Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description

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The gravitational-wave signal from inspiralling neutron-star–neutron-star (or black-hole–neutron-star) binaries will be influenced by tidal coupling in the system. An important science goal in the gravitational-wave detection of these systems is to obtain information about the equation of state of neutron star matter via the measurement of the tidal polarizability parameters of neutron stars. To extract this piece of information will require accurate analytical descriptions both of the motion and the radiation of tidally interacting binaries. We improve the analytical description of the late inspiral dynamics by computing the next-to-next-to-leading order relativistic correction to the tidal interaction energy. Our calculation is based on an effective-action approach to tidal interactions, and on its transcription within the effective-one-body formalism. We find that second-order relativistic effects (quadratic in the relativistic gravitational potential $u = G(m_1 + m_2)/(c^2 r)$) significantly increase the effective tidal polarizability of neutron stars by a distance-dependent amplification factor of the form $1 + \alpha_1 u + \alpha_2 u^2 + \cdots$ where, say for an equal-mass binary, $\alpha_1 = 5/4 = 1.25$ (as previously known) and $\alpha_2 = 85/14 \simeq 6.07143$ (as determined here for the first time). We argue that higher-order relativistic effects will lead to further amplification, and we suggest a Padé-type way of resumming them. We recommend testing our results by comparing resolution-extrapolated numerical simulations of inspiralling-binary neutron stars to their effective one body description.

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I. INTRODUCTION

Inspiralling binary neutron stars are among the most promising sources for the advanced versions of the currently operating ground-based gravitational-wave (GW) detectors LIGO/Virgo/GEO. These detectors will be maximally sensitive to the inspiral part of the GW signal, which will be influenced by tidal interaction between two neutron stars. An important science goal in the detection of these systems (and of the related mixed black-hole–neutron-star systems) is to obtain information about the equation of state of neutron-star matter via the measurement of the tidal polarizability parameters of neutron stars. The analytical description of tidally interacting compact-binary systems (made of two neutron stars or one black hole and one neutron star) has been initiated quite recently [1–8]. In addition, these analytical descriptions have been compared to accurate numerical simulations [5, 9–11], and have been used to estimate the sensitivity of GW signals to the tidal polarizability parameters [11–15].

Here, we shall focus on one aspect of the analytical description of tidally interacting *relativistic* binary systems, namely the role of the higher-order post-Newtonian (PN) corrections in the tidal interaction energy, as described, in particular, within the effective one body (EOB) formalism [16–19]. Indeed, the analysis of Ref. [5], which compared the prediction of the EOB formalism for the binding energy of tidally interacting neutron stars to (nonconformally flat) numerical simulations of quasi-equilibrium circular sequences of binary neutron stars [20, 21], suggested the importance of higher-order PN corrections to tidal effects, beyond the first post-Newtonian (1PN) level, and their tendency to *significantly increase* the "effective tidal polarizability" of neutron stars.

In the EOB formalism, the gravitational binding of a binary system is essentially described by a certain "radial potential" $A(r)$. In the tidal generalization of the EOB formalism proposed in Ref. [5], the EOB radial potential $A(r)$ is written as the sum of three contributions,

$$A(r) = A^{\rm BBH}(r) + A_A^{\rm tidal}(r) + A_B^{\rm tidal}(r)\,, \qquad (1.1)$$

where $A^{\rm BBH}(r)$ is the radial potential describing the dynamics of binary black holes, and where $A_A^{\rm tidal}(r)$ and $A_B^{\rm tidal}(r)$ are the additional radial potentials associated, respectively, with the tidal deformations of body A and body B. [For binary neutron-star systems both $A_A^{\rm tidal}$ and $A_B^{\rm tidal}$ are present, while for mixed neutron-star–black-hole systems only one term, corresponding to the neutron star, is present; see following]. Here, we consider a binary system of (gravitational) masses m_A and m_B, and denote

$$M \equiv m_A + m_B\,, \quad \nu \equiv \frac{m_A\, m_B}{(m_A+m_B)^2}\,. \qquad (1.2)$$

[A labelling of the two bodies by the letters A and B will be used in this Introduction for writing general formulas. We shall later use the alternative labelling $A = 1$, $B = 2$ when explicitly dealing with the metric generated by the two bodies.] The binary black-hole (or point mass) potential $A^{\rm BBH}(r)$ is known up to the third post-Newtonian (3PN) level [18], namely

$$A^{\rm BBH}_{\rm 3PN}(r) = 1 - 2\,u + 2\,\nu\,u^3 + a_4\,\nu\,u^4\,, \qquad (1.3)$$

where $a_4 = 94/3 - (41/32)\,\pi^2 \simeq 18.68790269$, and

$$u \equiv \frac{GM}{c^2\, r}\,, \qquad (1.4)$$

with c being the speed of light in vacuum and G the Newtonian constant of gravitation.

It was recently found [22, 23] that an excellent description of the dynamics of binary black-hole systems is obtained by augmenting the 3PN expansion Eq. (1.3) with additional fourth post-Newtonian (4PN) and fifth post-Newtonian (5PN) terms, and by Padé resumming the corresponding 5PN Taylor expansion.

The tidal contributions $A^{\rm tidal}_{A,B}(r)$ can be decomposed according to multipolar order ℓ, and type, as

$$A^{\rm tidal}_A(r) = \sum_{\ell\geq 2}\Big\{ A^{(\ell)\,\rm LO}_{A\,\rm electric}(r)\,\widehat{A}^{(\ell)}_{A\,\rm electric}(r) + A^{(\ell)\,\rm LO}_{A\,\rm magnetic}(r)\,\widehat{A}^{(\ell)}_{A\,\rm magnetic}(r) + \dots \Big\}\,. \qquad (1.5)$$

Here, the label "electric" refers to the gravito-electric tidal polarization induced in body A by the tidal field generated by its companion, while the label "magnetic" refers to a corresponding gravito-magnetic tidal polarization. On the other hand, the label LO refers to the leading-order approximation (in powers of u) of each (electric or magnetic) multipolar radial potential. For instance, the gravito-electric contribution at multipolar order ℓ is equal to [5]

$$A^{(\ell)\,\rm LO}_{A\,\rm electric}(r) = -\kappa^{(\ell)}_A\, u^{2\ell+2} \qquad (1.6)$$

where

$$\kappa^{(\ell)}_A = 2\,k^{(\ell)}_A\,\frac{m_B}{m_A}\left(\frac{R_A\,c^2}{G(m_A+m_B)}\right)^{2\ell+1}\,. \qquad (1.7)$$

Here, R_A denotes the radius of body A, and $k^{(\ell)}_A$ denotes a dimensionless "tidal Love number". [Note that $k^{(\ell)}_A$ was denoted k^A_ℓ in our previous work. Here we shall always put the multipolar index ℓ within parentheses to avoid ambiguity with our later use of the labelling $A, B = 1, 2$ for the two bodies.] The corresponding leading-order radial potential of the gravito-magnetic type is proportional to $u^{2\ell+3}$ (instead of $u^{2\ell+2}$), and to $j^{(\ell)}_A R^{2\ell+1}_A$, where $j^{(\ell)}_A$ denotes a dimensionless "magnetic tidal Love number". It was found [3, 4] that both types of Love numbers have a strong dependence upon the *compactness* $\mathcal{C}_A \equiv G\,m_A/(c^2 R_A)$ of the tidally deformed body, and that both $k^{(\ell)}_A$ and $j^{(\ell)}_A$ contain a factor $1-2\,\mathcal{C}_A$, so that they would formally vanish in the limit where body A becomes as compact as a black hole (i.e. $\mathcal{C}_A \to \mathcal{C}_{\rm BH} = \frac{1}{2}$). This is consistent with the decomposition Eq. (1.1), where the binary black-hole radial potential $A^{\rm BBH}(r)$ is the only remaining contribution when one formally takes the limit where both $\mathcal{C}_A$ and $\mathcal{C}_B$ tend to the black-hole value $\mathcal{C}_{\rm BH} = 1/2$. Finally, the supplementary factors $\widehat{A}^{(\ell)}_{A\,\rm electric}(r)$ and $\widehat{A}^{(\ell)}_{A\,\rm magnetic}(r)$ denote the distance-dependent *amplification factors* of the leading-order tidal interaction by higher-order PN effects. They have the general form

$$\widehat{A}^{(\ell)}_{A\,\rm electric}(r) = 1 + \alpha^{A(\ell)}_{1\,\rm electric} u + \alpha^{A(\ell)}_{2\,\rm electric} u^2 + \dots\,, \qquad (1.8)$$

$$\widehat{A}^{(\ell)}_{A\,\rm magnetic}(r) = 1 + \alpha^{A(\ell)}_{1\,\rm magnetic} u + \dots\,, \qquad (1.9)$$

where u is defined by Eq. (1.4).

The main aim of the present investigation will be to compute the electric-type amplification factors $\widehat{A}^{(\ell)}_{A\,\rm electric}$, for $\ell = 2$ (quadrupolar tide) and $\ell = 3$ (octupolar tide), at the *second order* in u, i.e. to compute both $\alpha^{A(\ell)}_{1\,\rm electric}$ and $\alpha^{A(\ell)}_{2\,\rm electric}$. We shall also compute the magnetic-type amplification factor $\widehat{A}^{(\ell)}_{A\,\rm magnetic}$, for $\ell = 2$, at the first order in u.

The analytical value of the first-order electric amplification coefficient $\alpha^{A(\ell)}_{1\,\rm electric}$ was computed some time ago for $\ell = 2$ (see Ref. [29] in [5]) and was reported in Eq. (38) of [5], namely

$$\alpha^{A(\ell=2)}_{1\,\rm electric} = \frac{5}{2}\,X_A\,, \qquad (1.10)$$

where $X_A \equiv m_A/(m_A+m_B)$ is the mass fraction of body A. The analytical result (1.10) has been recently confirmed [6]. On the other hand, several comparisons of the analytical description of tidal effects with the results of numerical simulations have indicated that the amplification factor $\widehat{A}^{(\ell=2)}_{A\,\rm electric}(r)$ is larger that its 1PN value $1+\alpha^{A(\ell=2)}_{1\,\rm electric}\,u$, and have suggested that the higher-order coefficients $\alpha^{A(\ell)}_{2\,\rm electric}, \dots$ take large, positive values. More precisely, the analysis of Ref. [5] suggested (when taking into account the value (1.10) for α_1) a value of order $\alpha^{A(\ell=2)}_{2\,\rm electric} \sim +40$ (for the equal-mass case) from a comparison with the numerical results of Refs. [20, 21] on quasi-equilibrium adiabatic sequences of binary neutron stars. Recently, a comparison with dynamical simulations of inspiralling binary neutron stars confirmed the need for such a large value of $\alpha^A_{2\,\rm electric}$ [9, 10]. [Note that, while the comparison to the highest resolution numerical data suggests the need of even larger values of

$\alpha_{2\,\mathrm{electric}}^{A(\ell=2)}$, of order $+100$, the comparison to approximate resolution-extrapolated data call only for α_2 values of order $+40$. See Fig. 13 in [10].]

II. EFFECTIVE ACTION APPROACH TO TIDAL EFFECTS

A. Finite-size effects and nonminimal worldline couplings

It was shown long ago [24], using the technique of matched asymptotic expansions, that the motion and radiation of N (non-spinning) compact objects can be described, up to the 5PN approximation, by an effective action of the type

$$S_0 = \int \frac{d^D x}{c} \, \frac{c^4}{16\,\pi\,G} \, \sqrt{g}\, R(g) + S_{\mathrm{point\,mass}} \, , \qquad (2.1)$$

where $R(g)$ represents the scalar curvature associated with the metric $g_{\mu\nu}$, with determinant $-g$, and where

$$S_{\mathrm{point\,mass}} = - \sum_A \int m_A \, c^2 \, d\tau_A \qquad (2.2)$$

is the leading order skeletonized description of the compact objects, as point masses. Here $d\tau_A$ denotes the proper time along the worldline $y_A^\mu(\tau_A)$ of A, namely $d\tau_A \equiv c^{-1}(-g_{\mu\nu}(y_A)\, dy_A^\mu \, dy_A^\nu)^{1/2}$. To give meaning to the notion of point mass sources in General Relativity one needs to use a covariant regularization method. The most convenient one is *dimensional regularization*, i.e. analytic continuation in the value of the spacetime dimension $D = 4 + \varepsilon$, with $\varepsilon \in \mathbb{C}$ being continued to zero only at the end of the calculation. The consistency and efficiency of this method has been shown in the calculations of the motion [25, 26] and radiation [27] of binary black holes at the 3PN approximation.

It was also pointed out in Ref. [24] that finite-size effects (linked to tidal effects, and the fact that neutron stars have, contrary to black holes, non-zero Love numbers $k_A^{(\ell)}$) enter at the 5PN level. In effective field theory, finite-size effects are treated by augmenting the point-mass action of Eq. (2.2) by nonminimal worldline couplings involving higher-order derivatives of the field [28–30]. In a gravitational context this means considering worldline couplings involving the 4-velocity $u_A^\mu \equiv dy_A^\mu/d\tau_A$ (satisfying $g_{\mu\nu}\, u_A^\mu \, u_A^\nu = -c^2$) together with the Riemann tensor $R_{\alpha\beta\mu\nu}$ and its covariant derivatives. To classify the possible worldline scalars that can be constructed one can appeal to the relativistic theory of tidal expansions [31–33]. In the notation of Refs. [32, 33] the tidal expansion of the "external metric" felt by body A can be entirely expressed in terms of two types of external tidal gradients evaluated along the central worldline of this body: the gravito-electric $G_L^A(\tau_A) \equiv G_{a_1 \ldots a_\ell}^A(\tau_A)$ and gravito-magnetic $H_L^A(\tau_A) \equiv H_{a_1 \ldots a_\ell}^A(\tau_A)$ symmetric trace-free (spatial) tensors, together with their time-derivatives. [The spatial indices $a_i = 1, 2, 3$ refer to a local frame $X_A^0 \equiv c\,\tau_A$, X_A^a attached to body A.] This implies that the most general nonminimal worldline action has the form

$$\begin{aligned} S_{\mathrm{nonminimal}} &= \sum_A \sum_{\ell \geq 2} \left\{ \frac{1}{2} \, \frac{1}{\ell!} \, \mu_A^{(\ell)} \int d\tau_A (G_L^A(\tau_A))^2 \right. \\ &+ \frac{1}{2} \, \frac{\ell}{\ell+1} \, \frac{1}{\ell!} \, \frac{1}{c^2} \, \sigma_A^{(\ell)} \int d\tau_A (H_L^A(\tau_A))^2 \\ &+ \frac{1}{2} \, \frac{1}{\ell!} \, \frac{1}{c^2} \, \mu_A'^{(\ell)} \int d\tau_A (\dot G_L^A(\tau_A))^2 \\ &+ \frac{1}{2} \, \frac{\ell}{\ell+1} \, \frac{1}{\ell!} \, \frac{1}{c^4} \, \sigma_A'^{(\ell)} \int d\tau_A (\dot H_L^A(\tau_A))^2 \\ &+ \left. \ldots \vphantom{\int} \right\} , \end{aligned} \qquad (2.3)$$

where $\dot G_L^A(\tau_A) \equiv d\,G_L^A/d\tau_A$, and where the ellipsis refer either to higher proper-time derivatives of G_L^A and H_L^A, or to higher-than-quadratic invariant monomials made from G_L^A, H_L^A and their proper-time derivatives. For instance, the leading-order non-quadratic term would be

$$\int d\tau_A \; G_{ab}^A \; G_{bc}^A \; G_{ca}^A \, . \qquad (2.4)$$

Note that the allowed monomials in G_L, H_L and their time derivatives are restricted by symmetry constraints. When considering a non-spinning neutron star (which is symmetric under time and space reflections) one should only allow monomials invariant under time and space reversals. For instance $G_{ab}\, \dot G_{ab}$ and $G_{ab}\, H_{ab}$ are not allowed.

B. Tidal coefficients

The electric-type tidal moments G_L^A are normalized in a Newtonian way, i.e. such that, in lowest PN order, they reduce to the usual Newtonian tidal gradients: $G_L^A = [\partial_L \, U(X^a)]_{X^a=0} + O\left(\frac{1}{c^2}\right)$, where $U(X)$ is the Newtonian potential and $\partial_L \equiv \partial_{i_1} \partial_{i_2} \ldots \partial_{i_\ell}$ represents multiple ordinary space derivatives. The magnetic-type ones H_L^A are defined (in lowest PN order) as repeated gradients of the gravitomagnetic field $c^3 g_{0a}$. With these normalizations the coefficients $\mu_A^{(\ell)}$ and $\sigma_A^{(\ell)}$ in the nonminimal action in Eq. (2.3) both have dimensions $[\mathrm{length}]^{2\ell+1}/G$. They are related to the dimensionless Love numbers $k_A^{(\ell)}$ and $j_A^{(\ell)}$, and to the radius of body A, via [3]

$$G \mu_A^{(\ell)} = \frac{1}{(2\ell-1)!!} \, 2 \, k_A^{(\ell)} \, R_A^{2\ell+1} \, , \qquad (2.5)$$

$$G \sigma_A^{(\ell)} = \frac{\ell-1}{4(\ell+2)} \, \frac{1}{(2\ell-1)!!} \, j_A^{(\ell)} \, R_A^{2\ell+1} \, . \qquad (2.6)$$

Note that the coefficients associated with the first time derivatives of G_L^A and H_L^A have dimensions $G\mu_A'^{(\ell)} \sim [\text{length}]^{2\ell+3} \sim G\sigma_A'^{(\ell)}$. The nonminimal action in Eq. (2.3) has a double ordering in powers of R_A and in powers of $1/c^2$. The lowest-order terms in the R_A expansion are proportional to R_A^5 and correspond to the electric and magnetic quadrupolar tides, as measured by G_{ab}^A and H_{ab}^A, respectively.

C. Tidal tensors

We have written the most general nonminimal action Eq. (2.3) in terms of the irreducible symmetric trace-free spatial tensors [with respect to the local space associated with the worldline $y_A^\mu(\tau_A)$] describing the tidal expansion of the "external metric" felt by body A, as defined in Ref. [32]. These tidal tensors played a useful role in simplifying the (1PN-accurate) relativistic theory of tidal effects. In our present investigation, it will be convenient to express them in terms of the Riemann tensor and its covariant derivatives. Eq. (3.40) in Ref. [32] shows that (in the case where one can neglect corrections proportional to the covariant acceleration of the worldline) the first two electric spatial tidal tensors, G_{ab} and G_{abc}, are simply equal (modulo a sign) to the non-vanishing spatial components (in the local frame) of the following spacetime tensors (evaluated along the considered worldline)

$$G_{\alpha\beta} \equiv -\, R_{\alpha\mu\beta\nu}\, u^\mu\, u^\nu\,, \tag{2.7}$$

$$G_{\alpha\beta\gamma} \equiv -\,\mathrm{Sym}_{\alpha\beta\gamma}(\nabla^\perp_\alpha\, R_{\beta\mu\gamma\nu})\, u^\mu\, u^\nu\,. \tag{2.8}$$

Here the notation $G_{\alpha\beta}$ for (minus) the electric part of the curvature tensor should not be confused with the Einstein tensor, $\mathrm{Sym}_{\alpha\beta\gamma}$ denotes a symmetrization (with weight one) over the indices $\alpha\,\beta\,\gamma$, while $\nabla^\perp_\alpha \equiv P(u)^\mu_\alpha\,\nabla_\mu$ denotes the projection of the spacetime gradient ∇_μ orthogonally to u^μ ($P(u)^\mu_\nu \equiv \delta^\mu_\nu + c^{-2}\,u^\mu\,u_\nu$). [Note that in the Newtonian limit $u^\mu \simeq c\,\delta^\mu_0$ so that the Newtonian limit of $G_{\alpha\beta}$ is $-\,c^2\,R_{\alpha 0\beta 0}$, where the factor c^2 cancels the $O(1/c^2)$ order of the curvature tensor.] By contrast, the presence of the extra term $-\,3\,c^{-2}\,E^*_{\langle a}\,E^*_{b\rangle}$ on the right-hand side of Eq. (3.40) in Ref. [32] shows that the $\ell = 4$ electric spatial tidal tensor $G_{abcd} = \partial_{\langle abc}\,E^*_{d\rangle}$ would differ from the symmetrized spatial projection of $(\nabla_\alpha\nabla_\beta\,R_{\gamma\mu\delta\nu})\,u^\mu\,u^\nu$ by a term proportional to $G_{\langle\alpha\gamma}\,G_{\beta\delta\rangle}$. (Here, the angular brackets denote a (spatial) symmetric trace-free projection.) In addition, the electric time derivatives, such as $\dot G_{ab}$ can be replaced by corresponding spacetime tensors such as $u^\mu\,\nabla_\mu\,G_{\alpha\beta}$. Similarly to Eqs. (2.7), (2.8), one finds that the $\ell = 2$ and $\ell = 3$ magnetic tidal tensors (as defined in Refs. [32, 33]) are equal to the nonvanishing local-frame spatial components of the spacetime tensors

$$H_{\alpha\beta} \equiv +\,2\,c\,R^*_{\alpha\mu\beta\nu}\,u^\mu\,u^\nu\,, \tag{2.9}$$

$$H_{\alpha\beta\gamma} \equiv +\,2\,c\,\mathrm{Sym}_{\alpha\beta\gamma}(\nabla^\perp_\alpha\,R^*_{\beta\mu\gamma\nu})\,u^\mu\,u^\nu\,, \tag{2.10}$$

where $R^*_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\,\epsilon_{\mu\nu\rho\sigma}\,R^{\rho\sigma}{}_{\alpha\beta}$ is the dual of the curvature tensor, $\epsilon_{\mu\nu\rho\sigma}$ denoting here the Levi-Civita tensor (with $\epsilon_{0123} = +\sqrt{g}$). Note the factor $+2$ entering the link between the magnetic tidal tensors $H_{\alpha\beta},\ldots$ (normalized as in Refs. [32, 33]) and the dual of the curvature tensor, which contrasts with the factor -1 entering the corresponding electric tidal-tensor links, Eqs. (2.7), (2.8). (The definition of $\mathcal{B}^A_{\alpha\beta}$ in the text below Eq. (5) of Ref. [5] should have included such a factor 2 in its right-hand side. On the other hand, the corresponding magnetic-quadrupole tidal action, Eq. (13) there, was computed with H_{ab} and was correctly normalized.) Let us also note that the expressions in Eqs. (2.7)–(2.10) assume that the Ricci tensor vanishes (e.g. to ensure the tracelessness of $G_{\alpha\beta}$). One could have, alternatively, defined $G_{\alpha\beta}$ etc. by using the Weyl tensor $C_{\alpha\mu\beta\nu}$ instead of $R_{\alpha\mu\beta\nu}$. However, as discussed in Ref. [29], the terms in an effective action which are proportional to the (unperturbed) equations of motion (such as Ricci terms) can be eliminated (modulo contact terms) by suitable field redefinitions.

D. Covariant description of tidal interactions

Finally, the covariant form of the effective action describing tidal interactions reads

$$S_{\rm tot} = S_0 + S_{\rm point\,mass} + S_{\rm nonminimal} \tag{2.11}$$

where S_0 and $S_{\rm point\,mass}$ are given by Eqs. (2.1), (2.2), and where the covariant form of the nonminimal worldline couplings starts as

$$\begin{aligned} S_{\rm nonminimal} \;=\; \sum_A \Big\{ & \frac{1}{4}\,\mu_A^{(2)} \int d\tau_A\, G^A_{\alpha\beta}\, G_A^{\alpha\beta} \\ +\; & \frac{1}{6\,c^2}\,\sigma_A^{(2)} \int d\tau_A\, H^A_{\alpha\beta}\, H_A^{\alpha\beta} \\ +\; & \frac{1}{12}\,\mu_A^{(3)} \int d\tau_A\, G^A_{\alpha\beta\gamma}\, G_A^{\alpha\beta\gamma} \\ +\; & \frac{1}{4\,c^2}\,\mu_A'^{(2)} \int d\tau_A (u_A^\mu\nabla_\mu G^A_{\alpha\beta})(u_A^\nu\nabla_\nu G_A^{\alpha\beta}) \\ +\; & \ldots \Big\}\,, \end{aligned} \tag{2.12}$$

where $G_A^{\alpha\beta} \equiv g^{\alpha\mu}\,g^{\beta\nu}\,G^A_{\mu\nu}$, etc. [evaluated along the A worldline].

