The calculus of multivectors on noncommutative jet spaces

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Abstract. The Leibniz rule for derivations is invariant under cyclic permutations of the co-multiples within the derivations' arguments. We now explore the implications of this fundamental principle, developing the calculus of variations on the infinite jet spaces for maps from sheaves of free associative algebras over commutative manifolds to the quotients of free associative algebras over the linear relation of equivalence under cyclic shifts. In the frames of such variational noncommutative symplectic geometry, we prove the main properties of the Batalin–Vilkovisky Laplacian and variational Schouten bracket. As a by-product of this intrinsically regularised picture, we show that the structures that arise in the classical variational Poisson geometry of infinite-dimensional integrable systems – such as the KdV, NLS, KP, or 2D Toda – do actually not refer to the graded commutativity assumption.

Frustra fit per plura quod potest fieri per pauciora.
William of Ockham (*1285–1349*)

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Introduction. Let $\mathbb{F}$ be a free associative algebra over $\Bbbk := \mathbb{R}$ and suppose $a_1, \ldots, a_k \in \mathbb{F}$. Denote by $\mathfrak{t}$ the counterclockwise cyclic shift of co-multiples in the product $a_1 \circ \ldots \circ a_k$,

$$\mathfrak{t}\,(a_1 \circ \ldots \circ a_{k-1} \circ a_k) \stackrel{\text{def}}{=} a_k \circ a_1 \circ \ldots a_{k-1}.$$

For the sake of definition, assume now that a given derivation $\partial\colon \mathbb{F} \to \mathbb{F}$ is such that its values at $a_1, \ldots, a_k$ do not leave that set. By the Leibniz rule, the derivation is cyclic-shift invariant:

$$\partial\bigl(\mathfrak{t}\,(a_1 \circ \ldots \circ a_k)\bigr) = \mathfrak{t}\,\bigl(\partial(a_1 \circ \ldots \circ a_k)\bigr). \tag{1}$$

Indeed, both sides of the equality above are given by the sum

$$\partial(a_k) \circ a_1 \circ \ldots \circ a_{k-1} + a_k \circ \partial(a_1) \circ \ldots a_k + a_k \circ a_1 \circ \ldots \circ \partial(a_{k-1}),$$

up to the sequential order in which these k summands follow each other (see Fig. 1). This observation is generalised in an obvious way to the case when the elements of

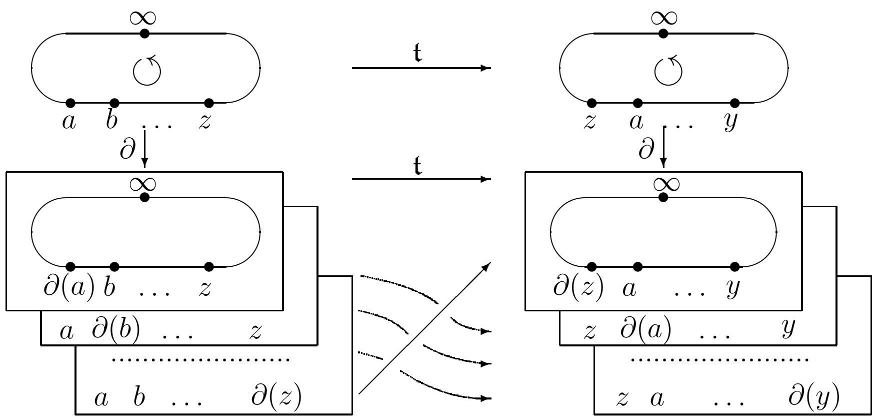


FIGURE 1. The cyclic-shift invariance of derivations.

algebra $\mathbb{F}$ are graded by some Abelian group, each element $a_1, \ldots, a_k$ is homogeneous with respect to the grading, and $\partial\colon \mathbb{F} \to \mathbb{F}$ is a graded derivation (i.e., not necessarily preserving the set $\{a_1, \ldots, a_k\}$ at hand).

