Motivic Cohomology Spectral Sequence and Steenrod Algebra

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Abstract. For an odd prime number p, it is shown that differentials d_n in the motivic cohomology spectral sequence with p-local coefficients vanish unless $p-1$ divides $n-1$. We obtain an explicit formula for the first non-trivial differential d_p, expressing it in terms of motivic Steenrod p-power operations and Bockstein homomorphisms. Finally, we construct examples of varieties, having non-trivial differentials d_p in their motivic spectral sequences.

The motivic cohomology spectral sequence is an algebro-geometric analogue of the Atiyah–Hirzebruch spectral sequence in topology. For smooth varieties it has the second term consisting of motivic cohomology groups and converges to algebraic K-theory.

The spectral sequence was initially constructed for fields by Bloch and Lichtenbaum in their unpublished preprint [BL]. Further, two constructions for varieties were given in papers of Friedlander and Suslin [FS] and Grayson [Gr]. The equivalence of their approaches was established in [Su2].

The behavior of differentials in the motivic cohomology spectral sequence is quite similar to the topological case. Being taken with rational coefficients the sequence collapses at its E_2-term (see [GS]).

The next natural question in the row is to describe possible non-trivial differentials in the spectral sequence with p-local coefficients.

In topology the goal was achieved by Buchstaber [Bu]. In the current paper we establish the parallel result for the motivic cohomology spectral sequence. Philosophically, our approach is quite similar to Buchstaber's one, but the technique is certainly rather different.

The strategy of the proof is the following: First, I show, using Adams operations, that the first non-trivial differential may appear only in E_p-term (Proposition 1.2). Then, computing the motivic Steenrod algebra in the corresponding degree, it is possible to show that the differential in question is proportional to some concrete cohomological operation. Finally, to check that the proportionality coefficient is not 0, I construct an example of a variety such that the differentials d_p in its motivic spectral sequences are non-trivial (Theorem 4.2).

The computation of the p-local Steenrod algebra is based on Voevodsky's result on the structure of the motivic Steenrod algebra with finite coefficients. Since the latter result is proven only for fields of characteristic 0, we should restrict ourself to this case. Besides this reference, all arguments work properly for arbitrary perfect fields of the characteristic prime to p.

First version of this paper was written many years ago (see [Ya]) and contained a gap. The approach used that time led to the necessity to prove triviality of some derived limits in the motivic homotopy category. Unfortunately, it seems that no one knows how to attack that question.

Finally, after years I was able to bypass this problem, using p-cyclotomic coefficients. The idea to do it suddenly appeared during my visit to the University of Nottingham. Passing to this new proof strategy drastically changed the text.

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Notation remarks. Everythere in the paper k denotes a field. For results of Section 3 one should assume that k is of characteristic 0. For Section 4, it is sufficient to assume that k is perfect.

We denote the category of smooth separated schemes of finite type over a base field (smooth varieties) by Sm/k.

We fix a prime number p and denote by $\mathbb{Z}_{(p)}$ the localization of the ring of integers outside the prime ideal (p). We also denote by $\mathbb{Z}/p^\infty$ the p-cyclotomic group, i.e. $\varinjlim \mathbb{Z}/p^n\mathbb{Z}$.

Except for the two examples in Section 3 we always assume that $p > 2$.

$\mathbb{A}^n$ (resp. $\mathbb{P}^n$) always denotes affine (resp. projective) space of dimension n.

A short exact sequence of Abelian groups $0 \to X \to Y \to Z \to 0$ is often denoted by $[X, Y, Z]$, and the corresponding Bockstein homomorphism by $\beta[X, Y, Z]$.

1. Differentials and Adams operations

As it was shown in [FS], for any $X \in Sm/k$ there exists the Motivic Cohomology Spectral Sequence:

$$E_2^{i,j} = H^{i-j,-j}(X) \Rightarrow K_{-i-j}(X), \tag{1.1}$$

starting from the motivic cohomology groups $H^{*,*}$ and converging to the algebraic K-groups of the variety X. Differentials in this spectral sequence are: $d_n \colon E_n^{i,j} \to E_n^{i+n,j-n+1}$ $(n \geq 2)$.

As it was shown in [GS], this spectral sequence taken with rational coefficients collapses immediately. On the other hand, its structure with integer coefficients becomes too tangled, because of the interrelation of different p-prime effects involved. The purpose of the current paper was to investigate the case of $\mathbb{Z}_{(p)}$-coefficients that allows to "distill" the p-prime effects. In this case one gets non-degenerated differentials of rather high degree and it is then interesting to interpret them in different terms, for example, as some cohomological operations. This can be summarized in the following

Theorem 1.1. *The motivic cohomology spectral sequence*

$$E_2^{i,j} = H^{i-j,-j}(X, \mathbb{Z}_{(p)}) \Rightarrow K_{-i-j}(X, \mathbb{Z}_{(p)})$$

has zero differentials d_n for $p-1 \nmid n-1$. The first non-trivial differential d_p coincides, up to the multiplication by a non-zero $\mathbb{Z}/p$-constant with the operation $\mathfrak{B}P^1 r$, where P^1 is the first $\mathbb{Z}/p$ motivic Steenrod power, $\mathfrak{B} = \beta[\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$, and r denotes the coefficient reduction operation corresponding to the residue map $\mathbb{Z}_{(p)} \to \mathbb{Z}/p$.

In the current section we prove the first statement of the theorem[1]. Then, in Section 2 the differential in question will be interpreted as a bistable motivic cohomology operation of degree $(2p-1, p-1)$ i.e., as an element of the corresponding Steenrod algebra, which is computed in Section 3. Finally, in Section 4 we construct an example of a variety for which the corresponding differential in the spectral sequence is non-trivial that completes the proof of the Theorem.

Proposition 1.2. $d_n = 0$ *for* $p-1 \nmid n-1$.

Proof. As it was shown in [GS], for an integer k such that $\frac{1}{k} \in \mathbb{Z}_{(p)}$ the Adams operations ψ_k in $K_*(X, \mathbb{Z}_{(p)})$ can be represented as operations acting on the whole motivic cohomology spectral sequence. Moreover,

[1]see also Merkurjev [Me], who treated, using similar technique, the case of the Brown–Gersten–Quillen spectral sequence.

its action on the E_2-term is given by the relation: $\psi_k(\alpha) = k^{-q}\alpha$ for $\alpha \in H^{*,q}(X)$. Therefore, all topological arguments proposed by Buchstaber [Bu] work in this case as well. Since Adams operations commute with differentials, we have for every integer $n > 1$:

$$d_n\psi_k = \psi_k d_n \colon H^{*,*}(X) \to H^{*+2n-1,*+n-1}(X).$$

Hence, one has: $(k^{n-1} - 1)d_n = 0$. Let us define a number $M(i)$ as the greatest common divisor of the following sequence:

$$M(i) := \underset{k>1}{\text{g.c.d.}}\{k^N(k^i - 1)\}, \tag{1.2}$$

where $N \gg i$. One can easily verify that these numbers are well-defined. The integers $M(i)$ are sometimes called Kervaire–Milnor–Adams constants[2] and their values are presented in Lemma 1.4 below. Obviously, $M(n-1)d_n = 0$. Since for $p - 1 \nmid n - 1$, we have: $p \nmid M(n-1)$, the differentials of these degrees vanish. □