In principle, one can then derive the influence of tidal interaction on the motion and radiation of binary systems by solving the equations of motion following from the action of Eqs. (2.11), (2.12). More precisely, this action implies both a dynamics for the worldlines where the geodesic equation is modified by tidal forces [coming from $\delta S_{\rm nonminimal}/\delta\, y_A^\mu(\tau_A)$], and modified Einstein equations for the gravitational field of the type

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = \frac{8\pi\, G}{c^4} \left\{T^{\text{point mass}}_{\mu\nu} + T^{\text{nonminimal}}_{\mu\nu}\right\}, \tag{2.13}$$

where the new tidal sources $T^{\mu\nu}_{\text{nonminimal}}(x) = (2c/\sqrt{g})\, \delta S^{\text{nonminimal}}/\delta\, g_{\mu\nu}(x)$ are, essentially, sums of derivatives of worldline Dirac-distributions:

$$T_{\text{nonminimal}}(x) \sim \sum_A \sum_\ell \partial^\ell\, \delta(x - y_A)\,.$$

E. A simplifying, general property of *reduced* actions

The task of solving the coupled dynamics of the worldlines and of the gravitational field, both being modified by tidal effects, at the second post-Newtonian (2PN) level, i.e. at the next-to-next-to-leading order in tidal effects, and then of computing the looked for higher-order terms in the amplification factors of Eqs. (1.8), (1.9) is quite non-trivial. Happily, one can drastically simplify the needed work by using a general property of *reduced actions*. Indeed, we are interested here in knowing the influence of tidal effects on the *reduced* dynamics of a compact binary, that is, the dynamics of the two worldlines $y^\mu_A(\tau)$, $y^\mu_B(\tau)$, obtained after having "integrated out" the gravitational field (i.e., after having explicitly solved $g_{\mu\nu}(x)$ as a functional of the two worldlines). When considering, as we do here, the *conservative* dynamics of the system (without radiation reaction), it can be obtained from a *reduced action*, which is traditionally called the "Fokker action". See Ref. [28] and references therein for a detailed discussion (using a diagrammatic approach) of Fokker actions (at the 2PN level, and with the inclusion of scalar couplings in addition to the pure Einsteinian tensor couplings). If we denote the fields mediating the interaction between the worldlines $y = \{y_A, y_B\}$ as φ (in our case $\varphi = g_{\mu\nu}$), the reduced worldline action $S_{\text{red}}[y]$ (a functional of the worldlines y) that corresponds to the complete action $S[\varphi, y]$ describing the coupled dynamics of y and φ is formally defined as:

$$S_{\text{red}}\,[y] \equiv S\,[\varphi_{\text{sol}}\,[y],\, y]\,, \tag{2.14}$$

where $\varphi_{\text{sol}}\,[y]$ is the functional of y obtained by solving the φ-field equation,

$$\delta\, S\,[\varphi, y]/\delta\varphi = 0\,, \tag{2.15}$$

considered as an equation for φ, with given source-worldlines. (This must be done with time-symmetric boundary conditions and, in the case of $g_{\mu\nu}$, the addition of a suitable gauge-fixing term; see Ref. [28] for details.)

Having recalled the concept of reduced (or Fokker) action, let us now consider the case where the complete action is of the form

$$S\,[\varphi, y] = S^{(0)}[\varphi, y] + \epsilon\, S^{(1)}[\varphi, y]\,, \tag{2.16}$$

where ϵ denotes a "small parameter". In our case, ϵ can be either a formal parameter associated with all the nonminimal tidal terms in $S_{\text{nonminimal}}$, Eq. (2.12), or, more concretely, any of the tidal parameters entering Eq. (2.12): $\mu^{(\ell=2)}_A$, $\mu^{(\ell=2)}_B$, etc. As said previously, when turning on ϵ, the equations of motion, and therefore the solutions of both φ and y get perturbed by terms of order ϵ : $\varphi = \varphi^{(0)} + \epsilon\, \varphi^{(1)} + \ldots$, $y = y^{(0)} + \epsilon\, y^{(1)} + \ldots$, but a simplification occurs when considering the reduced action Eq. (2.14). Indeed, it is true that the field equation (2.15) for φ gets modified into

$$0 = \frac{\delta\, S\,[\varphi, y]}{\delta\, \varphi} = \frac{\delta\, S^{(0)}[\varphi, y]}{\delta\, \varphi} + \epsilon\, \frac{\delta\, S^{(1)}[\varphi, y]}{\delta\, \varphi}\,, \tag{2.17}$$

so that its solution $\varphi_{\text{sol}}\,[y]$ gets perturbed:

$$\varphi_{\text{sol}}\,[y] = \varphi^{(0)}_{\text{sol}}\,[y] + \epsilon\, \varphi^{(1)}_{\text{sol}}\,[y] + O(\epsilon^2)\,. \tag{2.18}$$

However, when inserting the perturbed solution of Eq. (2.18) into the complete, perturbed action of Eq. (2.16), one finds

$$\begin{aligned} S_{\text{red}}\,[y] &= S\,[\varphi^{(0)}_{\text{sol}}\,[y] + \epsilon\, \varphi^{(1)}_{\text{sol}}\,[y] + O(\epsilon^2), y] \\ &= S\,[\varphi^{(0)}_{\text{sol}}\,[y], y] + \epsilon\, \varphi^{(1)}_{\text{sol}}\,[y]\, \frac{\delta\, S}{\delta\, \varphi}\,[\varphi^{(0)}_{\text{sol}}\,[y], y] + O(\epsilon^2) \\ &= S\,[\varphi^{(0)}_{\text{sol}}\,[y], y] \\ &\quad + \epsilon\, \varphi^{(1)}_{\text{sol}}\,[y]\, \frac{\delta\, S^{(0)}}{\delta\, \varphi}\,[\varphi^{(0)}_{\text{sol}}\,[y], y] + O(\epsilon^2) \\ &= S\,[\varphi^{(0)}_{\text{sol}}\,[y], y] + O(\epsilon^2)\,, \end{aligned} \tag{2.19}$$

because, by definition, $\varphi^{(0)}_{\text{sol}}$ is a solution of $\delta\, S^{(0)}/\delta\, \varphi = 0$. Note that, in Eq. (2.19), while the functional S is the *complete, perturbed* action, the functional argument is the *unperturbed* solution. Decomposing the functional S into its unperturbed plus perturbed parts [see Eq. (2.16)] then leads to the final result:

$$\begin{aligned} S_{\text{red}}\,[y] &= S^{(0)}[\varphi^{(0)}_{\text{sol}}\,[y], y] + \epsilon\, S^{(1)}[\varphi^{(0)}_{\text{sol}}\,[y], y] + O(\epsilon^2) \\ &= S^{(0)}_{\text{red}}\,[y] + \epsilon\, S^{(1)}[\varphi^{(0)}_{\text{sol}}\,[y], y] + O(\epsilon^2)\,. \end{aligned} \tag{2.20}$$

In words: the order $O(\epsilon)$ perturbation

$$\epsilon\, S^{(1)}_{\text{red}}\,[y] \equiv S_{\text{red}}\,[y] - S^{(0)}_{\text{red}}\,[y]$$

of the *reduced* action is correctly obtained, modulo terms of order $O(\epsilon^2)$, by replacing in the $O(\epsilon)$ perturbation

$$\epsilon\, S^{(1)}\,[\varphi, y]$$

of the *complete* (unreduced) action the field φ by its *unperturbed* solution $\varphi^{(0)}_{\text{sol}}\,[y]$.

In our case, the ordering parameter ϵ is either the collection $\mu^{(2)}_A, \mu^{(2)}_B, \mu^{(3)}_A, \mu^{(3)}_B, \ldots, \sigma^{(2)}_A\, c^{-2}, \ldots, \mu'^{(2)}_A\, c^{-2}, \ldots$, or the corresponding sequence of powers of R_A and R_B : $R^5_A, R^5_B, R^7_A, R^7_B, \ldots$ The terms quadratic in ϵ would therefore involve at least *ten* powers of the radii (and would mix with higher-than-quadratic worldline contributions akin to (2.4)). Neglecting such terms, we conclude that the higher-PN corrections to the tidal effects are correctly obtained by replacing in Eq. (2.12), considered as a functional of $g_{\mu\nu}(x)$ and $y_A^\mu(\tau_A)$, the metric $g_{\mu\nu}(x)$ by the *point-mass metric* obtained by solving Einstein's equations with point-mass sources. [This was the method used by one of us (T.D.) to compute the 1PN coefficient of Eq. (1.10) from the calculation by Damour, Soffel and Xu of the 1PN-accurate value of G_{ab} [34, 35].]

III. THE 2PN POINT-MASS METRIC AND ITS REGULARIZATION

A. Form of the 2PN point-mass metric

The result of the last Section allows one to compute the tidal corrections to the reduced action for two tidally interacting bodies A, B with the same accuracy at which one knows the metric generated by two (structureless) point masses $m_A, y_A^\mu; m_B, y_B^\mu$. The metric generated by two point masses has been the topic of many works over many years. It has been known (in various forms and gauges) at the 2PN approximation for a long time [36–38]. Here, we shall use the convenient, explicit harmonic-gauge form of Ref. [39], with respect to the (harmonic) coordinates $x^\mu = (x^0 \equiv ct, x^i)$, i.e. the metric

$$ds^2 = g_{00}(dx^0)^2 + 2\, g_{0i}\, dx^0 dx^i + g_{ij}\, dx^i dx^j\,, \qquad (3.1)$$

where, at 2PN, the metric components are written as

$$\begin{aligned} g_{00} &= -1 + 2\,\epsilon^2\, V - 2\,\epsilon^4\, V^2 \\ &\quad + 8\,\epsilon^6 \left(\hat{X} + \delta^{ij}\, V_i V_j + \frac{1}{6}\, V^3 \right) + O(8)\,, \\ g_{0i} &= -\,4\,\epsilon^3\, V_i - 8\,\epsilon^5 \hat{R}_i + O(7)\,, \\ g_{ij} &= \delta_{ij}\left(1 + 2\,\epsilon^2\, V + 2\,\epsilon^4\, V^2\right) + 4\,\epsilon^4\, \hat{W}_{ij} + O(6)\,. \end{aligned} \qquad (3.2)$$

Here, as below, we sometimes use the alternative notation $\epsilon \equiv 1/c$ for the small PN parameter. We used also the shorthand notation $O(n) \equiv O(\epsilon^n) \equiv O(c^{-n})$.

The various 2PN brick potentials $V, V_i, \hat{W}_{ij}, \hat{R}_i$ and $\hat{X}$ are the (time-symmetric) solutions of

$$\begin{aligned} \Box\, V &= -\,4\,\pi\, G\sigma\,, \\ \Box\, V_i &= -\,4\,\pi\, G\sigma_i\,, \\ \Box\, \hat{W}_{ij} &= -\,4\,\pi\, G(\sigma_{ij} - \delta_{ij}\,\sigma_{kk}) - \partial_i V\, \partial_j V\,, \\ \Box\, \hat{R}_i &= -\,4\,\pi\, G(V\sigma_i - V_i\sigma) -\, 2\,\partial_k V\, \partial_i V_k - \frac{3}{2}\,\partial_t V\, \partial_i V\,, \\ \Box\, \hat{X} &= -\,4\,\pi\, G\, V\sigma_{ii} + 2\, V_i\, \partial_t\, \partial_i V + V\, \partial_t^2\, V + \frac{3}{2}\,(\partial_t V)^2 \\ &\quad -\, 2\,\partial_i V_j\, \partial_j V_i + \hat{W}_{ij}\, \partial_{ij} V\,, \end{aligned} \qquad (3.3)$$

where ∂_t denotes a time derivative (while we remind that ∂_i, for instance, denotes a spatial one), and where the compact-supported source terms are [40]

$$\sigma \equiv \frac{T^{00} + T^{ii}}{c^2}\,, \quad \sigma_i \equiv \frac{T^{0i}}{c}\,, \quad \sigma_{ij} \equiv T^{ij}\,, \qquad (3.4)$$

with $T^{\mu\nu}$ being the stress-energy tensor of two point masses:

$$T^{\mu\nu} = \mu_1(t)\, v_1^\mu(t)\, v_1^\nu(t)\, \delta(\boldsymbol{x} - \boldsymbol{y}_1(t)) + 1 \leftrightarrow 2\,, \qquad (3.5)$$

where

$$\mu_1(t) = m_1 \left[g^{-1/2} (g_{\mu\nu}\, v_1^\mu\, v_1^\nu / c^2)^{-1/2} \right]_1\,. \qquad (3.6)$$

Here, $v_1^\mu = \frac{dy_1^\mu}{dt} = (c, v_1^i)$ and the index 1 on the bracket in Eq. (3.6) refers to a regularized limit where the field point x^i tends towards the (point-mass) source point y_1^i. Note that, in this section, we shall generally label the two particles as (m_1, y_1^i), (m_2, y_2^i), instead of (m_A, y_A^i), (m_B, y_B^i) as above. The notation $1 \leftrightarrow 2$ means adding the terms obtained by exchanging the particle labels 1 and 2.

The explicit forms of the 2PN-accurate brick potentials V, V_i, $\hat{W}_{ij}$, $\hat{R}_i$, $\hat{X}$ were given in Ref. [39]. Their time-symmetric parts are recalled in Appendix A. These brick potentials are expressed as explicit functions of $\boldsymbol{r}_1 \equiv \boldsymbol{x} - \boldsymbol{y}_1$, $r_1 \equiv |\boldsymbol{r}_1|$, $\boldsymbol{n}_1 \equiv \boldsymbol{r}_1/r_1$, $\boldsymbol{r}_2 \equiv \boldsymbol{x} - \boldsymbol{y}_2$, etc., $\boldsymbol{y}_{12} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$, $r_{12} \equiv |\boldsymbol{y}_{12}|$, $\boldsymbol{n}_{12} \equiv \boldsymbol{y}_{12}/r_{12}$, $\boldsymbol{v}_{12} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2$, $(n_{12}\, v_1) \equiv \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$. Note the appearance of the auxiliary quantity S, which denotes the perimeter of the triangle defined by $\boldsymbol{x}$, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, viz

$$S \equiv r_1 + r_2 + r_{12}\,. \qquad (3.7)$$

In all the PN expressions, the spacetime points x^μ, y_1^μ, y_2^μ (and the velocities v_A^μ) are taken at the same instant t, i.e. $x^0 = y_1^0 = y_2^0 = ct$.

B. Regularization of the 2PN metric and of the 2PN tidal actions

Let us now discuss in more detail the crucial operation (already implicit in Sec. II above) of regularization of all the needed field quantities, such as $g_{\mu\nu}(x)$, $g(x)$, $R_{\mu\alpha\nu\beta}(x), \ldots$, when they are to be evaluated on a worldline: $x^\mu \to y_A^\mu$. As mentionned at the beginning of Sec. II, all the quantities $[G_{\mu\nu}(x)]_1, \ldots, [R_{\mu\alpha\nu\beta}(x)]_1$ are defined by dimensional continuation. It was shown long ago [24, 41] that, at 2PN, dimensional regularization is equivalent to the Riesz' analytic regularization, and is a technical shortcut for computing the physical answer obtained by the matching of asymptotic expansions. In addition, because of the restricted type of singular terms that appear at 2PN [see Eqs. (25), (30) and (33) in Ref. [24]], the analytic-continuation regularization turns out to be equivalent to Hadamard regularization (used, at

2PN, in Refs. [38, 39, 42]); see below. Here, it will be technically convenient to use Hadamard regularization (which is defined in $D = 4$) because the explicit form of Eqs. (A1)–(A5) of the 2PN metric that we shall use applies only in the physical dimension $D = 4$ and has lost the information about its dimensionally continued kin in $D = 4 + \varepsilon$.

Let us summarize here the (Hadamard-type) definition of the regular part of any field quantity $\varphi(x)$ (which might be a brick potential, $V(x), V_i(x), \ldots$, a component of the metric $g_{\mu\nu}(x)$, or a specific contribution to a tidal moment, $G_{\alpha\beta}, \ldots$). We consider the behavior of $\varphi(x)$ near particle 1, i.e. when $r_1 = |\boldsymbol{x} - \boldsymbol{y}_1| \to 0$. To ease the notation, we shall provisionally put the origin of the (harmonic) coordinate system at $\boldsymbol{y}_1$ (at some instant t), i.e. we shall assume that $\boldsymbol{y}_1 = 0$, so that $r_1 = |\boldsymbol{x}| \equiv r$ and $\boldsymbol{n}_1 = \boldsymbol{r}_1/r_1 = \boldsymbol{x}/r \equiv \boldsymbol{n}$. We consider the expansion of $\varphi(\boldsymbol{x})$ in (positive and negative) integer powers k of $r_1 = r$, and in spherical harmonics of the direction $\boldsymbol{n}_1 = \boldsymbol{n}$, say (for $k \in \mathbb{Z}$, $\ell \in \mathbb{N}$, $N \in \mathbb{N}$)

$$\varphi(\boldsymbol{x}) = \sum_{k \geq -N} \sum_{\ell \geq 0} r^k \, \hat{n}^L \, f_L^k \,, \tag{3.8}$$

where $\hat{n}^L \equiv \hat{n}^{a_1 \cdots a_\ell}$ denotes the symmetric trace-free projection of the tensor $n^L \equiv n^{a_1} \ldots n^{a_\ell}$. [The angular function $f_L^k \, \hat{n}^L$ is equivalent to a sum of $\sum_{m=-\ell}^{+\ell} c_m \, Y_{\ell m}$.] We (uniquely) decompose the field $\varphi(\boldsymbol{x})$ in a *regular* part (R) and a *singular* one (S),

$$\varphi(\boldsymbol{x}) = R\left[\varphi(\boldsymbol{x})\right] + S\left[\varphi(\boldsymbol{x})\right], \tag{3.9}$$

by defining ($n \in \mathbb{N}$)

$$R\left[\varphi(\boldsymbol{x})\right] \equiv \sum_{\ell \geq 0} \sum_{n \geq 0} r^{\ell+2n} \, \hat{n}^L \, f_L^{\ell+2n} \,, \tag{3.10}$$

$$S\left[\varphi(\boldsymbol{x})\right] \equiv \sum_{k \neq \ell+2n} r^k \, \hat{n}^L f_L^k \,. \tag{3.11}$$

Note that $R\left[\varphi(\boldsymbol{x})\right]$ can be rewritten as a sum of infinitely differentiable terms of the type $\hat{x}^L (\boldsymbol{x}^2)^n$. By contrast $S\left[\varphi(\boldsymbol{x})\right]$ is such that it (if N, in Eq. (3.8), is strictly positive), or, one of its (repeated) spatial derivatives, tends towards infinity as $r \to 0$. Note also that the $R+S$ decomposition commutes with linear combinations (with constant coefficients), as well as with spatial derivatives, in the sense that $R\left[a\,\varphi(\boldsymbol{x}) + b\,\psi(\boldsymbol{x})\right] = a\,R\left[\varphi(\boldsymbol{x})\right] + b\,R\left[\psi(\boldsymbol{x})\right]$, $S\left[a\,\varphi(\boldsymbol{x}) + b\,\psi(\boldsymbol{x})\right] = a\,S\left[\varphi(\boldsymbol{x})\right] + b\,S\left[\psi(\boldsymbol{x})\right]$, $R\left[\partial_i\,\varphi(\boldsymbol{x})\right] = \partial_i\,R\left[\varphi(\boldsymbol{x})\right]$ and $S\left[\partial_i\,\varphi(\boldsymbol{x})\right] = \partial_i\,S\left[\varphi(\boldsymbol{x})\right]$. By contrast, the $R+S$ decomposition (as defined above, in the Hadamard way) does not commute with nonlinear operations (e.g. $R\left[\varphi\,\psi\right] \neq R\left[\varphi\right] R\left[\psi\right]$), nor even with multiplication by a smooth (C^∞) function $f(\boldsymbol{x})$ (e.g. $R\left[f\varphi\right] \neq f\,R\left[\varphi\right]$). This is a well-known inconsistency of the Hadamard regularization, which created many ambiguities when it was used at the 3PN level [43, 44]. One might worry that our present calculation (which aims at regularizing nonlinear quantities quadratic in $R_{\mu\alpha\nu\beta} \sim \partial^2 g + g^{-1}\,\partial g\,\partial g$) might be intrinsically ambiguous already at the 2PN level. Actually, this turns out not to be the case because of the special structure of the 2PN metric which is at work in the Riesz-analytic-continuation derivation of the 2PN dynamics in Ref. [24]. This structure guarantees, in particular, that the Riemann tensor (or its derivatives) is regularized unambiguously.

C. On the special structure of the 2PN metric guaranteeing its unambiguous regularization

Let us first recall why the Riesz-analytic-continuation method, or, equivalently (when considering the regularization of the metric and its derivatives), the dimensional-continuation method, *is* consistent under nonlinear operations. The dimensional-continuation analog of Eqs. (3.9)–(3.11) consists of distinguishing, within $\varphi(\boldsymbol{x})$, the terms that (in dimension $4+\varepsilon$) contain powers of r of the type $r^{k-n\varepsilon}$, with $n = 1, 2, 3, \ldots$ [which define the ε-singular part of $\varphi(\boldsymbol{x})$], and the terms that are (formally) C^∞ in $4+\varepsilon$ dimensions [which define the ε-regular part of $\varphi(\boldsymbol{x})$]. It is then easily seen in dimensional continuation (simply by considering the continuation to large, negative values of the real part of ε) that the ε-singular terms give vanishing contributions when evaluated at $r \to 0$, and that they do so consistently in nonlinear terms such as $\partial\varphi\,\partial\psi$. Let us now indicate why the special structure of the 2PN metric ensures that the decomposition into ε-singular parts and ε-regular parts of the various brick potentials $V(x), V_i(x), \ldots$ coincides with their above-defined decomposition into Hadamard-singular ($S\left[V(x)\right], S\left[V_i(x)\right], \ldots$) and Hadamard-regular parts ($R\left[V(x)\right], R\left[V_i(x)\right], \ldots$) in the four-dimensional case. This is trivially seen to be the case for most of the 2PN contributions to the brick potentials (because one easily sees how those contributions smoothly evolve when analytically continuing the dimension). However, the most nonlinear contributions to the 2PN metric, namely the terms, say $\hat{X}^{(VVV)}$, in $\hat{X}$ that are generated by the cubically nonlinear terms contained in the last source term, $\hat{W}_{ij}^{(VV)}\,\partial_{ij} V$, on the right hand-side of the last Eq. (3.3) (where $\hat{W}_{ij}^{(VV)}$ is the part of $\hat{W}_{ij}$ generated by $-\partial_i V\,\partial_j V$) are more delicate to discuss. Actually, among the contribution $\hat{X}^{(VVV)}$, only the terms proportional either to $m_1^2 m_2$ or to $m_1 m_2^2$, i.e., the terms whose cubically nonlinear source $\sim \partial^2 V \Delta^{-1} \partial V \partial V$ involve two V potentials generated by one worldline and one V potential generated by the other worldline, such as $\hat{X}^{(V_1 V_1 V_2)} \propto m_1^2 m_2$, pose a somewhat delicate problem. More precisely, it is easily seen that the only dangerous part in $\hat{X}^{(V_1 V_1 V_2)}$, considered near the first worldline, is of the form $f(\boldsymbol{x})/r_1^{(2+2\varepsilon)}$ in dimension $4+\varepsilon$, where $f(\boldsymbol{x})$ denotes a smooth function. [Here, we add back the particle label indicating whether the expansions Eqs. (3.10), (3.11) refer to the first ($A = 1$), or the second ($A = 2$) particle. The appropriate label should be added both on r and n in Eqs. (3.10), (3.11): $r^k \hat{n}^L \to r_A^k \hat{n}_A^L$.] The problem is that the power of $1/r_1$ in this ε-singular term becomes an *even integer* when $\varepsilon \to 0$. When inserting the Taylor expansion of $f(\boldsymbol{x})$, say $f(\boldsymbol{x}) \sim \sum r_1^{\ell+2n} \hat{n}_1^L f_L^{\ell+2n}$, some of the terms in the ε-singular contribution $f(\boldsymbol{x})/r_1^{(2+2\varepsilon)}$ might be of the form $r_1^{\ell+2n'-2\varepsilon} \hat{n}_1^L$, with $n' = n-1 \geq 0$, and might then contribute to the Hadamard-regular part of $\hat{X}^{(V_1V_1V_2)}$ in the limit $\varepsilon \to 0$. This would mean that the Hadamard-regular part of $\hat{X}^{(V_1V_1V_2)}$ would not coincide with its ε-regular part. We already know from Refs. [38, 39]), which used Hadamard regularization to derive the 2PN-accurate dynamics and found the same result (modulo gauge effects) as the analytic-continuation derivation of Ref. [24], that this is not the case for the regularized values of $\hat{X}^{(V_1V_1V_2)}$ and of its first derivatives on the first worldline. [Indeed, these quantities enter the computation of the equations of motion.] On the other hand, the computations that we shall do here involve higher spatial derivatives of $\hat{X}$, and it is important to check that we can safely use Hadamard regularization to evaluate them. This can be proven by using the techniques explained in Ref. [24], based on iteratively considering the singular terms in $\hat{W}_{ij}^{(VV)}$ and $\hat{X}^{(VVV)}$ generated by the singular local behaviour (near the first worldline) of their respective source terms. One finds then that the smooth function $f(\boldsymbol{x})$ entering the dangerous terms $f(\boldsymbol{x})/r_1^{(2+2\varepsilon)}$ in $\hat{X}^{(V_1V_1V_2)}$ is of the *special* form $f(\boldsymbol{x}) \sim \sum c_\ell G_L r_1^\ell n_1^L$ in dimension $4+\varepsilon$, with $\ell \geq 1$, where $G_L \equiv \partial_L V_2$ denotes the ℓ-th tidal gradient (considered near the first worldline) of the V potential generated by the second worldline. When working (as we do) at the 2PN accuracy, we can take V at Newtonian order, and the gradients $G_L \simeq [\partial_L(Gm_2/r_2^{(1+\varepsilon)})]_1$ are then *traceless*: $G_L = G_{a_1a_2\cdots a_\ell} = G_{\langle a_1a_2\cdots a_\ell\rangle}$. As a consequence, it is immediately seen that, in the limit $\varepsilon \to 0$, the potentially dangerous term $f(\boldsymbol{x})/r_1^{(2+2\varepsilon)}$ in $\hat{X}^{(V_1V_1V_2)}$ *does not* give any contribution to the Hadamard-regular part of $\hat{X}$. This means that we can compute the ε-regularized reduced tidal action in (2.12) by replacing, from the start, the brick potentials $V(x), V_i(x), \ldots$, by their Hadamard-regularized counterparts, $R\,[V(x)], R\,[V_i(x)], \ldots$

