# On-shell Techniques and Universal Results in Quantum Gravity

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# On-shell Techniques and Universal Results in Quantum Gravity

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We compute the leading post-Newtonian and quantum corrections to the Coulomb and Newtonian potentials using the full modern arsenal of on-shell techniques; we employ spinor-helicity variables everywhere, use the Kawai-Lewellen-Tye (KLT) relations to derive gravity amplitudes from gauge theory and use unitarity methods to extract the terms needed at one-loop order. We stress that our results are *universal* and thus will hold in any quantum theory of gravity with the same low-energy degrees of freedom as we are considering. Previous results for the corrections to the same potentials, derived historically using Feynman graphs, are verified explicitly, but our approach presents a huge simplification, since starting points for the computations are compact and tedious index contractions and various complicated integral reductions are eliminated from the onset, streamlining the derivations. We also analyze the spin dependence of the results using the KLT factorization, and show how the spinless correction in the framework are easily seen to be independent of the interacting matter considered.

## CONTENTS

## I. INTRODUCTION

Unitarity based methods combined with the helicity formalism have proven exceptionally successful in gauge theory calculations at one loop (see *e.g.* [1, 2]). Such methods have so far been less frequently applied to general relativity [3–5], and quantum corrections to gravitational systems with massive matter have not been studied in this framework at all. However, such techniques are well-suited for effective field theory considerations in low energy quantum gravity. Early treatments of gravitational loops tended to focus on the ultraviolet divergences, but effective field theory methods have allowed us to separate these ultraviolet divergences from the universal reliable predictions of the low energy portion of the theory [6–12]. The unitarity methods deal directly with on-shell and low energy amplitudes, and products of on-shell tree amplitudes can therefore yield the low energy one-loop results in a conceptually simple manner.

In this paper, we apply new on-shell amplitude methods to the gravitational scattering of massive matter [6–9]. The unitarity cut for the leading quantum corrections involves the gravitational Compton amplitude, *i.e.* the two on-shell gravitons coupled to matter. For matter fields of all spins, this amplitude has a simple structure, as it is related to the square of the electromagnetic Compton amplitude (involving photons) [3, 13–16]. A useful observation for our calculation is that computing the massless two-particle cut gives us exactly everything we need. The cut of the amplitude is precisely one-to-one with the non-analytic parts of the amplitude that contributes to the long-distance leading corrections to the scattering potential at one-loop. Hence, we do not need to reconstruct the full amplitude - we only need to consider the terms contributing to the massless two-particle cut.

Moreover, there is an added bonus in using the cut and decomposing the amplitudes using KLT; in such a setup one can easily dissect the interaction between the two particles into a series of spin corrections; *i.e.* a coefficient for the spinless interaction and coefficients of spin-spin interactions, *etc.* It has been seen before in direct calculations [11] that such a series of spin corrections is always independent of the type of interacting matter. For example, if we disregard the spin couplings, fermions and bosons couple generically only through the energy of their currents. This observation appears however to be somewhat puzzling in the context of Feynman diagrams, because here the vertex rules (and even the diagrams that need to be calculated) differ greatly for different types of matter particles. In this paper our focus will be on the spinless interaction part of the series of spin corrections. We will demonstrate directly using the on-shell cut method and KLT that this coupling is always identical for any type of matter interaction, and in the non-relativstic limit only dependent on the masses of the interacting particles.

The classical and quantum corrections to the Newtonian potential can addressed by studying the scattering matrix element in the non-relativistic limit

$$\langle p_1, p_2|iT|p_3, p_4\rangle = -i\, M(q)\, (2\pi)^4\, \delta^{(4)}(p - p')\,, \tag{I.1}$$

where $\vec{q}$ is the momentum transfer. In momentum space (in the non-relativistic and free particle limit) we employ the following definition of the potential $V(q)$ from the amplitude

$$V(q) = \frac{M(q)}{4m_1 m_2}\,. \tag{I.2}$$

The one-loop diagrams produce modifications to the tree interaction leading to a potential of the form

$$V(q) = \frac{G_N m_1 m_2}{\vec{q}^{\,2}} \Big[-4\pi + C^{NP}\, G_N(m_1 + m_2)\sqrt{|\vec{q}|^2} + G_N \hbar\, \vec{q}^{\,2} \Big(C^{QG} \log(\vec{q}^{\,2}) + \tilde{C}^{QG}\Big)\Big]\,. \tag{I.3}$$

If this object is Fourier transformed to form a spatial potential, the term with the square-root yields the classical $G_N m/r$ general relativistic correction to the potential, and the term with the logarithm produces a long-distance $G_N\hbar/r^2$ quantum correction. The analytic correction without a logarithm will yield a short range $\delta^3(r)$ effect in the potential. The non-analytic terms (the square-root and the logarithm) arise from long-distance propagation of the massless gravitons, and hence are genuinely low-energy quantum predictions. These can be calculated in the effective field theory approach. The analytic correction $\tilde{C}^{QG}$, however is not a prediction of the low-energy theory as it is sensitive to the coefficients of higher curvature terms in the gravitational action.

Our work in the present paper will focus on the square-root and logarithmic non-analytic terms of the scattering potential.

The plan of the paper is as follows. In Sec. II we discuss the relations between the gravitational Compton amplitude to the square of the electromagnetic one. In Sec. III we compute the one-loop amplitude in the helicity formalism. Here we first calculate the electromagnetic case as a warm up before moving on to our primary interest of the gravitational interaction. In Sec. IV we evaluate the amplitude in the covariant harmonic gauge and compare with the Feynman approach used in earlier computations. In Sec. V we discuss the matter-independence of the non-analytic long-range contributions to the amplitude. Finally, Sec. VI contains our conclusions and discussion. In Appendix A we list the covariant Feynman rules and Appendix B discusses an alternative evaluation of the cut using dispersion relations.

## II. THE GRAVITATIONAL COMPTON AMPLITUDE

In this section we will show how one can represent the gravitational Compton scattering of two gravitons off a massive target of spin $s = 0, \frac{1}{2}, 1$ as the square of the QED (Abelian) Compton scattering. We will do this first using covariant amplitudes, and then more compactly using the helicity formalism. The advantage of this approach is that one can use the known expressions for the massive tree-level amplitudes in Yang-Mills and QED to obtain in a condensed way the massive tree-level amplitudes in gravity. As well, the connection between the gravity and the QED amplitude will be instrumental in deriving the matter-independence results in section V.

### A. Covariant notation

We will evaluate the one-loop amplitude by considering the unitarity cut across the graviton lines in section II C and III B, thus we need to construct the tree-level amplitudes for the emission of two gravitons.

The tree amplitudes needed in this analysis can be constructed in various way. One direct covariant approach is to use the background field vertices derived in [7, 8]. These vertices are listed in Appendix A. The vertex $\tau_1^{\mu\nu}(p_1, p_2)$ given in eq. (A.1) describes the emission of a graviton from a massive scalar exchange. Because the metric is realized through the stress-energy tensor, the vertex couples identically to quantum $h^{\mu\nu}$ or background fields $H^{\mu\nu}$ (as used in refs. [7, 8]). The vertex $\tau_2^{\mu\nu;\rho\sigma}(p_1, p_2)$ given in eq. (A.2) is the four point interaction between two massive scalars and two gravitons. Again the coupling between gravity and the scalar through the stress-energy tensor implies that these vertices are the same for quantum or background fields.

In order to compute the general relativity correction and the quantum correction arising from the one-loop diagram we need the tree-level amplitude for emitting two gravitons as

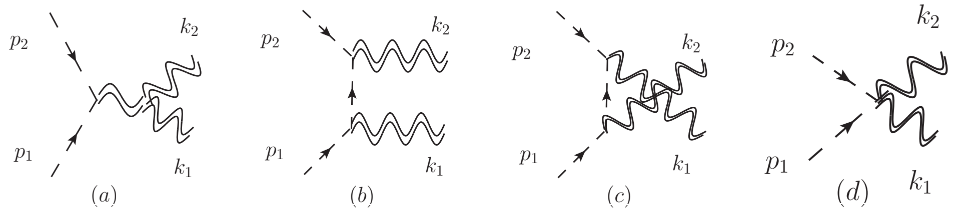


FIG. 1. The various contributions to the tree-amplitude $\phi + \phi \to 2$ *gravitons*: (a) $s$-channel, (b) $t$-channel, (c) $u$-channel, (d) contact term.

illustrated in figure 1. In the covariant approach using the background field vertices the tree-level amplitude for emitting two quantum gravitons $h$ of polarization $\epsilon_1^{\alpha\beta}(k_1)$ and $\epsilon_2^{\gamma\delta}(k_2)$ is given by (with all incoming external momenta)

$$\begin{aligned} iM^{\text{tree}}(p_1,p_2,k_1,k_2) &= \tau_1^{\mu\nu}(p_1,p_2)\,\frac{i\mathcal{P}_{\mu\nu;\rho\sigma}}{q^2+i\varepsilon}\,\tau_3^{\rho\sigma}{}_{\alpha\beta;\gamma\delta}(k_1,k_2,p_1+p_2)\,\epsilon_1^{\alpha\beta}(k_1)\epsilon_2^{\gamma\delta}(k_2)\\ &+\frac{\tau_{1\,\alpha\beta}(p_1,-p_1-k_1)\,i\,\tau_{1\,\gamma\delta}(p_1+k_1,p_2)}{(p_1+k_1)^2-m^2+i\varepsilon}\,\epsilon_1^{\alpha\beta}(k_1)\epsilon_2^{\gamma\delta}(k_2)\\ &+\frac{\tau_{1\,\gamma\delta}(p_1,-p_1-k_2)\,i\,\tau_{1\,\alpha\beta}(p_1+k_2,p_2)}{(p_1+k_2)^2-m^2+i\varepsilon}\,\epsilon_1^{\alpha\beta}(k_1)\epsilon_2^{\gamma\delta}(k_2)\\ &+\tau_{2\,\alpha\beta;\gamma\delta}(p_1,p_2)\,\epsilon_1^{\alpha\beta}(k_1)\epsilon_2^{\gamma\delta}(k_2)\,, \end{aligned} \tag{II.1}$$

with

$$\mathcal{P}_{\alpha\beta;\gamma\delta} = \frac{1}{2}\left[\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\beta\gamma}\eta_{\alpha\delta}-\eta_{\alpha\beta}\eta_{\gamma\delta}\right]\,, \tag{II.2}$$

in harmonic or de Donder gauge [17]. The three-graviton vertex $\tau_3^{\mu\nu}{}_{\alpha\beta;\gamma\delta}$, given in eq. (A.3), between two quantum fields $h$ and one background field $H$ differs from the vertex for three quantum gravitons derived by De Witt [18] and Sannan [19]. We have checked that the on-shell amplitude constructed with the three-graviton vertices derived in [18, 19] leads to the same answer as ours. Notice that its expression given in (A.3) is much simpler than the three-graviton vertex of these references.