Corollary 1.3. *The motivic spectral sequence with $\mathbb{Q}$-coefficients degenerates at E_2-term.*

Proof. Any differential vanishes after multiplication by an invertible number. □

Lemma 1.4. *For a prime p and a positive integer n denote by $\nu_p(n)$ the greatest dividing p-exponent[3] of n. The Kervaire–Milnor–Adams constants are determined by the following values. For $p > 2$*

$$\nu_p(M(i)) = \begin{cases} 1 + \nu_p(i) \text{ for } i \equiv 0 \mod p-1 \\ 0 \text{ else} \end{cases}$$

and for $p = 2$:

$$\nu_2(M(i)) = \begin{cases} 2 + \nu_2(i) \text{ for } i \equiv 0 \mod 2 \\ 1 \text{ else.} \end{cases}$$

Proof. See [Ad]. □

Corollary 1.5. *For $p > 2$, one has: $pd_p = 0$.*

Proof. Since, by Lemma 1.4, one has: $\nu_p M(p-1) = 1$, the corollary follows. □

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[2]Probably, after paper [KM]

[3]For example, for any positive integer n, one has: $n = 2^{\nu_2(n)}3^{\nu_3(n)}5^{\nu_5(n)}\ldots$

2. Differentials as cohomology operations

Let us give a brief explanation of the construction of motivic Eilenberg–Mac Lane spaces, following, almost literally, the exposition of [Vo3].

For a variety $X \in Sm/k$ consider the presheaf $\mathbb{Z}_{tr}(X)$ of abelian groups on the category Sm/k, which takes a variety U to the abelian group, generated by all cycles on $X \times U$, which are finite and equidimensional over U. For an Abelian group A we set $A_{tr} := A \otimes \mathbb{Z}_{tr}$ and define the presheaves

$$K^{pre}_{n,A} : U \mapsto A_{tr}(\mathbb{A}^n)(U)/A_{tr}(\mathbb{A}^n - \{0\})(U). \tag{2.1}$$

On the Nisnevitch site $(Sm/k)_{\mathrm{Nis}}$ one can sheafify $K^{pre}_{n,A}$. Applying to the resulting sheaves the functor, forgetting Abelian group structure, one obtains the family of pointed sheaves of sets $K_n(A)$ that play the role of Eilenberg–Mac Lane spaces in the homotopy category $\mathcal{H}o_{\mathbb{A}^1}$.

Alternatively, one can start from the presheaf $K^{pre}_{n,\mathbb{Z}}$ and obtain a complex $\mathbb{Z}(n)$ of sheaves of Abelian groups on $(Sm/k)_{\mathrm{Nis}}$ (see the construction in [VSF]). For any $i, j \in \mathbb{Z}$, a smooth scheme X, and an Abelian group A one defines motivic cohomology groups as hypercohomology groups $H^{i,j}(X, A) := \mathbf{H}^i(X_{\mathrm{Nis}}, A(j))$, where $A(j) = A \otimes \mathbb{Z}(j)$. Let $K(i, j, A)$ be a simplicial abelian group sheaf corresponding to the complex $A(j)[i]$. Applying again the forgetful functor one gets the simplicial sheaf of sets that defines an object of the motivic homotopy category $\mathcal{H}o_{\mathbb{A}^1}$ also denoted by $K(i, j, A)$. The sheaves $K(i, j, A)$ are $\mathbb{A}^1$-local [VD] and for any smooth scheme X one has: $H^{i,j}(X, A) = \mathrm{Hom}_{\mathcal{H}o_{\mathbb{A}^1}}(X_+, K(i, j, A))$. For any pointed simplicial sheaf $F_\bullet$ on $(Sm/k)_{\mathrm{Nis}}$ one can take the following definition of reduced motivic cohomology:

$$\tilde{H}^{i,j}(F_\bullet, A) = \mathrm{Hom}_{\mathcal{H}o_{\mathbb{A}^1}}(F_\bullet, K(i, j, A)). \tag{2.2}$$

It is shown in [VD] that there exists a weak equivalence between $K_n(A)$ and $K(2n, n, A)$, so the two constructions of Eilenberg–Mac Lane spaces agree.

Thus, it is reasonable to expect that natural transformations of motivic cohomology functors can be classified by cohomology groups of motivic Eilenberg-Mac Lane spaces. This statement, proven in [Vo] gives us the classification of motivic cohomology operations.

Let us recall that in $\mathbb{A}^1$-homotopy theory there exist two types of sphere objects and hence two different suspension functors. Among all the cohomological operations there are special ones that commute with both suspension isomorphisms. These operations are called bistable, and Voevodsky showed, using a simple trick [Vo3, Prop 2.6], that there exists a bijection between bistable operations and operations that *a priori* commute only with T-suspension. Here T means the Tate object $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$. Operations of the latter type we will call *stable.*

For every Eilenberg–Mac Lane space $K_n(A)$ one can choose a universal element ι_n in the group $H^{2n,n}(K_n(A))$, corresponding to the identical morphism of the space $K_n(A)$. Applying the T-suspension map to the element ι_n, one obtains the element $\Sigma_T \iota_n$, corresponding to some homotopy class $f_n \in [\Sigma_T K_n(A), K_{n+1}(A)]$. Finally, one can construct an inverse system of the groups $H^{i+2n,j+n}(K_n(A), B)$ as shown in the diagram below.

$$\begin{array}{ccc} H^{i+2n+2,j+n+1}(K_{n+1}(A),B) & \xrightarrow{f_n^*} & H^{i+2n+2,j+n+1}(\Sigma_T K_n(A),B) \\ & \searrow & \simeq \downarrow \Sigma_T \\ & & H^{i+2n,j+n}(K_n(A),B) \end{array} \tag{2.3}$$

Let us set:

$$\mathcal{OP}^{i,j}(A,B) = \lim_{\substack{\leftarrow \\ n}} H^{i+2n,j+n}(K_n(A),B). \tag{2.4}$$

It is shown in [Vo] that the groups $\mathcal{OP}^{i,j}(A,B)$ classify stable cohomological operations of degree $\{i,j\}$ from motivic cohomology groups with coefficients in A to ones with coefficients in B. In particular, the group $\mathcal{OP}^{*,*}(\mathbb{Z}/p, \mathbb{Z}/p)$ coincides with motivic Steenrod algebra $\mathcal{A}^{*,*}(\mathbb{Z}/p)$.

