Beltrami-Courant Differentials and G_∞-algebras

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Abstract. Using the symmetry properties of two-dimensional sigma models, we introduce a notion of the Beltrami-Courant differential, so that there is a natural homotopy Gerstenhaber algebra related to it. We conjecture that the generalized Maurer-Cartan equation for the corresponding L_∞ subalgebra gives solutions to the Einstein equations.

1. Introduction

The geometric and algebraic properties of two-dimensional sigma-models lead to a lot of important discoveries in mathematics. One of the most interesting topics, emerged this way in the last decade is the study of gerbes of chiral differential operators, which give the proper mathematical description of the simplest first-order sigma-models. In [17], it was shown that the classical actions of the standard second-order sigma-models can be reformulated under certain conditions (one of which is the introduction of complex structure) in terms of perturbed first-order ones. In the same article, it was also suggested that the conformal invariance conditions for the perturbed sigma model, which have the form of the Einstein (and higher order) equations, will have a homotopical meaning as generalized Maurer-Cartan equations for certain L_∞ algebra. In this paper, we show that there is a larger structure, namely of homotopy Gerstenhaber algebra, so that the desired L_∞ structure is a part of it.

The central object in the construction is the vertex algebroid with a Calabi-Yau structure and its classical limit, the Courant algeboid. In [37] we associated to every positively graded vertex operator algebra (VOA) the homotopy Gerstenhaber algebra, which, according to the work of [15], [13], [14], [31] can be extended to G_∞ algebra [30] and even to BV_∞ algebra [8], [9]. The relationship between vertex algebroid and vertex algebra is similar to the relationship between Lie algebra and its universal enveloping algebra. We show here that the correspondence constructed in [37] can be reformulated by consructing a functor from the category of vector algebroids to the category of G_∞-algebras. Another important observation of the article [37] is that one can construct a quasiclassical limit of the resulting G_∞ algebra, so that the operations become covariant, i.e. can be expressed via the operations of Courant algebroid only. This G_∞ algebra is much easier to grasp: its C_∞ and L_∞ subalgebras reduce to C_3 and L_3 algebras, where the

L_3-algebra is the extension of the L_3 algebra of Roytenberg and Weinstein [28]. An example of the above construction we need in this paper is the $G_\infty-$ algebra for the vertex algebroid on the space of holomorphic sections of $T^{(1,0)}M \oplus T^{*(1,0)}M$ and its antiholomorphic counterpart, so that the corresponding vertex algebra gives (locally) a description of the unperturbed first-order sigma-model. The appropriately completed tensor product of corresponding "holomorphic" and "antiholomorphic" homotopy Gerstenhaber algebras gives the homotopy Gertsenhaber algebra and we conjecture that this homotopy Gerstenhaber algebra can be extended to G_∞ algebra. The Maurer-Cartan elements for the resulting L_∞-subalgebra are parametrized by the perturbation terms of the first-order sigma model, i.e. by the sections of $\Gamma((T^{(1,0)}M \oplus T^{*(1,0)}M) \otimes (T^{(0,1)}M \oplus T^{*(0,1)}M)) \oplus \mathcal{C}(M)$. We call the sections from the first summand as $Beltrami-Courant\ differentials$, justifying that name by its symmetry transformations of the first-order sigma-model, which are very similar to the ones of Beltrami differentials on Riemann surfaces and by the fact that the infinitesimal formula is expressed algebraically via the operations on Courant algebroid. The sections of the second term in the summand will be called $normalized\ dilaton\ fields$.

It is possible to show that there is a subcomplex in the complex on which the homotopy Gerstenhaber algebra is defined, so that all higher operations vanish leaving the structure of Gerstenhaber algebra on this subcomplex (in fact, it is a BV-algebra [11]). We show that the Maurer-Cartan equation of the corresponding differential graded Lie algebra is equivalent to Einstein Equations with dilaton and B-field, if the bivector field from $\Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$, which parametrizes the Maurer-Cartan element, gives rise to the Hermitian metric. This allows to conjecture, using the relation between first and second-order sigma models that the generalized Maurer-Cartan equations (GMC) for L_∞-algebra on the full complex give Einstein equations with B-field and dilaton, parametrized by the Beltrami-Courant differential. We justify the conjecture by showing that the symmetries of GMC reproduce the infinitesimal diffeomorphisms and gauge transformations of a B-field up to the second order in the Beltrami-Courant differential.

The structure of the paper is as follows. In Section 2 we study the classic action functionals for first- and second-order sigma models and the relationship between their symmetries. This leads to the definition of the Beltrami-Courant differential and its symmetries, e.g. under diffeomorphism transformations. In Section 3, we discuss Vertex/Courant algebroids with the Calabi-Yau structure, related G_∞-algebras and their classical limits. In Section 4 we describe the relation of these algebras to Einstein equations with B-field and dilaton, parametrized by Beltrami differential. First we describe the simplest case, when the G_∞ algebra is reduced to the Gerstenhaber algebra, then we formulate the conjecture regarding more general

Einstein equations and support it by calculation of the symmetry transformations.

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2. Sigma-models and Beltrami-Courant Differentials

In this section, we introduce the first object of interest: Beltrami-Courant differential. We derive its definition from the symmetries of the classical sigma-model actions. Let Σ be some compact Riemann surface, M is a complex manifold of dimension d and $X : \Sigma \to M$ is some differentiable map. Let us consider the following action:

$$S_0 = \frac{1}{2\pi i h}\int_\Sigma (\langle p \wedge \bar\partial X\rangle - \langle \bar p \wedge \partial X\rangle), \tag{1}$$

where p and $\bar p$ belong to $\Omega^{(1,0)}(M) \otimes \Omega^{(1,0)}(\Sigma)$ and $\Omega^{(0,1)}M \otimes \Omega^{(0,1)}(\Sigma)$ correspondingly, and $\langle \cdot, \cdot \rangle$ stands for standard pairing. This action has the following symmetries (we write them in components in the infinitesimal form):

$$\begin{aligned} &X^i \to X^i - v^i(X), \quad p_i \to p_i + \partial_i v^k p_k,\\ &X^{\bar i} \to X^{\bar i} - v^{\bar i}(\bar X), \quad p_{\bar i} \to p_{\bar i} + \partial_{\bar i} v^{\bar k} p_{\bar k}. \end{aligned} \tag{2}$$

Here, the generators of the infinitesimal transformations $v, \bar v$ are the elements of $\Gamma(\mathcal{O}(T^{(1,0)}M))$ and $\Gamma(\mathcal{O}(T^{(0,1)}M))$ correspondingly, i.e. v^i ($v^{\bar i}$) are (anti)holomorphic. These symmetries illustrate invariance under the holomorphic coordinate transformations. There is another set of symmetries, induced by the (anti)holomorphic 1-forms. Let $\omega \in \mathcal{O}(T^{*(1,0)}M)$ and $\bar\omega \in \bar{\mathcal{O}}(T^{*(0,1)}M)$. Then the action (1) is invariant under the transformatiosn of $p, \bar p$:

$$p_i \to p_i - \partial X^k(\partial_k\omega_i - \partial_i\omega_k), \quad p_{\bar i} \to p_{\bar i} - \bar\partial X^{\bar k}(\partial_{\bar k}\omega_{\bar i} - \partial_{\bar i}\omega_{\bar k}). \tag{3}$$

We want to generalize the action (1) so that it would be invariant under the diffeomorphism transformations and nonholomorphic generalizations of (3). In order to do that, one has to introduce extra (perturbation) terms to the action (1). Let us see how it works with an example. Suppose $v^i, v^{\bar i}$ in the formulas (2) are not holomorphic anymore, then S_0 won't be invariant and there will be an extra contribution to S_0:

$$\delta S_0 = -\frac{1}{2\pi i h}\int (\langle \bar\partial v, p \wedge \bar\partial X\rangle + \langle \partial \bar v, \bar p \wedge \partial X\rangle). \tag{4}$$

