The zero locus of the infinitesimal invariant

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Abstract. Let ν be a normal function on a complex manifold X. The infinitesimal invariant of ν has a well-defined zero locus inside the tangent bundle TX. When X is quasi-projective, and ν is admissible, we show that this zero locus is constructible in the Zariski topology.

A. Introduction

1. Main result. Let $\mathcal{H}$ be a variation of Hodge structure of weight -1 on a Zariski open subset of a smooth complex projective variety X. We shall assume that $\mathcal{H}$ is polarizable and defined over $\mathbb{Z}$. We denote the Hodge filtration on the underlying flat vector bundle $\mathcal{H}_{\mathscr{O}}$ by the symbol $F^{\bullet}\mathcal{H}_{\mathscr{O}}$. Let ν be a normal function, that is to say, a holomorphic and horizontal section of the family of intermediate Jacobians $J(\mathcal{H})$. For any local lifting $\tilde{\nu}$ to a holomorphic section of $\mathcal{H}_{\mathscr{O}}$, we have

$$\nabla\tilde{\nu} \in \Omega_X^1 \otimes_{\mathscr{O}_X} F^{-1}\mathcal{H}_{\mathscr{O}},$$

which is independent of the choice of lifting modulo $\nabla(F^0\mathcal{H}_{\mathscr{O}})$. We are interested in the subset of the tangent bundle TX defined by the condition $\nabla\tilde{\nu} \in \nabla(F^0\mathcal{H}_{\mathscr{O}})$. Concretely, this is the set

$$I(\nu) = \left\{ (x,\xi) \in TX \;\middle|\; \nabla_{\xi}(\tilde{\nu} - \sigma)(x) = 0 \text{ for some } \sigma \in F^0\mathcal{H}_{\mathscr{O}} \right\}.$$

The following theorem describes the structure of $I(\nu)$ for admissible ν.

Theorem 1.1. *Suppose that ν is an admissible normal function on a Zariski open subset of a smooth complex projective variety X. Then $I(\nu)$ is constructible with respect to the Zariski topology on TX.*

Recall that a subset of an algebraic variety is *constructible* if it is a finite union of subsets that are locally closed in the Zariski topology. The proof of Theorem (1.1) is given in Section B. In Section C, we describe the relationship between this paper and the study of algebraic cycles via the approach to the Hodge conjecture by Green–Griffiths [GG07] using singularities of normal functions.

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B. Proof of the theorem

3. Algebraic description of the zero locus. Since X is a projective algebraic variety, it is possible to describe the zero locus $I(\nu)$ of the infinitesimal invariant of ν purely in terms of algebraic objects. In this section, we shall do this by a straightforward classical argument.

Using resolution of singularities, we may assume without loss of generality that ν is an admissible normal function on $X - D$, where X is a smooth projective variety, and $D \subseteq X$ is a divisor with normal crossings. Let $\mathcal{V}$ be the admissible variation of mixed Hodge structure with $\mathbb{Z}$-coefficients corresponding to ν; then $W_{-1}\mathcal{V} = \mathcal{H}$ and $W_0\mathcal{V}/W_{-1}\mathcal{V} \simeq \mathbb{Z}(0)$ by our choice of weights. The integrable connection $\nabla\colon \mathcal{V}_{\mathscr{O}} \to \Omega^1_{X-D} \otimes \mathcal{V}_{\mathscr{O}}$ on the underlying holomorphic vector bundle $\mathcal{V}_{\mathscr{O}}$ has regular singularities; because X is projective algebraic, it follows from [Del70] that $\mathcal{V}_{\mathscr{O}}$ and ∇ are algebraic. Admissibility implies that each Hodge bundle $F^p\mathcal{V}_{\mathscr{O}}$ is an algebraic subbundle of $\mathcal{V}_{\mathscr{O}}$; note that they satisfy $\nabla\bigl(F^p\mathcal{V}_{\mathscr{O}}\bigr) \subseteq \Omega^1_{X-D} \otimes F^{p-1}\mathcal{V}_{\mathscr{O}}$ because of Griffiths transversality.

To prove the constructibility of $I(\nu)$, our starting point is the exact sequence

$$0 \to F^0\mathcal{H}_{\mathscr{O}} \to F^0\mathcal{V}_{\mathscr{O}} \to \mathscr{O} \to 0 \tag{3.1}$$

of algebraic vector bundles on $X - D$. Let U be any affine Zariski open subset of $X - D$ with the following two properties: (1) both $F^0\mathcal{H}_{\mathscr{O}}$ and $F^{-1}\mathcal{H}_{\mathscr{O}}/F^0\mathcal{H}_{\mathscr{O}}$ restrict to trivial bundles on U; (2) there are coordinates $x_1, \dots, x_n \in \Gamma(U, \mathscr{O}_U)$, where $\Gamma(U, -)$ always denotes the space of all algebraic sections of an algebraic coherent sheaf. Since $X - D$ can be covered by finitely many such open subsets, it is clearly sufficient to show that $I(\nu) \cap TU$ is a constructible subset of TU.