Summarizing: The A-worldline part of the tidal action Eq. (2.12) can be obtained by computing all its elements ($d\tau_A = c^{-1}(-g_{\mu\nu}(y_A)\, dy_A^\mu\, dy_A^\nu)^{1/2}, G_{\alpha\beta}^A, \ldots$) within the A-regular metric $g_{\mu\nu}^{A-\text{reg}}(x)$ obtained by replacing each 2PN brick potential $V(x), V_i(x), \ldots$ by its A-Hadamard-regular part $R_A[V(x)], R_A[V_i(x)], \ldots$

As a check on our results (and on the many complicated algebraic operations needed to derive them) we have also re-computed the electric-quadrupole tidal Lagrangian, $L_{\mu_A^{(2)}} = \frac{1}{4}\,(d\tau_A/dt)\, G_{\alpha\beta}^A\, G_A^{\alpha\beta}$ by effecting the Hadamard regularization in a different way. Our alternative computation was done by separately Hadamard-regularizing each factor entering the Lagrangian, $L_{\mu_A^{(2)}}$, when it is expressed in terms of $d\tau_A/dt$, the contravariant metric, the covariant Riemann tensor, and the contravariant 4-velocity. More precisely, we first calculated $G_{\alpha\beta}(y_A)$ as $-R_A[R_{\alpha\mu\beta\nu}]\, R_A[u^\mu] R_A[u^\nu]$, then we computed $[G_{ab}^2](y_A) \equiv G_{\alpha\beta}(y_A) G_{\mu\nu}(y_A) R_A[g^{\alpha\mu}] R_A[g^{\beta\nu}]$, which we inserted into the expression of $L_{\mu_A^{(2)}}$ just written. The remaining factor, $(d\tau_A/dt)/4$, was taken to be $R_A[d\tau/dt]/4$. Note in passing that, while one can a priori prove that the alternative regularization of $G_{\alpha\beta}(y_A)$ (and subsequently $[G_{ab}^2](y_A) \equiv G_{\alpha\beta}(y_A) G_{\mu\nu}(y_A) R_A[g^{\alpha\mu}] R_A[g^{\beta\nu}]$) just explained, must coincide with the one explained above, namely $(G_{\alpha\beta}\, G^{\alpha\beta})[R_A(V), R_A(V_i), \ldots]$ (because both of them agree with the Riesz-analytic-regularization and/or dimensional-regularization) a different result would have been obtained if one had postponed the Hadamard regularization of the squared tidal quadrupole to the last moment, i.e. if one had computed $R_A[G_{\alpha\beta}\, G^{\alpha\beta}]$. [Such a difference occurs because of the appearance of a dangerous nonlinear mixing of Hadamard-regular and Hadamard-singular parts in $\partial_{ij} V \partial_{ij} \hat{X}^{(V_1V_1V_2)}$ (with the special structure of the delicate terms in $\hat{X}^{(V_1V_1V_2)}$ given above). This shows again the consistency problems of the Hadamard regularization, when it is used beyond the types of calculations where it is equivalent to the Riesz analytic regularization (or to dimensional regularization).]

D. Explicit rules for computing the regular parts of the 2PN brick potentials

Let us now give some indications on the computation of the regular parts of the various brick potentials $V(x), V_i(x), \ldots$

1. Regularizing V and V_i

The situation is very simple for the "linear potentials" V and V_i, which satisfy linear equations with delta-function sources [see Eqs. (3.3)]. Near, say, the particle $A = 1$, the A-regular parts of V and V_i are the terms in Eqs. (A1), (A2) which are generated by the source terms $\propto \delta(\boldsymbol{x} - \boldsymbol{y}_2)$ of the second particle. It is indeed easily seen [from the definition in Eq. (3.10)] that the 1-regular part of all the terms explicitly written in Eq. (A1) vanishes, while all the non-explicitly written terms obtained by the $1 \leftrightarrow 2$ exchange are regular near the particle 1. The same is true for V_i, Eq. (A2). A simple rule for obtaining these results is to note that, from the definition in Eq. (3.11), any term of the form

$$r_1^{2k+1}\, f(x)\,, \quad k \in \mathbb{Z}\,, \tag{3.12}$$

where $f(x)$ is a smooth function of x^μ (near $\boldsymbol{x} = \boldsymbol{y}_1$ at fixed instant t), and where the power of r_1 is *odd*, is purely singular.

The situation is more complicated for the higher-order potentials $\hat{W}_{ij}$ and $\hat{R}_i$, whose sources contain both compact terms $\propto \delta(\boldsymbol{x}-\boldsymbol{y}_A)$, and quadratically nonlinear noncompact ones $\propto \partial V\,\partial V$, and still more complicated for the $\hat{X}$ potential whose source even depends on the previous $\hat{W}_{ij}$ potential.

2. Regularizing $\hat{W}_{ij}$

The potential $\hat{W}_{ij}$ can be decomposed in powers of the masses. It contains terms proportional to m_1, m_2, m_1^2, m_2^2 and $m_1\,m_2$. It is easily seen that while the terms proportional to m_1 and m_1^2 are 1-singular, the terms proportional to m_2 and m_2^2 are 1-regular. It is more delicate to decompose the mixed terms $\propto m_1\,m_2$ into 1-regular (R_1) and 1-singular (S_1) parts. More precisely the $m_1\,m_2$ part of $\hat{W}_{ij}$ has the form

$$\hat{W}_{ij}^{[m_1 m_2]} = \hat{W}_{ij(0)}^{[m_1 m_2]} + \tilde{\hat{W}}_{ij(0)}^{[m_1 m_2]} \qquad (3.13)$$

where

$$\begin{aligned}\hat{W}_{ij}{}_{(0)}^{[m_1 m_2]} &= \frac{1}{r_{12}S}\delta^{ij} + \left\{\frac{1}{S^2}\left(n_1^{(i}n_2^{j)} + 2n_1^{(i}n_{12}^{j)}\right)\right.\\ &\quad \left. - n_{12}^i n_{12}^j\left(\frac{1}{S^2} + \frac{1}{r_{12}S}\right)\right\}\\ &\equiv \frac{1}{r_{12}S}P(n_{12})^{ij}\\ &\quad + \frac{1}{S^2}\left(n_1^{(i}n_2^{j)} + 2n_1^{(i}n_{12}^{j)} - n_{12}^i n_{12}^j\right), \qquad (3.14)\end{aligned}$$

$$\begin{aligned}\tilde{\hat{W}}_{ij}{}_{(0)}^{[m_1 m_2]} &= \frac{1}{r_{12}S}\delta^{ij} + \left\{\frac{1}{S^2}\left(n_2^{(i}n_1^{j)} - 2n_2^{(i}n_{12}^{j)}\right)\right.\\ &\quad \left. - n_{12}^i n_{12}^j\left(\frac{1}{S^2} + \frac{1}{r_{12}S}\right)\right\}\\ &\equiv \frac{1}{r_{12}S}P(n_{12})^{ij}\\ &\quad + \frac{1}{S^2}\left(n_2^{(i}n_1^{j)} - 2n_2^{(i}n_{12}^{j)} - n_{12}^i n_{12}^j\right), \qquad (3.15)\end{aligned}$$

and where $P(n_{12})^{ij} \equiv \delta^{ij} - n_{12}^i\,n_{12}^j$ denotes the projector orthogonal to the unit vector $\boldsymbol{n}_{12}$. [The decomposition in Eq. (3.13) simply corresponds to the decomposition of Eq. (A4) into an explicitly written term and its $1 \leftrightarrow 2$ counterpart.] Here we see that there appear (modulo x-independent factors, such as r_{12}^{-1}, n_{12}^i, $P(n_{12})^{ij}, \ldots$) terms of the type

$$\frac{1}{S}\,,\quad \frac{1}{S^2}\,,\quad \frac{n_1^i}{S^2}\,,\quad \frac{n_2^i}{S^2}\,,\quad \frac{n_1^i\,n_2^j}{S^2}\,, \qquad (3.16)$$

where we recall that $S \equiv r_1 + r_2 + r_{12}$. Near particle 1, n_2^i is a smooth function, while $n_1^i = r_1^i/r_1$ is the ratio of a smooth function ($r_1^i = x^i - y_1^i$) by r_1. In other words, the five terms listed in Eq. (3.16) are of *three* different types:

$$\frac{1}{S}\,,\quad \frac{f(x)}{S^2} \quad \text{and} \quad \frac{f(x)}{r_1\,S^2}\,, \qquad (3.17)$$

where $f(x)$ denotes a generic smooth function near particle 1. [As we always consider the neighborhood of particle 1, we do not add an index to $f(x)$ to recall that it is 1-regular, but might be singular near particle 2.] Because $S = r_1 + r_2 + r_{12}$ is a function of "mixed character" (partly regular and partly singular), it is not immediate to decompose the functions in Eq. (3.17) into 1-regular and 1-singular parts. [This mixed character of S is deeply linked with the fact that it enters the 2PN metric because of the basic fact that a solution of $\Delta g = r_1^{-1}\,r_2^{-1}$ is $g = \ln S$.] A simple (though somewhat brute-force) way of extracting the regular parts of the functions in Eq. (3.17) consists of decomposing S into

$$S \equiv S_0 + r_1 = S_0\left(1 + \frac{r_1}{S_0}\right)\,, \qquad (3.18)$$

with

$$S_0 \equiv r_2 + r_{12}\,, \qquad (3.19)$$

(note that S_0 is a smooth function near particle 1), and then expanding S^{-n} in powers of r_1/S_0. Namely

$$\frac{1}{S} = \frac{1}{S_0}\left(1 - \frac{r_1}{S_0} + \frac{r_1^2}{S_0^2} - \frac{r_1^3}{S_0^3} + \ldots\right)\,, \qquad (3.20)$$

$$\frac{1}{S^2} = \frac{1}{S_0^2}\left(1 - 2\,\frac{r_1}{S_0} + 3\,\frac{r_1^2}{S_0^2} - 4\,\frac{r_1^3}{S_0^3} + \ldots\right)\,, \qquad (3.21)$$

and more generally

$$\begin{aligned}\frac{1}{S^n} = \frac{1}{S_0^n}\Bigg(1 &- n\frac{r_1}{S_0} + \frac{(n+1)n}{2}\left(\frac{r_1}{S_0}\right)^2\\ &- \frac{(n+2)(n+1)n}{3!}\left(\frac{r_1}{S_0}\right)^3\\ &+ \frac{(n+3)(n+2)(n+1)n}{4!}\left(\frac{r_1}{S_0}\right)^4 + \ldots\Bigg)\,,\\ n = 1, 2, \ldots & \qquad (3.22)\end{aligned}$$

Using these expansions, together with the rule that terms of the form in Eq. (3.12) are purely singular, it is easy to derive the following results for the 1-regular parts of functions of the type in Eq. (3.17), and, more generally, of the types $f(x)/S$, $f(x)/S^2$, $f(x)/(r_1\,S)$ and $f(x)/(r_1\,S^2)$:

$$\left(\frac{f(x)}{S}\right)_R = \frac{f(x)}{S_0}\left(1 + \left(\frac{r_1}{S_0}\right)^2 + \left(\frac{r_1}{S_0}\right)^4 + \left(\frac{r_1}{S_0}\right)^6 + \ldots\right)$$

$$\equiv f(x) \left(\frac{1}{S}\right)_R , \quad (3.23)$$

$$\left(\frac{f(x)}{S^2}\right)_R = \frac{f(x)}{S_0^2}\left(1 + 3\left(\frac{r_1}{S_0}\right)^2 + 5\left(\frac{r_1}{S_0}\right)^4 + 7\left(\frac{r_1}{S_0}\right)^6 + \dots\right) \equiv f(x)\left(\frac{1}{S^2}\right)_R , \quad (3.24)$$

$$\left(\frac{f(x)}{r_1 S}\right)_R = -\frac{f(x)}{S_0^2}\left(1 + \left(\frac{r_1}{S_0}\right)^2 + \left(\frac{r_1}{S_0}\right)^4 + \dots\right) \equiv f(x)\left(\frac{1}{r_1 S}\right)_R , \quad (3.25)$$

$$\left(\frac{f(x)}{r_1 S^2}\right)_R = -\frac{f(x)}{S_0^3}\left(2 + 4\left(\frac{r_1}{S_0}\right)^2 + 6\left(\frac{r_1}{S_0}\right)^4 + 8\left(\frac{r_1}{S_0}\right)^6 + \dots\right) \equiv f(x)\left(\frac{1}{r_1 S^2}\right)_R . \quad (3.26)$$

Here, we use a lower R subscript $(\varphi(x))_R$ to denote the 1-regular part of a function $\varphi(x)$ (above denoted as $R\,[\varphi(x)]$). (We omit decorating R with a label 1, but one should remember that we are always talking about the 1-regular part of $\varphi(x)$.)

Note that, as indicated, all the terms above have the simple property that the regular-projection operator R *commutes* with the multiplication by a smooth function, e.g. $R\,[f(x)\,S^{-1}] = f(x)\,R\,[S^{-1}]$. Beware that this property is true only for the special singular terms considered here. We shall later see that more-complicated singular terms (entering the $\hat{X}$ potential) do not satisfy this simple commutativity property.

Note that the number of terms one needs to retain in the above expansions depends on the quantity one wants to evaluate on the first worldline. For instance, when evaluating $G^1_{\alpha\beta}$, which involves the curvature tensor, and therefore two spatial derivatives of the metric (and, in particular, of $R\,[\hat{W}_{ij}]$), we need to include enough terms to ensure that $R\,[\hat{W}_{ij}]$ is C^2 near $\boldsymbol{x} = \boldsymbol{y}_1$. Actually, we shall push our calculations up to the level of $G^1_{\alpha\beta\gamma}$, which depends on the first covariant derivative of the curvature tensor, and we shall therefore need all the brick potentials to be at least C^3 near $\boldsymbol{x} = \boldsymbol{y}_1$.

The application of the above results yields the following explicit expressions for the 1-regular part of the two separate $O(m_1 m_2)$ delicate contributions to $\hat{W}_{ij}$ [defined in Eqs. (3.13)–(3.15)]:

$$\begin{aligned}[\hat{W}_{ij}{}^{[m_1m_2]}_{(0)}]_R &= \frac{P(n_{12})^{ij}}{r_{12}}\frac{1}{S_0}\left(1 + \frac{r_1^2}{S_0^2} + \frac{r_1^4}{S_0^4} + \dots\right) \\ &+ \frac{\mathbf{r}_1^{(i}n_2^{j)}}{S_0^2}\left(-\frac{2}{S_0} - 4\frac{r_1^2}{S_0^3} - 6\frac{r_1^4}{S_0^5} + \dots\right) \\ &+ 2\frac{\mathbf{r}_1^{(i}n_{12}^{j)}}{S_0^2}\left(-\frac{2}{S_0} - 4\frac{r_1^2}{S_0^3} - 6\frac{r_1^4}{S_0^5} + \dots\right) \\ &- n_{12}^i n_{12}^j \frac{1}{S_0^2}\left(1 + 3\frac{r_1^2}{S_0^2} + 5\frac{r_1^4}{S_0^4} + \dots\right) , \end{aligned} \quad (3.27)$$

$$\begin{aligned}[\tilde{\hat{W}}_{ij}{}^{[m_1m_2]}_{(0)}]_R &= \frac{P(n_{12})^{ij}}{r_{12}}\frac{1}{S_0}\left(1 + \frac{r_1^2}{S_0^2} + \frac{r_1^4}{S_0^4} + \dots\right) \\ &+ \frac{n_2^{(i}\mathbf{r}_1^{j)}}{S_0^2}\left(-\frac{2}{S_0} - 4\frac{r_1^2}{S_0^3} - 6\frac{r_1^4}{S_0^5} + \dots\right) \\ &- 2n_2^{(i}n_{12}^{j)}\frac{1}{S_0^2}\left(1 + 3\frac{r_1^2}{S_0^2} + 5\frac{r_1^4}{S_0^4} + \dots\right) \\ &- n_{12}^i n_{12}^j \frac{1}{S_0^2}\left(1 + 3\frac{r_1^2}{S_0^2} + 5\frac{r_1^4}{S_0^4} + \dots\right) . \end{aligned} \quad (3.28)$$

3. Regularizing $\hat{R}_i$

As the potential $\hat{R}_i$ has a source of the same type as $\hat{W}_{ij}$ (namely $\delta(\boldsymbol{x} - \boldsymbol{y}_A)$ terms plus a non-compact term quadratic in the V potentials), the calculation of its regular part can be done in exactly the same way as $\hat{W}_{ij}$. $\hat{R}_i$ contains terms $\propto m_1^2, m_2^2$ and $m_1 m_2$. The $O(m_1^2)$ piece is purely singular, the $O(m_2^2)$ one is purely regular, while the $O(m_1\, m_2)$ one is a mix of regular and singular terms. As above, we can decompose the $m_1\, m_2$ part of $\hat{R}_i$ in two pieces, say

$$\hat{R}_i^{[m_1m_2]} = \hat{R}_{i(0)}^{[m_1m_2]} + \tilde{\hat{R}}_{i(0)}^{[m_1m_2]} , \quad (3.29)$$

where

$$\begin{aligned}\hat{R}_{i(0)}^{[m_1m_2]} &= n_{12}^i\left\{-\frac{(n_{12}v_1)}{2S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right) - \frac{2(n_2v_1)}{S^2} + \frac{3(n_2v_2)}{2S^2}\right\} \\ &+ n_1^i\frac{1}{S^2}\left(2(n_{12}v_1) - \frac{3(n_{12}v_2)}{2} + 2(n_2v_1) - \frac{3(n_2v_2)}{2}\right) \\ &+ v_1^i\left(\frac{1}{r_1r_{12}} + \frac{1}{2r_{12}S}\right) - v_2^i\frac{1}{r_1r_{12}} , \end{aligned} \quad (3.30)$$

$$\tilde{\hat{R}}_{i(0)}^{[m_1m_2]} = -n_{12}^i\left\{\frac{(n_{12}v_2)}{2S}\left(\frac{1}{S} + \frac{1}{r_{12}}\right) - \frac{2(n_1v_2)}{S^2} + \frac{3(n_1v_1)}{2S^2}\right\}$$

$$+ n_2^i \frac{1}{S^2}\left(-2(n_{12}v_2) + \frac{3(n_{12}v_1)}{2} + 2(n_1v_2) - \frac{3(n_1v_1)}{2}\right) + v_2^i\left(\frac{1}{r_2r_{12}} + \frac{1}{2r_{12}S}\right) - v_1^i\frac{1}{r_2r_{12}}\,. \quad (3.31)$$

Applying the above results then yields the following expressions for the 1-regular parts of these quantities:

$$[\hat{R}_{i\,(0)}^{[m_1m_2]}]_R = n_{12}^i\left\{\left[-\frac{(n_{12}v_1)}{2} - 2(n_2v_1) + \frac{3(n_2v_2)}{2}\right]\left(\frac{1}{S^2}\right)_R - \frac{(n_{12}v_1)}{2r_{12}}\left(\frac{1}{S}\right)_R\right\} + \left(2(n_{12}v_1) - \frac{3(n_{12}v_2)}{2} + 2(n_2v_1) - \frac{3(n_2v_2)}{2}\right)\mathbf{r}_1^i\left(\frac{1}{r_1S^2}\right)_R + \frac{v_1^i}{2r_{12}}\left(\frac{1}{S}\right)_R\,, \quad (3.32)$$

$$[\tilde{\hat{R}}_{i\,(0)}^{[m_1m_2]}]_R = -n_{12}^i\left\{\frac{(n_{12}v_2)}{2}\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right) - 2(\mathbf{r}_1v_2)\left(\frac{1}{r_1S^2}\right)_R + \frac{3}{2}(\mathbf{r}_1v_1)\left(\frac{1}{r_1S^2}\right)_R\right\}$$

$$+ n_2^i\left[\left(-2(n_{12}v_2) + \frac{3(n_{12}v_1)}{2}\right)\left(\frac{1}{S^2}\right)_R + \left(2(\mathbf{r}_1v_2) - \frac{3(\mathbf{r}_1v_1)}{2}\right)\left(\frac{1}{r_1S^2}\right)_R\right] + v_2^i\left(\frac{1}{r_2r_{12}} + \frac{1}{2r_{12}}\left(\frac{1}{S}\right)_R\right) - v_1^i\frac{1}{r_2r_{12}}\,. \quad (3.33)$$

One should substitute the expansions in Eqs. (3.23)–(3.26) into these results to get their explicit forms.

4. Regularizing $\hat{X}$

Finally, we come to the most complicated 2PN brick potential, namely $\hat{X}$. It contains contributions proportional to $m_1^2, m_1\,m_2, m_2^2$; $m_1^3, m_1^2\,m_2, m_1\,m_2^2$ and m_2^3 (see Eq. (A5)). The terms in $m_1^2, m_2^2, m_1^3, m_2^3$ are easily dealt with (they are either purely singular or purely regular). Many, but not all, of the $m_1\,m_2$ terms can be dealt with in the same way as the $m_1\,m_2$ terms in $\hat{W}_{ij}$ and $\hat{R}_i$. If we again decompose $\hat{X}^{[m_1m_2]}$ in two pieces

$$\hat{X}^{[m_1m_2]} = \hat{X}^{[m_1m_2]}_{(0)} + \tilde{\hat{X}}^{[m_1m_2]}_{(0)}\,, \quad (3.34)$$

we have the following results for their regular parts:

$$\begin{aligned}[\hat{X}^{[m_1m_2]}_{(0)}]_R \;=\;& v_1^2\left[\left(\frac{1}{r_1S}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right] + v_2^2\left[\left(\frac{1}{r_1S}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right] - (v_1v_2)\left(2\left(\frac{1}{r_1S}\right)_R + \frac{3}{2r_{12}}\left(\frac{1}{S}\right)_R\right)\\
&-(n_{12}v_1)^2\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right) - (n_{12}v_2)^2\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right)\\
&+\frac{3(n_{12}v_1)(n_{12}v_2)}{2}\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right) + 2(n_{12}v_1)(\mathbf{r}_1v_1)\left(\frac{1}{r_1S^2}\right)_R\\
&-5(n_{12}v_2)(\mathbf{r}_1v_1)\left(\frac{1}{r_1S^2}\right)_R - (\mathbf{r}_1v_1)^2\left(\frac{1}{r_1^2S^2} + \frac{1}{r_1^3S}\right)_R + 2(n_{12}v_2)(\mathbf{r}_1v_2)\left(\frac{1}{r_1S^2}\right)_R\\
&+2(\mathbf{r}_1v_1)(\mathbf{r}_1v_2)\left(\frac{1}{r_1^2S^2} + \frac{1}{r_1^3S}\right)_R - (\mathbf{r}_1v_2)^2\left(\frac{1}{r_1^2S^2} + \frac{1}{r_1^3S}\right)_R - 2(n_{12}v_2)(n_2v_1)\left(\frac{1}{S^2}\right)_R\\
&+2(\mathbf{r}_1v_2)(n_2v_1)\left(\frac{1}{r_1S^2}\right)_R - \frac{3}{2}(\mathbf{r}_1v_1)(n_2v_2)\left(\frac{1}{r_1S^2}\right)_R\,,\end{aligned} \quad (3.35)$$

$$\begin{aligned}[\tilde{\hat{X}}^{[m_1m_2]}_{(0)}]_R \;=\;& v_2^2\left(\frac{1}{r_2r_{12}} + \frac{1}{r_2}\left(\frac{1}{S}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right) + v_1^2\left(-\frac{1}{r_2r_{12}} + \frac{1}{r_2}\left(\frac{1}{S}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right)\\
&-(v_1v_2)\left(\frac{2}{r_2} + \frac{3}{2r_{12}}\right)\left(\frac{1}{S}\right)_R - (n_{12}v_2)^2\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right)\\
&-(n_{12}v_1)^2\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right) + \frac{3(n_{12}v_2)(n_{12}v_1)}{2}\left(\left(\frac{1}{S^2}\right)_R + \frac{1}{r_{12}}\left(\frac{1}{S}\right)_R\right)\end{aligned}$$

$$-2(n_{12}v_2)(n_2v_2)\left(\frac{1}{S^2}\right)_R+5(n_{12}v_1)(n_2v_2)\left(\frac{1}{S^2}\right)_R$$
$$-(n_2v_2)^2\left(\left(\frac{1}{S^2}\right)_R+\frac{1}{r_2}\left(\frac{1}{S}\right)_R\right)-2(n_{12}v_1)(n_2v_1)\left(\frac{1}{S^2}\right)_R$$
$$+2(n_2v_2)(n_2v_1)\left(\left(\frac{1}{S^2}\right)_R+\frac{1}{r_2}\left(\frac{1}{S}\right)_R\right)-(n_2v_1)^2\left(\left(\frac{1}{S^2}\right)_R+\frac{1}{r_2}\left(\frac{1}{S}\right)_R\right)$$
$$+2(n_{12}v_1)(\mathbf{r}_1v_2)\left(\frac{1}{r_1S^2}\right)_R+2(n_2v_1)(\mathbf{r}_1v_2)\left(\frac{1}{r_1S^2}\right)_R-\frac{3}{2}(n_2v_2)(\mathbf{r}_1v_1)\left(\frac{1}{r_1S^2}\right)_R. \tag{3.36}$$

One should substitute the expansions in Eqs. (3.23)–(3.26) into the corresponding terms in Eqs. (3.35), (3.36). However, these equations involve new types of terms, not discussed above. These new terms are of the form

$$f(x)\left(\frac{1}{r_1^2\,S^2}+\frac{1}{r_1^3\,S}\right). \tag{3.37}$$

The fact that we have a specific combination of $r_1^{-2}\,S^{-2}$ and $r_1^{-3}\,S^{-1}$ simplifies things. Indeed, using the expansions in Eqs. (3.20), (3.21) above we have

$$\frac{f(x)}{r_1^2\,S^2}=\frac{f(x)}{S_0^4}\left[\frac{S_0^2}{r_1^2}-2\,\frac{S_0}{r_1}+3-4\,\frac{r_1}{S_0}+5\,\frac{r_1^2}{S_0^2}+\dots\right], \tag{3.38}$$

$$\frac{f(x)}{r_1^3\,S}=\frac{f(x)}{S_0^4}\left[\frac{S_0^3}{r_1^3}-\frac{S_0^2}{r_1^2}+\frac{S_0}{r_1}-1+\frac{r_1}{S_0}-\frac{r_1^2}{S_0^2}+\dots\right]. \tag{3.39}$$

When summing these two equations we see that the terms $\propto 1/r_1^2$ cancel. We shall deal later with these terms, which turn out to be delicate to handle but, anyway, in the sum of Eqs. (3.38) and (3.39), they cancel out. The remaining terms contain either an odd power of r_1 [and are therefore *purely* singular, Eq. (3.12)] or a *positive*, *even* power of r_1 (which makes them purely regular). As a consequence, the regular part of the combination of Eq. (3.37) reads

$$\left[f(x)\left(\frac{1}{r_1^2S^2}+\frac{1}{r_1^3S}\right)\right]_R=\frac{f(x)}{S_0^4}\left(2+4\left(\frac{r_1}{S_0}\right)^2+6\left(\frac{r_1}{S_0}\right)^4+8\left(\frac{r_1}{S_0}\right)^6+\dots\right)\equiv f(x)\left(\frac{1}{r_1^2S^2}+\frac{1}{r_1^3S}\right)_R. \tag{3.40}$$

Note that, thanks to the cancellation of the $1/r_1^2$ terms, we have again a property of commutativity $R\,[f(x)\,\varphi(x)]=f(x)\,R\,[\varphi(x)]$, for the special type of terms $\varphi(x)$ entering Eq. (3.37).

