Smooth approximation of plurisubharmonic functions on almost complex manifolds

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Abstract

This note establishes smooth approximation from above for J-plurisubharmonic functions on an almost complex manifold (X, J). The following theorem is proved. Suppose X is J-pseudoconvex, i.e., X admits a smooth strictly J-plurisubharmonic exhaustion function. Let u be an (upper semi-continuous) J-plurisubharmonic function on X. Then there exists a sequence $u_j \in C^\infty(X)$ of smooth strictly J-plurisubharmonic functions point-wise decreasing down to u.

In any almost complex manifold (X, J) each point has a fundamental neighborhood system of J-pseudoconvex domains, and so the theorem above establishes local smooth approximation on X.

This result was proved in complex dimension 2 by the third author, who also showed that the result would hold in general dimensions if a parallel result for continuous approximation were known. This paper establishes the required step by solving the obstacle problem.

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1. Introduction.

On any smooth almost complex manifold (X, J) there is a well-defined notion of J-plurisubharmonic functions of class C^2, namely those $u \in C^2(X)$ which satisfy the condition $i\partial\overline{\partial}u \geq 0$. This notion extends directly to the space of distributions $\mathcal{D}'(X)$ by requiring the current $i\partial\overline{\partial}u$ to be positive. It also extends to the space $\mathrm{USC}(X)$ of upper semi-continuous functions $u : X \to [-\infty, \infty)$ in several ways – using viscosity theory, or by requiring that the restrictions to J-holomorphic curves in X be subharmonic. These different extensions have been shown to be, in a precise sense, equivalent (see [16], [12]), and the space of such functions is denoted by $\mathrm{PSH}(X, J)$.

We say that a function $u \in C^2(X)$ is **strictly** J-plurisubharmonic if $i\partial\overline{\partial}u > 0$ at every point. The manifold X is then said to be ***J*-pseudoconvex** if it admits a smooth (proper) exhaustion function $\rho : X \to \mathbb{R}$ which is strictly J-plurisubharmonic. (See Remark 3.7 for other equivalent definitions.)

The main point of this paper is to establish the following (in §4).

THEOREM 4.1. (C^∞ Strict Approximation). *Suppose (X, J) is an almost complex manifold which is J-pseudoconvex, and let $u \in \mathrm{PSH}(X, J)$ be a J-plurisubharmonic function. Then there exists a decreasing sequence $\{u_j\} \subset C^\infty(X)$ of smooth strictly J-plurisubharmonic functions such that $u_j(x) \downarrow u(x)$ at each $x \in X$.*

Now on any almost complex manifold X every point x has a fundamental neighborhood system of J-pseudoconvex domains – namely, small balls about x in appropriate local coordinates. Consequently, as a special case of Theorem 4.1 we have local C^∞ strict approximation on X (see Corollary 4.2).

By this local regularization result a current $i\partial\bar{\partial}u \wedge i\partial\bar{\partial}v$ defined in [18] is a positive current for plurisubharmonic u, v in the Sobolev class $W^{1,2}_{loc}$, in particular for bounded plurisubharmonic u, v (see Proposition 4.2 and Proposition 5.2 there and compare with Corollary 2 in [19]). For an application of our global regularization result see Corollary 4.3, which concerns hulls of sets.

We note that in the case of plurisubharmonic functions on domains in $\mathbb{C}^n$, smoothing as in Theorem 4.1 is possible on all pseudoconvex, Reinhardt, and tube domains (see [7]), but there are smooth domains where not all plurisubharmonic functions are a limits of a decreasing sequence of smooth plurisubharmonic functions (see [6]).

Theorem 4.1 was proved in complex dimension 2 by the third author (in [19]), who pointed out that his work would establish the result in general dimensions provided one could prove a certain parallel *continuous* approximation theorem. The required continuous approximation result can be deduced from work of the first two authors on the obstacle problem – more precisely the Dirichlet problem with an obstacle function.

The discussion of this obstacle problem in [10] and [13] and its exact implementation in the context of almost complex analysis is somewhat scattered, and so, for clarity, we give a coherent exposition of the needed results in the first two sections of this note. Nevertheless, this note draws heavily on the work in [10], [12], [13], [18] and [19].