Proposition 2.1. *Consider motivic cohomology spectral sequence* $(E_*^{*,*}, d_*)$. *Let us fix an integer* $n > 1$ *and assume that for every* $1 < i < n$ *and any scheme* $X \in Sm/k$ *the differentials* $d_i \colon H^{*,*}(X) \to H^{*+2i-1,*+i-1}(X)$ *are trivial.*

Then, the differential d_n *is a stable cohomological operation of degree* $(2n-1, n-1)$ *on the category* Sm/k.

Proof. Since all the previous operations vanish, the differential d_n actually acts on the E_2-term of the spectral sequence. Due to the functoriality of the spectral sequence construction, it becomes an operation of degree $(2n-1, n-1)$ on the motivic cohomology. To prove the stability, one has to check the commutativity of the following diagram:

$$\begin{array}{ccc} \tilde{H}^{i,j}(X) & \xrightarrow{d_n} & \tilde{H}^{i+2n-1,j+n-1}(X) \\ \Sigma_T \downarrow & & \downarrow \Sigma_T \\ \tilde{H}^{i+2,j+1}(T \wedge X) & \xrightarrow{d_n} & \tilde{H}^{i+2n+1,j+n}(T \wedge X). \end{array} \tag{2.5}$$

Though the space $T \wedge X$ does not belong to Sm/k, its cohomology is a direct summand of the cohomology of the scheme $\mathbb{P}^1 \times X$ due to the existence of the retraction $\mathrm{Spec}(k) \to \mathbb{P}^1$. Actually, the space $T \wedge X$ happens to be $\mathbb{A}^1$-homotopically equivalent to $(\mathbb{P}^1, \infty) \wedge X$ that allows us to apply differentials to its cohomology groups.

Motivic cohomology groups of $T \wedge X$ are $(2,1)$-shifted cohomology groups of X and the isomorphism Σ_T is delivered by multiplication with the Tate element $\sigma_T = \Sigma_T(1) \in H^{2,1}(\mathbb{P}^1)$. Since the spectral sequence differentials satisfy the Leibnitz rule, to prove the commutativity of 2.5, it is suffices to verify that $d_n(\sigma_T) = 0$. This element should lie in the cohomology group of degree $(2n+1, n)$ that vanishes, since $2n+1 > 2n$. So, the result follows by the dimension reasons. □

3. Some calculations in the Steenrod algebra

Proposition 3.1. *For any $n > 0$ the natural functor $\mathcal{F}_n \colon \mathcal{A}b \to \mathcal{H}o_{\mathbb{A}^1}$ mapping a group A to the Eilenberg–Mac Lane space $K_n(A)$ sends short exact sequences $[A, B, C]$ to fibered squares*

$$\begin{array}{ccc} K_n(A) & \longrightarrow & K_n(B) \\ \downarrow & & \downarrow \\ * & \longrightarrow & K_n(C). \end{array}$$

Proof. We will use the identification $K_n(A) = K(2n, n, A)$ mentioned before and the construction of $K(2n, n, A)$ mentioned on page 5. As $A = B \times_C *$ in the category $\mathcal{A}b$, it is sufficient to check that the functor chain coming from A to $K(2n, n, A)$ preserves finite limits. It can be easily checked for the functor $A \mapsto A(n)$. The functor (Complexes) $\to$ (Simplicial groups) and the forgetful functor from simplicial groups to simplicial sets both admit left adjoints, therefore, they also preserve finite limits.

□

Corollary 3.2. *Every short exact sequence $[A, B, C]$ of Abelian groups induces coefficient long exact sequence in cohomology. This correspondence is functorial with respect to morphisms of short exact sequences. In particular, this implies functoriality of the corresponding Bockstein homomorphisms.*

From now on we assume that the base field has characteristic 0.

Our current aim is to compute the module of stable operations from cohomology with $\mathbb{Z}/p^\infty$ coefficients. We start with Voevodsky's computation of the motivic Steenrod algebra. In the sequel we often use abbreviation $\mathcal{OP}^l$ for $\mathcal{OP}^{l,p-1}$, as we always consider only operations of weight $p-1$.

Example 3.3 ([Vo3])**.** For $l > 0$, one has:

$$\mathcal{OP}^l(\mathbb{Z}/p, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p & \text{for } l = 2p-2 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{for } l = 2p-1 \\ \mathbb{Z}/p & \text{for } l = 2p \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the operations P^1, $\{\beta P^1, P^1\beta\}$, and $\beta P^1\beta$ make sets of $\mathbb{Z}/p$-generators in degrees $2p-2, 2p-1$, and $2p$, correspondingly.

Proposition 3.4. *For $l > 0$, $m > 0$ the $\mathbb{Z}/p$-module $\mathcal{OP}^l(\mathbb{Z}/p^m, \mathbb{Z}/p)$ is independent of m. The following operations can be choosen as generators in the corresponding degrees:*

Degree	$2p-2$	$2p-1$	$2p$
Generator(s)	$P^1 r_m$	$P^1\beta_m,\ \beta_1 P^1 r_m$	$\beta_1 P^1 \beta_m$

Here r_m is induced by the coefficient reduction $\mathbb{Z}/p^m \to \mathbb{Z}/p$ and $\beta_m = \beta[\mathbb{Z}/p, \mathbb{Z}/p^{m+1}, \mathbb{Z}/p^m]$ denotes the corresponding Bockstein homomorphism.

Proof. Repeating, almost literally, the arguments of Voevodsky's proof of the motivic Steenrod algebra structure theorem [Vo, Theorem 3.49, Proposition 3.55], one sees that for finitely generated groups of coefficients the operation modules under consideration are isomorphic to their topological counterparts. In the classical algebraic topology there is Cartan's calculation [Ca] stating that the group $H^*(K(\mathbb{Z}/p^s), \mathbb{Z}/p)$ is independent of $s > 0$. This concludes the first part of the statement.

There are two natural maps $\mathcal{OP}(\mathbb{Z}/p, \mathbb{Z}/p) \rightrightarrows \mathcal{OP}(\mathbb{Z}/p^m, \mathbb{Z}/p)$. First of them is induced by the coefficient reduction $\mathbb{Z}/p^m \to \mathbb{Z}/p$ and the second one, shifting the degree, is induced by the Bockstein morphism $\beta_m = \beta[\mathbb{Z}/p, \mathbb{Z}/p^{m+1}, \mathbb{Z}/p^m]$. These maps move the generators $P^1, \beta P^1$ (see Example 3.3) to the generators from the table above and take zeros on the rest part of the Steenrod algebra. One can directly verify that the obtained operations are all non-trivial and pairwise different, that implies the proposition. ☐