Concerning the $m_1\,m_2^2$ contribution to $\hat{X}$, it is the sum of

$$\hat{X}_{(0)}^{[m_1m_2^2]}=-\frac{1}{2r_{12}^3}+\frac{r_2}{2r_1r_{12}^3}-\frac{1}{2r_1r_{12}^2} \tag{3.41}$$

and

$$\tilde{\hat{X}}_{(0)}^{[m_1m_2^2]}=\frac{1}{2r_2^3}+\frac{1}{16r_1^3}+\frac{1}{16r_2^2r_1}-\frac{r_1^2}{2r_2^2r_{12}^3}+\frac{r_1^3}{2r_2^3r_{12}^2}-\frac{r_2^2}{32r_1^3r_{12}^2}-\frac{3}{16r_1r_{12}^2}+\frac{15r_1}{32r_2^2r_{12}^2}-\frac{r_1^2}{2r_2^3r_{12}^2}-\frac{r_1}{2r_2^3r_{12}}-\frac{r_{12}^2}{32r_2^2r_1^3}. \tag{3.42}$$

Using the rule of Eq. (3.12), we easily see that each term clearly is either purely regular or purely singular. Computing the regular part of $\hat{X}^{[m_1m_2]}$ is then easy.

The most delicate contribution to $\hat{X}$ is its $O(m_1^2\,m_2)$ one, which can again be written as the sum of

$$\hat{X}_{(0)}^{[m_1^2m_2]}=\frac{1}{2r_1^3}+\frac{1}{16r_2^3}+\frac{1}{16r_1^2r_2}-\frac{r_2^2}{2r_1^2r_{12}^3}+\frac{r_2^3}{2r_1^3r_{12}^3}-\frac{r_1^2}{32r_2^3r_{12}^2}-\frac{3}{16r_2r_{12}^2}+\frac{15r_2}{32r_1^2r_{12}^2}-\frac{r_2^2}{2r_1^3r_{12}^2}-\frac{r_2}{2r_1^3r_{12}}-\frac{r_{12}^2}{32r_1^2r_2^3} \tag{3.43}$$

and

$$\tilde{\hat{X}}_{(0)}^{[m_1^2m_2]}=-\frac{1}{2r_{12}^3}+\frac{r_1}{2r_2r_{12}^3}-\frac{1}{2r_2r_{12}^2}. \tag{3.44}$$

Actually the part $\tilde{\hat{X}}_{(0)}^{[m_1^2m_2]}$ is easy to discuss: Its regular part is obtained simply by discarding the term: $r_1/(2\,r_2\,r_{12}^3)$. Similarly, most of the terms in $\hat{X}_{(0)}^{[m_1^2m_2]}$ are easy to treat, being either purely regular or purely singular because of Eq. (3.12). However, the third, fourth, eighth and last terms in the right hand side of Eq. (3.43) are somewhat tricky. [These terms correspond to the "dangerous terms" in $\hat{X}^{(V_1V_1V_2)}$ that were discussed in Sec. III when making the link between the ε-regularization and the Hadamard one.] The third term is

$$Q\equiv\frac{1}{16\,r_1^2\,r_2}, \tag{3.45}$$

while the sum of the fourth, eighth and last terms reads

$$P\equiv-\frac{r_2^2}{2\,r_1^2\,r_{12}^3}+\frac{15\,r_2}{32\,r_1^2\,r_{12}^2}-\frac{r_{12}^2}{32\,r_1^2\,r_2^3}. \tag{3.46}$$

Both Q and P are of the form $f(x)/r_1^2$ (but we shall see that Q is special compared with P). The computation of the regular part of $f(x)/r_1^2$ is a bit subtle. It can, however, be done by brute force, namely by replacing the smooth function $f(x)$ by its Taylor expansion around $\boldsymbol{y}_1$:

$$\begin{aligned} f(x) &= f(\boldsymbol{y}_1) + r_1^i\,\partial_i\, f(\boldsymbol{y}_1) + \frac{1}{2!}\, r_1^i\, r_1^j\, \partial_{ij}\, f(\boldsymbol{y}_1) \\ &+ \frac{1}{3!}\, r_1^i\, r_1^j\, r_1^k\, \partial_{ijk}\, f(\boldsymbol{y}_1) + \dots \end{aligned} \tag{3.47}$$

When replacing $r_1^i \to r_1\, n_1^i$ and dividing by r_1^2, one sees that the regular part of $f(x)/r_1^2$ will only come from the terms $r_1^L \equiv r_1^{i_1 i_2 \dots i_\ell}$ with $\ell = 2, 4, 6, \dots$ Moreover, by decomposing $r_1^L = r_1^\ell\, n_1^L$ in irreducible tensorial parts, as in

$$r_1^{ij} = r_1^2\, n_1^{ij} = r_1^2 \left[n_1^{\langle ij \rangle} + \frac{1}{3}\,\delta^{ij} \right], \tag{3.48}$$

where $n_1^{\langle ij \rangle} \equiv \hat{n}_1^{ij} \equiv n_1^{ij} - \frac{1}{3}\,\delta^{ij}$ denotes the symmetric trace-free projection of $n_1^{ij} \equiv n_1^i\, n_1^j$, we see [in view of the definition in Eq. (3.10)] that only the pieces containing at least one Kroneker δ in the decomposition of n_1^L will contribute to the regular part. For instance, in the case $\ell = 2$, only the δ^{ij} in Eq. (3.47) will contribute to the regular part of $f(x)/r_1^2$. More generally, we have that $R[r_1^L/r_1^2] = (r_1^L - \hat{r}_1^L)/r_1^2$.

Applying this method yields the following result (here written with the simplified notation used around Eq. (3.8)) for the regular part of $f(x)/r_1^2$:

$$\begin{aligned} \left(\frac{f(x)}{r^2}\right)_R &= \frac{1}{6}\Delta f(0) + \frac{1}{10} x^i \partial_i \Delta f(0) + \frac{1}{28} \hat{x}^{ij} \partial_{ij} \Delta f(0) \\ &+ \frac{1}{120} r^2 \Delta^2 f(0) + \frac{1}{108} \hat{x}^{ijk} \partial_{ijk} \Delta f(0) \\ &+ \frac{1}{280} r^2 x^i \partial_i \Delta^2 f(0) + O(x^4)\,. \end{aligned} \tag{3.49}$$

As one sees in Eq. (3.49) (and as can be proven to all orders), all the terms on the right-hand side of Eq. (3.49) are derivatives of the Laplacian of $f(x)$ (taken at $\boldsymbol{x} = \boldsymbol{y}_1$). As a consequence, in the particular case where $\Delta f(x) = 0$, the regular part of $f(x)/r_1^2$ is exactly zero. This is the case for the term Q in $\hat{X}^{[m_1^2 m_2]}$, Eq. (3.45). [Let us point out in passing that the discussion in Sec. IIIC of the link between the ε-regularization and the Hadamard one essentially consisted in remarking that all the "dangerous" terms in $\hat{X}^{[m_1^2 m_2]}$ had this innocuous structure $f(x)/r_1^{(2+2\varepsilon)}$ with $\Delta f(x) = 0$.] Therefore, we have simply

$$Q_R = 0\,. \tag{3.50}$$

On the other hand, this is not the case for the term P, Eq. (3.46). The evaluation of the regular part of P needs to appeal to the result in Eq. (3.49) and yields (modulo terms of order $O(\boldsymbol{r}_1^4)$ that will not be needed in our calculations)

$$\begin{aligned} P_R &= -\frac{1}{2r_{12}^3}\left(\frac{r_2^2}{r_1^2}\right)_R + \frac{15}{32 r_{12}^2}\left(\frac{r_2}{r_1^2}\right)_R - \frac{r_{12}^2}{32}\left(\frac{1}{r_1^2 r_2^3}\right)_R \\ &= -\frac{3}{8 r_{12}^3} + \frac{1}{r_{12}^5}\left[\frac{3}{224} r_1^2 - \frac{15}{112} (\mathbf{r}_1 n_{12})^2\right] \\ &+ \frac{5}{12 r_{12}^6} (\mathbf{r}_1 n_{12}) \left[(\mathbf{r}_1 n_{12})^2 - \frac{3}{8} r_1^2\right]. \end{aligned} \tag{3.51}$$

Summarizing: We have explicitly displayed all the rules needed to compute (near particle 1) the *regular* parts of the various brick potentials $V, V_i, \hat{W}_{ij}, \hat{R}_i, \hat{X}$ entering the 2PN metric. By replacing $V \to V_R, \dots, \hat{X} \to \hat{X}_R$, in Eq. (3.2), we define a *regularized* version of the 2PN metric generated by two point masses, $g_{\mu\nu}^R(x) \equiv g_{\mu\nu}[V_R(x), \dots, \hat{X}_R(x)]$, which is smooth near particle 1.

IV. COMPUTATION OF THE INVARIANTS ENTERING THE TIDAL ACTION

As we explained above, when neglecting terms quadratic in the tidal parameters $\mu^{(\ell)}$, etc., the tidal part of the two-body action is simply obtained by evaluating the $S_{\rm nonminimal}$, Eq. (2.12), as a function of the worldlines, by replacing the metric $g_{\mu\nu}(x)$ entering the right-hand side of Eq. (2.12) by the (regular part of the) point-mass metric $g_{\mu\nu}^{\rm point\,mass}(x, y_1, y_2, m_1, m_2)$. This reduced action is a sum over the various tidal parameters, $\mu_A^{(\ell)}$, $\sigma_A^{(\ell)}$, $\mu_A^{\prime(\ell)}, \dots$. We can therefore compute separately the part of the reduced action associated with each of them. This is what we shall do in this section for the actions associated with the parameters $\mu_{A=1}^{(\ell=2)}$, $\mu_{A=1}^{(\ell=3)}$, $\sigma_{A=1}^{(\ell=2)}$ and $\mu_{A=1}^{\prime(\ell=2)}$. [We shall only explicitly write down the results for $A = 1$ but, evidently, they also yield the results for $A = 2$ by exchanging $1 \leftrightarrow 2$.]

First, let us note that each action, say, associated with the parameter μ_1 related to the first worldline, is of the form

$$\mu_1 \int dt\, L_{\mu_1}(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{4.1}$$

where the Lagrangian L_{μ_1} is the product of a geometrical invariant by $d\tau_1/dt$. For instance

$$L_{\mu_1^{(2)}} = \frac{1}{4}\frac{d\tau_1}{dt}\left[G_{\alpha\beta}\, G^{\alpha\beta}\right]_1 \equiv \frac{1}{4}\frac{d\tau_1}{dt}\left[G_{ab}^2\right]_1. \tag{4.2}$$

We shall separately evaluate each geometrical invariant, $G_{ab}^2, G_{abc}^2, \dots$, before multiplying it by the (regularized) proper-time redshift factor $d\tau_1/dt$ ("Einstein time dilation"). Note also that we systematically work with the *order-reduced* 2PN metric, i.e. the 2PN metric in which the higher time derivatives of $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$ have been expressed in terms of positions and velocities only, $(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$, by iterative use of the (harmonic-gauge) equations of motion. As was discussed long ago, such an order reduction of the action is allowed, when it is understood that it corresponds to a certain additional change of coordinate gauge [45–47]. As we shall ultimately be interested in computing gauge-invariant quantities associated with the EOB reformulation of the dynamics, we do not need to keep track of this coordinate change.

A. Explicit 2PN-accurate tidal actions for general orbits

Let us start by discussing the simplest (and physically most important) geometric invariant, namely the one associated with the electric-type quadrupolar tide, say

$$\begin{aligned} J_{2e} &\equiv [G_{ab}\, G_{ab}]_1 = [G_{\alpha\beta}\, G^{\alpha\beta}]_1 \\ &= [R_{\alpha\mu\beta\nu}\, R_{\gamma\kappa\delta\lambda}\, g^{\alpha\gamma}\, g^{\beta\delta}\, u^\mu\, u^\nu\, u^\kappa\, u^\lambda]_1\,, \quad (4.3) \end{aligned}$$

where $u_1^\mu \equiv dy_1^\mu/d\tau_1 = (c, \boldsymbol{v}_1)\, dt/d\tau_1$, and where the subscript $2e$ on J_{2e} refers to "$\ell = 2$ electric." Using two independently written codes (one based on the Maple system, and the other one based on the Mathematica software supplemented by the package xAct [48]) we have computed the right-hand side of Eq. (4.3) within the (regularized) 2PN metric. (Actually, as explained above, the Mathematica code alternatively regularized, à la Hadamard, the value of $G_{\alpha\beta}$ computed with the full (non-regularized) 2PN metric.)

As the PN expansion of the quadrupolar tidal tensor Eq. (2.7) starts as

$$\begin{aligned} G_{ab} &= -c^2\, R_{a0b0} + \dots \\ &= +\frac{1}{2}\, c^2 (\partial_{ab}\, g_{00} - \partial_{a0}\, g_{b0} - \partial_{b0}\, g_{a0} + \partial_{00}\, g_{ab}) \\ &\quad + \dots\,, \end{aligned}$$

one sees that the 2PN-accurate metric [i.e., the knowledge of g_{00} up to $O(1/c^6)$ terms included, of g_{0a} up to $O(1/c^5)$, and of g_{ab} up to $O(1/c^4)$] is exactly what is needed to be able to compute G_{ab} to the 2PN (fractional) accuracy, i.e., $G_{ab} = +\partial_{ab}V + c^{-2}(\dots) + c^{-4}(\dots)$. The same is true for the higher *electric* tidal moments $G_{abc}, \dots$. However, one can easily see that one loses a PN order when evaluating either the *magnetic* tidal moments $H_{ab}, H_{abc}, \dots$ or the time-differentiated electric one $\dot{G}_{ab}, \dots$. The result we obtained, for general orbits, is

$$\begin{aligned} J_{2e} &= \frac{6G^2 m_2^2}{r_{12}^6} \Big\{ 1 + \frac{1}{c^2} \Big(-3(n_{12}v_{12})^2 - 3(n_{12}v_2)^2 + 3v_{12}^2 \\ &\qquad - \frac{G}{r_{12}}(5m_1 + 6m_2) \Big) \\ &+ \frac{1}{c^4} \Big[3(n_{12}v_{12})^4 + 12(n_{12}v_2)^2(n_{12}v_{12})^2 + 6(n_{12}v_2)^4 \\ &\quad -9v_{12}^2(n_{12}v_{12})^2 - 6(n_{12}v_{12})^2(v_2v_{12}) \\ &\quad -6(n_{12}v_2)(n_{12}v_{12})(v_2v_{12}) - 3v_2^2(n_{12}v_{12})^2 \\ &\quad -9v_{12}^2(n_{12}v_2)^2 - 3v_2^2(n_{12}v_2)^2 + 6v_{12}^4 \\ &\quad +6v_{12}^2(v_2v_{12}) + 3(v_2v_{12})^2 + 3v_2^2v_{12}^2 \\ &\quad + \frac{Gm_1}{r_{12}} \Big(-\frac{109}{4}(n_{12}v_{12})^2 + \frac{41}{2}(n_{12}v_2)^2 + \frac{21}{4}v_{12}^2 \Big) \\ &\quad + \frac{Gm_2}{r_{12}} \Big(6(n_{12}v_{12})^2 + 21(n_{12}v_2)^2 - 6v_{12}^2 \Big) \\ &\quad + \frac{G^2}{r_{12}^2} \Big(\frac{365m_1^2}{28} + \frac{125m_1m_2}{2} + 21m_2^2 \Big) \Big] \Big\}\,. \quad (4.4) \end{aligned}$$

Similarly, we computed the further geometrical invariants "$\ell = 3$ electric"

$$J_{3e} \equiv \left[G_{abc}^2\right]_1 = \left[G_{\alpha\beta\gamma}\, G^{\alpha\beta\gamma}\right]_1\,, \quad (4.5)$$

and "$\ell = 2$ magnetic"

$$\begin{aligned} J_{2m} &\equiv \frac{1}{4}\, \left[H_{ab}^2\right]_1 = \frac{1}{4}\, \left[H_{\alpha\beta}\, H^{\alpha\beta}\right]_1 \\ &= c^2 \left[R^*_{\alpha\mu\beta\nu}\, R^*_{\gamma\kappa\delta\lambda}\, g^{\alpha\gamma}\, g^{\beta\delta}\, u^\mu\, u^\nu\, u^\kappa\, u^\lambda\right]_1 . (4.6) \end{aligned}$$

Note the factor $\frac{1}{4}$, introduced in the definition of J_{2m} to have $J_{2m} = (cR^*_{\alpha\mu\beta\nu}u^\mu u^\nu)^2$, analogously to $J_{2e} = (R_{\alpha\mu\beta\nu}u^\mu u^\nu)^2$. Let us also note in passing that, in evaluating J_{3e}, i.e., the square of the electric octupole $G_{\alpha\beta\gamma}$, Eq. (2.8), it is important to use the orthogonally projected covariant derivative $\nabla^\perp_\alpha$. If, instead of $(G_{\alpha\beta\gamma})^2$, one evaluates $(C_{\alpha\beta\gamma})^2$ where $C_{\alpha\beta\gamma} = \mathrm{Sym}_{\alpha\beta\gamma}\, \nabla_\alpha\, R_{\beta\mu\gamma\nu}\, u^\mu\, u^\nu$, one finds a result which differs from $(G_{\alpha\beta\gamma})^2$ by a term proportional to $J_{\dot{2}e} = (\dot{G}_{ab})^2$ [see Eq. (6.51)].

The results for these invariants (along general orbits) are

$$\begin{aligned} J_{3e} &= \frac{90G^2 m_2^2}{r_{12}^8} \Big\{ 1 + \frac{1}{c^2} \Big[-2(n_{12}v_{12})^2 - 4(n_{12}v_2)^2 + 3v_{12}^2 \\ &\qquad - \frac{G(4m_1 + 10m_2)}{r_{12}} \Big] \\ &+ \frac{1}{c^4} \Big[10(n_{12}v_2)^2(n_{12}v_{12})^2 + 10(n_{12}v_2)^4 \\ &\quad - \frac{14}{3}v_{12}^2(n_{12}v_{12})^2 - 4(n_{12}v_{12})^2(v_2v_{12}) \\ &\quad -12v_{12}^2(n_{12}v_2)^2 - 4(n_{12}v_2)(n_{12}v_{12})(v_2v_{12}) \\ &\quad -2v_2^2(n_{12}v_{12})^2 - 4v_2^2(n_{12}v_2)^2 \\ &\quad + \frac{17}{3}v_{12}^4 + 6v_{12}^2(v_2v_{12}) + 3(v_2v_{12})^2 + 3v_2^2v_{12}^2 \\ &\quad + \frac{Gm_1}{r_{12}} \Big(-32(n_{12}v_{12})^2 + 2(n_{12}v_2)(n_{12}v_{12}) \\ &\qquad +22(n_{12}v_2)^2 + \frac{16}{3}v_{12}^2 \Big) \\ &\quad + \frac{Gm_2}{r_{12}} \Big(12(n_{12}v_{12})^2 + 45(n_{12}v_2)^2 - 18v_{12}^2 \Big) \\ &\quad + \frac{G^2}{r_{12}^2} \Big(9m_1^2 + \frac{259m_1m_2}{3} + 54m_2^2 \Big) \Big] \Big\}\,, \quad (4.7) \end{aligned}$$

and

$$J_{2m} = \frac{18G^2m_2^2}{r_{12}^6}\Big\{ -(n_{12}v_{12})^2 + v_{12}^2 + \frac{1}{c^2}\Big[(n_{12}v_{12})^4 + 4(n_{12}v_2)^2(n_{12}v_{12})^2 -3v_{12}^2(n_{12}v_{12})^2 - 2(n_{12}v_{12})^2(v_2v_{12}) - v_2^2(n_{12}v_{12})^2 -2(n_{12}v_2)(n_{12}v_{12})(v_2v_{12}) - 3v_{12}^2(n_{12}v_2)^2 +2v_{12}^4 + 2v_{12}^2(v_2v_{12}) + (v_2v_{12})^2 + v_2^2v_{12}^2 +\frac{2G}{r_{12}}\Big(\frac{m_1}{3}+m_2\Big)\Big((n_{12}v_{12})^2 - v_{12}^2\Big)\Big]\Big\}. \quad (4.8)$$

The result Eq. (4.4), after multiplication by the redshift factor

$$\frac{d\tau_1}{dt} = \left(-g_{00} - 2\,g_{0i}\,\frac{v_1^i}{c} - g_{ij}\,\frac{v_1^i\,v_1^j}{c^2}\right)^{1/2}, \quad (4.9)$$

which evaluates to (we use again the notation $\epsilon \equiv 1/c$, and henceforth often set Newton's constant to one)

$$\frac{d\tau_1}{dt} = 1 - \epsilon^2\left(\frac{1}{2}\,v_1^2 + \frac{m_2}{r_{12}}\right) + \epsilon^4\left[-\frac{1}{8}v_1^4 + \frac{m_2}{r_{12}}\left(\frac{1}{2}(n_{12}v_2)^2 - \frac{3}{2}v_1^2 + 4(v_1v_2) - 2v_2^2\right) + \frac{m_2}{2r_{12}^2}(3m_1+m_2)\right], \quad (4.10)$$

provides the $O(\mu_1^{(\ell=2)})$ piece ("gravitoelectric tidal quadrupole") of the reduced two-body action at the 2PN approximation level; i.e., including tidal correction terms that are $(v/c)^4$ smaller than the leading order tidal Lagrangian which is simply given by $J_{2e}^{(0)} = 6\,m_2^2/r_{12}^6$. Similarly, multiplying the results of Eqs. (4.7) and (4.8) by the redshift factor in Eq. (4.10) provides the reduced tidal actions associated with $J_{3e} \equiv [G_{abc}^2]_1$ and $J_{2m} \equiv \frac{1}{4}[H_{ab}^2]_1$, at the 2PN approximation for the electric-octupole term J_{3e}, and at the 1PN approximation for the magnetic-quadrupole term J_{2m}.