By our choice of U, the tangent bundle TU is trivial; let $\xi_1, \dots, \xi_n \in \Gamma(TU, \mathscr{O}_{TU})$ be the coordinates in the fiber direction corresponding to the algebraic vector fields $\partial/\partial x_1, \dots, \partial/\partial x_n$. Let $q = \operatorname{rk} F^0\mathcal{H}_{\mathscr{O}}$ and $p = \operatorname{rk} F^{-1}\mathcal{H}_{\mathscr{O}} \geq q$; we can then choose algebraic sections $e_1, \dots, e_p \in \Gamma(U, F^{-1}\mathcal{H}_{\mathscr{O}})$ such that $e_1, \dots, e_q \in \Gamma(U, F^0\mathcal{H}_{\mathscr{O}})$ are a frame for $F^0\mathcal{H}_{\mathscr{O}}$, and $e_1, \dots, e_p$ are a frame for $F^{-1}\mathcal{H}_{\mathscr{O}}$. For $i = 1, \dots, q$, we get

$$\nabla e_i = \sum_{k=1}^n \sum_{j=1}^p dx_k \otimes a^k_{i,j} e_j$$

with certain functions $a^k_{i,j} \in \Gamma(U, \mathscr{O}_U)$. Let $\tilde{\nu} \in \Gamma(U, F^0\mathcal{V}_{\mathscr{O}})$ be any lifting of the element $1 \in \Gamma(U, \mathscr{O}_X)$; then $\nabla\tilde{\nu} \in \Gamma(U, \Omega^1_U \otimes F^{-1}\mathcal{H}_{\mathscr{O}})$ can be written in the form

$$\nabla\tilde{\nu} = \sum_{k=1}^n \sum_{j=1}^p dx_k \otimes f^k_j e_j$$

for certain functions $f^k_j \in \Gamma(U, \mathscr{O}_U)$. By definition, a point $(x, \xi) \in TU$ lies in the zero locus $I(\nu)$ of the infinitesimal invariant iff there are holomorphic functions $\varphi_1, \dots, \varphi_q$ defined in a small open ball around $x \in U$, such that

$$\nabla\left(\tilde{\nu} - \sum_{i=1}^q \varphi_i e_i\right)$$

vanishes at the point (x, ξ). When expanded, this translates into the condition that

$$\sum_{k=1}^n \sum_{j=1}^p \xi_k f^k_j(x) e_j = \sum_{i=1}^q \sum_{k=1}^n \xi_k \frac{\partial\varphi_i}{\partial x_k}(x) e_i + \sum_{i=1}^q \varphi_i(x) \sum_{k=1}^n \sum_{j=1}^p \xi_k a^k_{i,j}(x) e_j.$$

This is a system of p linear equations in the $q(n+1)$ complex numbers

$$\varphi_i(x) \quad \text{and} \quad \frac{\partial\varphi_i}{\partial x_k}(x) \qquad (\text{for } 1 \leq i \leq q \text{ and } 1 \leq k \leq n),$$

with coefficients in the ring $\Gamma(TU, \mathscr{O}_{TU})$ of regular functions on TU. The proof of Proposition 5.3 shows that the set of points $(x, \xi) \in TU$, where this system has a solution, is a constructible subset of TU. This completes the proof of Theorem 1.1.

4. A more sophisticated description. For some purposes, it is better to have a natural extension of $I(\nu)$ to the entire cotangent bundle TX, without modifying the ambient variety X. In this section, we indicate how such an extension can be constructed using the theory of mixed Hodge modules [Sai90].

We begin by recalling how one associates a short exact sequence of the form

$$0 \to F_0\mathcal{M} \to F_0\mathcal{N} \to \mathscr{O}_X \to 0 \tag{4.1}$$

to the given admissible normal function; here $F_0\mathcal{M}$ and $F_0\mathcal{N}$ are algebraic coherent sheaves on X, and all three morphisms are morphisms of algebraic coherent sheaves.

