(c^2 \cdot +, -, -, -)$. Construct the auxiliary vector bundle ζ of complex dimension one over the base $M^{3,1}$ and then declare that the complex values $\psi(\boldsymbol{x}) \in \mathbb{C}$ at $\boldsymbol{x} \in M^{3,1}$ for its sections $\psi \in \Gamma_{\mathbb{C}}(\zeta)$ are *equivalent* if one value is obtained from the other after the multiplication by an element $g = e^{\boldsymbol{i}\Lambda} \in G = U(1)$, here $\boldsymbol{i}^2 = -1$ and Λ is a real-valued function of its arguments (see footnote 19 on p. 65) so that $|g| = 1$. In other words, the transformation $g \in G$ determines the pointwise equivalence $\psi \sim g\psi$ of sections for arbitrary $\psi \in \Gamma_{\mathbb{C}}(\zeta)$ and all $g \in G$. The Lie group G is called the structure group for the bundle ζ; the section $\psi = s(\boldsymbol{x})$ is the scalar massive field. (The way in which we parametrized $G = \{g = \exp(\boldsymbol{i}\Lambda)\}$ fixes the isomorphism $\mathfrak{g} = T_eG \simeq \boldsymbol{i}\mathbb{R}$ for the —here, commutative— Lie algebra $\mathfrak{g}$ of the Lie group G: the factor $\boldsymbol{i}$ is brought in by hand.)

The tangent vectors $\partial/\partial x^\mu$ on $M^{3,1}$, $1 \leqslant \mu \leqslant n = 3{+}1$, lift to the covariant derivatives ∇_μ in the bundle ζ by the rule

$$\nabla_\mu\big|_{\boldsymbol{x}} = \left.\frac{\partial}{\partial x^\mu}\right|_{\boldsymbol{x}} + \boldsymbol{i}\mathcal{A}_\mu(\boldsymbol{x}), \qquad \boldsymbol{i}\mathcal{A}\big|_{\boldsymbol{x}} \in \mathfrak{g}. \tag{6.3}$$

Under a shift $\boldsymbol{x} \mapsto \boldsymbol{x} + \varepsilon\,\Delta\boldsymbol{x} + o(\varepsilon)$ of the base point $\boldsymbol{x} \in U \subseteq M^{3,1}$, the linear part of the respective change of a section $\psi = s(\boldsymbol{x})$ first, acquires the contribution from the pointwise change of its values (this is the first term in the right-hand side of (6.3)) and second, experiences the infinitesimally close-to-unit transformation $g_\varepsilon = \mathbf{1} + \varepsilon\boldsymbol{i}\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + o(\varepsilon) \in G$, $\varepsilon > 0$.

The reparametrization rule for the G-connection one-form

$$\boldsymbol{i}\boldsymbol{\mathcal{A}} = \sum_{\mu=0}^{3} \boldsymbol{i}\mathcal{A}_\mu \,\mathrm{d}x^\mu$$

under the gauge transformations $\psi \mapsto g\psi$ in the fibres of ζ is obtained by a standard geometric reasoning. Namely, let us denote by $\boldsymbol{i}\boldsymbol{\mathcal{A}}^e$ the initially given connection one-form and by ∇^e_μ the covariant derivatives (6.3) for the transformed section $g\psi$, and we denote by $\boldsymbol{i}\boldsymbol{\mathcal{A}}^g$ the (yet unknown) connection one-form that serves for the derivation of the initial section ψ. We now require that the following diagram is commutative:

$$\begin{tikzcd}
g\psi \arrow[r, "\boldsymbol{\mathcal{A}}^e \leftrightarrow \nabla^e_\mu"] & \nabla^e_\mu(g\psi) \arrow[d, shift left, "g^{-1}"] \\
\psi \arrow[u, "g"] \arrow[r, dashed, "\boldsymbol{\mathcal{A}}^g \leftrightarrow \nabla^g_\mu"'] & \nabla^g_\mu(\psi). \arrow[u, shift left, "g"]
\end{tikzcd} \tag{6.4}$$

So, we postulate that

$$\left(\frac{\partial}{\partial x^\mu} + \boldsymbol{i}\mathcal{A}^e_\mu\right)\left(e^{\boldsymbol{i}\Lambda}\psi\right)(\boldsymbol{x}) = e^{\boldsymbol{i}\Lambda}\left(\frac{\partial}{\partial x^\mu} + \boldsymbol{i}\mathcal{A}^g_\mu\right)(\psi)(\boldsymbol{x}).$$

This implies the equality

$$\left(\frac{\partial}{\partial x^\mu} + \boldsymbol{i}\mathcal{A}^g_\mu\right)\psi = e^{-\boldsymbol{i}\Lambda}\left[e^{\boldsymbol{i}\Lambda}\frac{\partial}{\partial x^\mu} + \boldsymbol{i}\mathcal{A}^e_\mu e^{\boldsymbol{i}\Lambda}\right]\psi + e^{-\boldsymbol{i}\Lambda}\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(e^{\boldsymbol{i}\Lambda}\right)\psi,$$

where $\frac{\mathrm{d}}{\mathrm{d}x^\mu}$ is the *total* derivative. We thus obtain the (non-tensorial) transformation rule

$$\boldsymbol{i}\mathcal{A}^g_\mu = e^{-\boldsymbol{i}\Lambda}\,\boldsymbol{i}\mathcal{A}^e_\mu\, e^{\boldsymbol{i}\Lambda} + e^{-\boldsymbol{i}\Lambda}\cdot\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(e^{\boldsymbol{i}\Lambda}\right)$$

for the $U(1)$-connection one-form. Obviously, diagram (6.4) determines the universal formula[87, 103]

$$\boldsymbol{\mathcal{A}} \mapsto g^{-1}\boldsymbol{\mathcal{A}}\,g + g^{-1}\,\mathrm{d}g, \quad \boldsymbol{\mathcal{A}} \in \Lambda^1(M^n)\otimes\mathfrak{g}, \tag{6.5}$$

where the element $g \in G$ generates the local gauge transformation of the field ψ at a point $\boldsymbol{x} \in M^n$.

Remark 6.2. The gauge transformations of the field ψ, as well as the induced reparametrization (6.5) of the connection one-form $\boldsymbol{\mathcal{A}}$, are performed independently over different points of the base manifold M^n. However, let us suppose that the "field"

$g|_{\boldsymbol{x}\in M^n} \in G$ of the gauge group elements is (piecewise) continuously differentiable so that the application of the de Rham differential on M^n makes sense.

Second, the value $g|_{\boldsymbol{x}}$ may nontrivially depend on the jet $j_\infty(\boldsymbol{\mathcal{A}})(\boldsymbol{x})$, i.e., not only on the value of the connection field but also on its derivatives up to some finite but arbitrarily high order (see the definition of $C^\infty(J^\infty(\pi))$ in Lecture 1). Moreover, we claim that $g\colon (\boldsymbol{x}, [\boldsymbol{\mathcal{A}}]) \to g|_{\boldsymbol{x}} \in G$ is a nonlinear integro-differential operator with respect to $\boldsymbol{\mathcal{A}}$ (see below). Still we leave the theory to be *local* with respect to the points $\boldsymbol{x}$ of the base manifold: it always suffices to know the values of a section $\boldsymbol{\mathcal{A}}$ in an arbitrarily small neighbourhood of $\boldsymbol{x} \in M^n$ in order to determine the effective value $g|_{\boldsymbol{x}}$ at that point.

We recall that the component $\mathcal{A}_0$ of $\boldsymbol{\mathcal{A}}$ is the scalar potential of the electric field and the triple $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ is the vector potential of the magnetic field. Let us introduce the curvature tensor

$$\mathcal{F} = \mathrm{d}\boldsymbol{\mathcal{A}} + \boldsymbol{\mathcal{A}} \wedge \boldsymbol{\mathcal{A}} \in \Lambda^2(M^n) \otimes \mathfrak{g}, \tag{6.6}$$

where the multiplication in the right-hand side is the wedge product of the one-forms combined with the commutator in $\mathfrak{g}$. Because the Lie algebra $\mathfrak{u}(1) \simeq \boldsymbol{i}\mathbb{R}$ is commutative, we have $\mathcal{F} = \mathrm{d}\boldsymbol{\mathcal{A}}$ for Maxwell's electromagnetic field strength tensor:

$$\mathcal{F}_{\mu\nu} = \frac{\partial}{\partial x^\mu}(\mathcal{A}_\nu) - \frac{\partial}{\partial x^\nu}(\mathcal{A}_\mu) \equiv \mathcal{A}_{\nu;\mu} - \mathcal{A}_{\mu;\nu}.$$

Exercise 6.1. Find the transformation law for the $\mathfrak{g}$-valued two-form (6.6) under the gauge transformations $g|_{\boldsymbol{x}} \in G$.

From now on, we suppose that some gauge of $\boldsymbol{\mathcal{A}}$ is fixed, see (6.5), and instead, let us study the geometry of the *equations of motion* upon the unknowns $\boldsymbol{\mathcal{A}}$. To this end, we construct the infinite jet bundle $J^\infty(\pi)$ for the bundle $\pi\colon T^*M^{3,1} \otimes \mathfrak{g} \to M^{3,1}$. (Under the slightly misleading isomorphism $\mathfrak{g} \simeq \boldsymbol{i}\mathbb{R}$ which appears in Maxwell's abelian case, the part of the fibres that comes from the Lie algebra $\mathfrak{g}$ drops out and there remains only the cotangent bundle; however, the full setup is manifest for the Yang–Mills theories with other structure groups G.)

Exercise 6.2. Show that $\varkappa(\pi) \simeq \overline{\Lambda}^1(\pi) \otimes \mathfrak{g}$ whenever the gauge of $\boldsymbol{\mathcal{A}}$ is fixed in the Maxwell field.

From a phenomenological reasoning we know that the Lagrangian of the Maxwell field is

$$\mathcal{L} = \mathrm{const} \cdot \int \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = \mathrm{const} \cdot \int \langle *\mathcal{F} \wedge \mathcal{F} \rangle \in \overline{H}^n(\pi), \tag{6.7}$$

where the Hodge star $*$ maps the elements of a basis in $\mathfrak{g}$ to the respective elements of the dual basis in $\mathfrak{g}^*$ (which is followed by the coupling $\langle\, ,\, \rangle$) and the mapping of the horizontal differential forms under $*$ is such that $*\eta \wedge \eta = \mathrm{dvol}(M^{3,1})$. We also recall that the action $\mathcal{L} \sim \int \mathrm{tr}\langle *\mathcal{F}^a \wedge \mathcal{F}_a \rangle$ of the Yang–Mills equations with a structure group G of dimension greater than one contains the sum over the elements, which are indexed by a, of a basis in the Lie algebra $\mathfrak{g}$.

Maxwell's equations are the Euler–Lagrange equations $\delta\mathcal{L}/\delta\boldsymbol{\mathcal{A}} = 0$:

$$\overline{\mathrm{d}} * \overline{\mathrm{d}}\boldsymbol{\mathcal{A}} = 0 \quad \Longleftrightarrow \quad \mathcal{E}_{\mathrm{EL}} = \{F^j = \frac{\mathrm{d}}{\mathrm{d}x^i}\mathcal{F}^{ij} = 0\}. \tag{6.8}$$

Exercise 6.3. Using the Cartesian coordinates in space, verify that these Euler–Lagrange equations **are** indeed the Maxwell equations upon the electric field $\mathbf{E}$ and magnetic field $\boldsymbol{H}$.

Exercise 6.4. Show that the left-hand sides F^j of Maxwell's equations of motion (6.8) with the Lagrangian (6.7) belong to the horizontal module $P_0 = \overline{\Lambda}^3(\pi) \otimes \mathfrak{g}^*$ of $\mathfrak{g}^*$-valued horizontal three-forms.

For semisimple complex Lie algebras $\mathfrak{g}$ it is customary to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by using the nondegenerate Killing form (in fact, for $\mathfrak{g} = \mathfrak{u}(1) \simeq \boldsymbol{i}\mathbb{R}$ this duality is practically invisible).

The Noether relation between Maxwell's equations is obvious:

$$\overline{\mathrm{d}}^2 \equiv 0 \quad \Longrightarrow \quad -\overline{\mathrm{d}}(\overline{\mathrm{d}} * \overline{\mathrm{d}}\boldsymbol{\mathcal{A}}) = -\overline{\mathrm{d}}\boldsymbol{F} \equiv 0 \text{ on } J^\infty(\pi)$$

(the overall minus sign is chosen by intention). We have thus reached the top-degree horizontal four-forms in the bundle over the Minkowski space-time $M^{3,1}$:

$$\Phi[\boldsymbol{F}] = -\overline{\mathrm{d}}\boldsymbol{F} \in P_1 \simeq \overline{\Lambda}^4(\pi) \otimes \mathfrak{g}^*.$$

Now take any section $p(\boldsymbol{x}, [\boldsymbol{\mathcal{A}}]) \in \widehat{P_1} \simeq \mathcal{F}(\pi) \otimes \mathfrak{g}$ from the $\langle\, ,\, \rangle$-dual module and also note that $\overline{\mathrm{d}}^\dagger = -\overline{\mathrm{d}}$. Then the Second Noether Theorem states that the generating section

$$\varphi = \overline{\mathrm{d}}(p(\boldsymbol{x}, [\boldsymbol{\mathcal{A}}])$$

of the evolutionary vector field $\partial_\varphi^{(\boldsymbol{\mathcal{A}})}$ is an infinitesimal gauge symmetry of Maxwell's equations (6.8). Moreover, the extension of this evolutionary field *as is* from the equation $\mathcal{E}_{\mathrm{EL}}^\infty$ onto $J^\infty(\pi)$ is a Noether symmetry: indeed, it preserves the Lagrangian off-shell.

By integrating such infinitesimal deformations of the dependent variable $\boldsymbol{\mathcal{A}}$ over a finite interval $\varepsilon \in [0, \varepsilon_0)$ of values of the natural parameter along the would-be integral trajectories, we obtain the *finite* transformation

$$\mathcal{A}_\mu\big|_{\varepsilon=0} \mapsto \mathcal{A}_\mu + \frac{\mathrm{d}}{\mathrm{d}x^\mu}\Lambda(\boldsymbol{x}, [\boldsymbol{\mathcal{A}}]\big|_{\varepsilon\in[0,\varepsilon_0)})$$

of the electromagnetic potential.[19]

Remark 6.3. The scalar massive complex field ψ in the auxiliary bundle ζ over $M^{3,1}$ itself satisfies the relativistic-invariant **Klein–Gordon equation**

$$(\square - m^2)\psi = 0, \tag{6.9}$$

where $\square$ is the D'Alembertian for the Minkowski metric $g_{\mu\nu}$ on $M^{3,1}$. This equation encodes the equality $E^2 = \mathbf{p}^2c^2 + m^2c^4$, which relates the full energy E to the momentum $\mathbf{p}$ and the mass m (the system of units in (6.9) is such that the speed of light is $c := 1$ and also $\hbar := 1$).

[19]In a major part of the vastest literature devoted to the gauge freedom in Maxwell's electromagnetism it is proclaimed that the finite gauge transformations of the potential depend on the point $\boldsymbol{x} \in M^{3,1}$ of space-time. In the remaining part of the sources, the dependence on the initial value $\boldsymbol{\mathcal{A}}|_{\varepsilon=0}$ is also anticipated.

We shall continue the study of gauge systems in Lecture 11. Before that, we have to learn first the concept of nonlocalities (which we do in the next lecture) and then, using the techniques of the calculus of variational multivectors in the horizontal jet bundles over the space $J^\infty(\pi)$, we introduce the concept of Lie algebroids over the spaces of jets and represent those structures in terms of the homological evolutionary vector fields (see Lectures 8 and 10).

Problem 6.1. Show that the Klein–Gordon equation (6.9) is Euler–Lagrange with respect to the action (we raise the indexes in the derivatives by using the metric on the space-time $M^{3,1}$)

$$S = \int \left(\tfrac{1}{2}\psi_{;\mu}\overline{\psi}^{;\mu} - \tfrac{1}{2}m^2\psi\overline{\psi}\right) \mathrm{dvol}(M^{3,1}).$$

- Observing that the density L of S contains only the bilinear real combinations of complex-conjugate factors, deduce that the gauge transformation $\psi \mapsto \exp(\boldsymbol{i}\Lambda)\psi$ of the field leaves the action S invariant (and thus, this mapping is a finite local Noether symmetry).
- Demonstrate that the four components of the corresponding conserved current $\boldsymbol{J}$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}x^\mu}\right|_{\mathcal{E}} (J^\mu) \doteq 0 \text{ on Eq. (6.9)},$$

are equal to

$$J^\mu = \boldsymbol{i}\left(\frac{\partial L}{\partial \overline{\psi}_{;\mu}}\overline{\psi} - \frac{\partial L}{\partial \psi_{;\mu}}\psi\right), \quad \mu = 0, 1, 2, 3.$$

By definition, the integral

$$Q := \int_{x^0=\mathrm{const}} J^0 \, \mathrm{d}^3\boldsymbol{x}$$

is called the **electric charge**. The baryon charge, strangeness, or the charm are defined in precisely the same way for the now multi-component fields $\boldsymbol{\psi}$ from the isotopy or aroma symmetries etc. Remarkably, the electric charge Q and its $SU(2)$- or $SU(3)$-analogs are always quantised.

Problem 6.2. How does the number of relations in the Noether identity for the Yang–Mills equation upon the G-connection depend on the dimension of the structure Lie group G?

Problem 6.3. Calculate the operators that produce the gauge symmetries for the Yang–Mills equations with the structure groups $U(1)$, $SU(2)$, or $SU(3)$ and show that such symmetries produce improper conservation laws although they do preserve the Lagrangian (prove!).

Problem 6.4. Find all the Noether relations between the Yang–Mills equations with the simple complex structure Lie group G.

7. Nonlocalities

This lecture concludes the first part of the course: we now study the natural classes of vector bundles $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ over the infinite prolongations of differential equations $\mathcal{E} \subseteq J^k(\pi)$, provided that the total spaces $\widetilde{E}$ are differential equations as well and thus carry the Cartan distribution.[20] In particular, we address the jet-bundle analogs of the —canonically existing— tangent and cotangent bundles for smooth manifolds. More generally, we reveal the way to introduce the nonlocalities, i. e., the variables which are nonlocal with respect to the differential calculus on the jet space $J^\infty(\pi)$ over the base M^n. This geometric picture brings together the theory of coservation laws (if $n = 2$), recursion operators for (co)symmetries, Hamiltonian and symplectic operators (which we define in the noncommutative setup in Lecture 9), but not only: the non-abelian nonlocalities that determine zero-curvature representations for partial differential equations, or Bäcklund (auto)transformations between systems of PDE (see the last Lecture 12) do belong to the same class of nonlocal geometries.

It is important that the approach and the arising techniques convert various classification and reconstruction problems of the geometry of differential equations into practical algorithms.

Definition 7.1. Let $\mathcal{E} \subseteq J^k(\pi)$ be a formally integrable differential equation in the jet space over the n-dimensional manifold M^n.

A *covering* over the equation $\mathcal{E}$ is the triad $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{C}}, \tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty)$, where the *covering equation* $\widetilde{\mathcal{E}}$ is endowed with the n-dimensional Cartan distribution $\widetilde{\mathcal{C}}$ and τ is such that the tangent map $\tau_{*,\widetilde{\theta}}$ at each point $\widetilde{\theta} \in \widetilde{\mathcal{E}}$ is an isomorphism of the Cartan plane $\widetilde{\mathcal{C}}_{\widetilde{\theta}}$ to the Cartan plane $\mathcal{C}_{\tau(\widetilde{\theta})}$ at the point $\tau(\widetilde{\theta}) \in \mathcal{E}^\infty$. The dimension of the fibre in the bundle $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ is the *dimension* of the covering.

In practice, the construction of a covering over $\mathcal{E}$ means the introduction of new nonlocal variables such that the compatibility conditions for their mixed derivatives with respect to the base variables lie inside the initial system $\widetilde{\mathcal{E}}^\infty$. Whenever the covering is indeed realized as the vector bundle $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$, the forgetful map τ discards the nonlocalities.

Example 7.1. Consider the Korteweg–de Vries equation $\mathcal{E}_{\mathrm{KdV}} = \{u_t = -\frac{1}{2}u_{xxx} + 3uu_x\}$ and extend the set $(x, t, u, u_x, \ldots, u_k, \ldots)$ of its internal coordinates with the nonlocality v such that

$$v_x = u, \quad v_t = -\tfrac{1}{2}u_{xx} + \tfrac{3}{2}u^2, \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathcal{E}_{\mathrm{KdV}}}(v_x) \doteq \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{\mathcal{E}_{\mathrm{KdV}}}(v_t) \text{ on } \mathcal{E}^\infty_{\mathrm{KdV}}.$$

We see that the new variable $v \in \mathbb{R}$ satisfies the potential Korteweg–de Vries equation $v_t = -\frac{1}{2}v_{xxx} + \frac{3}{2}v_x^2$ and $u = v_x$ is the substitution which transforms the solutions to the solutions.

The coverings over differential equations are realized in coordinates as follows. As far as the local topological properties are concerned, the manifold $\widetilde{\mathcal{E}}$ is the product

[20] The definition of a covering τ over $\mathcal{E}^\infty$, see below, makes sense for the empty equation $\mathcal{E} = \{0 = 0\}$ such that $\mathcal{E}^\infty \simeq J^\infty(\pi)$.

$\mathcal{E}^\infty \times_M W$, where $W \subseteq \mathbb{R}^{\widetilde{m}}$ is an open domain and $1 \leqslant \widetilde{m} \leqslant +\infty$, so that the mapping $\tau\colon \mathcal{E}^\infty \times W \to \mathcal{E}^\infty$ is the projection. The distribution $\widetilde{\mathcal{C}}$ on $\widetilde{\mathcal{E}}$ is the extra structure, it is described by the vector fields

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i} = \frac{\mathrm{d}}{\mathrm{d}x^i}\Big|_{\mathcal{E}^\infty} + \sum_{j=1}^{\widetilde{m}} N_i^{\ j} \cdot \frac{\partial}{\partial v^j}, \quad i = 1, \ldots, n,$$

where $v^1, \ldots, v^{\widetilde{m}}$ are the local coordinates on W and the coefficients $N_i^{\ j} \in C^\infty(\widetilde{\mathcal{E}}) = \bigcup\limits_{\ell \in \mathbb{N}\cup\{0\}} C^\infty(\mathcal{E}^{(\ell)}) \otimes_{C^\infty(M)} C^\infty(W)$ are smooth "nonlocal" differential functions.

Exercise 7.1. Show that the Frobenius integrability condition $[\widetilde{\mathcal{C}}, \widetilde{\mathcal{C}}] \subseteq \widetilde{\mathcal{C}}$ for the distribution $\widetilde{\mathcal{C}}$ is equivalent to the set of relations $\left[\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}, \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^j}\right] \doteq 0$ on $\widetilde{\mathcal{E}}$ for $i, j = 1, \ldots, n$. In turn, these relations are equivalent to the flatness equation $\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(N_j^{\ k}) \doteq \frac{\widetilde{d}}{dx^i}(N_i^{\ k})$ on $\widetilde{\mathcal{E}}$ for the connection $\frac{\partial}{\partial x^i} \mapsto \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}$ and $1 \leqslant k \leqslant \widetilde{m}$, here i, j as above.

The coordinates v^k are called the *nonlocal variables*. The covering equation $\widetilde{\mathcal{E}}$ is described in the chosen system of coordinates x^i, u^j_σ, and v^k as follows:

$$\widetilde{\mathcal{E}} = \begin{cases} \boldsymbol{F} = 0, \ldots, \frac{d^{|\rho|}}{d\boldsymbol{x}^\rho}(\boldsymbol{F}) = 0, \ldots, & |\rho| \geqslant 0, \\ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v^k) = N_i^{\ k}(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}), \end{cases}$$

which is the combination of the infinite prolongation $\mathcal{E}^\infty$ and the rules to differentiate the nonlocalities, respectively. The underlying equation $\mathcal{E}$ ensures the compatibility of the mixed derivatives of the nonlocal variables: $\left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i} \circ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^j}\right)(v^k) - \left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^j} \circ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}\right)(v^k) \doteq 0$ by virtue of $\mathcal{E}^\infty$.

Example 7.2 ($n = 2$)**.** The dimension $n = 2$ of the base manifold M^n matches the Cartan connection one-forms with the conserved currents, which are horizontal forms on $\mathcal{E}^\infty$ of degree $n - 1$. This is why the geometry of differential equations is so rich at $n = 2$ in comparison with the higher dimensions of the base.

If $n = 2$, each conserved current $\eta = \eta_1 dx^1 + \eta_2 dx^2$ for an equation $\mathcal{E}$ yields the covering $\tau_\eta\colon \widetilde{\mathcal{E}}_\eta \to \mathcal{E}^\infty$. Namely, let $\widetilde{\mathcal{E}}_\eta = \mathcal{E}^\infty \times_M \mathbb{R}$ with the local coordinate v along $\mathbb{R}$, and set

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v) := (-)^{i+1}\eta_{3-i}(\boldsymbol{x}, [\boldsymbol{u}]), \quad i = 1, 2. \tag{7.1}$$

Then the compatibility condition for the mixed derivatives of v is equivalent to the conservation of η by virtue of $\mathcal{E}^\infty$:

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i} \circ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^j}(v) \doteq \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^j} \circ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v) \text{ on } \widetilde{\mathcal{E}}_\eta \Longleftrightarrow \bar{d}\big|_{\mathcal{E}^\infty}(\eta) \doteq 0 \text{ on } \mathcal{E}^\infty.$$

Simultaneously, we notice that the current η becomes *trivial* on $\widetilde{\mathcal{E}}_\eta$ even if this was not so on $\mathcal{E}^\infty$: indeed, we have that $\eta = \sum\limits_{i=1}^{2} \mathrm{d}x^i \cdot \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v)$ because $\eta_i = (-)^{i+1}\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^{3-i}}(v)$, $i = 1, 2$. In other words, we thus obtain the method of trivializing the conservation laws for a given system $\mathcal{E}$.

Remark 7.1. It is readily seen from (7.1) that the total derivatives of the new variables in $\widetilde{\mathcal{E}}_\eta$ do *not* depend on those new variables; such nonlocalities are called *abelian* (see Lecture 12).

We finally notice that in any dimension n, the covering equation $\widetilde{\mathcal{E}}$ may acquire new conservation laws that do not amount to any local conserved currents for the underlying system $\mathcal{E}^\infty$ alone. The equivalence classes of such continuity relations (up to the image of the horizontal differential $\widetilde{\mathrm{d}} = \sum_{i=1}^{n} \mathrm{d}x^i \cdot \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}$ in $\overline{\Lambda}^{n-1}(\widetilde{\mathcal{E}})$) are the τ-nonlocal conservation laws for the system $\mathcal{E}$. However, we recall from the example above that a nontrivial conserved current for $\mathcal{E}$ can become trivial in the nonlocal setup.

The shortcut from the conserved currents for $\mathcal{E}$ to the coverings over $\mathcal{E}^\infty$ is specific for $n = 2$ only. Nevertheless, over every equation in any dimension there exist two canonical coverings (by default, infinite-dimensional) which play the rôle of the tangent and cotangent bundles over smooth manifolds. These are the ℓ-covering and the $\ell^\dagger$-covering, respectively; here the symbol "ℓ" refers to the linearization operator and to the linear determining equations (3.3) for symmetries versus (4.2) for cosymmetries.

Let $\mathcal{E} = \{\boldsymbol{F} = 0\}$ be a formally integrable system of r differential equations upon sections of the bundle π with m-dimensional fibres; by definition, the projection $\mathcal{E}^\infty \to M^n$ is an epimorphism.

Tangent covering $\mathbb{T}$ *over* $\mathcal{E}^\infty$. First, consider the tangent bundle $\mathbb{T}\pi\colon E^{n+m} \times \mathbb{R}^m \to M^n$ to the (vector) bundle π over the base M^n, here $\mathbb{R}^m \simeq T_\theta(\pi^{-1}(\boldsymbol{x}))$ for all $\boldsymbol{x} \in M^n$ and $\theta \in \pi^{-1}(\boldsymbol{x})$; by definition, the sections of $\mathbb{T}\pi$ transform as vectors under the reparameterizations in the total space E^{n+m} of π. Construct the infinite jet space $J^\infty(\mathbb{T}\pi)$ for this bundle: Whenever $\boldsymbol{U}$ is the m-tuple of coordinates along the tangent space $\mathbb{R}^m$, the jet variables along the corresponding part of $\mathbb{T}\pi_\infty\colon J^\infty(\mathbb{T}\pi) \to M^n$ are $\boldsymbol{U}_\varnothing \equiv \boldsymbol{U}$, $\boldsymbol{U}_{\boldsymbol{x}}$, $\ldots$, $\boldsymbol{U}_\sigma$, $\ldots$ for all $|\sigma| \geqslant 0$. (Note that the horizontal jet bundle $\overline{J^\infty}(\pi^*_\infty(\pi)) \to J^\infty(\pi) \to M^n$ yields exactly the same construction for vector bundles π, see Lecture 3.) Let us take the infinite prolongation $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ and, at all points $\theta^\infty \in \mathcal{E}^\infty$, canonically restrict the new jet variables $\boldsymbol{U}$ by the linear equations

$$\mathcal{V} = \{\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}(\boldsymbol{U}) = 0\} \tag{3.3$'$}$$

and take its infinite prolongation $\mathcal{V}^\infty$. We thus obtain the covering $\mathbb{T}\colon \mathbb{T}\mathcal{E}^\infty \to \mathcal{E}^\infty$, where

$$\mathbb{T}\mathcal{E}^\infty = \mathcal{E}^\infty \cap \mathcal{V}^\infty \subseteq J^\infty(\mathbb{T}\pi)$$

and in coordinates we have

$$\mathbb{T}\mathcal{E}^\infty = \left\{ \frac{d^{|\sigma|}}{d\boldsymbol{x}^\sigma}(\boldsymbol{F}) = 0,\ \frac{d^{|\rho|}}{d\boldsymbol{x}^\rho}(\ell_{\boldsymbol{F}}(\boldsymbol{U})) = 0 \ \middle|\ |\sigma| \geqslant 0,\ |\rho| \geqslant 0 \right\}.$$

By definition, this is the *ℓ-covering* over $\mathcal{E}^\infty$. The new variables $\boldsymbol{U}$ are called the *linearized variables* associated with the m-tuple $\boldsymbol{u}$ for the bundle π; the new variables imitate the symmetries of $\mathcal{E}$, c. f. (3.3). We denote by $\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}$ the restrictions of the total derivatives $\frac{\mathrm{d}}{\mathrm{d}x^i}$ on $J^\infty(\mathbb{T}\pi)$ to the covering equation $\mathbb{T}\mathcal{E}^\infty$.

The construction of the tangent covering will be used further in the definition of recursions for symmetry algebras of differential equations.

Cotangent covering $\mathbb{T}^*$ *over* $\mathcal{E}^\infty$. We now consider the auxilliary vector bundle ξ of rank r over M^n such that $\boldsymbol{F} \in \Gamma(\pi_\infty^*(\xi))$, see Lecture 2. Let us take the dual bundle $\widehat{\xi}$ over M^n and construct the infinite jet space $J^\infty(\pi \times_M \widehat{\xi})$; we denote by $\boldsymbol{p} = (p_1, \ldots, p_r)$ the r-tuple of fibre coordinates in $\widehat{\xi}$ so that $\boldsymbol{p}_\sigma$ become the jet variables, here $|\sigma| \geqslant 0$ and $\boldsymbol{p}_\varnothing \equiv \boldsymbol{p}$. For a given formally integrable equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$, we impose upon $\boldsymbol{p}$ the equation

$$\mathcal{W} = \{\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger}(\boldsymbol{p}) = 0\} \tag{4.2$'$}$$

and take its infinite prolongation $\mathcal{W}^\infty$. The system

$$\mathbb{T}^*\mathcal{E} = \mathcal{E}^\infty \cap \mathcal{W}^\infty \subseteq J^\infty(\pi \times_M \widehat{\xi})$$

is the total space in the *cotangent covering* ($\ell^\dagger$*-covering*) $\mathbb{T}^*$ over $\mathcal{E}^\infty$. The coordinate notation reads

$$\mathbb{T}^*\mathcal{E} = \left\{ \frac{d^{|\sigma|}}{d\boldsymbol{x}^\sigma}(\boldsymbol{F}) = 0,\ \frac{d^{|\rho|}}{d\boldsymbol{x}^\rho}(\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger}(p))) = 0 \;\middle|\; |\sigma| \geqslant 0,\ |\rho| \geqslant 0 \right\}. \tag{7.2}$$

The fibre variables $\boldsymbol{p}$ imitate the cosymmetries for $\mathcal{E}$, c. f. (4.2). We denote by $\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}$ the restrictions of the total derivatives $\frac{\mathrm{d}}{\mathrm{d}x^i}$ on $J^\infty(\pi \times_M \widehat{\xi})$ to the covering equation $\mathbb{T}^*\mathcal{E}$.

Exercise 7.2. Prove that the cotangent covering equation (7.2) is the infinite prolongation of the Euler–Lagrange equation with the action $S = \int \boldsymbol{p} \cdot \boldsymbol{F}\, \mathrm{d}\boldsymbol{x}$.

Remark 7.2. If the equation $\mathcal{E} = \{0 = 0\}$ is empty, we label the m copies of it by the unknowns $u^1, \ldots, u^m$ and thus identify $\xi \simeq \pi$ so that the m-fold zero becomes $\boldsymbol{0} \in \Gamma(\pi_\infty^*(\xi))$ and the objects $\boldsymbol{p}$ transform as the coefficients of Cartan's one-forms. Then the covering equation $\mathbb{T}^*\mathcal{E}$ over $\mathcal{E}^\infty = J^\infty(\pi)$ is the horizontal jet space $\overline{J^\infty}(\widehat{\pi}_\pi)$, see Lecture 4 and [83].

Remark 7.3. The *neighbour* of the space $J^\infty(\pi \times_M \widehat{\xi})$ is the horizontal jet superspace $J^\infty(\Pi\widehat{\xi}_\pi)$ with the reversed $\mathbb{Z}_2$-parity of the fibre variables $\boldsymbol{b}_\sigma = \Pi\boldsymbol{p}_\sigma$; the prolongation $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ remains intact. The parity reversion $\Pi\colon \boldsymbol{p} \rightleftarrows \boldsymbol{b}$ is well-defined because the determining equation(4.2) is linear and hence it does not feel the parity of the adjoint linearization's argument.

This technique, which is the transition to the odd neighbours of vector spaces (here, the fibres in $\widehat{\xi}$) will be crucial for us in the entire Part II of this course.

7.1. **Nonlocal symmetries.**

Definition 7.2. Let $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ be a covering. A local symmetry of the covering equation $\widetilde{\mathcal{E}}$ is a τ-*nonlocal symmetry* of the system $\mathcal{E}$.

Suppose first that the evolutionary field $\partial_\varphi^{(\boldsymbol{u})}$ with $\varphi(\boldsymbol{x}, [\boldsymbol{u}]) \in \Gamma\bigl(\pi_\infty^*(\pi)\bigr)$ is a symmetry of the underlying equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ upon sections of a vector bundle π,

$$\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}(\varphi) \doteq 0 \text{ on } \mathcal{E}^\infty.$$

Two substantially different cases are possible:

(1) the restriction of the field $\partial_\varphi^{(\boldsymbol{u})}$ onto $\mathcal{E}^\infty$ can be extended to a symmetry $\widetilde{\mathcal{X}}$ of the covering equation $\widetilde{\mathcal{E}} \simeq \mathcal{E}^\infty \times W$,

(2) the converse: there is no such lift, which means that the field $\partial^{(\boldsymbol{u})}_{\varphi}$ at hand preserves $\mathcal{E}^\infty$ but propagates the total space to the one-parametric family $\widetilde{\mathcal{E}}_t \simeq \mathcal{E}^\infty \times W_t$ of equations that cover $\mathcal{E}^\infty$ at all t.

Lecture 12 is devoted to the deformation theory of nonlocal structures over differential equations. In the meantime, we focus on the first option.

Let $\mathcal{E} = \{\boldsymbol{F} = 0\}$ be the underlying equation and

$$\mathcal{W} = \left\{ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v^j) = N_i^{\ j}(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}),\ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant \widetilde{m} \right\}$$

be the rules to differentiate the nonlocalities $\boldsymbol{v}$ in the covering $\tau\colon \mathcal{E}^\infty \times \mathcal{W} \to \mathcal{E}^\infty$. It is readily seen that, for the field $\partial^{(\boldsymbol{u})}_{\varphi}$ to be a part of the infinitesimal symmetry $\widetilde{\mathcal{X}}$ acting on $\mathcal{E}^\infty$ **and** the nonlocalities, the determining equation

$$\widetilde{\mathcal{X}}(\boldsymbol{F}) \doteq 0, \quad \widetilde{\mathcal{X}}\left(\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(v^j) - N_i^{\ j} \right) \doteq 0 \text{ on } \widetilde{\mathcal{E}}$$

must have a solution (which is not always the case). The composition of the would-be lifting $\widetilde{\mathcal{X}}$ for the given fragment $\partial^{(\boldsymbol{u})}_{\varphi}$ is clear:

$$\widetilde{\mathcal{X}} = \partial^{(\boldsymbol{u})}_{\varphi(\boldsymbol{x},[\boldsymbol{u}])} + \sum_{j=1}^{\widetilde{m}} \phi^j(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}) \frac{\partial}{\partial v^j},$$

where the coefficients ϕ^j belong to the ring $C^\infty(\widetilde{\mathcal{E}})$ and transform appropriately under the changes of coordinates.

From now on, we study the symmetries of $\mathcal{E}$ which are truly nonlocal: we allow the dependence of the components φ^i in $\varphi = {}^{\mathrm{t}}(\varphi^1, \ldots, \varphi^m)$ on the nonlocalities,

$$\varphi = \varphi(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}) \quad \text{so that} \quad \widetilde{\partial}^{(\boldsymbol{u})}_{\varphi} = \sum_{j=1}^{m} \sum_{|\sigma| \geqslant 0} \frac{\widetilde{\mathrm{d}}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}(\varphi^j) \cdot \frac{\partial}{\partial u^j_\sigma}.$$

Otherwise speaking, we let $\varphi \in \Gamma((\pi_\infty|_{\mathcal{E}^\infty} \circ \tau)^*(\pi))$ for the vector bundle π.

Proposition 7.1. The full generating section ${}^{\mathrm{t}}(\varphi, \phi) \in \Gamma((\pi_\infty \circ \tau)^*(\pi \times_{M^n} (\pi_\infty \circ \tau)))$ of the τ-nonlocal symmetry

$$\widetilde{\mathcal{X}} = \widetilde{\partial}^{(\boldsymbol{u})}_{\varphi} + \sum_{\tilde{\jmath}=1}^{\widetilde{m}} \phi^{\tilde{\jmath}} \cdot \frac{\partial}{\partial v^{\tilde{\jmath}}}$$

for $\mathcal{E} = \{\boldsymbol{F} = 0\}$ satisfies the determining equations

$$\widetilde{\ell}^{(\boldsymbol{u})}_{\boldsymbol{F}}(\varphi) \doteq 0 \text{ on } \widetilde{\mathcal{E}},$$
$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(\phi^{\tilde{\jmath}}) \doteq \widetilde{\mathcal{X}}(N_i^{\ \tilde{\jmath}}) \text{ on } \widetilde{\mathcal{E}},$$

here $1 \leqslant i \leqslant n$, $1 \leqslant \tilde{\jmath} \leqslant \widetilde{m}$.

By definition, the first component $\varphi(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v})$ of the pair ${}^{\mathrm{t}}(\varphi, \phi)$ is called the *τ-shadow* of a τ-nonlocal symmetry for the underlying equation $\mathcal{E}$ in the covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$.

Remark 7.4. In general, not all τ-shadows φ satisfying the determining equation $\widetilde{\ell}_{\boldsymbol{F}}^{(\boldsymbol{u})}(\varphi) \doteq 0$ on $\widetilde{\mathcal{E}}$ can be extended —in the same covering τ— to the pair ${}^{\mathrm{t}}(\varphi, \phi)$ that yields a τ-nonlocal symmetry of $\mathcal{E}$. Nevertheless, the τ-shadows present an independent interest because, for a specific class of coverings over the initial system $\mathcal{E}$ and under extra assumptions about the sections φ (see below), these objects appear in the practical problem of finding recursion or Hamiltonian differential operators for the system at hand.

7.2. **Recursion differential operators.** Consider the infinite-dimensional tangent covering $\mathbb{T}\mathcal{E} \to \mathcal{E}^\infty$ over a formally integrable differential equation $\mathcal{E}$: we have that

$$\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0, \quad \widetilde{\ell}_{\boldsymbol{F}}^{(\boldsymbol{u})}(\boldsymbol{U}) = 0.$$

The linearized variables $\boldsymbol{U}$ imitate the local symmetries of $\mathcal{E}$ in the sense that they belong to the kernel of the linearization for $\boldsymbol{F}$ but do not have any internal structure, in contrast with the solutions $\varphi(\boldsymbol{x}, [\boldsymbol{u}])$ of (3.3). However, we notice that every expression $R[\boldsymbol{U}]$ which is linear in the nonlocal variables $\boldsymbol{U}_\sigma$, $|\sigma| \geqslant 0$, tautologically defines the linear differential operator R on the space $\operatorname{sym}\mathcal{E}$. We are now interested in finding the linear mappings whose values are again symmetries of the same equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$, that is, the values stay in the kernel of the linearization $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\Big|_{\mathcal{E}^\infty}$ with respect to $\boldsymbol{u}$.