In view of their complexity, the results of Eqs. (4.4), (4.7), (4.8), which provide the action for general orbits, are not very useful as they are. In what follows, we shall extract the physically most useful information they contain by: (i) focusing our attention on *circular orbits* and (ii) reformulating our results in terms of the EOB description of binary systems. Note in passing that though circular orbits are only special solutions of binary dynamics, they are the ones of prime physical importance in many situations, most notably radiation-reaction-driven inspiralling binary systems.

B. Tidal actions along circular orbits

In the following, we shall therefore restrict our attention to circular motions. [However, we shall show below how this restricted result can crucially inform the EOB description of tidally interacting binary systems.] We shall also focus on the *relative* dynamics in the center of mass frame. As we see in Eqs. (4.4), (4.7), (4.8), (4.10), the various Lagrangians depend only on the relative position $\boldsymbol{y}_{12} = \boldsymbol{y}_1 - \boldsymbol{y}_2$ and start depending on (individual) velocities only at 1PN (for general orbits), and even at 2PN for the invariants themselves (in the case of circular orbits). This implies that we shall not really need to use to its full 2PN accuracy the relation between center-of-mass variables $\boldsymbol{y}_1^{\rm CM}, \boldsymbol{y}_2^{\rm CM}, \boldsymbol{v}_1^{\rm CM}, \boldsymbol{v}_2^{\rm CM}$, and relative ones $\boldsymbol{y}_{12}, \boldsymbol{v}_{12}$, namely (in the circular case)

$$y_1^i = \left[X_2 + 3\left(\frac{M}{r_{12}\,c^2}\right)^2\nu\,X_{12}\right]y_{12}^i\,,$$

$$y_2^i = \left[-X_1 + 3\left(\frac{M}{r_{12}\,c^2}\right)^2\nu\,X_{12}\right]y_{12}^i\,, \quad (4.11)$$

and the corresponding velocity relations obtained by time-differentiating them, using the fact that $\boldsymbol{y}_{12} = r_{12}\,\boldsymbol{n}_{12}$ where r_{12} is constant and $\boldsymbol{n}_{12}$ rotates with an angular velocity given by

$$\Omega^2 = \frac{M}{r_{12}^3}\left[1 + \epsilon^2(\nu-3)\,\frac{M}{r_{12}} +\epsilon^4\left(6 + \frac{41}{4}\,\nu + \nu^2\right)\left(\frac{M}{r_{12}}\right)^2\right]. \quad (4.12)$$

Here and below we use the notation

$$X_1 \equiv \frac{m_1}{M}\,,\quad X_2 \equiv \frac{m_2}{M}\,,\quad \nu \equiv X_1\,X_2\,,\quad X_{12} \equiv X_1 - X_2\,, \quad (4.13)$$

(recall that $M \equiv m_1 + m_2$ so that $X_1 + X_2 = 1$). In our calculations, the $\epsilon^4 = 1/c^4$ contributions in Eqs. (4.11), (4.12) do not matter, and can be neglected from the start.

Using such an additional circular (and center-of-mass) reduction, we get a much simplified result for the electric-quadrupole invariant J_{2e}, Eq. (4.3), namely,

$$J_{2e}^{\rm (circ)} = \frac{6M^2X_2^2}{r_{12}^6}\left[1 + \epsilon^2\frac{(X_1-3)M}{r_{12}} -\epsilon^4\frac{M^2}{28r_{12}^2}(713X_1^2 - 805X_1 - 336)\right]. \quad (4.14)$$

In a similar manner, one gets much simplified results for the other (subleading) geometrical invariants of tidal significance, namely the magnetic quadrupolar term J_{2m}, Eq. (4.6), the electric octupolar term J_{3e}, Eq. (4.5), and also for the time-differentiated electric-quadrupole coupling, say,

$$J_{\dot{2}e} \equiv \left[\dot{G}_{ab}^2\right]_1 = \left[(u^\mu\,\nabla_\mu\,G_{\alpha\beta})(u^\nu\,\nabla_\nu\,G^{\alpha\beta})\right]_1\,. \quad (4.15)$$

Among these invariants, the 2PN accurate metric allows one (as for G_{ab}^2) to calculate to 2PN fractional accuracy only the electric-octupole term J_{3e}. The other ones can be computed only at 1PN fractional accuracy because of their "magnetic," or "$\partial_0 = c^{-1}\partial_t$" character. Our explicit "circular" results for J_{2m}, J_{3e} and $J_{\dot{2}e}$ are

$$J_{2m}^{(\mathrm{circ})} = \frac{18X_2^2M^3}{r_{12}^7}\left[1+\epsilon^2\frac{M}{3r_{12}}(3X_1^2+X_1-9)\right], \tag{4.16}$$

$$J_{3e}^{(\mathrm{circ})} = \frac{90X_2^2M^2}{r_{12}^8}\left[1+\epsilon^2(6X_1-7)\frac{M}{r_{12}} -\epsilon^4\frac{M^2}{3r_{12}^2}(61X_1^2+4X_1-98)\right], \tag{4.17}$$

$$J_{\dot{2}e}^{(\mathrm{circ})} = \frac{18X_2^2M^3}{r_{12}^9}\left[1+\epsilon^2(X_1^2-7)\frac{M}{r_{12}}\right]. \tag{4.18}$$

To complete the above results, and allow one to compute the corresponding associated Lagrangians, let us note that the circular value of the redshift factor is

$$\frac{d\tau_1}{dt} \equiv \frac{1}{\Gamma_1} = 1 - \frac{M(X_1-1)(X_1-3)}{2r_{12}}\epsilon^2 + \frac{M^2(X_1-1)}{8r_{12}^2}(3X_1^3-9X_1^2+13X_1-3)\epsilon^4. \tag{4.19}$$

Let us also quote the value of the inverse redshift factor, Γ_1 (analog to a Lorentz γ-factor $\gamma = 1/\sqrt{1-\boldsymbol{v}^2/c^2}$), namely

$$\Gamma_1 \equiv \frac{dt}{d\tau_1} = 1 + \frac{M(X_1-1)(X_1-3)}{2r_{12}}\epsilon^2 - \frac{M^2}{8r_{12}^2}(X_1-1)(X_1^3+5X_1^2-17X_1+15)\epsilon^4. \tag{4.20}$$

V. EOB DESCRIPTION OF THE TIDAL ACTION

We have computed above the effective actions associated with the tidal parameters $\mu_1^{(2)}, \mu_1^{(3)}, \sigma_1^{(2)}$ and $\mu_1'^{(2)}$. Before the restriction to circular motions (in the center-of-mass frame) they have the general form

$$\mu_1 \int dt\, L_{\mu_1}(\boldsymbol{y}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2), \tag{5.1}$$

where μ_1 denotes a generic tidal parameter, and $\boldsymbol{y}_{12} = \boldsymbol{y}_1 - \boldsymbol{y}_2$. In this section we discuss how one can describe the actions of Eq. (5.1) within the EOB formalism. Let us recall that the EOB formalism [16–19] replaces the (possibly higher-order) Lagrangian dynamics of two particles by the Hamiltonian dynamics of an "effective particle" embedded within some "effective external potentials." For non-spinning [66] bodies, the original (velocity-dependent) two-body interactions become reformulated (and simplified by means of a suitable contact transformation in phase space) in terms of three "EOB potentials": $A(r_{\mathrm{eff}})$, $\bar{B}(r_{\mathrm{eff}})$ and $Q(r_{\mathrm{eff}}, p^{\mathrm{eff}})$. The first two potentials, $A(r_{\mathrm{eff}})$ and $\bar{B}(r_{\mathrm{eff}})$, parametrize an "effective metric"

$$\begin{aligned} ds_{\mathrm{eff}}^2 &= g_{\mu\nu}(x_{\mathrm{eff}})\, dx_{\mathrm{eff}}^\mu\, dx_{\mathrm{eff}}^\nu \\ &= -A(r_{\mathrm{eff}})\, c^2\, dt_{\mathrm{eff}}^2 + \bar{B}(r_{\mathrm{eff}})\, dr_{\mathrm{eff}}^2 \\ &\quad + r_{\mathrm{eff}}^2(d\theta_{\mathrm{eff}}^2 + \sin^2\theta_{\mathrm{eff}}\, d\varphi_{\mathrm{eff}}^2), \end{aligned} \tag{5.2}$$

and its associated Hamilton-Jacobi equation, while the third potential $Q(r_{\mathrm{eff}}, p^{\mathrm{eff}})$ (which necessarily appears at 3PN [18]), describes additional contributions to the (Hamilton-Jacobi) mass-shell condition,

$$0 = \mu^2 + g_{\mathrm{eff}}^{\mu\nu}(x_{\mathrm{eff}})\, p_\mu^{\mathrm{eff}}\, p_\nu^{\mathrm{eff}} + Q(r_{\mathrm{eff}}, p^{\mathrm{eff}}) \tag{5.3}$$

[where $\mu \equiv m_1\, m_2/M \equiv \nu M$ is the reduced mass of the binary system], that are higher than quadratic in the effective momentum p^{eff}. Following the EOB-simplifying philosophy of Ref. [18], we shall assume that the third potential has been reduced (by a suitable canonical transformation) to a form where it vanishes with the radial momentum p_r^{eff}.

In addition, EOB theory introduces a *dictionary* between the original dynamical variables (positions, momenta, angular momentum, energy) and the effective ones. A crucial entry of this dictionary is a non-trivial transformation between the original "real" energy, i.e., the value of the original (total) Hamiltonian H, and the "effective" energy $-p_0^{\mathrm{eff}} \equiv H_{\mathrm{eff}}$ entering the mass-shell condition of Eq. (5.3). Because of this transformation, the final EOB-form of the (original, real) Hamiltonian reads (here we set $c=1$ for simplicity)

$$H^{\mathrm{EOB}}(\boldsymbol{x}_{\mathrm{eff}}, \boldsymbol{p}_{\mathrm{eff}}) = M\sqrt{1+2\,\nu\left(\frac{H_{\mathrm{eff}}}{\mu}-1\right)}, \tag{5.4}$$

where $H_{\mathrm{eff}} = H_{\mathrm{eff}}(\boldsymbol{x}_{\mathrm{eff}}, \boldsymbol{p}_{\mathrm{eff}})$ is given by

$$H_{\mathrm{eff}} = \sqrt{A(r_{\mathrm{eff}})\left(\mu^2 + \frac{\boldsymbol{J}_{\mathrm{eff}}^2}{r_{\mathrm{eff}}^2} + \frac{(p_r^{\mathrm{eff}})^2}{\bar{B}(r_{\mathrm{eff}})} + Q(r_{\mathrm{eff}}, p^{\mathrm{eff}})\right)}. \tag{5.5}$$

Here $\boldsymbol{J}_{\mathrm{eff}} \equiv \boldsymbol{x}_{\mathrm{eff}} \times \boldsymbol{p}_{\mathrm{eff}}$ denotes the effective orbital angular momentum, which, by the EOB dictionary, is actually identified with the original, total (center of mass) orbital angular momentum $\boldsymbol{J}$ of the binary system: $\boldsymbol{J}_{\mathrm{eff}} \equiv \boldsymbol{J}$.

A. EOB reformulation of tidal actions: general orbits

Let us now discuss what the various possible methods are for reformulating an original action of the type $L_0(\boldsymbol{y}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2, \ldots) + \mu_1\, L_{\mu_1}(\boldsymbol{y}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2)$ [where μ_1

stands for a sum over a collection of tidal parameters $\mu_1^{(2)}, \mu_2^{(2)}, \mu_1^{(3)}, \mu_2^{(3)}, \ldots]$ into corresponding μ_1-deformations of EOB potentials: $A_0(r_{\rm eff}) + \mu_1 A_{\mu_1}(r_{\rm eff})$, $\bar{B}_0(r_{\rm eff}) + \mu_1 \bar{B}_{\mu_1}(r_{\rm eff})$, $Q_0(r_{\rm eff}, p^{\rm eff}) + \mu_1 Q_{\mu_1}(r_{\rm eff}, p^{\rm eff})$. The main difficulty in finding the perturbed EOB potentials A_{μ_1}, $\bar{B}_{\mu_1}$, and Q_{μ_1} that encode the dynamics of L_{μ_1} is that such a dynamical equivalence is obtained only after some a priori unknown phase-space contact transformation between the EOB phase-space coordinates, say $\xi_{\rm eff} = (\boldsymbol{x}_{\rm eff}, \boldsymbol{p}_{\rm eff})$, and the original (harmonic-coordinate-related) ones, say $\xi_h = (\boldsymbol{y}_{12}, \boldsymbol{v}_{12})$. For simplicity, we assume that we have already performed the reduction of the original harmonic-coordinate dynamics to its center-of-mass version, in which one can express $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in terms of the relative velocity $\boldsymbol{v}_{12} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2$ and of $\boldsymbol{y}_{12} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$. On the other hand, we do not immediately assume that the original Lagrangian dynamics is expressed in Hamiltonian form. (Let us recall that, as was found long ago [24, 49], starting at the 2PN level, the harmonic-coordinate dynamics *does not* admit an ordinary Lagrangian, $L(y, \dot{y})$, but only a higher-order one, $L(y, \dot{y}, \ddot{y})$. In order to express the 2PN dynamics in Hamiltonian form, one already needs some (higher-order) contact transformation. However, this transformation is well-known, e.g., Ref. [46], and we do not need to complicate our discussion by explicitly mentionning it. Nonetheless, it will be taken into account in our calculations below.)

The transformation T between $\xi_{\rm eff}$ and ξ_h will have the general structure

$$\xi_h = T_0(\xi_{\rm eff}) + \mu_1 T_{\mu_1}(\xi_{\rm eff}) \,. \qquad (5.6)$$

The unperturbed part $T_0(\xi_{\rm eff})$ is known from the previous EOB work [16, 18], but the $O(\mu_1)$ perturbed part $T_{\mu_1}(\xi_{\rm eff})$ is unknown, and, actually, is part of the problem which must be solved for reformulating the (perturbed) harmonic-coordinate dynamics in EOB form. This means, in particular, that it would not be correct to try to compute A_{μ_1}, $\bar{B}_{\mu_1}$ and Q_{μ_1}, simply by replacing in the tidal action in Eq. (5.1) the harmonic variables ξ_h by their unperturbed expression $T_0(\xi_{\rm eff})$ in terms of the effective variables $\xi_{\rm eff}$.

For the general case of non-circular orbits, a universal, correct method for transforming the original Lagrangian $L(\xi_h) = L_0(\xi_h) + \mu_1 L_{\mu_1}(\xi_h)$ in EOB form consists (as explained in Ref. [16]) of the following steps: (i) to transform the original Lagrangian $L(\xi_h)$ in Hamiltonian form $H(\xi_H) = H_0(\xi_H) + \mu_1 H_{\mu_1}(\xi_H)$, where $\xi_H = (q, p)$ are canonical coordinates; (ii) to extract the gauge-invariant content of $H(\xi_H)$ by expressing it in terms of *action variables* $I_a = \frac{1}{2\pi} \oint p_a \, dq_a$, which yields the *Delaunay* Hamiltonian $H(\boldsymbol{I}) = H_0(\boldsymbol{I}) + \mu_1 H_{\mu_1}(\boldsymbol{I})$; (iii) to do the same thing for the EOB Hamiltonian, i.e. to compute, as a functional of the unknown EOB potentials, its Delaunay form $H_{\rm EOB}(\boldsymbol{I}) = H_0^{\rm EOB}(\boldsymbol{I}) + \mu_1 H_{\mu_1}^{\rm EOB}(\boldsymbol{I})$; and finally (iv) to identify the known $H(\boldsymbol{I})$ to $H_{\rm EOB}(\boldsymbol{I})$, which depends on the unknown functions A_{μ_1}, $\bar{B}_{\mu_1}$, Q_{μ_1}. This last step yields (functional) equations for A_{μ_1}, $\bar{B}_{\mu_1}$, Q_{μ_1} and thereby allows one to determine them. [In practice, the functional dependence on $A, \bar{B}, Q$ is replaced by a much simpler parameter-dependence by using the method of undetermined coefficients for parametrizing general forms of $A, \bar{B}, Q$.] An alternative (and equally universal) method for transforming $L(\xi_h)$ in EOB form (as used in Ref. [18]) is to add the transformation $\xi_h = T(\xi_{\rm eff})$ to the list of unknowns (using the method of undetermined coefficients), and to directly solve the set of constraints for $T, A, \bar{B}$ and Q coming from the requirement that $H_{\rm EOB}(\xi_{\rm eff}, A, \bar{B}, Q) = H(T(\xi_{\rm eff}))$. [One must then take into account that T is constrained to be a *canonical* transformation.]

B. EOB reformulation of tidal actions: circular orbits

The 2PN-accurate results, given for several tidal interactions in the case of general orbits, in the previous section, can, in principle, be transformed within the EOB format by using any of the two methods we just explained. However, from the point of view of current astrophysical applications, one is mainly interested in knowing the EOB description of (quasi)-*circular* motions. In this case, we know *a priori* that it is only the A radial potential which matters. Knowing this, the question arises how to compute the tidal perturbation A_{μ_1} of the EOB A potential in the most efficient manner, possibly without having to go through the rather involved, general universal methods recalled above. Fortunately, it is possible to do so by using the following facts.

The first useful fact concerns the relation between the tidal perturbation (in harmonic coordinates) of the Lagrangian of the binary system, say

$$\delta L^h(y_h, v_h) = \mu_1 L_{\mu_1}(\boldsymbol{y}_{12}, \boldsymbol{v}_1, \boldsymbol{v}_2) \,, \qquad (5.7)$$

and the corresponding tidal perturbation (in harmonic-related phase-space coordinates) of the Hamiltonian, say

$$\delta H^h(y_h, p_h) \equiv H^h_{\rm full}(y_h, p_h) - H^h_{\text{tidal-free}}(y_h, p_h) \,. \qquad (5.8)$$

[Strictly speaking, as we recalled above, the harmonic-related $(q, p) = (y_h, p_h)$ phase-space coordinates involve a supplementary $O(1/c^4)$ gauge transformation linked to the order reduction of $L_{\rm 2PN}(y, \dot{y}, \ddot{y})$ into $L^{\rm red}_{\rm 2PN}(y', \dot{y}')$.] Note that here and in the following the notation $\delta Q(\xi)$ will always refer to the tidal contribution to some function of specified variables, i.e. $\delta Q(\xi) \equiv Q_{\rm full}(\xi) - Q_{\text{tidal-free}}(\xi)$. One has to be careful about which variables are fixed as, for instance, the transformation between Lagrangian $(q, \dot{q})$ and Hamiltonian (q, p) coordinates does contain a tidal contribution [because $\delta^{\rm tidal} L(y_h, \dot{y}_h)$ does depend on velocities]. This being made clear, we have the well-known universal result about first-order deformations of Lagrangians by small parameters, $L(q, \dot{q}) =$

$L_0(q,\dot q) + \mu_1 L_{\mu_1}(q,\dot q)$, namely

$$\delta H^h(y_h, p_h) = -\,\delta L^h(y_h, v_h) \qquad (5.9)$$

which follows from the properties of the Legendre transform.

Let us now apply the second method recalled above for transforming the "harmonic" Hamiltonian $H^h_{\rm full}(y_h,p_h) = H^h_0(y_h,p_h) + \delta H^h(y_h,p_h)$ (where the index 0 refers to the *unperturbed*, tidal-free dynamics) into its corresponding EOB form $H^{\rm EOB}_{\rm full}(x_{\rm EOB}, p_{\rm EOB})$, defined in Eqs. (5.4), (5.5) above. [For clarity, we denote here the effective-one-body phase-space coordinates by $x_{\rm EOB}, p_{\rm EOB}$, instead of $x_{\rm eff}, p_{\rm eff}$ as above.] The crucial point is that the EOB potentials entering the definition of $H^{\rm EOB}_{\rm full}$ must be the *full*, tidally-completed values of $A, \bar B$ and Q, e.g.

$$\begin{aligned} A_{\rm full}(r_{\rm EOB}) &= A_0(r_{\rm EOB}) + \mu_1 A_{\mu_1}(r_{\rm EOB}) \\ &\equiv A_0(r_{\rm EOB}) + \delta A(r_{\rm EOB})\,. \end{aligned} \qquad (5.10)$$

In other words $\delta H^{\rm EOB}(x_{\rm EOB}, p_{\rm EOB}) \equiv H^{\rm EOB}_{\rm full}(x_{\rm EOB}, p_{\rm EOB}) - H^{\rm EOB}_0(x_{\rm EOB}, p_{\rm EOB})$ is obtained by varying the functions $A, \bar B$ and Q (i.e. $A(r_{\rm EOB}) = A_0(r_{\rm EOB}) + \delta A(r_{\rm EOB})$, etc.) in the definition in Eqs. (5.4), (5.5) of $H^{\rm EOB}_{\rm full}[x_{\rm EOB}, p_{\rm EOB}; A(r_{\rm EOB}), \bar B(r_{\rm EOB}), Q(r_{\rm EOB}, p^{\rm EOB})]$.

This second method for mapping $H^h_{\rm full}(\xi_h)$ into $H^{\rm EOB}_{\rm full}(\xi_{\rm EOB})$ [where $\xi_h \equiv (y_h, p_h)$, $\xi_{\rm EOB} \equiv (x_{\rm EOB}, p_{\rm EOB})$] consists of looking for a full, i.e., perturbed, (time-independent) contact transformation $\xi_h = T_{\rm full}(\xi_{\rm EOB}) = T_0(\xi_{\rm EOB}) + \mu_1 T_{\mu_1}(\xi_{\rm EOB})$ that transforms $H^h_{\rm full}(\xi_h)$ into $H^{\rm EOB}_{\rm full}(\xi_{\rm EOB})$, i.e., such that

$$H^h_{\rm full}[T_{\rm full}(\xi_{\rm EOB})] = H^{\rm EOB}_{\rm full}(\xi_{\rm EOB})\,. \qquad (5.11)$$

Rewriting the full transformation $T_{\rm full}$ as the composition $T' \circ T_0$ of the known unperturbed (tidal-free) contact transformation $\xi^0_h = T_0(\xi_{\rm EOB})$ mapping $H^h_0(\xi^0_h)$ into $H^{\rm EOB}_0(\xi_{\rm EOB})$ with an *unknown* near-identity additional transformation, $\xi_h = T'(\xi^0_h) = \xi^0_h + \mu_1\{G_{\mu_1}(\xi^0_h), \xi^0_h\}$ [where $\{f,g\}$ denotes a Poisson bracket and where $\mu_1 G_{\mu_1}(\xi^0_h)$ is the first-order *generating* function associated with the *canonical* transformation T'], and expanding all functions in Eq. (5.11) into unperturbed plus tidal contributions ($H^h = H^h_0 + \delta H^h$, $T = (1+\delta T')\circ T_0$, $H^{\rm EOB} = H^{\rm EOB}_0 + \delta H^{\rm EOB}$), leads to the condition

$$\begin{aligned} \left[\delta H^h(\xi^0_h) + \{\delta G(\xi^0_h), H^h(\xi^0_h)\}\right]_{\xi^0_h = T_0(\xi_{\rm EOB})} \\ = \delta H^{\rm EOB}(\xi_{\rm EOB})\,, \end{aligned} \qquad (5.12)$$

where $\delta G(\xi^0_h) = \mu_1 G_{\mu_1}(\xi^0_h)$.

In general, $\delta G(\xi^0_h)$ is part of the unknown functions that must be looked for when writing the condition in Eq. (5.12). However, another simplifying fact occurs in the case where one focusses on *circular* motions: The supplementary term $\{\delta G, H^h\}$ happens to *vanish*. Indeed, $\delta G(\xi^0_h)$ is a scalar function and the Poisson bracket $\{\delta G, H^h\}$ is equal to the time derivative of $\delta G(\xi^0_h)$ along the H^h-dynamical flow, which clearly vanishes along circular motions. This allows one to conclude that, along circular motions, we have the simple condition

$$\left[\delta H^h(\xi^0_h)\right]^{\rm circ}_{\xi^0_h = T_0(\xi_{\rm EOB})} = \left[\delta H^{\rm EOB}(\xi_{\rm EOB})\right]^{\rm circ}\,, \qquad (5.13)$$

where the left-hand side is, in principle, fully known.