Therefore, to compensate this term, it makes sense to add extra terms to the action of the form

$$\delta S_\mu = -\frac{1}{2\pi i h}\int (\langle \mu, p \wedge \bar\partial X\rangle + \langle \bar\mu, \partial X \wedge \bar p\rangle), \tag{5}$$

where $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}(M))$, $\bar{\mu} \in \Gamma(T^{(0,1)}M \otimes T^{*(1,0)}(M))$, so that upon the $(v, \bar{v})$ transformations $\mu, \bar{\mu}$ should be modified as follows:

$$\mu \to \mu - \bar{\partial} v + \dots, \quad \bar{\mu} \to \bar{\mu} - \partial \bar{v} + \dots, \tag{6}$$

where dots stand for terms higher in μ and $\bar{\mu}$. Continuing and further applying this approach to the non(anti)holomorphic generalizations of the transformations (2), (3) we find that we have to add the following terms to S_0, such that the resulting action is:

$$\begin{aligned}\tilde{S} = \frac{1}{2\pi i h}\int_\Sigma (\langle p \wedge \bar{\partial} X\rangle - \langle \bar{p} \wedge \partial X\rangle -\\ \langle \mu, p \wedge \bar{\partial} X\rangle - \langle \bar{\mu}, \partial X \wedge \bar{p}\rangle - \langle b, \partial X \wedge \bar{\partial} X\rangle),\end{aligned} \tag{7}$$

where $b \in \Gamma(T^{*(1,0)}M \otimes T^{*(0,1)}M)$. The resulting symmetry transformations generated by $(v, \bar{v})$ can be written as follows:

$$\begin{aligned}&\mu^i_{\bar{j}} \to \mu^i_{\bar{j}} - \partial_{\bar{j}} v^i + v^k \partial_k \mu^i_{\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} \mu^i_{\bar{j}} + \mu^i_{\bar{k}} \partial_{\bar{j}} v^{\bar{k}} - \mu^k_{\bar{j}} \partial_k v^i + \mu^i_{\bar{l}} \mu^k_{\bar{j}} \partial_k v^{\bar{l}},\\ &b_{i\bar{j}} \to b_{i\bar{j}} + v^k \partial_k b_{i\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} b_{i\bar{j}} + b_{i\bar{k}} \partial_{\bar{j}} v^{\bar{k}} + b_{l\bar{j}} \partial_i v^l + b_{i\bar{k}} \mu^k_{\bar{j}} \partial_k v^{\bar{k}} + b_{l\bar{j}} \bar{\mu}^{\bar{k}}_i \partial_{\bar{k}} v^l,\end{aligned} \tag{8}$$

and the formula for the transformation of $\bar{\mu}$ can be obtained from the one of μ by formal complex conjugation. This leads to the symmetry of the action $\tilde{S}$ if

$$\begin{aligned}&X^i \to X^i - v^i(X, \bar{X}), \quad p_i \to p_i + p_k \partial_i v^k - p_k \mu^k_{\bar{l}} \partial_i v^{\bar{l}} - b_{j\bar{k}} \partial_i v^{\bar{k}} \partial X^j,\\ &X^{\bar{i}} \to X^{\bar{i}} - v^{\bar{i}}(X, \bar{X}), \quad \bar{p}_{\bar{i}} \to \bar{p}_{\bar{i}} + \bar{p}_{\bar{k}} \partial_{\bar{i}} v^{\bar{k}} - \bar{p}_{\bar{k}} \bar{\mu}^{\bar{k}}_l \partial_{\bar{i}} v^l - b_{\bar{j}k} \partial_{\bar{i}} v^k \bar{\partial} X^{\bar{j}}.\end{aligned} \tag{9}$$

Therefore, the resulting action is invariant under the action of the infinitesimal diffeomorphism group. The component formulas (8) were first discovered in [10]. Similarly, we obtain that the transformations

$$\begin{aligned}&b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}} \omega_i - \partial_i \omega_{\bar{j}} + \mu^i_{\bar{j}}(\partial_i \omega_k - \partial_k \omega_i) +\\ &\bar{\mu}^{\bar{s}}_i(\partial_{\bar{j}} \omega_{\bar{s}} - \partial_{\bar{s}} \omega_{\bar{j}}) + \bar{\mu}^{\bar{i}}_j \mu^s_{\bar{k}}(\partial_s \omega_{\bar{i}} - \partial_{\bar{i}} \omega_s)\end{aligned} \tag{10}$$

accompanied with

$$\begin{aligned}&p_i \to p_i - \partial X^k(\partial_k \omega_i - \partial_i \omega_k) - \partial_{\bar{r}} \omega_i \partial X^{\bar{r}} - \bar{\mu}^{\bar{s}}_k \partial_i \omega_{\bar{s}} \partial X^k,\\ &p_{\bar{i}} \to p_{\bar{i}} - \bar{\partial} X^{\bar{k}}(\partial_{\bar{k}} \omega_{\bar{i}} - \partial_{\bar{i}} \omega_{\bar{k}}) - \partial_r \omega_{\bar{i}} \bar{\partial} X^r - \mu^s_{\bar{k}} \partial_i \omega_s \bar{\partial} X^{\bar{k}}.\end{aligned} \tag{11}$$

leave $\tilde{S}$ invariant. Hereinafter, it is useful to consider $\mu, \bar{\mu}, b$ as matrix elements of $\tilde{\mathbb{M}} \in \Gamma((T^{(1,0)}M \oplus T^{*(1,0)}M) \otimes (T^{(0,1)}M \oplus T^{*(0,1)}M)$, i.e.

$$\tilde{\mathbb{M}} = \begin{pmatrix} 0 & \mu \\ \bar{\mu} & b \end{pmatrix}. \tag{12}$$

For simplicity of notation let us define $E = TM \oplus T^*M$, also $\mathcal{E} = T^{(1,0)}M \oplus T^{*(1,0)}M$ and $\bar{\mathcal{E}} = T^{(0,1)}M \oplus T^{*(0,1)}M$, so that $E = \mathcal{E} \oplus \bar{\mathcal{E}}$.

Let $\alpha \in \Gamma(E)$, i.e. $\alpha = (v, \bar{v}, \omega, \bar{\omega})$, where $v, \bar{v}$ are the elements of $\Gamma(T^{(1,0)}M)$ and $\Gamma(T^{(0,1)}M)$ correspondingly and $\omega \in \Omega^{(1,0)}(M)$, $\bar{\omega} \in \Omega^{(0,1)}(M)$. Next, we introduce operator the operator $D : \Gamma(E) \to \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, such that

$$D\alpha = \begin{pmatrix} 0 & \bar\partial v \\ \partial \bar v & \partial\bar\omega - \bar\partial\omega \end{pmatrix}.$$

Then the transformation of $\tilde{\mathbb{M}}$ under (8), (10) can be expressed by the following formula:

$$\tilde{\mathbb{M}} \to \tilde{\mathbb{M}} - D\alpha + \phi_1(\alpha, \tilde{\mathbb{M}}) + \phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}). \tag{13}$$

The second operation $\phi_1(\alpha, \tilde{\mathbb{M}})$ can be described as follows. Let us consider

$$\begin{aligned} &\xi \in J^\infty(\mathcal{O}_M) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})) \oplus J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}_M), \\ &\mathbb{L} \in J^\infty(\mathcal{O}(\mathcal{E})) \otimes J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}})), \end{aligned} \tag{14}$$

where $J^\infty(E)$, for any bundle E over M stands for the corresponding ∞-jet bundle of E. In other words, let

$$\begin{aligned} \xi &= \sum_J f^J \otimes \bar b^J + \sum_K b^K \otimes \bar f^K, \\ \mathbb{L} &= \sum_I a^I \otimes \bar a^I, \end{aligned} \tag{15}$$

where $a^I, b^J \in J^\infty(\mathcal{O}(\mathcal{E}))$, $f^I \in J^\infty(\mathcal{O}_M)$ and $\bar a^I, \bar b^J \in J^\infty(\bar{\mathcal{O}}(\bar{\mathcal{E}}))$, $\bar f^I \in J^\infty(\bar{\mathcal{O}}_M)$. Then we can introduce the operation $\phi_1(\xi, \mathbb{L})$ as follows:

$$\phi_1(\xi, \mathbb{L}) = \sum_{I,J} [b^J, a^I]_D \otimes \bar f^J \bar a^I + \sum_{I,K} f^K a^I \otimes [\bar b^K, \bar a^I]_D, \tag{16}$$

where $[\cdot,\cdot]_D$ is a Dorfman bracket, see e.g. [26] or the next section. Completing the tensor products in (14), we find that the operation ϕ_1 can be induced on $\alpha \in \Gamma(E)$ and $\tilde{\mathbb{M}} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$. One can explicitly check that (16) leads to the part of (8) and (10), linear in α and $\tilde{\mathbb{M}}$. The last part, bilinear in $\tilde{\mathbb{M}}$, also has an algebraic meaning of a similar kind: returning back to the notation (15), we find that on the jet counterparts of $\alpha, \tilde{\mathbb{M}}$, i.e. on $\xi, \mathbb{L}$ the expression for ϕ_2 is:

$$\begin{aligned} &\phi_2(\xi, \mathbb{L}, \mathbb{L}) = \\ &\frac{1}{2} \sum_{I,J,K} \langle b^I, a^K \rangle a^J \otimes \bar a^J(\bar f^I) \bar a^K + \frac{1}{2} \sum_{I,J,K} a^J(f^I) a^K \otimes \langle \bar b^I, \bar a^K \rangle \bar a^J, \end{aligned} \tag{17}$$

where $\bar a^J(\bar f^I)$, $a^J(f^I)$ correspond to the action of the differential operator, associated to the vector field, on a function ($\bar a^J(\bar f^I)$, $a^J(f^I)$ are set to be zero if $\bar a^J$, a^J are 1-forms). At the same time, the operation ϕ_2 has the following simple description:

$$\phi_2(\alpha, \tilde{\mathbb{M}}, \tilde{\mathbb{M}}) = \tilde{\mathbb{M}} \cdot D\alpha \cdot \tilde{\mathbb{M}} \tag{18}$$

if we consider $\tilde{\mathbb{M}}$ as an element of $End(\Gamma(E))$.