The polarizable variation of Hodge structure $\mathcal{H}$ extends uniquely to a polarizable Hodge module with strict support equal to X. We denote by $\mathcal{M}$ the underlying regular holonomic $\mathscr{D}_X$-module; it is the minimal extension of the flat vector bundle $\mathcal{H}_{\mathscr{O}}$. It has a good filtration $F_\bullet\mathcal{M}$ by $\mathscr{O}_X$-coherent subsheaves, and $F_k\mathcal{M}$ is an extension of the Hodge bundle $F^{-k}\mathcal{H}_{\mathscr{O}}$. Since X is a complex projective variety, each $F_k\mathcal{M}$ is an algebraic coherent sheaf, and $\mathcal{M}$ is an algebraic $\mathscr{D}_X$-module.

Because the normal function ν is admissible, the corresponding variation of mixed Hodge structure extends uniquely to a mixed Hodge module on X; in fact, this condition is equivalent to admissibility [Sai96, p. 243]. Let $\mathcal{N}$ denote the underlying regular holonomic $\mathscr{D}_X$-module, and $F_\bullet\mathcal{N}$ its Hodge filtration; as before, $\mathcal{N}$ is an algebraic $\mathscr{D}_X$-module, and each $F_k\mathcal{N}$ is an algebraic coherent sheaf. We have an exact sequence of regular holonomic $\mathscr{D}_X$-modules

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathscr{O}_X \to 0,$$

in which all three morphisms are strict with respect to the Hodge filtration; in particular, (4.1) is an exact sequence of algebraic coherent sheaves on X. Because $\mathcal{N}$ is a filtered $\mathscr{D}_X$-module, we have $\mathbb{C}$-linear morphisms $\mathscr{T}_X \otimes F_k\mathcal{N} \to F_{k+1}\mathcal{N}$; note that they are not $\mathscr{O}_X$-linear.

We can use the exact sequence in (4.1) to construct an extension of the zero locus $I(\nu)$ to all of X. Inside the tangent bundle TX, we define a subset

$$\tilde{I}(\nu) = \left\{ (x, \xi) \in TX \;\middle|\; (\xi \cdot \sigma)(x) = 0 \text{ for some } \sigma \in F_0\mathcal{N} \text{ with } \sigma \mapsto 1 \right\},$$

where the notation "$\sigma \in F_0\mathcal{N}$" means that σ is a holomorphic section of the sheaf $F_0\mathcal{N}$, defined in some open neighborhood of the point $x \in X$.

Lemma 4.2. *We have $\tilde{I}(\nu) = I(\nu)$ over the Zariski open subset of X where the variation of Hodge structure $\mathcal{H}$ is defined.*

Proof. This is obvious from the definitions. □

Denote by $p\colon TX \to X$ the projection. The pullback p^*T_X of the tangent sheaf has a tautological global section θ; in local holomorphic coordinates $x_1, \dots, x_n$ on X, and corresponding coordinates $(x_1, \dots, x_n, \xi_1, \dots, \xi_n)$ on TX, it is given by the formula

$$\theta(x_1, \dots, x_n, \xi_1, \dots, \xi_n) = \xi_1 \frac{\partial}{\partial x_1} + \dots + \xi_n \frac{\partial}{\partial x_n}.$$

Let $\tilde{\mathcal{M}}$ denote the pullback of $\mathcal{M}$ to a filtered $\mathscr{D}$-module on the tangent bundle; because p is smooth, we have $\tilde{\mathcal{M}} = p^*\mathcal{M}$ and $F_k\tilde{\mathcal{M}} = p^*F_k\mathcal{M}$. Similarly define $\tilde{\mathcal{N}}$.

Lemma 4.3. *In the notation introduced above, we have*

$$\tilde{I}(\nu) = \left\{ (x,\xi) \in TX \;\middle|\; (\theta \cdot \tilde{\sigma})(x,\xi) = 0 \text{ for some } \tilde{\sigma} \in F_0\tilde{\mathcal{N}} \text{ with } \tilde{\sigma} \mapsto 1 \right\}.$$

Proof. The set on the right-hand side clearly contains $\tilde{I}(\nu)$. To prove that the two sets are equal, suppose that we have $(\theta \cdot \tilde{\sigma})(x,\xi) = 0$ for some holomorphic section $\tilde{\sigma}$ of $F_0\tilde{\mathcal{N}}$, defined in a neighborhood of the point $(x,\xi) \in TX$. Since $F_0\tilde{\mathcal{N}} = p^* F_0\mathcal{N}$, we can write $\tilde{\sigma} = \sum_k f_k \cdot p^*\sigma_k$ for suitably chosen $f_k \in \mathscr{O}_{TX}$ and $\sigma_k \in F_0\mathcal{N}$. Define $\sigma = \sum_k f_k(-,\xi)\sigma_k$; then $\sigma \in F_0\mathcal{N}$ and $\sigma \mapsto 1$. A brief calculation in local coordinates shows that