C. Link between the circular tidal action and the tidal contribution to the EOB A potential

Let us now evaluate the right-hand side of Eq. (5.13). When restricting the definition of Eqs. (5.4), (5.5) of the EOB Hamiltonian to circular motions, the terms $(p^{\rm EOB}_r)^2/\bar B$ and $Q(r_{\rm EOB}, p^{\rm EOB})$ disappear (because one works with a gauge-reduced Q which vanishes with $p^{\rm EOB}_r$). As a consequence, $H^{\rm circ}_{\rm EOB}(r_{\rm EOB}, J)$ only depends on the A potential. The difference, $\delta H^{\rm circ}_{\rm EOB} \equiv H^{\rm circ}_{\rm EOB}[r_{\rm EOB}, J, A_{\rm full}] - H^{\rm circ}_{\rm EOB}[r_{\rm EOB}, J, A_0]$, can then be simply computed by varying A ($A_{\rm full} = A_0 + \delta A$) within $H^{\rm circ}_{\rm EOB}[A]$. To write explicitly the result of this variation, it is convenient to work with dimensionless variables. We can replace the two phase-space variables $r_{\rm EOB}$, $p^{\rm EOB}_\varphi \equiv J$ that enter $H^{\rm circ}_{\rm EOB}$ by their dimensionless counterparts

$$u \equiv \frac{GM}{c^2\, r_{\rm EOB}} \equiv \frac{G(m_1+m_2)}{c^2\, r_{\rm EOB}}\,, \qquad (5.14)$$

and

$$j \equiv \frac{c\,J}{GM\,\mu} \equiv \frac{c\,J}{G\,m_1\,m_2}\,. \qquad (5.15)$$

In terms of these variables, the explicit expression of $\left[H^{\rm EOB}_{\rm full}\right]^{\rm circ}$ reads

$$\begin{aligned} &\left[H^{\rm EOB}_{\rm full}(u,j)\right]^{\rm circ} \\ &= M\,c^2\sqrt{1 + 2\,\nu\left(-1 + \sqrt{A(u)(1+j^2\,u^2)}\right)}\,. \end{aligned} \qquad (5.16)$$

Varying $A(u)$ in Eq. (5.16) then yields the following explicit expression for the right-hand side of Eq. (5.13):

$$\left[\delta H^{\rm EOB}(u,j)\right]^{\rm circ} = \frac{1}{2}\,M\,\nu\,c^2\,\sqrt{\frac{1+j^2\,u^2}{A(u)\left[1+2\,\nu\left(-1+\sqrt{A(u)(1+j^2\,u^2)}\right)\right]}}\;\delta A(u)\,. \tag{5.17}$$

In addition, one must take into account the constraint coming from the reduction to circular motions, namely, from $\dot p_r^{\rm EOB} = -\partial H^{\rm EOB}/\partial\, r_{\rm EOB}$, the fact that $\partial_u[A(u)(1+j^2\,u^2)] = 0$, i.e. the fact that j^2 is the following function of u (using a prime to denote the u-derivative):

$$j^2 = j^2_{\rm circ}(u) \equiv -\frac{A'(u)}{(u^2\,A(u))'}\,. \tag{5.18}$$

Note that this relation depends on the value of the radial potential $A(u)$. If one is considering the full, tidally-perturbed circular motions one must use $A_{\rm full}(u) = A_0 + \delta A$ in Eq. (5.18). On the other hand, as we are now interested in considering the (first-order) tidal perturbations δH^h and $\delta H^{\rm EOB}$, and their link in Eq. (5.13), we can evaluate $\delta H^{\rm EOB}_{\rm circ}$ with sufficient accuracy by replacing in the coefficient of $\delta A(u)$, on the right-hand side of Eq. (5.17), $A(u)$ and j^2 by their unperturbed, tidal-free expressions $A_0(u)$ and $j^2_{A_0}(u)$ (obtained by replacing $A \to A_0$ on the right-hand side of Eq. (5.18). [This remark applies to several other results below; notably Eqs. (5.20) and (5.22)].

Combining our results of Eqs. (5.9), (5.13) and (5.17), we finally get a very simple link between the tidal variation of the harmonic-coordinate Lagrangian $\delta L(y_h, v_h)$ and the corresponding tidal variation $\delta A(u)$ of the EOB A potential, namely,

$$\delta A(u) = -\,\frac{2}{M\,\nu\,c^2}\,\sqrt{F(u)}\;[\delta L(y_h, v_h)]^{\rm circ}_{r_h = T_0(u)}\;, \tag{5.19}$$

where

$$F(u) \equiv \left[\frac{A(u)}{1+j^2\,u^2}\left(1+2\,\nu\left(-1+\sqrt{A(u)(1+j^2\,u^2)}\right)\right)\right]^{\rm circ}_{A=A_0}\,. \tag{5.20}$$

Here, the superscript “circ” means that j^2 must be replaced by $j^2_{\rm circ}(u)$, Eq. (5.18). (Note that the replacement $A \to A_0$ indicated as a subscript must be done both in the explicit occurrence of A in Eq. (5.20) and in the definition in Eq. (5.18) of $j^2_{\rm circ}(u)$). Finally, if we introduce the short-hand notation

$$\tilde A(u) \equiv A(u) + \frac{1}{2}\,u\,A'(u)\,, \tag{5.21}$$

$F(u)$, Eq. (5.20), can be rewritten in the explicit form

$$F(u) = \tilde A(u)\left[1+2\,\nu\left(-1+\frac{A(u)}{\sqrt{\tilde A(u)}}\right)\right]\,, \tag{5.22}$$

which is valid along circular orbits, and applies for any relevant (exact or approximate) value of the A potential. On the other hand, as we computed δL only to the 2PN fractional accuracy, it is sufficient to use a value of $F(u)$ which is also only fractionally 2PN-accurate. One might think *a priori* that this would mean using for $A(u)$ in Eq. (5.22) the tidal-free approximation $A_0(u)$ truncated at the 2PN order, namely $A_0^{\rm 2PN}(u) = 1 - 2\,u + 2\,\nu\,u^3$. However, the contribution $2\,\nu\,u^3 = 2\,\nu(GM/(c^2\,r_{\rm EOB}))^3$ is $O(1/c^6)$ compared to one, which is the leading-order value of $F(u)$, which starts as $F(u) = 1 + O(u) = 1 + O(1/c^2)$. The same consideration applies to $\tilde A(u)$. [The situation would have been different if $F(u)$ had been, say, $\propto A'(u)$.] This means that, at the 2PN fractional accuracy, we can use the value of $F(u)$ obtained from the leading-order, “Schwarzschild-like” value of $A_0(u)$, namely $A_0^{\rm 1PN}(u) = 1 - 2\,u$. The corresponding $\tilde A$ function is then: $\tilde A_0^{\rm 1PN}(u) = 1 - 3\,u$, so that

$$F^{\rm 2PN}(u) = (1-3\,u)\left[1+2\,\nu\left(-1+\frac{1-2\,u}{\sqrt{1-3\,u}}\right)\right]\,. \tag{5.23}$$

Consistently with the fractional 2PN accuracy, and remembering, that $u = O(1/c^2)$, we could as well use the 2PN-accurate series expansion of Eq. (5.23), say $F^{\rm 2PN}(u) = 1 + f_1(\nu)\,u + f_2(\nu)\,u^2 + O(u^3)$. However, it is better to retain the information contained in Eq. (5.23) that, in the test-mass limit $\nu \to 0$ (where $A_0(u) \to 1 - 2\,u$), the exact value of $F(u)$ becomes $1 - 3\,u$ (see later).

There remains only one missing piece of information to be able to use our result in Eq. (5.19) for computing the various tidal contributions to $A(u)$. We need to work out the explicit form of the unperturbed transformation T_0 between $r_{\rm EOB}$ and r_h.

A first method for getting the transformation T_0 (at

2PN) is to compose the transformation $\xi_h^0 \to \xi_{\rm ADM}$ (obtained at 2PN in Ref. [46], and at 3PN in Ref. [50]) with the transformation $\xi_{\rm ADM} \to \xi_{\rm EOB}$ (obtained at 2PN in Ref. [16], and at 3PN in Ref. [18]). For our present purpose, it is enough to restrict these transformations to the circular case, i.e. to transformations $r_h \to r_{\rm ADM}$ and $r_{\rm ADM} \to r_{\rm EOB}$.

The transformation $h \to$ ADM starts at 2PN, i.e., $\boldsymbol{y}_A^h = \boldsymbol{x}_A^{\rm ADM} + c^{-4}\, Y_A^{\rm 2PN}(\boldsymbol{x}^{\rm ADM}, \boldsymbol{p}^{\rm ADM})$, with $Y_A^{\rm 2PN}(\boldsymbol{x}^{\rm ADM}, \boldsymbol{p}^{\rm ADM})$ given, e.g., in Eq. (4.5) of Ref. [50]. Its circular, and center-of-mass, reduction (with $\boldsymbol{n}_{12} \cdot \boldsymbol{p}_A = 0, \boldsymbol{p}_1 = -\boldsymbol{p}_2 \equiv \boldsymbol{p}$, and $(\boldsymbol{p}/\mu)^2 = GM/r_{12} + O(1/c^2)$) yields at 2PN

$$r_{12}^h = r_{12}^{\rm ADM} \left[1 + \left(\frac{1}{4} + \frac{29}{8}\, \nu \right) \left(\frac{GM}{c^2\, r_{12}} \right)^2 \right] . \qquad (5.24)$$

On the other hand the transformation ADM $\to$ EOB starts at 1PN. To determine the corresponding radial transformation $r_{12}^{\rm ADM} \to r^{\rm EOB}$, one could think of using Eq. (6.22) of Ref. [16]. However, this equation needs to be completed by the knowledge of the circularity condition relating $(\boldsymbol{p}^{\rm ADM}/\mu)^2$ to $GM/r_{12}^{\rm ADM}$ at the 1PN level included. This 1PN-accurate circularity condition can, e.g., be obtained from combining the 1PN-accurate $r^{\rm ADM} = r^{\rm ADM}(j)$ relation given in Ref. [51] (see below), with the fact that (setting $u_{\rm ADM} \equiv GM/(c^2\, r_{\rm ADM})$) $(\boldsymbol{p}_{\rm ADM}/(\mu c))^2 = j^2\, u_{\rm ADM}^2$. This yields $(\boldsymbol{p}_{\rm ADM}/(\mu c))^2 = u_{\rm ADM} + 4\, u_{\rm ADM}^2$, and therefrom the relation between $r_{\rm ADM}$ and $r_{\rm EOB}$.

Another method (which we have checked to give the same result) for determining the $r_{12}^{\rm ADM} \to r_{\rm EOB}$ transformation does not need to use Eq. (6.22) of Ref. [16]. It consists of directly eliminating the dimensionless angular momentum j between the two relations $r^{\rm ADM} = r^{\rm ADM}(j)$ and $r^{\rm EOB} = r^{\rm EOB}(j)$. The former relation was derived at 3PN in Ref. [51] and reads, at 2PN,

$$r_{12}^{\rm ADM} = \frac{GM}{c^2}\, j^2 \left[1 - \frac{4}{j^2} - \frac{1}{8}\, (74 - 43\, \nu)\, \frac{1}{j^4} \right] , \qquad (5.25)$$

while the latter one is obtained by inverting the 2PN-accurate version of Eq. (5.18), namely, using $A_{\rm 2PN}(u) = 1 - 2\, u + 2\, \nu\, u^3$:

$$\frac{1}{j^2} = \frac{u(1 - 3\, u + 5\, \nu\, u^3)}{1 - 3\, \nu\, u^2} . \qquad (5.26)$$

Inserting Eq. (5.26) into Eq. (5.25) yields (at 2PN)

$$\frac{GM}{c^2\, r_{12}^{\rm ADM}} = u \left[1 + u + \left(\frac{5}{4} - \frac{19}{8}\, \nu \right) u^2 \right] . \qquad (5.27)$$

Then, combining Eq. (5.27) and Eq. (5.24) yields the looked for transformation $r^{\rm EOB} \to r_{12}^h$, at 2PN accuracy,

$$r_{12}^h + \frac{GM}{c^2} = r^{\rm EOB} \left(1 + 6\, \nu \left(\frac{GM}{c^2\, r^{\rm EOB}} \right)^2 \right) , \qquad (5.28)$$

or, setting $u_h \equiv GM/(c^2\, r_{12}^h)$ by analogy with $u \equiv GM/(c^2\, r_{\rm EOB})$,

$$u_h = \frac{u}{1 - u}\, (1 - 6\, \nu\, u^2) . \qquad (5.29)$$

We have written the transformation of Eqs. (5.28), (5.29) so as to exhibit the exact form of the transformation $r_h \to r_{\rm EOB}$ in the extreme mass ratio limit $\nu \to 0$, namely $r_h = r_{\rm EOB} - GM/c^2 + O(\nu)$.

Summarizing: The (first-order) tidal contribution $\delta A(u) = \mu_1 A_{\mu_1}(u)$ to the main EOB radial potential, associated with any tidal parameter μ_1 ($= \mu_1^{(2)}, \mu_2^{(2)}, \mu_1^{(3)}, \ldots$), is given in terms of the corresponding harmonic-coordinate tidal contribution to the action $\delta L(y_h, v_h) = \mu_1 L_{\mu_1}(y_h, v_h)$, for circular motion, by Eq. (5.19), where $F(u)$ is given (at 2PN) by Eq. (5.23), and where the transformation between the harmonic radial separation r_{12}^h and the EOB radial coordinate $r_{\rm EOB} \equiv GM/(c^2\, u)$ is given by Eqs. (5.28) or (5.29).

VI. EOB DESCRIPTION OF TIDAL ACTIONS

A. Tidal actions for comparable-mass systems

We have explained in the previous section how to convert each contribution $\sim \mu_1\, L_{\mu_1}(y_h, v_h)$ to the (reduced) tidal action into a corresponding additional contribution $\mu_1 A_{\mu_1}(u)$ to the main EOB radial potential $A(u)$. For instance, if we consider the dominant tidal parameter, i.e. the electric quadrupolar one, $\mu_1^{(\ell=2)}$ (or $\mu_2^{(\ell=2)}$, after exchanging $1 \leftrightarrow 2$), the combination of the result of Eq. (4.2) for the associated Lagrangian, with Eq. (5.19) yields

$$\mu_1^{(2)} A_{\mu_1^{(2)}}(u) = -\frac{1}{2\, c^2}\, \frac{\mu_1^{(2)}}{M\nu}\, \sqrt{F(u)}\, \frac{d\tau_1}{dt}\, [G_{\alpha\beta}\, G^{\alpha\beta}]_1 . \qquad (6.1)$$

In other words, apart from a (negative) numerical coefficient, and the rescaled tidal parameter $\mu_1^{(2)}/(M\nu)$ (where $M\nu = \mu = m_1\, m_2/(m_1 + m_2)$ is the reduced mass of the system), the corresponding tidal contribution to $A(u)$ is the product of *three factors*: $\sqrt{F(u)}$, $d\tau_1/dt$ and the geometrical invariant associated with the considered tidal parameter, e.g., $[G_{\alpha\beta}\, G^{\alpha\beta}]_1$ for the electric quadrupole along the first worldline. In addition, two of these factors, $d\tau_1/dt$ and the geometrical invariant, must be re-expressed as functions of the EOB coordinates by using Eq. (5.28).

Let us start by applying this procedure to the dominant tidal action term: the electric-quadrupole one in Eq. (6.1). We have given above, in Eq. (4.14), the 2PN-accurate value of $J_{2e} \equiv [G_{\alpha\beta}\, G^{\alpha\beta}]_1$ in harmonic coordinates. Using the transformation of Eq. (5.29) to replace $1/r_{12}^h$ in terms of $1/r_{\rm EOB}$ leads to

$$J_{2e}^{\rm (circ)} \;=\; \frac{6 M^2 X_2^2}{r_{\rm EOB}^6} \left[1 + \epsilon^2 \frac{(X_1 + 3) M}{r_{\rm EOB}} \right.$$

$$+\epsilon^4 \frac{M^2}{28 r_{\rm EOB}^2}(295X_1^2-7X_1+336)\Bigg] \,. \quad (6.2)$$

In addition the reexpression of the time-dilation factor $d\tau_1/dt$, Eq. (4.19), in terms of $1/r_{\rm EOB}$ yields

$$\frac{d\tau_1}{dt} = \frac{1}{\Gamma_1} = 1-\frac{1}{2}(X_1-1)(X_1-3)u\epsilon^2 + \frac{3}{8}u^2(X_1-1)(X_1^3-3X_1^2+3X_1+3)\epsilon^4 \,. \quad (6.3)$$

Their product yields the electric-quadrupole tidal Lagrangian (stripped of its prefactor $\frac{1}{4}\mu_1^{(2)}$) in EOB coordinates, at the 2PN accuracy, namely

$$G_{ab}^2 \frac{d\tau_1}{dt} = \frac{J_{2e}^{(\rm circ)}}{\Gamma_1} = \frac{6(X_1-1)^2u^6}{M^4}\hat{\mathcal{L}}_{2e} \,, \quad (6.4)$$

where

$$\hat{\mathcal{L}}_{2e} = 1-\frac{1}{2}u(X_1^2-6X_1-3)\epsilon^2 + \frac{u^2}{56}(21X_1^4-112X_1^3+744X_1^2+238X_1+357)\epsilon^4 \,. \quad (6.5)$$

Adding the further factor $\sqrt{F(u)}$, as well as the prefactor, leads to the corresponding contribution to the EOB A potential, namely

$$\mu_1^{(2)} A_{\mu_1^{(2)}}(u) = A^{(2)\rm LO}_{1\,\rm electric}(r_{\rm EOB})\,\hat{A}^{(2)}_{1\,\rm electric}(u) \,, \quad (6.6)$$

where

$$A^{(2)\rm LO}_{1\,\rm electric}(r_{\rm EOB}) = -\frac{3\,G^2}{c^2}\frac{\mu_1^{(2)}M}{\nu}\frac{X_2^2}{r_{\rm EOB}^6} \,, \quad (6.7)$$

and

$$\hat{A}^{(2)}_{1\,\rm electric}(u) = \sqrt{F(u)}\,\hat{\mathcal{L}}_{2e} = 1+\alpha_1^{2e}\,u+\alpha_2^{2e}\,u^2+O(u^3) \,, \quad (6.8)$$

with

$$\alpha_1^{2e} = \frac{5}{2}X_1 \,, \quad (6.9)$$

$$\alpha_2^{2e} = \frac{337}{28}X_1^2+\frac{1}{8}X_1+3 \,. \quad (6.10)$$

The leading-order (i.e., Newtonian-level) A potential of Eq. (6.7) is equivalent to Eqs. (1.6) and (1.7) above (i.e., Eqs. (23), (25) of Ref. [5]), using the link

$$G\mu_A^{(\ell)} = \frac{1}{(2\ell-1)!!}\,2\,k_A^{(\ell)}\;R_A^{2\ell+1} \,. \quad (6.11)$$

The term of order u (i.e., 1PN) in the relativistic amplification factor $\hat{A}^{(2)}_{1\,\rm electric}(u)$, Eq. (6.8), coincides with the result computed some time ago (see Eq. (38) in Ref. [5]). By contrast, the (2PN) term of order u^2 in $\hat{A}^{(2)}_{1\,\rm electric}(u)$ is the main new result of our present work. Let us discuss its properties.

Similarly to the 1PN coefficient $\alpha_1^{2e}=\frac{5}{2}\,X_1$, which was positive, and monotonically increasing (from 0 to 5/2) as $X_1\equiv m_1/M$ varies between 0 and 1, the 2PN coefficient α_2^{2e} is also positive, and increases as X_1 varies between 0 and 1. When $X_1=0$ (i.e. in the limit $m_1\ll m_2$), α_2^{2e} takes the value $+3$, while when $X_1=1$ (i.e., in the limit $m_1\gg m_2$), it takes the value

$$\alpha_2^{2e}(X_1=1) = \frac{849}{56} = 15.16071429 \,. \quad (6.12)$$

Note that this is about 5 times larger than its value when $X_1=0$. Of most interest (as neutron stars are expected to have rather similar masses $\sim 1.4\,M_\odot$) is the equal-mass value of α_2^{2e}, which is

$$\alpha_2^{2e}\left(X_1=\frac{1}{2}\right) = \frac{85}{14} = 6.071428571 \,. \quad (6.13)$$

In other words, the distance-dependent amplification factor of the electric quadrupole reads, in the equal-mass case

$$\left[\hat{A}^{(2)}_{1\,\rm electric}(u)\right]^{\rm equal\text{-}mass} = 1+\frac{5}{4}\,u+\frac{85}{14}\,u^2+O(u^3) = 1+1.25\,u+6.071429\,u^2 + O(u^3) \,. \quad (6.14)$$

We will comment further on these results for $\hat{A}^{(2)}_{1\,\rm electric}(u)$ and on the recent comparisons between numerical simulations and the EOB description of tidal interactions below. For the time being, let us give the corresponding results of our analysis for some of the sub-leading tidal interactions.

The EOB-coordinate value of the electric octupole invariant, $J_{3e}^{(\rm circ)}$, Eq. (4.17), reads

$$J_{3e}^{(\rm circ)} = \frac{90X_2^2M^2}{r_{\rm EOB}^8}\Bigg[1+\epsilon^2(6X_1+1)\frac{M}{r_{\rm EOB}} +\epsilon^4\frac{M^2}{3r_{\rm EOB}^2}(83X_1^2+14X_1+17)\Bigg] \,. \quad (6.15)$$

Its corresponding action (stripped of its prefactor) is

$$G_{abc}^2\frac{d\tau_1}{dt} = \frac{J_{3e}^{(\rm circ)}}{\Gamma_1} = \frac{90X_2^2u^8}{M^6}\hat{\mathcal{L}}_{3e} \quad (6.16)$$

with

$$\hat{\mathcal{L}}_{3e} = 1-\frac{1}{2}(X_1^2-16X_1+1)u\epsilon^2 + \frac{1}{24}(9X_1^4-108X_1^3+994X_1^2-56X_1+73)u^2\epsilon^4 \quad (6.17)$$

while the corresponding contribution to the EOB A potential reads

$$\mu_1^{(3)}A_{\mu_1^{(3)}}(u) = A^{(3)\rm LO}_{1\,\rm electric}(r_{\rm EOB})\,\hat{A}^{(3)}_{1\,\rm electric}(u) \,, \quad (6.18)$$

where

$$A^{(3)\rm LO}_{1\,\rm electric}(r_{\rm EOB}) = -\frac{15\,G^2}{c^2}\frac{\mu_1^{(3)}\,M}{\nu}\frac{X_2^2}{r_{\rm EOB}^8}\,, \qquad (6.19)$$

and

$$\hat{A}^{(3)}_{1\,\rm electric}(u) = \sqrt{F(u)}\,\hat{\mathcal{L}}_{3e} = 1 + \alpha_1^{3e}\,u + \alpha_2^{3e}\,u^2 + O(u^3)\,, \qquad (6.20)$$

with

$$\alpha_1^{3e} = \frac{15}{2}X_1 - 2\,, \qquad (6.21)$$

$$\alpha_2^{3e} = \frac{110}{3}X_1^2 - \frac{311}{24}X_1 + \frac{8}{3}\,. \qquad (6.22)$$

Here, both results in Eqs. (6.21) and (6.22) are new. Note that, contrary to the quadrupolar case where α_1 and α_2 were always both positive (so that $\hat{A}^{(2)}_{1\,\rm electric}(u)$ was always an *amplification* factor) the electric-octupole factor $\hat{A}^{(3)}_{1\,\rm electric}(u)$ is smaller than 1 (for large separations) when $X_1 < \frac{4}{15} \simeq 0.2667$. Moreover, while the X_1-variation of α_1^{3e} is monotonic (going from -2 to $\frac{11}{2}$ as X_1 increases from 0 to 1), $\alpha_2^{3e}(X_1)$ first decreases from $\alpha_2^{3e}(0) = \frac{8}{3} = 2.666667$ to $\alpha_2^{3e}(X_1^{\rm min}) = 42853/28160 = 1.521768$ as X_1 increases from 0 to $X_1^{\rm min} = 311/1760 = 0.1767046$, before increasing as X_1 goes from $X_1^{\rm min}$ to 1, to reach the final value $\alpha_2^{3e}(1) = 211/8 = 26.375$ for $X_1 = 1$. Note, however, that when (as expected) the two masses are nearly equal the factor $\hat{A}^{(3)}_{1\,\rm electric}(u)$ is an amplification factor. In particular, its equal-mass value is

$$\left[\hat{A}^{(3)}_{1\,\rm electric}(u)\right]^{\rm equal\text{-}mass} = 1 + \frac{7}{4}\,u + \frac{257}{48}\,u^2 + O(u^3) = 1 + 1.75\,u + 5.354167\,u^2 + O(u^3) \qquad (6.23)$$

which is similar to its corresponding quadrupolar counterpart, Eq. (6.14).

Let us finally give the corresponding results for the magnetic quadrupole and time-differentiated electric quadrupole. For the magnetic quadrupole (at the 1PN fractional accuracy), we found

$$\frac{1}{4}H_{ab}^2 \equiv J_{2m}^{(\rm circ)} \qquad (6.24)$$
$$= \frac{18X_2^2M^3}{r_{\rm EOB}^7}\left[1 + \epsilon^2\frac{M}{3r_{\rm EOB}}(3X_1^2 + X_1 + 12)\right]\,,$$

$$\frac{1}{4}H_{ab}^2\,\frac{d\tau_1}{dt} \equiv \frac{18\,X_2^2}{M^4}\,u^7\hat{\mathcal{L}}_{2m}\,, \qquad (6.25)$$

$$\hat{\mathcal{L}}_{2m} = 1 + \frac{1}{6}(X_1+3)(3X_1+5)u\epsilon^2\,, \qquad (6.26)$$

$$\hat{A}^{(2)}_{1\,\rm magnetic}(u) = \sqrt{F(u)}\,\hat{\mathcal{L}}_{2m} = 1 + \alpha_1^{2m}\,u + O(u^2)\,, \qquad (6.27)$$

with

$$\alpha_1^{2m} = X_1^2 + \frac{11}{6}X_1 + 1\,. \qquad (6.28)$$

Here $\alpha_1^{2m}(X_1)$ is always positive, and monotonically increases from $\alpha_1^{2m}(0) = 1$ to $\alpha_1^{2m}(1) = \frac{23}{6} = 3.833333$, its equal-mass value being $\alpha_1^{2m}\left(\frac{1}{2}\right) = \frac{13}{6} = 2.166667$.