It is interesting to note that the work in [18] and [19] also involves solving the Dirichlet problem for the (almost) complex Monge-Ampère operator. In this case, however, the solutions are taken in the smooth category using results in [17], where the techniques are quite different from the viscosity methods employed in [10], [12], [13]. The idea of using the Monge-Ampère equation to approximate J-plurisubharmonic functions is probably due to J.-P. Rosay.

Remark. The main proof in this paper consists of combining a Richberg-type theorem (cf. [18, Thm. 3.1], [11, Thm. 9.10]) with the continuous approximation theorem which follows from solving the obstacle problem. The method applies generally to give smooth approximation of F-subharmonic functions whenever these two components can be established. An example is given in Appendix B where smooth approximation is established for subsolutions of the complex Hessian equations on a Kähler manifold.

2. The Obstacle Problem and Continuous Approximation for General Potential Theories.

We refer the reader to [10] or [13] for the concepts and terminology employed in this section.

Let $J^2(X) \to X$ be the bundle of 2-jets of real-valued functions on a manifold X. There is a natural splitting $J^2(X) = \mathbb{R} \times J^2_{\rm red}(X)$ where the first factor corresponds to the value of the function.

Consider a subequation of the form $F = \mathbb{R} \times F_0$ with $F_0 \subset J^2_{\rm red}(X)$. For a domain $\Omega \subset\subset X$, let $F(\overline{\Omega})$ denote the set of $u \in {\rm USC}(\overline{\Omega})$ such that $u\big|_\Omega$ is F-subharmonic (i.e., $u\big|_\Omega$ is a viscosity F-subsolution, cf. [2], [3]).

THEOREM 2.1. (The Obstacle Problem). *Suppose that:*

(1) F_0 is locally affinely jet-equivalent to a constant coefficient (reduced) subequation $\mathbf{F}_0$,

(2) F_0 has a monotonicity cone M_0 and X carries a C^2 strictly M-subharmonic function ψ where $M = \mathbb{R} \times M_0$,

(3) $g \in C(X)$, and

(4) $\Omega \subset\subset X$ is a domain with smooth boundary $\partial\Omega$ which is both F- and $\widetilde{F}$-strictly convex.

Then the function

$$h(x) \ \equiv \ \sup_{u \in \mathcal{F}[g]} u(x), \tag{2.1}$$

where $\mathcal{F}[g] \ \equiv \ \{u(x) : u \in F(\overline{\Omega}) \text{ and } u \leq g \text{ on } \overline{\Omega}\}$, *satisfies:*

(i) $h \in C(\overline{\Omega}) \cap F(\overline{\Omega})$,

(ii) $h \leq g$ *on* $\overline{\Omega}$

(iii) $h\big|_{\partial\Omega} \ = \ g\big|_{\partial\Omega}$

Furthermore,

(v) h is the Perron function, and $\mathcal{F}[g]$ is the Perron family, for the Dirichlet problem for the subequation

$$F^g \ \equiv (\mathbb{R}_- + g) \times F_0 \quad \text{on } \Omega$$

with boundary function $\varphi \equiv g\big|_{\partial\Omega}$.

(vi) Comparison holds for F^g on X.

COROLLARY 2.2. (Continuous Strict Approximation). *Suppose $u \in F(\overline{\Omega})$.*

(a) Then there exists a sequence of functions $u_j \in C(\overline{\Omega}) \cap F(\overline{\Omega})$ decreasing down to u on $\overline{\Omega}$. In fact, if $\{g_j\} \subset C(\overline{\Omega})$ is any sequence of continuous functions decreasing down to u, the $\{u_j\} \subset C(\overline{\Omega}) \cap F(\overline{\Omega})$ can be chosen so that

$$u \ \leq\ u_j \ \leq\ g_j \qquad \forall\, j. \tag{2.2}$$

(b) Moreover, given $\epsilon_j \downarrow 0$, the sequence $\{u_j + \epsilon_j \psi\}$ also decreases down to u on $\overline{\Omega}$, and on each compact subset of Ω, the functions $\{u_j + \epsilon_j \psi\}$ are c-strict for some $c > 0$.

See 2.3 below for a definition and discussion of c-strictness.