We have also checked that our amplitude correctly satisfies the relation to the QED amplitude [20] which we discuss below in the context of the helicity formalism.

### B. Massive trees amplitude in gravity from Yang-Mills tree amplitudes

A different approach is to construct the gravity amplitudes by applying the KLT method to the emission of two gluons from massive scalars.

The KLT relation between massless four-point gravity amplitudes and Yang-Mills amplitudes reads [21, 22]

$$iM_s^{\rm tree}(p_1,p_2,k_1,k_2) = \frac{\kappa_{(4)}^2}{2}(p_1\cdot k_1)\, A_s^{\rm tree}(p_1,p_2,k_2,k_1)\tilde A_0^{\rm tree}(p_1,k_2,p_2,k_1)\,. \tag{II.3}$$

Where $M_s^{\rm tree}(p_1,p_2,k_1,k_2)$ is the tree-level scattering between a matter field $X^s$ of spin $s=0,\frac{1}{2},1$ and gravitons $X^s(p_1)g(k_1)\to X^s(-p_2)g(-k_2)$ with $p_1+p_2+k_1+k_2=0$, given by the sum of diagrams in fig. 1. We use $\kappa_{(4)}^2 = 32\pi G_N$. The gauge theory amplitude $A_s^{\rm tree}(p_1,p_2,k_2,k_1)$ is the tree-level scattering amplitude between a matter field $\phi^s$ of spin $s=0,\frac{1}{2},1$ and gluons $\phi^s(p_1)(\phi^s(p_2))^*\to g(-k_1)g(-k_2)$. The amplitude $A_0^{\rm tree}(p_1,k_2,p_2,k_1)$ is the tree-level scattering between a scalar matter field $\phi^0$ and gluons $\phi^0(p_1)g(k_1)\to\phi^0(-p_2)g(-k_2)$.

The color-ordered Yang-Mills amplitudes satisfy the amplitude relation [22]

$$\tilde A_s^{\rm tree}(p_1,p_2,k_2,k_1) = \frac{p_1\cdot k_2}{k_1\cdot k_2}\,\tilde A_s^{\rm tree}(p_1,k_2,p_2,k_1)\,, \tag{II.4}$$

allowing us to express the amplitude in (II.3) in the following manner,

$$iM_s^{\rm tree}(p_1,p_2,k_1,k_2) = \frac{\kappa_{(4)}^2}{2e^2}\frac{(p_1\cdot k_1)\,p_1\cdot k_2}{k_1\cdot k_2}\,A_s^{\rm tree}(p_1,k_2,p_2,k_1)\tilde A_0^{\rm tree}(p_1,k_2,p_2,k_1)\,. \tag{II.5}$$

We will now explain that these amplitude relations are valid in the same form replacing massless fields $X^s$ with massive matter fields $\tilde X^s$. The general form of these massless amplitudes for $n$-point color-ordered gauge theory amplitudes $A_n^{\rm tree}(\sigma)$ and the $n$-point gravity amplitudes $M_n^{\rm tree}$ takes the form [21, 23, 24]

$$\begin{aligned} M^{\rm tree} = \sum_{\sigma,\gamma\in\mathfrak{S}_{n-3}} & \mathcal{S}[\sigma(2,\cdots,n-2)|\gamma(2,\cdots,n-2)]|_{k_1}\times \\ & \times A^{\rm tree}(1,\sigma(2,\cdots,n-2),n-1,n)A^{\rm tree}(n,n-1,\gamma(2,\cdots,n-2),1)\,. \end{aligned} \tag{II.6}$$

with the momentum kernel given by the expression

$$S[i_1,\dots,i_r|j_1,\dots,j_r]|_p = \prod_{t=1}^{r}(p\cdot k_{i_r} + \sum_{s>t}^{r}\theta(i_r,i_s)\,k_{i_r}\cdot k_{i_s})\,. \tag{II.7}$$

Here $\theta(i_t,i_s)$ equals 1 if the ordering of the legs $i_r$ and $i_s$ is opposite in the sets $\{i_1,\dots,i_r\}$ and $\{j_1,\dots,j_r\}$, and 0 if the ordering is the same.

This relation can be rewritten in various equivalent way thanks to the annihilation property satisfied by the color-ordered gauge theory amplitudes [24, 25]

$$\sum_{n\in\mathfrak{S}_{n-2}} \mathcal{S}[\sigma(2,\cdots,n-1)|\gamma(2,\cdots,n-2)]|_{k_1}\times A^{\rm tree}(1,\sigma(2,\cdots,n-1),n)=0; \qquad \forall\gamma\in\mathfrak{S}_{n-2}\,, \tag{II.8}$$

generalizing the relation in eq. (II.4).

These relations have been derived for a number of different types of matter including, massless scalars, vectors (gluons or photons), and gravitons [26]. The derivation shows that the relation is the same in any space-time dimensions. However, the key point is that a massive scalar in four dimensions is equivalent to a massless scalar in higher dimensions. Therefore, an amplitude between massive scalars and gravitons in four dimensions, can be seen as a tree-level amplitude between massless scalars in higher dimensions with gravitons polarized in four-dimensions. In this higher-dimensional setup the relation between gravity and gauge theory can be applied.

The validity of the amplitude relations with massive scalars and gravitons also follows directly from string theory. The case of tachyons was already considered in [27]. The relations in [21, 24] relies on the monodromy properties of the colored-ordered open string amplitudes

$$A_{\alpha'}(i_1,\ldots,i_n)=\int_{x_{i_1}<\cdots<x_{i_n}} f(x_i-x_j)\,\prod_{\substack{i,j=1\\ i\neq j}}^{n}(x_i-x_j)^{2\alpha' k_i\cdot k_j}\,\prod_{i=1}^{n}dx_i\,. \tag{II.9}$$

The monodromy property however does not depend on detailed expression of the function $f(x_i-x_j)$ and are derived from momentum conservation $\sum_i k_i=0$ and the phase factor that arises when going around the branch cut given the factors $(x_i-x_j)^{2\alpha' k_i\cdot k_j}$. The phase factor is not affected by any integer shifts of $2\alpha' k_i\cdot k_j$ arising *e.g.* from a pole from a massive scalar in the function $f(x_i-x_j)$. Thus the massive field theory amplitude relations obtained by considering the $\alpha'\to 0$ limit, satisfy the same properties as explained in [24] as corresponding massless ones.

Assured that the KLT relation applies for various types of matter fields, massless and massive, we will now study the case of four-point amplitudes describing the emission of two gravitons from a matter field of spin $s$.

### C. Application of KLT to the gravitational Compton scattering: Reduction to QED amplitudes

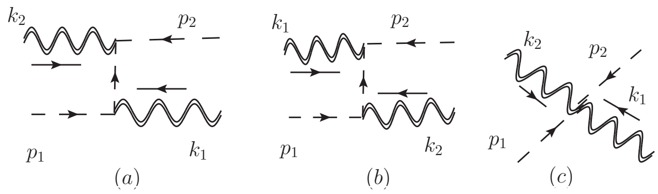


FIG. 2. Compton scattering given by the $t$-channel contribution in $(a)$, the $u$-channel contribution in $(b)$ and the contact term in $(c)$.

Our starting point for deriving the gravitational Compton amplitude is the KLT expression from the previous section

$$iM_s^{\rm tree}(p_1,p_2,k_1,k_2) = \frac{\kappa_{(4)}^2}{2e^2}\frac{p_1\cdot k_1\, p_1\cdot k_2}{k_1\cdot k_2}\, A_s^{\rm tree}(p_1,k_2,p_2,k_1)\tilde A_0^{\rm tree}(p_1,k_2,p_2,k_1)\,, \tag{II.10}$$

where the gravity amplitude is expressed as a product of Yang-Mills amplitudes without a $s$-channel pole and we thus have no Yang-Mills diagrams involving the non-Abelian three-gluon vertex. This KLT representation of the gravitational Compton scattering is key to the reduction of the amplitude to a product of QED amplitudes that we will consider in this section. The color ordered-amplitude $A_s^{\rm tree}(p_1,k_2,p_2,k_1)$ represents the scattering of two gauge boson from a spin $s$ matter field depicted in figure 2

$$A_s^{\rm tree}(p_1,k_2,p_2,k_1) = e^2\left(\frac{n_t^s}{p_1\cdot k_1}+\frac{n_u^s}{p_1\cdot k_2}+n_{ct}^s\right)\,, \tag{II.11}$$

where $p_1^2 = p_2^2 = m^2$ are the momenta of the massive particles and $k_1^2 = k_2^2 = 0$ are the momentum of the gluons with all incoming momenta $p_1+p_2+k_1+k_2=0$.

We will now explain that we can always express the amplitude $A_s^{\rm tree}(p_1,k_2,p_2,k_1)$ solely in terms of QED (abelian) Compton scattering amplitudes. The $t$- and $u$-channel diagrams in figure 2(a)-(b) are composed of three-point amplitudes between two matter fields of the same spin $s$ of the same flavor and one gauge boson. The coupling of a matter fields of spin 0 of the same species and one gauge boson is given by

$$e\,(p_1-p_2)_\mu\,(2\pi)^4\,\delta(p_1+p_2+k_1)\,, \tag{II.12}$$

or for particles of spins $\frac{1}{2}$ of the same species and one gauge boson

$$e\gamma_\mu\,(2\pi)^4\,\delta(p_1+p_2+k_1)\,. \qquad \text{(II.13)}$$

Finally the coupling between two massive spin 1 fields of the same species and one gauge boson is given by

$$-\,e\,(g_{\mu\nu}(k_1-p_2)_\rho+g_{\nu\rho}(p_2-p_1)_\mu+g_{\rho\mu}(p_1-k_1)_\nu)\,(2\pi)^4\,\delta(p_1+p_2+k_1)\,, \qquad \text{(II.14)}$$

where in all cases $e$ is the coupling constant.

There is no quartic coupling between two spinorial fields and one gauge boson and the four-point interaction in fig. 2(c) between two scalars (without flavor changing) and two gauge bosons

$$-\,ie^2 g_{\mu\nu}\,(2\pi)^4\,\delta(p_1+p_2+k_1+k_2) \qquad \text{(II.15)}$$

is the same in an non-Abelian as in an Abelian theory.

The four-point interaction between two massive vectors of the same species and two gauge bosons is in an non-Abelian theory given by

$$-\,ie^2\sum_e[f_{abe}f_{ecc}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})+f_{ace}f_{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\rho\nu})+f_{ace}f_{ebc}(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\rho}g_{\sigma\nu})]$$
$$\times\;(2\pi)^4\delta^4(p_1+p_2+k_1+k_2)\,. \qquad \text{(II.16)}$$

By antisymmetry of the structure constant $f_{ecc}=0$ the interaction reduces to

$$-\,ie^2\,(\sum_e f_{ace}f_{ebc})[(2g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\sigma\nu})]\times\;(2\pi)^4\delta^4(p_1+p_2+k_1+k_2)\,, \qquad \text{(II.17)}$$

which has the same kinematic part as the Abelian one.