Definition 7.3 (provisional)**.** A total differential operator $R\colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ is a *local differential recursion operator* for the system $\mathcal{E}$.

Remark 7.5. Most of the recursions which one encounters in practice are nonlocal. In Lecture 12 we formulate a more general definition of the recursions for the symmetry algebras; that concept properly grasps the introduction of the nonlocalities.

Exercise 7.3. Prove that $R(\boldsymbol{U})$ is a $\mathbb{T}$-shadow for $\mathcal{E}$ whenever $R\colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ is a local differential recursion operator for $\mathcal{E}$.

Exercise 7.4. Establish the converse to the previous exercise: If a $\mathbb{T}$-shadow $R(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{U}])$ is linear in $\boldsymbol{U}_\sigma$, then it determines the linear total differential operator R which is a local recursion on the Lie algebra $\operatorname{sym}\mathcal{E}$.

Corollary 7.2. Finding local differential recursion operators R for an equation $\mathcal{E}$ amounts to finding the $\mathbb{T}$-shadows which are linear in the linearized variables along the fibre in the tangent covering $\mathbb{T}\mathcal{E} \to \mathcal{E}^\infty$.

Remark 7.6. With just elementary modifications, the same reasoning allows us to interpret the *Noether* operators $A\colon \operatorname{cosym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ (in particular, the Hamiltonian differential operators, see Lecture 9) as those $\mathbb{T}$-shadows for the equation $\mathcal{E}$ which are linear in the variables $\boldsymbol{p}_\sigma$ along the fibre of the cotangent covering $\mathbb{T}^*\mathcal{E} \to \mathcal{E}^\infty$.

Yet even more: let us notice that every linear differential equation is obtained by the application of some linear differential operator to the unknowns. We also note that, on the other hand, the linearization of the left-hand sides for such linear equations with respect to the unknowns canonically coincide with those linear operators:

$$\mathcal{W} = \{L(\boldsymbol{U}) = 0\} \quad \Rightarrow \quad \ell_{L(\boldsymbol{U})}^{(\boldsymbol{U})} = L. \tag{7.3}$$

We finally observe that the **iso**morphism of the Cartan distributions works in both ways in the construction of a covering over a differential equation.

Summarizing, for a given system $\mathcal{E} = \{\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0\}$ we let the associated linearized system $\mathcal{W} = \{\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}(\boldsymbol{U}) = 0\}$ be the *equation* containing the formal parameters $\boldsymbol{u}_\sigma$, $|\sigma| \geqslant 0$, and then we consider the *covering* over it,

$$\tau\colon \mathcal{W}^\infty \times \mathcal{E}^\infty \to \mathcal{W}^\infty, \tag{7.4}$$

which reinstates the "parameters" $\boldsymbol{u}_\sigma$ as the genuine derivatives of $\boldsymbol{u}$ and also prescribes the substitution rules for some of such variables by virtue of the system $\mathcal{E}^\infty$.

Remark 7.7. The upside-down covering (7.4) is such that the $\boldsymbol{U}_\sigma$-linear τ-shadows $R = R(\boldsymbol{x}, [\boldsymbol{u}], [\boldsymbol{U}])$ satisfy the determining equation (see (7.3))

$$L(R) = \ell_{L(\boldsymbol{U})}^{(\boldsymbol{U})}(R) \doteq 0 \text{ on } \mathbb{T}\mathcal{E}, \text{ here } L = \ell_{\boldsymbol{F}}^{(\boldsymbol{u})}.$$

But let us recall that the calculation of the linearization operator L for $\mathcal{E} = \{\boldsymbol{F} = 0\}$ is equivalent to Step 3 in the algorithm for the calculation of symmetries of a given system, e. g., by using any suitable software (see p. 31 in Lecture 3).

Corollary 7.3 (see [48])**.** If a computer program is capable of finding higher infinitesimal symmetries for differential equations, then the same program suits well for the search of recursion operators.

- If the computer program is also capable of writing the determining equation for the cosymmetries, see (4.2), then that program fits for finding linear differential operators which propagate the cosymmetries.
- Under the above assumptions and after minimal efforts, the same program is ready to the search for Noether operators $\operatorname{cosym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$ and also the symplectic operators (subject to some further restriction by their definition) $\operatorname{sym}\mathcal{E} \to \operatorname{cosym}\mathcal{E}$.

Indeed, we view the former as the shadows for the linearized system $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}(\boldsymbol{U}) = 0$ and the latter as the shadows for the adjoint linearized system $\ell_{\boldsymbol{F}}^{(\boldsymbol{u})\dagger}(\boldsymbol{p}) = 0$; in both cases we assume that the shadows are linear in $\boldsymbol{p}_\sigma$ and $\boldsymbol{U}_\sigma$, respectively.

7.3. **Nonlocal recursion operators.** It is habitual for important equations of mathematical physics to have no *nontrivial* (i. e., other than the identity $\mathrm{id}\colon \varphi \mapsto \varphi$) local recursion operators. Instead, they often admit the nonlocal recursions which involve the integration such as taking the inverse of $\mathrm{d}/\mathrm{d}x^i$. To describe such nonlocal structures, we combine the approaches which we already know: (1) the introduction of "some" nonlocalities in a covering $\tau\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$ (e. g., by trivializing the conserved currents if $n = 2$), and (2) the construction of the tangent covering $\mathbb{T}\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}}^\infty$ and then, finding those $\mathbb{T}\widetilde{\mathcal{E}}$-shadow solutions φ of the determining equation $\widetilde{\ell}_{\boldsymbol{F}}^{(u)}(\varphi) \doteq 0$ on $\mathbb{T}\widetilde{\mathcal{E}}$ which are linear in the linearized variables along the fibre in the tangent covering.

Example 7.3. Consider the Korteweg–de Vries equation $\mathcal{E}_{\mathrm{KdV}} = \{F = u_t + \frac{1}{2}u_{xxx} - 3uu_x = 0\}$. It admits the local Noether operators

$$\widehat{A}_1^{\mathrm{KdV}} = \frac{\mathrm{d}}{\mathrm{d}x} \quad \text{and} \quad \widehat{A}_2^{\mathrm{KdV}} = -\tfrac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x\colon \ \operatorname{cosym}\mathcal{E}_{\mathrm{KdV}} \to \operatorname{sym}\mathcal{E}_{\mathrm{KdV}}.$$

Consequently, if we find a well-defined way to invert the first operator $\widehat{A}_1^{\text{KdV}}$, their composition $R = \widehat{A}_2^{\text{KdV}} \circ (\widehat{A}_1^{\text{KdV}})^{-1}$ will — possibly! — produce new local symmetries for some previously known symmetries of this equation.

We note that the skew-adjoint operator $\widehat{A}_1^{\text{KdV}}$ and $\widehat{A}_2^{\text{KdV}}$ are the renowned first and second Hamiltonian operators for the KdV equation (see Lecture 9). Moreover, they are *compatible*, and besides, the first Poisson cohomology vanishes for the restriction of the Gerstenhaber–Poisson differential, arising from $\widehat{A}_1^{\text{KdV}}$, to the space $\operatorname{span}\langle \mathcal{H}_i,\ i \geqslant 0\rangle$ of descendants from the Casimir $\mathcal{H}_0 = \int u\,dx$ for $\widehat{A}_1^{\text{KdV}}$. These properties of the two structures make the *Magri scheme* work for $\mathcal{E}_{\text{KdV}}$, producing the infinite tower of the Hamiltonians $\mathcal{H}_k$ in involution and the commutative hierarchy of the higher symmetries $\varphi_k = \widehat{A}_2^{\text{KdV}}(\delta\mathcal{H}_k/\delta u) = \widehat{A}_1^{\text{KdV}}(\delta\mathcal{H}_{k+1}/\delta u)$ for $\mathcal{E}_{\text{KdV}}$, here $k \geqslant 0$. This will be discussed in full detail in Lecture 9 (see also Lecture 12). Meanwhile, we observe that the symmetries $\varphi_k = \Bigl(\widehat{A}_2^{\text{KdV}} \circ (\widehat{A}_1^{\text{KdV}})^{-1}\Bigr)^k (u_x)$ remain local for all $k \in \mathbb{N}$ (i. e., they stay in $\operatorname{sym}\mathcal{E}_{\text{KdV}}$), and this can be rigorously *proved*.[21]

The empiric construction of the recursion R for the KdV equation is justified as follows. Obviously, the equation $\frac{\mathrm{d}}{\mathrm{d}x}(u) = \frac{\mathrm{d}}{\mathrm{d}x}(-\frac{1}{2}u_{xx} + \frac{3}{2}u^2)$ is itself the continuity relation for the current with the density $h_0 = u$. Let us trivialize this current by introducing the nonlocality v such that

$$v_x = u,\ v_t = -\tfrac{1}{2}u_{xx} + \tfrac{3}{2}u^2 \text{ and } (v_t)_x \doteq (v_x)_t \text{ on } \mathcal{E}_{\text{KdV}}.$$

Next, we double the number of unknowns by the introduction of the linearized variables U for u and V for v on $\mathbb{T}\widetilde{\mathcal{E}} \to \widetilde{\mathcal{E}} \to \mathcal{E}^\infty$, satisfying the system

$$\begin{aligned} U_t &= -\tfrac{1}{2}U_{xxx} + 3Uu_x + 3uU_x,\\ V_x &= U, \quad V_t = -\tfrac{1}{2}U_{xx} + 3uU. \end{aligned}$$

It is instructive to compare this system with Eq. (3.3) for the covering equation $\widetilde{\mathcal{E}}$ over $\mathcal{E}_{\text{KdV}}$.

In this nonlocal setup, the determining equation

$$\widetilde{\ell}_F^{(u)}(\mathcal{R}) \doteq 0 \text{ on } \mathbb{T}\widetilde{\mathcal{E}}$$

does acquire the nontrivial (i. e., $\mathcal{R} \neq U$) solution

$$\mathcal{R} = -\tfrac{1}{2}U_{xx} + 2uU + u_xV,$$

which is linear with respect to the linearized variables U_σ and V over $\widetilde{\mathcal{E}}$.

The correspondence between the solution $\mathcal{R}$ and the mapping $R = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2u + u_x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}$ is informally stated as follows. Because the linearized variables U_σ faithfully imitate the local symmetries for $\mathcal{E}_{\text{KdV}}$ (i. e., the solutions of (3.3)), the matching of the local parts in $\mathcal{R}$ and in R defined on $\operatorname{sym}\mathcal{E}_{\text{KdV}}$ is trivial. To interpret the nonlocality, we

[21] We remark that, just the way they are, the nonlocal recursions map τ-shadows (in particular, true local symmetries) to τ'-shadows in some coverings τ and τ' over the underlying system $\mathcal{E}$. One has to inspect whether the nonlocal mappings still can produce local symmetries of $\mathcal{E}$ for some classes of the seed symmetries. (For example, this is *not* the case for the scaling symmetry $xu_x + 3tu_t + 2u$ of $\mathcal{E}_{\text{KdV}}$ and the recursion for it as above.) If such locality is experimentally observed, it remains to prove it. This can be a nontrivial algebro-arithmetic problem; its solutions often involve *ad hoc* techniques which are specific to the equation at hand, see [76].

deduce from the covering relation $V_x = U$ that the object V is such that its derivative stands for the local symmetry of $\mathcal{E}_{\text{KdV}}$.

The general approach patterns upon the all-encompassing Example 7.3 with the construction of a recursion for the KdV equation. If the nontrivial ($\mathcal{R} \neq U \Leftrightarrow R \neq \text{id}$) recursion operators are not available, one keeps on covering the system at hand by the layers of *abelian* nonlocal variables (e. g., by trivializing the conserved currents if $n = 2$). We note that the covering equations $\widetilde{\mathcal{E}}$ can gain the conserved currents which did not initially show up for the underlying system because their density and/or flux explicity depend on the new, nonlocal variables; the general situation with the coverings over $\widetilde{\mathcal{E}}$ versus $\mathcal{E}$ is analogous in all dimensions n. It is this dependence which arranges the abelian nonlocalities along the layers

$$\widetilde{\mathcal{W}}_{(\ell)} = \Big\{ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(\boldsymbol{v}^{(1)}) = \boldsymbol{N}_i^{(1)}(\boldsymbol{x}, [\boldsymbol{u}]), \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(\boldsymbol{v}^{(2)}) = \boldsymbol{N}_i^{(2)}(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}^{(1)}), \ \ldots\ , \\ \frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(\boldsymbol{v}^{(\ell+1)}) = \boldsymbol{N}_i^{(\ell+1)}(\boldsymbol{x}, [\boldsymbol{u}], \boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(\ell)}),\ 1 \leqslant i \leqslant n,\ \ell \in \mathbb{N} \Big\}.$$

Having attained each successive layer $\widetilde{\mathcal{W}}_{(\ell)}$ of the nonlocal variables, one checks whether there appears a nontrivial solution $\mathcal{R}$ for the determining equation $\widetilde{\ell}_{\boldsymbol{F}}^{(\boldsymbol{u})}(\mathcal{R}) \doteq 0$ on the tangent covering over the system $\mathcal{E}^\infty \times \widetilde{\mathcal{W}}_{(\ell)}$.

The correspondence between the linearized variables and the operators which apply to the symmetries of $\mathcal{E}$ is recursively obtained by resolving the linearizations

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x^i}(\mathcal{V}^{(\ell+1)}) = \widetilde{\ell}_{\boldsymbol{N}_i}^{(\boldsymbol{u}, \boldsymbol{v}^{(1)}, \ldots, \boldsymbol{v}^{(\ell)})}({}^{\mathrm{t}}(\boldsymbol{U}, \boldsymbol{V}^{(1)}, \ldots, \boldsymbol{V}^{(\ell)}))$$

of the relations $\widetilde{\mathcal{W}}_{(\ell)}$ with respect to the linearized nonlocal variables $\boldsymbol{V}^{(\ell+1)}, \boldsymbol{V}^{(\ell)}, \ldots, \boldsymbol{V}^{(1)}$. At each step, this procedure is analogous to the definition of negative numbers from the set of natural numbers and zero.

We conclude that the practical search for the (shadows of nonlocal) recursion operators for symmetry algebras of differential equations is sufficiently algorithmic. The software [65] for $\mathbb{Z}_2$-graded evolutionary systems was developed to pursue exactly this goal (see the on-line companion [59]); many other packages are also available for various analytic environments (e.g., [93]). They greatly facilitate the philatelistic activity of producing many symmetries for various classes of differential equations.

Problem 7.1. Find the first-order nonlocal recursion operator for the Burgers equation $u_t = u_{xx} + uu_x$ in the covering where a unique local conservation law for that equation is trivialized.

Problem 7.2. Let $n = 2$ and $\mathcal{E}$ be an *evolution* equation upon the unknown $u(x, t)$. Suppose that one layer of abelian nonlocalities (whose derivatives are local differential functions on $\mathcal{E}^\infty$) is enough to find a nontrivial τ-nonlocal recursion. Prove that the associated nonlocal differential recursion operator always admits the decomposition

$$R = \langle \text{local part} \rangle + \sum_\alpha \varphi_\alpha \cdot \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1} \cdot \psi_\alpha,$$

where φ_α is a τ-shadow and ψ_α is the generating section of a local conservation law for $\mathcal{E}$.

By definition, such recursion operators are called *weakly nonlocal*. The weak nonlocality of the recursions is often helpful in the proofs of the locality of the symmetries which these operators produce from the given seed symmetries for the equation $\mathcal{E}$.

(Check whether the known recursions for the KdV and Burgers equations, see above, are indeed weakly nonlocal.)

Part II. Hamiltonian theory

All science is either physics or stamp collecting.
Ernest Rutherford

The second part of this course is oriented towards theoretical physics: we describe the pre-requisites to the BRST/BV- and deformation quantisation techniques. In fact, both concepts realize the fundamental construction of the *differential*, i. e., the parity-reversing evolutionary vector field $\boldsymbol{Q}$ that squares to zero on the appropriate odd horizontal jet superbundles $\overline{J^\infty}(\Pi\xi_\pi)$ over the infinite jet space $J^\infty(\pi)$ for the bundle π of the physical fields.[50] The arising $\boldsymbol{Q}$-cohomology theories are the Becchi–Rouet–Stora–Tyutin (BRST) or the Batalin–Vilkovisky (BV) and the Poisson–Lichnerowicz cohomologies, respectively.

The presence of the odd fibre bundles over $J^\infty(\pi)$, the use of the parity-reversing differential $\boldsymbol{Q}$, and the introduction of the odd Poisson bracket (the Schouten bracket, or the *antibracket*) means that from now on, we let the entire geometry be $\mathbb{Z}_2$-graded. One could view the content of Part I as its $\overline{0}$-component with respect to the parity, which takes the values $\overline{0}$ or $\overline{1}$.

Moreover, we abandon the cradle category of smooth manifolds and smooth fibre bundles.[17, 21, 111] We pass instead to the spaces $J^\infty(\pi^{\mathrm{nC}})$ of maps from the source manifolds (primarily, from the space-time) to free associative algebras factored by the relations of equivalence under the cyclic permutations in words of any length;[49, 71, 101] this generalization of the "smooth" geometry is furthered to the setup of purely noncommutative sources and targets. Nevertheless, we shall explain why such noncommutative picture is still not the full description of the quantum world but only a component of it.

Our reasoning will be essentially based on the postulated existence of just one functional, the *Hamiltonian* $\mathcal{H} \in \overline{H}^n(\pi^{\mathrm{nC}})$ and/or the *master-action* $\boldsymbol{S} \in \overline{H}^n(\Pi\xi_\pi^{\mathrm{nC}})$ that specifies the entire dynamics via the (odd) Poisson bracket.

8. The calculus of noncommutative multivectors

In this lecture we outline the notions and concepts of the calculus of (non)commutative variational multivectors over the spaces of infinite jets of mappings from commutative (non)graded smooth manifolds M^n to the factors $\mathcal{A}$ of associative algebras (possibly, noncommutative or without unit) over the equivalence under the cyclic permutations of the letters in the words. This setup[49] is the proper noncommutative generalization of the jet bundle geometry for manifolds, which we studied in Part I of the course; simultaneously, the definitions and concepts of the calculus can be easily extended to the purely noncommutative geometry of the sources and targets of the mappings.

We formulate the basics of the theory over such noncommutative jet bundles; this direction of research was pioneered in [101]. If, at the end of the day, the target algebra $\mathcal{A}$ is proclaimed (graded-)commutative (and if it satisfies the "smoothness" assumptions), we restore the standard, Gel'fand–Dorfman's calculus of variational multivectors.[100, 78] Alternatively, under the shrinking of the source manifold M^n, which may be our space-time, to a point (or by postulating that the image of M^n in $\mathcal{A}$ is a given element whenever the map $M^n \to \mathcal{A}$ is constant), we reproduce the noncommutative symplectic geometry of [71]. Here, we observe the parallel between the formalism at hand and the concept of closed strings.

We follow all our earlier conventions and preserve the notation. However, let us denote by $\boldsymbol{a}$ the m-tuples of unknowns in order to emphasize their attribution to the associative algebras.

This lecture is structured as follows. We first rephrase — in the noncommutative setup — the notion of the cotangent covering $\mathbb{T}J^\infty(\pi)$ over the noncommutative jet space $J^\infty(M^n \to \mathcal{A})$, and then we formulate the definition of the noncommutative Schouten bracket $[\![\,,\,]\!]$ as the odd Poisson bracket[101] (see also [6, 16]). We relate the odd evolutionary vector fields $\boldsymbol{Q}^{\boldsymbol{\xi}}$ to the operations $[\![\boldsymbol{\xi}, \cdot]\!]$, where $\boldsymbol{\xi}$ is a noncommutative variational multivector.[75] In these terms, we debate the hamletian "presence" or "absence" of the Leibniz rule for the Schouten bracket $[\![\,,\,]\!]$. We affirm the shifted-graded skew-symmetry of $[\![\,,\,]\!]$ and directly verify the Jacobi identity, which stems from the usual Leibniz rule for the derivations $\boldsymbol{Q}^{\boldsymbol{\xi}}$ acting on the bracket $[\![\boldsymbol{\eta}, \boldsymbol{\omega}]\!]$.

The spectral approach $N^m \leftrightarrow C^\infty(N^m)$ to smooth manifolds N^m views certain algebras as the (non)commutative analogs of the rings of smooth functions on N^m (here, along the m-dimensional fibres of the bundles $\pi\colon E^{m+n} \to M^n$). Adopting this point of view, we extend the jet bundle formalism to noncommutative geometry.

Let $\mathsf{Free}\,(a^1, \ldots, a^m)$ be a free associative algebra over $\mathbb{R}$ with the m generators $a^1, \ldots, a^m$; in the meanwhile, let this algebra be not graded. We deal with all the generators uniformly, in the spirit of operads, even if one of these elements is the unit. In what follows, the algebras we deal with could be noncommutative and non-unital.

By definition, the multiplication $\cdot$ in $\mathsf{Free}\,(a^1, \ldots, a^m)$ obeys the identity $(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$ for the concatenation of any words a_1, a_2 and a_3 written in the alphabet $a^1, \ldots, a^m$. For pedagogical reasons, we discard all other relations (if any) in the algebras at hand, leaving them free, except for the following rule: We postulate that all words $a \in \mathsf{Free}\,(a^1, \ldots, a^m)$ of length $\lambda(a)$ determine the equivalence classes under the *cyclic*

permutations t of the letters:[71]

$$a \sim \mathsf{t}(a) \sim \ldots \sim \mathsf{t}^{\lambda(a)-1}(a) \sim \frac{1}{\lambda(a)} \sum_{i=1}^{\lambda(a)} \mathsf{t}^{i-1}(a).$$

Otherwise speaking, for any words $a_1, a_2 \in \mathsf{Free}\,(a^1, \ldots, a^m)$ we set

$$a_1 \cdot a_2 \sim a_2 \cdot a_1$$

(no $\mathbb{Z}_2$-grading in the free associative algebra!) and denote

$$\mathcal{A} = \mathsf{Free}\,(a^1, \ldots, a^m)/\sim\ .$$

If a word consists of just one letter, its equivalence under the cyclic permutations is trivial. If both words a_1 and a_2 are some generators of $\mathsf{Free}\,(a^1, \ldots, a^m)$, then the equivalence encodes the commutativity. However, at the length three the true noncommutativity starts and it then does not retract to the commutative setup. For instance, we have that

$$xxyzz \sim xyzzy \sim yzzxx \sim zzxxy \sim zxxyz \quad \nsim xyxzz$$

for any $x, y, z \in \mathsf{Free}\,(a^1, \ldots, a^m)$.

The cyclic invariance for the classes in $\mathcal{A}$ is fundamental. On one hand, the Leibniz rule is invariant under the cyclic permutations. On the other hand, the main constructions of the calculus of variational multivectors, which we introduce in this lecture, survive under the transition from the (graded-)commutative world to the cyclic-invariant noncommutative world; in fact, the proof of the basic properties of the Schouten bracket does not refer to the commutativity. Thirdly, we note that the arising structures have a striking similarity with the topological closed string theory (in its unusual formulation **over** the space-time $M^{3,1}$): Namely, the operations — such as the multiplication of functions, the commutation of vector fields, or the Schouten bracket of multivectors and the Poisson bracket of the Hamiltonian functionals — amount to the pairs of topological pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

Remark 8.1. By convention, the oriented circles $\mathbb{S}^1$ which carry the letters from the alphabet of $\mathcal{A}$ also carry the marked point which is denoted by $\boldsymbol{\infty}$. Thus, each cyclic-invariant word is a necklace such that the infinity $\boldsymbol{\infty}$ is its lock; finitely many symbols are thread on the circle and are always read from the infinity on their left to the infinity on their right. The elements of the vector space $\mathcal{A}$ are formal sums over $\mathbb{R}$ of such words. The non-graded letters $a^1, \ldots, a^m$ can freely pass through the point $\boldsymbol{\infty}$ without contribution to the sign of the coefficient in front of a word.

The natural multiplication $\times$ on the space $\mathcal{A}$ of cyclic words is as follows:

$$a_1 \times a_2 = \frac{1}{\lambda(a_1)\lambda(a_2)} \sum_{i=1}^{\lambda(a_1)} \sum_{j=1}^{\lambda(a_2)} \mathsf{t}^{i-1}(a_1) \cdot \mathsf{t}^{j-1}(a_2), \qquad a_1, a_2 \in \mathcal{A}. \tag{8.1}$$

Namely, by taking the sum over all possible positions of the infinity $\boldsymbol{\infty}$ between the letters in each of the words, we detach the locks at $\boldsymbol{\infty}$ and, preserving the orientation, join the loose ends of the first string that carries the word a_1 with the two ends of the other string for a_2. We thus endow the space $\mathcal{A}$ with the structure of an algebra over

the ground field $\mathbb{R}$. The normalization by $(\lambda(a_1)\lambda(a_2))^{-1}$ matches the setup with the purely commutative world.

Example 8.1. Consider the free algebra $\mathsf{Free}\,(a^1,\ldots,a^4)$ and let the two words be a^1a^2 and a^3a^4. Then we have that

$$a^1a^2 \times a^3a^4 = \tfrac{1}{4}(a^1a^2a^3a^4 + a^2a^1a^3a^4 + a^3a^1a^2a^4 + a^3a^2a^1a^4) = a^3a^4 \times a^1a^2.$$

Exercise 8.1. Show that this multiplication $\times$ in $\mathcal{A}$ is commutative but in general not associative.

Remark 8.2. Instead of the circles $\mathbb{S}^1$, one could proceed by the Leibniz rule while taking the extra sum over the generators of the fundamental group $\pi_1(\Sigma)$ for less primitive oriented topological spaces, other than the necklace $\mathbb{S}^1$. This generalization of the geometry can be obviously continued from the shifts along the loops in $\pi_1(\Sigma)$ to the higher homotopy groups $\pi_i(\Sigma)$ and the symmetries of their generators.

Remark 8.3. The physical model that involves this (non)commutative geometry is as follows. Each cyclic word describes the string-like field $a(\boldsymbol{x})$ which co-exists in the states a, $\mathsf{t}(a)$, $\ldots$, $\mathsf{t}^{\lambda(a)-1}(a)$ at the points of the space-time $M^{3,1}$. The interaction $|a_1\rangle \otimes |a_2\rangle \mapsto |a_1 \times a_2\rangle$ is such that it does not matter whether $|a_1\rangle$ scatters on $|a_2\rangle$ or vice versa because the multiplication $\times$ in $\mathcal{A}$ is commutative. Then the new word $a_1 \times a_2$ simplifies under the factorization over the extra relations in the algebra (let us repeat that we do not address them here, leaving $\mathcal{A} = \mathsf{Free}\,(a^1,\ldots,a^m)/\sim$). It is thus possible that several synonyms describe the same particle; this picture patterns upon Gamov's DNA-alphabet and his concept of synonyms for the gene code. Independently, the pure states $a^1(\boldsymbol{x}),\ldots,a^m(\boldsymbol{x})$ are borrowed from the quark model.

In this and next lectures we pursue the goal of restoring the *associativity* of the scattering, *i.e.*, the independence of the out-going state from the order of two pairwise collisions. The deformation quantisation technique[10] is the key to the solution of the triangle equation (9.6) on p. 92. Thus the origin of the matching for the two channels in the associative scattering[118] is purely quantum.

The presence of the space-time M^n (and thus, the possibility to encounter the derivatives $\boldsymbol{a}_\sigma = \frac{d^{|\sigma|}}{d\boldsymbol{x}^\sigma}(\boldsymbol{a})$ of the m pure states $\boldsymbol{a}$) motivates the introduction of the noncommutative jet bundles $J^\infty(M^n \to \mathcal{A})$.

Consider the set of smooth maps from the oriented n-dimensional real manifold M^n to the algebra $\mathcal{A}$, which is a vector space of suitable (possibly, infinite) dimension. Constructing the infinite jet space $J^\infty(M^n \to \mathcal{A}) =: J^\infty(\pi^{\mathrm{nC}})$ as in Lecture 1, we enlarge the alphabet $a^1,\ldots,a^m$ of $\mathcal{A}$ by the jet variables[22] $\boldsymbol{a}_\sigma$ of all finite orders $|\sigma|$, here $a^j_\varnothing \equiv a^j$ for $1 \leqslant j \leqslant m$, and by the base variables $\boldsymbol{x}$ (but we expect the theory to be invariant w.r.t. translations along M^n). We denote by $\mathcal{F}(\pi^{\mathrm{nC}})$ the $C^\infty(M^n)$-algebra of the cyclic-invariant words written in such alphabet. For example,

$$a^1a^2_{xx}a^1_x \sim a^2_{xx}a^1_xa^1 \sim a^1_xa^1a^2_{xx} \quad \nsim a^1a^1_xa^2_{xx} \in \mathcal{F}(\pi^{\mathrm{nC}}).$$

[22]A more expansive notation would be

$$a^1_{\sigma_1}\cdot\vec{\mathrm{e}}_1 + \ldots + a^m_{\sigma_m}\cdot\vec{\mathrm{e}}_m,$$

where for each i we have that $a^i_{\sigma_i} \in \mathbb{R}$ are the coordinates with $|\sigma_i| \geqslant 0$ and $\vec{\mathrm{e}}_i$ is the vector in a basis; for the sake of brevity, we shall not write those vectors explicitly.

Remark 8.4. We leave the domain M^n of the maps to be a usual smooth manifold of dimension n. However, one could easily pass to the purely noncommutative geometry of the underlying space-time in π^{nC} by taking n (commuting) derivations $D_1, \ldots, D_n$ of the algebra $\mathcal{A}$ and extending the alphabet of $\mathcal{A}$ by the symbols $D_\sigma(a^j)$ for $1 \leqslant j \leqslant m$ and all multi-indexes σ, here $D_\varnothing(a^j) \equiv a^j$. The crucial observation is that, due to the Leibniz rule, every derivation of $\mathcal{A}$ does respect the postulated cyclic invariance. Therefore, the latter appears as the weakest but the most natural hypothesis in the (non)commutative differential calculus (c. f. Problem 9.2 on p. 95).

We now upgrade the definitions from Lecture 1 to the noncommutative jet space $J^\infty(\pi^{\mathrm{nC}})$; we emphasize that all the constructions and the statements which we establish in what follows for the new, wider setup obviously remain true under the stronger, more restrictive assumptions of the (graded-)commutativity.

The total derivative w.r.t. x^i, $1 \leqslant i \leqslant n$, on $J^\infty(\pi^{\mathrm{nC}})$ is

$$\overrightarrow{\mathrm{d}}/\mathrm{d}x^i = \overrightarrow{\partial}/\partial x^i + \sum_{|\sigma|\geqslant 0} \boldsymbol{a}_{\sigma+1_i} \overrightarrow{\partial}/\partial \boldsymbol{a}_\sigma;$$

the evolutionary derivation $\partial_\varphi^{(\boldsymbol{a})} = \sum\limits_{|\sigma|\geqslant 0} \bigl(\overrightarrow{\mathrm{d}}^{|\sigma|}/\mathrm{d}\boldsymbol{x}^\sigma\bigr)(\varphi) \cdot \overrightarrow{\partial}/\partial \boldsymbol{a}_\sigma$ acts from the left by the Leibniz rule.[23]

Denote by $\overline{\Lambda}^n(\pi^{\mathrm{nC}})$ the space of horizontal forms of the highest (n-th) degree and by $\overline{H}^n(\pi^{\mathrm{nC}})$ the respective cohomology w.r.t. the horizontal differential $\overline{\mathrm{d}} = \sum\limits_{i=1}^n \mathrm{d}x^i \cdot \overrightarrow{\mathrm{d}}/\mathrm{d}x^i$; the Cartan differential on $J^\infty(\pi^{\mathrm{nC}})$ is $\mathrm{d}_{\mathcal{C}} = \mathrm{d}_{\mathrm{dR}} - \overline{\mathrm{d}}$; by convention, the volume form $\mathrm{d}\boldsymbol{x}$ plays the rôle of the marked point on the rings which carry the highest horizontal forms $h\bigl(\boldsymbol{x}, [\boldsymbol{a}]\bigr)\,\mathrm{d}\boldsymbol{x} \in \overline{\Lambda}^n(\pi^{\mathrm{nC}})$. Denote by $\langle\, ,\, \rangle$ the $\overline{\Lambda}^n(\pi^{\mathrm{nC}})$-valued coupling between the spaces of variational covectors $\boldsymbol{p}$ and evolutionary vectors $\partial_\varphi^{(\boldsymbol{a})}$. By default, we pass to the cohomology and, using the integration by parts, normalize the (non)commutative covectors as follows, $\boldsymbol{p}\bigl(\boldsymbol{x}, [\boldsymbol{a}]\bigr) = \sum_{j=1}^m \langle\text{word}\rangle \cdot \mathrm{d}_{\mathcal{C}} a^j \cdot \langle\text{word}\rangle$; one then can freely push $\mathrm{d}_{\mathcal{C}}\boldsymbol{a}$ left- or rightmost using the cyclic invariance. (Due to the structure of $\partial_\varphi^{(\boldsymbol{a})}$, the value $\langle \boldsymbol{p}, \varphi\rangle$ is well-defined irrespective of the normalization, the value $\langle \boldsymbol{p}, \partial_\varphi^{(a)}\rangle$ is well defined irrespective of the normalization of the multi-index (which becomes empty) of $\mathrm{d}_{\mathcal{C}} a^j_\varnothing$ in $\boldsymbol{p}$.) We note that the derivatives $\partial/\partial a^j_\sigma$ and the differentials $\mathrm{d}_{\mathcal{C}} a^j_\sigma$ play the rôle of the marked points $\boldsymbol{\infty}$ in the cyclic words $\partial_\varphi^{(\boldsymbol{a})}$ and $\boldsymbol{p}$, respectively. The coupling $\langle \boldsymbol{p}, \partial_\varphi^{(\boldsymbol{a})}\rangle$ of the two oriented circles $\mathbb{S}^1$ carrying $\boldsymbol{p}$ and $\partial_\varphi^{(\boldsymbol{a})}$ goes along the usual lines of detaching the locks at the marked points and then joining the loose ends, preserving the orientation. (Again, this constitutes the pair of topological pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.)

Let $A\colon \boldsymbol{p} \to \partial^{(a)}_{A(\boldsymbol{p})}$ be a Noether noncommutative linear matrix operator in total derivatives, containing –apart from the total derivatives– the operators $(a\cdot)$ and $(\cdot a)$ of left- and right-multiplication by open words $a \in \mathcal{A}$ that are always read from left to right. The adjoint operator $\overrightarrow{A}^\dagger$ is defined from the equality $\langle \boldsymbol{p}_1, A(\boldsymbol{p}_2)\rangle = \langle \boldsymbol{p}_2, \overrightarrow{A}^\dagger(\boldsymbol{p}_1)\rangle$, in which we first integrate by parts and then transport the even covector $\boldsymbol{p}_2$ around the circle. Let us remark that the operators are "measured from the comma" in the

[23]By construction, the noncommutative evolutionary derivations determine the flat deformations of the sheafs of the algebras $\mathcal{A}$ on the base manifolds M^n.

coupling $\langle\,,\,\rangle$: namely, the *right* multiplication in the counterclockwise-acting operator $\overrightarrow{A}^{\dagger}$ corresponds to the *left* multiplication in the clockwise-acting $\overleftarrow{A}^{\dagger}$, and vice versa.

Lemma 8.1. For each evolutionary vector field $\partial_{\varphi}^{(\boldsymbol{a})}$, the induced velocity $\dot{\boldsymbol{p}} = \mathrm{L}_{\partial_{\varphi}^{(\boldsymbol{a})}}(\boldsymbol{p})$ equals $\dot{\boldsymbol{p}} = \partial_{\varphi}^{(\boldsymbol{a})}(\boldsymbol{p}) + (\boldsymbol{p})\big(\overleftarrow{\ell_{\varphi}^{(\boldsymbol{a})\,\dagger}}\big)$, where $\ell_{\varphi}^{(\boldsymbol{a})\,\dagger}$ is the adjoint to the linearization, which is $\ell_{\varphi}^{(\boldsymbol{a})}(\delta\boldsymbol{a}) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big|_{\varepsilon=0}\varphi\big(\boldsymbol{x}, [\boldsymbol{a} + \varepsilon\,\overrightarrow{\delta\boldsymbol{a}}]\big)$, and where $\partial_{\varphi}^{(\boldsymbol{a})}$ acts on $\boldsymbol{p}$ componentwise.

The proof is straightforward.

8.1. **Noncommutative multivectors.** The covectors $\boldsymbol{p}\big(\boldsymbol{x}, [\boldsymbol{a}]\big)$ were even. We reverse their parity, $\Pi\colon \boldsymbol{p} \mapsto \boldsymbol{b}\big(\boldsymbol{x}, [\boldsymbol{a}]\big)$, preserving the topology of the bundles but postulating that all objects are at most polynomial in finitely many derivatives of $\boldsymbol{b}$. Next, we consider the noncommutative variational cotangent superspace $\overline{J^\infty}\big(\Pi\widehat{\pi}_{\pi}^{\mathrm{nC}}\big) = J^\infty(\Pi\widehat{\pi}^{\mathrm{nC}}) \times_{M^n} J^\infty(\pi^{\mathrm{nC}})$, see [100, 78] and [50]. In effect, we declare that $\boldsymbol{b}$, $\boldsymbol{b}_{\boldsymbol{x}}$, $\boldsymbol{b}_{\boldsymbol{xx}}$, $\ldots$, $\boldsymbol{b}_\tau$ are the extra, odd jet variables on top of the old, even $\boldsymbol{a}_\sigma$'s. The total derivatives $\overrightarrow{\mathrm{d}}/\mathrm{d}x^i$ obviously lift onto $\overline{J^\infty}\big(\Pi\widehat{\pi}_{\pi}^{\mathrm{nC}}\big)$ as well as $\overline{\mathrm{d}}$ that yields the cohomology $\overline{H}^n\big(\Pi\widehat{\pi}_{\pi}^{\mathrm{nC}}\big) = \overline{\Lambda}^n\big(\Pi\widehat{\pi}_{\pi}^{\mathrm{nC}}\big)/(\mathrm{im}\,\overline{\mathrm{d}})$. The two components of the evolutionary vector fields $\boldsymbol{Q} = \partial_{\varphi^{\boldsymbol{a}}}^{(\boldsymbol{a})} + \partial_{\varphi^{\boldsymbol{b}}}^{(\boldsymbol{b})}$ now begin with $\dot{\boldsymbol{a}} = \varphi^{\boldsymbol{a}}\big(\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]\big)$ and $\dot{\boldsymbol{b}} = \varphi^{\boldsymbol{b}}\big(\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]\big)$, c. f. Lemma 8.1.

The definition of noncommutative variational k-vectors, their evaluation on k covectors, the definition of the noncommutative variational Schouten bracket, and its inductive calculation are two pairs of distinct concepts.

We would like to define a noncommutative k-vector $\boldsymbol{\xi}$, $k \geqslant 0$, as a cohomology class in $\overline{H}^n\big(\Pi\widehat{\pi}_{\pi}^{\mathrm{nC}}\big)$ whose density is k-linear in the odd $\boldsymbol{b}$'s or their derivatives. This is inconsistent because for every k the given k-vector can in fact be cohomologically trivial. There are two ways out: either by the direct inspection of the values of the multivectors on all k-tuples of the covectors (the calculation is defined in formula (8.2) below, but this option is inconvenient) or by using the normalization of $\boldsymbol{\xi}$.

Definition 8.1. A noncommutative k-vector $\boldsymbol{\xi}$ is the horizontal cohomology class of the element

$$\boldsymbol{\xi} = \langle \boldsymbol{b}, A(\boldsymbol{b}, \ldots, \boldsymbol{b})\rangle / k!\ ,$$

where the noncommutative total differential operator A depends on $(k-1)$ odd entries and may have $\boldsymbol{a}$-dependent coefficients.

Integrating by parts and pushing the letters of the word $\boldsymbol{\xi}$ along the circle, we infer that $\langle \boldsymbol{b}_1, A(\boldsymbol{b}_2, \ldots, \boldsymbol{b}_k)\rangle\big|_{\boldsymbol{b}_i := \boldsymbol{b}} = \langle \boldsymbol{b}_2, A^{\dagger}_{\circlearrowleft}(\boldsymbol{b}_3, \ldots, \boldsymbol{b}_k, \boldsymbol{b}_1)\rangle\big|_{\boldsymbol{b}_i := \boldsymbol{b}}$; note that, each time an odd variable $\boldsymbol{b}_\tau$ reaches a marked point $\boldsymbol{\infty}$ on the circle, it counts the $k-1$ other odd variables whom it overtakes and reports the sign $(-)^{k-1}$ (in particular, $A^{\dagger}_{\circlearrowleft} = A^{\dagger} = -A$ if $k = 2$).

Definition 8.2. The value of the k-vector $\boldsymbol{\xi}$ on k arbitrary covectors $\boldsymbol{p}_i$ is

$$\boldsymbol{\xi}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k) = \frac{1}{k!}\sum_{s \in S_k} (-)^{|s|} \langle \boldsymbol{p}_{s(1)}, A(\boldsymbol{p}_{s(2)}, \ldots, \boldsymbol{p}_{s(k)})\rangle\ . \tag{8.2}$$

We emphasize that, in contrast with [47], we shuffle the arguments but never swap their slots, which are built into the cyclic word $\boldsymbol{\xi}$.