Proof of Corollary 2.2. Pick $g_j \in C(\overline{\Omega})$ with $g_j \downarrow u$. Let u_j denote the solution of the obstacle problem for g_j. Then $u_j \in C(\overline{\Omega}) \cap F(\overline{\Omega})$ and $u_j \leq g_j$. Since u is in the Perron family $\mathcal{F}[g_j]$, we have (2.2). This proves Part (a). Part (b) follows from (a) and hypothesis (2). ∎

Proof of Theorem 2.1. The following is proved in [10] but not stated explicitly as a theorem. It is however stated explicitly as Theorem 8.1.2 in [13] and the proof is given there based on results in [10]

THEOREM 8.1.2 in [13]. *Suppose F is a subequation on a manifold X which is locally affinely jet-equivalent to a constant coefficient subequation. Suppose there exists a C^2 strictly M-subharmonic function on X where M is a monotonicity cone for F. Then for every domain $\Omega \subset\subset X$ whose boundary is strictly F- and $\widetilde{F}$-convex, both existence and uniqueness hold for the Dirichlet problem. That is, for every $\varphi \in C(\partial\Omega)$ there exists a unique F-harmonic function $u \in C(\overline{\Omega})$ with $u\big|_{\partial\Omega} = \varphi$.*

The adaptation to the general Obstacle Problem is given in Section 8.6 of [13]. What follows is a more detailed version of that argument.

By assumption we know that $F = \mathbb{R} \times F_0$ is affinely jet equivalent to the constant coefficient equation $\mathbb{R} \times \mathbf{F}_0 \subset \mathbb{R} \times \mathbf{J}^2_{\text{red}}$, with a jet equivalence which is the identity on the first factor. Hence the subequation

$$F^g \ \equiv\ \{r \leq g(x)\} \times F_0$$

is locally affinely jet equivalent to the subequation

$$\mathbf{F}^g \ \equiv\ \{r \leq g(x)\} \times \mathbf{F}_0$$

We now consider the affine jet equivalence

$$\Phi : \mathbb{R} \times \mathbf{J}^2_{\text{red}} \ \longrightarrow\ \mathbb{R} \times \mathbf{J}^2_{\text{red}}$$

given by

$$\Phi(r, J) \ \equiv\ (r - g(x), J).$$

Applying this gives the local equivalence

$$\Phi : \mathbf{F}^g \longrightarrow \{r \le 0\} \times \mathbf{F}_0 \equiv \mathbb{R}_- \times \mathbf{F}_0,$$

and so composing this with the first equivalence shows that F^g is locally affinely jet-equivalent to the constant coefficient subequation $\mathbb{R}_- \times \mathbf{F}_0$.

Now observe that if M_0 is a monotonicity cone for F_0, then $M_- \equiv \mathbb{R}_- \times M_0$ is a monotonicity cone for F^g.

Note also that if ψ is strictly M-subharmonic function, then so is $\psi - c$ for any constant $c \le 0$ because M satisfies the basic negativity condition (N). Given a domain $\Omega \subset\subset X$, we may therefore assume that $\psi < 0$ on a neighborhood of $\overline{\Omega}$. In this case, ψ is also M_--strictly subharmonic on $\overline{\Omega}$.

Since F^g is locally jet-equivalent[1] to a constant coefficient subequation, local weak comparison holds for F^g. This is Theorem 10.1 in [10] and follows from the Theorem on Sums. Local weak comparison implies weak comparison (Theorem 8.3 in [10]). Now using Theorems 9.5 and 9.2 we have that comparison holds for F^g on X.

The Dirichlet Problem for F^g-harmonics would now be solvable for arbitrarily prescribed boundary data $\varphi \in C(\partial\Omega)$, (by either Theorem 12.4 in [10] or Theorem 8.1.2 above) if one could prove that the boundary is strictly F^g and $\widetilde{F^g}$ convex.

However, this is not true in general, and in fact existence fails for a boundary function $\varphi \in C(\partial\Omega)$ unless $\varphi \le g\big|_{\partial\Omega}$. Nevertheless, *if $\partial\Omega$ is both F and $\widetilde{F}$ strictly convex, then existence holds for each boundary function $\varphi \le g\big|_{\partial\Omega}$*. Section 8.6 in [13] provides a proof of this.