We can thus conclude that the amplitudes $A_s^{\rm tree}$ with $s=0,\frac{1}{2}$ appearing in the factorization of the gravity amplitudes in (II.5) can be thought of as QED amplitudes for Compton scattering off massive matter fields[1].

The numerators of the QED Compton amplitudes $A_s^{\rm tree}(p_1,k_2,p_2,k_2)$ are given by

$$n_t^0 = 2\epsilon_1\cdot p_1\;\epsilon_2\cdot p_2\,, \qquad \text{(II.18)}$$

$$n_t^{\frac{1}{2}} = \frac{1}{2}\,\bar u(-p_2)\not\epsilon_2(\not p_1+\not k_1+m)\not\epsilon_1 u(p_1)\,, \qquad \text{(II.19)}$$

$$n_t^1 = 2[(h_1\cdot h_2)\,(\epsilon_1\cdot p_1)\,(\epsilon_2\cdot p_2)-h_1\cdot F_1\cdot F_2\cdot h_2$$

[1] The representation of the massive gravitational Compton scattering of a massive matter field of spin $s=0,1/2,1$ in terms of (Abelian) Compton amplitude was already noticed in [14, 20] and [28]. It would be interesting to understand if this factorization using purely Abelian interactions can be achieved with other types of gravitational amplitudes.

$$- (h_1 \cdot F_2 \cdot h_2)\, (\epsilon_1 \cdot p_1) - (h_1 \cdot F_1 \cdot h_2)(\epsilon_2 \cdot p_2)] \,, \tag{II.20}$$

and with similar expressions for $n_u^s$ with the exchange of $p_1$ and $p_2$ and finally[2]

$$n_{ct}^0 = 2\epsilon_1 \cdot \epsilon_2 \,, \tag{II.21}$$

$$n_{ct}^{\frac{1}{2}} = 0 \,, \tag{II.22}$$

$$n_{ct}^1 = -2 h_1 \cdot h_2 \, \epsilon_1 \cdot \epsilon_2 \,. \tag{II.23}$$

We have here made use of the notation $h_1 {\cdot} F_1 {\cdot} h_2 = h_1^\mu h_2^\nu F_{1\,\mu\nu}$ and $h_1 {\cdot} F_1 {\cdot} F_2 {\cdot} h_2 = h_1^\mu F_{1\mu\rho} F_2{}^\rho{}_\nu h_2^\nu$ with $F_{i\,\mu\nu} = k_{i\,\mu} \epsilon_{i\,\nu} - \epsilon_{i\,\nu} h_{i\,\mu}$ defining the field-strengths of the photons. With the given numerators factors we have checked that the spin 0 amplitude constructed from (II.5) correctly reproduces the covariant expression in (II.1).

One important consequence of the factorization of the gravitational Compton amplitude into a product of two Compton amplitudes is that it gives a rationale for the value $g = 2$ of the classical gyromagnetic momenta for all types of matter fields, as shown in ref. [15]. An evaluation of the Compton amplitude for massive particles shows that amplitude has a pole for $m = 0$ with residue $(g - 2)^2$. The two derivative nature of the gravitational interaction forbids the present of a singularity of the gravitational Compton amplitude when the mass of the particles goes to zero. Therefore the KLT relation in (II.5) implies that the right hand side cannot have a pole in the zero mass limit for generic values of the momenta. This implies the natural classical value $g = 2$ for all types of matter fields.

## D. Helicity tree amplitudes for QED and Gravity

### *1. The QED amplitudes*

In this section we compare the QED amplitudes in (II.11) with the scattering of two gluons off a massive scalar derived using the helicity formalism (see ref. [29]). We use here $e^2 = 1$. We have

$$A_0^{\rm tree}(p_1, p_2, k_2^+, k_1^+) = -\frac{m^2\, [k_1\, k_2]^2}{k_1 \cdot k_2\, 2k_1 \cdot p_1}, \qquad A_0^{\rm tree}(p_1, p_2, k_2^-, k_1^+) = \frac{\langle k_2 | p_1 | k_1 ]^2}{k_1 \cdot k_2\, 2k_1 \cdot p_1}\,, \tag{II.24}$$

[2] Notice that this is not a BCJ parameterization [22] because the numerators do not satisfying a dual Jacobi identity. One can define a set of BCJ numerators as $\tilde n_s^s = 2(n_t^s + n_u^s) + t n_{ct}^s$ and $\tilde n_t^s = -2n_t^s - t n_{ct}^s$ and $\tilde n_u^s = -2n_u^s$, satisfying $\tilde n_s^s + \tilde n_t^s + \tilde n_u^s = 0$. Other expressions are possible, depending on the distribution of contact terms amongst the pole terms.

with

$$A_0^{\text{tree}}(p_1, p_2, k_2^-, k_1^-) = (A_0^{\text{tree}}(p_1, p_2, k_2^+, k_1^+))^*\,,$$
$$A_0^{\text{tree}}(p_1, p_2, k_2^+, k_1^-) = (A_0^{\text{tree}}(p_1, p_2, k_2^-, k_1^+))^*\,. \tag{II.25}$$

It is immediate to check that the Compton scalar amplitude $A_0^{\text{tree}}(p_1, k_2, p_2, k_1)$ is related to the helicity amplitudes by the excepted monodromy relations $(p_1 \cdot k_2)\, A_0^{\text{tree}}(p_1, k_2, p_2, k_1) = (k_1 \cdot k_2) A_0^{\text{tree}}(p_1, p_2, k_2, k_1)$ and read

$$A_0^{\text{tree}}(p_1, k_2^+, p_2, k_1^+) = -\frac{m^2 e^2\, [k_1\, k_2]^2}{4(p_1 \cdot k_1)\,(p_1 \cdot k_2)}\,,$$
$$A_0^{\text{tree}}(p_1, k_2^-, p_2, k_1^+) = e^2\, \frac{\langle k_2|p_1|k_1]^2}{4\,(k_1 \cdot p_1)(p_1 \cdot k_2)}\,. \tag{II.26}$$

This expression is (although not manifestly) symmetric under the exchanges of $k_1$ and $k_2$ and $p_1$ and $p_2$.

### *2. The gravity amplitudes*

Using the relation in (II.3) we can write the expression for the four-point amplitudes for the emission of two gravitons. In this situation, we have

$$M_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^+) = \frac{\kappa_{(4)}^2}{16}\, \frac{1}{(k_1 \cdot k_2)}\, \frac{m^4\, [k_1\, k_2]^4}{(k_1 \cdot p_1)(k_1 \cdot p_2)}\,,$$
$$M_0^{\text{tree}}(p_1, p_2, k_1^-, k_2^+) = \frac{\kappa_{(4)}^2}{16}\, \frac{1}{(k_1 \cdot k_2)}\, \frac{\langle k_1|p_1|k_2]^2\, \langle k_1|p_2|k_2]^2}{(k_1 \cdot p_1)(k_1 \cdot p_2)}\,, \tag{II.27}$$

with

$$M_0^{\text{tree}}(p_1, p_2, k_1^-, k_2^-) = (M_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^+))^*\,,$$

and

$$M_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^-) = (M_0^{\text{tree}}(p_1, p_2, k_1^-, k_2^+))^*.$$

We have checked that these expressions match the covariant ones and the expression obtained from (II.5). The massive amplitude $M_0^{\text{tree}}(p_1, p_2, k_1^+, k_2^-)$ reproduces the one given in [4, eq. (5.4)] and its massless limit reproduces the results of [4, eq. (4.5)]. We note that using the KLT factorization property to construct the amplitudes that go into the cut avoids having to deal with tensor contractions of the complicated triple graviton vertex, which is a normal tedious feature of any off-shell Feynman diagram computation.

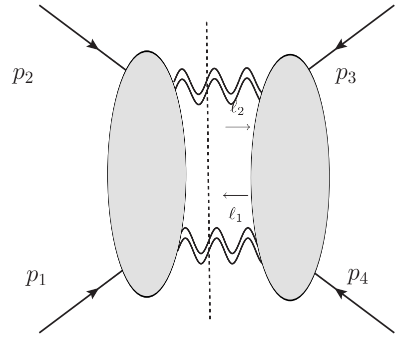


FIG. 3. The cut considered. The loop momenta are flowing clockwise. And the on-shell conditions are $\ell_1^2 = 0$ and $\ell_2^2 = (\ell_1 + k_1 + k_2)^2 = 0$. Solid lines are massive and wiggly lines are massless.

## III. THE ONE-LOOP AMPLITUDE IN THE HELICITY FORMALISM

In this section, we obtain the non-analytic terms that give the leading classical and quantum corrections to the scattering potential for QED and for general relativity. For this purpose we do not need to reconstruct the full amplitude, but only identify those terms in the cut that lead to non-analytic contributions, *i.e.* $C^{NP}$, the classical correction from general relativity, and $C^{GQ}$, the quantum gravity correction to Newton's potential in (I.3). We obtain these respectively from the coefficients of the non-analytic $1/\sqrt{-q^2}$ and $\log(-q^2)$ contributions in cut.

To extract the non-analytic parts of the amplitude, we will proceed as in ref. [30]. Instead of evaluating the phase-space integrals directly we simply reinstate the off-shell cut propagators but impose strictly the on-shell cut condition everywhere in the numerator. We thus evaluate the following types of expressions

$$M^{1-\text{loop}}\big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\sum_{\lambda_1,\lambda_2} M^{\text{tree}}_{\lambda_1\lambda_2}(p_1, p_2, -\ell_2^{\lambda_2}, \ell_1^{\lambda_1})(M^{\text{tree}}_{\lambda_1\lambda_2}(p_3, p_4, \ell_2^{\lambda_2}, -\ell_1^{\lambda_1}))^*}{\ell_1^2\ell_2^2}\Big|_{cut}, \tag{III.1}$$

with $\ell_1^2 = \ell_2^2 = 0$ and where $\lambda_1$ and $\lambda_2$ are the helicities of the massless particles (gravitons/photons) across the cut. In this formula, we are using the notation $|_{cut}$ to indicate the cut is taken in this integral. Whenever we discuss the discontinuity singularity it is understood that we are on the cut, although we will not explicitly indicate this in the integral for simplicity. This procedure allows us to directly identify the box, triangle and bubble integral functions which contribute to the amplitude, and use them to identify the non-analytic terms which we are seeking[3].

We will illustrate our discontinuity cut method by first calculating the case of the Coulomb potential. Here the cut is a little simpler and it is easier to demonstrate the techniques. In the next subsection we will then use the cut technique in the case of pure gravity.

### A. The one-loop correction to Coulomb potential

In this section we will compute the quantum correction to the Coulomb potential between two spin 0 particles of the same charge but non-zero masses $m_1$ and $m_2$.

We are constructing the one-loop amplitude by computing its discontinuity cut across the massless photon lines (double wavy-line in figure 3). We not are interested in reconstructing the full one-loop amplitude but only the parts that contain the infra-red logarithms and square-root contributions.