The group inclusions $i_m\colon \mathbb{Z}/p^m \hookrightarrow \mathbb{Z}/p^{m+1}$ induce maps in cohomology: $i_m^*\colon H^{*+n,*+n}(K_n(\mathbb{Z}/p^{m+1})) \to H^{*+n,*+n}(K_n(\mathbb{Z}/p^m))$. Passing to the projective limit on n, one obtains the inverse system:

$$\mathcal{OP}^l(\mathbb{Z}/p, \mathbb{Z}/p) \xleftarrow{i_1^*} \mathcal{OP}^l(\mathbb{Z}/p^2, \mathbb{Z}/p) \leftarrow \cdots$$

Corollary 3.5.

$$\lim_{\substack{\leftarrow \\ m}} \mathcal{OP}^l(\mathbb{Z}/p^m, \mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p \text{ for } l = 2p-1, 2p \\ 0 \qquad\qquad \text{otherwise.} \end{cases}$$

Proof. One can easily verify that the induced maps

$$i_m^* \colon \mathcal{OP}^*(\mathbb{Z}/p^{m+1}, \mathbb{Z}/p) \to \mathcal{OP}^*(\mathbb{Z}/p^m, \mathbb{Z}/p)$$

act on the above generators as follows: $i_m^*(P^1 r_{m+1}) = 0$, $i_m^*(P^1\beta_{m+1}) = P^1\beta_m$, $i_m^*(\beta_1 P^1 r_{m+1}) = 0$, and $i_m^*(\beta_1 P^1 \beta_{m+1}) = \beta_1 P^1 \beta_m$. Therefore, only the elements of the form $\{X\beta_1 \leftarrow X\beta_2 \leftarrow \ldots\}$ "survive" in the projective limit. The corollary follows immediately. □

Lemma 3.6. *Let* $X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \cdots$ *be a sequence of abelian groups. Then, for an abelian group* W*, one has:*

$$\lim_{\substack{\leftarrow \\ i}} \mathcal{OP}^{*,*}(X_i, W) \simeq \mathcal{OP}^{*,*}(\lim_{\substack{\rightarrow \\ i}} X_i, W).$$

Proof. The system $\{X_i, \varphi_i\}$ induces the projective system of groups:

$$\mathcal{OP}^{*,*}(X_1, W) \xleftarrow{\varphi_1^\sharp} \mathcal{OP}^{*,*}(X_2, W) \xleftarrow{\varphi_2^\sharp} \cdots$$

Let $\alpha \in \lim_{\leftarrow} \mathcal{OP}^{*,*}(X_i, W)$. Equivalently, consider the system of operations $\{\alpha_i \in \mathcal{OP}^{*,*}(X_i, W)\}$ endowed with induced morphisms $\varphi_i^\sharp$ such that $\alpha_i = \varphi_i^\sharp(\alpha_{i+1})$. For a variety Y let us also consider an element x such that $x \in H^{*,*}(Y, \lim_{\rightarrow} X_i) \simeq \lim_{\rightarrow} H^{*,*}(Y, X_i)$. This element defines a system $\{x_j \in H^{*,*}(Y, X_j)\}_{j>N(x)}$ such that $\varphi_*^j(x_j) = x_{j+1}$. Obviously, $\alpha(\varphi_*(x)) = \varphi^\sharp(\alpha)(x)$. Let us now define $\check{\alpha} \in \mathcal{OP}^{*,*}(\lim_{\rightarrow} X_i, W)$, setting

$$\check{\alpha}(x) := \alpha_N(x_N) = \alpha_{N+1}(x_{N+1}) = \cdots$$

In order to construct the opposite map, let us start with an operation $\gamma \in \mathcal{OP}^{*,*}(\lim_{\rightarrow} X_i, W)$ and construct for any index j an operation $\hat{\gamma}_j$ given by the through map $H^{*,*}(-, X_j) \to H^{*,*}(-, \lim_{\rightarrow} X_j) \xrightarrow{\gamma} H^{*,*}(-, W)$, where the first arrow is canonical and the second is given by the operation γ. These operations fit together to make an element of the projective system and, therefore, the operation $\hat{\gamma} \in \lim_{\substack{\leftarrow \\ i}} \mathcal{OP}^{*,*}(X_i, W)$. One can easily verify that given constructions are mutually inverse. □

Corollary 3.7. *The* $\mathbb{Z}/p$*-module* $\mathcal{OP}^*(\mathbb{Z}/p^\infty, \mathbb{Z}/p)$ *has two generators* $P^1\beta_\infty, \beta_1 P^1 \beta_\infty$*, lying in degrees 2p-1, 2p, correspondingly. Here* $\beta_\infty = \beta[\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty]$.

Proof. Application of Lemma 3.6 to Corollary 3.5 shows that the module $\mathcal{OP}^*(\mathbb{Z}/p^\infty, \mathbb{Z}/p)$ has generators only in degrees $2p-1$ and $2p$.

Natural identification
$$\beta_\infty = \lim_{\overleftarrow{m}} \beta_m$$
completes the proof. □

Let us now move back from p-cyclotomic coefficients to p-local. We will need some auxillary results. Now we are temporarily passing to the stable homotopy category of T-spectra. Denote by $\mathcal{S}$ the suspension spectrum of the final object pt. Let us consider the multiplication by p morphism $\mathcal{S} \xrightarrow{\times p} \mathcal{S}$ and denote its cone by $\mathcal{P}$.

Lemma 3.8. *The spectrum $\mathcal{P}$ is Spanier–Whitehead self-dual up to the 1-shift. That means, for arbitrary spectra U, V there exists a natural isomorphism*
$$[U \wedge \mathcal{P}, V] \simeq [U, V \wedge \mathcal{P}[1]].$$

Proof. Obviously, the spectrum $\mathcal{S}$ is self dual. Then, the Lemma follows from [Sw, 14.29–14.33]. All the arguments, used in *loc.cit.* are of formal nature and can be easily translated in the language of $\mathbb{A}^1$-homotopy theory. □

Corollary 3.9. *Assume that the morphism of spectra $f\colon X \to Y$ induces an isomorphism of all homotopy group sheaves with $\mathbb{Z}/p$-coefficients. Denote by C_pX the cone of the morphism $X \simeq \mathcal{S} \wedge X \xrightarrow{\times p} \mathcal{S} \wedge X \simeq X$. Then, the induced morphism of cones $C_f\colon C_pX \to C_pY$ is a homotopy weak equivalence.*

Proof. Consider the following diagram:
$$\begin{array}{ccc} [\mathcal{S}, C_pX[1]] & \xleftarrow{\simeq} & [\mathcal{P}, X] \\ {\scriptstyle C_f}\downarrow & & \downarrow{\scriptstyle \simeq} \\ [\mathcal{S}, C_pY[1]] & \xleftarrow{\simeq} & [\mathcal{P}, Y] \end{array}$$
The right vertical arrow is an isomorphism by the assumption. Both the horizontal arrows are duality isomorphisms by Lemma 3.8. This proves the Corollary. □

Corollary 3.10. *Let $f\colon X \to Y$ be a morphism of spaces inducing a weak equivalence of p-cones $C_f\colon C_pX \xrightarrow{\simeq} C_pY$. Then, for any cohomology theory E represented by an Ω_T-spectrum $\mathcal{E}$, the map:*
$$E^{*,*}(Y, \mathbb{Z}/p) \xrightarrow{f^*} E^{*,*}(X, \mathbb{Z}/p)$$
is an isomorphism.