Here we give a proof but with attention restricted to the case at hand where $\varphi = g\big|_{\partial\Omega}$. The Perron family for F^g with this boundary data consists of those functions $u \in \mathrm{USC}(\overline{\Omega})$ which are F-subharmonic on Ω and satisfy the additional constraint that $u \le g$ on Ω. The dual subequation to F^g is $\widetilde{F^g} = [(\mathbb{R}_- - g) \times J^2_{\mathrm{red}}(X)] \cup \widetilde{F}$. Since $\widetilde{F^g} \subset \widetilde{F}$, the $\partial\Omega$ is strictly $\widetilde{F^g}$-convex if it is strictly $\widetilde{F}$-convex. However, $\partial\Omega$ can never be strictly F^g-convex, as defined in Definition 11.10 of [10], because $(\overrightarrow{F_\lambda})_x = \emptyset$ for $\lambda > g(x)$),

Nevertheless, the only place that this hypothesis is used in proving Theorem 8.1.2 for H is in the barrier construction which appears in the proof of Proposition F in [10]. With $\varphi(x_0) = g(x_0)$, the barrier $\beta(x)$ as defined in (12.1) in [10] is not only F-strict near x_0 but also automatically F^g-strict since $\beta < g$ in a neighborhood of x_0. ∎

[1]See Appendix A for a discussion of jet-equivalence.

Definition 2.3. (Strictness). Let $F \subset J^2(X)$ be a subequation. A function $u \in F(\Omega)$ is **strictly F-subharmonic** (or simply **strict**) if for any $\varphi \in C_0^\infty(\Omega)$, there exists $\epsilon > 0$ such that $u + \epsilon\varphi \in F(\Omega)$.

Note that a C^2-function $u \in F(\Omega)$ is strict iff $J_x^2 u \in \mathrm{Int} F\ \forall\, x \in \Omega$.

In [10] there is the following related concept of c-strictness for $c > 0$. Equip $J^2(X)$ with a bundle metric (induced, say, from a riemannian metric on X), and for $x \in X$, define $F_x^c \equiv \{J \in F_x : \mathrm{dist}_x(J, \sim F) \geq c\}$ where dist_x denotes the distance in the fibre. A function $u \in F(\Omega)$ is said to be **c-strict** on a compact set $K \subset \Omega$ if u is F^c-subharmonic on a neighborhood of K. The constant c depends on the choice of bundle metric, but the condition of being c-strict on K for some $c > 0$ does not. Strictness, as defined above, is equivalent to being locally c-strict on Ω. (This is proved, though not explicitly stated, in §7 of [10].)

Remark 2.4. The main conclusion of Theorem 2.1 above can be stated in more appealing and succinct terms. Let us call the function h, defined in (2.1), the **largest F-subharmonic minorant of** g. Then we have the following abbreviated version of Theorem 2.1 and Corollary 2.2.

THEOREM 2.5. *Suppose $X, F = \mathbb{R} \times F_0$ and Ω are as in Theorem 2.1. Then given $g \in C(\overline{\Omega})$, the largest F-subharmonic minorant of g on $\overline{\Omega}$ is continuous and equals g on the boundary of Ω.*

Moreover, given $u \in F(\overline{\Omega})$ there exists a sequence $\{u_j\} \subset C(\overline{\Omega}) \cap F(\overline{\Omega})$ decreasing down to u (with each u_j strict).

3. Strict Continuous Approximation of Plurisubharmonic Functions on Almost Complex Manifolds

Let (X, J) be an almost complex manifold, and let $F(J) \subset J^2_{\mathrm{red}}(X)$ be the subequation defining the upper semi-continuous J-plurisubharmonic functions on X. (It is shown in [12] that all the different basic definitions of these functions are, in a precise sense, equivalent).[2]

Proposition 4.5 in the paper [12] proves that the subequation $F(J)$ is locally jet equivalent to a constant coefficient reduced subequation (in fact to the standard subequation $F(J_0) \cong \{i\partial\overline{\partial}u \geq 0\}$ determined by a standard parallel J_0).