Let us notice that we could generalize $\tilde{\mathbb{M}}$ in the following way: in the matrix expression for $\tilde{\mathbb{M}}$ let us fill in the empty spot, i.e. let us add extra element $g \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$. Then the modified $\tilde{\mathbb{M}}$, i.e. $\mathbb{M} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$ can be expressed as follows:

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}. \tag{19}$$

The corresponding action functional is:

$$\begin{aligned} S_{fo} = \frac{1}{2\pi i h} \int_\Sigma (\langle p \wedge \bar{\partial} X \rangle + \langle \bar{p} \wedge \partial X \rangle - \\ -\langle g, p \wedge \bar{p} \rangle - \langle \mu, p \wedge \bar{\partial} X \rangle - \langle \bar{\mu}, \bar{p} \wedge \partial X \rangle - \langle b, \partial X \wedge \bar{\partial} X \rangle). \end{aligned} \tag{20}$$

It turns out that the symmetries of this action functional can be described by the same formula (13), where algebraic meaning of the operations on the jet level is given by the same formulas (16), (17), and the formula (18) is also valid. In Appendix, one can find the explicit component formulas for the infinitesimal symmetries of the action S_{fo}. The reason for introducing the g-term in the action functional is as follows. If the matrix $\{g^{i\bar{j}}\}$ is invertible, then using elementary variational calculus, one can find that the critical points for S_{fo} are the same as for the second-order action functional:

$$\begin{aligned} &S_{so} = \\ &\frac{1}{2\pi h} \int d^2 z (g_{i\bar{j}} (\bar{\partial} X^i - \mu^i_{\bar{k}} \bar{\partial} X^{\bar{k}})(\partial X^{\bar{j}} - \bar{\mu}^{\bar{j}}_k \partial X^k) - b_{i\bar{j}} \partial X^i \bar{\partial} X^{\bar{j}}), \end{aligned} \tag{21}$$

which can be re-expressed as

$$S_{so} = \frac{1}{4\pi h} \int d^2 z (G_{\mu\nu} + B_{\mu\nu}) \partial X^\mu \bar{\partial} X^\nu, \tag{22}$$

where G is a symmetric tensor and B is antisymmetric, indices μ, ν run through the set $\{i, \bar{j}\}$. The expression for G and B via $\mathbb{M}$ is given by:

$$\begin{aligned} G_{s\bar{k}} &= g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} + g_{s\bar{k}} - b_{s\bar{k}}, \quad B_{s\bar{k}} = g_{\bar{i}j} \bar{\mu}^{\bar{i}}_s \mu^j_{\bar{k}} - g_{s\bar{k}} - b_{s\bar{k}}, \\ G_{si} &= -g_{i\bar{j}} \bar{\mu}^{\bar{j}}_s - g_{s\bar{j}} \bar{\mu}^{\bar{j}}_i, \quad G_{\bar{s}\bar{i}} = -g_{\bar{s}j} \mu^j_{\bar{i}} - g_{\bar{i}j} \mu^j_{\bar{s}}, \\ B_{si} &= g_{s\bar{j}} \bar{\mu}^{\bar{j}}_i - g_{i\bar{j}} \bar{\mu}^{\bar{j}}_s, \quad B_{\bar{s}\bar{i}} = g_{\bar{i}j} \mu^j_{\bar{s}} - g_{\bar{s}j} \mu^j_{\bar{i}}, \end{aligned} \tag{23}$$

where $\{g_{i\bar{j}}\}$ stands for the inverse matrix of $\{g^{i\bar{j}}\}$. Such parametrization of the second-order action in the case when M is a Riemann surface was first introduced in [25], [38]. The symmetries of the action functional S_{fo} transform into infinitesimal diffeomorphism transformations and the 2-form B symmetry

$$\begin{aligned} &G \to G - L_{\mathbf{v}} G, \quad B \to B - L_{\mathbf{v}} B, \\ &B \to B - 2d\boldsymbol{\omega}, \end{aligned} \tag{24}$$

if $\alpha = (\mathbf{v}, \boldsymbol{\omega})$, so that $\mathbf{v} \in \Gamma(TM)$, $\boldsymbol{\omega} \in \Omega^1(M)$, i.e. the symmetries of S_{so}.

Let us formulate this as a theorem.

Theorem 1.1. *Let $\mathbb{M} \in \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$, parametrized as in (19), so that its $\Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$ part is given by $\{g^{i\bar{j}}\}$, which is invertible, then the* *infinitesimal diffeomorphism transformations of the resulting symmetric and antisymmetric tensors G and B (see (23)), as well as the B-tensor shift by exact 2-forms are encoded in the formula*

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}), \tag{25}$$

where $\alpha \in \Gamma(E)$ and operations ϕ_1, ϕ_2 are defined above.

Note that if $\{G_{\mu\nu}\}$ is invertible and real, it gives rise to the metric tensor. Therefore, since $\mathbb{M}$ parametrizes both G and B, and transforms according to (25) under diffeomorphisms, it is analogous to Beltrami differential on the Riemann surface. So, from now on we will call the elements of $\Gamma(\mathcal{E} \otimes \bar{\mathcal{E}})$ as Beltrami-Courant differentials, since, as we see in the following sections, they are described by means of the Courant algebroid [16] structure on $\mathcal{E}, \bar{\mathcal{E}}$.

3. Vertex algebroids, G_∞-algebra and quasiclassical limit

In this section, we describe the constructions of the article [37] with some modifications and refer the reader to this article for some of the details.

Each of the terms in the classical action S_0 from which we started the previous section, leads to the quantum theory which is well described locally on open neighborhoods of M by means of vertex algebra generated by operator products

$$X^i(z)p_j(w) \sim \frac{h\delta^i_j}{z-w}, \quad X^{\bar{i}}(\bar{z})p_{\bar{j}}(\bar{w}) \sim \frac{h\delta^{\bar{i}}_{\bar{j}}}{\bar{z}-\bar{w}} \tag{26}$$

and globally by means of gerbes of chiral differential operators on M [22], [20]. Each of the corresponding vertex algebras, which provide the local description, form a $\mathbb{Z}_+$-graded vector space $V = \sum_{n=0}^{+\infty} V_n$, so that it is determined (see [20]) by means of a vertex algebroid. In our case, the vertex algebroid is described by means of the sheaf $\mathcal{V} = \mathcal{O}(\mathcal{E}) \otimes \mathbb{C}[h] \equiv \mathcal{O}(\mathcal{E})^h$ (resp. $\bar{\mathcal{O}}(\bar{\mathcal{E}})^h$), of vector spaces V_1, as well as the sheaf of V_0 spaces, which coincides with the structure sheaf $\mathcal{O}_M \otimes \mathbb{C}[h] = \mathcal{O}_M^h$ (resp. $\bar{\mathcal{O}}_M^h$), with certain algebraic operations between them.

Let us define a vertex alebroid (see e.g. [20], [2]) and then study our concrete case in detail.

A *vertex $\mathcal{O}_M$-algebroid* is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with a pairing $\mathcal{O}_M \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{V}$, i.e. $f \otimes v \mapsto f * v$ such that $1 * v = v$, equipped with a structure of a Leibniz $\mathbb{C}[h]$-algebra $[\ ,\ ] : \mathcal{V} \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{V}$, a $\mathbb{C}[h]$-linear map of Leibniz algebras $\pi : \mathcal{V} \to \Gamma(TM)$, which usually is referred to as an anchor, a symmetric $\mathbb{C}[h]$-bilinear pairing $\langle\ ,\ \rangle : \mathcal{V} \otimes_{\mathbb{C}[h]} \mathcal{V} \to \mathcal{O}_M^h$ a $\mathbb{C}$-linear map

$\partial : \mathcal{O}_M \to \mathcal{V}$ such that $\pi \circ \partial = 0$, which satisfy the relations

$$
\begin{aligned}
& f*(g*v)-(fg)*v = \pi(v)(f)*\partial(g)+\pi(v)(g)*\partial(f),\\
& [v_1, f*v_2] = \pi(v_1)(f)*v_2 + f*[v_1,v_2],\\
& [v_1,v_2]+[v_2,v_1] = \partial(\langle v_1,v_2\rangle), \quad \pi(f*v)=f\pi(v),\\
& \langle f*v_1, v_2\rangle = f\langle v_1,v_2\rangle - \pi(v_1)(\pi(v_2)(f)),\\
& \pi(v)(\langle v_1,v_2\rangle) = \langle [v,v_1],v_2\rangle + \langle v_1,[v,v_2]\rangle,\\
& \partial(fg) = f*\partial(g)+g*\partial(f),\\
& [v,\partial(f)] = \partial(\pi(v)(f)), \quad \langle v,\partial(f)\rangle = \pi(v)(f),
\end{aligned}
\tag{27}
$$

where $v, v_1, v_2 \in \mathcal{V}$, $f, g \in \mathcal{O}_M^h$.

The correspondence between vertex algebroid on each neighborhood U is similar to the correspondence between Lie algebra and its universal enveloping algebra: for more details see [20].