In the cut in Eq. (III.1) we have the following on-shell kinematic relations $p_1 + p_2 + p_3 + p_4 = 0$, $p_1^2 = p_2^2 = m_1^2$ and $p_3^2 = p_4^2 = m_2^2$. We define the momentum transfer $q$ from $q = p_1 + p_2 = -(p_3 + p_4)$. We have in the static non-relativistic limit $p_1, -p_2 \simeq (m_1, \vec{0})$ and $p_4, -p_3 \simeq (m_2, \vec{0})$, and furthermore that (in the mostly minus metric)

$$\begin{aligned} s &= (p_1 + p_2)^2 \simeq -\vec{q}^{\,2}\,, \\ t &= (p_1 + p_4)^2 \simeq (m_1 + m_2)^2\,, \\ u &= (p_1 + p_3)^2 \simeq (m_1 - m_2)^2 + \vec{q}^{\,2}\,. \end{aligned} \tag{III.2}$$

The tree-level helicity amplitudes are given in (II.26) hence the discontinuity of the one-loop amplitude takes the form

$$M^{1-\text{loop}}\Big|_{disc} = \frac{e^4}{16} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathcal{N}}{\ell_1^2 \ell_2^2 \prod_{i=1}^4 (p_i \cdot \ell_1)}\,. \tag{III.3}$$

We deduce that

$$\frac{1}{\ell_1 \cdot p_1\, \ell_1 \cdot p_2} = -\frac{2}{s} \left( \frac{1}{\ell_1 \cdot p_1} + \frac{1}{\ell_1 \cdot p_2} \right),$$

[3] By considering only this two-particle discontinuity across the massless momenta, we do not have enough information to reconstruct the full amplitude. To achieve this, we would need to consider all the discontinuities across the massive legs and evaluate the cut to all orders in $\epsilon$ with $D \equiv 4 - 2\epsilon$. However, the discontinuities across the massive propagators will not contribute to the leading order massless threshold, not will higher order terms from an $\epsilon$ expansion of the cut. Thus we will ignore all these contributions here as they are not important for our analysis.

$$\frac{1}{\ell_1\cdot p_3\,\ell_1\cdot p_4} = \frac{2}{s}\left(\frac{1}{\ell_1\cdot p_3} + \frac{1}{\ell_1\cdot p_4}\right), \tag{III.4}$$

using that $q = p_1 + p_2 = \ell_2 - \ell_1 = -p_3 - p_4$ and $\ell_1 \cdot q = -s/2$. This allows us to express the one-loop cut as a sum of integrals with numerator $\mathcal{N}$

$$M^{1-\text{loop}}\big|_{disc} = -\frac{e^4}{4}\sum_{i=1}^{2}\sum_{j=3}^{4}\int \frac{d^D\ell}{(2\pi)^D}\,\frac{\mathcal{N}}{s^2\ell_1^2\ell_2^2(p_i\cdot\ell_1)(p_j\cdot\ell_1)}\,. \tag{III.5}$$

where we will distinguish between the cases of the photons having the same helicity on each side of the cut (this is traditionally in the literature called a singlet contribution) or opposite helicity (called a non-singlet contribution).

For the singlet cut the numerator is given by

$$\mathcal{N}^{\text{singlet}} = m_1^2 m_2^2 s^2\,. \tag{III.6}$$

Giving a contribution from the singlet cut of only scalar boxes

$$M^{\text{singlet}} = -e^4\,2m_1^2m_2^2(I_4(s,t) + I_4(s,u))\,, \tag{III.7}$$

Here we have in $D = 4 - 2\epsilon$ using the normalization of ref. [8] that

$$\begin{aligned} I_4(s,t) &= \int \frac{d^D\ell}{(2\pi)^D}\,\frac{1}{\ell^2(\ell+q)^2\,((\ell+p_1)^2 - m_1^2)((\ell-p_4)^2 - m_2)}\,,\\ I_4(s,u) &= \int \frac{d^D\ell}{(2\pi)^D}\,\frac{1}{\ell^2(\ell+q)^2((\ell+p_1)^2 - m_1^2)((\ell-p_3)^2 - m_2)}\,, \end{aligned} \tag{III.8}$$

where $w = p_1 \cdot p_4 - m_1 m_2 =$ and $W = -p_1 \cdot p_3 - m_1 m_2$ and it should be remarked that $w = W - \frac{1}{2}\,q^2$. In the non-relativistic limit where $w \to 0$ and to leading order in $q^2$, we have [8]

$$\begin{aligned} I_4(s,t) + I_4(s,u) &= \frac{\log(-\vec{q}^2)}{96\pi^2 m_1^2 m_2^2}\\ I_4(s,u) - I_4(s,u) &= \frac{\log(-\vec{q}^2)}{8\pi^2 m_1^2 m_2^2\,q^2}\,. \end{aligned} \tag{III.9}$$

Thus the singlet cut amplitude in (III.7) in the non-relativistic limit gives

$$M^{\text{singlet}}(q) \simeq -\frac{e^4}{(4\pi)^2}\,\frac{1}{3}\log(\vec{q}^{\,2})\,, \tag{III.10}$$

to leading order in $q^2 \sim -\vec{q}\cdot\vec{q}$.

For the non-singlet cut contribution the numerator is given by

$$\mathcal{N}^{\text{non}-\text{singlet}} = \frac{1}{2}(\text{tr}_-(\ell_2 p_1 \ell_1 p_4)^2 + \text{tr}_+(\ell_2 p_1 \ell_1 p_4)^2)\,, \tag{III.11}$$

where the traces are defined by $\mathrm{tr}_\pm(abcd) = 2(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c) \pm 2i\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma$. Expanding the traces we see that one can rewrite the numerator in terms of two contributions $\mathcal{N}^{\rm non-singlet} \equiv \mathcal{E}^2 - 4\mathcal{O}$ where

$$\begin{aligned}\mathcal{E} &:= 2(\ell_1\cdot p_1\, \ell_2\cdot p_3 - \ell_1\cdot\ell_2\, p_1\cdot p_3 + \ell_1\cdot p_3\, \ell_2\cdot p_1)\,,\\ \mathcal{O} &:= (\epsilon^{\mu\nu\rho\sigma}\ell_{1\,\mu} p_{1\,\nu} \ell_{2\,\rho} p_{3\,\sigma})^2\,.\end{aligned} \tag{III.12}$$

This leads in the non-relativistic approximation to a rather simple form for the numerator

$$\mathcal{N}^{\rm non-singlet} \simeq (s\, m_1^2 + 4(p_1\cdot\ell_1)^2)\,(s\, m_2^2 + 4(p_4\cdot\ell_1)^2)\,. \tag{III.13}$$

Evaluating the contributions from the non-singlet cut (in the non-relativistic limit) lead to the following combinations of scalar box, triangle and bubble integrals to leading order

$$\begin{aligned}M^{\rm non-singlet} = -e^4 \Big(& 2m_1^2 m_2^2 (I_4(s,t) + I_4(s,u)) + m_1^2 (I_3(p_1,q,m_1) + I_3(p_2,q,m_2))\\ &+ m_2^2(I_3(-p_3,q,m_2) + I_3(-p_4,q,m_2)) + I_2(q)\Big)\,.\end{aligned} \tag{III.14}$$

The scalar triangle and bubbles integrals are defined following the conventions of ref. [8]

$$\begin{aligned}I_3(p,q,m) &:= \int \frac{d^D\ell}{(2\pi)^D}\, \frac{1}{\ell^2(\ell+q)^2((\ell+p)^2 - m^2)}\,,\\ I_2(q) &:= \int \frac{d^D\ell}{(2\pi)^D}\, \frac{1}{\ell^2(\ell+q)^2}\,.\end{aligned} \tag{III.15}$$

Where we in the non-relativistic limit have

$$I_3(p,q,m) \sim -\frac{i}{32\pi^2 m^2}\, \left(\log(-q^2) + S(m)\right)\,, \tag{III.16}$$

$$I_2(q) \sim \frac{i}{16\pi^2}\, \log(-q^2)\,, \tag{III.17}$$

defining $S(m) = -\pi^2 m/|\vec{q}|$.

Thus the contribution from the non-singlet cut amplitude in (III.14) yields in the non-relativistic limit $w \to 0$ (to the first order in $q^2 \sim -\vec{q}^{\,2}$)

$$M^{\rm non-singlet}(q) \simeq \frac{e^4}{(4\pi)^2}\, \left(\frac{8}{3}\, \log(-q^2) - \pi^2 \frac{m_1+m_2}{|\vec{q}|}\right) \tag{III.18}$$

Summing (III.10) and (III.18) we obtain the total amplitude

$$M^{\rm non-rel.}(q) \simeq \frac{e^4}{(4\pi)^2}\, \left(\frac{7}{3}\, \log(\vec{q}^{\,2}) - \pi^2 \frac{m_1+m_2}{|\vec{q}|}\right)\,, \tag{III.19}$$

and the one-loop correction to the non-relativistic potential is given by

$$V^{\rm one-loop}(q) = \frac{M^{\rm non-rel.}(q)}{4m_1 m_2} = \frac{e^4}{8\pi^2 m_1 m_2}\, \left(\frac{7}{3}\, \log(\vec{q}^{\,2}) - \pi^2 \frac{m_1+m_2}{|\vec{q}|}\right)\,. \tag{III.20}$$

This reproduces the result of [31, eqs. (4.50a), (4.51a), (4.54)] and [32], although we want to point out the huge simplicity of our cut derivation.

### B. The one-loop correction to Newton potential

In this section we will perform the evaluation of the correction to the Newton potential using the on-shell cut in the helicity formalism. This computation will as expected not require any ghost contributions.

Proceeding as in the QED case, the cut discontinuity of the amplitude can be expressed as a sum of integrals with numerator $\mathcal{N}$

$$M^{1-\text{loop}}\big|_{disc} = -\frac{\kappa_{(4)}^4}{16\, s^4} \sum_{i=1}^{2}\sum_{j=3}^{4} \int \frac{d^D\ell}{(2\pi)^D} \frac{\mathcal{N}}{\ell_1^2\ell_2^2(p_i\cdot\ell_1)(p_j\cdot\ell_1)}\,. \tag{III.21}$$

We will evaluate the amplitude in the static non-relativistic limit (III.2).

As in the QED case, we will here as well distinguish between the cases of the graviton having the same helicity on each side of the cut (singlet) or opposite helicity (non-singlet), and we separate the numerator factor $\mathcal{N}$ in these two contributions.

The singlet-cut numerator is easily evaluated and gives

$$\mathcal{N}^{\text{singlet}} = 2m_1^4\, m_2^4\, (\ell_1\cdot\ell_2)^4 = \frac{m_1^4 m_2^4}{8}\, s^4\,, \tag{III.22}$$

therefore its contribution to the one-loop amplitude is given by scalar boxes only

$$M^{\text{singlet}}(q) = -\frac{\kappa_{(4)}^4}{16}\, m_1^4 m_2^4\, (I_4(s,t) + I_4(s,u))\,. \tag{III.23}$$

This is readily evaluated in the non-relativistic limit to give

$$M^{\text{singlet}}(q) \simeq -G_N^2\, \frac{4m_1^2 m_2^2}{3}\, \log(\vec{q}^{\,2})\,, \tag{III.24}$$

where we have made use of the relation $\kappa_{(4)}^2 = 32\pi G_N$.