$$(\xi \cdot \sigma)(x) = (\theta \cdot \tilde{\sigma})(x,\xi) = 0,$$

and so we get $(x,\xi) \in \tilde{I}(\nu)$ as desired. □

The next step is to show that $\tilde{I}(\nu)$ is the zero locus of a holomorphic section of an analytic coherent sheaf on TX. Let $\mathscr{F}$ denote the analytic coherent sheaf on TX obtained by taking the quotient of $F_1\tilde{\mathcal{M}}$ by the analytic coherent subsheaf generated by $\theta \cdot F_0\tilde{\mathcal{M}}$. For any local holomorphic section $\sigma \in F_0\tilde{\mathcal{N}}$ with $\sigma \mapsto 1$, we have $\theta \cdot \sigma \in F_1\tilde{\mathcal{M}}$, and the image of $\theta \cdot \sigma$ in the quotient sheaf $\mathscr{F}$ is independent of the choice of σ, due to the exactness of (4.1). In this manner, we obtain a global holomorphic section s of the sheaf $\mathscr{F}$.

Lemma 4.4. *$\tilde{I}(\nu)$ is the zero locus of the section s of the coherent sheaf $\mathscr{F}$.*

Proof. If $(x,\xi) \in \tilde{I}(\nu)$, then we have $(\theta \cdot \tilde{\sigma})(x,\xi) = 0$ for some choice of $\tilde{\sigma} \in F_0\tilde{\mathcal{N}}$ with $\sigma \mapsto 1$; in particular, $s(x,\xi) = 0$. Conversely, suppose that we have $s(x,\xi) = 0$ for some point $(x,\xi) \in TX$. By definition of $\mathscr{F}$, we can then find local sections $\tilde{\sigma}_k \in F_0\tilde{\mathcal{M}}$ and local holomorphic functions $f_k \in \mathscr{O}_{TX}$, such that

$$\theta \cdot \tilde{\sigma} - \sum_k f_k \theta \cdot \tilde{\sigma}_k$$

vanishes at the point (x,ξ). Set $a_k = f_k(x,\xi) \in \mathbb{C}$; then

$$\theta \cdot \left(\tilde{\sigma} - \sum_k a_i \tilde{\sigma}_k \right) = \theta \cdot \tilde{\sigma} - \sum_k a_k \theta \cdot \tilde{\sigma}_k$$

also vanishes at (x,ξ), and this shows that $(x,\xi) \in \tilde{I}(\nu)$. □

Despite the analytic definition, both $\mathscr{F}$ and s are actually algebraic objects.

Lemma 4.5. *$\mathscr{F}$ is an algebraic coherent sheaf on TX, and $s \in \Gamma(TX,\mathscr{F})$ is an algebraic global section.*

Proof. Each $F_k\tilde{\mathcal{M}} = p^* F_k\mathcal{M}$ is an algebraic coherent sheaf on TX, and since the tautological section $\theta \in \Gamma(TX, p^*\mathscr{T}_X)$ is clearly algebraic, it follows that $\mathscr{F}$ is an algebraic coherent sheaf. To show that the global section $s \in \Gamma(TX,\mathscr{F})$ is algebraic, observe that we have an exact sequence of algebraic coherent sheaves

$$0 \to F_0\tilde{\mathcal{M}} \to F_0\tilde{\mathcal{N}} \to \mathscr{O}_{TX} \to 0;$$

indeed, (4.1) is exact, and $p\colon TX \to X$ is a smooth affine morphism. At every point $(x,\xi) \in TX$, we can therefore find an algebraic section $\sigma \in F_0\tilde{\mathcal{N}}$, defined in a Zariski open neighborhood of (x,ξ), such that $\sigma \mapsto 1$. This clearly implies that s, which is locally given by the image of $\theta \cdot \sigma$ in $\mathscr{F}$, is itself algebraic. □

To prove Theorem 1.1, it is clearly sufficient to show that the set $\tilde{I}(\nu)$ is constructible in the Zariski topology on TX. Lemma 4.4 and Lemma 4.5 reduce the problem to the following general result in abstract algebraic geometry: On any algebraic variety, the zero locus of a section of a coherent sheaf is constructible (but not, in general, Zariski closed). This fact is certainly well-known, but since it was surprising to us at first, we have decided to include a simple proof in the following section.