Let us concentrate on the case when $\mathcal{V} = \mathcal{O}(\mathcal{E})^h$. Explicitly, if $f \in \mathcal{O}_M$, $v, v_1, v_2 \in \mathcal{O}(T^{(1,0)}M)$, $\omega, \omega_1, \omega_2 \in \mathcal{O}(T^{*(1,0)}M)$, then locally in the neighborhood with the coordinates $\{X^i\}$

$$
\begin{aligned}
& \partial f = df, \quad \pi(v)f = -hv(f), \quad \pi(\omega)=0,\\
& f*v = fv + hdX^i\partial_i\partial_j f v^j, \quad f*\omega = f\omega,\\
& [v_1,v_2] = -h[v_1,v_2]_D - h^2 dX^i\partial_i\partial_k v_1^s\partial_s v_2^k,\\
& [v,\omega] = -h[v,\omega]_D, \quad [\omega,v] = -h[\omega,v]_D, \quad [\omega_1,\omega_2]=0,\\
& \langle v,\omega\rangle = -h\langle v,\omega\rangle^s, \quad \langle v_1,v_2\rangle = -h^2\partial_i v_1^j\partial_j v_2^i, \quad \langle\omega_1,\omega_2\rangle = 0,
\end{aligned}
\tag{28}
$$

where $\langle\cdot,\cdot\rangle^s$ is a standard pairing on $\mathcal{E}$ and $[\cdot,\cdot]_D$ is the Dorfman bracket:

$$
\begin{aligned}
& [v_1,v_2]_D = [v_1,v_2]^{Lie}, \quad [v,\omega]_D = L_v\omega,\\
& [\omega,v]_D = -i_v d\omega, \quad [\omega_1,\omega_2]_D = 0.
\end{aligned}
\tag{29}
$$

In [37], it was shown that given a holomorphic volume form on the open neighborhood U of M, one can associate a homotopy Gerstenhaber algebra to the vertex algebroid on U (although the main emphasis of [37] was on C_∞ part of it). This was done by considering semi-infinite complex associated to the vertex algebra: due to the results of[15], [14], [13], [31], there is a structure of G_∞ algebra attached to it if the central charge of the corresponding Virasoro algebra is 26. Using this fact and considering the subcomplex corresponding to the elements of total conformal weight zero, we find out that the central charge condition can be dropped. The resulting complex $(\mathcal{F}^{\cdot}, Q)$

appears to be much shorter that original semi-infinite one:

(30)
$$\begin{tikzcd}
 & \mathcal{V} \arrow[ddr, "-\frac{1}{2}h\mathrm{div}"] & \mathcal{V} \arrow[ddr, "\frac{1}{2}h\mathrm{div}"] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{O}_M^h \arrow[uur, "\partial"] & \mathcal{O}_M^h \arrow[uur, "\partial"'] \arrow[r, "id"'] & \mathcal{O}_M^h & \mathcal{O}_M^h .
\end{tikzcd}$$

Here $\mathcal{F}^0 \cong \mathcal{O}_M^h \cong \mathcal{F}_h^3$, $\mathcal{F}_h^1 \cong \mathcal{O}_M^h \oplus \mathcal{V} \cong \mathcal{F}_h^3$, and div stands for divergence operator with respect to the nonvanishing volume form applied to sections of $\Gamma(U, T^{(1,0)}(M))$. Appropriate analogue of operator div in the case of general vertex algebroid is called $Calabi - Yau\ structure$ on vertex algebroid [20] (since e.g. in our case to be defined globally M should possess a non-vanishing holomorphic volume form). According to [37], this complex has a bilinear operation, which satisfies the Leibniz identity with respect to Q, it is also homotopy commutative and associative, and can be described by the following table:

(31) $$(a_1, a_2)_h =$$

$a_2 \backslash a_1$	u_1	A_1	v_1	$\tilde{A}_1$	$\tilde{v}_1$	$\tilde{u}_1$
u_2	u_1u_2	A_1u_2 $+\pi(A_1)u_2$	v_1u_2	$\tilde{A}_1u_2$	$\tilde{v}_1u_2$	$\tilde{u}_1u_2$
A_2	u_1A_2	$-[A_1, A_2]+$ $\frac{1}{2}\langle A_1, A_2\rangle$	$-v_1A_2+$ $\pi(\tilde{v}_1)A_2$	$\frac{1}{2}\langle \tilde{A}_1, A_2\rangle$	$-\pi(A_2)\tilde{v}_1$	0
v_2	$u_1\tilde{u}_2$	A_1v_2	0	$-\pi(\tilde{A}_1)v_2$	$-\tilde{v}_1v_2$	0
$\tilde{A}_2$	$u_1\tilde{A}_2$	$\frac{1}{2}\langle A_1, \tilde{A}_2\rangle$	$-\pi(\tilde{A}_2)v_1$	0	0	0
$\tilde{v}_2$	$u_1\tilde{u}_2$	$\pi(A_1)\tilde{v}_2$	$-v_1\tilde{v}_2$	0	0	0
$\tilde{u}_2$	$u_1\tilde{u}_2$	0	0	0	0	0

,

where $u_i \in \mathcal{F}_h^0$, $(v_i, A_i) \in \mathcal{F}_h^1$, $(\tilde{v}_i, \tilde{A}_i) \in \mathcal{F}_h^2$, $\tilde{u}_i \in \mathcal{F}_h^3$.

We note that there is an operator $\mathbf{b}$ of degree -1 on $(\mathcal{F}_h^{\cdot}, Q)$ which anti-commutes with Q:

(32)
$$\begin{tikzcd}
 & \mathcal{V} & \mathcal{V} \arrow[l, "-id"'] & \\
 & \bigoplus & \bigoplus & \\
\mathcal{O}_M^h & \mathcal{O}_M^h \arrow[l, "id"'] & \mathcal{O}_M^h & \mathcal{O}_M^h \arrow[l, "-id"']
\end{tikzcd}$$

This operator gives rise to the bracket operation

(33) $$(-1)^{|a_1|}\{a_1, a_2\}_h = \mathbf{b}(a_1, a_2)_h - (\mathbf{b}a_1, a_2)_h - (-1)^{|a_1|}(a_1\mathbf{b}a_2)_h,$$

which satisfies quadratic relations together with $(\cdot,\cdot)_h$ and Q, which follows from the properties of the vertex algebra [15]. On the cohomology of Q these relations turn into defining properties of Gerstenhaber algebra. Namely, the following Proposition holds.

Proposition 3.1.[37] *Symmetrized versions of operations (31) together with (33) satisfy the relations of the homotopy Gerstenhaber algebra, which follows from these relations:*

$$
\begin{aligned}
(34)\quad & Q(a_1,a_2)_h = \\
& (Qa_1,a_2)_h+(-1)^{|a_1|}(a_1,Qa_2)_h,\\
& (a_1,a_2)_h-(-1)^{|a_1||a_2|}(a_2,a_1)_h=\\
& Qm(a_1,a_2)+m(Qa_1,a_2)+(-1)^{|a_1|}m(a_1,Qa_2),\\
& Q(a_1,a_2,a_3)_h+(Qa_1,a_2,a_3)_h+(-1)^{|a_1|}(a_1,Qa_2,a_3)_h+\\
& (-1)^{|a_1|+|a_2|}(a_1,a_2,Qa_3)_h=((a_1,a_2)_h,a_3)_h-(a_1,(a_2,a_3)_h)_h\\
& \{a_1,a_2\}+(-1)^{(|a_1|-1)(|a_2|-1)}\{a_2,a_1\}=\\
& (-1)^{|a_1|-1}(Qm'_h(a_1,a_2)-m'_h(Qa_1,a_2)-(-1)^{|a_2|}m'_h(a_1,Qa_2)),\\
& \{a_1,(a_2,a_3)_h\}_h=(\{a_1,a_2\}_h,a_3)_h+(-1)^{(|a_1|-1)||a_2|}(a_2,\{a_1,a_3\}_h)_h,\\
& \{(a_1,a_2)_h,a_3\}_h-(a_1,\{a_2,a_3\}_h)_h-(-1)^{(|a_3|-1)|a_2|}(\{a_1,a_3\}_h,a_2)_h=\\
& (-1)^{|a_1|+|a_2|-1}(Qn'_h(a_1,a_2,a_3)-n'_h(Qa_1,a_2,a_3)-\\
& (-1)^{|a_1|}n'_h(a_1,Qa_2,a_3)-(-1)^{|a_1|+|a_2|}n'_h(a_1,a_2,Qa_3),\\
& \{\{a_1,a_2\}_h,a_3\}_h-\{a_1,\{a_2,a_3\}_h\}_h+\\
& (-1)^{(|a_1|-1)(|a_2|-1)}\{a_2,\{a_1,a_3\}_h\}_h=0,
\end{aligned}
$$

where m_h, m'_h are some bilinear operations of degrees -1, -2 correspondingly and n_h, n'_h are trilinear operations of degree -1, -2 correspondingly. There exist higher homotopies which turn this homotopy Gerstenhaber algebra into G_∞ algebra.

The last part of the Proposition follows from the results of [13], [31], [14] where it was show that the symmetrized versions of $(\cdot,\cdot)_h$, $\{,\}_h$ can be continued to the G_∞ algebra [30].