8.2. **Noncommutative Schouten bracket.**

Exercise 8.2. Show that the commutative concatenation (8.1) of the *densities* of two multivectors provides an ill-defined product in $\overline{H}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$.

The genuine multiplication in the algebra $\overline{H}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$ is the odd Poisson bracket (the *antibracket*). We fix the Dirac ordering $\delta\boldsymbol{a}\wedge\delta\boldsymbol{b}$ over each $\boldsymbol{x}$ in $\overline{J^\infty}\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)\to M^n$; note that $\delta\boldsymbol{a}$ is a covector and $\delta\boldsymbol{b}$ is an odd vector[2, §37] so that their coupling equals $+1\cdot\mathrm{d}\boldsymbol{x}$.

Definition 8.3. The noncommutative variational Schouten bracket of two multivectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is

$$[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]=\langle\overrightarrow{\delta\boldsymbol{\xi}}\wedge\overleftarrow{\delta\boldsymbol{\eta}}\rangle. \tag{8.3}$$

In coordinates, this yields

$$[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]=\bigl[\overrightarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{a}\cdot\overleftarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{b}-\overrightarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{b}\cdot\overleftarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{a}\bigr],$$

where (1) all the derivatives are thrown off the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ via the integration by parts, then (2) the letters $\boldsymbol{a}_\sigma$, $\boldsymbol{b}_\tau$, $\delta\boldsymbol{a}$, and $\delta\boldsymbol{b}$, which are thread on the two circles $\delta\boldsymbol{\xi}$ and $\delta\boldsymbol{\eta}$, spin along these rosaries so that the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ match in all possible combinations, and finally, (3) the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ detach from the circles and couple, while the loose ends of the two remaining open strings join and form the new circle.

Lemma 8.2. The Schouten bracket is shifted-graded skew-symmetric: if $\boldsymbol{\xi}$ is a k-vector and $\boldsymbol{\eta}$ an ℓ-vector, then $[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]=-(-)^{(k-1)(\ell-1)}[\![\boldsymbol{\eta},\boldsymbol{\xi}]\!]$.

Proof. It is obvious that the brackets $[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]$ and $[\![\boldsymbol{\eta},\boldsymbol{\xi}]\!]$ contain the same summands which can differ only by signs; now it is our task to calculate these factors and show that the same signs appear at all the summands simultaneously. Let us compare the terms

$$\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{a}}\langle\delta\boldsymbol{a},\delta\boldsymbol{b}\rangle\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{b}}\qquad\text{and}\qquad-\frac{\overrightarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{b}}\langle\delta\boldsymbol{a},\delta\boldsymbol{b}\rangle\frac{\overleftarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{a}}$$

containing the variation of a k-vector $\boldsymbol{\xi}$ with respect to the even variables $\boldsymbol{a}$ and the variation of an ℓ-vector $\boldsymbol{\eta}$ with respect to the odd entries $\boldsymbol{b}$. We first note that the left-to-right transportation of the differential $\delta\boldsymbol{b}$ along the queue of $\ell-1$ odd elements $\boldsymbol{b}_\tau$ in the variation of $\boldsymbol{\eta}$ produces the sign $(-)^{\ell-1}$. The variations disappear in the coupling $\langle\delta\boldsymbol{a},\delta\boldsymbol{b}\rangle=+1$, so it remains to carry, via the infinity ∞, the object $\overrightarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{b}$ of parity $(-)^{\ell-1}$ around the object $\overleftarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{a}$, which remains k-linear in the odd variables after the variation with respect to the even $\boldsymbol{a}$. This yields the sign $(-)^{(\ell-1)\cdot k}$. Therefore, the overall difference in the sign between the terms $\overrightarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{a}\cdot\overleftarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{b}$ and $\overrightarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{b}\cdot\overleftarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{a}$ equals

$$-(-)^{\ell-1}\cdot(-)^{(\ell-1)\cdot k}=-(-)^{(k+1)(\ell-1)}=-(-)^{(k-1)(\ell-1)}.$$

Due to the $(k\leftrightarrow\ell)$-symmetry of the exponent, the same sign factor matches the other pair of summands in the Schouten bracket, namely, $\overrightarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{b}\cdot\overleftarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{a}$ and $\overrightarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{a}\cdot\overleftarrow{\delta\boldsymbol{\xi}}/\delta\boldsymbol{b}$. The proof is complete. □

Define the evolutionary vector field $\boldsymbol{Q}^{\boldsymbol{\xi}}$ on $\overline{J^\infty}\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$ by the rule $\boldsymbol{Q}^{\boldsymbol{\xi}}(\boldsymbol{\eta}) = [\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]$, whence $\boldsymbol{Q}^{\boldsymbol{\xi}} = -\partial^{(\boldsymbol{a})}_{\vec{\delta}\boldsymbol{\xi}/\delta\boldsymbol{b}} + \partial^{(\boldsymbol{b})}_{\vec{\delta}\boldsymbol{\xi}/\delta\boldsymbol{a}}$. The normalization $\boldsymbol{\xi} = \bigl\langle \boldsymbol{b}, A(\boldsymbol{b},\ldots,\boldsymbol{b})\bigr\rangle/k!$ determines $\boldsymbol{Q}^{\boldsymbol{\xi}} = -(-)^{k-1}\frac{1}{(k-1)!}\partial^{(\boldsymbol{a})}_{A(\boldsymbol{b},\ldots,\boldsymbol{b})} + (-)^{k-1}\frac{1}{k!}\partial^{(\boldsymbol{b})}_{\vec{\ell}^{(\boldsymbol{a})\dagger}_{A(\boldsymbol{b}_2,\ldots,\boldsymbol{b}_k)}(\boldsymbol{b}_1)}\bigr|_{\boldsymbol{b}_i:=\boldsymbol{b}}$.

Exercise 8.3. Show that $\boldsymbol{Q}^{\frac{1}{2}\langle\boldsymbol{b},A(\boldsymbol{b})\rangle} = \partial^{(\boldsymbol{a})}_{A(\boldsymbol{b})} - \frac{1}{2}\partial^{(\boldsymbol{b})}_{\vec{\ell}^{(\boldsymbol{a})\dagger}_{A(\boldsymbol{b})}(\boldsymbol{b})}$, see [50, 78].

Remark 8.5 (Is $[\![\,,\,]\!]$ a bi-derivation?). Freeze the coordinates, fix the volume form on M^n, and choose any representatives $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ of the cohomology classes in $\overline{H}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$. The derivation $\boldsymbol{Q}^{\boldsymbol{\xi}}$ acts on the word $\boldsymbol{\eta}$ by the graded Leibniz rule, inserting $\frac{\vec{\mathrm{d}}^{|\sigma|}}{\mathrm{d}\boldsymbol{x}^\sigma}\bigl(\boldsymbol{Q}^{\boldsymbol{\xi}}(\boldsymbol{q})\bigr)$ instead of each letter $\boldsymbol{q}_\sigma$ (here $\boldsymbol{q}$ is $\boldsymbol{a}$ or $\boldsymbol{b}$). Next, promote the letter $\boldsymbol{q}$ to the zero- or one-vector $\boldsymbol{q}\cdot\mathrm{d}\boldsymbol{x} \in \overline{\Lambda}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$ and use the Leibniz rule again to expand the entries

$$[\![\boldsymbol{\xi},\boldsymbol{q}]\!] = (\boldsymbol{\xi})\overleftarrow{\boldsymbol{Q}^{\boldsymbol{q}}} + \text{trivial terms}.$$

This argument shows that the Schouten bracket

$$[\![\,,\,]\!]\colon \overline{\Lambda}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)\times\overline{\Lambda}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)\to\overline{\Lambda}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$$

is always a derivation of *any* of the two arguments but, in general, is *not* a bi-derivation of its two arguments simultaneously.

Besides, the normalization of the final result is a must in order to let us compare any given multivectors; for example, the usual commutator of one-vectors is always transformed to $\bigl\langle\boldsymbol{b}, -\bigl(\partial^{(\boldsymbol{a})}_{\varphi_1}(\varphi_2) - \partial^{(\boldsymbol{a})}_{\varphi_2}(\varphi_1)\bigr)\bigr\rangle = [\![\langle\boldsymbol{b},\varphi_1\rangle,\langle\boldsymbol{b},\varphi_2\rangle]\!]$, with no derivatives falling on $\boldsymbol{b}$. At this point, the Leibniz rule — under the multiplication $\times$ of the densities by elements from $\mathcal{F}(\pi^{\mathrm{nC}})$ — is in general irreparably lost.

Proposition 8.3. The equality

$$\bigl[\boldsymbol{Q}^{\boldsymbol{\xi}},\boldsymbol{Q}^{\boldsymbol{\eta}}\bigr] = \boldsymbol{Q}^{[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]} \tag{8.4}$$

correlates the graded commutator of the graded evolutionary vector fields with the the noncommutative variational Schouten bracket of two multivectors.

Proof. To verify Eq. (8.4), it suffices to inspect the composition of the variations

$$\vec{\delta}([\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]) = \frac{\vec{\delta}([\![\boldsymbol{\xi},\boldsymbol{\eta}]\!])}{\delta\boldsymbol{a}}\cdot\delta\boldsymbol{a} + \frac{\vec{\delta}([\![\boldsymbol{\xi},\boldsymbol{\eta}]\!])}{\delta\boldsymbol{b}}\cdot\delta\boldsymbol{b}.$$

They determine the two terms in $\boldsymbol{Q}^{[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]}$ and, moreover, contain the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ standing at the right of the variational derivatives (which may themselves bear the total derivatives at their left due to the integration by parts).

Suppose that initially the k-vector $\boldsymbol{\xi}$ was written in red ink and $\boldsymbol{\eta}$ in black. Let us also agree that the colour of the ink is preserved in all the formulas which involve $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (say, the red vector field $\boldsymbol{Q}^{\boldsymbol{\xi}}$ inserts, by the Leibniz rule, the red sub-words in the black word $\boldsymbol{\eta}$). It is readily seen that some of the variations $\delta\boldsymbol{a}$ or $\delta\boldsymbol{b}$ in $\vec{\delta}([\![\boldsymbol{\xi},\boldsymbol{\eta}]\!])$ are red and some are black; for indeed, they stem from the argument $[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]$ by the Leibniz rule, whereas some letters in the argument are red and some are black. We now study the superposition of the two Leibniz rules: one for the variation $\vec{\delta}$, which drags the differentials to the *right*, and the other rule for the evolutionary fields $\boldsymbol{Q}^{\boldsymbol{\xi}}$ and $\boldsymbol{Q}^{\boldsymbol{\eta}}$, which act on the arguments from the *left*. Consider first the application of

$\boldsymbol{Q}^{\boldsymbol{\xi}}$ to (without loss of generality) the $\boldsymbol{a}$-component $\overrightarrow{\delta\boldsymbol{\eta}}/\delta\boldsymbol{b}$ of the generating section for $\boldsymbol{Q}^{\boldsymbol{\eta}}$ in the left-hand side of (8.4). The field $\boldsymbol{Q}^{\boldsymbol{\xi}}$ is evolutionary, so it dives under all the total derivatives and acts, by the Leibniz rule, on the letters of the argument. That argument is "almost" $\boldsymbol{\eta}$ from which, by a yet another Leibniz rule for the Cartan differential, the black variation $\delta_{\boldsymbol{\eta}}\boldsymbol{b}$ of one $\boldsymbol{b}$ is taken out and transported to the right. Consequently, the graded derivation $\boldsymbol{Q}^{\boldsymbol{\xi}}$ never overtakes that odd object $\delta_{\boldsymbol{\eta}}\boldsymbol{b}$, hence no extra sign is produced (for the even variation $\delta_{\boldsymbol{\eta}}\boldsymbol{a}$ in $\overrightarrow{\delta\boldsymbol{\eta}}$, no signs would appear at all, but such permutations also never occur). The red field $\boldsymbol{Q}^{\boldsymbol{\xi}}$ acts on the black coefficient of $\delta_{\boldsymbol{\eta}}\boldsymbol{b}$: on one hand, it differentiates — one after another — all letters in all the terms. (Note that, in particular, by the Leibniz rule it replaces the black variables $\boldsymbol{b}_\tau$ by their velocities, except for the variables which turned into the variations $\delta_{\boldsymbol{\eta}}\boldsymbol{b}$. But such elements are duly processed in the other summands of the Leibniz formula, in which other odd letters $\boldsymbol{b}_\tau$ yield the variations.) On the other hand, none of the red $\boldsymbol{b}$'s from $\boldsymbol{\xi}$ in $\boldsymbol{Q}^{\boldsymbol{\xi}}$ shows up in the form of the differential at the right. To collect the red variations $\delta_{\boldsymbol{\xi}}\boldsymbol{b}$, which contribute to the rest of the $\boldsymbol{a}$-component of the generating section of the evolutionary field

$$[\boldsymbol{Q}^{\boldsymbol{\xi}},\boldsymbol{Q}^{\boldsymbol{\eta}}]=\boldsymbol{Q}^{\boldsymbol{\xi}}\circ\boldsymbol{Q}^{\boldsymbol{\eta}}-(-)^{(k-1)(\ell-1)}\boldsymbol{Q}^{\boldsymbol{\eta}}\circ\boldsymbol{Q}^{\boldsymbol{\xi}},$$

we repeat the above reasoning for the term $\boldsymbol{Q}^{\boldsymbol{\eta}}\circ\boldsymbol{Q}^{\boldsymbol{\xi}}$ in the graded commutator (c. f. Lemma 8.2).

It only remains to notice that the variations $\delta\boldsymbol{b}$ of all the letters $\boldsymbol{b}$ (red or black) are now properly counted. Comparing the object

$$\boldsymbol{Q}^{\boldsymbol{\xi}}(\overrightarrow{\delta\boldsymbol{\eta}}/\delta_{\boldsymbol{\eta}}\boldsymbol{b})-(-)^{(k-1)(\ell-1)}\boldsymbol{Q}^{\boldsymbol{\eta}}(\overrightarrow{\delta\boldsymbol{\xi}}/\delta_{\boldsymbol{\xi}}\boldsymbol{b})$$

with

$$\overrightarrow{\delta}(\boldsymbol{Q}^{\boldsymbol{\xi}}(\boldsymbol{\eta}))/\delta_{\boldsymbol{\eta}}\boldsymbol{b}-(-)^{(k-1)(\ell-1)}\overrightarrow{\delta}(\boldsymbol{Q}^{\boldsymbol{\eta}}(\boldsymbol{\xi}))/\delta_{\boldsymbol{\xi}}\boldsymbol{b}=\overrightarrow{\delta}([\![\boldsymbol{\xi},\boldsymbol{\eta}]\!])/\delta\boldsymbol{b},$$

we conclude that the two expressions coincide. The same holds for the variation of $[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]$ with respect to $\boldsymbol{a}$, which is also composed by the sum of red variations $\delta_{\boldsymbol{\xi}}\boldsymbol{a}$ and black $\delta_{\boldsymbol{\eta}}\boldsymbol{a}$. This implies equality (8.4). □

Corollary 8.4. The Leibniz rule $\boldsymbol{Q}^{\boldsymbol{\xi}}\bigl([\![\boldsymbol{\eta},\boldsymbol{\omega}]\!]\bigr)=[\![\boldsymbol{Q}^{\boldsymbol{\xi}}(\boldsymbol{\eta}),\boldsymbol{\omega}]\!]+(-)^{(k-1)(\ell-1)}[\![\boldsymbol{\eta},\boldsymbol{Q}^{\boldsymbol{\xi}}(\boldsymbol{\omega})]\!]$, where $\boldsymbol{\omega}\in\overline{H}^n\bigl(\Pi\widehat{\pi}_\pi^{\mathrm{nC}}\bigr)$, is the Jacobi identity

$$[\![\boldsymbol{\xi},[\![\boldsymbol{\eta},\boldsymbol{\omega}]\!]]\!]=[\![[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!],\boldsymbol{\omega}]\!]+(-)^{(k-1)(\ell-1)}[\![\boldsymbol{\eta},[\![\boldsymbol{\xi},\boldsymbol{\omega}]\!]]\!] \tag{8.5}$$

for the noncommutative variational Schouten bracket $[\![\,,\,]\!]$.

Problem 8.1. Prove Lemma 8.1.

In the following two problems we analyse the possibility to generalize —to the (non)commutative jet bundle setup— the equivalent definitions of the Schouten bracket for ordinary manifolds. (Throughout this course, we accept the odd Poisson bracket (8.3) as the standard definition, and also use its reformulation as in (8.4).)

Problem 8.2. Prove that the space $\mathbf{D}_k(\pi^{\mathrm{nC}})$ of the variational (non)commutative k-vectors is not equal to the exterior power $\bigwedge^k\mathbf{D}_1(\pi^{\mathrm{nC}})$ of the spaces of one-vectors

for $k \geqslant 1$. In other words, in presence of the base M^n which yields the horizontal cohomology, the variational multivectors no longer split.

Consequently, the formula

$$[\![X_1 \wedge \ldots \wedge X_k, Y_1 \wedge \ldots \wedge Y_\ell]\!] = \\ = \sum_{\substack{1 \leqslant i \leqslant k \\ 1 \leqslant j \leqslant \ell}} (-)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_k \wedge Y_1 \wedge \ldots \wedge \widehat{Y_j} \wedge \ldots \wedge Y_\ell,$$

which was one of the equivalent definitions of the Schouten bracket $[\![\,,\,]\!]$, becomes obsolete (here $X_i, Y_j \in \mathbf{D}_1(N^m)$).

What was the value of the Schouten bracket

$$[\![X_1 \wedge \ldots \wedge X_k, f]\!]$$

in terms of the values $X_i(f)$ for a smooth function $f \in C^\infty(N^m)$, i. e., a zero-vector ?

Let $\boldsymbol{\xi}$ be a k-vector and, for the time being, $\boldsymbol{p}$ be a true covector. Denote by $\boldsymbol{\xi}(\boldsymbol{p})$ the $(k-1)$-vector which, whenever it is evaluated on its $k-1$ arguments, amounts to the value of $\boldsymbol{\xi}$ on the same $(k-1)$-tuple with $\boldsymbol{p}$ inserted in the *rightmost* slot. We have that $\boldsymbol{\xi}(\boldsymbol{p}_k)(\boldsymbol{p}_{k-1})\ldots(\boldsymbol{p}_1) = \boldsymbol{\xi}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$, where the value of the right-hand side is defined in (8.2).

Another definition of (almost) the value of the variational (non)commutative Schouten bracket $\mathbf{D}_k(\pi^{\text{nC}}) \times \mathbf{D}_\ell(\pi^{\text{nC}}) \to \mathbf{D}_{k+\ell-1}(\pi^{\text{nC}})$ on its arguments is recursive.[78]

Problem 8.3. Let $\boldsymbol{\xi}$ be a k-vector and $\boldsymbol{\eta}$ an ℓ-vector, and let $\boldsymbol{p}$ be the even fibre coordinate in the horizontal jet bundle $\overline{J^\infty}(\widehat{\pi}^{\text{nC}}_{\pi})$ so that $\boldsymbol{p}$ imitates the true covector. Set

$$[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!](\boldsymbol{p}) = \frac{\ell}{k+\ell-1}[\![\boldsymbol{\xi}, \boldsymbol{\eta}(\boldsymbol{p})]\!] + (-)^{\ell-1}\frac{k}{k+\ell-1}[\![\boldsymbol{\xi}(\boldsymbol{p}), \boldsymbol{\eta}]\!],$$

also putting $[\![\partial^{(\boldsymbol{a})}_\varphi, \mathcal{H}]\!] = \partial^{(\boldsymbol{a})}_\varphi(\mathcal{H})$ for $\mathcal{H} \in \overline{H}^n(\pi^{\text{nC}})$.

Prove that the object $[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!](\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k+\ell-1})$, defined recursively in this way, is *almost* the value of the odd Poisson bracket (8.3) on $k+\ell-1$ covectors: the calculation produces the expression in which the applications of the evolutionary fields $\partial^{(\boldsymbol{a})}_\varphi$, for any φ, to the variables $\boldsymbol{p}_i$ must be first replaced by zero, and only then the genuine sections $\boldsymbol{p}_i(\boldsymbol{x}, [\boldsymbol{a}])$ plugged in.

On top of that, inspect whether the two constructions (the standard Definition 8.3 in contrast to this recursive procedure) match identically or the respective values $[\![\boldsymbol{\xi}, \boldsymbol{\eta}]\!](\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k+\ell-1})$ in the two cases differ by the factorial coefficients $\frac{k!\ell!}{(k+\ell-1)!}$ (due to the normalization in Definition 8.1 by the volume $|S_k|$). (Hint: Consider the Schouten bracket of two bi-vectors and evaluate it on three covectors.)

9. Noncommutative Poisson formalism

In this lecture we consider the (non)commutative variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$, which satisfy the classical master-equation[33] $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] = 0$ and determine the Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ on the spaces $\overline{H}^n(\pi^{\mathrm{nC}})$ of the Hamiltonian functionals. With the Poisson structures we associate the class of Hamiltonian evolution differential equations and formalize the complete integrability of such (non)commutative systems. Finally, we return to the problem of restoring — at the quantum level, regarding a given Poisson structure as the linear correction term in the expansion in the Planck constant $\hbar$ for the commutative nonassociative multiplication $\times$ on $\mathcal{A}$, see (8.1), — the *associative* product $\star$ on the space $\overline{H}^n(\pi^{\mathrm{nC}})$ of $\mathcal{A}$-valued functionals for the bundle π^{nC}. We conclude that the three parts of the current exposition correspond to the static, kinematics, and dynamics in the physical model which we introduced in the previous lecture. However, we do not yet derive the full set of equations which govern the interactions of the noncommutative closed string-like fields.

9.1. **Poisson brackets.** Let us endow the space $\overline{H}^n(\pi^{\mathrm{nC}})$ of the functionals for the jet space $J^\infty(\pi^{\mathrm{nC}})$ with the variational Poisson algebra structure. For this we notice that each skew-adjoint noncommutative linear total differential operator $A\colon \boldsymbol{p} \mapsto \partial^{(\boldsymbol{a})}_{A(\boldsymbol{p})}$ yields the bivector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$. Let $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ be zero-vectors, i. e., $\mathcal{H}_i = \int h_i(\boldsymbol{x}, [\boldsymbol{a}])\,\mathrm{d}\boldsymbol{x} \in \overline{H}^n(\pi^{\mathrm{nC}})$. By definition, put

$$\{\mathcal{H}_i, \mathcal{H}_j\}_{\boldsymbol{\mathcal{P}}} := \boldsymbol{\mathcal{P}}\bigl(\overrightarrow{\delta\mathcal{H}_i}/\delta\boldsymbol{a}, \overrightarrow{\delta\mathcal{H}_j}/\delta\boldsymbol{a}\bigr), \tag{9.1}$$

which equals

$$\bigl\langle \overrightarrow{\delta\mathcal{H}_i}/\delta\boldsymbol{a}, A\bigl(\overrightarrow{\delta\mathcal{H}_j}/\delta\boldsymbol{a}\bigr)\bigr\rangle = \partial^{(\boldsymbol{a})}_{A(\overrightarrow{\delta\mathcal{H}_j}/\delta\boldsymbol{a})}(\mathcal{H}_i) \pmod{\operatorname{im}\overline{\mathrm{d}}}.$$

The bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ is bilinear and skew-symmetric by construction.

Exercise 9.1. Represent the calculation of the bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ as the consecutive construction of two pairs of topological pants, and draw the graphical interpretation of the bracket's skew-symmetry.

Exercise 9.2. Show that, in general, the bracket $\{\,,\,\}\colon \overline{H}^n(\pi^{\mathrm{nC}}) \times \overline{H}^n(\pi^{\mathrm{nC}}) \to \overline{H}^n(\pi^{\mathrm{nC}})$ does not restrict as a *bi*-derivation to the horizontal cohomology with respect to $\overline{\mathrm{d}}$.

Definition 9.1. Bracket (9.1) is *Poisson* if it satisfies the Jacobi identity[24]

$$\sum_{\circlearrowright} \bigl\{\{\mathcal{H}_1, \mathcal{H}_2\}_A, \mathcal{H}_3\bigr\}_A = 0, \tag{9.2}$$

which also is

$$\sum_{s\in S_3} (-)^{|s|} \partial^{(\boldsymbol{a})}_{A(\overrightarrow{\delta\mathcal{H}}_{s(3)}/\delta\boldsymbol{a})} \bigl(\tfrac{1}{2}\bigl\langle \overrightarrow{\delta\mathcal{H}}_{s(1)}/\delta\boldsymbol{a}, A\bigl(\overrightarrow{\delta\mathcal{H}}_{s(2)}/\delta\boldsymbol{a}\bigr)\bigr\rangle\bigr) = 0;$$

the operator A in $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$ is then called a *Hamiltonian operator*.

[24] By its origin, the Jacobi identity (9.2) is the slice proportional to $\hbar$ in the scattering equation (9.6) when the full quantum geometry is restored[68] (see below).

Remark 9.1. The tempting notation $\partial^{(\boldsymbol{a})}_{A(\boldsymbol{b})}(\boldsymbol{\mathcal{P}})\bigl(\bigotimes^3 \overrightarrow{\delta}\mathcal{H}_i/\delta\boldsymbol{a}\bigr)=0$ is illegal by Lemma 8.1 that forbids us to set $\dot{\boldsymbol{p}}\equiv 0$ at will so that the vector field $\partial^{(\boldsymbol{a})}_{\varphi}$ would be ill-defined on $\overline{J^\infty}\bigl(\Pi\widehat{\pi}^{\mathrm{nC}}_{\pi}\bigr)$.

Exercise 9.3. Show that the Jacobi identity (9.2) is equivalent to the classical (non)-commutative master-equation

$$\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}}(\boldsymbol{\mathcal{P}})=[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!]=0$$

upon the Poisson bi-vector $\boldsymbol{\mathcal{P}}$.

Let us consider several examples[15] of Hamiltonian differential operators and Poisson bi-vectors.

Exercise 9.4. Prove that every skew-adjoint linear total differential operator $A\colon \boldsymbol{p}\mapsto \partial^{(\boldsymbol{a})}_{A(\boldsymbol{p})}$ whose coefficients belong to the ground ring $C^\infty(M^n)$ –in particular, with constant coefficients– is always a Hamiltonian operator.

Continuing the line of reasoning from Example 7.3 on p. 73 in Lecture 7, let us briefly return to the commutative setup and recognize the first and second Poisson structures for the commutative KdV equation $w_t=-\frac{1}{2}w_{xxx}+3ww_x$.

Exercise 9.5. Prove that the operators $A_1^{\mathrm{KdV}}=\frac{\mathrm{d}}{\mathrm{d}x}$ and $A_2^{\mathrm{KdV}}=-\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3}+\frac{\mathrm{d}}{\mathrm{d}x}\circ w+w\cdot\frac{\mathrm{d}}{\mathrm{d}x}$ are Hamiltonian, and also prove that their arbitrary linear combinations are Hamiltonian operators as well.[89]

- Represent the commutative KdV equation in the form

$$w_t=A_1^{\mathrm{KdV}}\left(\frac{\delta\mathcal{H}_2}{\delta w}\right)=A_2^{\mathrm{KdV}}\left(\frac{\delta\mathcal{H}_1}{\delta w}\right)$$

and find the Hamiltonian functionals $\mathcal{H}_1$ and $\mathcal{H}_2$.

- Calculate the odd evolutionary vector fields $\boldsymbol{Q}$ which correspond to the first and second Poisson structures for the Korteweg–de Vries equation.

Let us note that the compatibility condition for two Poisson structures is nontrivial because the Jacobi identity is quadratic with respect to the bracket.

Definition 9.2. Two (non)commutative Poisson bi-vectors are (*Poisson*)-*compatible* — and are said to constitute a *Poisson pencil* — if their arbitrary linear combination remains Poisson.

Exercise 9.6. Show that two Poisson bi-vectors $\boldsymbol{\mathcal{P}}_1$ and $\boldsymbol{\mathcal{P}}_2$ are compatible if and only if they satisfy the equation $[\![\boldsymbol{\mathcal{P}}_1,\boldsymbol{\mathcal{P}}_2]\!]=0$.

Next, we consider the *non*commutative KdV equation, which we encode by the evolutionary vector field as follows:

$$a_t=\partial^{(a)}_{\varphi}(a)=\left(\bigl(-\tfrac{1}{2}a_{xxx}+\tfrac{3}{2}(a_xa+aa_x)\bigr)\frac{\partial}{\partial a}+\ldots\right)(a). \tag{9.3}$$

Example 9.1 ([101])**.** The noncommutative differential operators $A_1^{\mathrm{ncKdV}}=\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}$ and

$$A_2^{\mathrm{ncKdV}}=-\frac{1}{2}\frac{\overrightarrow{\mathrm{d}}^3}{\mathrm{d}x^3}+\frac{1}{2}\Bigl(\Bigl[a,\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}(\cdot)\Bigr]_+ +\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}\circ[a,\cdot]_+\Bigr)+\frac{1}{2}\Bigl[a,\Bigl(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}\Bigr)^{-1}\circ[a,\cdot]\Bigr],$$

where $[\,,\,]$ is the commutator, $[\alpha,\beta]=\alpha\cdot\beta-\beta\cdot\alpha$, and $[\,,\,]_+$ is the anticommutator, $[\alpha,\beta]_+=\alpha\cdot\beta+\beta\cdot\alpha$, are compatible Hamiltonian operators.

Exercise 9.7. Represent the noncommutative KdV equation (9.3) by using the evolutionary vector field in the images of these two operators,

$$\partial_\varphi^{(a)}=\partial^{(a)}_{A_1^{\text{ncKdV}}\left(\overrightarrow{\delta\mathcal{H}_2^{\text{nC}}}/\delta a\right)}=\partial^{(a)}_{A_2^{\text{ncKdV}}\left(\overrightarrow{\delta\mathcal{H}_1^{\text{nC}}}/\delta a\right)},$$

and find the Hamiltonians $\mathcal{H}_1^{\text{nC}}$ and $\mathcal{H}_2^{\text{nC}}$.

- Does equation (9.3) possess any conserved densities of higher orders, other than the densities of $\mathcal{H}_1^{\text{nC}}$ and $\mathcal{H}_2^{\text{nC}}$?

Remark 9.2. The extension of a given commutative differential equation to the noncommutative (e. g., the cyclic-invariant) setup is a nontrivial problem if one requires that the richness of the original equation's geometry must be preserved. Meaningful noncommutative generalizations are known for the Burgers, KdV, modified KdV, the nonlinear Schrödinger equation (NLS), and for several other commutative integrable systems. Nevertheless, we stress that this transition between the two worlds may not be appropriately called a "quantisation."

9.2. **Completely integrable systems.**

Exercise 9.8. From the Jacobi identity (8.5) for the Schouten bracket deduce that, whenever $\boldsymbol{\mathcal{P}}$ is a Poisson bi-vector, the corresponding odd evolutionary vector field $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}}$ is a differential: $(\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}})^2=0$.

- Reformulate the compatibility of the Poisson structures in terms of the graded commutators of the respective differentials.

Exercise 9.9. Prove that the odd evolutionary vector fields $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}_i}$ for the Korteweg–de Vries equation ($i=1,2$), see Exercise 9.5, are indeed the differentials.

The Poisson differentials $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}}$ give rise to the Poisson(–Lichnerowicz) cohomology[34] groups $\mathrm{H}^k_{\boldsymbol{\mathcal{P}}}$, $k\geqslant 0$. The group $\mathrm{H}^0_{\boldsymbol{\mathcal{P}}}$ is composed by the Casimirs $\mathcal{H}_0\in\overline{H}^n(\pi^{\text{nC}})$ such that $[\![\boldsymbol{\mathcal{P}},\mathcal{H}_0]\!]=0$. The first Poisson cohomology group $\mathrm{H}^1_{\boldsymbol{\mathcal{P}}}$ consists of the Hamiltonian evolutionary vector fields $\partial_\varphi^{(\boldsymbol{a})}$ without Hamiltonian functionals: $[\![\boldsymbol{\mathcal{P}},\partial_\varphi^{(\boldsymbol{a})}]\!]=0$ but $\partial_\varphi^{(\boldsymbol{a})}\neq[\![\boldsymbol{\mathcal{P}},\mathcal{H}]\!]$ for any $\mathcal{H}\in\overline{H}^n(\pi^{\text{nC}})$. The second group $\mathrm{H}^2_{\boldsymbol{\mathcal{P}}}$ contains the nontrivial deformations of the Poisson bi-vector $\boldsymbol{\mathcal{P}}$, i. e., those shifts $\boldsymbol{\mathcal{P}}\mapsto\boldsymbol{\mathcal{P}}+\varepsilon\cdot\boldsymbol{\omega}+\overline{o}(\varepsilon)$ preserving the classical master equation $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!]=0$ which are not generated by the bi-vector $\boldsymbol{\mathcal{P}}$ itself: $\boldsymbol{\omega}\neq[\![\boldsymbol{\mathcal{P}},X]\!]$ for any one-vector X.

The calculation of the Poisson–Lichnerowicz cohomology groups, which is a yet another nontrivial problem, is performed[18, 35] by catching them in between the known de Rham cohomologies of the jet bundles $J^\infty(\pi^{\text{nC}})$. However, if some cohomological obstructions vanish (see below) and the cocycles *are* coboundaries, then this implies the existence of infinitely many Hamiltonians in involution and the presence of hierarchies of commuting flows. This is the renowned (Lenard–)Magri scheme.[89]

Theorem 9.1. *Let $\boldsymbol{\mathcal{P}}_1$ and $\boldsymbol{\mathcal{P}}_2$ be two (non)commutative variational Poisson bi-vectors on the jet space $J^\infty(\pi^{\text{nC}})$, suppose that they are compatible: $[\![\boldsymbol{\mathcal{P}}_1,\boldsymbol{\mathcal{P}}_2]\!]=0$, and assume that the first Poisson–Lichnerowicz cohomology group $\mathrm{H}^1_{\boldsymbol{\mathcal{P}}_1}$ with respect to the differential $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}_1}=[\![\boldsymbol{\mathcal{P}}_1,\cdot\,]\!]$ vanishes. Let $\mathcal{H}_0\in\mathrm{H}^0_{\boldsymbol{\mathcal{P}}_1}\subseteq\overline{H}^n(\pi^{\text{nC}})$ be a Casimir of $\boldsymbol{\mathcal{P}}_1$.*

Then for any integer $k > 0$ there is a Hamiltonian functional $\mathcal{H}_k \in \overline{H}^n(\pi^{\mathrm{nC}})$ such that

$$[\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_{k-1}]\!] = [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_k]\!]. \tag{9.4}$$

Moreover, let $\mathcal{H}_0^{(\alpha)}$ and $\mathcal{H}_0^{(\beta)}$ be any two Casimirs (generally, distinct) for the bi-vector $\boldsymbol{\mathcal{P}}_1$ and construct the two infinite sequences of the functionals $\mathcal{H}_i^{(\alpha)}$ and $\mathcal{H}_j^{(\beta)}$ by using (9.4), here $i, j \geqslant 0$. Let $\partial^{(\boldsymbol{a})}_{\varphi_i^{(\alpha)}} := [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_i^{(\alpha)}]\!]$ and similarly, $\partial^{(\boldsymbol{a})}_{\varphi_j^{(\beta)}} := [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_j^{(\beta)}]\!]$. Then for all i, j and α, β,

- *the Hamiltonians $\mathcal{H}_i^{(\alpha)}$ and $\mathcal{H}_j^{(\beta)}$ Poisson-commute with respect to each of the Poisson brackets, $\{\,,\,\}_{\boldsymbol{\mathcal{P}}_1}$ and $\{\,,\,\}_{\boldsymbol{\mathcal{P}}_2}$;*
- *the evolutionary derivations $\partial^{(\boldsymbol{a})}_{\varphi_i^{(\alpha)}}$ and $\partial^{(\boldsymbol{a})}_{\varphi_j^{(\beta)}}$ commute;*
- *the density of $\mathcal{H}_i^{\alpha)}$ is conserved by virtue of each evolution equation $\boldsymbol{a}_{t_j^{(\beta)}} = \partial^{(\boldsymbol{a})}_{\varphi_j^{(\beta)}}(\boldsymbol{a})$.*

Existence proof. The main homological equality (9.4) is established by induction on k. Let us recall that specifically for any bi-vectors $\boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2$ and $\boldsymbol{\mathcal{P}}_3$, Jacobi identity (8.5) for the Schouten bracket $[\![\,,\,]\!]$ acquires the form

$$[\![[\![\boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2]\!], \boldsymbol{\mathcal{P}}_3]\!] + [\![[\![\boldsymbol{\mathcal{P}}_2, \boldsymbol{\mathcal{P}}_3]\!], \boldsymbol{\mathcal{P}}_1]\!] + [\![[\![\boldsymbol{\mathcal{P}}_3, \boldsymbol{\mathcal{P}}_1]\!], \boldsymbol{\mathcal{P}}_2]\!] = 0. \tag{9.5}$$

Hence we start with a Casimir $\mathcal{H}_0$ for $\boldsymbol{\mathcal{P}}_1$ and obtain that

$$0 = [\![\boldsymbol{\mathcal{P}}_2, 0]\!] = [\![\boldsymbol{\mathcal{P}}_2, [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_0]\!]]\!] = -[\![\boldsymbol{\mathcal{P}}_1, [\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_0]\!]]\!] \mod [\![\boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2]\!] = 0,$$

using Jacobi identity (9.5). The first Poisson cohomology $\mathrm{H}^1_{\boldsymbol{\mathcal{P}}_1} = 0$ is trivial by an assumption of the theorem, hence the closed element $[\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_0]\!]$ in the kernel of $[\![\boldsymbol{\mathcal{P}}_1, \cdot]\!]$ is exact: $[\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_0]\!] = [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_1]\!]$ for some $\mathcal{H}_1$. For $k \geqslant 1$, we have

$$[\![\boldsymbol{\mathcal{P}}_1, [\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{\mathcal{P}}_2, [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{\mathcal{P}}_2, [\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_{k-1}]\!]]\!] = 0$$

using (9.5) and by $[\![\boldsymbol{\mathcal{P}}_2, \boldsymbol{\mathcal{P}}_2]\!] = 0$. Consequently, by $\mathrm{H}^1_{\mathcal{P}_1} = 0$ we have that $[\![\boldsymbol{\mathcal{P}}_2, \mathcal{H}_k]\!] = [\![\boldsymbol{\mathcal{P}}_1, \mathcal{H}_{k+1}]\!]$, and we thus proceed infinitely. □

Remark 9.3. We emphasize that the inductive step, which is the existence of the next, $(k+1)$-th Hamiltonian functional in involution with all the preceding ones, is possible if and only if the seed $\mathcal{H}_0$ is a Casimir,[25] and therefore the Hamiltonian operators A_i in the bi-vectors $\boldsymbol{\mathcal{P}}_i = \frac{1}{2}\langle \boldsymbol{b}, A_i(\boldsymbol{b})\rangle$ are restricted onto the linear subspace which is spanned in the space of variational covectors by the Euler derivatives of the descendants of $\mathcal{H}_0$, i. e., of the Hamiltonians of the hierarchy. We note that the image under A_2 of a generic section from the domain of operators A_1 and A_2 can not be resolved w.r.t. A_1 by (9.4).

For example, the image $\operatorname{im} A_2^{\mathrm{KdV}}$ of the second Hamiltonian operator for the Korteweg–de Vries equation is not entirely contained in the image of the first structure for the generic values of the arguments. But on the linear subspace of descendants $\mathcal{H}_k$ of the Casimir $\int w\,\mathrm{d}x$ for A_1^{KdV}, the inclusion $\operatorname{im} A_2^{\mathrm{KdV}} \subseteq \operatorname{im} A_1^{\mathrm{KdV}}$ is attained.

[25]The Magri scheme starts from any two Hamiltonians $\mathcal{H}_{k-1}, \mathcal{H}_k \in \overline{H}^n(\pi)$ that satisfy (9.4), but we operate with the maximal subspaces of the space of functionals such that the sequence $\{\mathcal{H}_k\}$ can not be extended with any local quantities at $k < 0$.

Exercise 9.10. Prove that the descendants $\mathcal{H}_i^{(\alpha)}$ and $\mathcal{H}_j^{(\alpha)}$ of the same Casimir $\mathcal{H}_0^{(\alpha)}$ Poisson-commute with respect to the bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}_1}$ or $\{\,,\,\}_{\boldsymbol{\mathcal{P}}_2}$ for any $i,j \geqslant 0$. (HINT: Consider separately the two cases, $i - j \equiv 0 \mod 2$ and $i - j \equiv 1 \mod 2$.)

Exercise 9.11. Derive the commutation of the evolutionary fields $\partial^{(\boldsymbol{a})}_{\varphi_i^{(\alpha)}}$ for a fixed α and all $i \geqslant 0$ from the Jacobi identity for the Poisson bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}_1}$.

Definition 9.3. The bi-Hamiltonian evolutionary differential equations which satisfy the hypotheses of Theorem 9.1 and possess as many non-extendable sequences of local Hamiltonians in involution as the number of the unknowns are called the (infinite-dimensional) *completely integrable systems*.

The (non)commutative Korteweg–de Vries equation is the best-known example[15] of a completely integrable partial differential equation.

9.3. **Deformation quantisation.** Before we address the problem of deformation of the non-associative commutative multiplication $\times$ to an associative non-commutative product $\star$, let us clarify a fine geometric aspect which itself determines two distinct concepts in the problem of transition from manifolds to jet spaces. This will also be very important in the next lecture.