Furthermore, $F(J)$ is a convex cone subequation and in particular it satisfies $F(J) + F(J) \subset F(J)$. Therefore, $F(J)$ is a monotonicity cone for itself. A C^2-function ψ is strictly J-plurisubharmonic (i.e., strictly $F(J)$-subharmonic) if $i\partial\overline{\partial}\psi > 0$ on X.

[2] It is also shown at the end of section 7 in [12] that the various notions of $F(J)$-harmonic (including the notion of being maximal and continuous) are equivalent.

Definition 3.1. A domain $\Omega \subset\subset X$ is called **strictly J-pseudoconvex** if it has a global C^2 defining function ψ which is strictly J-plurisubharmonic on a neighborhood of $\overline{\Omega}$. Let $\widetilde{F}(J)$ denote the dual subequation. One checks that

$$F(J) + F(J) \subset F(J) \quad \Rightarrow \quad \widetilde{F}(J) + F(J) \subset \widetilde{F}(J) \quad \Rightarrow \quad F(J) \subset \widetilde{F}(J),$$

so if $\partial\Omega$ is strictly $F(J)$-convex, it is automatically strictly $\widetilde{F}(J)$-convex.

Thus, as a special case of Theorem 2.5 we have the following.

THEOREM 3.2. *Let $\Omega \subset\subset X$ be a strictly J-pseudoconvex domain in an almost complex manifold (X, J). Let $g \in C(\overline{\Omega})$. Then the largest J-plurisubharmonic minorant of g is continuous.*
Moreover, given $u \in \mathrm{PSH}(\overline{\Omega})$ there exists a sequence $\{u_j\} \subset C(\overline{\Omega}) \cap \mathrm{PSH}(\overline{\Omega})$ decreasing down to u (with each u_j strict).

We now address the global question of continuous approximation of J-plurisubharmonic functions on X.

Definition 3.3. An almost complex manifold (X, J) is **J-pseudoconvex** if it has a global C^2 strictly J-plurisubharmonic exhaustion function. (See Remark 3.7 below for equivalent definitions.)

It is standard that a strictly J-pseudoconvex domain Ω is itself J-pseudoconvex.

THEOREM 3.4. *Suppose X is a J-pseudoconvex manifold. Then for each $u \in \mathrm{PSH}(X)$ there exists a sequence of continuous strictly J-plurisubharmonic functions $u_j \in C(X)$ decreasing down to u on X.*

Proof. We shall adapt a part of the proof of the Theorem 1 from [19]. Take a decreasing sequence of continuous functions $\{g_k\}$ converging down to u. We begin with a result in smooth topology.

Claim 3.5. Let h be an arbitrary continuous function on X, and suppose that $\rho : X \to \mathbb{R}$ is a C^2 (proper) exhaustion function. Then there exists a convex function $\chi \in C^\infty(\mathbb{R})$ with $\chi' \geq 1$ so that

$$\chi(\rho(x)) \;\geq\; h(x) \qquad \text{for all } x \in X.$$

Proof. Set $\psi(t) \equiv \sup\{h(x) : \rho(x) \leq t\}$ and note that

$$\chi(\rho(x)) \;\geq\; h(x) \;\; \forall\, x \in X \qquad \Longleftrightarrow \qquad \chi(t) \;\geq\; \psi(t) \;\; \forall\, t \in \mathrm{range}(\rho).$$

This reduces the claim to a one-variable claim. To establish this, assume that $\mathrm{range}(\rho) = [0, \infty)$ and replace ψ by a smooth function which is larger. Then choose $\chi \in C^\infty([0, \infty))$ to have $\chi(0) = \psi(0)$, $\chi'(0) \geq \max\{\psi'(0), 1\}$ and $\chi'' \geq \max\{\psi'', 0\}$. ∎

Now let $\rho \in C^\infty(X)$ be a strictly J-plurisubharmonic exhaustion function. For any smooth convex, increasing function $\chi \in C^\infty(\mathbb{R})$, with $\chi' \geq 1$, the composition $\chi \circ \rho$ is also a smooth strictly J-plurisubharmonic exhaustion. Thus, by Claim 3.5, with h taken to be g_1 plus any exhaustion function for X, we can assume ρ is chosen so that

$$\lim_{z\to\infty} (\rho(z) - g_1(z)) = +\infty \tag{3.1}$$

where $\lim_{z\to\infty}$ denotes the limit in the one-point compactification of X.