Proof. By the previous lemma and corollary, one has:

$$[Y, \mathcal{E} \wedge \mathcal{P}[1]] \simeq [C_pY, \mathcal{E}] \overset{C_f}{\simeq} [C_pX, \mathcal{E}] \simeq [X, \mathcal{E} \wedge \mathcal{P}[1]].$$

The outside groups are isomorphic (after some shift) to the cohomology groups $E^{*,*}(Y, \mathbb{Z}/p)$ and $E^{*,*}(X, \mathbb{Z}/p)$, correspondingly. □

Lemma 3.11. *For any $n > 0$ all homotopy group sheaves $\pi_*(K_n(\mathbb{Q}), \mathbb{Z}/p)$ vanish.*

Proof. Consider the multiplication by p self-morphism $K_n(\mathbb{Q}) \to K_n(\mathbb{Q})$. It is a weak equivalence, since $\mathbb{Q}$ is divisible. On the other hand, this morphism induces zero maps on homotopy groups with $\mathbb{Z}/p$-coefficients, that concludes the proof. □

Below $\sigma_T = \Sigma_T(1)$ denotes the element of the group $H^{2,1}(T)$ obtained by suspension of 1.

Lemma 3.12. *The following sets are bijective:*

(1) *bistable operations $\mathcal{OP}^{i,j}(A, B)$;*
(2) *the collection of natural transformations*

$$\varphi_n \colon H^{2n-1,n}(-, A) \to H^{2n-1+i,n+j}(-, B)$$

given for $n \geq 0$ such that $\varphi_{n+1}(x \wedge \sigma_T) = \varphi_n(x) \wedge \sigma_T$;
(3) *the collection of motivic cohomology classes*

$$\psi_n \in H^{2n-1+i,n+j}(\Omega_S K_n(A), B)$$

such that the restriction of ψ_{n+1} to $\Omega_S K_n(A) \wedge T$ is $\psi_n \wedge \sigma_T$.

Proof. To obtain the bijection between the first and the second sets one just rewrite (up to the index change) Proposition 2.6 of [Vo3]. We give here an explicit construction of the projective system corresponding to the collection of (3). We shall construct the morphisms $\gamma_n \colon \Sigma_T \Omega_S K_n \to \Omega_S K_{n+1}$. Let us start from the structure morphisms $\alpha_n \colon \Sigma_T K_n \to K_{n+1}$ in the Eilenberg-Mac Lane spectrum. Taking the dual ones, applying the Ω_S functor, and swapping two loop functors, we obtain the collection of morphisms:

$$\tilde{\gamma_n} \colon \Omega_S K_n \to \Omega_S \Omega_T K_{n+1} \simeq \Omega_T \Omega_S K_{n+1}.$$

Dualizing the latter morphisms, we have the desired ones.

To get the bijection between (2) and (3) we just claim that the space $\Omega_S K_n(A)$ represents motivic cohomology group $H^{2n-1,n}(-, A)$. The rest is parallel to Proposition 2.7 of *loc. cit.* □

Proposition 3.13. *The Bockstein homomorphism* $B = \beta[\mathbb{Z}_{(p)}, \mathbb{Q}, \mathbb{Z}/p^\infty]$ *induces the isomorphism of* $\mathbb{Z}/p$*-modules:*

$$\mathcal{OP}^*(\mathbb{Z}_{(p)}, \mathbb{Z}/p) \underset{\simeq}{\xrightarrow{B^*}} \mathcal{OP}^{*+1}(\mathbb{Z}/p^\infty, \mathbb{Z}/p).$$

The group $\mathcal{OP}^*(\mathbb{Z}_{(p)}, \mathbb{Z}/p)$ *is* $\mathbb{Z}/p$ *in degrees* $* = 2p-2, 2p-1$ *and trivial otherwise. One can canonically take operations* $P^1 r, \beta_1 P^1 r \in \mathcal{OP}^*(\mathbb{Z}_{(p)}, \mathbb{Z}/p)$ *as generators in the corresponding degrees.*

Proof. Using [Vo3, Proposition 2.6] and Lemma 3.12 one can see that operations on both sides of the proposition statement can be seen as the collection of elements $\chi_n \in H^{2n+*,n+p-1}(K_n(\mathbb{Z}_{(p)}), \mathbb{Z}/p)$ on the left-hand-side and $\psi_n \in H^{2n+*,n+p-1}(\Omega_S K_n(\mathbb{Z}/p^\infty), \mathbb{Z}/p)$ on the right-hand-side, correspondingly, where both the collections satisfy compatibility conditions of Lemma 3.12.

Consider the short exact sequence of abelian groups

$$0 \to \mathbb{Z}_{(p)} \to \mathbb{Q} \to \mathbb{Z}/p^\infty \to 0.$$

By Proposition 3.1 for every $n > 0$ it gives a fibered square of motivic Eilenberg-Mac Lane spaces:

(3.1)
$$\begin{array}{ccc} K_n(\mathbb{Z}_{(p)}) & \longrightarrow & K_n(\mathbb{Q}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & K_n(\mathbb{Z}/p^\infty) \end{array}$$

In the homotopy category it determines a morphism $\theta_n\colon \Omega_S K_n(\mathbb{Z}/p^\infty) \to K_n(\mathbb{Z}_{(p)})$.

By functoriality, one can check that for $n > 0$ diagrams

$$\begin{array}{ccc} \Sigma_T K_n(\mathbb{Z}_{(p)}) & \xleftarrow{\theta_n} & \Sigma_T \Omega_S K_n(\mathbb{Z}/p^\infty) \\ {\scriptstyle \alpha_n}\downarrow & & \downarrow{\scriptstyle \gamma_n} \\ K_{n+1}(\mathbb{Z}_{(p)}) & \xleftarrow{\theta_{n+1}} & \Omega_S K_{n+1}(\mathbb{Z}/p^\infty) \end{array}$$

are commutative in the homotopy category. Therefore, the family θ_n determines the map of sets:

$$\Theta\colon \mathcal{OP}^m(\mathbb{Z}_{(p)}, \mathbb{Z}/p) \to \mathcal{OP}^{m+1}(\mathbb{Z}/p^\infty, \mathbb{Z}/p).$$

By the construction, it is also clear that the map Θ coinside with the map B^* induced by the Bockstein homomorphism $B = \beta[\mathbb{Z}_{(p)}, \mathbb{Q}, \mathbb{Z}/p^\infty]$.