5. Zero loci of sections of coherent sheaves. In this section, we carefully define the "zero locus" for sections of coherent sheaves, and show that it is always constructible in the Zariski topology. This is obviously a local problem, and so it suffices to consider the case of affine varieties. Let R be a commutative ring with unit; to avoid technical complications, we shall also assume that R is Noetherian. For any prime ideal $\mathfrak{p} \subseteq R$, we denote by the symbol

$$\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$$

the residue field at $\mathfrak{p}$; it is isomorphic to the field of fractions of the local ring $R_{\mathfrak{p}}$. Let $X = \operatorname{Spec} R$ be the set of prime ideals of the ring R, endowed with the Zariski topology. For any ideal $I \subseteq R$, the set

$$V(I) = \{ \mathfrak{p} \in X \mid \mathfrak{p} \supseteq I \}$$

is closed in the Zariski topology on X, and any closed subset is of this form; likewise, for any element $f \in R$, the set

$$D(f) = \{ \mathfrak{p} \in X \mid \mathfrak{p} \not\ni f \}$$

is an open subset, and these open sets form a basis for the Zariski topology.

Definition 5.1. A subset of X is called *constructible* if it is a finite union of subsets of the form $D(f) \cap V(I)$.

Here is how this algebraic definition is related to constructibility on complex algebraic varieties. Suppose that R is a $\mathbb{C}$-algebra of finite type. Let $X(\mathbb{C})$ be the set of all maximal ideals of R, endowed with the classical topology; it is an affine complex algebraic variety, and the inclusion mapping $X(\mathbb{C}) \hookrightarrow X$ is continuous.

Definition 5.2. A subset of $X(\mathbb{C})$ is called *constructible* (in the Zariski topology) if it is the set of maximal ideals in a constructible subset of X.

Any coherent sheaf on $X = \operatorname{Spec} R$ is uniquely determined by the finitely generated R-module of its global sections; conversely, any finitely generated R-module M defines a coherent sheaf on X, and hence by restriction to the subset $X(\mathbb{C})$ an algebraic coherent sheaf $\mathscr{F}_M$ on $X(\mathbb{C})$. Its fiber at the point corresponding to a maximal ideal $\mathfrak{m} \subseteq R$ is the finite-dimensional $\mathbb{C}$-vector space

$$M \otimes_R \kappa(\mathfrak{m}) = M_{\mathfrak{m}}/\mathfrak{m}M_{\mathfrak{m}}.$$

Similarly, any element $m \in M$ defines an algebraic global section s_m of the sheaf $\mathscr{F}_M$. Obviously, s_m vanishes at the point corresponding to a maximal ideal $\mathfrak{m} \subseteq R$ if and only if m goes to zero in $M \otimes_R \kappa(\mathfrak{m})$. Thus if we define

$$Z(M, m) = \{ \mathfrak{p} \in X \mid m \text{ goes to zero in } M \otimes_R \kappa(\mathfrak{p}) \},$$

then the zero locus of s_m on $X(\mathbb{C})$ is precisely the set of maximal ideals in $Z(M, m)$. Thus the desired result about zero loci of sections of coherent sheaves is a consequence of the following general theorem in commutative algebra.

Proposition 5.3. *Let R be a commutative Noetherian ring with unit. Then for any finitely generated R-module M, and any $m \in M$, the set $Z(M,m)$ is constructible.*

Proof. We are going to construct a finite covering

$$\operatorname{Spec} R = \bigcup_{k=1}^{n} D(f_k) \cap V(I_k)$$

with $f_1, \dots, f_n \in R$ and $I_1, \dots, I_n \subseteq R$, such that for every $k = 1, \dots, n$, one has

$$Z(M,m) \cap D(f_k) \cap V(I_k) = D(f_k) \cap V(I_k + J_k),$$

for a certain ideal $J_k \subseteq R$. This is sufficient, because it implies that

$$Z(M,m) = \bigcup_{k=1}^{n} D(f_k) \cap V(I_k + J_k)$$

is a constructible subset of $\operatorname{Spec} R$.

Since M is finitely generated and R is Noetherian, we may find a presentation

$$R^{\oplus q} \xrightarrow{A} R^{\oplus p} \twoheadrightarrow M, \tag{5.4}$$

in which A is a $p \times q$-matrix with entries in R. Let $y \in R^{\oplus p}$ be any vector mapping to $m \in M$. Then $Z(M,m)$ is the set of $\mathfrak{p} \in \operatorname{Spec} R$ such that the equation $y = Ax$ has a solution over the field $\kappa(\mathfrak{p})$.