By (3.1) the sets $U_k \equiv \{\rho > g_1 + k\}$ provide a fundamental neighborhood system for the point at infinity. Since ρ is an exhaustion, we have that $\{\rho - k \geq t\} \subset U_k$ if t is sufficiently large. By Sard's Theorem we may choose such t to be a regular value t_k of $\rho - k$. Then $\Omega_k \equiv \{\rho - k < t_k\}$ is a strictly J-pseudoconvex domain, and

$$\rho - k \ > \ g_1 \ (\geq \ g_k) \quad \text{on a neighborhood of} \quad \sim \Omega_k. \tag{3.2}$$

Hence,

$$\widetilde{g}_k \ \stackrel{\text{def}}{=} \ \max\{g_k, \rho - k\} \ = \ \rho - k \quad \text{on a neighborhood of} \quad \sim \Omega_k. \tag{3.3}$$

Now let u_k be the largest J-psh minorant of $\tilde{g}_k$ on Ω_k, and note that u_k is continuous by Theorem 3.2. By (3.3) we have $\tilde{g}_k = \rho - k$ on a neighborhood of $\sim \Omega_k$. Since $\rho - k$ is J-psh, and u_k is the largest J-psh minorant of $\widetilde{g}_k$, we have $u_k = \rho - k$ on a neighborhood of $\sim \Omega_k$. Thus we can extend u_k as a J-psh function to all of X by setting $u_k = \rho - k$ on $\sim \Omega_k$.

Note that since $\widetilde{g}_k \equiv \max\{g_k, \rho - k\}$, $g_{k+1} \leq g_k$, and $g_k \downarrow u$, one has

$$\widetilde{g}_{k+1} \ \leq \ \widetilde{g}_k \qquad \text{and} \qquad \widetilde{g}_k \ \downarrow \ u. \tag{3.4}$$

By definition

$$u_k \leq \widetilde{g}_k \qquad \text{and} \qquad u_k = \widetilde{g}_k \ \text{ on } \ \sim \Omega_k. \tag{3.5}$$

Now since $u_{k+1} \leq \widetilde{g}_{k+1}$, and since u_k is the largest J-psh minorant of $\widetilde{g}_k$ on $\overline{\Omega}_k$, we have by (3.4) that $u_{k+1} \leq u_k$ on $\overline{\Omega}_k$. On the complement $\sim \Omega_k$, we have $u_k = \widetilde{g}_k$ and so $u_{k+1} \leq u_k$ again by (3.4) and (3.5). Hence,

$$u_{k+1} \ \leq \ u_k \qquad \text{on } X. \tag{3.6}$$

Since $u \leq \widetilde{g}_k$ is J-psh and u_k is the largest such minorant on $\overline{\Omega}_k$, we have that $u \leq u_k$ on $\overline{\Omega}_k$. On the complement $\sim \Omega_k$, we have $u_k = \widetilde{g}_k$ and so $u \leq u_k$ there as well. Hence,

$$u \ \leq \ u_k \qquad \text{and} \qquad u_k \ \downarrow \ u \qquad \text{on } X.$$

In other words $\{u_k\}$ is a decreasing sequence of continuous J-psh functions decreasing down to u on X, and we can replace u_k with $u_k + \frac{1}{k}\rho$ to make u_k strict. ∎

Remark 3.7. (Equivalent Definitions of J-Pseudoconvexity). In defining J-pseudoconvexity it is enough to assume the existence of a *continuous* strictly J-plurisubharmonic exhaustion function $\rho : X \to \mathbb{R}$. This follows from the extension of Richberg's Theorem to almost complex manifolds (Theorem 3.1 in [18]). Such manifolds are called *almost Stein manifolds* in [4].

J-Pseudoconvex manifolds (X, J) can also be characterized in terms of the hulls of compact sets (see (4.1) below) by requiring that:

(i) There exists some $u \in \mathrm{PSH}^\infty(X, J)$ which is strict, and

(ii) For every compact $K \subset X$, the hull $\widehat{K}_{C^\infty}$ is compact.