Remark 9.4. We have by now generalized many notions and constructions of differential calculus, taking them in the world of usual smooth manifolds N^m and adapting them to the world of the jet spaces for maps $M^n \to N^m$ of (non)commutative manifolds. We observe that, in the apparent absence of the source manifold M^n, the ring $C^\infty(N^m)$ of functions coincides with the space of highest (but now $n = 0$) horizontal forms, whence the Hamiltonians on Poisson manifolds are often treated as "functions" in the literature.

However, let us point to the *two* approaches to manifolds (on top of the spectral approach $N^m \leftrightarrow \mathcal{A} = C^\infty(N^m)$ which we employ in Part II of the course). Namely, one can view a manifold N^m as the set of points. Alternatively, one can view the same manifold as the set of maps of a point to it, i. e., as the space of sections for the bundle $\pi\colon N^m \to \{\mathrm{pt}\}$ over the zero-dimensional manifold with N^m at hand as the fibre; in effect, the points of N^m mark the sections and simultaneously are the graphs of such sections.

This alternative provides two different interpretations for the algebra of functions of points $\boldsymbol{a} \in N^m$: Under the blow-up of the base point $\{\mathrm{pt}\}$ in $\pi\colon N^m \to \{\mathrm{pt}\}$ to the n-dimensional manifold M^n with $n > 0$, which yields the jet bundle $J^\infty(M^n \to N^m)$, the old functions can either remain the $\mathcal{A}$-valued *functions of points* of the total space, i. e., the elements $h \in \mathcal{F}(\pi^{\mathrm{nC}})$ of the $C^\infty(M^n)$-algebra of differential functions on $J^\infty(\pi^{\mathrm{nC}})$, or become the *functions of sections* of the bundle π^{nC}, i. e., the functionals $\mathcal{H} = \int h(\boldsymbol{x}, [\boldsymbol{a}])\mathrm{d}\boldsymbol{x} \in \overline{H}^n(\pi^{\mathrm{nC}})$ from the n-th horizontal cohomology group of $J^\infty(\pi^{\mathrm{nC}})$ so that the evaluation of the integral[26] $\mathcal{H}(\boldsymbol{s}) = \int_{M^n} (j_\infty(\boldsymbol{s})^* h)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ gives the element of $\mathcal{A}$ for each $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$. In other words, the sections $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$ play the rôle of points and the functionals are their $\mathcal{A}$-valued functions.

[26]The integral converges under the proper assumptions about the classes of sections $\Gamma(\pi^{\mathrm{nC}})$; the observations of the short-range nuclear force hint us to consider, e. g., the sections with finite supports.

We notice that the first option is local in the space-time: the reference to a point $\boldsymbol{x} \in M^n$ is preserved after the plugging of the jet $j_\infty(\boldsymbol{s})(\boldsymbol{x})$ of a section $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$ at $\boldsymbol{x}$ in the function $h \in \mathcal{F}(\pi^{\mathrm{nC}})$, which determines the $\mathcal{A}$-valued function on the manifold M^n. Conversely, the second option is space-time nonlocal: it involves the integration over the base M^n, thus losing the information about its points.[27] We see that it is then meaningless to inspect the value $\mathcal{H}(\boldsymbol{s}) \in \mathcal{A}$ at a point $\boldsymbol{x} \in M^n$.

In what follows, we consider both concepts and formulate the respective problems of the deformation quantisation for

(1) the multiplication $\times$, which is $(f \times g)(\theta^\infty_{\mathrm{nC}}) = f(\theta^\infty_{\mathrm{nC}}) \times g(\theta^\infty_{\mathrm{nC}})$ for $f, g \in \mathcal{F}(\pi^{\mathrm{nC}})$, in the algebra $\mathcal{F}(\pi^{\mathrm{nC}})$ of the $\mathcal{A}$-valued differential functions on the jet space $J^\infty(\pi^{\mathrm{nC}}) \ni \theta^\infty_{\mathrm{nC}}$, and for
(2) the multiplication $\times$, which is $(F \times G)(\boldsymbol{s}) = F(\boldsymbol{s}) \times G(\boldsymbol{s})$ for $F, G \in \overline{H}^n(\pi^{\mathrm{nC}})$, in the space $\overline{H}^n(\pi^{\mathrm{nC}})$ of the $\mathcal{A}$-valued integral functionals on the space $\Gamma(\pi^{\mathrm{nC}})$ of sections.[28]

We keep on studying the second option because the expansion (9.7) in $\hbar$ for the associative product $\star$ contains the Poisson bracket (9.8) in the $\hbar$-linear term. Whereas the definition of a Poisson bracket makes no difficulty for the *functionals* F and G (see (9.1–9.2) above), the introduction of the Poisson structure on the space $\mathcal{F}(\pi^{\mathrm{nC}})$ of *functions* is new. This is a purely noncommutative effect which shows up only in the quantisation; in the (graded-)commutative setup, which we analysed in Part I, this Poisson bracket was always equal to zero, whence it was never taken into account.

The multiplication $\times$ in the algebra $\mathcal{A}$ is commutative but not associative; it induces the product $(f \times g)(\theta^\infty_{\mathrm{nC}}) = f(\theta^\infty_{\mathrm{nC}}) \times g(\theta^\infty_{\mathrm{nC}})$ on the space $\mathcal{F}(\pi^{\mathrm{nC}}) \ni f, g$, here $\theta^\infty_{\mathrm{nC}} \in J^\infty(\pi^{\mathrm{nC}})$. We now pose the problem[10, 28, 68] of the construction of the new multiplication $\star$ in the algebra $\mathcal{F}(\pi^{\mathrm{nC}})$ of differential functions on the noncommutative jet space $J^\infty(\pi^{\mathrm{nC}})$: We expect the (non)commutative non-symplectic variational generalization of the Moyal star-product to be *associative*[118] (though in general not commutative) at all points $\theta^\infty_{\mathrm{nC}} \in J^\infty(\pi^{\mathrm{nC}})$:

$$f \star (g \star h) = (f \star g) \star h \tag{9.6}$$

for all $f, g, h \in \mathcal{F}(\pi^{\mathrm{nC}})$. To this end, for any $f, g \in \mathcal{F}(\pi^{\mathrm{nC}})$ we allow

$$\begin{aligned}\star\colon (f \otimes g)(\theta^\infty_{\mathrm{nC}}) \mapsto (f \star g)(\theta^\infty_{\mathrm{nC}}) = f(\theta^\infty_{\mathrm{nC}}) \times g(\theta^\infty_{\mathrm{nC}}) + \mathrm{const} \cdot \hbar B_1(f, g)(\theta^\infty_{\mathrm{nC}}) +{}\\ + \hbar^2 B_2(f, g)(\theta^\infty_{\mathrm{nC}}) + \ldots,\end{aligned} \tag{9.7}$$

where the Planck constant $\hbar$ appears as the deformation parameter.[29]

[27]The second option looks less physical because the integral over the space-time M^n involves the values at the points which are very remote in the past and future. On one hand, the observed values of the physical fields $\boldsymbol{s}(\boldsymbol{x})$ at those times must be zero (indeed, before the birth and after the possible decay of the particle in a reaction), but this would mean the 'memory" of the Universe and the full determinism for its future states, including all collisions.

[28]It is a priori not obvious why the object $F \times G$ exists for all $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$ and if so, that it is local and belongs to $\overline{H}^n(\pi^{\mathrm{nC}})$.

[29]The normalization of the constant in front of $\hbar \cdot B_1$ by the unit (as in [68]) corresponds to the imaginary time in the metric tensor. Perhaps, the convention of [16] with $\frac{\boldsymbol{i}\hbar}{2}$ is more appropriate (here $\boldsymbol{i}^2 = -1$ and the deformation's ground field is $\mathbb{C}$).

Remark 9.5. As soon as the entire setup is based on the formal power series $\mathcal{A}[[\hbar]]$ with respect to the Planck constant, the "gauge" reparametrizations

$$\boldsymbol{a} \mapsto \boldsymbol{a} + \hbar\Delta_1(\boldsymbol{a}) + \hbar^2\Delta_2(\boldsymbol{a}) + \dots\ , \quad 1 \leqslant i \leqslant m,$$

of the generators of the algebra $\mathcal{A}$ induce the transformations

$$f(\theta^\infty_{\mathrm{nC}}) \mapsto f' = f(\theta^\infty_{\mathrm{nC}}) + \hbar\nabla_1(f)(\theta^\infty_{\mathrm{nC}}) + \hbar^2\nabla_2(f)(\theta^\infty_{\mathrm{nC}}) + \dots$$

of the values of the functions $f \in \mathcal{F}(\pi^{\mathrm{nC}})$ at the points $\theta^\infty_{\mathrm{nC}} \in J^\infty(\pi^{\mathrm{nC}})$. Let us use this ambiguity in order to eliminate the symmetric part of the bilinear term $B_1(\cdot,\cdot)$ in expansion (9.7).

Exercise 9.12. Decompose $B_1 = B_1^+ + B_1^-$ to the symmetric and, respectively, antisymmetric parts. Using the commutativity of the multiplication $\times$ in the algebra $\mathcal{A}$, show that the antisymmetric component B_1^- stays invariant under the "gauge" reparametrizations, whereas the symmetric part B_1^+ can always be trivialized.

(Hint: For convenience, restrict the operation B_1^+ on the even diagonal $f = g$ and denote by $\boldsymbol{r}$ the variable along it (c. f. Lecture 8 where the *odd* variables $\boldsymbol{b}$ were introduced). By using induction over the order of the derivatives along the m generators of $\mathcal{A}$, show that the equation

$$B_1^+(\boldsymbol{r},\boldsymbol{r}) - 2\boldsymbol{r}\cdot\nabla_1(\boldsymbol{r}) + \nabla_1(\boldsymbol{r}\cdot\boldsymbol{r}) = 0$$

admits a solution $\nabla_1(\cdot)$ at all orders.)

Consequently, we may assume ab initio that the term $B_1(\cdot,\cdot)$ in (9.7) is skew-symmetric.

Lemma 9.2. Whenever the multiplication $\star$ is associative, the bracket

$$\{\ ,\ \}_\star\colon f\otimes g \mapsto \frac{1}{\mathrm{const}\cdot 2\hbar}\,(f\star g - g\star f)\big|_{\hbar:=0} = B_1^-(f,g) \tag{9.8}$$

is Poisson.

Proof. The bracket $\{\ ,\ \}_\star$ is obviously bilinear and antisymmetric. Taking six times the associativity equation (9.6) for f, g, and $h \in \mathcal{F}(\pi^{\mathrm{nC}})$ (evaluated at $\theta^\infty_{\mathrm{nC}} \in J^\infty(\pi^{\mathrm{nC}})$; here we omit the argument $\theta^\infty_{\mathrm{nC}}$), we obtain that

$$\begin{aligned} &f\star(g\star h) = (f\star g)\star h, \quad g\star(h\star f) = (g\star h)\star f, \quad h\star(f\star g) = (h\star f)\star g,\\ &f\star(h\star g) = (f\star h)\star g, \quad g\star(f\star h) = (g\star f)\star h, \quad h\star(g\star f) = (h\star g)\star f. \end{aligned}$$

Let us now subtract the second line from the first, divide the difference by const $\cdot 2\hbar$ and then set $\hbar := 0$. This yields that

$$\sum_{\circlearrowleft}\{f,\{g,h\}_\star\}_\star = 0,$$

which is the Jacobi identity for $\{\ ,\ \}_\star$ at all $\theta^\infty_{\mathrm{nC}} \in J^\infty(\pi^{\mathrm{nC}})$. $\square$

Consequently, the associative noncommutative multiplications $\star$ on the space $\mathcal{F}(\pi^{\mathrm{nC}})$ of functions are marked at least by the Poisson brackets standing at the first order in $\hbar$; it is still possible that there are some extra free parameters at the higher powers of $\hbar$.

Conjecture 9.3. The commutative but not associative multiplication $\times$ in the algebra $\mathcal{F}(\pi^{\mathrm{nC}})$ of differential functions **can** be deformed via (9.7) at all orders in the Planck constant $\hbar$ to the associative but not commutative star-product $\star$.

We remark that the would-be expansion in $\hbar$ for the product $f \star g$ is independent of either a section $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$ or a pair $\boldsymbol{s}_1 \otimes \boldsymbol{s}_2$ of sections in the evaluations $(f \star g)(\boldsymbol{s})(\boldsymbol{x})$ and $(f \star g)(\boldsymbol{s}_1 \otimes \boldsymbol{s}_2)(\boldsymbol{x})$, respectively. In a sense, formula (9.7) deals with all the sections simultaneously.

We now consider the second option which we stated in Remark 9.4. Let us notice that the entire set of properties and assertions which we accumulated so far for the problem of deformation of the product $\times$ in the algebra $\mathcal{F}(\pi^{\mathrm{nC}})$ of differential functions can be literally transferred to the other operation, $\times\colon (F \otimes G)(\boldsymbol{s}) \mapsto (F \times G)(\boldsymbol{s}) = F(\boldsymbol{s}) \times G(\boldsymbol{s})$ in the space $\overline{H}^n(\pi^{\mathrm{nC}}) \ni F, G$ of the $\mathcal{A}$-valued integral functionals on the space of sections $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$. So, suppose that $\boldsymbol{\mathcal{P}}$ is a variational Poisson bi-vector and $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ is the Poisson bracket which it determines on $\overline{H}^n(\pi^{\mathrm{nC}})$.

Conjecture 9.4. For **every** Poisson bi-vector $\boldsymbol{\mathcal{P}}$, the commutative nonassociative multiplication $\times$ on the space $\mathcal{A}$ of values of the functionals $F, G, H \in \overline{H}^n(\pi^{\mathrm{nC}})$ at the sections $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$ **can** be deformed to the associative,

$$(F \star G) \star H = F \star (G \star H), \tag{9.9}$$

but not commutative multiplication $\star$ which is

$$\star\colon (F \otimes G)(\boldsymbol{s}) \mapsto F(\boldsymbol{s}) \times G(\boldsymbol{s}) + \mathrm{const} \cdot \hbar \int_{M^n} \{F, G\}_{\boldsymbol{\mathcal{P}}}(\boldsymbol{s})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \bar{o}(\hbar), \tag{9.10}$$

meaning that the bi-differential terms at all higher powers of the Planck constant $\hbar$ exist and are expressed in terms of the bi-vector $\boldsymbol{\mathcal{P}}$ and/or its (variational) derivatives (and, moreover, the formulas for such terms do not depend on the choice of a section $\boldsymbol{s} \in \Gamma(\pi^{\mathrm{nC}})$).

For $M^n = \{\mathrm{pt}\}$ and $\mathcal{A} = \mathbb{R}$ for the algebra $C^\infty(N^m)$ of real-valued functions on a smooth manifold N^m, this conjecture was proven in the paper [68], c. f. [67].

Remark 9.6. The canonical formulation of the Formality Conjecture was motivated by the Mirror symmetry.[69] In this course we have developed the toy model of the noncommutative closed string-like fields in which the deformation quantisation problem appears naturally. We expect that the graph technique,[70] which was used in the original proof of the conjecture for the explicit calculation of the terms $B_k(\cdot,\cdot)$ at all orders $k > 0$ in the series (9.10), can be carried over to the geometry of noncommutative jet spaces.

We also recall that the graph technique admits an interpretation in terms of the Feynman path integral for a Poisson sigma-model ([16], see also [1, 117]). In the next lecture we further the formalism of homological evolutionary fields $\boldsymbol{Q}$, bearing in mind, in particular, the cohomological approach to this class of nonlinear systems.

Problem 9.1. Derive the transformation rule for Hamiltonian differential operators under the changes of coordinates on the jet space $J^\infty(\pi^{\mathrm{nC}})$.

Problem 9.2 (The substitution principle)**.** Let the vector bundle π over M^n be commutative.

- Show that locally, each covector $\boldsymbol{p}$ whose coefficients $p_i(x) \in C^\infty(M^n)$ are smooth functions on the base M^n of the jet bundle $J^\infty(\pi)$ can always be represented as the variational derivative $\delta\mathcal{H}/\delta\boldsymbol{a}$ of a functional $\mathcal{H} \in \overline{H}^n(\pi)$.
- Prove that, whenever some identity in total derivatives involves the variational covectors $\boldsymbol{p}$ and holds if their components belong to the ground ring $C^\infty(M^n)$ — instead of the full ring $\mathcal{F}(\pi)$, — then this identity is valid for **all** covectors with arbitrary coefficients.
- Deduce from the above that it suffices to check the identities in total derivatives involving the variational covectors on the *exact* sections $\boldsymbol{p} = \delta\mathcal{H}/\delta\boldsymbol{a}$ only.

Under which assumptions on the bundle π^{nC} does this *substitution principle* remain valid for the *non*commutative jet bundle?

(HINT: The base of the bundle π^{nC} can also be a noncommutative manifold, see Remark 8.4 on p. 81.)

Problem 9.3. Let $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$ be any Poisson bi-vector. By inspecting the coefficient of $\partial/\partial\boldsymbol{a}$ in the evolutionary vector field $(\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}})^2$ prove that the image of the Hamiltonian operator A is closed under commutation, $[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A$, and calculate the Lie algebra structure $[\,,\,]_A$ which is induced on the domain of A. (This concept will be central in the next lecture.)

- Show that the commutator $\bigl[[\![\boldsymbol{\mathcal{P}}, \mathcal{H}_1]\!], [\![\boldsymbol{\mathcal{P}}, \mathcal{H}_2]\!]\bigr]$ of two Hamiltonian vector fields is again a Hamiltonian vector field and find its Hamiltonian functional.

(The bracket $[\![\mathcal{H}_2, [\![\boldsymbol{\mathcal{P}}, \mathcal{H}_1]\!]]\!]$, which equals $\{\mathcal{H}_1, \mathcal{H}_2\}_{\boldsymbol{\mathcal{P}}}$, is called the *derived bracket*[72] for $\mathcal{H}_1$ and $\mathcal{H}_2$.)

Problem 9.4. Prove that the Hamiltonian functionals $\mathcal{H}_i^{(\alpha)}$ and $\mathcal{H}_j^{(\beta)}$ which belong to the sequences of descendants from two *different* Casimirs $\mathcal{H}_0^{(\alpha)} \neq \mathcal{H}_0^{(\beta)}$ in the Magri scheme Poisson-commute with respect to both Poisson brackets, and show that the evolutionary vector fields $[\![\boldsymbol{\mathcal{P}}_k, \mathcal{H}_i^{(\alpha)}]\!]$ and $[\![\boldsymbol{\mathcal{P}}_\ell, \mathcal{H}_j^{(\beta)}]\!]$ also commute, here $k, \ell = 1, 2$.

We now unveil the geometry of the completely integrable Drinfel'd–Sokolov hierarchies[22] which are related to the nonperiodic 2D Toda chains associated with the root systems of semi-simple complex Lie algebras (see [62]; we also refer to [64] for a pedagogical exposition of the rank two cases). In the following three problems we bring together many examples and exercises scattered in this course. (Also, these problems shed more light on the geometry of the Darboux-integrable Liouville-type 2D Toda chains;[112, 119] we continue their study in the next lecture.)

Problem 9.5. Suppose that two evolutionary systems $\boldsymbol{u}_t = \varphi(\boldsymbol{x}, [\boldsymbol{u}])$ and $\mathfrak{m}_t = \Psi(\boldsymbol{x}, [\mathfrak{m}])$ are related by a differential substitution $\mathfrak{m} = \mathfrak{m}(\boldsymbol{x}, [\boldsymbol{u}])$ so that $\mathfrak{m}_t = \partial_\varphi^{(\boldsymbol{u})}(\mathfrak{m}(\boldsymbol{x}, [\boldsymbol{u}]))$. Suppose further that the substitution is itself such that adjoint linearization $\ell_{\mathfrak{m}}^{(\boldsymbol{u})\dagger}$ is a Hamiltonian operator for the equation $\mathfrak{m}_t = -(\ell_{\mathfrak{m}}^{(\boldsymbol{u})})^\dagger\left(\frac{\delta\mathcal{H}(\boldsymbol{x},[\mathfrak{m}])}{\delta\mathfrak{m}}\right)$. Derive the representation[19, 56]

$$\boldsymbol{u}_t = \frac{\delta}{\delta\mathfrak{m}}\mathcal{H}(\boldsymbol{x}, [\mathfrak{m}]), \quad \mathfrak{m}_t = -\frac{\delta}{\delta\boldsymbol{u}}\mathcal{H}(\boldsymbol{x}, [\mathfrak{m}[\boldsymbol{u}]])$$

for the dynamics of the pair $(\boldsymbol{u}, \mathfrak{m})$ of canonically conjugate coordinates.

• Consider (1) the Korteweg–de Vries equation upon the momentum $w = -\frac{1}{2}v_x$ and the potential KdV equation upon the coordinate v;

(2) the modified KdV equation upon the momentum $\mathfrak{m} = -\frac{1}{2}u_x$ and the potential modified KdV equation $u_t = (u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x})(u_x^2 - u_{xx})$ upon the coordinate u (see Problems 3.1–3.1 on p. 39).

Problem 9.6 (correlated Magri's schemes)**.** Consider the potential modified Korteweg–de Vries equation

$$u_t = (u_x + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}x})(u_x^2 - u_{xx})$$

and the KdV equation

$$w_t = -\tfrac{1}{2}w_{xxx} + 3ww_x,$$

and check that they are related by the Miura substitution[95] $w = u_x^2 - u_{xx}$.

• Show that the potential KdV equation upon v such that $w = -\frac{1}{2}v_x$ is bi-Hamiltonian. (Hint: Both structures are nonlocal.)

• Show that the (potential) modified KdV equation is bi-Hamiltonian (we put $\mathfrak{m} = -\frac{1}{2}u_x$ and denote by $\widehat{B}_1^{\text{mKdV}}$, $\widehat{B}_2^{\text{mKdV}}$ the Poisson pencil for the modified KdV equation and by B_1^{pmKdV}, B_2^{pmKdV} the pencil of nonlocal structures for the potential modified KdV equation).

• Show that the (potential) modified KdV hierarchy with the coordinate u and the momentum $\mathfrak{m}$ (see Problem 9.5) and the (potential) KdV hierarchy upon the canonically conjugate pair (v, w) **share** the Hamiltonian via the Miura substitution $w = w[\mathfrak{m}]$ so that

$$\mathcal{H}_{i+1}^{\text{(p)mKdV}}[u] = \mathcal{H}_i^{\text{(p)KdV}}[w[\mathfrak{m}[u]]],$$

with the only exception for the lowest level in the mKdV hierarchy, the Casimir $\mathcal{H}_0^{\text{(p)mKdV}} = \int u\,\mathrm{d}x$.

• Show that the *junior* structure $\widehat{B}_1^{\text{mKdV}}$ for the mKdV hierarchy induces the *second* Hamiltonian operator A_2^{KdV} via the factorization[56]

$$A_2^{\text{KdV}} = \ell_w^{(\mathfrak{m})} \circ \widehat{B}_1^{\text{mKdV}} \circ (\ell_w^{(\mathfrak{m})})^{\dagger}. \tag{9.11}$$

In this setup, recognize and interpret the operator $\square = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ which generates the Noether symmetries of the Liouville equation. (We remark that identity (9.11) fixes the nonlocalities and prescribes the factorization of the nonlocal second Hamiltonian operator $\widehat{B}_2^{\text{mKdV}}$ for the modified KdV hierarchy.)

Problem 9.7. The potential modified KdV hierarchy starts with the translation $\varphi_0 = u_x$ along x. Construct the nonlocal recursion operator

$$R_{\text{pmKdV}} = B_2^{\text{pmKdV}} \circ (B_1^{\text{pmKdV}})^{-1} \colon \varphi_k \mapsto \varphi_{k+1}$$

and apply it to the right-hand side φ_{-1} of the nonlocal transcription $u_y = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-1}\exp(2u)$ for the hyperbolic Liouville equation $u_{xy} = \exp(2u)$. Compare the seed section φ_0 and the image $R_{\text{pmKdV}}(\varphi_{-1})$.

Problem 9.8. What are the hypotheses on the bundle π^{nC} and the class $\Gamma(\pi^{\text{nC}})$ of the admissible sections under which the multiplication $(F \times G)(\boldsymbol{s}) = F(\boldsymbol{s}) \times G(\boldsymbol{s})$ on $\overline{H}^n(\pi^{\text{nC}}) \ni F, G$ does not depend on the choice of a section $\boldsymbol{s} \in \Gamma(\pi^{\text{nC}})$?

10. Lie algebroids over jet spaces

Let us further the concept of homological evolutionary vector fields $\boldsymbol{Q}$ on the horizontal jet superspaces $\overline{J^\infty}(\Pi\xi_\pi^{\mathrm{nC}})$, creating a uniform approach to, e. g., the Poisson differentials $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}}$, which we began to study in the previous lecture, or the BRST- and BV-differentials, which are used in the quantisation of gauge systems (see the next lecture). Namely, we now define the Lie algebroids over the infinite jet spaces $J^\infty(\pi^{\mathrm{nC}})$ and represent these structures in terms of the differentials $\boldsymbol{Q}$. We extend — in a nontrivial way — the classical notion of Lie algebroids over smooth manifolds; this transition from the world of manifolds N^m to the world of jet spaces $J^\infty(M^n \to N^m)$ is not immediate because the old definition stops working whenever $M^n \neq \{\mathrm{pt}\}$, c. f. Remark 9.4.

We note in passing that the construction of Lie algebroids appears — often in disguise — in various models of differential geometry and mechanics on manifolds (particularly, in Poisson geometry[91]). Firstly, Lie algebras are toy examples of Lie algebroids over a point. The other standard examples are the tangent bundle and the Poisson algebroid structure of the cotangent bundle to a Poisson manifold.[73]

We also duly recall that Lie algebroids over the spaces of finite jets of sections for the *tangent* bundle $\pi\colon TM \to M^n$ were defined in [81]. However, here we let the bundle be arbitrary: we analyse the general case when the base and fibre dimensions may be not related. Hence we illustrate the concept not only by the Poisson–Lie algebroids which are specified by (non)commutative Hamiltonian operators, but also recognizing the Lie algebroid structure of the hyperbolic 2D Toda systems associated with semisimple complex Lie algebras. The third example of Lie algebroids is contained in the next lecture, where we address the BRST- and BV-differentials for the gauge models.

This lecture is organized as follows. We first review the classical definition of Lie algebroids and analyse its properties. It then becomes clear why it is impossible to transfer these structures literally from the usual smooth manifolds to the infinite jet bundles. Namely, it is the Leibniz rule which is lost in the commutators for the (almost never used explicitly) $\mathcal{F}(\pi^{\mathrm{nC}})$-module structure of the space of generating sections for the evolutionary vector fields. Because of this, we find an appropriate consequence of the now-obsolete classical definition and take it as the new definition. Here we analyse the geometry of involutive distributions of operator-valued evolutionary vector fields (in particular, we establish the properties of the bi-differential Christoffel symbols which emerge from the structure constants for the closed algebras). Next, we represent the variational Lie algebroids over jet spaces $J^\infty(\pi^{\mathrm{nC}})$ in terms of the differentials $\boldsymbol{Q}$ on the associated superspaces. Finally, by using the $\boldsymbol{Q}$-structure over the space $J^\infty(\pi^{\mathrm{nC}})$ also endowed with a bi-vector $\boldsymbol{\mathcal{P}}$, we derive a yet another convenient criterion for the verification whether a given differential operator is Hamiltonian.

We remark that in this lecture the constructions of the induced bundles $\pi_\infty^*(\xi)$ and the horizontal jet superbundles $\overline{J^\infty}(\Pi\xi_\pi^{\mathrm{nC}}) \to J^\infty(\pi^{\mathrm{nC}})$ reveal their full strength.

10.1. **Classical Lie algebroids.** Let N^m be a smooth real m-dimensional manifold ($1 \leqslant m \leqslant +\infty$) and denote by $\mathcal{F} = C^\infty(N^m)$ the ring of smooth functions on it. The space $\varkappa = \Gamma(TN)$ of sections of the tangent bundle TN is an $\mathcal{F}$-module. Simultaneously, the space $\varkappa$ is endowed with the natural Lie algebra structure $[\,,\,]$ which is the

commutator of vector fields,

$$[X, Y] = X \circ Y - Y \circ X, \qquad X, Y \in \Gamma(TN). \tag{10.1}$$

As usual, we regard the sections of the tangent bundle as first order differential operators with zero free term.

The $\mathcal{F}$-module structure of the space $\Gamma(TN)$ manifests itself for the generators of $\varkappa$ through the Leibniz rule

$$[f\,X, Y] = (f\,X) \circ Y - f \cdot Y \circ X - Y(f) \cdot X, \qquad f \in \mathcal{F}. \tag{10.2}$$

The coefficient $-Y(f)$ of the vector field X in the last term of (10.2) belongs again to the prescribed ring $\mathcal{F}$. (We use the parentheses in such a way that the Leibniz rule (10.2) matches formula (10.10) for evolutionary vector fields.)

Let $\xi\colon \Omega^{m+d} \to N^m$ be another vector bundle over N and suppose that its fibres are d-dimensional. Again, the space $\Gamma\Omega$ of sections of the bundle ξ is a module over the ring $\mathcal{F}$ of smooth functions on the manifold N^m.

Definition 10.1 ([113])**.** A *Lie algebroid* over a manifold N^m is a vector bundle $\xi\colon \Omega^{d+m} \to N^m$ whose space of sections $\Gamma\Omega$ is equipped with a Lie algebra structure $[\,,\,]_A$ together with a morphism of the bundles $A\colon \Omega \to TN$, called the *anchor*, such that the Leibniz rule

$$[f \cdot \mathcal{X}, \mathcal{Y}]_A = f \cdot [\mathcal{X}, \mathcal{Y}]_A - \bigl(A(\mathcal{Y})f\bigr) \cdot \mathcal{X} \tag{10.3}$$

holds for any $\mathcal{X}, \mathcal{Y} \in \Gamma\Omega$ and any $f \in C^\infty(N^m)$.

Essentially, the anchor in a Lie algebroid is a specific fibrewise-linear mapping from a given vector bundle over a smooth manifold M^m to its tangent bundle.

Lemma 10.1 ([41])**.** The anchor A maps the bracket $[\,,\,]_A$ for sections of the vector bundle ξ to the Lie bracket $[\,,\,]$ for sections of the tangent bundle to the manifold N^m.

Remarkably, the assertion of Lemma 10.1 is often *postulated* (for convenience, rather than derived) as a part of the definition of a Lie algebroid, e. g., see [113, 115] vs [41, 73].

Proof. This property is a consequence of the Leibniz rule (10.3) and the Jacobi identity for the Lie algebra structure $[\,,\,]_A$ in $\Gamma\Omega$. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma\Omega$ and $f \in \mathcal{F}_\Omega$ be arbitrary. Then, by definition, we have the Jacobi identity

$$[[\mathcal{X}, \mathcal{Y}]_A, f \cdot \mathcal{Z}]_A + [[\mathcal{Y}, f \cdot \mathcal{Z}]_A, \mathcal{X}]_A + [[f \cdot \mathcal{Z}, \mathcal{X}]_A, \mathcal{Y}]_A = 0.$$

Now using (10.3), we obtain

$$\begin{aligned} A([\mathcal{X}, \mathcal{Y}]_A)(f) \cdot \mathcal{Z} + f \cdot [[\mathcal{X}, \mathcal{Y}]_A, \mathcal{Z}]_A + [A(\mathcal{Y})(f) \cdot \mathcal{Z}, \mathcal{X}]_A + [f \cdot [\mathcal{Y}, \mathcal{Z}]_A, \mathcal{X}]_A \\ - [A(\mathcal{X})(f) \cdot \mathcal{Z}, \mathcal{Y}]_A + [f \cdot [\mathcal{Z}, \mathcal{X}]_A, \mathcal{Y}]_A = 0, \end{aligned}$$

whence we obviously deduce that

$$\begin{aligned} f \cdot ([[\mathcal{X}, \mathcal{Y}]_A, \mathcal{Z}]_A + [[\mathcal{Y}, \mathcal{Z}]_A, \mathcal{X}]_A + [[\mathcal{Z}, \mathcal{X}]_A, \mathcal{Y}]_A) \\ + (A([\mathcal{X}, \mathcal{Y}]_A)(f) - A(\mathcal{X})(A(\mathcal{Y})(f)) + A(\mathcal{Y})(A(\mathcal{X})(f))) \cdot \mathcal{Z} = 0, \end{aligned}$$

and the assertion easily follows by using the Jacobi identity for $[\,,\,]_A$ one more time. $\square$

Corollary 10.2. The anchors A are the Lie algebra homomorphisms:

$$\begin{array}{ccc} \mathcal{X}\otimes_{\mathbb{R}}\mathcal{Y} & \xrightarrow{[\,,]_A} & [\mathcal{X},\mathcal{Y}]_A \\ \Big\downarrow{\scriptstyle A\times A} & & \Big\downarrow{\scriptstyle A} \\ A(\mathcal{X})\otimes_{\mathbb{R}}A(\mathcal{Y}) & \xrightarrow{[\,,]} & [A(\mathcal{X}),A(\mathcal{Y})], \qquad \mathcal{X},\mathcal{Y}\in\Gamma\Omega. \end{array}$$

The push-forward of the bracket $[\,,\,]_A$ *does not* produce[30] any new bracket in the space of sections $\Gamma(TN)$. Indeed, the Lie algebra $(\Gamma\Omega, [\,,\,]_A)$ is isomorphic to the quotient of a Lie subalgebra in $\mathfrak{g} = (\Gamma(TN), [\,,\,])$ over the kernel $\ker A$ of the homomorphism A.

Corollary 10.3. In the above notation, the Leibniz rule (10.3) is the following commutative diagram:

$$\begin{array}{ccc} f\cdot\mathcal{X}\otimes_{\mathbb{R}}\mathcal{Y} & \xrightarrow{[\,,]_A} & [f\mathcal{X},\mathcal{Y}]_A = \text{Eq. (10.3)} \\ \Big\downarrow{\scriptstyle A\times A} & & \Big\downarrow{\scriptstyle A} \\ f\cdot A(\mathcal{X})\otimes_{\mathbb{R}}A(\mathcal{Y}) & \xrightarrow{[\,,]} & [fA(\mathcal{X}),A(\mathcal{Y})] = \text{Eq. (10.2)}, \end{array}$$

where the matching with (10.2) is $X = A(\mathcal{X})$ and $Y = A(\mathcal{Y})$. Consequently, the structure of a Lie algebroid over N^m *does not* extend the set of endomorphisms $\{f\cdot\colon \mathfrak{g}\to\mathfrak{g},\ f\in C^\infty(N^m)\}$ for the Lie algebra $\mathfrak{g}$ of vector fields on the manifold N^m. Indeed, the multiplication of a section $\mathcal{X}\in\Gamma\Omega$ by a ring element $f\in\mathcal{F}$ corresponds to the multiplication of the vector field $X = A(\mathcal{X})$ by the same ring element f.

Lemma 10.4 ([113])**.** Equivalently, a Lie algebroid structure on Ω is a homological vector field Q on $\Pi\Omega$ (take the fibres of Ω, reverse their parities, and thus obtain the new super-bundle $\Pi\Omega$ over N^m). These homological vector fields, which are the differentials on $C^\infty(\Pi\Omega) = \Gamma(\bigwedge^\bullet\Omega^*)$, equal

$$Q = A_i^\alpha(q) b^i \frac{\partial}{\partial q^\alpha} - \frac{1}{2} b^i\, c_{ij}^k(q)\, b^j \frac{\partial}{\partial b^k}, \qquad [Q,Q]=0 \iff 2Q^2 = 0, \tag{10.4}$$

where

- (q^α) is a system of local coordinates near a point $q\in N^m$,
- (p^i) are local coordinates along the d-dimensional fibres of Ω and (b^i) are the respective coordinates on $\Pi\Omega$, and
- $[e_i,e_j]_A = c_{ij}^k(q)e_k$ gives the structure constants for a d-element local basis (e_i) of sections in $\Gamma\Omega$ over the point q, and $A(e_i) = A_i^\alpha(q)\cdot\partial/\partial q^\alpha$ is the image of e_i under the anchor A.

Sketch of the proof. The anti-commutator $[Q,Q] = 2Q^2$ of the odd vector field Q with itself is a vector field. Its coefficient of $\partial/\partial q^\alpha$ vanishes because A is the Lie algebra homomorphism by Lemma 10.1. The equality to zero of the coefficient of $\partial/\partial b^q$ is achieved in three steps. First, we notice that the application of the second term in (10.4) to itself by the graded Leibniz rule for vector fields yields the overall numeric

[30]This scheme, which underlies the Inönü–Wigner contractions of Lie algebras,[45, 98] is still able to generate new, non-isomorphic Lie algebras from a given one by passing to the limits $\epsilon\to+0$ in parametric families A_ϵ, $\epsilon\in(0,1]$, with a nontrivial analytic behaviour at $\epsilon=0$, see Example 10.1.

factor $\frac{1}{4}$, but it doubles due to the skew-symmetry of the structure constants c_{ij}^k. Second, we recognize the right-hand side of Leibniz rule (10.3), with $c_{ij}^k(q)$ for $f \in C^\infty(N^m)$, in the term

$$\frac{1}{2}\sum\nolimits_{ijn} b^i b^j b^n \left(-A_n^\alpha \frac{\partial}{\partial q^\alpha}\bigl(c_{ij}^q(q)\bigr) + c_{ij}^\ell c_{\ell n}^q\right) \cdot \frac{\partial}{\partial b^q}. \tag{10.5}$$

Third, we note that the *cyclic* permutation $i \to j \to n \to i$ of the odd variables b^i, b^j, b^n does not change the sign of (10.5). We thus triple it by taking the sum over the permutations (and duly divide by three) and, as the coefficient of $\partial/\partial b^q$, obtain the Jacobi identity for $[\,,\,]_A$,

$$\tfrac{1}{2} \cdot \tfrac{1}{3} \cdot \sum_{\circlearrowleft} \bigl[[e_i, e_j]_A, e_n\bigr]_A = 0. \tag{10.6}$$

The zero in its right-hand side calculates the required coefficient. Thus, $Q^2 = 0$. $\square$

Let us consider two examples of classical Lie algebroids (we shall later discuss their analogs in the jet bundle framework). These examples will motivate our definition of variational Lie algebroids.

Example 10.1 (Tangent bundle)**.** Consider a fibrewise-acting diffeomorphism $A\colon TN \to TN$ whose restriction onto N^m is the identity and which thus acts by an isomorphism in the fibre over each point $q \in N^m$. Endow the space of sections $\Gamma(TN)$ for the copy $\xi\colon TN \to N^m$ of the tangent bundle, which will be the domain of A, with the Lie algebra structure by the formula[31] $[\mathcal{X}, \mathcal{Y}]_A := A^{-1}\bigl[A(\mathcal{X}), A(\mathcal{Y})\bigr]$, here $\mathcal{X}, \mathcal{Y} \in \Gamma(TN)$. Setting $\Omega = TN$ and choosing A for the anchor, we obtain the Lie algebroid over N^m. Indeed, the Leibniz rule (10.3) holds due to the $C^\infty(N)$-linearity of the anchor: $A(f(q) \cdot \mathcal{X}) = f(q) \cdot A(\mathcal{X})$. Obviously, the new Lie algebra $(\Gamma\Omega, [\,,\,]_A)$ is isomorphic to $\mathfrak{g}$. However, a non-isomorphic structure on $\Gamma\Omega$ can be obtained by a continuous contraction[32] of this Lie algebra:[45, 98] suppose further that there is a family A_ϵ, $\epsilon \in (0, 1]$, of such anchors with a nontrivial analytic behaviour as $\epsilon \to +0$. In the fibre over each point of N^m define the bracket $[\mathcal{X}, \mathcal{Y}]_0 := \lim_{\epsilon \to +0} A_\epsilon^{-1}\bigl[A_\epsilon(\mathcal{X}), A_\epsilon(\mathcal{Y})\bigr]$. Now, the resulting Lie algebra $(\Gamma\Omega, [\,,\,]_0)$ can be non-isomorphic to the initial algebra $\mathfrak{g}$, see [98] for further details.

Example 10.2 (Cotangent bundle)**.** Let $\boldsymbol{\mathcal{P}} \in \Gamma\bigl(\bigwedge^2(TN)\bigr)$ be a nondegenerate Poisson bi-vector field with the Schouten bracket $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] = 0$ vanishing on the manifold N^m. Using the coupling $\langle\,,\,\rangle\colon \Gamma(T^*N) \times \Gamma(TN) \to C^\infty(N)$ and the bi-vector $\boldsymbol{\mathcal{P}}$, we transfer the Lie algebra structure $[\,,\,]$ on $\Gamma(TN)$ to the bracket $[\,,\,]_{\boldsymbol{\mathcal{P}}}$ on $\Gamma(T^*N) \ni \boldsymbol{p}_1, \boldsymbol{p}_2$ and obtain the Koszul–Dorfman–Daletsky–Karasëv bracket[20]

$$[\boldsymbol{p}_1, \boldsymbol{p}_2]_{\boldsymbol{\mathcal{P}}} = \mathrm{L}_{\boldsymbol{\mathcal{P}}\boldsymbol{p}_1}(\boldsymbol{p}_2) - \mathrm{L}_{\boldsymbol{\mathcal{P}}\boldsymbol{p}_2}(\boldsymbol{p}_1) + \mathrm{d}_{\mathrm{dR}}\bigl(\boldsymbol{\mathcal{P}}(\boldsymbol{p}_1, \boldsymbol{p}_2)\bigr), \tag{10.7}$$

here L is the Lie derivative and d_{dR} is the de Rham differential on N^m.