We construct the desired covering of $\operatorname{Spec} R$ by looking at all possible minors of the matrix A. Fix an integer $0 \leq \ell \leq \min(p,q)$ and an $\ell \times \ell$-submatrix of A; to simplify the notation, let us assume that it is the $\ell \times \ell$-submatrix in the upper left corner of A. Let f be the determinant of the submatrix, and let I be the ideal generated by all minors of A of size $(\ell+1) \times (\ell+1)$; if $\ell = 0$, we set $f = 1$, and if $\ell = \min(p,q)$, we set $I = 0$. We can then make a coordinate change in $R^{\oplus q}$, invertible over the localization $R_f = R[f^{-1}]$, and arrange that

$$A = \begin{pmatrix} f & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & f & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f & 0 & \cdots & 0 \\ a_{\ell+1,1} & a_{\ell+1,2} & \cdots & a_{\ell+1,\ell} & a_{\ell+1,\ell+1} & \cdots & a_{\ell+1,q} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{p,1} & a_{p,2} & \cdots & a_{p,\ell} & a_{p,\ell+1} & \cdots & a_{p,q} \end{pmatrix} \in \operatorname{Mat}_{p \times q}(R).$$

Now let $J \subseteq R$ be the ideal generated by the elements

$$f y_i - \sum_{j=1}^{\ell} a_{i,j} y_j$$

for $i = \ell+1, \dots, p$. Then we have

$$Z(M,m) \cap D(f) \cap V(I) = D(f) \cap V(I+J).$$

Indeed, suppose that $\mathfrak{p}$ is any prime ideal with $f \notin \mathfrak{p}$ and $I \subseteq \mathfrak{p}$. Since $a_{i,j} \in \mathfrak{p}$ for every $\ell+1 \leq i \leq p$ and $\ell+1 \leq j \leq q$, the equation $y = Ax$ reduces over the field

$\kappa(\mathfrak{p})$ to the equations $y_i = f x_i$ for $i = 1, \dots, \ell$, and

$$y_i = \sum_{j=1}^{\ell} a_{i,j} x_j$$

for $i = \ell + 1, \dots, p$; they are obviously satisfied if and only if $J \subseteq \mathfrak{p}$.

We now obtain the assertion by applying the above construction of f, I, and J to all possible $\ell \times \ell$-submatrices of A. $\square$

Here is a simple example to show that, when the coherent sheaf is not locally free, the zero locus of a section need not be Zariski closed.

Example 5.5. Let $R = \mathbb{C}[x, y]$, let M be the ideal of R generated by x, y, and let $m = x$. Then M has a free resolution of the form $R \to R^{\oplus 2}$, and $\mathfrak{p} \in Z(M, m)$ if and only if the equations $1 + yf = 0$ and $xf = 0$ have a common solution $f \in \kappa(\mathfrak{p})$. A simple computation now shows that

$$Z(M, m) = \left\{ \mathfrak{p} \in \operatorname{Spec} R \;\middle|\; x \in \mathfrak{p} \text{ and } y \notin \mathfrak{p} \right\}.$$

As a subset of $\mathbb{C}^2$, the zero locus consists of the y-axis minus the origin; it is constructible, but not Zariski closed.

C. Relation to algebraic cycles

6. Green-Griffiths Program. Our interest in the algebraicity of $I(\nu)$ is motivated in part by the program [GG07] of Green and Griffiths to study the Hodge conjecture via singularities of normal functions. More precisely, given a smooth complex projective variety X, a very ample line bundle $L \to X$ and a non-torsion, primitive Hodge class ζ of type (n, n) on X, Griffiths and Green construct an admissible normal function

$$\nu_\zeta : P - \hat{X} \to J(\mathcal{H})$$

on the complement of the dual variety $\hat{X}$ in $P = \mathbb{P}H^0(X, \mathcal{O}(L))$. At each point $\hat{x} \in \hat{X}$, the cohomology class of ν_ζ localizes to an invariant

$$\operatorname{sing}_{\hat{x}}(\nu_\zeta) \in IH^1_{\hat{x}}(\mathcal{H})$$

called the *singularity* of ν_ζ at $\hat{x}$. A normal function ν_ζ is said to be *singular* if there is a point $\hat{x} \in \hat{X}$ at which $\operatorname{sing}_{\hat{x}}(\nu_\zeta)$ is non-torsion.