By Theorem 3.1 in [18] we have that the hulls $\widehat{K}_{C^0} = \widehat{K}_{C^\infty}$ agree (see Corollary 4.3 below). Therefore, J-Pseudoconvex manifolds can also be characterized by the requiring:

(i) There exists some $u \in \mathrm{PSH}^0(X, J)$ which is strict, and

(ii) For every compact $K \subset X$, the hull $\widehat{K}_{C^0}$ is compact.

For the proof one applies standard arguments (cf. [11, §4] or [9, Prop. 9.3]) to show that (i) and (ii) imply the existence of a strict PSH-exhaustion (in either case).

4. Strict Smooth Approximation of Plurisubharmonic Functions on Almost Complex Manifolds

THEOREM 4.1. **(C^∞ Strict Approximation).** *Suppose (X, J) is an almost complex manifold which is J-pseudoconvex, and let $u \in \mathrm{PSH}(X, J)$ be a J-plurisubharmonic function. Then there exists a decreasing sequence $\{u_j\} \subset C^\infty(X)$ of smooth strictly J-plurisubhrmonic functions such that $u_j(x) \downarrow u(x)$ at each $x \in X$.*

Proof. Apply Theorem 3.1 in [18] and Theorem 3.4 above. ∎

This generalizes Theorem 1 in [19] to arbitrary dimensions.

COROLLARY 4.2. **(Local C^∞ Strict Approximation).** *Let (X, J) be an arbitrary (smooth) almost complex manifold. Then every point $x \in X$ has a fundamental system of neighborhoods U with the property that for every $u \in \mathrm{PSH}(U, J)$ there is a decreasing sequence $\{u_j\} \subset C^\infty(U)$ of strictly J-plurisubharmonic functions such that $u_j \downarrow u$.*

Proof. Fix local coordinates in $\mathbb{C}^n$ for X near x so that J is C^1-close to the standard J_0 at the origin. Then $\chi(z) = |z|^2$ is strictly J-psh on the ball $B_\epsilon(0) = \{|z| < \epsilon\}$ for all $\epsilon > 0$ sufficiently small. It is standard that any domain which admits a C^2 strictly J-plurisubharmonic defining function, is J-pseudoconvex. ∎

One can also give a more direct proof of Corollary 4.2 based on Theorem 3.2 above and Theorem 3.1 in [18].

Another immediate consequence of the global approximation Theorem 4.1 is that all the various possible definitions of the hull of a set actually agree.

Given a compact set $K \subset X$ we define its ***J*-plurisubharmonic hull** to be the set

$$\widehat{K} \;\equiv\; \left\{ x \in X : u(x) \le \sup_K u \quad \forall\, u \in \mathrm{PSH}(X,J) \right\}. \tag{4.1}$$

One could also define $\widehat{K}_{C^0}$ and $\widehat{K}_{C^\infty}$ by replacing $\mathrm{PSH}(X,J)$ in (3.4) with $\mathrm{PSH}^0(X,J) \equiv \mathrm{PSH}(X,J) \cap C(X)$ and $\mathrm{PSH}^\infty(X,J) \equiv \mathrm{PSH}(X,J) \cap C^\infty(X)$ respectively.

Corollary 4.3. *Suppose (X,J) is J-pseudoconvex. Then for any compact $K \subset X$, one has $\widehat{K} \;=\; \widehat{K}_{C^0} \;=\; \widehat{K}_{C^\infty}$.*

Proof. Clearly $\widehat{K} \subset \widehat{K}_{C^0} \subset \widehat{K}_{C^\infty}$, so it suffices to show that $\widehat{K}_{C^\infty} \subset \widehat{K}$. Suppose that $x \notin \widehat{K}$. Then there exists $u \in \mathrm{PSH}(X,J)$ with $u \le 0$ on K and $u(x) = 1$. Replace u with $\max\{u, 0\}$. Let $\{u_j\}$ be the sequence given in Theorem 4.1. Then u_j converges uniformly to 0 on the compact set K and $u_j(x) \ge 1$ for all j. Hence, $x \notin \widehat{K}_{C^\infty}$. ∎

Appendix A. Affine Jet-Equivalence.