One of the central observations of [37] was that this G_∞ algebra has *quasiclassical* limit, which can be constructed as follows. Let $\mathcal{V}|_{h=0}=\mathcal{V}^0$ (in our example $\mathcal{V}^0=\mathcal{O}(\mathcal{E})$), then consider the subcomplex of $(\mathcal{F}^{\cdot}_h, Q)$, i.e. $(\mathcal{F}^{\cdot}, Q)\cong(\mathcal{F}^{\cdot}_1, Q)$, which is:

(35)
$$
\begin{tikzcd}
 & \mathcal{V}^0 \arrow[ddrrr, "-\frac{1}{2}h\mathrm{div}"] & & h\mathcal{V}^0 \arrow[ddrr, "\frac{1}{2}h\mathrm{div}"] & & \\
 & \bigoplus & & \bigoplus & & \\
\mathcal{O}_M \arrow[uur, "\partial"] & & h\mathcal{O}_M \arrow[uur, "\partial"] \arrow[rr, "\mathrm{id}"'] & & h\mathcal{O}_M & h^2\mathcal{O}_M
\end{tikzcd}
$$

It appears that

$$\begin{aligned} \mathcal{F}^i \otimes \mathcal{F}^j &\to \mathcal{F}^{i+j}[h], \quad \{\mathcal{F}_i, \mathcal{F}_j\} \to h\mathcal{F}_{i+j-1}[h], \\ \mathbf{b} : \mathcal{F}^i &\to h\mathcal{F}^{i-1}[h], \end{aligned} \tag{36}$$

so that

$$(\cdot,\cdot)_0 = \lim_{h\to 0}(\cdot,\cdot)_h, \quad \{\cdot,\cdot\}_0 = \lim_{h\to 0} h^{-1}\{\cdot,\cdot\}_h, \quad \mathbf{b}_0 = \lim_{h\to 0} h^{-1}\mathbf{b} \tag{37}$$

are well defined. The corresponding homotopy Gerstenhaber algebra is much less complicated: the corresponding L_∞ and C_∞ parts are only L_3 and C_3-algebras. Let us have a look in detail. On the level of the vertex algebroid of $\mathcal{O}(\mathcal{E})^h$, let us denote

$$\begin{aligned} &\lim_{h\to 0} h^{-1}[v_1, v_2] = [v_1, v_2]_0, \quad \lim_{h\to 0} h^{-1}\pi = \pi_0, \\ &\lim_{h\to 0} h^{-1}\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_0. \end{aligned} \tag{38}$$

Therefore, we can express the bilinear operations $(\cdot,\cdot)_0$ and $\{\cdot,\cdot\}_0$ on the complex

$$\begin{tikzcd} & \mathcal{O}(\mathcal{E}) \arrow[dr, "-\frac{1}{2}\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & \mathcal{O}(\mathcal{E}) \arrow[dr, "\frac{1}{2}\mathrm{div}"] \arrow[d, phantom, "\bigoplus"] & \\ \mathcal{O}_M \arrow[ur, "d"] & \mathcal{O}_M \arrow[ur, "d"] \arrow[r, "id"'] & \mathcal{O}_M & \mathcal{O}_M \end{tikzcd} \tag{39}$$

via the following tables:

$(a_1, a_2)_0=$

$a_2 \backslash a_1$	u_1	A_1	v_1	$\tilde{A}_1$	$\tilde{v}_1$	$\tilde{u}_1$
u_2	u_1u_2	A_1u_2 $+\pi_0(A_1)u_2$	v_1u_2	$\tilde{A}_1u_2$	$\tilde{v}_1u_2$	$\tilde{u}_1u_2$
A_2	u_1A_2	$-[A_1, A_2]_0-$ $\frac{1}{2}\langle A_1, A_2\rangle_0$	$-v_1A_2$	$\frac{1}{2}\langle\tilde{A}_1, A_2\rangle_0$	$-\pi_0(A_2)(\tilde{v}_1)$	0
v_2	$u_1\tilde{u}_2$	A_1v_2	0	0	$-\tilde{v}_1v_2$	0
$\tilde{A}_2$	$u_1\tilde{A}_2$	$\frac{1}{2}\langle A_1, \tilde{A}_2\rangle_0$	0	0	0	0
$\tilde{v}_2$	$u_1\tilde{u}_2$	$-\pi_0(A_1)\tilde{v}_2$	$-v_1\tilde{v}_2$	0	0	0
$\tilde{u}_2$	$u_1\tilde{u}_2$	0	0	0	0	0

$\{a_1, a_2\}_0=$

$a_2 \backslash a_1$	u_1	A_1	v_1	$\tilde{A}_1$	$\tilde{v}_1$	$\tilde{u}_1$
u_2	0	$-\pi_0(A_1)u_2$	0	$\pi_0(\tilde{A}_1)u_2$	0	0
A_2	0	$-[A_1, A_2]_0$	0	$-[\tilde{A}_1, A_2]_0$ $-\frac{1}{2}\langle\tilde{A}_1, A_2\rangle_0$	$-\pi_0(A_2)\tilde{v}_1$	$\pi_0(A_2)\tilde{u}_1$
v_2	0	$-\pi_0(A_1)v_2$	0	0	0	0
$\tilde{A}_2$	0	$-[A_1, \tilde{A}_2]_0$	0	$\langle\tilde{A}_1, \tilde{A}_2\rangle_0$	$-\pi_0(\tilde{A}_2)\tilde{v}_1$	0
$\tilde{v}_2$	0	$-\pi_0(A_1)\tilde{v}_2$	0	$-\pi_0(\tilde{A}_1)\tilde{v}_2$	0	0
$\tilde{u}_2$	$-\pi_0(A_1)\tilde{u}_2$	0	0	0	0	0

,

where $u_i \in \mathcal{F}_h^0$, $(v_i, A_i) \in \mathcal{F}_h^1$, $(\tilde{v}_i, \tilde{A}_i) \in \mathcal{F}_h^2$, $\tilde{u}_i \in \mathcal{F}_h^3$.

Let us summarize the results about the quasiclassical limit via Proposition.

Proposition 3.2.[37] *The operations $(\cdot,\cdot)_0$, $\{\cdot,\cdot\}_0$ satisfy the relations (34) so that their symmetrized versions satisfy the relations of G_∞ algebra which is the quasiclassical limit of G_∞ algebra considered in Proposition 3.1. The resulting C_∞ and L_∞ algebras are reduced to C_3 and L_3 algebras.*

The classical limits for the corresponding homotopies $m_h = m_0 + O(h)$ and $n_h = n_0 + O(h)$ are as follows. The commutativity homotopy m_0 is nonzero iff its both arguments belong to $\mathcal{F}_1$:

$$m_0 = -\langle A_1, A_2\rangle_0. \tag{40}$$

The associativity homotopy n_0 is nonzero only when all three elements belong to $\mathcal{F}_1$ or one of the first two belongs to $\mathcal{F}_2$ and the other belong to $\mathcal{F}_1$:

$$\begin{aligned} &n_0(A_1, A_2, A_3) = A_2\langle A_1, A_3\rangle_0 - A_1\langle A_2, A_3\rangle_0, \\ &n_0(A_1, \tilde{v}, A_2) = n_0(\tilde{v}, A_1, A_2) = -\tilde{v}\langle A_1, A_2\rangle_0. \end{aligned} \tag{41}$$

Notice, that in the quasiclassical limit we get rid of all noncovariant terms in the expression for the product and the bracket. This is very close to the classical limit procedure for vertex algebroid. Namely, using (38), one can obtain vertex algebroid from Courant algebroid.

The definition of Courant algebroid is as follows (see e.g. [16], [2]). A Courant $\mathcal{O}_M$-algebroid is an $\mathcal{O}_M$-module $\mathcal{Q}$ equipped with the structure of a Leibniz $\mathbb{C}$-algebra $[,]_0 : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \to \mathcal{Q}$, an $\mathcal{O}_M$-linear map of Leibniz algebras (the anchor map) $\pi_0 : \mathcal{Q} \to \Gamma(TM)$, a symmetric $\mathcal{O}_M$-bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{Q} \otimes_{\mathcal{O}_M} \mathcal{Q} \to \mathcal{O}_M$, a derivation $\partial : \mathcal{O}_M \to \mathcal{Q}$, which satisfy

$$\begin{aligned} &\pi \circ \partial = 0, \quad [q_1, fq_2]_0 = f[q_1, q_2] + \pi_0(q_1)(f)q_2, \\ &\langle[q, q_1], q_2\rangle + \langle q_1, [q, q_2]\rangle = \pi_0(q)(\langle q_1, q_2\rangle_0), \quad [q, \partial(f)]_0 = \partial(\pi_0(q)(f)), \\ &\langle q, \partial(f)\rangle = \pi_0(q)(f) \quad [q_1, q_2]_0 + [q_2, q_1]_0 = \partial(\langle q_1, q_2\rangle_0), \end{aligned} \tag{42}$$

where $f \in \mathcal{O}_M$ and $q, q_1, q_2 \in \mathcal{Q}$. In our case $\mathcal{Q} \cong \mathcal{O}(\mathcal{E})$, π_0 is just a projection on $\mathcal{O}(TM)$

$$[q_1, q_2]_0 = -[q_1, q_2]_D, \quad \langle q_1, q_2\rangle_0 = -\langle q_1, q_2\rangle^s, \quad \partial = d. \tag{43}$$

As we indicated earlier, both C_∞ and L_∞ parts of G_∞ algebra appear to be short. We expect this to happen with all the homotopies, i.e. it is natural to suggest the following.

Conjecture 3.1. *The G_∞ algebra of Proposition 3.2. has only bilinear and trilinear operations, i.e. it is a G_3 algebra.*

In the following, since we are interested only in the quasiclassical algebra on the complex $(\mathcal{F}^{\cdot}, Q)$, we will neglect the 0 subscript for all multilinear operations of this algebra.