Conjecture 6.1. *Let (X, L, ζ) be as above. Then, there exists an integer $k > 0$ such that after replacing L by L^k, the associated normal function ν_ζ is singular.*

Theorem 6.2. [GG07, BFNP, dCM09] *Conjecture* (6.1) *holds (for every even dimensional X and every non-torsion, primitive middle dimensional Hodge class ζ) if and only if the Hodge conjecture holds (for all smooth projective varieties).*

Now, as explained in part III of [GG07] one can also define a notion $\operatorname{sing}_{\hat{x}}(\delta\nu_\zeta)$ of the singularities of infinitesimal invariant $\delta\nu_\zeta$ of ν_ζ. Moreover,

$$\operatorname{sing}_{\hat{x}}(\delta\nu_\zeta) = \operatorname{sing}_{\hat{x}}(\nu_\zeta)$$

for $L \gg 0$. As a first attempt at constructing points at which ν_ζ is singular, observe that

$$Z(\nu_\zeta) = \{\, p \in P - \hat{X} \mid \nu_\zeta(p) = 0 \,\}$$

is an analytic subset of $P - \hat{X}$, and hence it is natural to ask if its closure is an algebraic subvariety of P which intersects $\hat{X}$ at some point where ν_ζ is singular. An affirmative answer is provided by the following two results:

Theorem 6.3. [BP, KNU11, Sch12] *If S is a smooth complex algebraic variety and $\nu : S \to J(\mathcal{H})$ is an admissible normal function then $Z(\nu)$ is an algebraic subvariety of S.*

Proposition 6.4. [Sch10] *Let ν_ζ be the normal function on $P \setminus \hat{X}$, associated to a non-torsion primitive Hodge class $\zeta \in H^{2n}(X, \mathbb{Z}) \cap H^{n,n}(X)$. Assume that $Z(\nu_\zeta)$ contains an algebraic curve C, and that $P = |L^d|$ for L very ample and $d \geq 3$. Then ν_ζ is singular at one of the points where the closure of C meets $\hat{X}$.*

The caveat here, which is illustrated in the example (6.5) below, is that there is no reason for $Z(\nu_\zeta)$ to contain a curve. The advantage of working with the infinitesimal invariant is that it is often easier to compute [Gri83], and will vanish along directions tangent to $Z(\nu)$. Of course, $I(\nu)$ will also contain the directions tangent to any m-torsion locus of ν, as well as potentially other components.

Question. Is there an analog of Proposition (6.4) for $I(\nu_\zeta)$?

Remark. The study of zero loci of normal function also arises in connection with the construction of the Bloch–Beilinson filtration on Chow groups. For a survey of results of this type, see [KP10].

The determination of a good notion of the expected dimension of the zero locus of a normal function is an important open problem in the study of algebraic cycles. In particular, in the Green–Griffiths setting, if a smooth projective variety has moduli, any reasonable expected dimension count is probably only valid at the generic point of the locus where the class ζ remains a Hodge class.

In the case of a smooth projective surface X, if $L = \mathcal{O}(D)$ is a very ample line bundle, then a Riemann–Roch calculation shows the expected dimension of the zero locus of the associated normal functions arising from the Green–Griffiths program (i.e., comparing the dimensions of the fiber and the base) is

$$-(D \cdot K_X) + \chi(\mathcal{O}_X) - 2$$

where K_X is the canonical bundle of X. For X of general type, on the basis of this calculation one would expect the zero locus to be empty for all sufficiently ample L. We close with a careful study of a simple example of normal function of Green–Griffiths type for which the naive expected dimension count is positive.

Example 6.5. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ viewed as the smooth quadric $Q = V(q) \subseteq \mathbb{P}^3$ defined by the vanishing of $q = x_0^2 + x_1^2 + x_2^2 + x_3^2$. Let L_α and L_β be the lines on Q defined by the equations

$$L_\alpha : t \mapsto [1, t, it, i], \qquad L_\beta : t \mapsto [1, t, -it, i]$$

Then, the difference $\zeta = [L_\alpha] - [L_\beta]$ is a primitive Hodge class on X. For future use, we also introduce the line

$$L_\gamma : t \mapsto [1, t, -it, -i]$$

which is parallel to L_α and intersects L_β at $t = \infty$.

Let $P = \mathbb{P}H^0(X, \mathcal{O}(2))$. Then, the associated normal function ν_ζ assigns to each smooth section

$$X_\sigma = V(q) \cap V(\sigma)$$

the class of $(L_\alpha - L_\beta) \cap V(\sigma)$ in the Jacobian of X_σ. A naive expected dimension count for the zero locus of ν_ζ can be obtained as follows: The dimension of P is $8 = 10 - 1 - 1$ since the space of quadratic forms on $\mathbb{C}^4$ has dimension 10, and we need mod out by Q and then projectivize. The adjunction formula shows the fibers X_σ to have genus 1. Accordingly, the graph of ν_ζ in the associated bundle of Jacobians $J \to P$ has codimension 1. Likewise, the zero section of J is also codimension 1, and so to first approximation the zero locus of ν_ζ in this case should have codimension 2 in J, which corresponds to a 7-dimensional subvariety of P.