The numerator for the non-singlet cut contribution is evaluated to

$$\mathcal{N}^{\text{non-singlet}} = \frac{1}{2}\left((\text{tr}_-(\ell_1 p_1 \ell_2 p_3))^4 + (\text{tr}_+(\ell_1 p_3 \ell_2 p_1))^4\right)\,. \tag{III.25}$$

The evaluation of this contribution is a bit more involved since the expression contains integrals with up to eight powers of loop momentum in the numerator. We note that in the gravity case the cut is not the square of the QED cut but the sum of the squares of the corresponding QED terms in the cut.

Decomposing the trace as in the QED case (keeping only terms that give a contribution in the non-relativistic limit) the numerator factor takes the form

$$\mathcal{N}^{\text{non-singlet}} = ((\mathcal{E}^2 - 4\mathcal{O})^2 - 16\mathcal{E}^2\mathcal{O})$$

$$= 16s^2(m_2\, p_1\cdot\ell_1 - m_1\, p_4\cdot\ell_1)^4$$
$$+ 24s(m_2\, p_1\cdot\ell_1 - m_1\, p_4\cdot\ell_1)^2(m_1m_2s + 4p_1\cdot\ell_1p_4\cdot\ell_1)^2$$
$$+ (m_1m_2s + 4p_1\cdot\ell_1p_4\cdot\ell_1)^4\,. \tag{III.26}$$

In the non-relativistic limit evaluating the discontinuity cut integrals leaves us with a sum of scalar boxes, scalar, linear and quadratic triangles and bubbles integral functions ranging from scalar to quartic, *i.e.*

$$M^{\rm non-singlet}(q) = M^{\rm non-singlet}_{\rm boxes}(q) + M^{\rm non-singlet}_{\rm triangles}(q) + M^{\rm non-singlet}_{\rm bubbles}(q)\,. \tag{III.27}$$

To the leading order in the non-relativistic limit, we have scalar box integral functions $M^{\rm boxes}$ given by

$$M^{\rm non-singlet}_{\rm boxes}(q) = -\frac{\kappa_{(4)}^4}{8}\,\big(m_1^4m_2^4(I_4(s,t)+I_4(s,u) + 2m_1^3m_2^3\,s(I_4(s,t)-I_4(s,u)\big)\,. \tag{III.28}$$

This gives in the non-relativistic limit (using (III.9))

$$M^{\rm non-singlet}_{\rm boxes} \simeq -G_N^2\,\frac{100m_1^2m_2^2}{3}\,\log(\vec q^{\,2})\,. \tag{III.29}$$

For the triangles, we have integrals from scalars up to quadratic terms $M^{\rm triangles}$,

$$M^{\rm non-singlet}_{\rm triangles} = -\frac{\kappa_{(4)}^2}{16}\Big[ \tag{III.30}$$
$$6m_1^4m_2^2(I_3(p_1)+I_3(p_2)) + 6m_1^2m_2^4(I_3(-p_3)+I_3(-p_4))$$
$$-2m_1^4(I_3(p_1,\{p_4\}) + I_3(p_2,\{p_4\})) + 8m_1^3m_2(I_3(p_1,\{p_4\}) - I_3(p_2,\{p_4\}))$$
$$+2m_2^4(I_3(-p_3,\{p_1\}) + I_3(-p_4,\{p_1\})) + 8m_1m_2^3(I_3(-p_3,\{p_1\}) - I_3(-p_4,\{p_1\}))$$
$$+4m_1^2(I_3(p_1,\{p_4,p_4\}) + I_3(p_2,\{p_4,p_4\})) + 4m_2^2(I_3(-p_3,\{p_1,p_1\}) + I_3(-p_4,\{p_1,p_1\}))$$
$$+\frac{4}{q^2}\Big(m_1^4(I_3(p_1,\{p_4,p_4\}) + I_3(p_2,\{p_4,p_4\})) + m_2^4(I_3(-p_3,\{p_1,p_1\}) + I_3(-p_4,\{p_1,p_1\}))\Big)\Big]\,,$$

with linear and quadratic triangles defined via

$$I_3(p,q,m;\{K_1,\dots,K_r\}) := \int \frac{d^D\ell}{(2\pi)^D}\,\frac{\prod_{i=1}^r \ell\cdot K_i}{\ell^2(\ell+q)^2((\ell+p)^2-m^2)}\,, \tag{III.31}$$

were we have $r=0$ for scalar triangles, $r=1$ for linear triangles and $r=2$ for quadratic triangles. We use here the short hand notation that $I_3(p_r,\cdots) = I_3(p_r,q,m_1,\cdots)$ for $r=1,2$ and $I_3(-p_r,\cdots) = I_3(-p_r,q,m_2,\cdots)$ for $r=3,4$.

Taking the non-relativistic limit leaves us with

$$I_3(p,q,m;K_1) \sim \frac{i}{32\pi^2m^2}\,\left[(K_1\cdot p)\left(-1-\frac{q^2}{2m^2}\right)\log(-q^2) + K_1\cdot q\,(\log(-q^2) + \frac{1}{2}\,S(m))\right]\,,$$

$$I_3(p,q,m;K_1,K_2) \sim \frac{i}{32\pi^2 m^2}\Big[(K_1\cdot q)(K_2\cdot q)(-\log(-q^2) - \frac{3}{8}S(m))$$
$$- (K_1\cdot p)(K_2\cdot p)\,\frac{q^2}{8m^2}(4\log(-q^2)+S(m))$$
$$+ ((K_1\cdot q)(K_2\cdot p)+(K_1\cdot p)(K_2\cdot q))\left(\frac{q^2+m^2}{2m^2}\log(-q^2)+\frac{3q^2}{16m^2}S(m)\right)$$
$$+ \frac{1}{8}K_1\cdot K^2\, q^2\,(2\log(-q^2)+S(m))\Big]\,, \qquad \text{(III.32)}$$

so that

$$M^{\rm non-singlet}_{\rm triangles}(q) \simeq G_N^2\, m_1^2 m_2^2\left(120\,\log(\vec q\cdot\vec q) - 24\pi^2\,\frac{m_1+m_2}{|\vec q|}\right)\,. \qquad \text{(III.33)}$$

To the leading order in the non-relativistic limit, the bubble contribution $M^{\rm non-singlet}_{\rm bubble}$ is given by

$$M^{\rm non-singlet}_{\rm bubbles} = -\frac{\kappa^4_{(4)}}{16}\Big[\frac{16}{s^2}I_2(q,\{p_1,p_1,p_4,p_4\}) \qquad \text{(III.34)}$$
$$- 4\Big(3m_1^2m_2^2 I_2(q) - m_2(2m_1+3m_2)I_2(q,\{p_1\}) + m_1(3m_1+2m_2)I_2(q,\{p_4\})$$
$$+ I_2(q,\{p_1,p_1\}) + I_2(q,\{p_4,p_4\}) + 3I_2(q,\{p_1,p_4\})\Big)$$
$$+ \frac{8}{s}\Big(3(m_2^2 I_2(q,\{p_1,p_1\}) + m_1^2 I_2(q,\{p_4,p_4\})) - 4m_1m_2 I_2(q,\{p_1,p_4\})$$
$$+ I_2(q,\{p_1,p_4,p_4\}) - I_2(q,\{p_1,p_1,p_4\})\Big)\,, \qquad \text{(III.35)}$$

where

$$I_2(q,\{K_1,\cdots,K_r\}) := \int \frac{d^D\ell}{(2\pi)^D}\,\frac{\prod_{i=1}^r \ell\cdot K_r}{\ell^2(\ell+q)^2}\,, \qquad \text{(III.36)}$$

with $r = 0,1,2,3,4$. The bubble integrals are all given by

$$I_2(q,\{K_1,\cdots,K_r\}) = I_2(q)\,P_r(q^2) + \text{rational part}\,, \qquad \text{(III.37)}$$

where $I_2(q)$ is the scalar bubble function given in (III.15) and $P_r(q,K_1,\ldots,K_r)$ is a polynomial. The rational part does not contribute to our analysis. The polynomials are given by

$$P_1(q,K_1) = -\frac{q\cdot K_1}{2}\,, \qquad \text{(III.38)}$$
$$P_2(q,K_1,K_2) = \frac{1}{12}\left(4q\cdot K_1\, q\cdot K_2 - q^2 K_1\cdot K_2\right)\,,$$
$$P_3(q,K_1,K_2,K_3) = \frac{1}{24}\Big(q^2\left(K_1\cdot K_3K_2\cdot q + K_1\cdot K_2K_3\cdot q + K_1\cdot qK_2\cdot K_3\right)$$
$$- 6K_1\cdot qK_2\cdot qK_3\cdot q)\Big)\,,$$
$$P_4(q,K_1,K_2,K_3,K_4) = \frac{1}{240}\Big((q^2)^2(K_1\cdot K_4K_2\cdot K_3 + K_1\cdot K_3K_2\cdot K_4 + K_1\cdot K_2K_3\cdot K_4)$$
$$- 6q^2(K_1\cdot qK_2\cdot qK_3\cdot K_4 + K_1\cdot K_4K_2\cdot qK_3\cdot q + K_1\cdot K_3K_2\cdot qK_4\cdot q+$$

$$K_1 \cdot K_2 K_3 \cdot q K_4 \cdot q + K_1 \cdot q K_2 \cdot K_4 K_3 \cdot q + K_1 \cdot q K_2 \cdot K_3 K_4 \cdot q)$$
$$+ 48 K_1 \cdot q K_2 \cdot q K_3 \cdot q K_4 \cdot q \Big) \,.$$

Leaving us with

$$M^{\text{non-singlet}}_{\text{bubbles}}(q) = G_N^2 \, \frac{788 m_1^2 m_2^2}{15} \, \log(\vec{q}^{\,2}) \,. \qquad \text{(III.39)}$$

Thus the total contribution is given by summing (III.24), (III.29), (III.33) and (III.39) yielding

$$M^{\text{total}}(q) = G_N^2 \, 4 m_1^2 m_2^2 \left( -6\pi^2 \frac{m_1 + m_2}{|\vec{q}|} + \frac{41}{5} \, \log(\vec{q}^{\,2}) \right) , \qquad \text{(III.40)}$$

leading to the one-loop correction to the non-relativistic potential

$$V^{\text{one-loop}}(q) = \frac{M^{\text{total}}(q)}{4 m_1 m_2} = G_N^2 m_1 m_2 \left( -6\pi^2 \frac{m_1 + m_2}{|\vec{q}|} + \frac{41}{5} \, \log(\vec{q}^{\,2}) \right) . \qquad \text{(III.41)}$$

This matches refs. [8, 9]. We point out that other computations can be carried out with much greater ease using the cut method as well, for example the mixed electromagnetic-gravitational scattering case, previous computed in refs. [33, 34].