To show Θ is a bijection, write down a long exact sequence of homotopy group sheaves with $\mathbb{Z}/p$ coefficients, corresponding to the fibered square 3.1. By Lemma 3.11, one has:

$$\pi_*(\Omega_S K_n(\mathbb{Z}/p^\infty), \mathbb{Z}/p) \xrightarrow{\simeq} \pi_*(K_n(\mathbb{Z}_{(p)}), \mathbb{Z}/p)$$

and Corollaries 3.9, 3.10 show that constructed maps give a bijection between projective families ψ and φ.

Finally, the morphism of short exact sequences

$$(r, \times p^{-1}, \mathrm{id}) \colon [\mathbb{Z}_{(p)}, \mathbb{Q}, \mathbb{Z}/p^\infty] \to [\mathbb{Z}/p, \mathbb{Z}/p^\infty, \mathbb{Z}/p^\infty]$$

implies the equality $rB = \beta_\infty$ that, together with exposition of groups $\mathcal{OP}^*(\mathbb{Z}/p^\infty, \mathbb{Z}/p)$ given above, supplies us with the desired set of generators. This proves the proposition. □

Denote by ${}_p\mathcal{A}^{*,*}$ the subgroup of the Steenrod algebra $\mathcal{A}^{*,*}$, made by operations, vanishing after multiplication by p.

Lemma 3.14. *The Bockstein homomorphism $\mathfrak{B} = \beta[\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$ induces the epimorphism:*

$$\mathcal{OP}^{2p-2}(\mathbb{Z}_{(p)}, \mathbb{Z}/p) \xrightarrow{\mathfrak{B}} {}_p\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)}).$$

Proof. Considering a long exact cohomology sequence, corresponding to the short exact coefficient sequence $[\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$ one can easily see that any p-torsion cohomological operation from $\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)})$ lifts to an element from $\mathcal{OP}^{2p-2}(\mathbb{Z}_{(p)}, \mathbb{Z}/p)$. □

Remark 3.15. In fact, the epimorphism in Lemma 3.14 is an isomorphism. In order to check, it is sufficient to exhibit a non-trivial p-torsion operation from $\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)})$. The examples of such operations are given in the next section.

Summarizing the results of Proposition 3.13, and Lemma 3.14, we obtain the following

Theorem 3.16. *Let $\mathcal{F} \colon H^{*,*}(-, \mathbb{Z}_{(p)}) \to H^{*+2p-1,*+p-1}(-, \mathbb{Z}_{(p)})$ be a non-trivial bistable cohomological operation on motivic cohomology. Let also $p\mathcal{F} = 0$. Then, the operation $\mathcal{F}$ coincides up to the multiplication by a non-zero constant[4] with the operation $\mathfrak{B}P^1 r$, where P^1 is the first $\mathbb{Z}/p$ motivic Steenrod power, $\mathfrak{B} = \beta[\mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}, \mathbb{Z}/p]$, and r denotes the corresponding coefficient reduction operation.*

[4]In fact, a non-zero element of $\mathbb{Z}/p$.

4. $d_p \neq 0$

The purpose of the current section is to construct a variety such that the corresponding motivic cohomology spectral sequence has a non-trivial differential. Although the previous deicussion makes us to consider only the case of $p > 2$ and char $k = 0$, in this section we decided to give more general statements. So, here we just have to assume that p is a prime number and the field k is perfect such that (char $k, p) = 1$. All coefficient rings are assumed to be $\mathbb{Z}_{(p)}$. Abusing the notation, we omit mentioning coefficients unless it is absolutely nesessary. k denotes a field of characteristic 0. From now on $\mathcal{D}$ always denotes a central simple algebra over k of degree p. We also denote by G the norm variety $SL_{1,\mathcal{D}}$ and by $X = SB(\mathcal{D})$ the corresponding Severi–Brauer variety. Let us mention that since G is a twisted form of SL_p, one has $\dim G = p^2 - 1$.

The variety G supplies us with the examples of non-trivial differentials, which are going to be considered in Theorem 4.2. Before we pass to the theorem statement, let us give another example[5], demonstrating the non-triviality of differentials d_2 in the case $p = 2$.

Example 4.1. Consider the motivic spectral sequence for the variety $\operatorname{Spec}\mathbb{Q}$. One can check that the Milnor symbol $\{-1,-1,-1,-1\} \in K_4^M(\mathbb{Q})$ is non-trivial of the order 2. This symbol lies in $E_2^{0,-4}$. On the other hand, the spectral sequence converges to $K_4(\mathbb{Q})$, the map $K_4^M(\mathbb{Q}) \to K_4(\mathbb{Q})$ should pass through the stable homotopy group of the sphere spectrum π_S^4. The latter group is trivial, therefore, one gets from the short exact sequence $E_2^{-2,-3} \to E_2^{0,-4} \to E_\infty^{0,-4}$ that the differential $d_2 \colon H^1(\operatorname{Spec}\mathbb{Q}, \mathbb{Z}(3)) \to E_2^{0,-4} = K_4^M(\mathbb{Q})$ is non-zero. This is, certainly, true with $\mathbb{Z}/2$ coefficients as well.

Theorem 4.2. *Consider the Motivic spectral sequence corresponding to the variety G*

$$E_2^{i,j} = H^{i-j,-j}(G) \Rightarrow K_{-i-j}(G).$$

The differential $d_p \colon E_p^{1,-2} \to E_p^{p+1,-p-1}$ is non-trivial.

Proof. We formulate, first, three conditions that imply, almost obviously, the theorem statement.

(1) $K_0(G) = \mathbb{Z}_{(p)} \cdot 1$, *where the class* 1 *lies in codimension* 0*;*
(2) $CH^{p+1}(G) \neq 0$;
(3) *For* $1 < n < p$ *the differentials* d_n *vanish.*

[5] The author is in debt to Chuk Weibel, who paid his attention to this example.

The first condition is proven in [Su, Theorem 6.1]. The third one is exactly the statement of Proposition 1.2. The rest of the paper will be devoted to the proof of the second one (Proposition 4.3).

To complete the proof of the theorem, we just mention that since motivic cohomology coincide with higher Chow groups, the term

$$E_2^{p+1,-p-1} = CH^{p+1}(G,0) = CH^{p+1}(G) \neq 0$$

by (2). On the other hand, $E_\infty^{p+1,-p-1} = 0$, since, by (1) the whole group $K_0(G)$ is concentrated in the term $E_\infty^{0,0} = \mathbb{Z}_{(p)}$. Hence, there should be a non-trivial differential, that kills the term $E_2^{p+1,-p-1}$. By (3) the only possibility is that $0 \neq d_p\colon E_p^{1,-2} \to E_p^{p+1,-p-1}$. □

Proposition 4.3. *For the algebra $\mathcal{D}$, one has: $CH^{p+1}(G) \neq 0$.*

Proof. Let us recall that $X = SB(\mathcal{D})$ denotes the Severi-Brauer variety of dimension $p-1$ corresponding to the algebra $\mathcal{D}$. For the projection map $G \times X \to G$ consider a filtration of the base by codimension of points and write down the corresponding spectral sequence (see Rost[Ro]):

$$E_1^{st}(n) = \coprod_{g \in G^{(s)}} H^t(X_{F(g)}, \mathcal{K}_{n-s}) \ \Rightarrow\ H^{s+t}(G \times X, \mathcal{K}_n), \tag{4.1}$$

where $X_{F(g)} = X \times \operatorname{Spec} F(g)$ is a fiber over the generic point g. This spectral sequence is a natural generalization of the Brown–Gersten–Quillen (BGQ) spectral sequence.