4. Homotopy Gerstenhaber algebra and Einstein equations

4.1. BV-subalgebra and a nontrivial example of Einstein equations. The homotopy Gerstenhaber algebra we studied in the previous section, has a subalgebra based on the following complex $(\mathcal{F}^{\cdot}_{sm}, \mathcal{Q})$.

$$\begin{tikzcd}
 & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "-\frac{1}{2}\mathrm{div}"] & \mathcal{O}(T^{(1,0)}M) \arrow[ddr, "\frac{1}{2}\mathrm{div}"] & \\
 & \bigoplus & \overset{0}{\bigoplus} & \\
\mathbb{C} \arrow[uur, "0"] & \mathbb{C} \arrow[uur] \arrow[r, "i"'] & \mathcal{O}_M & \mathcal{O}_M
\end{tikzcd} \tag{44}$$

It is just a Gerstenhaber algebra (with no higher homotopies), moreover it is a BV algebra, since $\mathbf{b}$ operator also preserves $(\mathcal{F}^{\cdot}_{sm}, Q)$. Therefore, we have the following Proposition.

Proposition 4.1. *Bilinear operations $(\cdot,\cdot)$, $\{\cdot,\cdot\}$ together with operator* $\mathbf{b}$ *generate the structure of BV algebra on $(\mathcal{F}^{\cdot}_{sm}, Q)$.*

Let us consider the ∞-jet version of the complex $(\mathcal{F}^{\cdot}_{sm}, Q)$: we substitute $\mathcal{O}_M$, $\mathcal{O}(T^{(0,1)}(M))$ by $J^\infty(\mathcal{O}_M)$, $J^\infty(\mathcal{O}(T^{(0,1)}(M))$. We denote the resulting complex as $(\mathcal{F}^{\cdot}_{sm,\infty}, Q)$. Then the completed tensor product

$$\mathbf{F}^{\cdot}_{sm,\infty} = \mathcal{F}^{\cdot}_{sm,\infty} \hat{\otimes} \bar{\mathcal{F}}^{\cdot}_{sm,\infty} \tag{45}$$

where $(\bar{\mathcal{F}}^{\cdot}_{sm,\infty}, \bar{Q})$ is the complex obtained from $(\mathcal{F}^{\cdot}_{sm,\infty}, Q)$ by complex conjugation. Complex $(\mathbf{F}^{\cdot}_{sm,\infty}, \mathcal{Q})$, where $\mathcal{Q} = Q + \bar{Q}$, is the jet version of the complex $(\mathbf{F}^{\cdot}_{sm}, \mathcal{Q})$, such that e.g. $\mathbf{F}^2_{sm} = \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M) \oplus \bar{\mathcal{O}}(T^{(0,1)}M) \oplus \mathcal{O}(T^{(1,0)}M) \oplus \mathcal{O}_M \oplus \bar{\mathcal{O}}_M \oplus \mathbb{C}$. Clearly, the complex $(\mathbf{F}^{\cdot}_{sm}, \mathcal{Q})$

carries a structure of BV algebra inherited from $(\mathcal{F}^{\cdot}_{sm,\infty}, Q)$ and its complex conjugation, so that

$$(-1)^{|a_1|}\{a_1,a_2\} = \mathbf{b}^-(a_1,a_2) - (\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^- a_2), \tag{46}$$

where $\mathbf{b}^- = \mathbf{b} - \bar{\mathbf{b}}$. Note, that the elements closed under $\mathbf{b}^-$ form a subalgebra in the differential graded algebra (DGLA), generated by $Q, \{\cdot,\cdot\}$. It turns out that the Maurer-Cartan equations of this DGLA and their symmetries have a very interesting meaning. To describe them, let us define some extra algebraic operations for convenience.

Let $g, h \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$ so that their components are $g^{i\bar{j}}\partial_i \otimes \partial_{\bar{j}}, h^{i\bar{j}}\partial_i \otimes \partial_{\bar{j}}$. Then one can define symmetric bilinear operation [17], [34]:

$$[[,]] : \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M) \otimes \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M) \to \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M) \tag{47}$$

written in components as follows:

$$[[g,h]]^{k\bar{l}} \equiv (g^{i\bar{j}}\partial_i\partial_{\bar{j}}h^{k\bar{l}} + h^{i\bar{j}}\partial_i\partial_{\bar{j}}g^{k\bar{l}} - \partial_i g^{k\bar{j}}\partial_{\bar{j}}h^{i\bar{l}} - \partial_i h^{k\bar{j}}\partial_{\bar{j}}g^{i\bar{l}}) \tag{48}$$

and looks much less complicated in the jet notation (see section 2). Namely, if $\tilde{\xi}\tilde{\eta} \in J^\infty(\mathcal{O}(T^{(1,0)}M) \otimes J^\infty(\mathcal{O}(T^{(0,1)}M)$, so that $\xi = \sum_I v^I \otimes \bar{v}^I$, $\eta = \sum_J w^J \otimes \bar{w}^J$, where $v^I, w^J \in J^\infty(\mathcal{O}(T^{(1,0)}M)$, $\bar{v}^I, \bar{w}^J \in J^\infty(\mathcal{O}(T^{(0,1)}M)$, then

$$[[\xi,\eta]] = \sum_{I,J}[v^I, w^J] \otimes [\bar{v}^I, \bar{w}^J]. \tag{49}$$

As noted in [17],[34], if bilinear tensor g is such that one can associate a Kähler metrics to it, then the Ricci tensor $R^{i\bar{j}}$ associated with such metric tensor is proportional to $[[g,g]]$, more precisely

$$R^{i\bar{j}}(g) = \frac{1}{2}[[g,g]]^{i\bar{j}}. \tag{50}$$

If the complex manifold M has a volume form Ω, such that in local coordinates $\Omega = e^f dX^1 \cdots \wedge dX^n \wedge dX^{\bar{1}} \wedge \dots dX^{\bar{n}}$. Let us denote the volume form which determines the differential $\mathcal{Q}$ as Ω', so that $f = -2\Phi'_0$, then Φ'_0 has to be locally a sum of holomorphic and antiholomorphic functions, i.e. it satisfies equation $\partial_i\partial_{\bar{j}}\Phi'_0 = 0$.

We will refer to the vector field $div_\Omega g$ such that $(div_\Omega g)^{\bar{j}} = \partial_i g^{i\bar{j}} + \partial_i f g^{i\bar{j}}$, $(div_\Omega g)^i = \partial_{\bar{j}} g^{i\bar{j}} + \partial_{\bar{j}} f g^{i\bar{j}}$ as the divergence of bivector field g with respect to the volume form Ω.

Now let the Maurer-Cartan element, closed under $\mathbf{b}^-$, namely the element of $\Gamma(T^{(1,0)}(M) \otimes T^{(0,1)}(M)) \oplus \mathcal{O}(T^{(0,1)}(M) \oplus \mathcal{O}(T^{(1,0)}(M) \oplus \mathcal{O}_M \oplus \bar{\mathcal{O}}_M)$ be defined by its components in the direct sum, i.e. as $(g, \bar{v}, v, \phi, \bar{\phi})$.

Then the following Theorem holds, which can be proven by direct calculation.

Theorem 4.1a. *The Maurer-Cartan equation for the differential graded Lie algebra on* $\mathbf{F}^{\cdot}_{sm}|_{\mathbf{b}^-=0}$ *generated by* $\mathcal{Q}$ *and* $\{\cdot,\cdot\}$ *imposes the following system of equations on* $g, \phi, \bar{\phi}$ *(*$\bar{v}, v$ *turn out to be auxialliary variables):*

1). *Vector field* $div_\Omega g$*, where* $\Omega = \Omega' e^{-2\phi+2\bar{\phi}}$ *is determined by* $f \equiv -2\Phi_0 = -2(\Phi'_0 + \phi - \bar{\phi})$ *and* $\partial_i \partial_{\bar{j}} \Phi_0 = 0$*, is such that its* $\Gamma(T^{(1,0)}M)$*,* $\Gamma(T^{(0,1)}M)$ *components are correspondingly holomorphic and antiholomorphic.*

2). *Bivector field* $g \in \Gamma(T'M \otimes T''M)$ *obeys the following equation:*

$$[[g,g]] + \mathcal{L}_{div_\Omega(g)} g = 0, \tag{51}$$

where $\mathcal{L}_{div_\Omega(g)}$ *is a Lie derivative with respect to the corresponding vector fields.*

3). $div_\Omega div_\Omega(g) = 0$.

The infinitesimal symmetries of the Maurer-Cartan equation coincide with the holomorphic coordinate transformations of the volume form and tensor $\{g^{i\bar{j}}\}$*.*

The constraints 1), 2), 3) coincide with the equations studied in [34], where it was shown that they are equivalent to Einstein equations, i.e. the following statement is valid.