Finally, for the time-differentiated electric quadrupole, we got

$$\dot{G}_{ab}^2 = J_{\dot{2}e}^{(\rm circ)} = \frac{18X_2^2M^3}{r_{\rm EOB}^9}\left[1 + \epsilon^2(X_1^2+2)\frac{M}{r_{\rm EOB}}\right]\,, \qquad (6.29)$$

$$\dot{G}_{ab}^2\,\frac{d\tau_1}{dt} = \frac{18\,X_2^2}{M^6}\,u^9\,\hat{\mathcal{L}}_{\dot{2}e}\,, \qquad (6.30)$$

$$\hat{\mathcal{L}}_{\dot{2}e} = 1 + \frac{1}{2}u\epsilon^2(X_1^2 + 4X_1 + 1)\,, \qquad (6.31)$$

$$\hat{A}^{(2)}_{1\dot{G}}(u) = \sqrt{F(u)}\,\hat{\mathcal{L}}_{\dot{2}e} = 1 + \alpha_1^{\dot{2}e}\,u + O(u^2)\,, \qquad (6.32)$$

with

$$\alpha_1^{\dot{2}e} = \frac{1}{2}(X_1+2)(2X_1-1)\,. \qquad (6.33)$$

B. Tidal actions of a tidally-deformable test mass

One of the characteristic features of the EOB formalism for point-mass systems is the natural incorporation of the exact test-mass limit $\nu \to 0$. Indeed, in this limit the effective metric in Eq. (5.2) describing the relative dynamics reduces to the Schwarzschild metric: $\lim_{\nu\to 0} A(u) = 1 - 2\,u = \left(\lim_{\nu\to 0}\bar{B}(u)\right)^{-1}$. Let us study the test-mass limit of tidal effects, with the aim of incorporating it similarly in their EOB description. When considering the nonminimal worldline action of particle 1, the simplest test-mass limit to study is the limit $m_1/m_2 \to 0$. [When considering tidal effects within body 2, the permutation $1 \leftrightarrow 2$ of our results below allow them to describe the limit $m_2/m_1 \to 0$. We leave to future work a study of the limit $m_2/m_1 \to 0$, when considering tidal effects taking place within body 1.] In the limit investigated here, one is considering a tidally deformable test-mass $(m_1, \mu_1^{(\ell)}, \ldots)$ moving around a large mass $m_2 \gg m_1$. The effective action of body 1 is then exactly obtained by evaluating the $A = 1$ contribution of the general (two-body) effective action of Eq. (2.12) within the background metric generated by the (non-tidally deformable) large mass m_2, at rest, i.e. within a Schwarzschild metric of mass m_2. The latter reads

$$ds^2(m_2) = -\left(1 - 2\,\frac{G\,m_2}{c^2\,r_s}\right)c^2\,dt^2 + \frac{dr_s^2}{1 - 2\,\frac{G\,m_2}{c^2\,r_s}} + r_s^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \qquad (6.34)$$

in "Schwarzschild", or areal, coordinates, and

$$ds^2(m_2) = -\frac{1-\frac{G\,m_2}{c^2\,r_h}}{1+\frac{G\,m_2}{c^2 r_h}}\,c^2\,dt^2 + \frac{1+\frac{G\,m_2}{c^2\,r_h}}{1-\frac{G\,m_2}{c^2 r_h}}\,dr_h^2 + \left(r_h + \frac{G\,m_2}{c^2}\right)^2 (d\theta^2 + \sin^2\theta\, d\varphi^2) \qquad (6.35)$$

in harmonic coordinates: $r_h = r_s - G\,m_2/c^2$. As a check on the results below (and on our codes), we have computed them both in Schwarzschild coordinates and in harmonic ones.

The geometrical invariants $J_{2e} = G_{ab}^2$, etc., take the following values in this Schwarzschild limit, and when considering as above circular motions (we again set G and c to one for simplicity):

$$\begin{aligned} G_{ab}^{(\mathrm{S})\,2} = \bar{J}_{2e}^{(\mathrm{S})} &= \frac{6m_2^2(m_2^2 + r_h^2 - m_2 r_h)}{(r_h - 2m_2)^2 (r_h + m_2)^6} \\ &\sim \frac{6m_2^2}{r_h^6}\left[1 - \frac{3m_2}{r_h} + \frac{12m_2^2}{r_h^2} + \ldots\right] \\ &= \frac{6u_S^6}{m_2^4}\frac{1}{(1-3u_S)}\left[1 + \frac{3u_S^2}{(1-3u_S)}\right] \\ &= \frac{6u_S^6}{m_2^4}\left[1 + 3u_S\frac{(1-2u_S)}{(1-3u_S)^2}\right], \qquad (6.36) \end{aligned}$$

$$\begin{aligned} \frac{1}{4}H_{ab}^{(\mathrm{S})\,2} = \bar{J}_{2m}^{(\mathrm{S})} &= \frac{18m_2^3(r_h - m_2)}{(r_h - 2m_2)^2 (r_h + m_2)^6} \\ &\sim \frac{18m_2^3}{r_h^7}\left[1 - \frac{3m_2}{r_h} + \frac{11m_2^2}{r_h^2} + \ldots\right] \\ &= \frac{18u_S^7}{m_2^4}\left[1 + \frac{u_S(4-9u_S)}{(1-3u_S)^2}\right], \qquad (6.37) \end{aligned}$$

$$\begin{aligned} G_{abc}^{(\mathrm{S})\,2} = \bar{J}_{3e}^{(\mathrm{S})} &= \frac{30m_2^2(r_h - m_2)(2m_2^2 + 3r_h^2 - 3m_2 r_h)}{(r_h - 2m_2)^2 (r_h + m_2)^9} \\ &\sim \frac{90m_2^2}{r_h^8}\left[1 - 7\frac{m_2}{r_h} + \frac{98}{3}\frac{m_2^2}{r_h^2} + \ldots\right] \\ &= \frac{90u_S^8}{m_2^6}\frac{(1-2u_S)}{(1-3u_S)}\left[1 + \frac{8u_S^2}{3(1-3u_S)}\right], \qquad (6.38) \end{aligned}$$

$$\begin{aligned} \dot{G}_{ab}^{(\mathrm{S})\,2} = \bar{J}_{\dot{2}e}^{(\mathrm{S})} &= \frac{18m_2^3(r_h - m_2)^2}{(r_h - 2m_2)^2 (r_h + m_2)^9} \\ &= \frac{18m_2^3}{r_h^9}\left[1 - 7\frac{m_2}{r_h} + 32\frac{m_2^2}{r_h^2} + \ldots\right] \\ &= \frac{18u_S^9}{m_2^6}\frac{(1-2u_S)^2}{(1-3u_S)^2}, \qquad (6.39) \end{aligned}$$

where $u_S \equiv G\,m_2/(c^2\,r_s)$. We have indicated above the expansions in powers of the inverse harmonic radius r_h as checks of our 2PN-accurate results, written in harmonic coordinates; see Eqs. (4.14), (4.16)–(4.18).

In the following, we shall focus on the transformation of the exact test-mass geometrical invariants above into corresponding contributions to the EOB A potential. As explained previously, Eqs. (5.19), (6.1), apart from the universal prefactor $-2/(M\,\nu\,c^2)$ and the specific original tidal coefficient multiplying the considered geometrical invariant (such as $\frac{1}{4}\,\mu_1^{(2)}$ for the electric quadrupole), the contribution to $A(u)$ associated with some given invariant is obtained by multiplying it by two extra factors: (i) the time-dilation factor $d\tau_1/dt$ and (ii) the EOB-rooted factor $\sqrt{F(u)}$. Let us discuss their values in the test-mass limit $m_1 \ll m_2$ that we are now considering.

The first factor is the square-root of

$$\left(\frac{d\tau_1}{dt}\right)^2 = 1 - \frac{2\,G\,m_2}{c^2\,r_s} - \frac{1}{c^2}\,r_s^2\left(\frac{d\varphi}{dt}\right)^2. \qquad (6.40)$$

Denoting, as above, $u_S \equiv G\,m_2/(c^2\;r_s)$, and using the well-known Kepler law for circular orbits in Schwarzschild coordinates, $\Omega^2 = G\,m_2/r_s^3$, simply yields

$$\left(\frac{d\tau_1}{dt}\right)^{\text{test-mass}}_{\text{circ}} = \sqrt{1 - 3\,u_S}\,. \qquad (6.41)$$

The exact test-mass limit of the second factor is obtained by taking the limit $\nu \to 0$ in the exact expression of Eq. (5.22). In this limit, $A(u) \to 1 - 2\,u$, so that $\tilde{A}(u) \to 1 - 3\,u$, and

$$\left(\sqrt{F(u)}\right)^{\text{test-mass}}_{\text{circ}} = \sqrt{1 - 3\,u}\,. \qquad (6.42)$$

In addition, as the EOB coordinates reduce to Schwarzschild coordinates in the test-mass limit $\nu \to 0$, and $M = m_1 + m_2 \to m_2$, we have simply

$$u_S \equiv \frac{G\,m_2}{c^2\,r_s} \to u \equiv \frac{GM}{c^2\,r_{\text{EOB}}}\,. \qquad (6.43)$$

In other words, the two extra factors in Eqs. (6.41), (6.42) become both equal to $\sqrt{1-3\,u}$. As a consequence the A contribution corresponding to the various geometrical invariants of Eqs. (6.36)–(6.39) is obtained (apart from a constant prefactor) by multiplying these invariants by $\left(\sqrt{1-3\,u}\right)^2 = 1 - 3\,u = 1 - 3\,u_S$. Including the universal factor $-2/(M\,\nu\,c^2)$ and the various tidal coefficients $\frac{1}{2}\,\frac{1}{\ell!}\,\mu_1^{(\ell)}$, $\frac{1}{2}\,\frac{\ell}{\ell+1}\,\frac{1}{\ell!}\,\frac{\sigma_1^{(\ell)}}{c^2}, \ldots$ (as well as the factor 4 in $H_{ab}^2 = 4J_{2m}$) yields the following exact, test-mass contributions

$$\mu_1^{(2)}\,A^{\text{test-mass}}_{\mu_1^{(2)}}(u) = -3\,\frac{G^2}{c^2}\,\frac{\mu_1^{(2)}}{m_1}\,\frac{(m_2)^2}{r_{\text{EOB}}^6}\left(1 + \frac{3\,u^2}{1 - 3\,u}\right), \qquad (6.44)$$

$$\mu_1^{(3)}\,A^{\text{test-mass}}_{\mu_1^{(3)}}(u) = -15\,\frac{G^2}{c^2}\,\frac{\mu_1^{(3)}}{m_1}\,\frac{(m_2)^2}{r_{\text{EOB}}^8}\,(1 - 2\,u) \times \left(1 + \frac{8}{3}\,\frac{u^2}{1 - 3\,u}\right), \qquad (6.45)$$

$$\sigma_1^{(2)}\, A^{\rm test\text{-}mass}_{\sigma_1^{(2)}}(u) = -\,24\,\frac{G^3}{c^4}\,\frac{\sigma_1^{(2)}}{m_1}\,\frac{(m_2)^3}{r_{\rm EOB}^7}\,\frac{1-2\,u}{1-3\,u}\,, \tag{6.46}$$

$$\mu_1'^{(2)}\, A^{\rm test\text{-}mass}_{\mu_1'^{(2)}}(u) = -\,9\,\frac{G^3}{c^4}\,\frac{\mu_1'^{(2)}}{m_1}\,\frac{(m_2)^3}{r_{\rm EOB}^9}\,\frac{(1-2\,u)^2}{1-3\,u}\,. \tag{6.47}$$

One easily sees that the various exact, test-mass amplification factors $\hat{A}(u)$ exhibited here are compatible with the $X_1 \to 0$ limit of the 2PN-expanded ones $\sim 1+\alpha_1 u+\alpha_2\, u^2+O(u^3)$ derived above.

C. Light-ring behavior of test-mass tidal actions

A striking feature of all the amplification factors present in Eqs. (6.44)–(6.47), such as

$$\hat{A}^{(2)\,\rm test\text{-}mass}_{1\,\rm electric}(u) = 1+3\,\frac{u^2}{1-3\,u}\,, \tag{6.48}$$

is that they all formally exhibit a pole $\propto 1/(1-3\,u)$ mathematically located at $3\,u=1$, i.e. corresponding to formally letting particle 1 tend to the last unstable circular orbit, located at $3\,G\,m_2/c^2$ ("light-ring" orbit). This behavior has a simple origin.

The invariant that is simplest to consider in order to see this is $J_{2e}=G_{ab}^2$. From Eq. (4.3) its covariant expression reads

$$G_{ab}^2 = R_{\alpha\mu\beta\nu}\, R^{\alpha\ \ \beta}_{\ \bullet\kappa\bullet\lambda}\; u^\mu\, u^\nu\, u^\kappa\, u^\lambda\,. \tag{6.49}$$

Let us study its mathematical behavior in the formal limit where particle 1 tends to the light-ring orbit. Using the language of Special Relativity, we consider the Schwarzschild coordinates as defining a "lab-frame." With respect to this lab-frame, particle 1 becomes ultra-relativistic as it approaches the light ring. More precisely, near the light ring the lab-frame components of the 4-velocity $u^\mu=(dt/d\tau_1)(c,v^i)$ tend towards infinity proportionally to $dt/d\tau_1=\Gamma_1=1/\sqrt{1-3\,u}$, while the lab-frame components of $R_{\alpha\mu\beta\nu}$ (and of the metric) stay finite. As G_{ab}^2 is quartic in the lab-frame components of u^μ, it will tend towards infinity like $\Gamma_1^4=(dt/d\tau_1)^4=(1-3\,u)^{-2}$. The corresponding contribution to $A(u)$ is obtained by multiplying G_{ab}^2 by the factor $(d\tau_1/dt)^2=\Gamma_1^{-2}=(1-3\,u)^{+1}$, which reduces the blow-up of G_{ab}^2 to the milder $(1-3\,u)^{-2+1}=(1-3\,u)^{-1}$ behavior that is apparent in Eqs. (6.44) or (6.48).

A different way of phrasing this result uses the law of transformation of the electric and magnetic components of the Weyl tensor, G_{ab} and H_{ab}, under a boost. Using, for instance, the fact that, under a boost with velocity $\beta=\tanh\varphi$ in the x direction, the complex tensor $F_{ab}=G_{ab}+i\,H_{ab}$ undergoes a complex rotation of angle $\psi=i\,\varphi$ in the yz plane [52], one easily finds that the transverse traceless components of F_{ab} (in the yz plane) acquire, under such a boost, a factor of order $\cos^2\psi=\cosh^2\phi=(1-\beta^2)^{-1}\equiv\Gamma_1^2$. Because of the special structure of the tensor $F_{ab}\propto{\rm diag}\,(-1,-1,2)$, with the third axis z labelling the radial direction, this reasoning shows that boosts in the radial (z) direction leave F_{ab} invariant. However, we are mainly interested here in boosts in a "tangential" direction, say x, associated with the fast motion of a circular orbit, and therefore orthogonal to the radial direction, which do introduce a factor Γ_1^2 in some of the boosted components of F_{ab}. For completeness, let us indicate that because of this special structure of F_{ab}, the invariant $J_{2e}=G_{ab}^2$ for general, *non-circular* orbits is equal to

$$J_{2e}=G_{ab}^2=\frac{6m_2^2}{r_s^6}\left(1+3\mathbf{u}_{\rm tg}^2+3\mathbf{u}_{\rm tg}^4\right)\,, \tag{6.50}$$

where $\mathbf{u}_{\rm tg}^2\equiv r_s^2((u^\theta)^2+\sin^2\theta(u^\phi)^2)$ is the square of the part of the 4-velocity u^μ that is tangent to the sphere. [The radial component of the 4-velocity brings no contribution to J_{2e}.]

The behavior near the light ring of the magnetic-quadrupole invariant $J_{2m}=\frac{1}{4}H_{ab}^2$ is understood in the same way as that of $J_{2e}=G_{ab}^2$. Concerning the other invariants, one can note that $J_{3e}=G_{abc}^2$ can be written as the sum

$$J_{3e}=G_{abc}^2=C_{\alpha\beta\gamma}\,C^{\alpha\beta\gamma}+\frac{1}{3\,c^2}\,J_{\dot{2}e} \tag{6.51}$$

where

$$C_{\alpha\beta\gamma}={\rm Sym}_{\alpha\beta\gamma}\,\nabla_\alpha\,R_{\beta\mu\gamma\nu}\,u^\mu\,u^\nu\,, \tag{6.52}$$

and

$$J_{\dot{2}e}=\dot{G}_{ab}^2=\dot{G}_{\alpha\beta}\,\dot{G}^{\alpha\beta} \tag{6.53}$$

with

$$\dot{G}_{\alpha\beta}=u^\lambda\,\nabla_\lambda\,R_{\alpha\mu\beta\nu}\,u^\mu\,u^\nu\,. \tag{6.54}$$

Similarly to G_{ab}^2, Eq. (6.49), the term $C_{\alpha\beta\gamma}^2$ in Eq. (6.51) is quartic in u^μ and is therefore expected to blow up like Γ_1^4. On the other hand, though $\dot{G}_{\alpha\beta}$, Eq. (6.54), is cubic in u^μ, it only blows up like Γ_1^2 (so that $J_{\dot{2}e}\sim\Gamma_1^4$ and $J_{3e}\sim C^2+J_{\dot{2}e}\sim\Gamma_1^4$) because of the special geodetic-precession properties of the proper-time derivative operator $\nabla/d\tau=u^\lambda\,\nabla_\lambda$ (see, e.g., Sec. 3.6 of Ref. [53]).

D. A suggested "resummed" version of comparable-mass tidal actions

Having understood that the formal pole-like behavior, $\sim(1-3\,u)^{-1}$, in the test-mass limit of the electric-quadrupole A potential is linked to simple boost properties of G_{ab} near the light-ring orbit, and knowing that the EOB formalism predicts the existence of a formal analog of the usual Schwarzschild light ring at the EOB dimensionless radius $\hat{r}_{\rm LR}\equiv 1/u_{\rm LR}$, defined as the solution of

$$\tilde{A}(u_{\rm LR})=0\,, \tag{6.55}$$

with $\tilde{A}(u)$ defined in Eq. (5.21), it is natural to expect the (unknown) exact two-body version of the electric-quadrupole A potential to mathematically exhibit an analogous pole-like behavior of the form $\sim (1-\hat{r}_{\rm LR}\,u)^{-1}$. As we shall discuss elsewhere, such a mathematical behavior, linked to considering (within the EOB-simplifying approach advocated in Ref. [18]) what would happen if one formally considered (unstable) circular orbits with $u \to u_{\rm LR}$, does not mean that there is a real physical singularity in the EOB dynamics near $u = u_{\rm LR}$, but it indicates that higher-than-2PN contributions to the electric-quadrupole amplification factor $\hat{A}^{(2)}_{1\,{\rm electric}}(u) = 1 + \alpha_1^{2e}\,u + \alpha_2^{2e}\,u^2 + \alpha_3^{2e}\,u^3 + \cdots$ will probably be slowly convergent, and will tend to amplify further the corresponding tidal interaction. Such an extra amplification might, for instance, be physically important in the last orbits of comparable-mass neutron-star binaries (which will reach contact for values of u smaller than $u_{\rm LR}$).

This leads us to suggest that a more accurate value (for $u < u_{\rm LR}$) of the electric-quadrupole amplification factor is the following "resummed" version of Eq. (6.8):

$$\hat{A}^{(2)}_{1\,{\rm electric}}(u) = 1 + \alpha_1^{2e}\,u + \alpha_2^{2e}\,\frac{u^2}{1-\hat{r}_{\rm LR}\,u}\,, \qquad (6.56)$$

where α_1^{2e} and α_2^{2e} are given by Eqs. (6.9) and (6.10), and where $\hat{r}_{\rm LR} \equiv 1/u_{\rm LR}$ is the solution of Eq. (6.55). Similar resummed versions of the other amplification factors can be defined by incorporating in their PN-expanded versions the formal light-ring behaviors exhibited by the exact test-mass results of Eqs. (6.44)–(6.47).

Let us finally discuss several possible approximate values for $\hat{r}_{\rm LR}$ in the proposed Eq. (6.56). The simplest approximation consists of using the "Schwarzschild" value $\hat{r}^S_{\rm LR} = 3$. However, a better value might be obtained by taking a solution of Eq. (6.55) that incorporates more physical effects. This might require solving Eq. (6.55) numerically, with $A(u)$ being the full A potential (containing both Padé-resummed two-point-mass effects and the various tidal contributions). In order to have a feeling for the modification of $\hat{r}_{\rm LR}$ brought by incorporating these changes, let us consider solving Eq. (6.55) when using the following approximation to the full A potential:

$$A_{\rm approx}(u) = 1 - 2\,u + 2\,\nu\,u^3 - \kappa\,u^6 \qquad (6.57)$$

where

$$\kappa \;=\; \kappa_1^{(2)} + \kappa_2^{(2)} = 2\,k_1^{(2)}\,\frac{m_2}{m_1}\left(\frac{R_1\,c^2}{G(m_1+m_2)}\right)^5 + 2\,k_2^{(2)}\,\frac{m_1}{m_2}\left(\frac{R_2\,c^2}{G(m_1+m_2)}\right)^5 . \qquad (6.58)$$

Here, the term $+2\,\nu\,u^3$ is the 2PN-accurate point-mass modification of $A(u)$, while the term $-\kappa\,u^6$ is the leading-order tidal modification. Note that they have opposite signs. The corresponding expression of $\tilde{A}(u)$ reads

$$\tilde{A}_{\rm approx}(u) = 1 - 3\,u + 5\,\nu\,u^3 - 4\,\kappa\,u^6\,. \qquad (6.59)$$

The corresponding value of $u_{\rm LR} \equiv 1/\hat{r}_{\rm LR}$ is the solution close to $1/3$ of the equation

$$u_{\rm LR} = \frac{1}{3}\left[1 + 5\,\nu\,u^3_{\rm LR} - 4\,\kappa\,u^6_{\rm LR}\right]\,. \qquad (6.60)$$

If we could treat both ν and κ as small deformation parameters, this would imply that, to first order in these two deformation parameters, the value of $u_{\rm LR}(\nu,\kappa)$ would be obtained by inserting the leading-order value $u_{\rm LR} \simeq 1/3$ in the right-hand side of Eq. (6.60). This would yield

$$u_{\rm LR}(\nu,\kappa) = \frac{1}{3}\left[1 + \frac{5}{3^3}\,\nu - \frac{4}{3^6}\,\kappa + O(\nu^2,\nu\kappa,\kappa^2)\right]\,, \qquad (6.61)$$

and

$$\hat{r}_{\rm LR}(\nu,\kappa) = 3\left[1 - \frac{5}{3^3}\,\nu + \frac{4}{3^6}\,\kappa + O(\nu^2,\nu\kappa,\kappa^2)\right]\,. \qquad (6.62)$$

Note that while comparable-mass corrections ($\propto \nu$) have the effect of decreasing $\hat{r}_{\rm LR}$, tidal ones ($\propto \kappa$) have the opposite effect of increasing $\hat{r}_{\rm LR}$. Let us focus on the tidal effects, and consider the equal-mass case with $R_1 = R_2$ and $k_1^{(2)} = k_2^{(2)}$. One has a first order increase of $\hat{r}_{\rm LR}$ equal to

$$\delta^{\rm tidal}\,\hat{r}_{\rm LR} \simeq 16\,k_1^{(2)}\left(\frac{R_1\,c^2}{6\,Gm_1}\right)^5 = 16\,k_1^{(2)}\,\frac{1}{(6\,\mathcal{C}_1)^5}\,, \qquad (6.63)$$

where $\mathcal{C}_1 \equiv Gm_1/(c^2\,R_1)$ denotes the common compactness of the two neutron stars. This simple approximate analytical formula shows that $\delta^{\rm tidal}\,\hat{r}_{\rm LR}$ is very sensitive to the value of the compactness of the neutron star. If $\mathcal{C}_1 = 1/6 = 0.166667$, i.e., $R_1 = 6\,Gm_1/c^2$ (roughly corresponding to a radius of 12 km for a $1.4\,M_\odot$ neutron star), then $\delta^{\rm tidal}\,\hat{r}_{\rm LR} = 1.44\,(k_1^{(2)}/0.09)$ will be of order 1 [the value $k_1^{(2)} = 0.09$ being typical for $\mathcal{C}_1 = 1/6$; see, e.g., Table II in Ref. [5]]. On the other hand, if $\hat{R}_1 \equiv R_1\,c^2/(Gm_1)$ is slightly smaller than 6, $\delta^{\rm tidal}\,\hat{r}_{\rm LR}$ will quickly become much smaller than 1, while if $\hat{R}_1$ is slightly larger than 6, $\delta^{\rm tidal}\,\hat{r}_{\rm LR}$ will quickly become formally large (thereby invalidating the first-order analytical estimate of Eq. (6.63), which assumed $\delta\,\hat{r}_{\rm LR} \ll 3$). These rough estimates indicate that in many cases, tidal effects on $\hat{r}_{\rm LR}$ will be quite important and will significantly increase the numerical value of $\hat{r}_{\rm LR}$. Note that an increased value of $\hat{r}_{\rm LR}$ will, in turn, *increase* the effect of the conjectured resummed 2PN contribution $\alpha_2^{2e}\,u^2/(1-\hat{r}_{\rm LR}\,u)$ to $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$.