[31] It is highly instructive to derive the standard three-term decomposition (10.15) of the bracket $[\,,\,]_A$ by noting that each vector field from $\mathfrak{g}$ acts on the image of the anchor A by the Leibniz rule.

[32] Of course, not only Lie algebras $\mathfrak{g}$ of vector fields can be contracted. Still, we recall that the physical origin for this procedure is taking the Galilean limit $c \to +\infty$ for the speed of light c in the Lie algebra of Poincaré group of Lorentz transformations at each point of the Minkowski space-time.

Exercise 10.1. By definition, set $\Omega = T^*N$ and determine the anchor $A\colon T^*N \to TN$ by using the bi-vector $\boldsymbol{\mathcal{P}}$ as follows: $\langle \boldsymbol{p}_1, A(\boldsymbol{p}_2)\rangle = \boldsymbol{\mathcal{P}}(\boldsymbol{p}_1, \boldsymbol{p}_2)$. Prove that the triad $(T^*N, [\,,\,]_{\boldsymbol{\mathcal{P}}}, A)$ is the (Poisson–)Lie algebroid (in particular, show that bracket (10.7) satisfies the Jacobi identity).

Remark 10.1. The de Rham differential $\mathrm{d_{dR}}$ on N^m is defined in the complex over the Lie algebra $\mathfrak{g} = \big(\Gamma(TN), [\,,\,]\big)$ by using the Cartan formula. If the Poisson bi-vector $\boldsymbol{\mathcal{P}}$ has the inverse symplectic two-form $\boldsymbol{\mathcal{P}}^{-1}$ such that

$$\boldsymbol{\mathcal{P}}^{-1}[X,Y] = [\boldsymbol{\mathcal{P}}^{-1}X, \boldsymbol{\mathcal{P}}^{-1}Y]_{\boldsymbol{\mathcal{P}}}, \tag{10.8}$$

then the differential $\mathrm{d_{dR}}$ is correlated by $\mathrm{d_{dR}} = [\![\boldsymbol{\mathcal{P}}^{-1}, \cdot]\!]_{\boldsymbol{\mathcal{P}}}$ with the shifted-graded Koszul–Schouten–Gerstenhaber bracket[73] $[\![\,,\,]\!]_{\boldsymbol{\mathcal{P}}}$, which extends (10.8) by the Leibniz rule to $\bigwedge^\bullet(T^*N)$ from $[\![\mathrm{d_{dR}}(h_1), h_2]\!]_{\boldsymbol{\mathcal{P}}} = \{h_1, h_2\}_{\boldsymbol{\mathcal{P}}}$ and $[\![\mathrm{d_{dR}}(h_1), \mathrm{d_{dR}}(h_2)]\!]_{\boldsymbol{\mathcal{P}}} = \mathrm{d_{dR}}\big(\{h_1,h_2\}_{\boldsymbol{\mathcal{P}}}\big)$, here $h_1, h_2 \in C^\infty(N^m)$. The differential $\mathrm{d_{dR}}$ on $\bigwedge^\bullet(T^*N)$ is intertwined[75] with the Poisson differential $\mathrm{d}_{\boldsymbol{\mathcal{P}}} = [\![\boldsymbol{\mathcal{P}}, \cdot]\!]$ on $\bigwedge^\bullet(TN)$ by the formula

$$\Big(\bigwedge^{k+1}\boldsymbol{\mathcal{P}}\Big)\big([\![\boldsymbol{\mathcal{P}}^{-1}, \Psi]\!]_{\boldsymbol{\mathcal{P}}}\big) + [\![\boldsymbol{\mathcal{P}}, \Big(\bigwedge^{k}\boldsymbol{\mathcal{P}}\Big)(\Psi)]\!] = 0, \qquad \forall\,\Psi \in \Gamma\Big(\bigwedge^k(T^*N)\Big). \tag{10.9}$$

Unfortunately, this magnificent correlation does not have an immediate analog on the jet bundles because neither the variational multivectors nor the variational k-forms split, whence the componentwise action of the k copies of $\boldsymbol{\mathcal{P}}$ becomes ill-defined.

10.2. **Variational Lie algebroids.** For consistency, we first summarize some notation. Let M^n be an n-dimensional orientable smooth real manifold, and let $\pi\colon E^{m+n} \xrightarrow[N^m]{} M^n$ be a vector bundle over it with m-dimensional fibres N^m; in this and next lecture we denote by $\boldsymbol{q} = (q^1, \ldots, q^m)$ the m-tuples of fibre coordinates.[33] We denote by $\boldsymbol{q}_\sigma$, $|\sigma| \geqslant 0$, the fibre coordinates in the infinite jet bundle $J^\infty(\pi) \xrightarrow{\pi_\infty} M^n$. The notation $[\boldsymbol{q}]$ stands for the differential dependence on $\boldsymbol{q}$ and its derivatives up to some finite order. We introduce the ring $\mathcal{F}(\pi) := C^\infty(J^\infty(\pi))$ of differential functions in a standard way (see Lecture 1).

Exercise 10.2. Show that, in contrast with classical formula (10.2), the Leibniz rule over the ring $\mathcal{F}(\pi)$ of functions (other than the constants) does not hold for the space $\varkappa(\pi)$ of generating sections φ for evolutionary vector fields $\partial^{(\boldsymbol{q})}_\varphi$ on $J^\infty(\pi)$.

Remark 10.2. Nevertheless, we recover the analog of (10.2) by introducing the left $\mathbb{F}(\pi)$-module structure on $\mathfrak{g}(\pi) = \big(\varkappa(\pi), [\,,\,]\big)$ for the new ring $\mathbb{F}(\pi)$ of differential recursion operators (see Lecture 7). Namely, for $\partial^{(\boldsymbol{q})}_{[\varphi_1,\varphi_2]} = \big[\partial^{(\boldsymbol{q})}_{\varphi_1}, \partial^{(\boldsymbol{q})}_{\varphi_2}\big]$, where φ_1, $\varphi_2 \in \varkappa(\pi)$, and a recursion $\boldsymbol{f} \in \mathbb{F}\colon \varkappa(\pi) \to \varkappa(\pi)$ we have that

$$\big[\boldsymbol{f}(\varphi_1), \varphi_2\big] = \partial^{(\boldsymbol{q})}_{\boldsymbol{f}(\varphi_1)}(\varphi_2) - \boldsymbol{f}\big(\partial^{(\boldsymbol{q})}_{\varphi_2}(\varphi_1)\big) - \big(\partial^{(\boldsymbol{q})}_{\varphi_2}(\boldsymbol{f})\big)(\varphi_1), \tag{10.10}$$

which goes in parallel with (10.2). Yet we emphasize that the ring $\mathcal{F} = C^\infty(N^m)$ of *functions* extends to the ring $\mathbb{F}(\pi)$ of recursion differential *operators* under the transition from the manifold N^m to the jet space $J^\infty(\pi)$ over M^n, whereas in the Poisson formalism

[33]For pedagogical reasons, we deal with the bundle π as if it were purely *commutative* and *not* graded, c. f. Lectures 8–9; likewise, we let π be a *vector* bundle.

the same ring $\mathcal{F}$ becomes the space $\overline{\Lambda}^n(\pi)$ of the highest horizontal differential *forms* (see Lecture 9).

Let $\xi\colon K \to M^n$ be another vector bundle over the same base M^n and take its pull-back $\pi_\infty^*(\xi)\colon K \times_{M^n} J^\infty(\pi) \to J^\infty(\pi)$ along π_∞. We recall that by definition, the $\mathcal{F}(\pi)$-module of sections $\Gamma\bigl(\pi_\infty^*(\xi)\bigr) = \Gamma(\xi) \otimes_{C^\infty(M^n)} \mathcal{F}(\pi)$ is called *horizontal*, see Lectures 2–3. Three horizontal $\mathcal{F}(\pi)$-modules are canonically associated with every jet space $J^\infty(\pi)$: these are the modules of sections $\varkappa(\pi)$ ($\xi = \pi$ for the evolutionary vector fields), $\widehat{\varkappa(\pi)}$ ($\xi = \widehat{\pi}$ for the space of variational covectors), and $\mathrm{Ber}(\pi)$ ($\xi = \bigwedge^n T^*M^n$ for the space of n-th horizontal forms on $J^\infty(\pi)$).

Remark 10.3. The structure of the fibres in the bundle ξ can be composite and their dimension infinite. With the following canonical example we formalize the 2D Toda geometry.[62] Namely, let $\zeta\colon I^{r+n} \underset{W^r}{\longrightarrow} \varSigma^n$ be a vector bundle with r-dimensional fibres in which $w = (w^1, \ldots, w^r)$ are local coordinates. Consider the infinite jet bundle $\xi_\infty\colon J^\infty(\xi) \to M^n$ and, by definition, set either $\xi = \zeta_\infty \circ \zeta_\infty^*(\zeta)\colon \varkappa(\xi) \to M^n$ or $\xi = \zeta_\infty \circ \zeta_\infty^*(\widehat{\zeta})\colon \widehat{\varkappa(\xi)} \to M^n$.

To avoid an inflation of formulas, we shall always briefly denote by $\Gamma\Omega\bigl(\xi_\pi\bigr)$ the horizontal $\mathcal{F}(\pi)$-module at hand.

The main objects of our study are the total differential operators (i. e., matrix, linear differential operators in total derivatives) that take values in the Lie algebra $\mathfrak{g}(\pi) = \bigl(\varkappa(\pi), [\,,\,]\bigr)$ of evolutionary vector fields on $J^\infty(\pi)$. Consider a total differential operator $A\colon \Gamma\Omega\bigl(\xi_\pi\bigr) \to \varkappa(\pi)$ whose image is closed under the commutation in $\mathfrak{g}(\pi)$:

$$[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A \iff \bigl[A(\boldsymbol{p}_1), A(\boldsymbol{p}_2)\bigr] = A\bigl([\boldsymbol{p}_1, \boldsymbol{p}_2]_A\bigr), \quad \boldsymbol{p}_1, \boldsymbol{p}_2 \in \Gamma\Omega\bigl(\xi_\pi\bigr). \tag{10.11}$$

The operator A transfers the Lie algebra structure $[\,,\,]\big|_{\operatorname{im} A}$ in a Lie subalgebra of $\mathfrak{g}(\pi)$ to the bracket $[\,,\,]_A$ in the quotient $\Gamma\Omega\bigl(\xi_\pi\bigr)/\ker A$.

Definition 10.2 ([60])**.** Let the above assumptions and notation hold. The triad

$$\bigl(\Gamma\Omega\bigl(\xi_\pi\bigr), [\,,\,]_A\bigr) \xrightarrow{A} \bigl(\varkappa(\pi), [\,,\,]\bigr) \tag{10.12}$$

is the *variational Lie algebroid* $\mathfrak{A}$ over the infinite jet space $J^\infty(\pi)$, and the Lie algebra homomorphism A is the *variational anchor*.

Essentially, the variational anchor in a variational Lie algebroid is a specific linear mapping from a given horizontal module of sections of an induced bundle over the infinite jet space $J^\infty(\pi)$ to the prescribed horizontal module of generating sections for evolutionary derivations on $J^\infty(\pi)$.

Example 10.3 (Tangent bundle)**.** In [63] we demonstrated that the dispersionless three-component Boussinesq system of hydrodynamic type admits a two-parametric family of finite deformations $[\,,\,]_{\boldsymbol{\epsilon}}$ for the standard bracket $[\,,\,]$ of its symmetries $\operatorname{sym}\mathcal{E}$. For this, we used two recursion differential operators $R_i\colon \operatorname{sym}\mathcal{E} \to \operatorname{sym}\mathcal{E}$, $i = 1, 2$, the images of which are closed under commutation and which are *compatible* in this sense, spanning the two-dimensional space of the operators $R_{\boldsymbol{\epsilon}}$ with involutive images. We obtain the new brackets $[\,,\,]_{\boldsymbol{\epsilon}}$ via (10.11) (c. f. [45, 98] and references therein): $\bigl[R_{\boldsymbol{\epsilon}}(\boldsymbol{p}_1), R_{\boldsymbol{\epsilon}}(\boldsymbol{p}_2)\bigr] = R_{\boldsymbol{\epsilon}}\bigl([\boldsymbol{p}_1, \boldsymbol{p}_2]_{\boldsymbol{\epsilon}}\bigr)$ for $\boldsymbol{p}_1$, $\boldsymbol{p}_2 \in \operatorname{sym}\mathcal{E}$ and $\boldsymbol{\epsilon} \in \mathbb{R}^2 \setminus \{0\}$.

Example 10.4 (Cotangent bundle)**.** Consider the second Hamiltonian operator $A_2^{\text{KdV}} = -\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + w\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\circ w$ for the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$. The image of A_2^{KdV} is closed under commutation: the bracket $[\,,\,]_{A_2^{\text{KdV}}}$ is (c. f. Problem 9.3)

$$[p_1, p_2]_{A_2^{\text{KdV}}} = \partial_{A_2^{\text{KdV}}(p_1)}(p_2) - \partial_{A_2^{\text{KdV}}(p_2)}(p_1) + \tfrac{\mathrm{d}}{\mathrm{d}x}(p_1)\cdot p_2 - p_1\cdot\tfrac{\mathrm{d}}{\mathrm{d}x}(p_2). \qquad (10.13)$$

Example 10.5 (The 'heavenly' Toda equation)**.** The generators of the Lie algebra of point symmetries for the $(2+1)$-dimensional 'heavenly' Toda equation $u_{xy} = \exp(-u_{zz})$ are $\varphi^x = \widehat{\square}^x\bigl(p(x)\bigr)$ or $\varphi^y = \widehat{\square}^y\bigl(\overline{p}(y)\bigr)$, where p and $\overline{p}$ are arbitrary smooth functions and $\widehat{\square}^x = u_x + \frac{1}{2}z^2\frac{\mathrm{d}}{\mathrm{d}x}$, $\widehat{\square}^y = u_y + \frac{1}{2}z^2\frac{\mathrm{d}}{\mathrm{d}y}$. The image of each operator is closed under commutation such that $[p_1, p_2]_{\widehat{\square}^x} = \partial_{\widehat{\square}^x(p_1)}(p_2) - \partial_{\widehat{\square}^x(p_2)}(p_1) + p_1\cdot\frac{\mathrm{d}}{\mathrm{d}x}(p_2) - \frac{\mathrm{d}}{\mathrm{d}x}(p_1)\cdot p_2$ for any $p_1(x)$ and $p_2(x)$, and similarly for $\widehat{\square}^y$.

Example 10.6. Two distinguished constructions of variational Lie algebroids emerge from the variational (co)tangent bundles over $J^\infty(\pi)$, see Remark 10.3 and Examples 10.3–10.4 above. These two cases involve an important intermediate component, which is the Miura substitution (c. f. Problem 9.6).

Namely, consider the induced fibre bundle $\pi_\infty^*(\zeta)$ and fix its section w. Making no confusion, we continue denoting by the same letter w the fibre coordinates in ζ and the fixed section $w\bigl(\boldsymbol{x}, [\boldsymbol{q}]\bigr) \in \Gamma\bigl(\pi_\infty^*(\zeta)\bigr)$, which is a nonlinear differential operator in $\boldsymbol{q}$.

Obviously, the substitution $w = w\bigl(\boldsymbol{x}, [\boldsymbol{q}]\bigr)$ converts the horizontal $\mathcal{F}(\zeta)$-modules to the submodules of horizontal $\mathcal{F}(\pi)$-modules.[34] By this argument, we obtain the modules

$$\varkappa(\zeta)\big|_{w(\boldsymbol{x},[\boldsymbol{q}])\colon J^\infty(\pi)\to\Gamma(\zeta)} \quad\text{and}\quad \widehat{\varkappa(\zeta)}\big|_{w(\boldsymbol{x},[\boldsymbol{q}])\colon J^\infty(\pi)\to\Gamma(\zeta)},$$

where the latter is the module of sections of the pull-back by π_∞^* for the $\overline{\Lambda}^n(\zeta)$-dual to the induced bundle $\zeta_\infty^*(\zeta)$. We emphasize that by this approach we preserve the correct transformation rules for the sections in $\varkappa(\zeta)$ or $\widehat{\varkappa(\zeta)}$ under the (unrelated!) reparametrizations of the fibre coordinates w and $\boldsymbol{q}$ in the bundles ζ and π, respectively.

We say that the linear operators $A\colon \varkappa(\zeta)\big|_{w(\boldsymbol{x},[\boldsymbol{q}])} \to \varkappa(\pi)$ and $A\colon \widehat{\varkappa}(\zeta)\big|_{w(\boldsymbol{x},[\boldsymbol{q}])} \to \varkappa(\pi)$ subject to (10.11) are the variational anchors of *first* and *second kind*, respectively. The recursion operators with involutive images are examples of the anchors of first kind. All Hamiltonian operators on jet spaces and, more generally, Noether operators with involutive images are variational anchors of second kind (c. f. the next lecture). The operators $\square$, $\overline{\square}$ that yield symmetries of the Euler–Lagrange Liouville-type systems are also anchors of second kind; they are 'non-skew-adjoint generalizations of Hamiltonian operators' [119] in exactly this sense.

Exercise 10.3. Show that under differential reparametrizations $\widetilde{\boldsymbol{q}} = \widetilde{\boldsymbol{q}}[\boldsymbol{q}]\colon J^\infty(\pi) \to \Gamma(\pi)$ and $\widetilde{w} = \widetilde{w}[w]\colon J^\infty(\zeta) \to \Gamma(\zeta)$, the operators A of first kind are transformed according to the formula $A \mapsto \widetilde{A} = \ell_{\widetilde{\boldsymbol{q}}}^{(\boldsymbol{q})} \circ A \circ \ell_w^{(\widetilde{w})}\Big|_{\substack{w=w(\boldsymbol{x},[\boldsymbol{q}])\\ \boldsymbol{q}=\boldsymbol{q}[\widetilde{\boldsymbol{q}}]}}$, whereas the operators of

[34] For instance, we have $\zeta = \pi$ for the recursion operators $\varkappa(\pi) \to \varkappa(\pi)$, see Example 10.3. The domains of Hamiltonian operators on the empty jet spaces are $\widehat{\varkappa(\pi)}$, c. f. Example 10.4. Here we set $w = \mathrm{id}\colon \Gamma(\pi) \to \Gamma(\zeta)$ in both cases.

second kind obey the rule $A \mapsto \widetilde{A} = \ell^{(\boldsymbol{q})}_{\widetilde{\boldsymbol{q}}} \circ A \circ \bigl(\ell^{(w)}_{\widetilde{w}}\bigr)^{\dagger}\Bigr|_{\substack{w=w(\boldsymbol{x},[\boldsymbol{q}]) \\ \boldsymbol{q}=\boldsymbol{q}[\widetilde{\boldsymbol{q}}]}}$. (The symbol † denotes the adjoint operator.)

Let us analyse the standard structure of the induced bracket $[\,,\,]_A$ on the quotient of $\Gamma\Omega\bigl(\xi_\pi\bigr)$ by $\ker A$. By the Leibniz rule, two sets of summands appear in the bracket of evolutionary vector fields $A(\boldsymbol{p}_1), A(\boldsymbol{p}_2)$ that belong to the image of the variational anchor A:

$$\bigl[A(\boldsymbol{p}_1), A(\boldsymbol{p}_2)\bigr] = A\bigl(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_2) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_2)}(\boldsymbol{p}_1)\bigr) + \bigl(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(A)(\boldsymbol{p}_2) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_2)}(A)(\boldsymbol{p}_1)\bigr). \quad (10.14)$$

In the first summand we have used the permutability of evolutionary derivations and total derivatives. The second summand hits the image of A by construction. Consequently, the Lie algebra structure $[\,,\,]_A$ on the domain of A equals

$$[\boldsymbol{p}_1, \boldsymbol{p}_2]_A = \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_2) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_2)}(\boldsymbol{p}_1) + \{\{\boldsymbol{p}_1, \boldsymbol{p}_2\}\}_A. \quad (10.15)$$

Thus, the bracket $[\,,\,]_A$, which is defined up to $\ker A$, contains the two standard summands and the bi-differential skew-symmetric part $\{\{\,,\,\}\}_A \in \mathcal{C}\mathrm{Diff}\bigl(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\bigr)$. For any $\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k \in \Gamma\Omega(\xi_\pi)$, the Jacobi identity for (10.15) is

$$\begin{aligned}
0 &= \sum_{\circlearrowright} \bigl[[\boldsymbol{p}_i, \boldsymbol{p}_j]_A, \boldsymbol{p}_k\bigr]_A = \sum_{\circlearrowright} \bigl[\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_i)}(\boldsymbol{p}_j) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_j)}(\boldsymbol{p}_i) + \{\{\boldsymbol{p}_i, \boldsymbol{p}_j\}\}_A, \boldsymbol{p}_k\bigr]_A \\
&= \sum_{\circlearrowright} \Bigl\{ \partial^{(\boldsymbol{q})}_{A\left(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_i)}(\boldsymbol{p}_j) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_j)}(\boldsymbol{p}_i)\right)}(\boldsymbol{p}_k) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_k)}\bigl(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_i)}(\boldsymbol{p}_j) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_j)}(\boldsymbol{p}_i)\bigr) \\
&\quad + \{\{\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_i)}(\boldsymbol{p}_j) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_j)}(\boldsymbol{p}_i), \boldsymbol{p}_k\}\}_A \\
&\quad + \partial^{(\boldsymbol{q})}_{A\left(\{\{\boldsymbol{p}_i,\boldsymbol{p}_j\}\}_A\right)}(\boldsymbol{p}_k) - \underline{\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_k)}\bigl(\{\{\boldsymbol{p}_i, \boldsymbol{p}_j\}\}_A\bigr)} + \{\{\{\{\boldsymbol{p}_i, \boldsymbol{p}_j\}\}_A, \boldsymbol{p}_k\}\}_A \Bigr\}. \quad (10.16)
\end{aligned}$$

The underlined summand contains derivations of the coefficients of $\{\{\,,\,\}\}_A$, which belong to $\mathcal{F}(\pi)$. Even if the action of evolutionary fields $\partial^{(\boldsymbol{q})}_{\varphi} = \varphi\frac{\partial}{\partial \boldsymbol{q}} + \ldots$ on the the arguments of A is set to zero (which makes sense, see section 10.3), these summands may not vanish. Note that the Jacobi identity for $[\,,\,]_A$ then amounts to

$$\sum_{\circlearrowright} \Bigl(-\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_k)}\bigl(\{\{\boldsymbol{p}_i, \boldsymbol{p}_j\}\}_A\bigr) + \{\{\{\{\boldsymbol{p}_i, \boldsymbol{p}_j\}\}_A, \boldsymbol{p}_k\}\}_A\Bigr) = 0. \quad (10.16')$$

Formulas (10.11) and (10.15), formula (10.16), and (10.16′) will play the same rôle in the proof of Theorem 10.6 (see below) as Lemma 10.1 and, respectively, formulas (10.6) and (10.3) [or (10.5)] did for the equivalence of the two classical definitions.

Let us, as it may seem, generalize the notion of Lie algebroids over infinite jet spaces by replacing the variational anchors with the N-tuples of differential operators whose images are subject to collective commutation closure. The linear spaces of such operators become the algebras with bi-differential structure constants. Whereas the Poisson formalism yields the canonical examples of Poisson–Lie algebroids in which the anchors are determined by the Hamiltonian operators, the construction of many "anchors," $N \geqslant 1$, is central for gauge theory. Namely, it is then known as the Berends–Burgers–van Dam hypothesis of the collective commutation closure for the images of the gauge symmetry generators,[14] resulting in the *closed algebras* of field transformations. In this

sense, the next lecture contains another canonical example of variational Lie algebroids, which are called the *gauge algebroids*.[3]

Suppose that $A_i\colon \Gamma\bigl(\pi_\infty^*(\xi_i)\bigr) \to \varkappa(\pi)$ are linear total differential operators that take values in the space of evolutionary vector fields for each $i = 1, \ldots, N$. That is, the domains of the N operators may be different but the target space is common to all of them. Assume further that the images of the N operators A_i are subject to the collective commutation closure,

$$[\operatorname{im} A_i, \operatorname{im} A_j] \subseteq \sum_{k=1}^{N} \operatorname{im} A_k. \tag{10.17}$$

The sum of the images in the right-hand side is not direct because in effect they can overlap (see Remark 9.3 on p. 90).

Exercise 10.4. The setup of *many* operators and their individual domains can always be reduced to one operator on the large domain as follows. Namely, take the Whitney sum over $J^\infty(\pi)$ of the induced bundles $\pi_\infty^*(\xi_i)$, which in down-to-earth terms means that we compose the new "tall" sections

$$\boldsymbol{p} = {}^{\mathrm{t}}\bigl(\boldsymbol{p}_1 \mid \boldsymbol{p}_2 \mid \ldots \mid \boldsymbol{p}_N\bigr) \in \Gamma\Bigl(\bigoplus_{k=1}^{N}{}_{J^\infty(\pi)}\, \pi_\infty^*(\xi_i)\Bigr) =: \Gamma\Omega\bigl(\xi_\pi\bigr), \tag{10.18a}$$

and understand the sections $\boldsymbol{p}$ as the arguments of the new "wide" matrix operator

$$A = \bigl(A_1 \mid A_2 \mid \ldots \mid A_N\bigr). \tag{10.18b}$$

Show that property (10.17) implies that condition (10.11) is satisfied, whence operator (10.18b) is the variational anchor in the variational Lie algebroid whose total space is the quotient of the sum over $J^\infty(\pi)$ of the initial domains for A_i by the kernel $\ker A = \bigl\{\boldsymbol{p} \in \Gamma\Omega\bigl(\xi_\pi\bigr) \mid \sum_{i=1}^{N} A_i(\boldsymbol{p_i}) = 0\bigr\}$.

Remark 10.4. The operator A transfers the commutator $[\,,\,]$ in the Lie algebra of evolutionary vector fields to the Lie algebra structure $[\,,\,]_A$ on the quotient of its domain by the kernel. It is obvious that for a given collection A_1, …, A_N of the operators with a *common* domain (and not necessarily involutive images, but this does not matter) the induced bracket $[\,,\,]_A$ may not determine a well-defined Lie structure on a single copy of that common domain: Indeed, it is the "tall" concatenations $\boldsymbol{p}$ of the N elements $\boldsymbol{p}_i$ from the same space for which the new bracket actually appears.

Assumption (10.17) gives rise to the bi-differential structure constants via

$$\bigl[A_i(\boldsymbol{p}_i), A_j(\boldsymbol{p}_j)\bigr] = \sum_{k=1}^{N} A_k\bigl(\mathbf{c}_{ij}^k(\boldsymbol{p}_i, \boldsymbol{p}_j)\bigr); \tag{10.19}$$

each $\mathbf{c}_{ij}^k$ is the equivalence class of bi-differential operators (possibly, with *non*constant, field-dependent coefficients) with both its arguments and its values taken modulo (at least, see Exercise 10.4) the kernels of the operators A_i, A_j and A_k, respectively. (The kernels can be sufficiently large unless the extra nondegeneracy assumptions are made; for the sake of brevity, we shall not emphasize the presence of such kernels in *all* formulas but rather in few ones, c. f. [60].)

Using the permutability of the evolutionary vector fields with the total derivatives and taking into account the Leibniz rule that always holds for the evolutionary derivations $\partial^{(\boldsymbol{q})}_{A_\ell(\boldsymbol{p}_\ell)}$, we obtain the canonical decomposition of the bi-differential structure constants,

$$\mathbf{c}^k_{ij}(\boldsymbol{p}_i,\boldsymbol{p}_j) = \partial^{(\boldsymbol{q})}_{A_i(\boldsymbol{p}_i)}(\boldsymbol{p}_j)\cdot\delta^k_j - \partial^{(\boldsymbol{q})}_{A_j(\boldsymbol{p}_j)}(\boldsymbol{p}_i)\cdot\delta^k_i + \Gamma^k_{ij}(\boldsymbol{p}_i,\boldsymbol{p}_j). \tag{10.20}$$

Obviously, the convention

$$\Gamma^1_{11} = \{\![\,,\,]\!\}_{A_1}$$

holds if $N = 1$. The symbols Γ^k_{ij} absorb the bi-linear action of $\mathbf{c}^k_{ij}$ by total differential operators on the arguments; the notation is justified by Theorem 10.5, see below. We emphasize that generally speaking, the symbols Γ^k_{ij} alone –that is, *without* the first two standard terms in (10.20)– do not determine any Lie algebra structure: the true Jacobi identity is given by (10.25) where the standard terms are taken into account properly.[35]

Remark 10.5. In general, we have that $\mathbf{c}^k_{ii} \neq 0 \pmod{\ker A_k}$ if $k \neq i$, that is, the image of an operator alone may not be involutive even if the operator (in particular, its domain of definition) is well defined regardless of the entire collection $A_1, \ldots, A_N$. Also, we note that for only one operator ($N = 1$) the Lie algebra of evolutionary vector fields in its image is, generally, non-abelian, which must not be confused with the abelian gauge theories (e. g., Maxwell's electrodynamics) corresponding to one-dimensional Lie groups, which are always commutative.

It is remarkable that the Yang–Mills theories with either abelian or non-abelian gauge groups constitute a regular class of examples with the field-independent (and at most constant, whenever nonzero) coefficients in the bi-differential Christoffel symbols $\Gamma^k_{ij}(\cdot,\cdot)$. On the other hand, *gravity* produces the drastically more involved structure of these symbols with the explicit dependence on the unknown fields in their coefficients. This is immanent also to the Liouville-type systems.

Exercise 10.5. Show that for any $i, j, k \in [1, \ldots, N]$ and for any $\boldsymbol{p}_i, \boldsymbol{p}_j \in \Gamma\Omega(\xi_\pi)$ we have that

$$\Gamma^k_{ij}(\boldsymbol{p}_i,\boldsymbol{p}_j) = -\Gamma^k_{ji}(\boldsymbol{p}_j,\boldsymbol{p}_i) \tag{10.21}$$

due to the skew-symmetry of the commutators in (10.19).

The method by which we introduced the symbols Γ^k_{ij} suggests that under reparametrizations $g\colon \boldsymbol{p}_i \mapsto g\,\boldsymbol{p}_i$ invoked by a differential change of the jet coordinates (whence the g's are linear operators in total derivatives) these symbols obey a proper analog of the standard rule (6.5) for the connection one-forms (see p. 63). This is indeed so.[61]

[35]It is likely that the restoration of the standard component in (10.20) for known bi-differential symbols Γ^k_{ij} is the cause of a fierce struggle in the modern double field theory and in the theories which are based on Courant-like brackets. We notice further that the bi-differential Christoffel symbols introduced in (10.20) must not be misunderstood as any connections $\boldsymbol{\mathcal{A}} = \mathcal{A}_i\,\mathrm{d}x^i$ in principal fibre bundles in gauge theories.

Theorem 10.5 (Transformations of Γ_{ij}^k)**.** *Let g be a reparametrization*[36] $\boldsymbol{p}^1 \mapsto \widetilde{\boldsymbol{p}}^1 = g\boldsymbol{p}^1$, $\boldsymbol{p}^2 \mapsto \widetilde{\boldsymbol{p}}^2 = g\boldsymbol{p}^2$ *of the tall sections* $\boldsymbol{p}^1, \boldsymbol{p}^2 \in \Gamma\Omega(\xi_\pi)$ *in the domain* (10.18a) *of operator* (10.18b)*. In this notation, the operators A_1, …, A_N are transformed by the formula $A_i \mapsto \widetilde{A}_i = A_i \circ g^{-1}\big|_{\boldsymbol{q}=\boldsymbol{q}(\boldsymbol{x},[\widetilde{\boldsymbol{q}}])}$. Then the bi-differential symbols $\Gamma_{ij}^k \in \mathcal{C}\mathrm{Diff}\big(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\big)$ are transformed according to the rule*

$$\begin{gathered}\Gamma_{ij}^k(\boldsymbol{p}_i^1, \boldsymbol{p}_j^2) \mapsto \widetilde{\Gamma}_{\widetilde{i}\widetilde{j}}^{\widetilde{k}}(\widetilde{\boldsymbol{p}}_{\widetilde{i}}^1, \widetilde{\boldsymbol{p}}_{\widetilde{j}}^2) \\ = \big(g \circ \Gamma_{\widetilde{i}\widetilde{j}}^{\widetilde{k}}\big)\big(g^{-1}\widetilde{\boldsymbol{p}}_{\widetilde{i}}^1, g^{-1}\widetilde{\boldsymbol{p}}_{\widetilde{j}}^2\big) + \delta_{\widetilde{i}}^{\widetilde{k}} \cdot \partial^{(\boldsymbol{q})}_{\widetilde{A}_{\widetilde{j}}(\widetilde{\boldsymbol{p}}_{\widetilde{j}}^2)}(g)\big(g^{-1}\widetilde{\boldsymbol{p}}_{\widetilde{i}}^1\big) - \delta_{\widetilde{j}}^{\widetilde{k}} \cdot \partial^{(\boldsymbol{q})}_{\widetilde{A}_{\widetilde{i}}(\widetilde{\boldsymbol{p}}_{\widetilde{i}}^1)}(g)\big(g^{-1}\widetilde{\boldsymbol{p}}_{\widetilde{j}}^2\big). \quad (10.22)\end{gathered}$$

Proof. Denote $A = A_i$ and $B = A_j$; without loss of generality we assume $i = 1$ and $j = 2$ and put $\boldsymbol{p} = \boldsymbol{p}_i^1$ and $\boldsymbol{r} = \boldsymbol{p}_j^2$. Let us calculate the commutators of vector fields in the images of A and B using two systems of coordinates in the domain. We equate the commutators straighforwardly, because the fibre coordinates in the images of the operators are not touched at all. So, we have, originally,

$$\begin{aligned}\big[A(\boldsymbol{p}),B(\boldsymbol{r})\big] &= B\big(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p})}(\boldsymbol{r})\big) - A\big(\partial^{(\boldsymbol{q})}_{B(\boldsymbol{r})}(\boldsymbol{p})\big) + A\big(\Gamma_{AB}^A(\boldsymbol{p},\boldsymbol{r})\big) + B\big(\Gamma_{AB}^B(\boldsymbol{p},\boldsymbol{r})\big) \\ &\quad + \sum_{k=3}^N A_k\big(\Gamma_{AB}^k(\boldsymbol{p},\boldsymbol{r})\big).\end{aligned}$$

On the other hand, we substitute $\widetilde{\boldsymbol{p}} = g\boldsymbol{p}$ and $\widetilde{\boldsymbol{r}} = g\boldsymbol{r}$ in $\big[\widetilde{A}(\widetilde{\boldsymbol{p}}), \widetilde{B}(\widetilde{\boldsymbol{r}})\big]$, whence, by the Leibniz rule, we obtain

$$\begin{aligned}\big[\widetilde{A}(\widetilde{\boldsymbol{p}}),\widetilde{B}(\widetilde{\boldsymbol{r}})\big] &= \widetilde{B}\big(\partial^{(\boldsymbol{q})}_{\widetilde{A}(\widetilde{\boldsymbol{p}})}(g)(\boldsymbol{r})\big) + \big(\widetilde{B} \circ g\big)\big(\partial^{(\boldsymbol{q})}_{\widetilde{A}(\widetilde{\boldsymbol{p}})}(\boldsymbol{r})\big) \\ &\quad - \widetilde{A}\big(\partial^{(\boldsymbol{q})}_{\widetilde{B}(\widetilde{\boldsymbol{r}})}(g)(\boldsymbol{p})\big) - \big(\widetilde{A} \circ g\big)\big(\partial^{(\boldsymbol{q})}_{\widetilde{B}(\widetilde{\boldsymbol{r}})}(\boldsymbol{p})\big) \\ &\quad + \big(A \circ g^{-1}\big)\big(\Gamma_{\widetilde{A}\widetilde{B}}^{\widetilde{A}}(g\boldsymbol{p}, g\boldsymbol{r})\big) + \big(B \circ g^{-1}\big)\big(\Gamma_{\widetilde{A}\widetilde{B}}^{\widetilde{B}}(g\boldsymbol{p}, g\boldsymbol{r})\big) \\ &\quad + \sum_{\widetilde{k}=3}^N \big(A_{\widetilde{k}} \circ g^{-1}\big)\big(\Gamma_{\widetilde{A}\widetilde{B}}^{\widetilde{k}}(g\boldsymbol{p}, g\boldsymbol{r})\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma_{AB}^A(\boldsymbol{p},\boldsymbol{r}) &= \big(g^{-1} \circ \Gamma_{\widetilde{A}\widetilde{B}}^{\widetilde{A}}\big)(g\boldsymbol{p}, g\boldsymbol{r}) - \big(g^{-1} \circ \partial_{B(\boldsymbol{r})}(g)\big)(\boldsymbol{p}), \\ \Gamma_{AB}^B(\boldsymbol{p},\boldsymbol{r}) &= \big(g^{-1} \circ \Gamma_{\widetilde{A}\widetilde{B}}^{\widetilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{r}) + \big(g^{-1} \circ \partial_{A(\boldsymbol{p})}(g)\big)(\boldsymbol{r}), \\ \Gamma_{AB}^k(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1} \circ \Gamma_{\widetilde{A}\widetilde{B}}^{k}\big)(g\boldsymbol{p}, g\boldsymbol{r}) \qquad \text{for } k \geqslant 3.\end{aligned}$$

Acting by g on these equalities and expressing $\boldsymbol{p}^1 = g^{-1}\widetilde{\boldsymbol{p}}^1$, $\boldsymbol{p}^2 = g^{-1}\widetilde{\boldsymbol{p}}^2$, we obtain (10.22) and conclude the proof. □

We finally notice that the nature of the arguments $\boldsymbol{p}_i$ is firmly fixed by the reparametrization rules $\boldsymbol{p}_i \mapsto g\,\boldsymbol{p}_i$. Therefore, the isolated components of the sections $\boldsymbol{p}_i$, which

[36]Under an invertible change $\widetilde{\boldsymbol{q}} = \widetilde{\boldsymbol{q}}\big(\boldsymbol{x}, [\boldsymbol{q}]\big)$ of fibre coordinates, the variational covectors are transformed by the inverse of the adjoint linearization $g = \big[\big(\ell_{\widetilde{\boldsymbol{q}}}^{(\boldsymbol{q})}\big)^\dagger\big]^{-1}$, whereas for variational vectors, $g = \ell_{\widetilde{\boldsymbol{q}}}^{(\boldsymbol{q})}$ is the linearization.

one may be tempted to treat as “functions” and by this fully neglect their geometry, may not be well-defined as true *functions*. We repeat that this primitivization discards much of the information about the setup and shadows the geometry which we have addressed so far.

10.3. **Homological evolutionary vector fields.** We now represent the variational Lie algebroids using the homological evolutionary vector fields $\boldsymbol{Q}$ on the horizontal infinite jet super-bundles which we naturally associate with Definition 10.2. To achieve this goal, we make two preliminary steps.

We first notice that it is not the horizontal modules $\Gamma\bigl(\pi_\infty^*(\xi_i)\bigr)$ of sections $\boldsymbol{p}_i\bigl(\boldsymbol{x},[\boldsymbol{q}]\bigr)$ of the induced fibre bundles $\pi_\infty^*(\xi_i)$ which we really need for the representation of the geometry at hand in terms of the homological vector fields, but it is the product $\overline{J^\infty}(\xi_\pi) := J^\infty(\xi) \times_{M^n} J^\infty(\pi)$ of the infinite jet spaces for π and $\xi = \bigoplus_{i=1}^N \xi_i$ over M^n. This means that we operate with the jet coordinates $\boldsymbol{p}_{i;\tau}$ corresponding to the multi-indices τ instead of the derivatives $\frac{\mathrm{d}^{|\tau|}}{\mathrm{d}\boldsymbol{x}^\tau}\boldsymbol{p}_i\bigl(\boldsymbol{x},[\boldsymbol{q}]\bigr)$, whence the linear differential operators A_i become linear vector-functions of the jet variables $\boldsymbol{p}_{i;\tau}$. (In practice, this often means also that the number of the “fields” *doubles* at exactly this moment.) The following diagram endows the total space $\overline{J^\infty}(\xi_\pi)$ with the Cartan connection $\nabla_{\mathcal{C}}$,

$$
\begin{tikzcd}
J^\infty(\xi) \times_{M^n} J^\infty(\pi) \arrow[r, shift left] \arrow[d, shift right] & J^\infty(\pi) \arrow[l, dotted, shift left, "\nabla_{\mathcal{C}}^{\pi}"] \arrow[d, shift right, "\pi_\infty"'] \\
J^\infty(\xi) \arrow[r, shift left, "\xi_\infty"] \arrow[u, dotted, shift right, "\nabla_{\mathcal{C}}^{\xi}"'] & M^n. \arrow[l, dotted, shift left, "\nabla_{\mathcal{C}}^{\xi}"] \arrow[u, dotted, shift right, "\nabla_{\mathcal{C}}^{\pi}"']
\end{tikzcd}
\tag{10.23}
$$

This justifies the application of the total derivatives to the variables $\boldsymbol{p}$ (including each $\boldsymbol{p}_i$ alone) and, from now on, allows us to consider on $\overline{J^\infty}(\xi_\pi)$ the evolutionary vector fields with their sections depending (non)linearly on the fields $\boldsymbol{q}$ and the variables $\boldsymbol{p}_i$.

The conversion of the sections $\boldsymbol{p}_i$ to jet variables creates the miracle: the kernel $\ker A$ of the operator A becomes a linear subspace in $\overline{J^\infty}(\xi_\pi)$ for each point of $J^\infty(\pi)$. This is a significant achievement because it allows us to operate with the quotient spaces, which we anticipated when the bi-differential structure constants $\mathbf{c}_{ij}^k$ were introduced in (10.20).