## IV. THE ONE-LOOP AMPLITUDE IN HARMONIC GAUGE

We can also use the discontinuity cut technique to evaluate the potential using the covariant notation, in harmonic gauge. This has two interesting features. One is that this gauge requires ghost fields, and we will see that the discontinuity from the ghosts must be added in order to obtain the full result. In addition, this calculation lets us make direct contact with the Feynman diagram approach in harmonic gauge [8, 9]. We will describe in this section how one can compare with the individual diagrams of the effective field theory calculation.

### A. The graviton and ghost contributions

Our starting point is the tree-level amplitude which takes the generic form

$$M^{\text{tree}}(p_1, p_2, k_1, k_2) = M^{\text{tree}}_{\mu\nu,\rho\sigma}(p_1, p_2, k_1, k_2) \epsilon^{\mu\nu}(k_1) \epsilon^{\rho\sigma}(k_2) \,. \qquad \text{(IV.1)}$$

When we take the discontinuity across the massless graviton lines we use the harmonic gauge polarization sum $\mathcal{P}_{\alpha\beta,\gamma\delta}$ given in Eq. II.2. This yields the expression for the on-shell discontinuity (in $D = 4 - 2\epsilon$ dimensions)

$$M^{1-loop}\big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{M^{\rm tree}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1)\mathcal{P}^{\mu\nu,\alpha\beta}\mathcal{P}^{\rho\sigma,\gamma\delta}(M^{\rm tree}_{\alpha\beta,\gamma\delta}(p_4,p_3,\ell_2,-\ell_1))^*}{\ell_1^2\ell_2^2}\,. \tag{IV.2}$$

A significant simplification in evaluating the discontinuity across the cut in (IV.2) is due to the following remarkable identities noticed in [8]

$$\tau_{2\,\mu\nu,\rho\sigma}(p_1,p_2,m)\mathcal{P}^{\mu\nu}{}_{\alpha\beta}\mathcal{P}^{\rho\sigma}{}_{\gamma\delta} = \tau_{2\,\alpha\beta,\gamma\delta}(p_1,p_2,m)\,, \tag{IV.3}$$

$$\tau^{\rho\sigma}_{3\,\mu\nu,\rho\sigma}(k_1,k_2,q)\mathcal{P}^{\mu\nu}{}_{\alpha\beta}\mathcal{P}^{\rho\sigma}{}_{\gamma\delta} = \tau^{\rho\sigma}_{3\,\alpha\beta,\gamma\delta}(k_1,k_2,q)\,. \tag{IV.4}$$

The identification of the boxes, triangles and bubbles is not as neat as in the helicity approach, and we do not display the intermediate formulas. Performing the index contraction with Mathematica and taking the non-relativistic limit as described in [8] we obtain for the contribution of the cut in eq. (IV.2)

$$M^{\rm disc}(q) \simeq G_N^2\, 4m_1^2m_2^2\left(-\frac{26}{3}\log(\vec{q}^{\,2}) - 6\pi^2\frac{m_1+m_2}{|\vec{q}|}\right)\,. \tag{IV.5}$$

Since we used the harmonic gauge in this covariant computation we need to include the

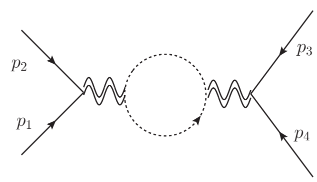


FIG. 4. The ghost contribution from the vacuum polarization of the graviton

extra graph of figure 4 from the contribution of the ghost to the vacuum polarization of the graviton. The ghost Lagrangian for the de Donder harmonic gauge used in this work reads [17, 35]

$$\mathcal{S}^{\rm ghost} = \int d^4x\,\sqrt{g}\,\eta^{*\,\mu}\,(\nabla^\lambda\nabla_\lambda\eta_\mu + \nabla^\lambda\nabla_\mu\eta_\lambda - \nabla_\mu\nabla_\lambda\eta^\lambda)\,. \tag{IV.6}$$

Evaluating the graph in figure 4 leads to the contribution in the non-relativistic limit

$$M^{\rm ghost}(q) \simeq G_N^2\,\frac{1012}{15}m_1^2m_2^2\log(\vec{q}^{\,2})\,. \tag{IV.7}$$

Summing the contributions in (IV.5) and (IV.7) leads to the result given by the helicity computation (III.40) and verifies again ref. [8, eq. (44)]

$$M^{1-\rm loop}(q) \simeq G_N\,4m_1^2m_2^2\left(-6\pi^2\frac{m_1+m_2}{|\vec{q}|} + \frac{41}{5}\,\log(\vec{q}^{\,2})\right)\,. \tag{IV.8}$$

By way of comparison, we note that the helicity amplitude calculation of the previous section corresponds to a sum over the physical helicities

$$M^{1-loop}\big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{M^{\rm tree}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1)\mathcal{S}^{\mu\nu,\alpha\beta}\mathcal{S}^{\rho\sigma,\gamma\delta}(M^{\rm tree}_{\alpha\beta,\gamma\delta}(p_4,p_3,\ell_2,-\ell_1))^*}{\ell_1^2\ell_2^2}\,, \quad \text{(IV.9)}$$

where $\mathcal{S}_{\mu\nu,\rho\sigma}$ arises from the axial-gauge polarization sum

$$\mathcal{S}_{\mu\nu,\rho\sigma} := \sum_{\lambda=\pm1} \epsilon^{\lambda\lambda}_{\mu\nu}(k)(\epsilon^{\lambda\lambda}_{\rho\sigma}(k))^* = \frac{1}{2}\left(S_{\mu\rho}S_{\nu\sigma} + S_{\nu\rho}S_{\mu\sigma} - S_{\mu\nu}S_{\rho\sigma}\right)\,, \quad \text{(IV.10)}$$

with $S_{\mu\nu}$ the axial-gauge spin 1 polarization sum

$$S_{\mu\nu} := \sum_{\lambda=\pm1} \epsilon^{\lambda}_{\mu}(k)(\epsilon^{\lambda}_{\nu}(k))^* = -\eta_{\mu\nu} + \frac{(q_{\rm ref})_\mu k_\nu + (q_{\rm ref})_\nu k_\mu}{q_{\rm ref}\cdot k}\,, \quad \text{(IV.11)}$$

where $(q_{\rm ref})_\mu$ is an arbitrary massless reference momentum. That this sum includes only the two transverse modes can be seen from the condition

$$\eta^{\mu\rho}\eta^{\nu\sigma}\mathcal{S}_{\mu\nu,\rho\sigma} = 2\,, \quad \text{(IV.12)}$$

corresponding to the normalization condition for the two polarization vectors $\epsilon^{\lambda\lambda}_{\mu\nu}(k)$. Our work therefore confirms the expected gauge invariance of the quantum correction.

### B. Comparison with the Feynman graph approach

One useful feature of this method is that one can confirm the analysis of ref. [8] diagram by diagram. Squaring the tree amplitude shown in Fig. 1 leads to discontinuities with the same topology of all the Feynman diagrams evaluated in [8]. Evaluating these individually confirms not only the total result, but also the result of each of the separate diagrams.[4] The advantage of doing the diagrams by the unitarity approach is that one does not have to worry about symmetry factors between Feynman graphs, it is automatically taken care of by the cut.

The precise relation with the analysis of [8] is the following. We decompose the expression for the tree in (II.1) in a sum of three contributions. The first contribution corresponds to the sum of the graph in figure 1(a)

$$M^{(a)}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1) = \tau_1^{\alpha\beta}(p_1,p_2)\,\frac{i\mathcal{P}_{\alpha\beta;\gamma\delta}}{q^2+i\varepsilon}\,\tau_3{}^{\gamma\delta}{}_{\mu\nu;\rho\sigma}(k_1,k_2,p_1+p_2)\,, \quad \text{(IV.13)}$$

[4] In [8] the result for each diagrams has been divided by $4m_1m_2$, whereas in this work the amplitudes are not divided by this factor.

the second contribution corresponds to the graphs in figure 1(b) and (c) and is given by

$$M^{(b)+(c)}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1) = \frac{\tau_{1\,\mu\nu}(p_1,-p_1-\ell_1)\,i\,\tau_{1\,\rho\sigma}(-p_2+\ell_2,p_2)}{2\,p_1\cdot\ell_1+i\varepsilon} + \frac{\tau_{1\,\rho\sigma}(p_1,-p_1+\ell_2)\,i\,\tau_{1\,\mu\nu}(-p_2+\ell_1,p_2)}{-2\,p_1\cdot\ell_2+i\varepsilon}\,. \tag{IV.14}$$

The third contribution corresponding to the graph in figure 1(d)

$$M^{(d)}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1) = \tau_{2\,\mu\nu,\rho\sigma}(p_1,p_2)\,. \tag{IV.15}$$

In the cut we get a total of of six different contributions from the multiplication of the trees. Multiplication of the contributions of type (IV.14) on both sides of the cut gives the discontinuity of the box diagram of [8, sec. 3.2]. Multiplying the contribution (IV.14) and (IV.13) leads to the the discontinuity of the vertex correction contributions in figure 5(a) and 5(b) of [8, sec. 3.5]. Multiplying the contribution (IV.14) and (IV.13) leads to the discontinuity of the triangle contribution of [8, sec. 3.3]. Multiplying the contribution (IV.13) and (IV.13) on both side of the cut gives the discontinuity of the vacuum graph contribution in figure 6(a) of [8, sec. 3.6] without the ghost contribution from figure 4. Multiplying the contribution (IV.13) and (IV.15) leads to the discontinuity of the vertex correction contributions in figure 5(c) and 5(d) of [8, sec. 3.5]. Finally multiplying the contribution (IV.15) on both sides of the cut leads to the discontinuity of the double-seagull diagrams of [8, sec. 3.4].

## V. MATTER-INDEPENDENCE OF THE QUANTUM CORRECTIONS

In this section we will address the previously noted matter-independence of the coefficients $C^{NP}$ and $C^{QG}$. It was found in ref. [11] that the values of these coefficients are independent of the type of the external matter under consideration.

Within the unitarity-based methods, the logic for matter-independence is quite simple. The on-shell gravitational Compton amplitude has a generic form in the low-energy limit. Therefore the discontinuity is matter-independent in the low energy limit, and since we can extract the quantum correction from the discontinuity, the leading quantum corrections also inherits this matter-independence.

That the on-shell gravitational Compton amplitude is matter-independent can be argued for in various ways. Weinberg [36] has shown that the corresponding electromagnetic amplitude is matter-independent using only gauge invariance. It then follows that the on-shell gravitational amplitude is also matter-independent because the latter can be expressed as the square of the electromagnetic amplitude as discussed in Section II. Alternatively, as Weinberg also noted, we know that the electromagnetic amplitude can be expressed by an effective Lagrangian, whose non-relativistic limit is determined by the charge and magnetic moment. In the gravitational case, there is also a low-energy effective Lagrangian for a massive system, described by its energy-momentum and spin [37, 38]. This yields the leading couplings of two gravitons to the heavy particle, which is equivalent to the low-energy limit of our gravitational Compton amplitudes.