VII. SUMMARY AND CONCLUSIONS

Using an effective action technique, we have shown how to compute the additional terms in the *reduced* (Fokker) two-body Lagrangian $L(\boldsymbol{y}_1,\boldsymbol{y}_2,\dot{\boldsymbol{y}}_1,\dot{\boldsymbol{y}}_2)$ that are linked to tidal interactions. Thanks to a general property of perturbed Fokker actions [explained at the end of Sec. II, see Eq. (2.20)], the additional tidal terms are correctly obtained (to first order in the tidal perturbations) by replacing in the complete, unreduced action $S[g_{\mu\nu}; y_1, y_2]$ the gravitational field $g_{\mu\nu}$ by the solution of Einstein's equations generated by two structureless point masses $m_1, \boldsymbol{y}_1; m_2, \boldsymbol{y}_2$. This allowed us to compute in a rather straightforward manner the reduced tidal action at the 2PN fractional accuracy by using the known, explicit form of the 2PN-accurate two-point-mass metric [36–39]. The main technical subtlety in this calculation is the regularization of the self-field effects associated with the computation of the various nonminimal tidal-action terms $\sim \int d\tau (R_{\alpha\mu\beta\nu}\, u^\mu\, u^\nu)^2 + \ldots$, where, e.g., $R_{\alpha\mu\beta\nu}(x; y_1, y_2)$ is to be evaluated on one of the worldlines that generate the metric $g_{\mu\nu}$ (so that $R_{\alpha\mu\beta\nu}(y_1; y_1, y_2)$ is formally infinite). We explained in detail (in Sec. III) one (algorithmic) way to perform this regularization, using Hadamard regularization (which is equivalent to dimensional regularization at the 2PN level). We then computed the regular parts of the brick potentials that parametrize the 2PN metric, from which we derived the regularized values of several of the geometrical invariants entering the nonminimal worldline tidal action terms. [See Eqs. (4.4)–(4.10) for the 2PN-accurate Lagrangians (for general orbits) of the three leading tidal terms (electric quadrupole, electric octupole and magnetic quadrupole)]. We then focussed on the most physically useful information contained in these actions, namely the corresponding contributions to the EOB main radial potential, $A(u)$, with $u = G(m_1 + m_2)/(c^2\, r_{\rm EOB})$. Our Eqs. (5.19), (5.20), (5.28) gave the explicit transformation between the previously derived harmonic-coordinates tidal Lagrangians and their corresponding contributions to the EOB A potential. Using this transformation, we could finally explicitly compute the most important tidal contributions to the EOB A potential to a higher accuracy than had been known before: namely, we computed the quadrupolar ($\ell = 2$) and octupolar ($\ell = 3$) gravito-electric tidal contributions to 2PN fractional accuracy, i.e., with the inclusion of a relativistic distance-dependent factor of the type $u^{2\ell+2}(1 + \alpha_1\, u + \alpha_2\, u^2)$ [see Eqs. (6.6)–(6.10) and (6.18)–(6.22)]. We also computed the quadrupolar gravito-magnetic tidal contribution, as well as a newly introduced time-differentiated electric quadrupolar tidal term, to 1PN fractional accuracy [see Eqs. (6.25)–(6.28), (6.30)–(6.33)]. Of most interest among these results is the obtention of the 2PN coefficient α_2^{2e} entering the distance-dependence of the electric quadrupolar term. We found that this coefficient, Eq. (6.10), is always positive and varies between $+\,3$ and $+\,15.16071$ as the mass fraction $X_1 = m_1/(m_1 + m_2)$ of the considered tidally deformed body varies between 0 and 1. In the equal-mass case, $m_1 = m_2$, i.e. $X_1 = \frac{1}{2}$, we found that $\alpha_2^{2e} = 6.07143$. This value shows that, when the neutron stars near their contact, 2PN effects are comparable to 1PN ones. Indeed, contact occurs when the separation $r \simeq R_1 + R_2 = Gm_1/(c^2\, \mathcal{C}_1) + Gm_2/(c^2\, \mathcal{C}_2)$ (where $\mathcal{C}_A \equiv Gm_A/(c^2\, R_A)$, $A = 1, 2$, are the two compactnesses). In the equal-mass case (with $\mathcal{C}_1 = \mathcal{C}_2$), this shows that, at contact, $u = G(m_1 + m_2)/(c^2\, r)$ is approximately equal to $u_{\rm contact} \simeq \mathcal{C}_1$. If we consider as typical neutron star a star of mass $1.4\, M_\odot$ and radius 12 km, we expect $\mathcal{C}_1 \sim 1/6$, i.e. $u_{\rm contact} \sim 1/6$. The successive PN contributions to the distance-dependent amplification factor $\hat{A}^{(2)2{\rm PN}}_{1\,{\rm electric}}(u) = 1 + \alpha_1^{2e}\, u + \alpha_2^{2e}\, u^2$ of the electric quadrupolar tidal interaction for the first body then becomes, at contact,

$$\hat{A}^{(2)2{\rm PN}}_{1\,{\rm electric}}(u_{\rm contact}) \simeq 1 + \alpha_1^{2e}\, \mathcal{C}_1 + \alpha_2^{2e}\, \mathcal{C}_1^2 \sim 1 + \frac{1.25}{6} + \frac{6.07143}{6^2}\,, \qquad (7.1)$$

where one sees that the 2PN ($O(u^2)$) contribution is numerically comparable to the 1PN one. This suggests that the PN-expanded form of the tidal amplification factor $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$ is slowly converging and could get comparable or even larger contributions from higher powers of u (i.e., 3PN and higher terms). In order to get a feeling about the possible origin of this slow convergence of the PN expansion, we followed the approach of Ref. [54], i.e., we looked for the existence of a nearby pole (in the complex u plane) within the formal analytic continuation of the considered function $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$. [Ref. [54] considered the energy flux F as a function of $x = (GM\,\Omega/c^3)^{2/3}$; it pointed out that $F(x)$ had (in the test-mass limit) a pole at the light-ring value $x = 1/3$ and recommended improving the PN expansion of $F(x)$ (for $x < 1/3$) by a Padé-type resummation incorporating the existence of this pole in $F(x)$.] By computing the exact test-mass limit of the function $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$, we found that it formally exhibits a pole located at the light-ring value $u_{\rm LR}^{\rm test\ mass} = 1/3$ [see Eq. (6.44)]. Such a pole is also present in other amplification factors [see Eqs. (6.45)–(6.47)], and we discussed its origin. [Note that two equal-mass neutron stars will get in contact before reaching this pole. However the idea here is that the hidden presence of this pole in the analytical continuation of the function $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$ is behind the bad convergence of the Taylor expansion of this function in powers of u.] This led us to suggest that one might get an improved value of the tidal amplification factor $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$ by formally incorporating the presence of this pole in the following Padé-resummed manner:

$$\hat{A}^{(2)}_{1\,{\rm electric}}(u) = 1 + \alpha_1^{2e}\, u + \alpha_2^{2e}\, \frac{u^2}{1 - \hat{r}_{\rm LR}\, u}\,, \qquad (7.2)$$

where $\hat{r}_{\rm LR} \equiv 1/u_{\rm LR}$ is the (EOB-defined) dimensionless light-ring radius, i.e., the solution of Eq. (6.55), with $\tilde{A}(u)$ defined by Eq. (5.21). Let us point out that Eq. (7.2) is equivalent to saying that the 2PN coefficient α_2^{2e} becomes replaced by the effective distance-dependent coefficient $\alpha_2^{\rm eff}(u) \equiv \alpha_2^{2e}/(1 - \hat{r}_{\rm LR}\, u)$. Note that $\alpha_2^{\rm eff}(u) > \alpha_2^{2e}$. In particular, for the "typical" compactness $\mathcal{C}_1 = \mathcal{C}_2 \sim 1/6$ considered above, and when using the unperturbed value of $\hat{r}_{\rm LR}$, i.e. $\hat{r}_{\rm LR}^{(0)} = 3$, the effective value $\alpha_2^{\rm eff}(u)$ will, at contact (i.e. when $u = u_{\rm contact} \simeq \mathcal{C}_1 \sim 1/6$), be equal to $\alpha_2^{\rm eff}(u_{\rm contact}) \simeq \alpha_2^{2e}/(1-3\,\mathcal{C}_1) \sim \alpha_2^{2e}/(1-3/6) \sim 2\,\alpha_2^{2e} \sim 12$. We recalled in the Introduction that several comparisons between the analytical (EOB) description of tidal effects and numerical simulations of tidally interacting binary neutron stars [5, 9, 10] have suggested the need for significant amplification factors $\hat{A}^{(2)}_{1\,{\rm electric}}(u)$ parametrized by rather large values of α_2^{2e}. However, up to now, the numerical results that have been used have been affected by numerical errors that have not been fully controlled. In particular, in the recent comparisons [9, 10], one did not have in hand sufficiently many simulations with different resolutions for being able to compute and subtract the finite-resolution error. We hope that a more complete analysis will be performed soon (see, in this respect, Refs. [61, 62]). We recommend comparing resolution-extrapolated numerical data to the pole-improved amplification factor of Eq. (7.2). As discussed in Sec. VI, it might be necessary to use as value of $\hat{r}_{\rm LR}$ the improved estimate obtained from the full (tidally modified) value of the A potential. This suggests (especially for compactnesses $\mathcal{C}_1 \lesssim 1/6$) as discussed above that $\hat{r}_{\rm LR}$ might be significantly larger than 3, thereby further amplifying the effective value of α_2^{2e} during the last stages of the inspiral.

The present study has focused on the 2PN tidal effects in the interaction Hamiltonian. There is also a 2PN tidal effect in the radiation reaction, which has contributions from various tidally modified multipolar waveforms. The tidal contribution to each (circular) multipolar gravitational waveform can be parametrized (following Refs. [5, 10]) as an additional term of the form

$$h_{\ell m}^{\rm tidal}(x) = \sum_J h_{\ell m}^{(J)\,{\rm LO}}(x)\,\hat{h}_{\ell m}^{(J)\,{\rm tail}}(x)\,\hat{h}_{\ell m}^{(J)\,{\rm PN}}(x)\,, \quad (7.3)$$

where $x \equiv (G(m_1+m_2)\,\Omega/c^3)^{2/3}$; J labels the various tidal geometrical invariants, such as $J_{2e} \equiv G_{\alpha\beta}\,G^{\alpha\beta}$; $h_{\ell m}^{(J)\,{\rm LO}}(x)$ denotes the leading-order (i.e., Newtonian-order) tidal waveform; $\hat{h}_{\ell m}^{(J)\,{\rm tail}}(x)$ the effect of tails [55, 56] and their resummed EOB form [57]; while

$$\hat{h}_{\ell m}^{(J)\,{\rm PN}}(x) = 1 + \beta_1^{(J\ell m)} x + \beta_2^{(J\ell m)} x^2 + \dots \quad (7.4)$$

denotes the effect of higher PN contributions. The 1PN coefficient $\beta_1^{(J_{2e}22)}$ is known [7, 15]. The other 1PN coefficients needed for deriving a 2PN-accurate flux can be obtained from applying the simple 1PN-accurate formalism of Eq. [40]. It is more challenging to compute the 2PN coefficient $\beta_2^{(J_{2e}22)}$. Indeed, this requires applying the 2PN-accurate version [56] of the Blanchet-Damour-Iyer wave-generation formalism [40, 58–60] to the tidal-modified Einstein equations (2.13). Let us, however, note that although from a PN point of view, the 2PN coefficient $\beta_2^{(J_{2e}22)}$ contributes to the phasing of coalescing binaries at the same formal level as the dynamical 2PN coefficient α_2^{2e} determined above, it has been found in Refs. [9, 15] that (if $\beta_2^{(J_{2e}22)} \sim \alpha_2^{2e}$) it has a significantly smaller observable effect.

Let us finally point out that our general result in Eq. (2.20) also opens the possibility of computing the 3PN coefficient α_3^{2e} in the PN-expanded amplification factor of the electric quadrupolar tidal interaction $\hat{A}^{(2)}_{1\,{\rm electric}}(u) = 1 + \alpha_1^{2e}\,u + \alpha_2^{2e}\,u^2 + \alpha_3^{2e}\,u^3 + O(u^4)$. This computation would, however, be much more involved than the calculation of α_2^{2e} because of the technical subtleties in the regularization of self-field effects at the 3PN level [43, 63–65] that necessitate using dimensional regularization [25, 26] instead of Hadamard regularization.

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Appendix A: Explicit forms of the (time-symmetric) 2PN-accurate brick potentials

The explicit forms of the (time-symmetric) 2PN-accurate brick potentials V, V_i, etc. are [39]

$$\begin{aligned} V &= \frac{Gm_1}{r_1} + \frac{Gm_1}{c^2}\left(-\frac{(n_1v_1)^2}{2r_1} + \frac{2v_1^2}{r_1} + Gm_2\left(-\frac{r_1}{4r_{12}^3} - \frac{5}{4r_1r_{12}} + \frac{r_2^2}{4r_1r_{12}^3}\right)\right) \\ &+ \frac{Gm_1}{c^4 r_1}\left(\frac{3(n_1v_1)^4}{8} - \frac{3(n_1v_1)^2 v_1^2}{2} + 2v_1^4\right) \end{aligned}$$

$$
\begin{aligned}
&+\ \frac{G^2m_1m_2}{c^4}\left\{v_1^2\left(\frac{3r_1^3}{16r_{12}^5}-\frac{37r_1}{16r_{12}^3}-\frac{1}{r_1r_{12}}-\frac{3r_1r_2^2}{16r_{12}^5}+\frac{r_2^2}{r_1r_{12}^3}\right)\right.\\
&\qquad +v_2^2\left(\frac{3r_1^3}{16r_{12}^5}+\frac{3r_1}{16r_{12}^3}+\frac{3}{2r_1r_{12}}-\frac{3r_1r_2^2}{16r_{12}^5}+\frac{r_2^2}{2r_1r_{12}^3}\right)\\
&\qquad +(v_1v_2)\left(-\frac{3r_1^3}{8r_{12}^5}+\frac{13r_1}{8r_{12}^3}-\frac{3}{r_1r_{12}}+\frac{3r_1r_2^2}{8r_{12}^5}-\frac{r_2^2}{r_1r_{12}^3}\right)\\
&\qquad +(n_{12}v_1)^2\left(-\frac{15r_1^3}{16r_{12}^5}+\frac{57r_1}{16r_{12}^3}+\frac{15r_1r_2^2}{16r_{12}^5}\right)\\
&\qquad +(n_{12}v_2)^2\left(-\frac{15r_1^3}{16r_{12}^5}-\frac{33r_1}{16r_{12}^3}+\frac{7}{8r_1r_{12}}+\frac{15r_1r_2^2}{16r_{12}^5}-\frac{3r_2^2}{8r_1r_{12}^3}\right)\\
&\qquad +(n_{12}v_1)(n_{12}v_2)\left(\frac{15r_1^3}{8r_{12}^5}-\frac{9r_1}{8r_{12}^3}-\frac{15r_1r_2^2}{8r_{12}^5}\right)\\
&\qquad +(n_1v_1)(n_{12}v_1)\left(-\frac{3r_1^2}{2r_{12}^4}+\frac{3}{4r_{12}^2}+\frac{3r_2^2}{4r_{12}^4}\right)+(n_1v_2)(n_{12}v_1)\left(\frac{3r_1^2}{4r_{12}^4}+\frac{2}{r_{12}^2}\right)\\
&\qquad +(n_1v_1)(n_{12}v_2)\left(\frac{3r_1^2}{2r_{12}^4}+\frac{13}{4r_{12}^2}-\frac{3r_2^2}{4r_{12}^4}\right)+(n_1v_2)(n_{12}v_2)\left(-\frac{3r_1^2}{4r_{12}^4}-\frac{3}{2r_{12}^2}\right)\\
&\qquad \left.+(n_1v_1)^2\left(-\frac{r_1}{8r_{12}^3}+\frac{7}{8r_1r_{12}}-\frac{3r_2^2}{8r_1r_{12}^3}\right)+\frac{(n_1v_1)(n_1v_2)r_1}{2r_{12}^3}\right\}\\
&+\ \frac{G^3m_1^2m_2}{c^4}\left(-\frac{r_1^3}{8r_{12}^6}+\frac{5r_1}{8r_{12}^4}+\frac{3}{4r_1r_{12}^2}+\frac{r_1r_2^2}{8r_{12}^6}-\frac{5r_2^2}{4r_1r_{12}^4}\right)\\
&+\ \frac{G^3m_1m_2^2}{c^4}\left(-\frac{r_1^3}{32r_{12}^6}+\frac{43r_1}{16r_{12}^4}+\frac{91}{32r_1r_{12}^2}-\frac{r_1r_2^2}{16r_{12}^6}-\frac{23r_2^2}{16r_1r_{12}^4}+\frac{3r_2^4}{32r_1r_{12}^6}\right)+O(6)+1\leftrightarrow 2\ , \qquad \text{(A1)}
\end{aligned}
$$

$$
\begin{aligned}
V_i\ &=\ \frac{Gm_1v_1^i}{r_1}+n_{12}^i\frac{G^2m_1m_2}{c^2r_{12}^2}\left((n_1v_1)+\frac{3(n_{12}v_{12})r_1}{2r_{12}}\right)\\
&+\ \frac{v_1^i}{c^2}\left\{\frac{Gm_1}{r_1}\left(-\frac{(n_1v_1)^2}{2}+v_1^2\right)+G^2m_1m_2\left(-\frac{3r_1}{4r_{12}^3}+\frac{r_2^2}{4r_1r_{12}^3}-\frac{5}{4r_1r_{12}}\right)\right\}\\
&+\ v_2^i\frac{G^2m_1m_2r_1}{2c^2r_{12}^3}+O(4)+1\leftrightarrow 2\ , \qquad \text{(A2)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{W}_{ij}\ &=\ \delta^{ij}\left(-\frac{Gm_1v_1^2}{r_1}-\frac{G^2m_1^2}{4r_1^2}+\frac{G^2m_1m_2}{r_{12}S}\right)+\frac{Gm_1v_1^iv_1^j}{r_1}+\frac{G^2m_1^2n_1^in_1^j}{4r_1^2}\\
&+\ G^2m_1m_2\left\{\frac{1}{S^2}\left(n_1^{(i}n_2^{j)}+2n_1^{(i}n_{12}^{j)}\right)-n_{12}^in_{12}^j\left(\frac{1}{S^2}+\frac{1}{r_{12}S}\right)\right\}+O(2)+1\leftrightarrow 2\ , \qquad \text{(A3)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{R}_i\ &=\ G^2m_1m_2n_{12}^i\left\{-\frac{(n_{12}v_1)}{2S}\left(\frac{1}{S}+\frac{1}{r_{12}}\right)-\frac{2(n_2v_1)}{S^2}+\frac{3(n_2v_2)}{2S^2}\right\}\\
&+\ n_1^i\left\{\frac{G^2m_1^2(n_1v_1)}{8r_1^2}+\frac{G^2m_1m_2}{S^2}\left(2(n_{12}v_1)-\frac{3(n_{12}v_2)}{2}+2(n_2v_1)-\frac{3(n_2v_2)}{2}\right)\right\}\\
&+\ v_1^i\left\{-\frac{G^2m_1^2}{8r_1^2}+G^2m_1m_2\left(\frac{1}{r_1r_{12}}+\frac{1}{2r_{12}S}\right)\right\}-v_2^i\frac{G^2m_1m_2}{r_1r_{12}}+O(2)+1\leftrightarrow 2\ , \qquad \text{(A4)}
\end{aligned}
$$

$$
\hat{X}\ =\ \frac{G^2m_1^2}{8r_1^2}\left((n_1v_1)^2-v_1^2\right)+G^2m_1m_2v_1^2\left(\frac{1}{r_1r_{12}}+\frac{1}{r_1S}+\frac{1}{r_{12}S}\right)
$$

$$
\begin{aligned}
&+\ G^2 m_1 m_2 \left\{ v_2^2 \left(-\frac{1}{r_1 r_{12}} + \frac{1}{r_1 S} + \frac{1}{r_{12} S} \right) - \frac{(v_1 v_2)}{S} \left(\frac{2}{r_1} + \frac{3}{2 r_{12}} \right) - \frac{(n_{12} v_1)^2}{S} \left(\frac{1}{S} + \frac{1}{r_{12}} \right) \right. \\
&\qquad - \frac{(n_{12} v_2)^2}{S} \left(\frac{1}{S} + \frac{1}{r_{12}} \right) + \frac{3 (n_{12} v_1)(n_{12} v_2)}{2S} \left(\frac{1}{S} + \frac{1}{r_{12}} \right) + \frac{2 (n_{12} v_1)(n_1 v_1)}{S^2} \\
&\qquad - \frac{5 (n_{12} v_2)(n_1 v_1)}{S^2} - \frac{(n_1 v_1)^2}{S} \left(\frac{1}{S} + \frac{1}{r_1} \right) + \frac{2 (n_{12} v_2)(n_1 v_2)}{S^2} \\
&\qquad + \frac{2 (n_1 v_1)(n_1 v_2)}{S} \left(\frac{1}{S} + \frac{1}{r_1} \right) - \frac{(n_1 v_2)^2}{S} \left(\frac{1}{S} + \frac{1}{r_1} \right) - \frac{2 (n_{12} v_2)(n_2 v_1)}{S^2} \\
&\qquad \left. + \frac{2 (n_1 v_2)(n_2 v_1)}{S^2} - \frac{3 (n_1 v_1)(n_2 v_2)}{2 S^2} \right\} + \frac{G^3 m_1^3}{12 r_1^3} \\
&+\ G^3 m_1^2 m_2 \left(\frac{1}{2 r_1^3} + \frac{1}{16 r_2^3} + \frac{1}{16 r_1^2 r_2} - \frac{r_2^2}{2 r_1^2 r_{12}^3} + \frac{r_2^3}{2 r_1^3 r_{12}^3} - \frac{r_1^2}{32 r_2^3 r_{12}^2} - \frac{3}{16 r_2 r_{12}^2} + \frac{15 r_2}{32 r_1^2 r_{12}^2} \right. \\
&\qquad \left. - \frac{r_2^2}{2 r_1^3 r_{12}^2} - \frac{r_2}{2 r_1^3 r_{12}} - \frac{r_{12}^2}{32 r_1^2 r_2^3} \right) + G^3 m_1 m_2^2 \left(-\frac{1}{2 r_{12}^3} + \frac{r_2}{2 r_1 r_{12}^3} - \frac{1}{2 r_1 r_{12}^2} \right) \\
&\qquad + O(2) + 1 \leftrightarrow 2\ . \qquad \text{(A5)}
\end{aligned}
$$

Here $\boldsymbol{r}_1 \equiv \boldsymbol{x} - \boldsymbol{y}_1$, $r_1 \equiv |\boldsymbol{r}_1|$, $\boldsymbol{n}_1 \equiv \boldsymbol{r}_1/r_1$, $\boldsymbol{r}_2 \equiv \boldsymbol{x} - \boldsymbol{y}_2$, etc., $\boldsymbol{y}_{12} \equiv \boldsymbol{y}_1 - \boldsymbol{y}_2$, $r_{12} \equiv |\boldsymbol{y}_{12}|$, $\boldsymbol{n}_{12} \equiv \boldsymbol{y}_{12}/r_{12}$, $\boldsymbol{v}_{12} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2$, $(n_{12}\, v_1) \equiv \boldsymbol{n}_{12} \cdot \boldsymbol{v}_1$. In addition, the notation $1 \leftrightarrow 2$ means adding the terms obtained by exchanging the particle labels 1 and 2, while the quantity S denotes the perimeter of the triangle defined by $\boldsymbol{x}$, $\boldsymbol{y}_1$ and $\boldsymbol{y}_2$, viz.

$$
S \equiv r_1 + r_2 + r_{12}\, . \qquad \text{(A6)}
$$

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[66] What is important for the current discussion (i.e. the application to tidal interactions) is the absence of preferred directions in the *orbital* dynamics $\boldsymbol{y}, \boldsymbol{v}$ or $\boldsymbol{y}, \boldsymbol{p}_y$. This is the case for the *reduced* tidal actions (generated by orbital-induced tidal moments).