A local affine jet-equivalence is a local isomorphism of the 2-jet bundle $\mathbf{J}(\mathbb{R}^n) = \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$ which is of the form:

$$r' = r + r_0(x), \quad p' = k(x)p + p_0(x), \quad A' = h(x)Ah(x)^t + L_x(p) + A_0(x)$$

where

$r_0(x)$ takes values in $\mathbb{R}$,

$p_0(x)$ takes values in $\mathbb{R}^n$,

$A_0(x)$ takes values in $\mathrm{Sym}^2(\mathbb{R}^n)$,

(i.e., $J_0(x) \equiv (r_0(x), p_0(x), A_0(x))$ is a section of $\mathbf{J}(\mathbb{R}^n)$)

and

$k(x)$ and $h(x)$ take values in $\mathrm{GL}_n(\mathbb{R})$, while

L_x takes values in $\mathrm{Hom}\left(\mathbb{R}^n, \mathrm{Sym}^2(\mathbb{R}^n)\right)$

The regularity conditions on the jet-equivalence required in the proof of Theorem 10.1 in [10] are:

(1) k, h and L are Lipschitz continuous, and

(2) J_0 is continuous.

For the second jet equivalence in our application to the Obstacle Problem, $g \equiv h \equiv Id$ and $J_0(x) = (r_0(x), 0, 0)$, so our obstacle function $g(x) = -r_0(x)$ need only be continuous.

Appendix B. Σ_m-Subharmonic Functions.

As noted in Remark 1.3, for any subequation F, smooth approximation for F-subharmonic functions can be proved whenever continuous approximation and a Richberg-type theorem can be established for F. In this appendix we give just such a result for the complex hessian subequations on a Kähler manifold.

Let X be a complex manifold of dimension n with a fixed Kähler form ω. We say that a function $u \in \mathcal{C}^2(\Omega)$ is Σ_m-subharmonic on a domain $\Omega \subset\subset X$ if $(dd^c u)^k \wedge \omega^{n-k} \geq 0$ for $k = 1, \ldots, m$. We say that a locally integrable function

$$u : \Omega \to [-\infty, +\infty)$$

is Σ_m-subharmonic ($u \in \Sigma_m(\Omega)$) if u is upper semicontinuous and

$$dd^c u \wedge dd^c u_1 \wedge \ldots \wedge dd^c u_{m-1} \wedge \omega^{n-m} \geq 0,$$

for any $\mathcal{C}^2$ Σ_m-subharmonic functions $u_1, \ldots u_{m-1}$ (they are defined in [1] for $\omega = \omega_{st} = dd^c(|z|^2)$ in $\mathbb{C}^n$ and in [5] and [14] for general Kähler form). This is just the subequation $F \equiv \Sigma_m$ defined on X by the condition that the first m elementary symmetric functions of the complex hessian satisfy $\sigma_\ell(\mathrm{Hess}_{\mathbb{C}} u) \geq 0$ for $\ell = 1, ..., m$ (compare Example 18.1 in [10] and Lemma 7 in [20]).

A Richberg-type theorem for Σ_m was proved in [20] (Theorem 2). Lu and Nguyen proved in [15] that on compact Kähler manifolds any quasi-Σ_m-subharmonic function can be approximated from above by smooth quasi-Σ_m-subharmonic functions (a function u is quasi-Σ_m-subharmonic if the function $u + \rho$ is Σ_m-subharmonic where ρ is local potential for ω). Actually their global result implies that locally it is possible to regularize Σ_m-subharmonic functions. However, in the same way as in Theorem 4.1, we can prove a slightly stronger result.

THEOREM B.1. *Let X be a Σ_m-pseudoconvex Kähler manifold. Let u be a Σ_m-subharmonic function on X. Then there exists a decreasing sequence $u_j \in \mathcal{C}^\infty(X)$ of Σ_m-subharmonic functions such that $u_j \downarrow u$.*

By Σ_m**-pseudoconvex** we mean that X has a global $\mathcal{C}^2$ strictly Σ_m-subharmonic exhaustion function. In particular Stein manifolds are Σ_m-pseudoconvex.

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