In this section we will provide general arguments for the matter-independence of the coefficients $C^{NP}$ and $C^{QG}$ based on the KLT amplitude relation. We will here only consider the spinless contribution to the correction of the classical non-relativistic potential. An more general analysis of the spin multipole expansion will be done elsewhere.

### A. The spin 1 case

In the non-relativistic limit the orthogonality conditions on the spin 1 polarizations, $p_1 \cdot h_1 = p_2 \cdot h_2 = 0$, imply that

$$h_1^0 \simeq \frac{1}{m} \vec{h}_1 \cdot \vec{p}_1, \qquad h_2^0 \simeq \frac{1}{m} \vec{h}_2 \cdot \vec{p}_2 \,. \tag{V.1}$$

Using the relation $(\vec{u} \times \vec{v}) \cdot (\vec{x} \times \vec{y}) = (\vec{u} \cdot \vec{x})(\vec{v} \cdot \vec{y}) - (\vec{u} \cdot y)(\vec{v} \cdot \vec{x})$ we have the following multipole decomposition

$$h_1 \cdot h_2 \simeq -S \left(1 + \frac{q^2}{6m^2}\right) - \frac{i}{2m^2} \vec{S} \cdot (\vec{p}_1 \times \vec{p}_2) + \frac{1}{m^2} \vec{p}_1 \cdot \underline{Q} \cdot \vec{p}_2 \,, \tag{V.2}$$

where $S = \vec{h}_1 \cdot \vec{h}_2$ is the spinless singlet, $\vec{S} := i\vec{h}_1 \times \vec{h}_2$ is the spin vector, and $\underline{Q}^{ij} = \frac{1}{2} (h_1^i h_2^j + h_1^j h_2^i) - \frac{1}{3} \delta^{ij} (\vec{h}_1 \cdot \vec{h}_2)$ is the (traceless) quadrupole tensor. We have used that in the non-relativistic limit $q^2 = (p_1 + p_2)^2 \simeq -2\vec{p}_1 \cdot \vec{p}_2$.

In the non-relativistic limit we can perform an $1/m$ expansion of the Compton tree amplitudes. The Compton scattering of a massive spin 1 vector given in section II C reads

$$\begin{aligned} A_1^{\text{tree}}(p_1, k_2, p_2, k_1) = &-(h_1 \cdot h_2)\, A_0^{\text{tree}}(p_1, k_2, p_2, k_1) \\ &- \frac{h_1 \cdot F_1 \cdot F_2 \cdot h_2 + (h_1 \cdot F_2 \cdot h_2)\,(\epsilon_1 \cdot p_1) + (h_1 \cdot F_1 \cdot h_2)\,(\epsilon_2 \cdot p_2)}{2 p_1 \cdot k_1} \\ &- \frac{h_1 \cdot F_2 \cdot F_1 \cdot h_2 + (h_1 \cdot F_1 \cdot h_2)\,(\epsilon_1 \cdot p_1) + (h_1 \cdot F_2 \cdot h_2)\,(\epsilon_2 \cdot p_2)}{2 p_1 \cdot k_2} \end{aligned} \tag{V.3}$$

To leading order in $1/m$ the amplitude approximates to

$$A_1^{\rm tree}(p_1,k_2,p_2,k_1) \simeq S\, A_0^{\rm tree}(p_1,k_2,p_2,k_1) \\ - \frac{h_1^i F_{1\,ij} {F_2}^j{}_k h_2^k + i(\vec S\cdot\vec B_2)\,(\epsilon_1\cdot p_1) + i(\vec S\cdot\vec B_1)\,(\epsilon_2\cdot p_2)}{2p_1\cdot k_1} \\ - \frac{h_1^i F_{2\,ij} {F_1}^j{}_k h_2^k + i(\vec S\cdot\vec B_1)\,(\epsilon_1\cdot p_1) + i(\vec S\cdot\vec B_2)\,(\epsilon_2\cdot p_2)}{2p_1\cdot k_2}\,. \tag{V.4}$$

The first line receives a contribution from the spin-independent operator $S$ and the last two lines from the spin-orbit and quadrupole operator. The indices $i,j,k=1,2,3$ run over the spatial components.

The singlet spin-independent contribution $S=\vec h_1\cdot\vec h_2$ in this amplitude is multiplied by the scalar Compton amplitude. Using the KLT relation the same property is true for the gravitational Compton amplitude. Therefore the spin-independent contribution of the one-loop correction to Coulomb's potential QED and Newton's potential in gravity, will be the same as the one finds for scalar scattering, even with spin 1 external states.

## B. The spin $\frac{1}{2}$ case

For the spin $\frac{1}{2}$ matter we have a similar decomposition in terms of a spin-independent piece and a spin-orbit part. The spin $\frac{1}{2}$ amplitude takes the form

$$A_{\frac12}^{\rm tree}(p_1,k_2,p_2,k_1) = \frac{n_t^{\frac12}}{p_1\cdot k_1} + \frac{n_u^{\frac12}}{p_1\cdot k_2}\,. \tag{V.5}$$

The expression for $n_t^{\frac12}$ is given in eq. (II.19) with an equivalent expression for $n_u^{\frac12}$ with the exchange of the labels $k_1$ and $k_2$.

We start by rewriting these numerators factors using the identity $(\not p_1+m)\gamma^\mu u(p_1) = 2p_1^\mu u(p_1)$, which is a consequence of the equation of motion $(\not p_1 - m)u(p_1)=0$, to get[5]

$$n_t^{\frac12} = 2\bar u(-p_2)\,\not\epsilon_2 u(p_1)\,(\epsilon_1\cdot p_1) - \frac{2}{3}\varepsilon^{\mu\nu\rho\lambda}\,\epsilon_{2\,\mu}F_{1\,\nu\rho}S_\lambda - 2\bar u(-p_2)\gamma^\nu u(p_1)\,\epsilon_2^\mu F_{1\,\mu\nu}\,. \tag{V.6}$$

Here we have introduced the spin vector

$$S^\mu := \frac{i}{2}\bar u(-p_2)\gamma_5\gamma^\mu u(p_1)\,. \tag{V.7}$$

[5] Where we used that $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$ and $\gamma_5=-i\varepsilon_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}$, and $\gamma^{\mu\nu\rho}=-\frac{i}{3!}\varepsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\lambda$.

Using Gordon's identities one gets that [32]

$$\bar{u}(-p_2)\gamma^\mu u(p_1) = \frac{1}{1-\frac{q^2}{4m^2}} \left( S\, \frac{p_1^\mu - p_2^\mu}{2m} + \frac{i}{m^2}\varepsilon^{\mu\nu\rho\sigma} p_{1\,\nu} p_{2\,\rho} S_\sigma \right) , \tag{V.8}$$

where $S = \bar{u}(-p_2)u(p_1)$ is the spinless singlet.

Since our spinor are normalized according to $\bar{u}(p)u(p) = 2m$, following the conventions of [39], the non-relativistic limit gives

$$S^\mu \simeq -2m \left( 0, \vec{S} := \frac{1}{2}\, \xi_2^\dagger \vec{\sigma} \xi_1 \right) , \tag{V.9}$$

$$S \simeq -2m \left( \xi_2^\dagger \xi_1 + \frac{i}{m^2} \vec{S} \cdot (\vec{p}_1 \times \vec{p}_2) \right) . \tag{V.10}$$

In this limit, the numerator factor approximates to

$$n_t^{\frac{1}{2}} \simeq (\xi_2^\dagger \xi_1)\ (2(\epsilon_2 \cdot p_2)(\epsilon_1 \cdot p_1) + 2(p_1 \cdot k_1)(\epsilon_1 \cdot \epsilon_2)) + \frac{2m}{3} \varepsilon^{\mu\nu\rho} \epsilon_{2\mu} F_{1\nu\rho} S_i\, . \tag{V.11}$$

Therefore the leading $1/m$ expansion of the spin $\frac{1}{2}$ Compton scattering takes the form

$$A_{\frac{1}{2}}^{\text{tree}}(p_1, k_2, p_2, k_1) = (\xi_2^\dagger \xi_1)\, A_0^{\text{tree}}(p_1, k_2, p_2, k_1) + \frac{2m}{3} \varepsilon^{\mu\nu\, i} S_i \left( \frac{\epsilon_{2\mu} F_{1\nu\rho}}{p_1 \cdot k_1} + \frac{\epsilon_{1\mu} F_{2\nu\rho}}{p_1 \cdot k_2} \right) . \tag{V.12}$$

We observe that the spin-independent part is again equal to the scalar amplitude and the spin-orbit part is identical to the one derived for spin 1 amplitudes. Using the KLT relation the same property is true for the gravitational Compton amplitude. Therefore the spin-independent contribution of the one-loop correction to Coulomb's potential in QED and in Newton's potential in gravity, will be the same as the one finds for scalar scattering, even with massive fermionic external states.

## VI. CONCLUSION

In this paper we have computed the leading classical and quantum corrections to the Coulomb and Newton potentials. This has been done using modern techniques employing spinor-helicity variables and on-shell unitarity methods at one-loop order for the first time.

This approach greatly simplifies the evaluation of these corrections. It is possible to compare our computation directly to previous Feynman diagram computations by staying in a covariant formalism, and explicitly put in the ghost loop contribution. By doing so, we have verified the gauge invariance of the quantum correction. Such unitarity based methods also emphasize that the quantum correction come from only the low energy limit of the on-shell gravitational amplitudes, and are insensitive to the unknown high energy behavior of the full theory of quantum gravity.

We also considered matter-independence properties of the results for the non-analytic contributions, and we showed directly using the KLT formalism that the spinless corrections to the amplitude theoretically has to be manifestly independent of the nature of the interacting particles as have been observed in the literature previously [11]. Such matter-independence statements for low energy quantum gravity appears to be equivalent to previously noted statements at low energy in QED [36]. The results are low-energy theorems of quantum gravity.

The ultimate and ultraviolet safe theory of quantum gravity is still not known, however it is gratifying to learn that it is possible to compute universal results in the theory of quantum gravity. They are universal in the sense that any theory having the same low-energy spectrum of particles will have the same answer for the leading corrections independent of what the high-energy completion might turn out to be. Although quantum gravity is at times an *exhaustive* discipline [35] is important to realize that the treament using modern on-shell methods presents a huge advantage in efficiency. For example it might be possible to apply some of our techniques to the recent paper [40] and more generally it might be of interest to reconsider many historical computations in the light of new computational methods. The recent progress in computational techniques will here most likely allow an extended analysis.

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## Appendix A: Vertices and Propagators

We will here list the Feynman rules which are employed in our calculation. For the derivation of these forms, see [7, 41]. Our convention differs from these work by having all incoming momenta.