Theorem 4.1b. *If tensor* $\{g^{i\bar{j}}\}$ *parametrises Hermitean metric, then the conditions 1), 2), 3) on* g *and* Φ_0 *from Theorem 4.1a are equivalent to Einstein equations*

$$\begin{aligned} R^{\mu\nu} &= \frac{1}{4} H^{\mu\lambda\rho} H^{\nu}_{\lambda\rho} - 2\nabla^\mu \nabla^\nu \Phi, \\ &\nabla_\mu H^{\mu\nu\rho} - 2(\nabla_\lambda \Phi) H^{\lambda\nu\rho} = 0, \\ &4(\nabla_\mu \Phi)^2 - 4\nabla_\mu \nabla^\mu \Phi + R + \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} = 0, \end{aligned} \tag{52}$$

where $H = dB$ *is a 3-form, so that metric* G*, 2-form* B *and the dilaton field* $\Phi \in \mathcal{C}(M)$ *are expressed as follows:*

$$\begin{aligned} &G_{i\bar{k}} = g_{i\bar{k}}, \quad B_{i\bar{k}} = -g_{i\bar{k}}, \quad \Phi = \log \sqrt{g} + \Phi_0, \\ &G_{ik} = G_{\bar{i}\bar{k}} = G_{ik} = G_{\bar{i}\bar{k}} = 0, \end{aligned} \tag{53}$$

where by g under the square root we denote the determinant of $\{g_{i\bar{j}}\}$. In other words, (52) are equivalent to the following system:

$$\partial_i\partial_{\bar{k}}\Phi_0 = 0, \quad \partial_{\bar{p}} d_{\bar{l}}^{\Phi_0} g^{\bar{l}k} = 0, \quad \partial_p d_l^{\Phi_0} g^{\bar{k}l} = 0,$$
$$2g^{r\bar{l}}\partial_r\partial_{\bar{l}} g^{i\bar{k}} - 2\partial_r g^{i\bar{p}}\partial_{\bar{p}} g^{r\bar{k}} - g^{i\bar{l}}\partial_{\bar{l}} d_s^{\Phi_0} g^{s\bar{k}} - g^{r\bar{k}}\partial_r d_{\bar{j}}^{\Phi_0} g^{\bar{j}i} +$$
$$\partial_r g^{i\bar{k}} d_{\bar{j}}^{\Phi_0} g^{\bar{j}r} + \partial_{\bar{p}} g^{\bar{k}i} d_n^{\Phi_0} g^{n\bar{p}} = 0,$$
$$d_i^{\Phi_0} d_{\bar{j}}^{\Phi_0} g^{\bar{j}i} = 0, \tag{54}$$

where $d_i^{\Phi_0} g^{\bar{j}i} \equiv (\partial_i - 2\partial_i\Phi_0) g^{i\bar{j}}$, $d_{\bar{j}}^{\Phi_0} g^{\bar{j}i} \equiv (\partial_{\bar{j}} - 2\partial_{\bar{j}}\Phi_0) g^{i\bar{j}}$, which is the component form of the conditions 1), 2) and 3).

4.2. Physical motivation for the main conjectures. In this subsection, we give the physics motivation for the generalization of the result of Section 4.1: namely, we want to extend Theorem 4.1. to the case of full complex $\mathcal{F}^{\cdot}$. For more details we refer the reader to the paper [17], [34]. In Section 2, we considered the equivalence of two action functionals S_{fo} and S_{so}. On the quantum level the object of primary interest is the path integral

$$\int [dp][d\bar{p}][dX][d\bar{X}]e^{-S}. \tag{55}$$

In the case of $S = S_0$ the quantum theory corresponding to this path integral, is described by the gerbes of chiral differential operators (and locally just by vertex algebras), as it was already mentioned in Section 3. However, this action should be modified to accomodate the holomorphic volume form on M, otherwise the Virasoro element in the corresponding vertex algebras wouldn't be globally defined. On the level of action functionals, one has to add an extra term to S_0, namely

$$S_0 \to S_0 + \int_\Sigma \sqrt{\gamma} R^{(2)}(\gamma)\phi(X), \tag{56}$$

where $e^{-2\phi}$ is the density for the volume form on M, so that $\partial_i\partial_{\bar{j}}\phi = 0$, γ is a metric on Σ and $R^{(2)}(\gamma)$ is its curvature. Let us add a similar term to its perturbed version S_{fo}, i.e. $\int_\Sigma \sqrt{\gamma} R^{(2)}(\gamma)\Phi_0(X)$ with no restrictions on Φ_0, and the resulting action will be denoted as $S_{fo}^{\Phi_0}$. We will call Φ_0 a *normalized dilaton field*.

The integration over $p, \bar{p}$ leads to the following (see [17]):

$$\int [dp][d\bar{p}][dX][d\bar{X}]e^{-S_{fo}^{\Phi_0}} = \int [dX][d\bar{X}]e^{-S_{so}+\int R^{(2)}(\gamma)(\Phi_0(X)+\sqrt{g})}. \tag{57}$$

This heuristic formula gives the proper correspondence between the first- and second-order actions on the quantum level. The relation of those to Einstein equations is as follows. Analysing the path integral in the right hand side involves regularization procedure which leads to the broken scale (and conformal) invariance. In order to make the model conformally invariant, one has to impose a sequence of constraints, appearing for the vanishing of the β-function (see [3], [4], [24]). The β-function is the function depending on G-metric, 2-form B, dilaton

$$\Phi = \Phi_0 + \sqrt{g} \tag{58}$$

and parameter h. At the zeroth order in h vanishing of the β-function leads to Einstein equations (52). Vanishing of the coefficients of higher powers in h lead to the equations involving higher number of derivatives and higher powers in Ricci curvature. It was noted that the linearized form of Einstein equations and their symmetries can be obtained as the closedness condition for the elements of degree 2 in the semi-infinite (BRST) complex associated to the Virasoro module corresponding to the conformal field theory described by S_{so} with the flat metric. One of the statements of String Field Theory is that the full conformal invariance conditions can be obtained from Maurer-Cartan equations for some L_∞-algebra on BRST complex [39], so that the full metric, B-field and dilaton can be restored from the Maurer-Cartan element. The symmetries of the maurer-cartan equations correspond to the h-corrected diffeomorphism symmetries and the exact shifts of the antisymmetri tensor B.

The complex corresponding to the flat metric does not have any simple algebraic structure on it (because it is not a vertex algebra) and it is complicated to construct such algebraic operations explicitly. On the contrary, for the first-order model we start from the vertex algebra and we have related G_∞ structure due to [15], [14], [13], [31]. Using the results of [37], we are able to reduce it to much smaller complex and find the quasiclassical limit. We claim that the Maurer-Cartan equation corresponding to its L_∞-subalgebra of the quasiclassical limit of this G_∞ algebra reproduces Einstein equations and their symmetries, where the metric, 2-form B and the dilaton Φ are expressed by means of (23), (58). In subsection 4.1., we obtained this correspondence in the case when only one of the perturbing terms was present in S_{fo}, namely $\langle g, p \wedge \bar{p} \rangle$. In the next subsection, we extend the statement of Theorem 4.1 to the case of general S_{fo}.

4.3. Main Conjectures. Following the ideas of Section 4.1, we want to repeat the construction in the case of the complex $(\mathcal{F}^{\cdot}, \mathcal{Q})$. Namely, we consider its jet version $(\mathcal{F}^{\cdot}_\infty, \mathcal{Q})$ and its complex conjugate $(\bar{\mathcal{F}}^{\cdot}_\infty, \mathcal{Q})$, so that

$$\mathbf{F}^{\cdot}_\infty = \mathcal{F}^{\cdot}_\infty \hat{\otimes} \bar{\mathcal{F}}^{\cdot}_\infty. \tag{59}$$

It is the jet version of the complex $(\mathbf{F}^{\cdot}, \mathcal{Q})$, such that e.g. the subspace of degree 1 is as follows: $\mathbf{F}^1 \cong \Gamma(E) \oplus \mathcal{C}(M) \oplus \mathcal{C}(M)$. As in the section 4.1, the divergence operator which determines $\mathcal{Q}$-operator, is based on the volume form, given in the local coordinates as $e^{-2\Phi_0'(X)} dX^1 \dots dX^n \wedge dX^{\bar{1}} \dots dX^{\bar{n}}$, so that $\partial_i \partial_{\bar{j}} \Phi_0' = 0$.

We can give $\mathbf{F}^{\cdot}$ the structure of the homotopy Leibniz bracket by means of formula which is the same as in Section 4.1:

$$(-1)^{|a_1|}\{a_1, a_2\} = \mathbf{b}^-(a_1, a_2) - (\mathbf{b}^- a_1, a_2) - (-1)^{|a_1|}(a_1 \mathbf{b}^- a_2), \tag{60}$$

however now we have higher homotopies. We also note that $\mathbf{F}^{\cdot}|_{\mathbf{b}^-=0} \equiv \mathbf{F}^{\cdot}_-$ is invariant under $\{\cdot,\cdot\}$. Let us formulate the first part of the main conjecture.