Second, we take the fibres of the vector bundles ξ_i (hence, of $\xi_{i,\infty}$) and reverse their parity, $\Pi\colon \boldsymbol{p}_i \rightleftarrows \boldsymbol{b}_i$, the entire underlying jet space $J^\infty(\pi)$ remaining intact. This produces the horizontal infinite jet superbundle $\overline{J^\infty}(\Pi\xi_\pi) := J^\infty(\Pi\xi) \times_{M^n} J^\infty(\pi) \to J^\infty(\pi)$ with odd fibres, the coordinates there being $\boldsymbol{b}_{i;\tau}$ for $i = 1, \ldots, N$ and $|\tau| \geqslant 0$. (In a special class of geometries, see Lecture 11, we shall recognize the variables $\boldsymbol{b}_i$ as the *ghosts*, which are denoted usually by $\boldsymbol{\gamma}_i$.) The operators A_i tautologically extend to $\overline{J^\infty}(\Pi\xi_\pi)$ and become fibrewise-linear functions in $\boldsymbol{b}_{i;\tau}$; the equivalence classes of the bi-differential Christoffel symbols Γ_{ij}^k are represented by bi-linear functions on that superspace.

Theorem 10.6 ([50])**.** *Whatever representatives in the equivalence classes of the bi-differential symbols* Γ^k_{ij} *be taken, the odd-parity evolutionary vector field*

$$\boldsymbol{Q} = \partial^{(\boldsymbol{q})}_{\sum_{i=1}^N A_i(\boldsymbol{b}_i)} - \frac{1}{2}\sum_{k=1}^N \partial^{(\boldsymbol{b}_k)}_{\sum_{i,j=1}^N \Gamma^k_{ij}(\boldsymbol{b}_i,\boldsymbol{b}_j)} \tag{10.24}$$

is homological:

$$\boldsymbol{Q}^2 = 0 \quad (\mathrm{mod}\ \sum_{k=1}^N \partial^{(\boldsymbol{b}_k)}_{\varphi_k(\boldsymbol{b}\otimes\boldsymbol{b}\otimes\boldsymbol{b})} \mid \sum_{k=1}^N A_k(\varphi_k) = 0).$$

Example 10.7. Let us give at once a minimally possible nontrivial illustration. The operator $\square = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ for the Liouville equation $u_{xy} = \exp(2u)$ specifies the differential

$$\boldsymbol{Q} = \partial^{(u)}_{\square(b)} + \partial^{(b)}_{bb_x};$$

the equality of the even velocity bb_x of the odd variable b to the respective velocity in the field (see Exercise 9.5 on p. 88)

$$\boldsymbol{Q} = \partial^{(w)}_{A_2^{\mathrm{KdV}}(b)} + \partial^{(b)}_{bb_x}$$

for the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx} + 3ww_x$ is no coincidence!

We prefer to prove the theorem straightforwardly, *not* passing from many arguments of many operators to the tall sections of wide operators, see (10.18). This attests that the nature of the variables $\boldsymbol{b}_i$ at different i's may be entirely uncorrelated.

Proof. The halved anticommutator $\frac{1}{2}[\boldsymbol{Q},\boldsymbol{Q}] = \boldsymbol{Q}^2$ of the odd vector field $\boldsymbol{Q}$ with itself is an evolutionary vector field. In this anticommutator, the coefficient of $\partial/\partial\boldsymbol{q}$, which determines the coefficients of $\partial/\partial\boldsymbol{q}_\sigma$ at all $|\sigma| \geqslant 0$, is equal to

$$\partial^{(\boldsymbol{q})}_{\sum_{i=1}^N A_i(\boldsymbol{b}_i)}\Bigl(\sum_{j=1}^N A_j(\boldsymbol{b}_j)\Bigr) - \frac{1}{2}\sum_{k,\ell=1}^N \delta^\ell_k \cdot A_\ell\Bigl(\sum_{i,j=1}^N \Gamma^k_{ij}(\boldsymbol{b}_i,\boldsymbol{b}_j)\Bigr);$$

the evaluation (8.2) of this coefficient at any two "tall" sections $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ gives zero by the definition of the symbols Γ^k_{ij}.

Second, let us consider the Jacobi identity for the Lie algebra of evolutionary vector fields with the generating sections belonging to the images of the operators $A_1, \ldots, A_N$ viewed as the fibrewise-linear functions on the total space of the bundle $\overline{J^\infty(\xi_\pi)} \to J^\infty(\pi)$:

$$\begin{aligned}
0 &= \sum_{\substack{\circlearrowright\\(imn)}} \bigl[A_i(\boldsymbol{p}_i), \sum_k A_k\bigl(\Gamma^k_{mn}(\boldsymbol{p}_m,\boldsymbol{p}_n)\bigr)\bigr]\\
&= \sum_{\substack{\circlearrowright\\(imn)}} \sum_k A_k\Bigl\{\partial^{(\boldsymbol{q})}_{A_i(\boldsymbol{p}_i)}\bigl(\Gamma^k_{mn}(\boldsymbol{p}_m,\boldsymbol{p}_n)\bigr) + \sum_\ell \Gamma^k_{i\ell}\bigl(\boldsymbol{p}_i, \Gamma^\ell_{mn}(\boldsymbol{p}_m,\boldsymbol{p}_n)\bigr)\Bigr\},
\end{aligned}$$

where we have relabelled the indices $\ell \rightleftarrows k$ in the second sum,

$$= -\sum_{\substack{\circlearrowright\\(imn)}} \sum_k A_k \Big\{ -\partial^{(\boldsymbol{q})}_{A_i(\boldsymbol{p}_i)} \big(\Gamma^k_{mn}(\boldsymbol{p}_m, \boldsymbol{p}_n)\big) + \sum_\ell \Gamma^k_{\ell i}\big(\Gamma^\ell_{mn}(\boldsymbol{p}_m, \boldsymbol{p}_n), \boldsymbol{p}_i\big) \Big\}. \tag{10.25}$$

Thirdly, the velocity of each odd variable $\boldsymbol{b}_k$ induced by the (for convenience, *not* halved) anticommutator $[\boldsymbol{Q}, \boldsymbol{Q}] = 2\boldsymbol{Q}^2$ is obtained as follows,

$$\begin{aligned}
&-\partial^{(\boldsymbol{q})}_{\sum_{i=1}^N A_i(\boldsymbol{b}_i)} \big(\sum_{m,n} \Gamma^k_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n)\big) + \frac{1}{2} \sum_{\ell=1}^N \partial^{(\boldsymbol{b}_\ell)}_{\sum\limits_{m,n} \Gamma^\ell_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n)} \Big(\sum_{j,i} \Gamma^k_{ji}(\boldsymbol{b}_j, \boldsymbol{b}_j) \Big) \\
&\quad = -\partial^{(\boldsymbol{q})}_{\sum_{i=1}^N A_i(\boldsymbol{b}_i)} \big(\sum_{m,n} \Gamma^k_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n)\big) \\
&\qquad\qquad + \frac{1}{2} \sum_{\ell,i} \Gamma^k_{\ell i} \Big(\sum_{m,n} \Gamma^\ell_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n), \boldsymbol{b}_i \Big) - \frac{1}{2} \sum_{i,\ell} \Gamma^k_{i\ell} \Big(\boldsymbol{b}_i, \sum_{m,n} \Gamma^\ell_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n) \Big) \\
&\quad = \sum_{i,m,n} \Big\{ -\partial^{(\boldsymbol{q})}_{A_i(\boldsymbol{b}_i)} \big(\Gamma^k_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n)\big) + \sum_{\ell=1}^N \Gamma^k_{\ell i}\big(\Gamma^\ell_{mn}(\boldsymbol{b}_m, \boldsymbol{b}_n), \boldsymbol{b}_i\big) \Big\}.
\end{aligned}$$

We notice that the extra sum over the three cyclic permutations of each fixed set of the indices i, m, and n does not produce any change of the signs because the cyclic permutations (of the respective three odd $\boldsymbol{b}$'s) are even. Consequently, by taking the sum of all possible Jacobi identities (10.25), we conclude that the sought-for coefficient is equal to

$$= \frac{1}{3} \sum_{i,m,n} \sum_{\substack{\circlearrowright\\(imn)}} \Big\{ \ldots \Big\} = 0 \quad (\text{mod } \boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N) \mid \sum_{k=1}^N A_k(\varphi_k) = 0).$$

This completes the proof. $\square$

Remark 10.6. The above construction of the homological vector field $\boldsymbol{Q}$ in absence of the equations of motion but with an a priori given collection A_1, …, A_N of generators (10.24) for the infinitesimal transformations of the model patterns upon the construction of nontrivial gauge theories with the zero action functional and the prescribed nontrivial gauge group, see [9, 110].

In the next lecture we shall consider the general case of gauge models[5] with a possibly nontrivial action $S(\boldsymbol{x}, [\boldsymbol{q}])$ and associate the evolutionary BRST-differentials $\boldsymbol{Q}$ and the BV-differentials $\boldsymbol{D}$ with such systems; these homological vector fields $\boldsymbol{D}$ are themselves Hamiltonian with respect to the full BV-action $\boldsymbol{S} = S + \ldots$ such that $\boldsymbol{D} = [\![\boldsymbol{S}, \cdot\,]\!]$.

Likewise, the correspondence $\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}} = [\![\boldsymbol{\mathcal{P}}, \cdot]\!]$ between the (non)commutative variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$ satisfying the classical master-equation $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] = 0$ is a regular source of the homological vector fields (see Lectures 8–9). Let us notice further that these fields are Hamiltonian: indeed, the bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$ then stands as the Hamiltonian functional that specifies the dynamics via the odd Poisson bracket $[\![\,,\,]\!]$.

We conclude this lecture with a convenient criterion[49] which states whether a given differential operator A is Hamiltonian and endows the space $\overline{H}^n(\pi)$ of functionals on $J^\infty(\pi)$ with a Poisson bracket.

Theorem 10.7. *A skew-adjoint linear total differential operator $A\colon \boldsymbol{p} \in P \mapsto \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p})}$ with involutive image is Hamiltonian if and only if*

$$\boldsymbol{Q}^A(\boldsymbol{\mathcal{P}}) = 0 \in \overline{H}^n(\Pi\widehat{\pi}_\pi),$$

where $\boldsymbol{Q}^A$ is the odd-parity vector field (10.4) *on $\overline{J^\infty}(\Pi\widehat{\pi}_\pi)$ and $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$ is the bi-vector.*

Remark 10.7. The equality $A = -A^\dagger$ of the mappings $A\colon P \to \varkappa(\pi)$ and $A^\dagger\colon \widehat{\varkappa(\pi)} \to \widehat{P}$ uniquely determines the domain $P = \widehat{\varkappa(\pi)}$ of A. We now see that the "only if" part of the assertion is Exercise 9.3 on p. 88 combined with Problem 9.3.

Proof. Because the operator A is skew-adjoint, the bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\langle \boldsymbol{b}, A(\boldsymbol{b})\rangle$ is well defined. By definition, put $\boldsymbol{Q}^A = \partial^{(\boldsymbol{q})}_{A(\boldsymbol{b})} - \frac{1}{2}\partial^{(\boldsymbol{b})}_{\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A}$, where the Christoffel symbol $\{\{\,,\,\}\}_A$ is such that $\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(A)(\boldsymbol{p}_2) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_2)}(A)(\boldsymbol{p}_1) = A(\{\{\boldsymbol{p}_1, \boldsymbol{p}_2\}\}_A)$ for any sections $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \widehat{\varkappa(\pi)}$. We note that if $\{\{\,,\,\}\}_A \equiv 0$, there is nothing to prove (see Exercise 9.4 on p. 88). From now on, we suppose that $\{\{\,,\,\}\}_A \not\equiv 0$.

Using the Leibniz rule, we deduce that

$$0 = \boldsymbol{Q}^A(\boldsymbol{\mathcal{P}}) = \tfrac{1}{2}\langle -\tfrac{1}{2}\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A, A(\boldsymbol{b})\rangle - \tfrac{1}{2}\langle \boldsymbol{b}, \partial^{(\boldsymbol{q})}_{A(\boldsymbol{b})}(A)(\boldsymbol{b})\rangle + \tfrac{1}{2}\langle \boldsymbol{b}, \tfrac{1}{2}A(\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A)\rangle,$$

where we integrate by parts in the first term of the right-hand side and obtain

$$\cong \tfrac{1}{2}\cdot\Big(-\tfrac{1}{2}\langle \boldsymbol{b}, A(\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A)\rangle + \langle \boldsymbol{b}, \underline{-\partial^{(\boldsymbol{q})}_{A(\boldsymbol{b})}(A)(\boldsymbol{b}) + \tfrac{1}{2}A(\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A)}\rangle\Big).$$

The evaluation (8.2) of the underlined factor in the second coupling at any two covectors yields zero so that there only remains

$$= (-\tfrac{1}{2})\cdot\langle \boldsymbol{b}, A(\tfrac{1}{2}\{\{\boldsymbol{b},\boldsymbol{b}\}\}_A)\rangle.$$

By the hypothesis of the theorem, the value of this tri-vector at any three covectors $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3 \in \widehat{\varkappa(\pi)}$ is equal to

$$-\frac{1}{2}\cdot\sum_{\circlearrowleft}\langle \boldsymbol{p}_1, A(\{\{\boldsymbol{p}_2,\boldsymbol{p}_3\}\}_A)\rangle = 0 \in \overline{H}^n(\pi). \tag{10.26}$$

We claim that the sum (10.26) is zero if and only if A is a Hamiltonian operator (i. e., whenever it induces the Poisson bracket $\{\,,\,\}_A$ which satisfies the Jacobi identity).

Indeed, define a bi-linear skew-symmetric bracket $\{\,,\}_A$ for any $\mathcal{H}_i, \mathcal{H}_j \in \overline{H}^n(\pi)$ by the rule

$$\{\mathcal{H}_i, \mathcal{H}_j\}_A := \big\langle \delta\mathcal{H}_i/\delta\boldsymbol{q}, A(\delta\mathcal{H}_j/\delta\boldsymbol{q})\big\rangle \cong \partial^{(\boldsymbol{q})}_{A(\delta\mathcal{H}_j/\delta\boldsymbol{q})}(\mathcal{H}_i).$$

Now let $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3 \in \overline{H}^n(\pi)$ be three arbitrary functionals and put $\boldsymbol{p}_i(\boldsymbol{x},[\boldsymbol{q}]) = \delta\mathcal{H}_i/\delta\boldsymbol{q}$ for $i = 1,2,3$. Then the left-hand side of the Jacobi identity for the bracket $\{\,,\,\}_A$ is, up to a nonzero constant, equal to

$$\begin{aligned}&\sum_{\circlearrowright} \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}\langle \boldsymbol{p}_2, A(\boldsymbol{p}_3)\rangle \\ &\quad = \sum_{\circlearrowright}\Big(\langle \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_2), A(\boldsymbol{p}_3)\rangle + \langle \boldsymbol{p}_2, \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(A)(\boldsymbol{p}_3)\rangle + \big\langle \boldsymbol{p}_2, A\big(\partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_3)\big)\big\rangle\Big). \end{aligned} \tag{10.27}$$

Let us integrate by parts in the last term of the right-hand side; at the same time, using the property $\ell^{(\boldsymbol{q})}_{\boldsymbol{p}_2} = \ell^{(\boldsymbol{q})\dagger}_{\boldsymbol{p}_2}$ of the exact covector $\boldsymbol{p}_2 = \delta\mathcal{H}_2/\delta\boldsymbol{q}$ (we assume that the topology of the bundle π matches the hypotheses of the Helmholtz theorem 5.1 on p. 55), we conclude that the first term in the right-hand side of (10.27) equals

$$\begin{aligned}\langle \ell^{(\boldsymbol{q})}_{\boldsymbol{p}_2}\big(A(\boldsymbol{p}_1)\big), A(\boldsymbol{p}_3)\rangle = \langle \ell^{(\boldsymbol{q})\dagger}_{\boldsymbol{p}_2}\big(A(\boldsymbol{p}_1)\big), A(\boldsymbol{p}_3)\rangle \cong \langle \ell^{(\boldsymbol{q})}_{\boldsymbol{p}_2}\big(A(\boldsymbol{p}_3)\big), A(\boldsymbol{p}_1)\rangle = \\ = \langle \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_3)}(\boldsymbol{p}_2), A(\boldsymbol{p}_1)\rangle.\end{aligned}$$

Continuing the equality from (10.27), we have

$$\cong \sum_{\circlearrowright}\Big(\langle \boldsymbol{p}_2, \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(A)(\boldsymbol{p}_3)\rangle + \underline{\langle \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_3)}(\boldsymbol{p}_2), A(\boldsymbol{p}_1)\rangle - \langle \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_3), A(\boldsymbol{p}_2)\rangle}\Big).$$

The $3+3$ underlined terms cancel in the sum, whence we obtain the equality

$$\begin{aligned}&= \sum_{\circlearrowright}\langle \boldsymbol{p}_1, \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_3)}(A)(\boldsymbol{p}_2)\rangle = \frac{1}{2}\sum_{\circlearrowright}\langle \boldsymbol{p}_1, \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_3)}(A)(\boldsymbol{p}_2) - \partial^{(\boldsymbol{q})}_{A(\boldsymbol{p}_2)}(A)(\boldsymbol{p}_3)\rangle \\ &= -\frac{1}{2}\cdot\sum_{\circlearrowright}\big\langle \boldsymbol{p}_1, A\big(\{\{\boldsymbol{p}_2,\boldsymbol{p}_3\}\}_A\big)\big\rangle.\end{aligned}$$

By (10.26), this expression equals zero.[37] Therefore, the Jacobi identity for the bracket $\{\,,\,\}_A$ holds, whence the operator A is Hamiltonian. The proof is complete. $\square$

Remark 10.8. Theorem 10.7 converts the *quadratic* equation $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] = 0 \Leftrightarrow (\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}})^2 = 0$ to the *bi-linear* condition $\boldsymbol{Q}^A(\boldsymbol{\mathcal{P}}) = 0$ upon the bi-vector $\boldsymbol{\mathcal{P}}$ and the *Q-structure* for the infinite jet superspace $\overline{J^\infty}(\Pi\widehat{\pi}_\pi)$, which is endowed with the odd Poisson bracket $[\![\,,\,]\!]$, c. f. [1].

Problem 10.1 ([20]). Prove that the variational derivative $\delta/\delta\boldsymbol{q}$ and a Hamiltonian differential operator A establish the Lie agebra homomorphisms

$$\big(\overline{H}^n(\pi), \{\,,\,\}_A\big) \longrightarrow \big(\widehat{\varkappa(\pi)}, [\,,\,]_A\big) \longrightarrow \big(\varkappa(\pi), [\,,\,]\big).$$

Problem 10.2. By a direct calculation — i. e., not appealing to formula (9.11) and the property $(\boldsymbol{Q}^{\boldsymbol{\mathcal{P}}})^2 = 0$ of the Poisson bi-vectors $\boldsymbol{\mathcal{P}}$ induced by that factorization, — demonstrate that the odd-parity evolutionary vector fields $\boldsymbol{Q}^{\square}$ are the differentials for

[37] By the substitution principle (which is Problem 9.2), if such an expression vanishes for the *exact* covectors $\boldsymbol{p}_i = \delta\mathcal{H}_i/\delta\boldsymbol{q}$, then it vanishes for *all* sections $\boldsymbol{p}_i \in \widehat{\varkappa(\pi)}$.

each root system of a semisimple complex Lie algebra of rank two and the generator $\square$ of symmetries for the associated nonperiodic 2D Toda chain. (The formulas for the anchors $\square$ and the Christoffel symbols $\{\!\{\,,\,\}\!\}_\square$ are contained in [64].)

Problem 10.3. Consider the generator $D_{\boldsymbol{\mathcal{A}}} = \overline{\mathrm{d}} + [\boldsymbol{\mathcal{A}}, \cdot\,]$ of the gauge symmetries for the Yang–Mills equations $D_{\boldsymbol{\mathcal{A}}} * D_{\boldsymbol{\mathcal{A}}}(\boldsymbol{\mathcal{A}}) = 0$, where $\boldsymbol{\mathcal{A}} = \mathcal{A}_i\,\mathrm{d}x^i$ is the connection one-form (containing the fields $\mathcal{A}_i$) in the principal G-bundle over $M^{3,1}$ and $*$ is the Hodge structure on the Minkowski space-time $M^{3,1}$ (e. g., let $G = U(1)$, $G = SU(2)$, or $G = SU(3)$). Assume that the Lie algebra $\mathfrak{g}$ of the Lie group G is complex semi-simple.

Suppose first that the image of $D_{\boldsymbol{\mathcal{A}}}$ is involutive and, were it indeed so, show that

- the total differential order of the bi-differential Christoffel symbol $\{\!\{\,,\,\}\!\}_{D_{\boldsymbol{\mathcal{A}}}}$ must be equal to zero;
- the differential order of the coefficients of the Christoffel symbol must be equal to zero;
- moreover, the coefficients of $\{\!\{\,,\,\}\!\}_{D_{\boldsymbol{\mathcal{A}}}}$ can not depend on the unknown variables $\boldsymbol{\mathcal{A}}$;
- but also, in view of the invariance of the Yang–Mills theory under translations along the base $M^{3,1}$, those coefficients can not depend on the points $\boldsymbol{x} \in M^{3,1}$.

Recall further that, by construction, the gauge parameters $\boldsymbol{p} \in \widehat{P_1}$ belong to the horizontal module $\widehat{P_1} \simeq \mathfrak{g} \otimes_{C^\infty(M^{3,1})} \mathcal{F}(\pi)$, where π is the fibre bundle over $M^{3,1}$ for the connection fields $\boldsymbol{\mathcal{A}}$, but let us identify $\mathfrak{g}^* \simeq \mathfrak{g}$ by using the Killing form of the semi-simple complex Lie algebra $\mathfrak{g}$.

- Prove that the image of the gauge symmetry generator $D_{\boldsymbol{\mathcal{A}}}$ **is** closed under commutation and the entries of the Christoffel symbol $\{\!\{\,,\,\}\!\}_{D_{\boldsymbol{\mathcal{A}}}}$ **are** equal to the structure constants of the Lie algebra $\mathfrak{g}$.

11. BRST- and BV-formalism

In this lecture we study the geometry of the Becchi–Rouet–Stora–Tyutin[11] (BRST) and Batalin–Vilkovisky[8] (BV) approach to the quantisation of the Euler–Lagrange gauge models with the action S and the structure group G. The main element of this technique is the representation of Lie algebroids over infinite jet spaces $J^\infty(\pi)$ in terms of the homological evolutionary vector fields $\boldsymbol{Q}$, which we introduced in the previous lecture. The BRST technique resolves[4, 39] the algebra of gauge-invariant observables (i. e., the functions or functionals which remain unchanged under the gauge transformations) as the zero cohomology of the complex endowed with the differential $\boldsymbol{Q}$. In turn, the BV-formalism extends the domain of definition for the differential $\boldsymbol{Q}$ to the Whitney sum over $J^\infty(\pi)$ of the induced fibre (super-)bundles containing not only the unknowns $\boldsymbol{q}$ (which are the gauge fields) and the field-dependent gauge parameters $\boldsymbol{p}$, but also the equations of motion, the (higher generations of the) Noether identities, and the elements of the modules which are dual to the equations and all such syzygies (see Lecture 2). By reversing the parity of the fibres in, roughly speaking, a half of the horizontal jet spaces for these induced bundles, we complete the classical BV-zoo[7, 29, 108] which is inhabited by the (anti)fields, (anti)ghosts, and possibly, the extra pairs of the variables which stem from the higher generations of the syzygies.

It is now clear that the gauge systems carry the excessive degrees of freedom which are not essential in the study of the physics described by such models. Firstly, all objects are littered with the identically-zero elements that contain the Noether identities. Second, it is the motion along the gauge orbits that yields the physically equivalent states. Neither of the two factors contributes to the values of the observables from $\mathcal{F}(\pi)^G$ or $\overline{H}^n(\pi)^G$. (Again, the two emerging sub-theories for the differential functions and the integral functionals do and, respectively, do not distinguish between the points of the source manifold M^n. But we are free to choose the right concept: a field theory over the space-time votes for the first option while the theory of strings within the space-time sets the preference for the other.) It is our intermediate goal to introduce a cohomology theory whose zeroth term would be the space of the observables of our choice. Such a theory will allow us to deal with the on-shell defined gauge-invariant objects staying off-shell in the jet space and not taking care about fixing the gauge.[36, 94]

For consistency, let us recall from Lecture 6 some basic facts about the Euler-Lagrange gauge systems. We know that every Euler–Lagrange system $\mathcal{E}_{\text{EL}} = \bigl\{\boldsymbol{F} = \delta S/\delta\boldsymbol{q} = 0 \mid S \in \overline{H}^n(\pi)\bigr\}$ contains as many equations as there are unknowns $\boldsymbol{q}$. By convention, such equations $F_i = 0$ are labelled by the respective fields q^i; let us remember that the horizontal module P_0 of the sections $\boldsymbol{F}$ is then $P_0 \simeq \widehat{\varkappa(\pi)}$.

The gauge systems do admit the differential constraints (the *syzygies* or the *Noether identities*) between the equations of motion,

$$\boldsymbol{\Phi}(\boldsymbol{x}, [\boldsymbol{q}], [\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{q}])]) \equiv 0 \text{ on } J^\infty(\pi), \qquad \boldsymbol{\Phi} \in P_1 = \Gamma\bigl(\xi^1_{\xi_\pi}\bigr).$$

The relations $\boldsymbol{\Phi}_{i+1}(\boldsymbol{x}, [\boldsymbol{q}], [\boldsymbol{F}], [\boldsymbol{\Phi}], \ldots, [\boldsymbol{\Phi}_i]) \equiv 0$ between the relations, whenever valid for all sections $\boldsymbol{s} \in \Gamma(\pi)$ identically in $\boldsymbol{q}$, $\boldsymbol{F}$, $\ldots$, $\boldsymbol{\Phi}_{i-1}$ (here $\boldsymbol{\Phi}_0 = \boldsymbol{F} \in P_0$), give rise to possibly several but always finitely many generations of the horizontal modules $P_i \ni \Phi_i$ for $i > 0$.

As in Lecture 6, we assume that the given action functional S determines the system $\mathcal{E}_{\text{EL}} = \{\boldsymbol{F} = 0\}$ of equations of motion with one generation of the constraints $\boldsymbol{\Phi}(\boldsymbol{x}, [\boldsymbol{q}], [\boldsymbol{F}]) = 0$ and that there are no further relations between the already known ones. However, if a given model is *reducible* — i. e., there is at least one more generation of the syzygies $\boldsymbol{\Phi}_2\bigl(\boldsymbol{x}, [\boldsymbol{q}], [\boldsymbol{F}], [\boldsymbol{\Phi}]\bigr) \equiv 0 \in P_2$ between the Noether identities $\boldsymbol{\Phi} \equiv 0 \in P_1$, — then each new generation of the relations, which are stored in the suitable horizontal modules P_i, $i \geqslant 2$, gives rise to the four neighbours:[38] P_i, ΠP_i, $\widehat{P}_i$, and $\Pi\widehat{P}_i$.

By the Second Noether Theorem 6.1, the relations $\boldsymbol{\Phi}\bigl(\boldsymbol{x}, [\boldsymbol{q}], [\boldsymbol{F}]\bigr) = 0$ between the equations $F_j = 0$, $1 \leqslant j \leqslant m$, yields the linear total differential operator $A\colon \widehat{P_1} \to \operatorname{sym}\mathcal{E}_{\text{EL}}$ that generates the gauge symmetries of the Euler–Lagrange system $\mathcal{E}_{\text{EL}}$. Such symmetries are parametrized by arbitrary field-dependent gauge parameters $\boldsymbol{p}\bigl(\boldsymbol{x}, [\boldsymbol{q}]\bigr) \in \widehat{P_1}$.

We assume further that the operator A *can* be extended to the operator that takes values in the larger space $\varkappa(\pi) \supseteq \operatorname{sym}\mathcal{E}_{\text{EL}}$; such extensions could in principle involve the improper symmetries which vanish on-shell. (However, let us emphasize that some gauge symmetries may be Noether and some Noether symmetries may be gauge, still the two concepts are distinct in general.) Let the columns $A_1, \ldots, A_N\colon \widehat{P_1} \to \varkappa(\pi)$ of the operator A be the entire collection of the gauge symmetry generators for the system $\mathcal{E}_{\text{EL}}$. For the sake of clarity, we assume the off-shell validity of the Berends–Burgers–van Dam hypothesis[14] about the collective commutation closure for the images of these operators, see (10.11). In other words, we postulate that the extensions of the gauge symmetry generators, which take their values in $\operatorname{sym}\mathcal{E}_{\text{EL}}$, are such that the improper symmetries do not occur explicitly in the off-shell involutivity condition (10.17).

Then the parity-reversed arguments $\boldsymbol{b} \in \Pi\widehat{P_1} = \Gamma\bigl(\pi_\infty^*(\Pi\widehat{\xi^1})\bigr)$ are known as the *ghosts*, usually denoted by $\boldsymbol{\gamma}$. The even elements of the dual module P_1 are[39] the *antighosts* $\boldsymbol{\gamma}^\dagger$. In the following diagram we summarize the notation and interpret the objects at hand in terms of the BRST- and BV-theory:[8, 26]

$$
\begin{tikzcd}
\text{ghosts} \arrow[r] & \boldsymbol{\gamma} = \boldsymbol{b} \in \Pi\widehat{P_1} \arrow[d, shift left] & & \boldsymbol{\gamma}^\dagger \in P_1 \arrow[d, leftrightarrow, "\text{id}"] & \text{antighosts} \arrow[l] \\
\text{gauge parameters} \arrow[r] & \boldsymbol{\epsilon} = \boldsymbol{p} \in \widehat{P_1} \arrow[u, shift left, "\Pi"] \arrow[rr, leftrightarrow, "*", "\langle\,,\rangle"'] & & \boldsymbol{\Phi}[\boldsymbol{F}] \in P_1 & \text{Noether identities} \arrow[l] \\
& & & \boldsymbol{q}^\dagger \in \Pi P_0 \arrow[d, shift left] & \text{antifields} \arrow[l] \\
\text{Noether symmetries} \arrow[r] & \varphi_{\mathcal{L}} \doteq \psi \in \varkappa(\pi) \simeq \widehat{P_0} \arrow[rr, leftrightarrow, "*", "\langle\,,\rangle"'] & & \boldsymbol{F} \in P_0 \simeq \widehat{\varkappa(\pi)} \arrow[u, shift left, "\Pi"] & \text{usual identification.} \arrow[l]
\end{tikzcd}
$$

[38]Thus, each generation yields the canonically conjugate, $\langle\,,\rangle$-dual pair of the *higher ghosts* $\mathbf{c}^i \in (\Pi)\widehat{P}_i$ and the higher antighosts $\mathbf{c}_i^\dagger \in (\Pi)P_i$ of opposite parity, see the diagram below.

[39]We do not fix any metric on the base manifold M^n and therefore not grasp the subtle difference between the commonly accepted transcripts $\boldsymbol{q}^\dagger$ vs $\boldsymbol{q}^*$ for the antifields and $\boldsymbol{\gamma}^\dagger$ vs $\boldsymbol{\gamma}^*$ for the antighosts (see, e. g., [16]). For the same reason, in this course we do not pay any *particular* attention to the upper or lower location of the indices.

Passing to the horizontal jet bundle $\overline{J^\infty}\bigl(\pi_\infty^*(\Pi\widehat{\xi^1})\bigr)$ over $J^\infty(\pi)$, we then apply Theorem 10.6 and construct the odd evolutionary vector field $\boldsymbol{Q}$. This is the *Becchi–Rouet–Stora–Tyutin* (BRST-) *differential*.[11] The variational Lie algebroid which is encoded by the evolutionary differential $\boldsymbol{Q}$ is the *gauge algebroid*.

This generalization of the odd cotangent covering $\Pi\mathbb{T}^*J^\infty(\pi)$ over the initial jet space $J^\infty(\pi)$ is again endowed with the odd Poisson bracket $[\![\,,\,]\!]$ (i. e., the Schouten bracket or the *antibracket*, see Lecture 8). Likewise, the action $S(\boldsymbol{x},[\boldsymbol{q}])$ for the equation of motion $\mathcal{E}_{\text{EL}}^\infty \subseteq J^\infty(\pi)$ can be extended to the full BV-action $\boldsymbol{S} = S + \ldots$ that satisfies the classical master equation $[\![\boldsymbol{S},\boldsymbol{S}]\!] = 0$. The new functional induces the BV-differential $\boldsymbol{D} = [\![\boldsymbol{S},\,\cdot\,]\!]$ that incorporates $\boldsymbol{Q}$ as the "longitudinal" component along the gauge orbits.

Let us first inspect the (co)vectorial nature of the objects at hand (by definition, the evolutionary fields $\partial_\varphi^{(\boldsymbol{q})}$ are vectors). This is essential for the future use of Dirac's convention about the bra- covectors $\langle\,|$ and -ket vectors $|\,\rangle$ in the definition of the antibracket by using the full set of the BV-variables which are now at our disposal.

Exercise 11.1. Consider the pair of variables $\boldsymbol{q} \longleftrightarrow \boldsymbol{q}^\dagger = \boldsymbol{F}^\dagger$, where $\Pi\colon \boldsymbol{F} \rightleftarrows \boldsymbol{F}^\dagger$; the odd objects $\boldsymbol{q}^\dagger$ are called the *antifields*. Show that under a reparametrization[40] of the variables $\boldsymbol{q}$, the variation $\delta\boldsymbol{q}^\dagger$ is an odd *vector* (c. f. [2, §37]).

• Second, let $\boldsymbol{\Phi} \in P_1$ be the column of the Noether identities and consider the corresponding pair $\boldsymbol{\gamma} \longleftrightarrow \boldsymbol{\gamma}^\dagger$. Show that the even *antighost* $\boldsymbol{\gamma}^\dagger$ transforms as the coefficients of a vector, whence $\delta\boldsymbol{\gamma}^\dagger$ is a covector, and that the variation $\delta\boldsymbol{\gamma}$ is a vector for the $\langle\,,\,\rangle$-dual odd ghost $\boldsymbol{\gamma}$.

This yields the odd-parity symplectic structure

$$\boldsymbol{\omega} = \delta\boldsymbol{q} \wedge \delta\boldsymbol{q}^\dagger + \delta\boldsymbol{\gamma}^\dagger \wedge \delta\boldsymbol{\gamma}$$

on the Whitney sum of the respective horizontal jet (super-)spaces and fixes the choice of the signs in the *antibracket* (see [120] and, e. g., [37, 116])

$$[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!] = \bigl\langle \overrightarrow{\delta\boldsymbol{\xi}} \wedge \overleftarrow{\delta\boldsymbol{\eta}} \bigr\rangle,$$

which straightforwardly extends to all the generations $i = 1, \ldots, \lambda$ of the ghost-antighost pairs.[41]

---

[40] We remark that this exercise is concerned with the identification of the left-hand sides of the Euler–Lagrange equations as variational covectors $\boldsymbol{F}\,\delta\boldsymbol{q}$; the variables $\boldsymbol{q}$ could describe, e. g., the coefficients of the connection one-form in a principal fibre bundle over the manifold M^n so that the *vector* bundle π involves as the unknowns the components of a *covector*!

[41] Likewise, the odd Laplacian is

$$\Delta_{\text{BV}} = \frac{\overrightarrow{\delta}}{\delta\boldsymbol{q}} \circ \frac{\overleftarrow{\delta}}{\delta\boldsymbol{q}^\dagger} + \frac{\overrightarrow{\delta}}{\delta\boldsymbol{\gamma}^\dagger} \circ \frac{\overleftarrow{\delta}}{\delta\boldsymbol{\gamma}}.$$

We have that $[\![\boldsymbol{\xi},\boldsymbol{\eta}]\!]=$

$$\Big(\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{q}}\langle\delta\boldsymbol{q},\delta\boldsymbol{q}^\dagger\rangle\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{q}^\dagger}-\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{q}^\dagger}\langle\delta\boldsymbol{q},\delta\boldsymbol{q}^\dagger\rangle\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{q}}\Big)+\Big(\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{\gamma}^\dagger}\langle\delta\boldsymbol{\gamma}^\dagger,\delta\boldsymbol{\gamma}\rangle\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{\gamma}}-\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{\gamma}}\langle\delta\boldsymbol{\gamma}^\dagger,\delta\boldsymbol{\gamma}\rangle\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{\gamma}^\dagger}\Big)$$
$$=\left[\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{q}}\cdot\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{q}^\dagger}-\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{q}^\dagger}\cdot\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{q}}\right]+\left[\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{\gamma}^\dagger}\cdot\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{\gamma}}-\frac{\overrightarrow{\delta\boldsymbol{\xi}}}{\delta\boldsymbol{\gamma}}\cdot\frac{\overleftarrow{\delta\boldsymbol{\eta}}}{\delta\boldsymbol{\gamma}^\dagger}\right].$$

The main miracle of the Batalin–Fradkin–Vilkovisky theory is the *existence* of the master-action[40, 74]

$$\boldsymbol{S}=S+\langle\boldsymbol{q}^\dagger,A_k(\boldsymbol{\gamma}_k)\rangle-\frac{1}{2}\left\langle\Gamma^k_{ij}(\boldsymbol{\gamma}_i,\boldsymbol{\gamma}_j),\boldsymbol{\gamma}^\dagger_k\right\rangle+\langle\text{correction terms}\rangle \tag{11.1}$$

that

(1) extends the action $S(\boldsymbol{x},[\boldsymbol{q}])\in\overline{H}^n(\pi)$ for the equations of motion and
(2) satisfies the classical master equation

$$[\![\boldsymbol{S},\boldsymbol{S}]\!]=0. \tag{11.2}$$

The possible necessity to introduce the correction terms of higher polynomial orders in the odd variables $\boldsymbol{q}^\dagger$ and $\boldsymbol{\gamma}$ or their derivatives —with an also possible dependence on $[\boldsymbol{\gamma}^\dagger]$ and the higher ghost–antighost pairs— is legitimate whenever the coefficients of the bi-differential Christoffel symbols Γ^k_{ij} explicitly depend on the fields $\boldsymbol{q}$.

Exercise 11.2. Show that *no* correction terms appear in the BV-action (11.1) for the Yang–Mills equations with $S=\text{const}\cdot\int\langle *\mathcal{F}\wedge\mathcal{F}\rangle$, see Problem 2.5 and 10.3, and then verify the classical master equation (11.2) by a direct calculation.

Remark 11.1. The theory guarantees the existence of a solution $\boldsymbol{S}$ to the classical master equation (11.2) in the form of a power series in the (derivatives of the) BV-variables $\boldsymbol{q}^\dagger$, $\boldsymbol{\gamma}\leftrightarrow\boldsymbol{\gamma}^\dagger$, and the higher ghost–antighost pairs $\mathbf{c}^\alpha\leftrightarrow\mathbf{c}^\dagger_\alpha$. Such solution $\boldsymbol{S}=S+\ldots$ can be obtained perturbatively because no obstructions appear at each step from a degree ℓ to $\ell+1$. In this sense, equation (11.2) is formally integrable.

Let us notice, however, that the series $\boldsymbol{S}$ truncates after a finite number of steps for many relevant systems (see Exercise 11.2). At the same time, we emphasize that a possible presence of infinitely many derivatives of unbounded orders, which is in contrast with our earlier convention about the differential functions (see Lecture 1) still does not imply that the theory becomes nonlocal, i. e., that it involves the values of certain sections at more than one point of the base M^n. Indeed, the formal power series (11.1) can be divergent almost everywhere.

Definition 11.1. The Batalin–Vilkovisky differential is the evolutionary vector field $\boldsymbol{D}_{\text{BV}}$ which acts by the rule

$$\boldsymbol{D}_{\text{BV}}(\,\cdot\,)\cong[\![\boldsymbol{S},\,\cdot\,]\!];$$

the domain of definition of the right-hand side is the space of functionals that depend on all the BV-variables.

Exercise 11.3. Show that $\boldsymbol{D}_{\text{BV}}=\boldsymbol{Q}+\ldots$ (in fact, the dots stand for the *Koszul–Tate differential*, which captures the Noether identities and the higher generations of the syzygies, plus the correction terms).

By construction, the functionals $\mathcal{F}$ and $\mathcal{F}+\varepsilon[\![\boldsymbol{S},\mathcal{G}]\!]$ now describe the same physics for any $\mathcal{G}$: indeed, we place both functionals in the same cohomology class within $\mathrm{H}^0_{[\![\boldsymbol{S},\,\cdot\,]\!]}$.

Remark 11.2. Strictly speaking, the homological evolutionary vector field $\boldsymbol{D}_{\mathrm{BV}}$ and the differential $[\![\boldsymbol{S},\,\cdot\,]\!]$ are defined on non-coinciding spaces. Indeed, the domain of definition of $\boldsymbol{D}_{\mathrm{BV}}$ is the category of all modules of sections of the induced bundles whose base is the total space for the Whitney sum of the horizontal jet spaces over $J^\infty(\pi)$ with the BV-coordinates along their fibres. For example, the application of $\boldsymbol{D}_{\mathrm{BV}}$ to a *function* $\boldsymbol{q}_x\boldsymbol{q}^\dagger(\boldsymbol{\gamma}^\dagger)^2\boldsymbol{\gamma}_{xxx}$ is well defined but the differential $[\![\boldsymbol{S},\,\cdot\,]\!]$ can act only on the *functional* $\int \boldsymbol{q}_x\boldsymbol{q}^\dagger(\boldsymbol{\gamma}^\dagger)^2\boldsymbol{\gamma}_{xxx}\,\mathrm{d}\boldsymbol{x}$.