For convenience we have drawn below the most important for us case $n = p+1$, setting, for shortness

$$\coprod_{g \in G^{(s)}} H^t(X_{F(g)}, \mathcal{K}_u) = R^s_{t,u}.$$

The E_1-term of this spectral sequence is concentrated in the strip given by the conditions: $0 \leq t \leq p-1$ and $s+t \leq n$.

$$\begin{array}{cccccc}
R^0_{p-1,p+1} & R^1_{p-1,p} & R^2_{p-1,p-1} & 0 & 0 & \\
\vdots & \vdots & & \ddots & 0 & 0 \\
R^0_{1,p+1} & R^1_{1,p} & \cdots & R^{p-1}_{1,2} & R^p_{1,1} & 0 \\
R^0_{0,p+1} & R^1_{0,p} & \cdots & & R^p_{0,1} & R^{p+1}_{0,0}
\end{array}
\qquad d_p : R^1_{p-1,p} \to R^{p+1}_{0,0}$$

Almost everythere we need to consider the spectral sequence for $n = p+1$. In these cases we will just omit the index $(p+1)$.

Assume, the following statements hold:

(1) $E_2^{p+1,0} = CH^{p+1}(G)$;

(2) $\operatorname{Coker}\left(H^p(G \times X, \mathcal{K}_{p+1}) \xrightarrow{\varphi} E_p^{1,p-1}\right) \neq 0$.

Then the proposition follows easily. Actually, just consider the boundary short exact sequence:

$$H^p(G \times X, \mathcal{K}_{p+1}) \xrightarrow{\varphi} E_p^{1,p-1} \xrightarrow{d_p} E_p^{p+1,0}.$$

As, by (2) φ is not an epimorphism, one has: $E_p^{p+1,0} \neq 0$, but by (1) and the dimension reasons, there exists an epimorphism $CH^{p+1}(G) = E_2^{p+1,0} \twoheadrightarrow E_p^{p+1,0}$ that proves the desired result.

The rest of the paper is devoted to the proof of auxillary statements. (1) is established in Lemma 4.4 right below, (2) is proven in Proposition 4.8. □

Lemma 4.4. *$E_2^{p+1,0} = CH^{p+1}(G)$.*

Proof. One has: $E_2^{p+1,0} = R_{0,0}^{p+1}/R_{0,1}^p$. Decoding the notation, we get:

$$E_2^{p+1,0} = \operatorname{Coker}\left(\coprod_{g \in G^{(p)}} F(g)^* \to \coprod_{g \in G^{(p+1)}} \mathbb{Z}\right) = CH^{p+1}(G) \tag{4.2}$$

that completes the proof. The same is, certainly, true with $\mathbb{Z}_{(p)}$ coefficients. □

Now we show that $E_2^{1,p-1} = E_p^{1,p-1}$. This allows us to compute the term $E_p^{1,p-1}$ in the explicit way.

Proposition 4.5. *Differential maps* $d_1^t \colon R_{t,t+1}^{p-t} \to R_{t,t}^{p-t+1}$ *are epimorphisms, provided that* $1 \le t \le p-1$. *Therefore, in these cases* $E_2^{p+1-t,t} = 0$.

Proof. Deciphering the notation, we have to prove that the maps

$$\coprod_{g \in G^{(p-t)}} H^t(X_{F(g)}, \mathcal{K}_{t+1}) \to \coprod_{g \in G^{(p+1-t)}} H^t(X_{F(g)}, \mathcal{K}_t)$$

are epimorphisms. The inner groups $H^t(X_{F(g)}, \mathcal{K}_{t+m})$ can be computed using the Brown–Gersten–Quillen spectral sequence. Writing down Gersten resolutions for different values of t one gets natural maps on the resolutions, induced by embedding of points of different codimensions. This implies natural maps of BGQ spectral sequences and, finally, natural maps of $\mathcal{K}$-cohomology groups

$$\cdots \to H^t(X_{F(g)}, \mathcal{K}_{t+m}) \to H^{t+1}(X_{F(g)}, \mathcal{K}_{t+1+m}) \to \cdots$$

By Statement 4.6, these maps are isomorphisms for $m = 0, 1$ and $1 \le t \le p-1$. By functoriality of the construction this implies that

$$E_2^{p+1-t,t} = R_{t,t}^{p-t+1}/R_{t,t+1}^{p-t} \simeq R_{p-1,p-1}^{p-t+1}/R_{p-1,p}^{p-t} = E_2^{p-t+1,p-1}(2p-t).$$

The rest follows from Lemma 4.7 below. □

In the proof of previous proposition we needed a result of Merkurjev and Suslin, which we reproduce here.

Statement 4.6 ([MS, Corollary 8.7.2])**.** *Let* $\bar{k}$ *be the algebraic closure of* k. *For a Severi–Brauer variety* X *of dimension* $p-1$, *set* $\bar{X} = X \times \operatorname{Spec} \bar{k}$. *Then*

$$H^i(X, \mathcal{K}_i) = CH^i(X) = p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} = CH^i(\bar{X}) \tag{4.3}$$

and

$$H^i(X, \mathcal{K}_{i+1}) = \operatorname{Nrd} \mathcal{D}^* \subset \bar{k}^* = H^i(\bar{X}, \mathcal{K}_{i+1}), \tag{4.4}$$

provided that $1 \le i \le p-1$. *(Here* Nrd *denotes the group of the reduced norms.)*

Lemma 4.7. *For* $n > p$, *one has:* $E_2^{n-p-1,p-1}(n) = 0$.