Conjecture 4.1a. *The structure of homotopy Gerstenhaber algebra on $\mathbf{F}^{\cdot}$ can be extended to G_∞-algebra, so that the subcomplex $\mathbf{F}^{\cdot}_-$ is invariant under L_∞ operations.*

Let us focus on the subcomplex $(\mathbf{F}^{\cdot}_-, \mathcal{Q})$. The space of Maurer-Cartan elements, i.e. the subspace of the elements of degree 2 is:

$$\mathbf{F}^2_- \cong \Gamma(\mathcal{E} \otimes \bar{\mathcal{E}}) \oplus \Gamma(E) \oplus \mathcal{C}(M) \oplus \mathcal{C}(M). \tag{61}$$

The elements of this space are defined by means of the components from the direct sum above, i.e. $\Psi = (\mathbb{M}, \eta, \phi, \bar{\phi})$. We will denote the difference $\phi - \bar{\phi} \equiv \Phi''_0$ and $\Phi_0 \equiv \Phi'_0 + \Phi''_0$. Let us formulate the second part of the main conjecture.

Conjecture 4.1b. *Let $\Psi = (\mathbb{M}, \eta, \phi, \bar{\phi})$ be the solution of the generalized Maurer-Cartan (GMC) equation for L_∞-algebra on $\mathbf{F}^{\cdot}_-$, so that*

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}, \tag{62}$$

Then the η-component is auxiliary and is expressed via $\mathbb{M}$ and $\phi, \bar{\phi}$. If $\{g^{i\bar{j}}\}$ is invertible, then G, B obtained from $\mathbb{M}$ via (23) together with $\Phi = \Phi_0 + \sqrt{g}$, where g is the determinant of $\{g_{i\bar{j}}\}$, satisfy the Einstein equations (52).

The space of infinitesimal symmetry generators of GMC equation, i.e. $\mathbf{F}^1$ is given by

$$\mathbf{F}^1_- \cong \Gamma(E) \oplus \mathcal{C}(M), \tag{63}$$

so that any element can be written in components as $\Lambda = (\xi, f)$.

The third part of the conjecture concerns the question how ξ, f are related to $\alpha \in \Gamma(E)$ in the transformation formula

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}) \tag{64}$$

from Section 1. First, to justify the statement of Conjecture 4.1b., we prove the following Proposition.

Proposition 4.2. *Let $\Lambda = (\xi, f) \in \mathbf{F}^1_-$ be the generator of the infinitesimal transformation of the solution of GMC equation. Then after the substitution $\xi = \alpha + \frac{1}{2}\mathbb{M} \cdot \alpha$ (where $\mathbb{M}$ is considered as an element of $End(\Gamma(E))$) the transformation of $\mathbb{M}$-component of the solution coincides with (64) up to the* *second order in* $\mathbb{M}$.

Proof. At first, we show that the expression

$$\Psi \to \Psi + \mathcal{Q}\Lambda' - \{\Lambda', \Psi\} + \frac{1}{2}\{\Lambda', \Psi, \Psi\} \tag{65}$$

gives the formula (64), where $\{\cdot,\cdot,\cdot\}$ is the homotopy for the Leibniz identity for $[\cdot,\cdot]$, $\Lambda' = (\alpha, s)$, $\Psi = (\mathbb{M}, \eta, \phi, \bar\phi)$. It is easy to check that for the 0th and 1st order in $\mathbb{M}$. To prove Proposition 4.2, we just need to check the term corresponding to trilinear operation. Let us return to the jet level, i.e. we assume that

$$\begin{aligned} \alpha &= \sum_J f^J \otimes \bar b^J + \sum_K b^K \otimes \bar f^K, \\ \mathbb{M} &= \sum_I a^I \otimes \bar a^I. \end{aligned} \tag{66}$$

Then the only terms contributing to the relevant part of $\{\Lambda', \Psi, \Psi\}$ are as follows:

$$\begin{aligned} &- \sum_{I,J,K} (m_0'(b^I, a^K), a^J) \otimes (\{\bar a^J, \bar f^I\}, \bar a^K) - \\ &\sum_{I,J,K} (\{a^J, f^I\}, a^K) \otimes (m_0'(\bar b^I, \bar a^K), \bar a^J). \end{aligned} \tag{67}$$

We see that modulo the necessary coefficient this coincides with the trilinear operation ϕ_2 (17). The statement of the Proposition 4.2 can be obtained from the antisymmetrization of $\{\cdot,\cdot\}$, and therefore of $\{\cdot,\cdot,\cdot\}$, so that the formula

$$\Psi \to \Psi + \mathcal{Q}\Lambda + \{\Psi, \Lambda\}^{asymm} + \frac{1}{2}\{\Psi, \Psi, \Lambda\}^{asymm} + \dots \tag{68}$$

corresponds to (64) if $\Lambda = (\alpha + \frac{1}{2}\mathbb{M}\cdot\alpha, s)$. ∎

We note, that the symmetry generated by f-part of $\mathbf{F}^1$ element does not affect metric B-field or dilaton. It is easy to check that on the level of 0th order in $\mathbb{M}$: the symmetry transformation corresponds to the shift of ϕ and $\bar\phi$ by f. One can check, similar to Proposition 4.2, that this symmetry remains redundant for the first and second order. We claim that these statements are exact, namely the following Conjecture is true.

Conjecture 4.1c. *Let* $\Lambda = (\xi, f) \in \mathbb{F}^1_-$, *be the generator of the infinitesimal symmetries of GMC equation (68). The corresponding transformation of* $\mathbb{M}$-*component of the solution of GMC coincide with (64) if* $\xi = \alpha + \frac{1}{2}\mathbb{M}\cdot\alpha$. *Under conditions of Conjecture 4.1b these transformations reproduce infinitesimal diffeomorphism transformations and shifts of B-field by exact 2-form, which are the symmetries of equations (52).*

5. Appendix

Here we give explicitly the formulas for the transformations of $\mathbb{M}$ (see Section 2):

$$\mathbb{M} \to \mathbb{M} - D\alpha + \phi_1(\alpha, \mathbb{M}) + \phi_2(\alpha, \mathbb{M}, \mathbb{M}) \tag{69}$$

of the matrix elements of Beltram-Courant differential

$$\mathbb{M} = \begin{pmatrix} g & \mu \\ \bar{\mu} & b \end{pmatrix}, \tag{70}$$

where $g \in \Gamma(T^{(1,0)}M \otimes T^{(0,1)}M)$, $\mu \in \Gamma(T^{(1,0)}M \otimes T^{*(0,1)}M)$, $\bar{\mu} \in \Gamma(T^{*(1,0)}M \otimes T^{(0,1)}M)$, $b \in \Gamma(T^{*(1,0)}M \otimes T^{*(0,1)}M)$. The explicit form of the transformations in components is (with the notations from Section 2):

$$\begin{aligned}
&g^{i\bar{j}} \to g^{i\bar{j}} + v^k \partial_k g^{i\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} g^{i\bar{j}} - g^{i\bar{k}} \partial_{\bar{k}} v^{\bar{j}} - g^{k\bar{j}} \partial_k v^i + \\
&g^{i\bar{k}} \bar{\mu}^{\bar{j}}_k \partial_{\bar{k}} v^k + g^{k\bar{j}} \mu^i_{\bar{k}} \partial_k v^{\bar{k}}, \\
&\mu^i_{\bar{j}} \to \mu^i_{\bar{j}} - \partial_{\bar{j}} v^i + v^k \partial_k \mu^i_{\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} \mu^i_{\bar{j}} + \mu^i_{\bar{k}} \partial_{\bar{j}} v^{\bar{k}} - \mu^k_{\bar{j}} \partial_k v^i + \\
&\mu^i_{\bar{l}} \mu^k_{\bar{j}} \partial_k v^{\bar{l}} + g^{i\bar{k}} b_{l\bar{j}} \partial_{\bar{k}} v^l, \\
&b_{i\bar{j}} \to b_{i\bar{j}} + v^k \partial_k b_{i\bar{j}} + v^{\bar{k}} \partial_{\bar{k}} b_{i\bar{j}} + b_{i\bar{k}} \partial_{\bar{j}} v^{\bar{k}} + b_{l\bar{j}} \partial_i v^l + \\
&b_{i\bar{k}} \mu^k_{\bar{j}} \partial_k v^{\bar{k}} + b_{l\bar{j}} \bar{\mu}^{\bar{k}}_i \partial_{\bar{k}} v^l,
\end{aligned} \tag{71}$$

$$\begin{aligned}
&g^{i\bar{j}} \to g^{i\bar{j}} + g^{i\bar{k}}(\partial_l \omega_{\bar{k}} - \partial_{\bar{k}} \omega_l) g^{l\bar{j}}, \\
&\mu^i_{\bar{l}} \to \mu^i_{\bar{l}} + g^{i\bar{j}}(\partial_{\bar{l}} \omega_{\bar{j}} - \partial_{\bar{j}} \omega_{\bar{l}}) + \mu^r_{\bar{l}} g^{i\bar{k}}(\partial_r \omega_{\bar{k}} - \partial_{\bar{k}} \omega_r), \\
&b_{i\bar{j}} \to b_{i\bar{j}} + \partial_{\bar{j}} \omega_i - \partial_i \omega_{\bar{j}} + \mu^i_{\bar{j}}(\partial_i \omega_k - \partial_k \omega_i) + \bar{\mu}^{\bar{s}}_i(\partial_{\bar{j}} \omega_{\bar{s}} - \partial_{\bar{s}} \omega_{\bar{j}}) + \\
&\bar{\mu}^{\bar{i}}_j \mu^s_{\bar{k}}(\partial_s \omega_{\bar{i}} - \partial_{\bar{i}} \omega_s).
\end{aligned} \tag{72}$$

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