Remark 11.3. The BRST-field $\boldsymbol{Q}$ and the BV-field $\boldsymbol{D}_{\mathrm{BV}}$ were introduced *formally* by postulating that they are differentials. Unfortunately, it follows from nowhere that these evolutionary vector fields should at the same time be *geometric*, i, e., why the values of the velocities of the BV-coordinates (such as the ghosts $\boldsymbol{\gamma}$) on true sections $\gamma = \Pi\boldsymbol{p}\big(\boldsymbol{x},[\boldsymbol{q}]\big)$ of the horizontal bundles over $J^\infty(\pi)$ do coincide — up to the necessity to recover the standard summands in (10.20) — with the velocities

$$\mathrm{L}_{\partial^{(\boldsymbol{q})}_{A(\Pi\boldsymbol{p}(\boldsymbol{x},[\boldsymbol{q}]))}}\big(\Pi\boldsymbol{p}(\boldsymbol{x},[\boldsymbol{q}])\big)$$

of those sections, induced by the evolution $\dot{\boldsymbol{q}} = A(\boldsymbol{\gamma})$ on the fields $\boldsymbol{q}$, c. f. Lemma 8.1 on p. 82.

Exercise 11.4. Show that the formal evolutionary vector fields (10.24) *are* geometric for all Hamiltonian operators (and from this deduce that Lemma 8.1 calculates the respective bi-differential Christoffel symbols).

Exercise 11.5. Suppose that the above conjecture is true and thus, all the homological evolutionary vector field realizations $\boldsymbol{Q}$ of variational Lie algebroids are geometric. Prove that every skew-adjoint linear total differential operator with involutive image in $\mathfrak{g}(\pi) = (\varkappa(\pi),[\,,\,])$ is a Hamiltonian operator (c. f. Theorem 10.7).

The Batalin–Fradkin–Vilkovisky method is a tool for the quantisation of gauge systems.[37] As usual, all the dependent variables become formal power series in the Planck constant $\hbar$; in particular, the full BV-action is now the formal power series $\boldsymbol{S}_\hbar$.

Exercise 11.6. List, count their number, and classify the conceptual difficulties of the theory which one encounters during the derivation of the *quantum master-equation*

$$\boldsymbol{i}\hbar\Delta_{\mathrm{BV}}\boldsymbol{S}_\hbar - \tfrac{1}{2}[\![\boldsymbol{S}_\hbar,\boldsymbol{S}_\hbar]\!] = 0$$

from the equality

$$\Delta_{\mathrm{BV}}\left(\exp\left(\frac{\boldsymbol{i}}{\hbar}\boldsymbol{S}_\hbar\right)\right) = 0,$$

which itself follows from the postulate

$$\langle 1\rangle = \int \exp\left(\frac{\boldsymbol{i}}{\hbar}\boldsymbol{S}_\hbar(\boldsymbol{x},[\boldsymbol{q}])\right)[D\boldsymbol{q}]$$

for the Feynman path integral.

Finally, let us recall from [16] a remarkable link between the application of the Batalin–Vilkovisky method to a nonlinear Poisson sigma model (c. f. [1, 117]) and to the calculation of Feynman's path integrals and, on the other hand, the Kontsevich deformation quantisation of the Poisson brackets on usual manifolds,[68] which we addressed in Lecture 9.

Let $\mathcal{D}$ be a disc in $\mathbb{R}^2$ and N^m be a Poisson manifold endowed with the Poisson bivector $\mathcal{P}^{ij}$. The fields in this model are the mappings $\mathcal{X}\colon \mathcal{D} \to N^m$ and the differential one-forms η on $\mathcal{D}$ taking values in the pull-back under $\mathcal{X}$ of the cotangent bundle over N^m; it is also supposed that at the boundary $\partial\mathcal{D}$ of the disc those forms vanish on all vector fields which are tangent to the boundary.

The action of the model is

$$S\left([\mathcal{X}], [\eta]\right) = \int_{\mathcal{D}} \Bigl(\eta_i(\boldsymbol{u}) \wedge \mathrm{d}\mathcal{X}^i(\boldsymbol{u}) + \tfrac{1}{2}\mathcal{P}^{ij}\big|_{\mathcal{X}(\boldsymbol{u})} \left(\eta_i(\boldsymbol{u}) \wedge \eta_j(\boldsymbol{u})\right) \mathrm{d}\boldsymbol{u}\Bigr). \tag{11.3}$$

Let $0, 1$, and ∞ be any three cyclically ordered points on the boundary $\partial\mathcal{D}$ and $f, g \in C^\infty(N)$ be functions (i. e., the Hamiltonians). Then the star-product $f \star g$ is the semi-classical expansion of the path integral:

$$(f \star g)(\boldsymbol{x}) = \int_{\mathcal{X}(\infty)=\boldsymbol{x}} f(\mathcal{X}(1))g(\mathcal{X}(0)) \exp\left(\frac{\boldsymbol{i}}{\hbar} S([\mathcal{X}], [\eta])\right) [D\mathcal{X}\, D\eta]. \tag{11.4}$$

Problem 11.1. Calculate the Noether identities Φ, gauge symmetry generators A, bi-differential Christoffel symbols $\{\!\{\ ,\ \}\!\}_A$, the BRST evolutionary vector field $\boldsymbol{Q}$, the full BV-action $\boldsymbol{S}$, and the Batalin–Vilkovisky differential $\boldsymbol{D}_{\mathrm{BV}}$ for the Poisson sigma model with action (11.3) (see also [30]).

- Calculate the path integral (11.4) and re-derive the Kontsevich formula for the star-product $f \star g$ as the sum over graphs.

Problem 11.2. Inspect the induced transformation of the (BRST-) homological vector field $\boldsymbol{Q}$ under the cylindric symmetry reduction of the Yang–Mills equation to the nonpreriodic 2D Toda chains (c. f. [88]), and reveal the origin of the integrals $w^1, \ldots, w^r$ for these exponential-nonlinear hyperbolic equations[112] in terms of the geometry of the full Yang–Mills models.

12. Deformations of nonlocal structures

This is the last lecture in the course: in it we bring together the concepts of gauge invariance and nonlocalities. Namely, we analyse the geometry of non-abelian coverings over differential equations and study the deformations of these structures. Our attention is focused on finding parametric families of

- Bäcklund (auto)transformations between equations, and
- Lie-algebra valued zero-curvature representations for nonlinear systems.

Bäcklund transformations allow us to produce new –e. g., multi-soliton– solutions from known ones (for instance, starting from very simple sections or from the solutions which describe a smaller number of solitons). On the other hand, the technique of Lax pairs, which is furthered to the notion of zero-curvature representations, allow us to solve the Cauchy problems for nonlinear partial differential equations by using the inverse scattering transform. A detailed exposition of the kinematic integrability approach would go far beyond the frames of our first acquaintance with it; indeed, here we only highlight its most prominent elements, placing them in the context of the already known material.

The notion of covering, which was introduced in Lecture 7, is omnipresent here. In the first part of this lecture we study Bäcklund (auto)transformations, which are the correlated pairs of coverings. Next, we address Gardner's deformations from a similar perspective and inspect how they are naturally generalized to the construction of gauge coverings in terms of flat connections in the principal fibre bundles over differential equations.

12.1. **Bäcklund (auto)transformations.** Intuitively, a Bäcklund transformation between two equations $\mathcal{E}_1 \subseteq J^{k_1}(\pi_1)$ and $\mathcal{E}_2 \subseteq J^{k_2}(\pi_2)$ is a system $\widetilde{\mathcal{E}}$ of differential relations which are imposed on the respective unknowns $\boldsymbol{u}^1$ and $\boldsymbol{u}^2$ in $\mathcal{E}_1$ and $\mathcal{E}_2$ and which are such that if a section $\boldsymbol{s}^1 \in \Gamma(\pi_1)$ solves the first equation $\mathcal{E}_1$ and two sections $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ satisfy those relations $\widetilde{\mathcal{E}}$ simultaneously, then $\boldsymbol{s}^2 \in \Gamma(\pi_2)$ is a solution of the equation $\mathcal{E}_2$, and vice versa.

Example 12.1. The Cauchy–Riemann equations $v^1_y = v^2_x$, $v^1_x = -v^2_y$ determine the Bäcklund autotransformation for the Laplace equation on $\mathbb{R}^2$ (here $\mathcal{E}_1 = \{\Delta v^1 = 0\}$ and $\mathcal{E}_2 = \{\Delta v^2 = 0\}$ are two copies of the Laplace equation).

Definition 12.1. Let $\mathcal{E}_i \subseteq J^{k_i}(\pi_i)$, $i = 1, 2$, be two formally integrable differential equations, $\mathcal{E}_i^\infty$ be their prolongations, and $\tau_i \colon \widetilde{\mathcal{E}} \to \mathcal{E}_i^\infty$ be the coverings with the same total space $\widetilde{\mathcal{E}}$. Then the diagram $\mathcal{B}(\widetilde{\mathcal{E}}; \tau_i; \mathcal{E}_i)$

$$\begin{array}{ccc} & \widetilde{\mathcal{E}} & \\ {}^{\tau_1}\swarrow & & \searrow^{\tau_2} \\ \mathcal{E}_1^\infty & & \mathcal{E}_2^\infty \end{array} \tag{12.1}$$

is called a *Bäcklund transformation* between the equations $\mathcal{E}_1$ and $\mathcal{E}_2$. Diagram (12.1) is called a Bäcklund *auto*transformation for the system $\mathcal{E}$ if $\mathcal{E}_1^\infty \simeq \mathcal{E}_2^\infty \simeq \mathcal{E}^\infty$.

Exercise 12.1. Consider the covering τ^+ of the Korteweg–de Vries equation $w^+_t = -\frac{1}{2}w^+_{xxx} + 3w^+w^+_x$ by the modified KdV equation $\mathfrak{m}_t = -\frac{1}{2}\mathfrak{m}_{xxx} + 3\mathfrak{m}^2\mathfrak{m}_x$, where the

projection τ^+ is determined by the Miura substitution $w^+ = 4\mathfrak{m}^2 + 2\mathfrak{m}_x$. Notice that the modified KdV equation $\mathcal{E}_{\text{mKdV}}$ admits the discrete symmetry $\mu\colon \mathfrak{m} \mapsto -\mathfrak{m}$. Denote by $\tau^- = \tau^+ \circ \mu\colon \mathcal{E}_{\text{mKdV}} \to \mathcal{E}^\infty_{\text{KdV}}$ another covering over the second copy of the Korteweg–de Vries equation $\mathcal{E}_{\text{KdV}}$ upon the unknown w^-. Then $\mathcal{B}(\mathcal{E}_{\text{mKdV}}; \tau^+, \tau^-; \mathcal{E}_{\text{KdV}})$ is a Bäcklund autotransformation for the equation $\mathcal{E}_{\text{KdV}}$. By eliminating the variable $\mathfrak{m}$ altogether, derive the relations between the unknowns w^+ and w^- and their derivatives.

Remark 12.1. We shall consider only those pairs of fibre bundles π_1 and π_2 which also have the common base M^n, assuming that it remains intact under the morphisms of bundles. We thus exclude from the further consideration those transformations which swap or mix the independent variables on the base with the unknowns along the fibres (e. g., as it happens in the *reciprocal transformations*).

Exercise 12.2. Show that every dispersionless two-component system $\boldsymbol{u}_t = f(\boldsymbol{u}) \cdot \boldsymbol{u}_x$ with two independent variables can be made *linear* by the hodograph transformation $u^1(t,x), u^2(t,x) \mapsto t(u^1,u^2), x(u^1,u^2)$.

Remark 12.2. A Bäcklund autotransformation $\mathcal{B}(\widetilde{\mathcal{E}}; \tau_i; \mathcal{E})$ for an equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ induces the *tangent* Bäcklund autotransformation $\mathbb{T}\mathcal{B}(\mathbb{T}\widetilde{\mathcal{E}}; \tau_i \otimes \mathbb{T}\tau_i; \mathbb{T}\mathcal{E})$ between the tangent coverings $\mathbb{T}\mathcal{E}$ over the two copies of $\mathcal{E}$ (the definition of $\mathbb{T}\mathcal{E}$, see Lecture 7, is based on the introduction of the "linearized" variables $\mathbf{U}$ on top of the unknowns $\boldsymbol{u}$ so that the linearized equations $\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}(\mathbf{U}) = 0$ hold).

Definition 12.2. The *recursion transformation* for symmetries $\varphi \in \ker \ell^{(\boldsymbol{u})}_{\boldsymbol{F}}\big|_{\mathcal{E}}$ of an equation $\mathcal{E} = \{\boldsymbol{F} = 0\}$ is a Bäcklund autotransformation for the linearized system $\ell^{(\boldsymbol{u})}_{\boldsymbol{F}}(\mathbf{U}) = 0$.

We conclude that an equation $\mathcal{E}$ does admit a recursion transformation for its symmetries if there is a Bäcklund autotransformation for solutions of $\mathcal{E}$.

Let us now study the construction of Bäcklund (auto)transformations in more detail. Let $\tau_i\colon \widetilde{\mathcal{E}}_i \to \mathcal{E}^\infty_i$ be two coverings ($i = 1,2$) and $\mu\colon \widetilde{\mathcal{E}}_1 \to \widetilde{\mathcal{E}}_2$ be a diffeomorphism of manifolds with distributions (i. e., the tangent mapping $\mu_*\colon \widetilde{\mathcal{C}}_{\tau_1} \to \widetilde{\mathcal{C}}_{\tau_2}$ is an isomorphism of the Cartan distribution at all points of $\widetilde{\mathcal{E}}_1$), though not necessarily a morphism of the coverings which would transform the underlying equations one into another and preserve the structure of the nonlocal fibres over them. Rather, the diffeomorphism μ could swap the fibre variables in π_1 with the nonlocal variables along the fibre of $\tau_1\colon \widetilde{\mathcal{E}} \to \mathcal{E}^\infty_1$; this guarantees that the resulting transformation of solutions to $\mathcal{E}_1$ does not amount to a mere finite symmetry.[43] Then the diagram $\mathcal{B}(\widetilde{\mathcal{E}}_1; \tau_1, \tau_2 \circ \mu; \mathcal{E}_i)$ is a Bäcklund transformation between the equations $\mathcal{E}_1$ and $\mathcal{E}_2$.

Example 12.2. Consider the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$ and construct the family of one-dimensional coverings $\tau_\lambda\colon \widetilde{\mathcal{E}}_\lambda \to \mathcal{E}^\infty_{\text{Liou}}$ over it: let us extend the total derivatives

$$\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{\mathcal{E}_{\text{Liou}}} + \widetilde{u}_x \frac{\partial}{\partial \widetilde{u}}, \qquad \frac{\widetilde{\mathrm{d}}}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\bigg|_{\mathcal{E}_{\text{Liou}}} + \widetilde{u}_y \frac{\partial}{\partial \widetilde{u}}, \qquad \left[\frac{\widetilde{\mathrm{d}}}{\mathrm{d}x}, \frac{\widetilde{\mathrm{d}}}{\mathrm{d}y}\right] = 0$$

by setting the derivatives of the nonlocal variable $\widetilde{u} = \widetilde{u}(\lambda)$ as follows:

$$\begin{aligned} \widetilde{u}_x &= u_x + \exp(-\lambda) \cdot \exp(\widetilde{u} + u), \\ \widetilde{u}_y &= -\, u_y + 2\exp(\lambda) \cdot \sinh(\widetilde{u} - u). \end{aligned} \tag{12.2}$$

Let the diffeomorphism μ be the swapping $u \rightleftarrows \widetilde{u}$ of the fibre variable u in $\mathcal{E}_{\text{Liou}}$ and the nonlocality $\widetilde{u}$ combined with the involution $x \mapsto -x$ and $y \mapsto -y$. Then the diagram $\mathcal{B}(\widetilde{\mathcal{E}}_\lambda; \tau_\lambda, \tau_\lambda \circ \mu; \mathcal{E}_{\text{Liou}})$ is a Bäcklund autotransformation for the Liouville equation. The relations in $\widetilde{\mathcal{E}}_\lambda$ are, obviously,

$$\begin{aligned} (\widetilde{u} - u)_x &= \exp(-\lambda) \cdot \exp(\widetilde{u} + u), \\ (\widetilde{u} + u)_y &= 2\exp(\lambda) \cdot \sinh(\widetilde{u} - u). \end{aligned} \tag{12.3}$$

For each $\lambda \in \mathbb{R}$, the unknown $\widetilde{u}$ satisfies the equation $\widetilde{u}_{xy} = \exp(2\widetilde{u})$ whenever u solves $u_{xy} = \exp(2u)$, and vice versa.

Exercise 12.3. Construct the one-parametric families of Bäcklund (auto)transformations for all possible choices of a pair from the following list:

- the Liouville equation $u_{xy} = \exp(2u)$,
- the wave equation $v_{xy} = 0$, and
- the scal$^+$-Liouville equation $\Upsilon_{xy} = \exp(-2\Upsilon)$ that describes the conformal metrics[42] of *positive* constant Gauss curvature $K = +1$.

Bäcklund (auto)transformations become the powerful generators of exact solutions to nonlinear systems when they appear in families $\mathcal{B}_\lambda(\widetilde{\mathcal{E}}_\lambda; \tau_i^\lambda; \mathcal{E}_i)$. In Lecture 7 we revealed a mechanism for the construction of families of coverings over differential equations: Suppose that φ is a τ_1-shadow that preserves $\mathcal{E}_1$ for all values of the nonlocalities but does *not* lift to a true τ_1-nonlocal symmetry of the equation $\widetilde{\mathcal{E}}_{\lambda=\lambda_0}$ in the covering τ_1. Consequently, this shadow spreads the Cartan differentials of the nonlocalities to a one-parametric family, which determines the law of deformation for the covering equations $\mathcal{E}_\lambda$.

Exercise 12.4. Show that it is the scaling symmetry

$$X = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \ldots \text{ (find this lifting explicitly!)}$$

of the Liouville equation that generates the one-parametric family of coverings (12.2); here λ is the natural parameter along the flow $A_\lambda = \exp(\lambda X)$.

Exercise 12.5. By noticing that the Galilean symmetry of the Korteweg–de Vries equation does not lift to the modified KdV equation (see Exercise 12.1), construct a one-parametric family of Bäcklund autotransformations for the KdV equation.

Remark 12.3. In the literature, the natural parameter λ along such integral trajectories often appears in disguise in the resulting families of the Bäcklund transformations: it is $k = \exp(\lambda)$ but not λ itself which is taken to mark the relations $\mathcal{B}_\lambda(\widetilde{\mathcal{E}}_\lambda; \tau_i^\lambda; \mathcal{E}_i^\infty)$.

[42]Bäcklund (auto)transformations were historically first discovered for the sine–Gordon equation in the context of propagation of the pseudospherical surfaces of negative Gauss curvature $K = -1$.

Exercise 12.6. Rescale the Korteweg–de Vries equation to $w_t + w_{xxx} + 6ww_x = 0$ and potentiate it by using the substitution $w = v_x$. Consider the family of Bäcklund autotransformations

$$\begin{aligned}\widetilde{v}_x + v_x &= \tfrac{1}{2}k - \tfrac{1}{2}(v - \widetilde{v})^2,\\ \widetilde{v}_t + v_t &= (v - \widetilde{v})(v_{xx} - \widetilde{v}_{xx}) - 2(v_x^2 + v_x\widetilde{v}_x + \widetilde{v}_x^2)\end{aligned}$$

for the potential KdV equation $v_t + v_{xxx} + 3v_x^2 = 0$. Now take $v = 0$, which obviously is a solution, plug it into the Bäcklund autotransformation, fixing a value $k_1 \in \mathbb{R}$ of the parameter, and integrate for $\widetilde{v}$. Then take the solution $\widetilde{v}(x,t)$ which is bounded at the zero of its argument(s) and construct $\widetilde{w}(x,t) = \widetilde{v}_x(x,t)$. Compare the outcome with the one-soliton solution of the KdV equation (see Problem 2.6 on p. 26).

Exercise 12.7. Let $k_1, \ldots, k_N \in \mathbb{R}$ be the wave numbers. Take an N-soliton solution $w_{k_1,\ldots,k_N}(x,t)$ of the KdV equation and plug it into the Bäcklund autotransformation (see the previous exercise) at the value $k_{N+1} \in \mathbb{R}$ of the parameter, and integrate. Is the new solution of the KdV equation a multi-soliton solution and if so, how many nonlinearly-interacting solitons does it describe and what are their wave numbers?

Another important property of the parametric families of Bäcklund (auto)transformations for equations $\mathcal{E}_1$ and $\mathcal{E}_2$ is the construction of the *Lamb diagrams*. Namely, the requirement that for any two values λ_1, λ_2 of the parameter the two consecutive transformations are permutable leads to a *compatible* system of differential relations

$$\begin{array}{ccc} \boldsymbol{u} \in \mathrm{Sol}(\mathcal{E}_1) & \xrightarrow{\lambda_1} & \boldsymbol{u}_1 \in \mathrm{Sol}(\mathcal{E}_2) \\ \lambda_2 \downarrow & & \downarrow \lambda_2 \\ \boldsymbol{u}_2 \in \mathrm{Sol}(\mathcal{E}_2) & \xrightarrow[\lambda_1]{} & \boldsymbol{u}_{12} \in \mathrm{Sol}(\mathcal{E}_1), \end{array} \tag{12.4}$$

which is a highly nontrivial fact. Moreover, the resulting solution $\boldsymbol{u}_{12}$ can be algebraically expressed in terms of the three previously found sections $\boldsymbol{u}, \boldsymbol{u}_1$, and $\boldsymbol{u}_2$. Such relations are called the *nonlinear superposition formulas*.

Exercise 12.8. Let $u(x,y)$ be a solution of the Liouville equation $u_{xy} = \exp(2u)$ and let u_1 and u_2 be the solutions which are related to the initial section $u(x,y)$ via Bäcklund autotransformation (12.3) as in Diagram (12.4) for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Show that there **exists** a unique solution $u_{12}(x,y)$ which makes the diagram commutative and moreover, that the section u_{12} satisfies the nonlinear superposition relation

$$\exp(u_{12}) = \exp(u) \cdot \frac{k_2 \exp(u_1) - k_1 \exp(u_2)}{k_2 \exp(u_2) - k_1 \exp(u_1)},$$

where we put $k_i := \exp(\lambda_i)$.

Exercise 12.9. Derive the nonlinear superposition formulas for solutions of the Korteweg–de Vries equation by using the one-parametric family of Bäcklund autotransformations for it (see Exercise 12.6).

Exercise 12.10. Derive all the nine nonlinear superposition formulas for solutions of the Liouville, scal$^+$-Liouville, and the wave equations (see Exercise 12.3) by considering the six one-parametric families of Bäcklund (auto)transformations between these three equations.

12.2. **Gardner's deformations.** The definition of an infinite-dimensional integrable system — such as the Korteweg–de Vries equation — is based on the presence of infinitely many integrals of motion which are in involution and which are arranged according to the Magri scheme (see Lecture 9). The Hamiltonian functionals $\mathcal{H}_i^{(j)}$ determine the hierarchy of commuting flows; in Lecture 7 we described the construction of the recursion differential operators that propagate the symmetries even without knowing the Hamiltonians themselves. (In fact, the presence of infinitely many commuting symmetries is a strong evidence, though of course not a guarantee, that the system at hand may also possess infinitely many conservation laws.) However, the technique of recursion differential operators is not capable of finding any reasonable recurrence relation between the Hamiltonians of the hierarchies. This shows that the recurrence relations between the Hamiltionians are much more informative than any recursions for the (co)symmetries. In what follows, we address this problem from a classical but almost forgotten perspective; not only the solution but the way for attaining it will have sound repercussions in the geometry of integrable systems.

Namely, we consider the technique of Gardner's deformations of completely integrable bi-Hamiltonian evolutionary systems. The essence of this procedure is that the generating vector-functions $\widetilde{\boldsymbol{u}}(\varepsilon)$ for the Hamiltonians of the hierarchies satisfy the auxiliary evolution equations that obey certain restrictions (see Lemma 12.1 below). Gardner's deformations yield the recurrence relations for densities of the Hamiltonians and also determine the parametric extensions of known systems, which serves as a generator of new integrable models.

Having advanced that far, we shall review the link between Gardner's deformations of completely integrable systems and the deformations of zero-curvature representations for kinematic-integrable systems. This link between the Poisson world of Magri schemes and the gauge world of flat connections in the principal fibre bundles over differential equations relies on the high-energy scattering governed by the Schrödinger equation.[90]

Let us start with a motivating example of the remarkable nonlinear substitution that signalled the beginning of the KdV-boom.

Example 12.3 (Gardner's deformation of the KdV equation)**.** Let $\varepsilon \in \mathbb{R}$ be a parameter. Consider the family of *Gardner's equations*[95]

$$\mathcal{E}_\varepsilon = \{\widetilde{w}_t = -\tfrac{1}{2}\widetilde{w}_{xxx} + 3\widetilde{w}\widetilde{w}_x + 3\varepsilon^2\widetilde{w}^2\widetilde{w}_x\} \tag{12.5}$$

which extend the Korteweg–de Vries equation

$$\mathcal{E}_0 = \{w_t = -\tfrac{1}{2}w_{xxx} + 3ww_x\}$$

at $\varepsilon = 0$ in such a way that at all $\varepsilon \in \mathbb{R}$ there is the *Miura contraction*

$$\mathfrak{m}_\varepsilon = \{w = \widetilde{w} \pm \varepsilon\widetilde{w}_x + \varepsilon^2\widetilde{w}^2\} \colon \mathcal{E}_\varepsilon \to \mathcal{E}_0.$$

Suppose that $\widetilde{w} = \sum\limits_{k=0}^{+\infty} \varepsilon^k \widetilde{w}_k$ is the formal power series expansion of the fibre variabes $\widetilde{w} = \widetilde{w}(\varepsilon)$ in the family of vector bundles π_ε over $\mathbb{R}^2 \ni (t, x)$. Because the Gardner equations $\mathcal{E}_\varepsilon$ express the conservation of some currents, the coefficients $\widetilde{w}_k$ of the expansion are termwise conserved. Now, using the Miura contraction $\mathfrak{m}_\varepsilon$ and equating the formal

power series in its right-hand side to $w = w \cdot 1 + 0 \cdot \varepsilon + 0 \cdot \varepsilon^2 + \ldots$, we obtain the *recurrence relation*

$$\widetilde{w}_0 = w,\ \widetilde{w}_1 = \mp w_x,\ \widetilde{w}_k = \mp\frac{\mathrm{d}}{\mathrm{d}x}(\widetilde{w}_{k-1}) - \sum_{i+j=k-2} \widetilde{w}_i\widetilde{w}_j,\ k \geqslant 2,$$

upon the integrals of motion $\widetilde{w}_k[w]$ for the Korteweg–deVries equation.

Exercise 12.11. Show that the densities $\widetilde{w}_{2k+1}[w]$ with odd indexes are trivial but all the conserved densities $\widetilde{w}_{2k}[w]$ are nontrivial (and determine the Hamiltonians $\mathcal{H}_k$ of the KdV hierarchy).

Definition 12.3 (provisional)**.** The pair $(\mathcal{E}_\varepsilon, \mathfrak{m}_\varepsilon \colon \mathcal{E}_\varepsilon \to \mathcal{E}_0)$, where $\mathcal{E}_\varepsilon$ is a continuity equation, is called the *Gardner deformation* of the evolutionary system $\mathcal{E}_0$.

Exercise 12.12. Find Gardner's deformations of

- the Kaup–Boussinesq system $u_t = uu_x + v_x,\ v_t = (uv)_x + u_{xxx}$, and
- the Boussinesq equation $u_t = v_x,\ v_t = u_{xxx} + uu_x$ (or $u_{tt} = u_{xxxx} + (\frac{1}{2}u^2)_{xx}$).

Derive the recurrence relations for densities of the Hamiltonians in the respective hierarchies and check that they are indeed nontrivial.

Let us note further that the bi-Hamiltonian hierarchies *share* the functionals with the modified hierarchies which are correlated with the former by the Miura substitutions (see Lecture 9): the integrals of motion are the same up to, possibly, finitely many lowest-order Casimirs for the modified hierarchies. This suggests a generalization[53] of the provisional definition.

Definition 12.4. The *Gardner deformation* for a system $\mathcal{E}'$ is the diagram

$$\begin{tikzcd} \mathcal{E}' \arrow[dr, "\tau"'] & & \mathcal{E}_\varepsilon \arrow[dl, "\mathfrak{m}_\varepsilon"] \\ & \mathcal{E}_0 & \end{tikzcd} \tag{12.6}$$

in which τ is a substitution[43] and $\mathfrak{m}_\varepsilon$ is the Miura contraction at all ε.

This understanding of the Gardner deformation problem extends the class of deformable systems (in particular, it now includes the systems which are not written as continuity relations, such as the potential KdV equation or the potential modified KdV equation).

Exercise 12.13. Find Gardner's deformations for the Kaup–Broer system

$$u_t = u_{xx} + uu_x + w_x,\ w_t = (uw)_x - w_{xx},$$

which is mapped by the substitution $w = v - u_x$ to the Kaup–Boussinesq equation.

[43] For example, τ can be an invertible reparametrization of the unknowns which destroys the form $u = \widetilde{u} + \varepsilon \cdot (\ldots)$ of the contraction $\mathfrak{m}_\varepsilon$; alternatively, it can be the Miura-type transformation of the hierarchies for $\mathcal{E}'$ and $\mathcal{E}_0$ that induces the senior Poisson structures for $\mathcal{E}_0$ via the junior structures for $\mathcal{E}'$, see (9.11) on p. 96.

We observe that Gardner's deformations (12.6) are dual to the notion of Bäcklund transformations (12.1). Moreover, discrete symmetries μ of the extensions $\mathcal{E}_\varepsilon$ induce Bäcklund autotransformations $\mathcal{E}_0 \xleftarrow{\mathfrak{m}_\varepsilon} \mathcal{E}_\varepsilon \xrightarrow{\mathfrak{m}_\varepsilon\circ\mu} \mathcal{E}_0$ for the equation $\mathcal{E}_0$. For example, a reversion of the sign of the parameter ε in the contraction $\mathfrak{m}_\varepsilon$ for the KdV equation (see (12.5)), followed by the elimination of the variable $\widetilde{u}(\varepsilon)$, provides the one-parametric family of Bäcklund autotransformations for the Korteweg–de Vries equation.

Furthermore, we claim that Gardner's deformations are the inhomogeneous generalizations of the infinitesimal symmetries for differential equations.

Lemma 12.1. Let $\mathcal{I} \subseteq \mathbb{R}$ be an open set such that for each $\varepsilon \in \mathcal{I}$ there is the Miura contraction $\mathfrak{m}_\varepsilon\colon \mathcal{E}_\varepsilon = \{\boldsymbol{F}_\varepsilon = 0\} \to \mathcal{E}_0 = \{\boldsymbol{F} = 0\}$. Let $\dot{\boldsymbol{u}}(\varepsilon)$ be the derivative of the formal power series $\widetilde{u} = \widetilde{u}(\varepsilon)$ with respect to ε. Then for every $\varepsilon_0 \in \mathcal{I}$ we have that

(i) the equality

$$\ell_{\boldsymbol{F}_{\varepsilon_0}}^{(\widetilde{u}(\varepsilon_0))}(\dot{\boldsymbol{u}}(\varepsilon_0)) + \frac{\partial \boldsymbol{F}_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon=\varepsilon_0}([\widetilde{u}(\varepsilon_0)]) \doteq 0 \quad \text{on } \mathcal{E}_{\varepsilon_0}$$

is the inhomogeneous generalization of the determining equation (3.3) for symmetries $\dot{\boldsymbol{u}} = \varphi(\boldsymbol{x}, [\boldsymbol{u}])$;

(ii) the evolution equation $\mathcal{E}_{\varepsilon_0} = \{\boldsymbol{F}_{\varepsilon_0} = 0\}$ and the Miura contraction satisfy the equality

$$\ell_{\boldsymbol{F}}^{(\boldsymbol{u})}\Big(\frac{\partial \mathfrak{m}_\varepsilon}{\partial \varepsilon}\bigg|_{\varepsilon_0}\Big) \doteq \partial^{(\widetilde{\boldsymbol{u}}(\varepsilon_0))}_{\partial \boldsymbol{F}_\varepsilon/\partial\varepsilon|_{\varepsilon=\varepsilon_0}}(\mathfrak{m}_{\varepsilon_0}),$$

which holds by virtue of any of the two equations $\mathcal{E}_0 = \{\boldsymbol{F}(\boldsymbol{x}, [\boldsymbol{u}]) = 0\}$ or $\mathcal{E}_\varepsilon = \{\boldsymbol{F}_\varepsilon(\boldsymbol{x}, [\widetilde{\boldsymbol{u}}(\varepsilon)]; \varepsilon) = 0\}$ at $\varepsilon = \varepsilon_0$ because the understanding of $\widetilde{\boldsymbol{u}}(\varepsilon)$ as a formal power series with coefficients depending on $[\boldsymbol{u}]$ and of $\boldsymbol{u}$ as the image of $\widetilde{\boldsymbol{u}}(\varepsilon)$ under $\mathfrak{m}_\varepsilon$ identifies the times t in both systems.

Remark 12.4. The construction of the extensions $\mathcal{E}_\varepsilon$ is a generator of a new, *adjoint* completely integrable systems; we refer to Problem 12.3 for their definition.

We finally recall that the integrals of motion can be found alternatively, via the calculation of the residues of fractional powers of the Lax operators (see below), that is, by using the formal calculus of pseudodifferential operators.[32] Still we note that such a calculation of the $(k+1)$-th residue does not take into account the already known residues at smaller indexes. On the other hand, when using Gardner's deformations, at each inductive step we can use all the previously obtained conserved densities. (Besides, there is no need to multiply the pseudodifferential operators by applying the Leibniz rule an always increasing number of times.) By this argument we understand Gardner's deformations as the transformation in the space of integrals of motion that maps the residues to the coefficients $\widetilde{u}_k$ of the generating functions $\widetilde{\boldsymbol{u}}(\varepsilon)$.

Still there is a deep intrinsic relation between Gardner's deformations and the eigenvalue problems for the Lax operators (or, more generally, nontrivial deformations of Lie-algebra valued zero-curvature representations for nonlinear systems). Namely, both approaches involve the vector field- and, respectively, matrix representations of Lie algebras, whereas the deformation parameter ε is inverse-proportional to the wave number k (the eigenvalue in the Sturm–Liouville problem $L\Psi = E\Psi$ is then $E = \hbar k^2$). Let us discuss these aspects in more detail.

12.3. **Zero-curvature representations.** Let us convert Gardner's deformation (12.5) to a very famous non-abelian covering over the Korteweg–de Vries equation $w_t = -\frac{1}{2}w_{xxx}+3ww_x$; in what follows, we capture the transformations of the x-derivatives for the nonlocalities and leave the control over the nontrivial correspondence of the time evolutions as an obligatory exercise. So, we have that

$$\pm\widetilde{w}_x = \frac{1}{\varepsilon}(w-\widetilde{w}) - \varepsilon\widetilde{w}^2.$$

The auxiliary formal power series $F = \varepsilon\cdot\widetilde{w}$ (the Stieltjes function) satisfies the Riccati ordinary differential equation

$$\pm F_x = w - \frac{1}{\varepsilon}F - F^2. \tag{12.7}$$

Let us fix the plus sign in (12.7) and use the projective substitution

$$F = P_x/P = \frac{\mathrm{d}}{\mathrm{d}x}(\log P), \tag{12.8}$$

whence we obtain that

$$-P_{xx} - \frac{1}{\varepsilon}P_x + w\cdot P = 0. \tag{12.9}$$

Let the *wave function* be

$$\Psi = \exp(\boldsymbol{i}kx)\cdot P,$$

where $k = 2\pi/\boldsymbol{\lambda}$ is the wave number for a wave length $\boldsymbol{\lambda}$. From (12.9) we conclude that the wave function Ψ satisfies the stationary *Schrödinger equation*

$$\left(-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + w(x,t)\right)\Psi = \hbar k^2\cdot\Psi,$$

where the system of units is such that $\hbar = 1$ and we put $\varepsilon = \frac{1}{2\boldsymbol{i}k}$: the limit $\varepsilon\to 0$ in the Miura contraction $\mathfrak{m}_\varepsilon$ corresponds to the short waves, $\boldsymbol{\lambda}\to 0$ and $k\to\infty$, and the scattering on the potential w at high energies.

Exercise 12.14. Restore the time-components in all the coverings and show that the wave function $\Psi(x,t)$ satisfies the system

$$L\Psi = \hbar k^2\cdot\Psi, \qquad \Psi_t = A\Psi,$$

where $L = -\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + w(x,t)$ is the Hill operator and ($\hbar := 1$)

$$A = (-L)_+^{\frac{3}{2}} = -\tfrac{1}{2}\mathrm{d}^3/\mathrm{d}x^3 + \tfrac{3}{4}w\cdot\mathrm{d}/\mathrm{d}x + \tfrac{3}{8}w_x$$

is the nonnegative part of the formal fractional power of $(-L)$; by a mere coincidence the operator A is a Hamiltonian operator yet it is not the second structure for the KdV equation upon $w(x,t)$.

The crucial postulate is $k_t = 0$, i.e., the time evolution is isospectral (hence the eigenvalues that correspond to the bound states are the integrals of motion).

Exercise 12.15. Derive the Lax equation[86]

$$(L_t + [L,A])\,\Psi = 0$$

upon the L-A-pair (the *Lax pair*).

Our next step is very logical: when the Riccati equation (12.7) and the projective substitution (12.8) appear on-stage, $\mathfrak{sl}_2(\mathbb{C})$ joins them. This opens the way from a concrete example of Gardner's deformation for the KdV equation to the general concept of $\mathfrak{g}$-valued zero-curvature representations for nonlinear systems $\mathcal{E}$ and Lie algebras $\mathfrak{g}$.

By definition, put $\lambda = k^2$ and, instead of the single variable Ψ, consider the two-dimensional covering over the KdV equation. Namely, let us denote Ψ by ψ_0 and set $\psi_1 = \Psi_x$ (note that the derivative $\frac{\mathrm{d}}{\mathrm{d}x}(\psi_1)$ is then expressed from the Lax equation $L\Psi = \lambda\Psi$). Hence we obtain

$$\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}_x = \begin{pmatrix}0 & 1\\ w-\lambda & 0\end{pmatrix}\begin{pmatrix}\psi_0\\ \psi_1\end{pmatrix}.$$

Exercise 12.16. Derive the formula for the time-derivative ${}^{\mathrm{t}}(\psi_0,\psi_1)_t$ and conclude that both (2×2)-matrices in the right-hand sides of the equations upon the nonlocalities belong to the tensor product $\mathfrak{sl}_2(\mathbb{C})\otimes\mathcal{F}(\mathcal{E}^\infty_{\mathrm{KdV}})$.

Remark 12.5. As soon as the Lie algebra $\mathfrak{g}$ at hand is recognized and its structure constants are known, we are able to pass freely between any representations of that algebra (for example, choosing its matrix representations or encoding its generators by using vector fields). The first option is traditional, whereas the use of vector field representations for $\mathfrak{g}$ in the fibres of the bundles over differential equations permits the following interpretation of the Lax pairs in terms of coverings.

Namely, let us recall that it is not the wave function itself but the structure Lie group G and its Lie algebra $\mathfrak{g}$ which are involved in the construction of the Lax equations upon the curvature and in the gauge transformations (c. f. Lecture 6 where we addressed the Maxwell and the Klein–Gordon equations).

Example 12.4. Choose the standard basis $\langle\mathfrak{e},\mathfrak{f},\mathfrak{h}\rangle$ in $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$ so that the commutation relations in the Lie algebra are

$$[\mathfrak{e},\mathfrak{h}] = -2\mathfrak{e},\quad [\mathfrak{f},\mathfrak{h}] = 2\mathfrak{f},\quad [\mathfrak{e},\mathfrak{f}] = \mathfrak{h}.$$

Consider the representations of $\mathfrak{g}$

$$\rho\colon \mathfrak{g}\to\{M\in\mathrm{Mat}(2,2)\mid \mathrm{tr}\,M=0\}\colon\ \rho(\mathfrak{e}) = \left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right),\ \rho(\mathfrak{f}) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\ \rho(\mathfrak{h}) = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$$

and

$$\varrho\colon \mathfrak{g}\to\mathrm{Der}(\mathbb{C},\mathrm{poly})\colon\ \varrho(\mathfrak{e}) = \frac{\partial}{\partial z},\quad \varrho(\mathfrak{f}) = -z^2\frac{\partial}{\partial z},\quad \varrho(\mathfrak{h}) = -2z\frac{\partial}{\partial z}.$$

Then the L-A-pair for the KdV equation specifies the family ($\lambda\in\mathbb{R}$) of the one-dimensional non-abelian coverings with the nonlocal variable z.

Exercise 12.17. Show that $\frac{\tilde{\mathrm{d}}}{\mathrm{d}x}(z) = 1 + (\lambda - w)\cdot z^2$, where $\lambda\in\mathbb{R}$.

- Calculate the derivative $\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}(z)$ and check that the mixed derivatives of z are equal by virtue of the equations of motion.