Proof. Consider now $G \times X$ as a group-variety over X. By Suslin's computations [Su], $H^*(G \times X, \mathcal{K}_*)$ becomes a module over $H^*(X, \mathcal{K}_*)$ generated by Chern classes c_j, where $c_j \in H^j(G \times X, \mathcal{K}_{j+1})$ for $j > 0$. In particular, this implies that $CH^i(G \times X) = 0$ for $i > p-1$. Therefore, the spectral sequence converges to zero in the nth diagonal. In particular, $E_\infty^{n-p-1,p-1}(n) = 0$. By the dimension reasons, there are no differentials, affecting the term $E_2^{n-p-1,p-1}(n)$. So that, one has: $E_2^{n-p-1,p-1}(n) = E_\infty^{n-p-1,p-1}(n) = 0$. □

Proposition 4.8. *The map* $\varphi\colon H^p(G \times X, \mathcal{K}_{p+1}) \to E_p^{1,p-1}$ *has non-trivial cokernel.*

Proof. Let us mention, first, that by the previous lemma, one has: $E_p^{1,p-1} = E_2^{1,p-1}$. Denote this group by V and consider the base-change commutative diagram corresponding to the morphism $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$, where $\bar{k}$ is the algebraic closure of k.

$$\begin{array}{ccc} H^p(G \times X, \mathcal{K}_{p+1}) & \xrightarrow{\varphi} & V \\ \chi \downarrow & & \psi \downarrow \\ H^l(\bar{G} \times \bar{X}, \mathcal{K}_{p+1}) & \xrightarrow{\bar{\varphi}} & \bar{V} \end{array} \tag{4.5}$$

The desired statement easily follows from the following three claims:

(1) Im χ is divisible by p;
(2) $\psi\colon V \to \bar{V}$ is an epimorphism;
(3) $\bar{V} = \mathbb{Z}_{(p)}$.

Assume that φ is an epimorphism. Since ψ is also an epimorphism, we can chose an element $x \in H^p(G \times X, \mathcal{K}_{p+1})$ such that $\psi\varphi(x) = 1$. Then, by (1), $1 = \bar{\varphi}\chi(x)$ is p-divisible. This gives a contradiction. We prove (1) in Lemma 4.9, (2) in Proposition 4.10 below. Finally, (3) appears in the proof of 4.10 as an indirect result. □

Lemma 4.9. Im χ *is divisible by* p.

Proof. This follows from the above mentioned (see the proof of Lemma 4.7) decomposition

$$H^p(G \times X, \mathcal{K}_{p+1}) = \coprod_{i>0} c_i CH^{p-i}(X) \tag{4.6}$$

and the fact that the map $CH^i(X) \to CH^i(\bar{X})$ is a multiplication by p due to Proposition 4.6. □

Proposition 4.10. *The map* $\psi\colon V \to \bar{V}$ *is an epimorphism.*

Proof. First, consider the BGQ spectral sequence converging to the K-groups of the Severi–Brauer variety X. Since $(p-1)!$ is invertible in the coefficient ring, this spectral sequence has no non-trivial differentials affecting the two highest diagonals. Moreover, if the base field is algebraically closed, all the differentials in the spectral sequence vanish (see [MS, 8.6.2]).

The infinity term of BGQ consists of consequent factor-filtration groups of $K(X)$. Taking into account the triviality of differentials mentioned in the previous paragraph, there exist boundary maps:

$$H^{p-1}(X, \mathcal{K}_{p-1+m}) \to K_m(X)^{(p-1)}, \tag{4.7}$$

where $m = 0, 1, 2$. These maps are isomorphisms for $m = 0, 1$. Provided that the base field is algebraically closed, they are isomorphisms also for $m = 2$.

Since $(p-1)!$ is invertible in the coefficient ring, the topological filtration coincide with the γ-filtration. The desired filtration is generated by the image of the corresponding γ-operation.

Later we need one additional definition. For a quasi-compact locally Noetherian scheme Y, let A be a sheaf of algebras on Y locally isomorphic in the étale topology on Y to the sheaf of split algebras $M_n(\mathcal{O}_Y)$. In other words, A is an Azumaya algebra on Y. Consider now the category $\mathcal{P}(Y; A)$, whose objects are sheaves of left A-modules, which are locally free coherent $\mathcal{O}_Y$-modules. Morphisms in this category are $\mathcal{O}_Y$-module ones, compatible with the A-module structure.

Definition 4.11. We set $K_*(Y; A) = K_*(\mathcal{P}(Y; A))$, where on the right-hand side we have Quillen's K-functor construction.

By Quillen's computation of K-groups of Severi–Brauer varieties, [Qu] one has isomorphisms: $K_m(X)^{(p-1)} \simeq K_m(\mathcal{D}^{\otimes(p-1)})$. Denoting $\mathcal{D}^{\otimes(p-1)}$ by $\mathfrak{D}$, we obtain the maps: $H^{p-1}(X_g, \mathcal{K}_{p-1+m}) \overset{\rho_m}{\to} K_m(F(g); \mathfrak{D})$ for $m = 0, 1, 2$, which are isomorphisms for $m = 0, 1$ and isomorphism for $m = 2$ provided that the base-field is algebraically closed. As a result, one gets the map of complexes ρ_*:

(4.8)

$$
\begin{array}{ccccc}
R^0_{p-1,p+1} & \longrightarrow & R^1_{p-1,p} & \longrightarrow & R^2_{p-1,p-1} \\
\downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_0} \\
K_2(F(G); \mathfrak{D}) & \longrightarrow & \coprod_{g \in G^{(1)}} K_1(F(g); \mathfrak{D}) & \longrightarrow & \coprod_{g \in G^{(2)}} K_0(F(g); \mathfrak{D}),
\end{array}
$$

inducing the epimorphism map $\tilde{\rho}$ on the middle-term homology groups. The latter map becomes an isomorphism after passing back to the algebraic closure. The middle-term homology group in the bottom line can be rewritten as $H^1(G, \mathcal{K}_2; \mathfrak{D})$. Let us consider the base-change diagram corresponding to the morphism $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$:

$$
\begin{array}{ccc}
V & \xrightarrow{\psi} & \bar{V} \\
{\scriptstyle \tilde{\rho}}\downarrow & & \| \\
H^1(G, \mathcal{K}_2; \mathfrak{D}) & \xrightarrow{\omega} & H^1(\bar{G}, \mathcal{K}_2; \bar{\mathfrak{D}}).
\end{array}
\tag{4.9}
$$

Observe now, that $\bar{G} = SL_n(\bar{k})$ and $H^1(\bar{G}, \mathcal{K}_2; \bar{\mathfrak{D}}) = H^1(SL_n, \mathcal{K}_2) = \mathbb{Z}_{(p)}$ with a natural choice of a generator, given by the first Chern class (see [Su, Theorem 2.7]). Consider another base-change diagram:

$$\begin{array}{ccc} H^1(G,\mathcal{K}_2;\mathfrak{D}) & \xrightarrow{\ \omega\ } & H^1(SL_n,\mathcal{K}_2) \\ \Big\uparrow{\scriptstyle c_1} & & {\scriptstyle \bar{c}_1}\Big\uparrow \\ K_1(G;\mathfrak{D}) & \xrightarrow{\ \ f\ \ } & K_1(SL_n) \end{array} \tag{4.10}$$

Consider the universal element $\alpha \in K_1(G;\mathfrak{D})$ defined as in [Su, Section 4]. It is constructed in such a way that its image $f(\alpha)$ in $K_1(SL_n)$ is the universal matrix element. Then, due to [Su, Theorem 2.7], $\bar{c}_1 f(\alpha) = 1$. Hence, the map ω is an epimorphism and so is ψ. □

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