To see that the zero locus of ν_ζ is in fact empty, let $Y \subset \mathbb{P}^3$ be a smooth quadric which intersects X in a smooth curve E. Let $\Lambda \subset X$ be a line of the form $\{z\} \times \mathbb{P}^1$ which intersects E in a pair of distinct points

$$e = (z, w), \qquad f = (z, w')$$

Let the line $\Upsilon = \mathbb{P}^1 \times \{w\}$ intersect E in the divisor $e + g$. Then, since every line on X is parallel to either Λ or Υ, it follows that $L_\alpha - L_\beta$ intersects E in a divisor which is linearly equivalent to

$$(e + f) - (e + g) \sim f - g$$

Accordingly, if ν_ζ vanishes at Y then $f \sim g$ and hence $\Lambda = \Upsilon$.

As a consequence of symmetries however, the 2-torsion locus of ν_ζ is non-zero. To be explicit, let $S = \mathbb{C} - \{-1, -i, 0, i, 1\}$ and $\mu : S \to P$ be the map which associates to a point $s \in S$ the quadric

$$Q_s = V(s^2x_0^2 + x_1^2 - x_2^2 - s^2x_3^2)$$

Then, for each $s \in S$, the associated curve $X_{\mu(s)}$ is smooth.

Let θ be the involution of $\mathbb{P}^3$ induced by the linear map

$$(c_0, c_1, c_2, c_3) \mapsto (-c_3, -c_2, c_1, c_0)$$

on $\mathbb{C}^4$. Then, the lines L_α and L_γ are the projectivizations of the $\pm i$-eigenspaces of this map, and hence are pointwise fixed under the action of θ. The involution θ also fixes the quadrics Q and Q_s, and hence the curve $X_{\mu(s)}$. Consequently, the fixed points of the action θ on $X_{\mu(s)}$ are exactly the four points

$$\alpha_1 = [1, is, -s, i], \qquad \alpha_2 = [1, -is, s, i]$$
$$\gamma_1 = [1, is, s, -i], \qquad \gamma_2 = [1, -is, -s, -i]$$

corresponding to the intersection of the lines L_α and L_γ with Q_s. The line L_β on the other hand intersects Q_s at the points

$$\beta_1 = [1, is, s, i], \qquad \beta_2 = [1, -is, -s, i]$$

which are interchanged under the action of θ.

Let

$$F_1 = sx_0 + ix_1 - x_2 + isx_3$$
$$F_2 = sx_0 - ix_1 + x_2 + isx_3$$
$$F_3 = ix_1 + x_2$$

Then, direct calculation shows that $V(F_1)$ is a plane passing through $\{\alpha_1, \alpha_2, \gamma_1\}$ which is also tangent to E_s at γ_1. Similarly, $V(F_2)$ is a plane passing through

$\{\alpha_1, \alpha_2, \gamma_2\}$ which is tangent to E_s at γ_2. Finally, $V(F_3)$ is a plane passing through $\{\beta_1, \beta_2, \gamma_1, \gamma_2\}$. Moreover, one can easily check that these planes have no additional points of intersection or tangency other than the ones listed above. Therefore, the rational function

$$F = (F_1 F_2)/F_3^2$$

on $\mathbb{P}^3$ restricts to a meromorphic function on E_s with divisor

$$(\alpha_1 + \alpha_2 + 2\gamma_1) + (\alpha_1 + \alpha_2 + 2\gamma_2) - 2(\beta_1 + \beta_2 + \gamma_1 + \gamma_2) = 2(\alpha_1 + \alpha_2) - 2(\beta_1 + \beta_2)$$

and hence $2\nu_\zeta$ vanishes along the image of μ.

Finally, to get a 7-dimensional subvariety of P as predicted above, observe that the group $SO(4)$ has dimension 6 and acts on $\mathbb{P}^3$ preserving the quadric Q. This action also fixes the integral Hodge class ζ, and hence acts on the 2-torsion locus. The orbit of S under the action of $SO(4)$ therefore provides a 7-dimensional complex analytic subvariety of P on which $2\nu_\zeta$ vanishes.

Remark. The infinitesimal invariant for the intersection of a generic quadric X and cubic Y in $\mathbb{P}^3$ is considered in [Gri83, Section 6d], where it is shown that in this case, the invariant determines the curve $C = X \cap Y$.

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