Remark 12.6. The transition from a finite-dimensional matrix Lie algebra $\mathfrak{g}\subseteq\mathfrak{gl}_k(\mathbb{C})$ to the covering whose structure is determined by a vector field representation of $\mathfrak{g}$ relies on the following general construction.

Namely, we first let $V := \mathbb{C}^k$ (it is possible that V is a local chart for a k-dimensional $\Bbbk$-manifold; as usual, the choice of the ground field $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ does not matter here). Let $\boldsymbol{v} = (v^1, \ldots, v^k)$ be the coordinates on V and put $\partial_{\boldsymbol{v}} = {}^{\mathrm{t}}(\partial/\partial v^1, \ldots, \partial/\partial v^k)$. For any element $g \in \mathfrak{g} \subseteq \mathfrak{gl}_k(\mathbb{C})$ we define by the formula $\mathrm{r}(g) \stackrel{\text{def}}{=} \boldsymbol{v} g \partial_{\boldsymbol{v}}$ its representation in the space of linear vector fields on V.

Exercise 12.18. Verify the equality $[\mathrm{r}(g), \mathrm{r}(h)] = \mathrm{r}([g,h])$ for all $g, h \in \mathfrak{g}$.

Second, at all points of the subset $v^1 \neq 0$ we consider the projection $p\colon v^i \mapsto \omega^i = \kappa v^i / v^1$, $\kappa \in \Bbbk$. We thus obtain the new coordinates $\boldsymbol{\omega} = (\kappa, \omega^2, \ldots, \omega^k)$; the differential $\mathrm{d}p$ yields the basis $\partial_{\boldsymbol{\omega}} = {}^{\mathrm{t}}(-\frac{1}{\kappa}\sum\limits_{i=2}^{k} \omega^i \cdot \partial/\partial\omega^i, \partial/\partial\omega^2, \ldots, \partial/\partial\omega^k)$ at each point of the two loci $v^1 \neq 0$. Now consider the vector field

$$\varrho(g) \stackrel{\text{def}}{=} \mathrm{d}p(\mathrm{r}(g));$$

in coordinates, we have $\varrho(g) = \boldsymbol{\omega}\, g\, \partial_{\boldsymbol{\omega}}$.

Exercise 12.19. Inspect that the commutation relations in the matrix Lie algebra $\mathfrak{g} \subseteq \mathfrak{gl}_k(\mathbb{C})$ remain valid for the respective *nonlinear* vector fields:

$$[\varrho(g), \varrho(h)] = \varrho([g,h])$$

for $g, h \in \mathfrak{g}$.

Summarizing, it is a matter of agreement or convenience whether the standard matrix arithmetic is used for the calculation of the commutators or true vector fields are commuted. In any case, the $\mathfrak{g}$-valued zero-curvature represrntations (which are an immediate generalization of the Lax pairs and which we study right now) belong to a particular class of coverings over differential equations so that all the concepts and techniques from the theory of nonlocalities, see Lecture 7, remain applicable.

Let G be a finite-dimensional matrix Lie group over $\mathbb{C}$, $\mathfrak{g}$ be its Lie algebra, and $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ be a differential equation. Consider the *principal G-bundle* over $\mathcal{E}^\infty$: we denote by $\boldsymbol{\psi}$ the wave function and by $g \in G$ the elements of the structure group that acts in the fibres by gauge transformations.[44] Endow the total space of the G-bundle over $\mathcal{E}^\infty$ with the structure of a covering by introducing in it the flat connection that restricts to the Cartan connection on $\mathcal{E}^\infty$ under the projection to that base of the G-bundle. Namely, let

$$\boldsymbol{\alpha} = \sum_{i=1}^{n} \alpha_i \, \mathrm{d}x^i$$

be a $\mathfrak{g}$-valued connection one-form with the coefficients $\alpha_i \in \mathcal{F}(\mathcal{E}^\infty) \otimes_{C^\infty(M^n)} \mathfrak{g}$. We require that the connection $\nabla_{\boldsymbol{\alpha}}$ is *flat*, i. e., its curvature equals zero.

Proposition 12.2. The following four statements are equivalent:

- the G-connection $\nabla_{\boldsymbol{\alpha}}$ determined by the $\mathfrak{g}$-valued one-form $\boldsymbol{\alpha}$ is flat;

[44]This construction was discussed in Lecture 6 in a simpler setup of the G-bundles over the base manifold M^n but not over the infinite prolongation $\mathcal{E}^\infty$ which itself fibres over M^n. As in Lecture 6, we recall that the generators of (in)finite gauge transformations are the *functions of their arguments*.

- the system of equations
$$\frac{\mathrm{d}}{\mathrm{d}x^j}(\alpha_i) - \frac{\mathrm{d}}{\mathrm{d}x^i}(\alpha_j) + [\alpha_i, \alpha_j] = 0$$
holds for all i, j such that $1 \leqslant i < j \leqslant n$;
- the connection one-form $\boldsymbol{\alpha}$ satisfies the Maurer–Cartan equation
$$\overline{\mathrm{d}}\boldsymbol{\alpha} \doteq \tfrac{1}{2}[\boldsymbol{\alpha}, \boldsymbol{\alpha}] \text{ on } \mathcal{E}^\infty, \tag{12.10}$$
where the horizontal differential $\overline{\mathrm{d}}$ acts by the Leibniz rule and the commutator of $\mathfrak{g}$-valued differential forms is the bracket in $\mathfrak{g}$ tensored with the wedge product in the space of forms;
- the mixed derivatives $\boldsymbol{\psi}_{x^i x^j} \doteq \boldsymbol{\psi}_{x^j x^i}$ coincide by virtue of $\mathcal{E}^\infty$ for any covariantly constant[45] wave function $\boldsymbol{\psi}$ such that $\boldsymbol{\psi}_{x^i} = \alpha_i \boldsymbol{\psi}$.

Exercise 12.20. Prove Proposition 12.2.

Definition 12.5. The G-bundle over $\mathcal{E}^\infty$ endowed with the $\mathfrak{g}$-valued horizontal one-form $\boldsymbol{\alpha}$ that satisfies the Maurer–Cartan equation (12.10) by virtue of $\mathcal{E}^\infty$ — and thus determines the flat connection — is called the *zero-curvature representation* for the differential equation $\mathcal{E}$.

Exercise 12.21. Derive the reparametrization formulas for the $\mathfrak{g}$-valued connection one-form $\boldsymbol{\alpha}$ under the gauge action of the structure Lie group G along the fibres (c. f. Eq. (6.5) on p. 63 and Theorem 10.5 on p. 106; pay due attention to the choice of the signs in the definition of the covariant derivatives and to the direction in which the elements g or $g^{-1} \in G$ act).

Exercise 12.22. Instead of using a matrix representation for the Lie group G and its Lie algebra $\mathfrak{g}$, supose that the same covering is given in a traditional way — in terms of vector fields — by using the construction from Remark 12.6. Describe the action of the group G by gauge transformations in that covering.

Our experience with the application of Bäcklund autotransformations to multi-soliton solutions of nonlinear models hints us that a greater profit is gained from the nonlocal structures when they appear in families. This remains true with zero-curvature representations. Suppose that $\boldsymbol{\alpha}(\lambda)$ is an analytic family of solutions to the Maurer–Cartan equation (12.10) at all $\lambda \in \mathcal{I}$ for a given system $\mathcal{E}$ and some open set $\mathcal{I} \subseteq \mathbb{R}$.

Definition 12.6. The parameter λ is *removable* if the zero-curvature representations $\boldsymbol{\alpha}(\lambda)$ are gauge-equivalent under the action of the Lie group G at all values $\lambda \in \mathcal{I}$ of the parameter. Otherwise, the parameter λ is *non-removable*.

By definition, put $\dot{\boldsymbol{\alpha}}(\lambda_0) = \frac{\partial}{\partial\lambda}\big|_{\lambda=\lambda_0} \boldsymbol{\alpha}(\lambda)$.

Theorem 12.3 ([92])**.** *The parameter λ is removable if and only if for each $\lambda \in \mathcal{I}$ there is a matrix $Q(\lambda) \in \mathcal{F}(\mathcal{E}^\infty) \otimes_{C^\infty(H^n)} \mathfrak{g}$, piecewise continuously differentiable in λ, such that*
$$\frac{\partial}{\partial\lambda}\alpha_i(\lambda) = \frac{\mathrm{d}}{\mathrm{d}x^i}Q(\lambda) - [\alpha_i(\lambda), Q(\lambda)], \quad 1 \leqslant i \leqslant n.$$

[45] The non-standard choice of the *minus* sign in the definition $\nabla_i = \frac{\mathrm{d}}{\mathrm{d}x^i} - \alpha_i$ of the covariant derivative is widely accepted in the literature on zero-curvature representations.

(*In other words, the evolution* $\dot{\boldsymbol{\alpha}}_\lambda = \overline{\partial}_{\boldsymbol{\alpha}(\lambda)} Q(\lambda)$ *does not leave the gauge cohomology class of* $\boldsymbol{\alpha}_{\lambda_0}$ *for any* $\lambda_0 \in \mathcal{I}$ *with respect to the differential* $\overline{\partial}_{\boldsymbol{\alpha}(\lambda)} = \overline{\mathrm{d}} - \mathrm{ad}_{\boldsymbol{\alpha}(\lambda)}$; *two* $\mathfrak{g}$-*valued horizontal one-forms are gauge-equivalent if and only if they belong to the same gauge cohomology class.*)

This theorem is the most effective instrument for checking whether the parameter λ in a given family $\boldsymbol{\alpha}(\lambda)$ of zero-curvature representations is or is not removable. However, the next step is much more subtle: if the parameter λ is already known to be non-removable, then this in general *does not imply* that a meaningful inverse scattering problem can be posed for $\mathcal{E}$ by using this family $\boldsymbol{\alpha}(\lambda)$ of connection one-forms. Namely, let us consider the most popular case of differential equations with two ($n = 2$) independent variables; recall that the conserved currents $\eta \in \overline{\Lambda}^{n-1}(\mathcal{E})$ are horizontal one-forms on the equation $\mathcal{E}$. It then can happen that the classes of gauge-equivalent solutions $\boldsymbol{\alpha}$ to the Maurer–Cartan equation are mixed — in a way which is hard to control or even recognize — with the conserved currents η which are tensored by some elements $\mathfrak{t}$ of the Lie algebra $\mathfrak{g}$ (provided that the obvious compatibility conditions hold for $\boldsymbol{\alpha}$, η, and $\mathfrak{t}$).

Exercise 12.23. Derive the compatibility condition upon $\mathfrak{g}$-valued zero-curvature representations $\boldsymbol{\alpha}$, conserved currents η, and elements $\mathfrak{t} \in \mathfrak{g}$ such that $\boldsymbol{\alpha} + \lambda \cdot \eta \cdot \mathfrak{t}$ remains a solution of the Maurer–Cartan equation (12.10) at all $\lambda \in \mathcal{I}$ from an open subset $\mathcal{I} \subseteq \mathbb{R}$.

Intuitively, the picture is analogous to the structure of Galilei's group of motion for $\mathbb{E}^3$: the translations are almost everywhere transversal to the orbits of subgroups of rotations around any point. Let us give an "extreme" example[80] when there is no zero-curvature representation at all.

Example 12.5. Consider the KdV equation $w_t + w_{xxx} + 6ww_x = 0$ and two conserved currents $\frac{\mathrm{d}}{\mathrm{d}t}(X_i) \doteq \frac{\mathrm{d}}{\mathrm{d}x}(T_i)$ for it: let $X_1 = w$, $T_1 = -3w^2 - w_{xx}$ and $X_2 = w^2$, $T_2 = -4w^3 - 2ww_{xx} + w_x^2$. Note that $[\mathfrak{e}, \mathfrak{e}] = 0$, where $\mathfrak{e} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is the upper-triangular element in the basis for the matrix representation of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Therefore, for each $\lambda \in \mathbb{R}$ we formally conclude that the one-form

$$\boldsymbol{\alpha}(\lambda) = (X_1 + \lambda \cdot X_2) \cdot \mathfrak{e}\, \mathrm{d}x + (T_1 + \lambda T_2) \cdot \mathfrak{e}\, \mathrm{d}t \tag{12.11}$$

belongs to the one-parametric family of $\mathfrak{sl}_2(\mathbb{C})$-valued zero-curvature representations for the Korteweg–de Vries equation.

Exercise 12.24. Show that the parameter λ in family (12.11) is non-removable under the gauge transformations by elements $g \in SL(2, \mathbb{C})$.

We say that a non-removable parameter λ in the family $\boldsymbol{\alpha}(\lambda)$ of $\mathfrak{g}$-valued zero-curvature representation for $\mathcal{E}$ is *spectral* if the Cauchy problem for the equation $\mathcal{E}$ **can** be solved by the inverse scattering on the basis of the given spectral problem $\nabla_{\boldsymbol{\alpha}(\lambda)} \boldsymbol{\psi} = 0$. The method relies on the analysis of the evolution of the reflection and transition coefficients and the behaviour of the eigenvalues for the bound, negative-energy states; when it works, this technique is one of the most powerful tools for solving nonlinear models of mathematical physics.

Problem 12.1. Construct a one-parametric family of Bäcklund autotransformations for the sine–Gordon equation $z_{xy} = \sin z$, explore which effect such transformations produce on its multi-soliton solutions, and derive the nonlinear superposition formulas by verifying that the Lamb diagrams are indeed commutative.

Problem 12.2. Find the Poisson pencil consisting of the local first and nonlocal second Hamiltonian operators for Gardner's extension $\mathcal{E}_\varepsilon$ of the Korteweg–de Vries equation, see (12.5); note that the second structure explicitly depends on the deformation parameter ε.

Problem 12.3 (adjoint hierarchies)**.** Assume that Gardner's extension $\mathcal{E}_\varepsilon$ for a system $\mathcal{E}_0$ is polynomial in ε. Suppose further that for every $\varepsilon \in \mathbb{R}$ the equation $\mathcal{E}_\varepsilon$ belongs to the hierarchy of (higher) flows (this can be achieved, for example, by plugging the Miura contraction $\mathfrak{m}_\varepsilon \colon \mathcal{E}_\varepsilon \to \mathcal{E}_0$ for $\boldsymbol{u}$ in the Hamiltonians $\mathcal{H}_i^{(j)}[\boldsymbol{u}]$ for the hierarchy of $\mathcal{E}_0$) and let the higher flows also be polynomial in ε.
- Show that the flows which stand as the top-degree coefficients of those polynomials in ε pairwise commute.

This determines the *adjoint hierarchy*.
- For example, the adjoint Korteweg–de Vries equation is $\dot{w} = w^2 w_x$. Find its bi-Hamiltonian structure.
- Find the Poisson pencil of local Hamiltonian operators for the adjoint Kaup–Boussinesq equation

$$\begin{aligned} \dot{u} &= uu_{xx} + u_x^2 + u_x v + uv_x, \\ \dot{v} &= vv_x - 2u_x u_{xx} - uu_{xxx} - u_x v_x - uv_{xx}. \end{aligned} \tag{12.12}$$

Note further that the dispersionless adjoint KdV equation, which we rescale to $\dot{w} = 3w^2 \cdot w_x$, is extended to the dispersionless bi-Hamiltonian modified KdV equation $\dot{w} = -\frac{1}{2} w_{xxx} + 3w^2 w_x$.
- Can the adjoint Kaup–Boussinesq system (12.12) be extended to a completely integrable system of higher order?
- Consider the Boussinesq equation $u_{tt} = u_{xxxx} + \left(\frac{1}{2}u^2\right)_{xx}$ and explore the integrability of the adjoint Boussinesq systems and their dispersionful extensions.

Problem 12.4. Prove Lemma 12.1.

Problem 12.5. Prove Theorem 12.3.
- Reformulate the assertion and the proof of Theorem 12.3 in terms of not matrix but vector field representations of the Lie algebra $\mathfrak{g}$.

Problem 12.6. By introducing the covariant derivatives $\widehat{D}_i = \frac{\mathrm{d}}{\mathrm{d}x^i} - \mathrm{ad}_{\alpha_i}$ and interpreting the Maurer–Cartan equation as the definition of conservation for non-abelian currents, develop — in the full parallel with the technique of generating sections for conservation laws, see Lecture 4 — the machinery of generating sections (or *characteristic elements*) for zero-curvature representations (see [92] and references therein).
- Recover the homotopy formula for the reconstruction of the $\mathfrak{g}$-valued connection one-forms $\boldsymbol{\alpha}$ from their characteristic elements.

Problem 12.7. Take the N-soliton solution of the Korteweg–de Vries equation,[96] see (2.3), and calculate the reflection and transition coefficients taking the profile

$w(x, t_0)$ at any fixed time t_0 as the potential in the Schrödinger equation.
• Solve the Cauchy problem for the Korteweg–de Vries equation for this initial datum by using the inverse spectral transform.[27, 31]

Problem 12.8. Construct the homological evolutionary vector field $\boldsymbol{Q}$ that captures the gauge degrees of freedom in the non-abelian setup of the "empty" principal G-bundle over a differential equation $\mathcal{E}^\infty$, i. e., when the Cartan connection is specified in the total space of such covering by the solutions of the Maurer–Cartan equation (12.10) and there are no other restrictions upon the sections of the G-bundle over $\mathcal{E}^\infty$.

References

[1] *Alexandrov M., Schwarz A., Zaboronsky O., Kontsevich M.* (1997) The geometry of the master equation and topological quantum field theory, *Int. J. Modern Phys.* **A12**:7, 1405–1429.

[2] *Arnol'd V. I.* (1996) Mathematical methods of classical mechanics. Grad. Texts in Math. **60**, Springer–Verlag, NY.

[3] *Barnich G.* (2010) A note on gauge systems from the point of view of Lie algebroids, AIP Conf. Proc. **1307** XXIX Workshop on Geometric Methods in Physics (June 27 – July 3, 2010; Białowieża, Poland), 7–18. `arXiv:math-ph/1010.0899`

[4] *Barnich G., Brandt F., Henneaux M.* (1995) Local BRST cohomology in the antifield formalism. I., II. *Commun. Math. Phys.* **174**:1, 57–91, 93–116. `arXiv:hep-th/9405109`

[5] *Barnich G., Brandt F., Henneaux M.* (2000) Local BRST cohomology in gauge theories, *Phys. Rep.* **338**:5, 439–569.

[6] *Barnich G., Henneaux M.* (1996) Isomorphisms between the Batalin–Vilkovisky antibracket and the Poisson bracket, *J. Math. Phys.* **37**:11, 5273–5296.

[7] *Batalin I. A., Fradkin E. S., Fradkina T. E.* (1990) Generalized canonical quantization of dynamical systems with constraints and curved phase space, *Nucl. Phys.* **B332**:3, 723–736.

[8] *Batalin I., Vilkovisky G.* (1981) Gauge algebra and quantization, *Phys. Lett.* **B102**:1, 27–31;
Batalin I. A., Vilkovisky G. A. (1983) Quantization of gauge theories with linearly dependent generators, *Phys. Rev.* **D29**:10, 2567–2582.

[9] *Baulieu L., Singer I. M.* (1988) Topological Yang–Mills symmetry. Conformal field theories and related topics (Annecy-le-Vieux, 1988), *Nuclear Phys. B Proc. Suppl.* **5B**, 12–19.

[10] *Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D.* (1978) Deformation theory and quantization. I, II. Deformations of symplectic structures, *Ann. Phys.* **111**:1, 61–110, 111–151.

[11] *Becchi C., Rouet A., Stora R.* (1976) Renormalization of gauge theories, *Ann. Phys.* **98**:2, 287–321;
Tyutin I. V. (1975) Gauge invariance in field theory and statistical mechanics, *Preprint* Lebedev FIAN no. 39.

[12] *Beilinson A., Drinfeld V.* (2004) Chiral algebras. AMS Colloq. Publ. **51**, AMS, Providence, RI.

[13] *Belavin A. A.* (1989) KdV-type equations and W-algebras, in: Integrable systems in quantum field theory and statistical mechanics. *Adv. Stud. Pure Math.* **19**, Acad. Press, Boston, MA, 117–125.

[14] *Berends F. A., Burgers G. J. H., van Dam H.* (1985) On the theoretical problems in constructing interactions involving higher-spin massless particles, *Nucl. Phys.* **B260**:2, 295–322.

[15] *Błaszak M.* (1998) Multi-Hamiltonian theory of dynamical systems, Springer, Berlin.

[16] *Cattaneo A. S., Felder G.* (2000) A path integral approach to the Kontsevich quantization formula, *Commun. Math. Phys.* **212**:3, 591–611.

[17] *Connes A.* (1994) Noncommutative geometry. Acad. Press, San Diego, CA.
[18] *De Sole A., Kac V. G.* (2011) The variational Poisson cohomology, 130 p. *Preprint* `arXiv:math-ph/1106.0082`; *De Sole A., Kac V. G.* (2011) Essential variational Poisson cohomology, 30 p. *Preprint* `arXiv:math-ph/1106.5882`
[19] *Dirac P. A. M.* (1967) Lectures on quantum mechanics. Belfer Grad. School of Science Monographs Ser. **2**. Acad. Press, Inc., NY;
De Donder Th. (1935) Theorie invariantive du calcul des variations (Nuov ed.) Gauthier–Villars, Paris.
[20] *Dorfman I. Ya.* (1993) Dirac structures, J. Whiley & Sons.
[21] *Douglas M. R., Nekrasov N. A.* (2001) Noncommutative field theory, *Rev. Mod. Phys.* **73**:4, 977–1029.
[22] *Drinfel'd V. G., Sokolov V. V.* (1985) Lie algebras and equations of Korteweg–de Vries type, *J. Sov. Math.* **30**, 1975–2035.
[23] *Dubrovin B. A.* (1996) Geometry of 2D topological field theories, *Lect. Notes in Math.* **1620** Integrable systems and quantum groups (Montecatini Terme, 1993), Springer, Berlin, 120–348.
[24] *Dubrovin B. A., Novikov S. P.* (1983) Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov–Whitham averaging method, *Sov. Math. Dokl.* **27**:3, 665–669; *Dubrovin B. A., Novikov S. P.* (1984) Poisson brackets of hydrodynamic type, *Sov. Math. Dokl.* **30**:3, 651–654.
[25] *Dubrovin B. A., Novikov S. P., Fomenko A. T.* (1986) Modern geometry. Methods and applications (2nd ed.) Nauka, Moscow.
[26] *Dubois-Violette M.* (1987) Systèmes dynamiques contraints: l'approche homologique, *Ann. Inst. Fourier (Grenoble)* **37**:4, 45–57.
[27] *Faddeev L. D., Takhtajan L. A.* (2007) Hamiltonian methods in the theory of solitons. Classics in Math. Springer, Berlin.
[28] *Fedosov B.* (1994) A simple geometric construction of deformation quantization, *J. Diff. Geom.* **40**(2), 213–238.
[29] *Fradkin E. S., Vilkovisky G. A.* (1975) Quantization of relativistic systems with constraints, *Phys. Lett.* **B55**:2, 224–226.
[30] *Fulp R., Lada T., Stasheff J.* (2003) Noether's variational theorem II and the BV formalism. Proc. 22nd Winter School "Geometry and Physics" (Srní, 2002), *Rend. Circ. Mat. Palermo* (2) Suppl. No. 71, 115–126.
[31] *Gardner C. S., Greene J. M., Kruskal M. D., Miura R. M.* (1967) Method for solving the Korteweg–de Vries equation, *Phys. Rev. Lett.* **19**, 1095–1097.
[32] *Gelfand I. M., Dikiĭ L. A.* (1975) Asymptotic properties of the resolvent of Sturm–Liouville equations, and the algebra of Korteweg–de Vries equations, *Russ. Math. Surveys* **30**:5, 77–113.
[33] *Gel'fand I. M., Dorfman I. Ja.* (1981) Schouten bracket and Hamiltonian operators, *Functional Anal. Appl.* **14**:3, 223–226.
[34] *Gerstenhaber M., Schack S.D.* (1988) Algebraic cohomology and deformation theory. Deformation theory of algebras and structures and applications (M. Gerstenhaber and M. Hazelwinkel, eds.) Kluwer, Dordrecht, 11–264.
[35] *Getzler E.* (2002) A Darboux theorem for Hamiltonian operators in the formal calculus of variations, *Duke Math. J.* **111**:3, 535–560.

[36] *Gitman D. M., Tyutin I. V.* (1990) Quantization of fields with constraints. Springer Ser. Nucl. Part. Phys., Springer-Verlag, Berlin.
[37] *Gomis J., París J., Samuel S.* (1995) Antibracket, antifields and gauge-theory quantization, *Phys. Rep.* **259**:1-2, 1–145.
[38] *Green M. B., Schwarz J. H., Witten E.* (1988) Superstring theory. **1**, **2**. 2nd ed. Cambridge Monographs on Math. Phys. CUP, Cambridge.
[39] *Henneaux M., Teitelboim C.* (1992) Quantization of gauge systems. Princeton University Press, Princeton, NJ.
[40] *Henneaux M.* (1985) Hamiltonian form of the path integral for theories with a gauge freedom, *Phys. Rep.* **126**:1, 1–66.
[41] *Herz J.-C.* (1953) Pseudo-algèbres de Lie. I, II. *C. R. Acad. Sci. Paris* **236**, 1935–1937, 2289–2291.
[42] *Igonin S.* (2006) Coverings and fundamental algebras for partial differential equations, *J. Geom. Phys.* **56**:6, 939–998.
[43] *Igonin S., Krasil'shchik J.* (2002) On one-parametric families of Bäcklund transformations. Lie groups, geometric structures and differential equations — one hundred years after Sophus Lie (Kyoto/Nara, 1999), *Adv. Stud. Pure Math.* **37**, Math. Soc. Japan, Tokyo, 99–114.
[44] *Igonin S., Verbovetsky A.* (2012) Symmetry-invariant solutions of PDEs and their generalizations (unpublished);
Kruglikov B. (2011) Symmetry, compatibility and exact solutions of PDEs, *Preprint* `arXiv:1111.5856` [math.DG]
[45] *Inönü E., Wigner E. P.* (1953) On the contraction of groups and their representations, *Proc. National Acad. Sci. USA* **39**, 510–524.
Saletan E. J. (1961) Contraction of Lie groups, *J. Math. Phys.* **2**, 1–21 (errat. 742).
Segal I. E. (1951) A class of operator algebras which are determined by groups, *Duke Math. J.* **18**, 221–265.
[46] *Kac V.* (1998) Vertex algebras for beginners (2nd ed.) Univ. Lect. Ser. **10**, AMS, Providence, RI.
[47] *Kassel C.* (1995) Quantum groups. NY, Springer–Verlag.
[48] *Kersten P., Krasil'shchik I., Verbovetsky A.* (2004) Hamiltonian operators and ℓ^*-coverings, *J. Geom. Phys.* **50**:1–4, 273–302.
[49] *Kiselev A. V.* (2012) On the noncommutative variational Poisson geometry, *Physics of Elementary Particles and Atomic Nuclei* n. 5 (in press), 4 p. `arXiv:math-ph/1112.5784`
[50] *Kiselev A. V.* (2012) Homological evolutionary vector fields in Korteweg–de Vries, Liouville, Maxwell, and several other models / *J. Phys. Conf. Ser.* **343**, Proc. 7th Int. workshop QTS-7 'Quantum Theory and Symmetries' (August 7–13, 2011; CVUT Prague, Czech Republic), 012058, 20 p. `arXiv:math-ph/1111.3272`
[51] *Kiselev A. V.*, Geometry of interaction, I: Classical mechanics (Lecture notes, in Russian), ISPU Press, Ivanovo (2011), 128 p.
[52] *Kiselev A. V.* (2008) Minimal surfaces associated with nonpolynomial contact symmetries, *J. Math. Sci.* **151**:4 'Hamiltonian & Lagrangian systems and Lie algebras,' 3133–3138 (transl. from: *Fundam. Appl. Math.* (2006) **12**:7, 93–100). `arXiv:math.DG/0603424`

[53] *Kiselev A. V.* (2007) Algebraic properties of Gardner's deformations for integrable systems, *Theor. Math. Phys.* **152**:1, 963–976. `arXiv:nlin.SI/0610072`

[54] *Kiselev A. V.* (2007) Associative homotopy Lie algebras and Wronskians, *J. Math. Sci.* **141**:1, 1016–1030 (transl. from: *Fundam. Appl. Math.* (2005) **11**:1, 159–180). `arXiv:math.RA/0410185`

[55] *Kiselev A. V.* (2006) Methods of geometry of differential equations in analysis of integrable models of field theory, *J. Math. Sci.* **136**:6 'Geometry of Integrable Models,' 4295–4377 (transl. from: *Fundam. Appl. Math.* (2004) **10**:1, 57–165). `arXiv:nlin.SI/0406036`

[56] *Kiselev A. V.* (2005) Hamiltonian flows on Euler-type equations, *Theor. Math. Phys.* **144**:1, 952–960. `arXiv:nlin.SI/0409061`

[57] *Kiselev A. V.* (2004/5) Geometric methods of solving boundary–value problems, *Note di Matematica* **23**:2, 99–111.

[58] *Kiselev A. V., Hussin V.* (2009) Hirota's virtual multi-soliton solutions of $N = 2$ supersymmetric KdV equations // *Theor. Math. Phys.* **159**:3, 832–840. `arXiv:nlin.SI/0810.0930`

[59] *Kiselev A. V., Krutov A.* (2009) Computing symmetries and recursion operators of evolutionary super-systems: a step-by-step informal introduction. *Preprint* `http://lie.math.brocku.ca/twolf/papers/cookbook.ps`, 10 p.

[60] *Kiselev A. V., van de Leur J. W.* (2011) Variational Lie algebroids and homological evolutionary vector fields, *Theor. Math. Phys.* **167**:3, 772–784. `arXiv:math.DG/1006.4227`

[61] *Kiselev A. V., van de Leur J. W.* (2011) Involutive distributions of operator-valued evolutionary vector fields and their affine geometry, Proc. 5th Int. workshop 'Group analysis of differential equations and integrable systems' (June 6–10, 2010; Protaras, Cyprus), 99–109. `arXiv:math-ph/0904.1555`

[62] *Kiselev A. V., van de Leur J. W.* (2010) Symmetry algebras of Lagrangian Liouville-type systems, *Theor. Math. Phys.* **162**:3, 149–162. `arXiv:nlin.SI/0902.3624`

[63] *Kiselev A. V., van de Leur J. W.* (2009) A family of second Lie algebra structures for symmetries of dispersionless Boussinesq system, *J. Phys. A*: *Math. Theor.* **42**:40, 404011 (8 p.) `arXiv:nlin.SI/0903.1214`

[64] *Kiselev A. V., van de Leur J. W.* (2009) A geometric derivation of KdV-type hierarchies from root systems, Proc. 4th Int. workshop 'Group analysis of differential equations and integrable systems' (October 26–30, 2008; Protaras, Cyprus), 87–106. `arXiv:nlin.SI/0901.4866`

[65] *Kiselev A. V., Wolf T.* (2007) Classification of integrable super-systems using the SsTools environment, *Comput. Phys. Commun.* **177**:3, 315–328. `arXiv:nlin.SI/0609065`

[66] *Kontsevich M.* (2011) Noncommutative identities, *Mathematische Arbeitstagung* 2011 (MPIM Bonn, Germany), 9 p.

[67] *Kontsevich M.* (1999) Operads and motives in deformation quantization. Moshé Flato (1937–1998). *Lett. Math. Phys.* **48**:1, 35–72.

[68] *Kontsevich M.* (2003) Deformation quantization of Poisson manifolds. I, *Lett. Math. Phys.* **66**:3, 157–216. `arXiv:q-alg/9709040`

[69] *Kontsevich M.* (1995) Homological algebra of mirror symmetry. Proc. ICM, Zürich, 1994. Vol. **1**, Birkhäuser, 120–139.

[70] *Kontsevich M.* (1994) Feynman diagrams and low-dimensional topology, First European Congress of Mathematics **II** (Paris, 1992), Progr. Math. **120**, Birkhäuser, Basel, 97–121.

[71] *Kontsevich M.* (1993) Formal (non)commutative symplectic geometry, The Gel'fand Mathematical Seminars, 1990-1992. (L. Corwin, I. Gelfand, and J. Lepowsky, eds) Birkhäuser, Boston MA, 173–187.

[72] *Kosmann-Schwarzbach Y.* (2004) Derived brackets, *Lett. Math. Phys.* **69**, 61–87.

[73] *Kosmann-Schwarzbach Y., Magri F.* (1990) Poisson–Nijenhuis structures, *Ann. Inst. H. Poincaré, ser. A: Phys. Théor.* **53**:1, 35–81.

[74] *Kostant B., Sternberg S.* (1987) Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras, *Ann. Phys.* **176**:1, 49–113.

[75] *Koszul J.-L.* (1985) Crochet de Schouten–Nijenhuis et cohomologie. The mathematical heritage of Élie Cartan (Lyon, 1984), Astérisque, hors serie, 257–271.

[76] *Krasil'shchik I. S.* (2002) A simple method to prove locality of symmetry hierarchies, *Preprint* DIPS-9/2002, 4 p.

[77] *Krasil'shchik I. S., Kersten P. H. M.* (2000) Symmetries and recursion operators for classical and supersymmetric differential equations, Kluwer, Dordrecht etc.

[78] *Krasil'shchik I., Verbovetsky A.* (2011) Geometry of jet spaces and integrable systems, *J. Geom. Phys.* **61**, 1633–1674. `arXiv:math.DG/1002.0077`,

[79] *Krasil'shchik I. S., Vinogradov A. M.*, eds. (1999) Symmetries and conservation laws for differential equations of mathematical physics. (Bocharov A. V., Chetverikov V. N., Duzhin S. V. *et al.*) AMS, Providence, RI.

[80] *Krutov A. O.* (2012) Deformations of equations and structures on them in the problems of mathematical physics. PhD dissertation (ISPU, Ivanovo, Russia), in progress.

[81] *Kumpera A., Spencer D.* (1972) Lie equations. I: General theory. Annals of Math. Stud. **73**. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo.

[82] *Kunzinger M., Popovych R. O.* (2009) Is a nonclassical symmetry a symmetry? Proc. 4th Int. workshop 'Group analysis of differential equations and integrable systems' (October 26–30, 2008; Protaras, Cyprus), 107–120.

[83] *Kupershmidt B. A.* (1980) Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms. Geometric methods in mathematical physics (Proc. NSF–CBMS Conf., Univ. Lowell, Mass., 1979), Lecture Notes in Math. **775**, Springer, Berlin, 162–218.

[84] *Lada T., Stasheff J.* (1993) Introduction to SH Lie algebras for physicists, *Internat. J. Theoret. Phys.* **32**:7, 1087–1103.

[85] *Landau L. D., Lifshitz E. M.* (1976) Course of theoretical physics **1**. Mechanics (3rd ed.) Pergamon Press, Oxford–New York–Toronto, Ontario, Canada.

[86] *Lax P.* (1968) Integrals of nonlinear equations of evolution and solitary waves, *Commun. Pure Appl. Math.* **21**, 467–490.

[87] *Lee J. M.* (2003) Introduction to smooth manifolds. Grad. Texts in Math. **218**, Springer–Verlag, NY.

[88] *Leznov A. N., Saveliev M. V.* (1979) Representation of zero curvature for the system of nonlinear partial differential equations $x_{\alpha,z\bar{z}} = \exp(Kx)_\alpha$ and its integrability, *Lett. Math. Phys.* **3**, 489–494;
Leznov A. N., Saveliev M. V. (1992) Group-theoretical methods for integration of nonlinear dynamical systems. Progr. in Phys. **15**. Birkhäuser Verlag, Basel.

[89] *Magri F.* (1978) A simple model of the integrable equation, *J. Math. Phys.* **19**:5, 1156–1162.

[90] *Manin Yu. I.* (1978) Algebraic aspects of nonlinear differential equations. Current problems in mathematics **11**, AN SSSR, VINITI, Moscow, 5–152 (in Russian).

[91] *Marsden J. E., Ratiu T. S.* (1999) Introduction to mechanics and symmetry. A basic exposition of classical mechanical systems (2nd ed.) Texts in Appl. Math. **17** Springer–Verlag, NY.

[92] *Marvan M.* (2004) Reducibility of zero curvature representations with application to recursion operators, *Acta Appl. Math.* **83**:1-2, 39–68.

[93] *Marvan M.* (2003) Jets. A software for differential calculus on jet spaces and diffieties, Opava. `http://diffiety.org/soft/soft.htm`

[94] *McCloud P.* (1994) Jet bundles in quantum field theory: the BRST-BV method, *Class. Quant. Grav.* **11**:3, 567–587.

[95] *Miura R. M.* (1968) Korteweg–de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation, *J. Math. Phys.* **9**:8, 1202–1204.

[96] *Miwa T., Jimbo M., Date E.* 2000 Solitons. Differential equations, symmetries and infinite-dimensional algebras. Cambridge Tracts in Math. **135**. CUP, Cambridge, UK.

[97] *Monastyrsky M.* (1993) Topology of gauge fields and condensed matter. Plenum Press, NY & Mir Publ., Moscow.

[98] *Nesterenko M., Popovych R.* (2006) Contractions of low-dimensional Lie algebras, *J. Math. Phys.* **47**:12, 123515, 45 pp.

[99] *Okounkov A.* (2000) Toda equations for Hurwitz numbers, *Preprint* `arXiv:math.AG/0004128`

[100] *Olver P. J.* (1993) Applications of Lie groups to differential equations, Grad. Texts in Math. **107** (2nd ed.), Springer–Verlag, NY.

[101] *Olver P. J., Sokolov V. V.* (1998) Integrable evolution equations on associative algebras, *Comm. Math. Phys.* **193**:2, 245–268.

[102] *Polyakov A. M.* (1987) Gauge fields and strings. Harwood Acad. Publ., Chur, Switzerland.

[103] *Postnikov M.* (1989) Smooth manifolds. Lectures in geometry. Semester III. Mir, Moscow;
Postnikov M. M. (1988) Differential geometry. Lectures in geometry. Semester IV. Nauka, Moscow.

[104] *Reyman A. G., Semenov–Tian-Shansky M. A.* (1994) Group-theoretical methods in the theory of finite dimensional integrable systems, in: Dynamical systems VII (V. I. Arnold and S. P. Novikov, eds.), Encyclopaedia of Math. Sci. **16**, Springer, Berlin, 116–225;
Reyman A. G., Semenov–Tian-Shansky M. A. (2003) Integrable systems: group-theoretic approach. Inst. Comp. Stud., Moscow etc. (in Russian).

[105] *Saunders D. J.* (1989) The geometry of jet bundles. London Math. Soc. Lect. Note Ser. **142**. CUP, Cambridge.

[106] *Saveliev M. V.* (1993) On the integrability problem of a continuous Toda system, *Theor. Math. Phys.* **92**:3, 1024–1031.

[107] *Schouten J. A., Struik D. J.* Einführung in die neueren Methoden der Differentialgeometries **1**, P. Nordhoff N. V., Groningen–Batavia, 1935.

[108] *Schwarz A.* (1993) Geometry of Batalin–Vilkovisky quantization, *Commun. Math. Phys.* **155**:2, 249–260.

[109] *Schwarz A. S.* (1993) Quantum field theory and topology. Fundam. Principles of Math. Sci. **307**. Springer–Verlag, Berlin.

[110] *Schwarz A. S.* (1979) The partition function of a degenerate functional, *Commun. Math. Phys.* **67**:1, 1–16.

[111] *Seiberg N., Witten E.* (1999) String theory and non-commutative geometry, *JHEP* 9909:032.

[112] *Shabat A. B.* (1995) Higher symmetries of two-dimensional lattices, *Phys. Lett.* **A200**:2, 121–133; *Shabat A. B.* (1996) First integrals of the infinite Toda lattice. Symmetries and integrability of difference equations (D. Levi, L. Vinet and P. Winternitz, eds.; Estérel, PQ, 1994), CRM Proc. Lecture Notes **9**, AMS, Providence, RI, 345–351.

[113] *Vaintrob A. Yu.* (1997) Lie algebroids and homological vector fields, *Russ. Math. Surv.* **52**:2, 428–429.

[114] *Vinogradov A. M.* (1984) The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory, II. The nonlinear theory, *J. Math. Anal. Appl.* **100**:1, 1–40, 41–129.

[115] *Voronov T.* (2002) Graded manifolds and Drinfeld doubles for Lie bialgebroids, in: Quantization, Poisson brackets, and beyond (Voronov T., ed.) Contemp. Math. **315**, AMS, Providence, RI, 131–168.

[116] *Witten E.* (1990) A note on the antibracket formalism, *Modern Phys. Lett.* **A5**:7, 487–494.

[117] *Witten E.* (1988) Topological sigma models, *Commun. Math. Phys.* **118**:3, 411–449.

[118] *Zamolodchikov A. B., Zamolodchikov Al. B.* (1979) Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models, *Ann. Physics* **120**:2, 253–291.

[119] *Zhiber A. V., Sokolov V. V.* (2001) Exactly integrable hyperbolic equations of Liouvillean type, *Russ. Math. Surveys* **56**:1, 61–101.

[120] *Zinn-Justin J.* (1975) Renormalization of gauge theories. Trends in Elementary Particle Theory (Lect. Notes in Phys. **37** H. Rollnick and K. Dietz eds), Springer, Berlin, 2–39;
Zinn-Justin J. (1976) Méthodes en théorie des champs / Methods in field theory. (École d'Été de Physique Théorique, Session XXVIII, tenue à Les Houches, 28 Juillet–6 Septembre, 1975; R. Balian and J. Zinn-Justin, eds) North-Holland Publ. Co., Amsterdam etc.