The propagators are given by

- The massive scalar propagator is $\dfrac{i}{q^2 - m^2 + i\varepsilon}$.

- The graviton propagator in harmonic gauge can be written in the form $\dfrac{i\mathcal{P}^{\alpha\beta,\gamma\delta}}{q^2 + i\varepsilon}$ where $\mathcal{P}^{\alpha\beta,\gamma\delta}$ is defined in (II.2).

In the background field methods used in [7, 41], one develops the metric into an expansion $g_{\mu\nu} = H_{\mu\nu} + \kappa_{(4)}\, h_{\nu\mu}$ where $H_{\mu\nu}$ is the classical background field and $h_{\mu\nu}$ is the quantum field. The relation between the vertices given below and the vertices derived by De Witt is discussed in sec. II A.

The vertices are given by

- The 2-scalar-1-graviton vertex $\tau_1^{\mu\nu}(p_1, p_2)$ is

$$\tau_1^{\mu\nu}(p_1, p_2) = \frac{i\kappa_{(4)}}{2} \left[ p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2}\eta^{\mu\nu} (p_1 + p_2)^2 \right] . \tag{A.1}$$

- The 2-scalar-2-graviton vertex $\tau_2^{\eta\lambda\rho\sigma}(p_1, p_2)$ is

$$\begin{aligned}\tau_2^{\eta\lambda\rho\sigma}(p_1, p_2) = -i\kappa_{(4)}^2 \Bigg[ & \left\{ \mathcal{P}^{\eta\lambda,\alpha\delta}\mathcal{P}^{\rho\sigma,\beta}{}_{\delta} + \frac{1}{4}\left\{ \eta^{\eta\lambda}\mathcal{P}^{\rho\sigma,\alpha\beta} + \eta^{\rho\sigma}\mathcal{P}^{\eta\lambda,\alpha\beta} \right\} \right\} (p_{1\,\alpha} p_{2\,\beta} + p_{2\,\alpha} p_{1\,\beta}) \\ & + \frac{1}{4}\mathcal{P}^{\eta\lambda,\rho\sigma} (p_1 + p_2)^2 \Bigg] . \end{aligned} \tag{A.2}$$

- The 3-graviton vertex, between two quantum fields and one classical field, derived via the background field method has the form [41], where $k+q+\pi=0$,

$$\begin{aligned}\tau_3{}^{\mu\nu}_{\alpha\beta\gamma\delta}(k,q) = -\frac{i\kappa_{(4)}}{2}\times\Bigg(&\mathcal{P}_{\alpha\beta\gamma\delta}\left[k^\mu k^\nu+\pi^\mu\pi^\nu+q^\mu q^\nu-\frac{3}{2}\eta^{\mu\nu}q^2\right]\\
&+2q_\lambda q_\sigma\left[I_{\alpha\beta}{}^{\sigma\lambda}I_{\gamma\delta}{}^{\mu\nu}+I_{\gamma\delta}{}^{\sigma\lambda}I_{\alpha\beta}{}^{\mu\nu}-I_{\alpha\beta}{}^{\mu\sigma}I_{\gamma\delta}{}^{\nu\lambda}-I_{\gamma\delta}{}^{\mu\sigma}I_{\alpha\beta}{}^{\nu\lambda}\right]\\
&+\Bigg[q_\lambda q^\mu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\nu\lambda}+\eta_{\gamma\delta}I_{\alpha\beta}{}^{\nu\lambda}\right)+q_\lambda q^\nu\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\lambda}+\eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\lambda}\right)\\
&-q^2\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\mu\nu}+\eta_{\gamma\delta}I_{\alpha\beta}{}^{\mu\nu}\right)-\eta^{\mu\nu}q_\sigma q_\lambda\left(\eta_{\alpha\beta}I_{\gamma\delta}{}^{\sigma\lambda}+\eta_{\gamma\delta}I_{\alpha\beta}{}^{\sigma\lambda}\right)\Bigg]\\
&+\Bigg[2q_\lambda\big(I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta\sigma}{}^{\nu}\pi^\mu+I_{\alpha\beta}{}^{\lambda\sigma}I_{\gamma\delta\sigma}{}^{\mu}\pi^\nu+I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta\sigma}{}^{\nu}k^\mu+I_{\gamma\delta}{}^{\lambda\sigma}I_{\alpha\beta\sigma}{}^{\mu}k^\nu\big)\\
&+q^2\left(I_{\alpha\beta\sigma}{}^{\mu}I_{\gamma\delta}{}^{\nu\sigma}+I_{\alpha\beta}{}^{\nu\sigma}I_{\gamma\delta\sigma}{}^{\mu}\right)+\eta^{\mu\nu}q_\sigma q_\lambda\left(I_{\alpha\beta}{}^{\lambda\rho}I_{\gamma\delta\rho}{}^{\sigma}+I_{\gamma\delta}{}^{\lambda\rho}I_{\alpha\beta\rho}{}^{\sigma}\right)\Bigg]\\
&+\Bigg\{(k^2+\pi^2)\big[\mathcal{P}_{\alpha\beta}{}^{\mu\sigma}\mathcal{P}_{\gamma\delta,\sigma}{}^{\nu}+\mathcal{P}_{\gamma\delta}{}^{\mu\sigma}\mathcal{P}_{\alpha\beta,\sigma}{}^{\nu}-\frac{1}{2}\eta^{\mu\nu}(\mathcal{P}_{\alpha\beta,\gamma\delta}-\eta_{\alpha\beta}\eta_{\gamma\delta})\big]\\
&+\left(\mathcal{P}_{\gamma\delta}{}^{\mu\nu}\eta_{\alpha\beta}\pi^2+\mathcal{P}_{\alpha\beta}{}^{\mu\nu}\eta_{\gamma\delta}k^2\right)\Bigg\}\Bigg)\,,\end{aligned} \tag{A.3}$$

where $I_{\alpha\beta,\gamma\delta}\equiv\mathcal{P}_{\alpha\beta,\gamma\delta}+\frac{1}{2}\,\eta_{\alpha\beta}\eta_{\gamma\delta}$. In section II A we explained that the on-shell tree level amplitudes obtained using these vertices are equivalent to the ones computed with the vertices given by De Witt [18] and Sannan [19]. We remark that the expression for $\tau_3$ is simpler than the three-graviton vertex in these references.

## Appendix B: Dispersion relations

In the main text, we calculated the unitarity cut by projecting it onto discontinuities of box, triangle and bubble integrals. A complementary method involves using the discontinuities to provide the input to a dispersion relation. We have carried this out in both the de Donder gauge (with ghosts) and using the helicity basis (which has only physical degrees of freedom). We briefly describe the dispersive treatment in this appendix.

The dispersive approach to potentials was pioneered by Feinberg and Sucher [31] for QED[6]. They argue for a dispersive representation of the scattering potential

$$V(s,q^2) = -\frac{1}{\pi}\int_0^\infty dt\frac{1}{t-q^2}\rho(s,t)+\text{R.H.cut}\,. \tag{B.1}$$

[6] We have already compared to their QED result in Section III.

where the right-hand cut involves only massive states and does not influence the low energy behavior of the amplitude. Depending on the ultimate high energy theory, this dispersion relation may require subtractions. However, an important point is that the subtraction constants are analytic functions of powers of $q^2$. The subtraction constants then are related to local, analytic terms in the effective Lagrangian [42], and cannot modify the non-analytic terms that come from the low energy end of the dispersion relation. In the case of gravity, the subtraction constants correspond to higher curvature terms in the gravitational action. If we are interested in the low-energy non-analytic terms we can use either subtracted or unsubtracted forms of the dispersion relation.

The spectral function $\rho(s,t)$ is formed by multiplying together the on-shell gravitational Compton amplitudes. In the axial gauge of the helicity basis we have only the physical degrees of freedom

$$\rho(s,t) = -\frac{1}{\pi}\int \frac{d\Omega_\ell}{4\pi}\, M^{\text{tree}}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1)\mathcal{S}^{\mu\nu,\alpha\beta}\mathcal{S}^{\rho\sigma,\gamma\delta}(M^{\text{tree}}_{\alpha\beta,\gamma\delta}(p_4,p_3,\ell_2,-\ell_1))^*\,, \qquad \text{(B.2)}$$

where $\mathcal{S}_{\mu\nu,\rho\sigma}$ is the polarization sum of Eq. IV.10. The graviton momenta in the numerator are taken to be on-shell. If we work in harmonic gauge we have a similar relation with the harmonic gauge polarization sum of Eq. II.2

$$\rho(s,t) = -\frac{1}{\pi}\int \frac{d\Omega_\ell}{4\pi}\, M^{\text{tree}}_{\mu\nu,\rho\sigma}(p_1,p_2,-\ell_2,\ell_1)\mathcal{P}^{\mu\nu,\alpha\beta}\mathcal{P}^{\rho\sigma,\gamma\delta}(M^{\text{tree}}_{\alpha\beta,\gamma\delta}(p_4,p_3,\ell_2,-\ell_1))^*\,. \qquad \text{(B.3)}$$

Of course, in the harmonic gauge we expect to also need to include ghost fields, and this will be verified.

Feinberg and Sucher describe how to do the angular phase-space integrals. It is useful to go to the frame where $p_1 = (\omega,\vec{p})$, $p_2 = (\omega,-\vec{p})$ with $\vec{p} = im_1\zeta_1\hat{p}$ and $\zeta_1 = \sqrt{1-t/4m_1^2}$. In the gravity case there are more momentum factors in the numerator than with QED, but the phase space integrals are simple generalizations of the ones described in [31]. After the phase-space integration, the spectral functions can be expanded at low-energy with the form

$$\rho(s,t) = a_1(s)\frac{1}{\sqrt{t}} + a_2(s) + .... \qquad \text{(B.4)}$$

yielding a potential function

$$V(s,q^2) = \frac{1}{\pi}[a_1(s)\frac{\pi}{\sqrt{-q^2}} + a_2(s)\ln(-q^2) + ....] \qquad \text{(B.5)}$$

which is to be evaluated in the non-relativistic limit.

We have carried out this program in both the helicity basis and in harmonic gauge. In the helicity basis, for simplicity we chose the reference momentum for $\ell_1$ to be $\ell_2$ and visa-versa. The covariant amplitudes were multiplied together, the phase-space integral done and the result was Taylor expanded at low energy using Mathematica. In the helicity basis, this directly reproduced both the classical and quantum non-analytic terms as described in the text. For the harmonic gauge calculation, ghost fields were needed and a separate spectral function for ghosts was included, with the sum of graviton and ghost effects again yielding the expected answer.

The main technical difference between the methods described in the text and this dispersive method is that in the latter method the phase space integral is explicitly calculated while in the former the discontinuity is used to identify the contributions of box, triangle and bubble diagrams. Of course, these yield the same results because the box, triangle and bubble diagrams respect the causality and analyticity properties that go into the dispersion relations.

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