How much (graded-) commutativity is really needed to make the calculus of variations in Lagrangian and Hamiltonian formalisms work, thus allowing for the Batalin–Vilkovisky technique for quantisation of gauge systems — and creating the cohomological approach to complete integrability of infinite-dimensional KdV-type systems ?[1]

We now claim that it is not the restrictive assumption of commutativity that shows through *arbitrary* permutations — but it is the linear equivalence $a \sim \mathfrak{t}(a)$ of words a, written in a given algebra's alphabet, with respect to the *cyclic* permutations $\mathfrak{t}$ that is sufficient for the structures of the calculus of iterated variations to be well defined. Introduced in this cyclic-invariant setup, the Batalin–Vilkovisky Laplacian Δ and variational Schouten bracket $[\![\,,\,]\!]$ are rigorously proven to satisfy the main identities such as the cocycle condition $\Delta^2 = 0$, see (2a–2d) below. Both the definitions and assertions

[1] We refer to [2, 3, 39, 15, 12, 4, 46, 47] or [19] and to [34, 37, 38, 8, 7, 10, 32, 33] or [17, 27], see also [28, 23] in both contexts.

are then literally valid in the sub-class of graded-commutative geometries; the reason for this is that the latter can be obtained from the former by using the postulated commutativity reduction at the end of the day when the proof is over. In brief, Fig. 1 portrays the immanent property of Leibniz rule, so that much of differential calculus is possible regardless of the commutativity but thanks to cyclic invariance (1).

The idea to establish the formal noncommutative symplectic geometry on the cyclic invariance, generalising the geometry of commutative symplectic manifolds, was introduced by Kontsevich in [30], cf. [14] and references therein. The quotient spaces of cyclic words were employed as target sets for maps from usual manifolds in [38] by Olver and Sokolov; several integrable equations of KdV-type were recovered in such noncommutative set-up.[2] Variations arising in the variational Poisson or Schouten brackets for integral functionals, their calculus was then pursued along the lines of [37], cf. [21]. The paper [38] initiated the classification and study of evolutionary ODE and PDE systems on associative algebras, which required the calculation of standard geometric structures for such models in jet spaces (e.g., see [40] in this context).

In this paper we futher that approach to noncommutative jet spaces.[3] Continuing the line of reasoning from [19, 20] where the intrinsic regularisation of Batalin–Vilkovisky formalism is revealed, we now verify the main identities for Δ and $[\![\,,\,]\!]$ in the variational noncommutative set-up of (homogeneous) local functionals F, G, H:

$$\Delta(F \times G) = \Delta F \times G + (-)^{|F|}[\![F, G]\!] + (-)^{|F|} F \times \Delta G, \tag{2a}$$

$$[\![F, G \times H]\!] = [\![F, G]\!] \times H + (-)^{(|F|-1)\cdot|G|} G \times [\![F, H]\!], \tag{2b}$$

$$\Delta\bigl([\![F, G]\!]\bigr) = [\![\Delta F, G]\!] + (-)^{|F|-1}[\![F, \Delta G]\!], \tag{2c}$$

$$\text{Jacobi}\bigl([\![\,,\,]\!]\bigr) = 0 \qquad \Longleftrightarrow \qquad \Delta^2 = 0. \tag{2d}$$

It is quite paradoxical that for a long time, these identities were proclaimed to be valid just formally [13, 15]; for it was believed that the Batalin–Vilkovisky technique would necessarily contain some divergencies or "infinite constants," whereas their manual regularisation appealed to surreal principles like "$\boldsymbol{\delta}(0) := 0$" for Dirac's $\boldsymbol{\delta}$-function (see [19] and references therein for discussion on the history of the problem).

Let us emphasize that through the use of noncommutativity we gain a deeper understanding of classical objects and structures such as the iterated variations of local functionals; it is by this that the intrinsic regularisation of the Batalin–Vilkovisky formalism was achieved (see [22] for illustration). In fact, it was enough to focus on the algebraic distinction between the commutative substrate manifold M^n (e.g., the Minkowski space-time underlying the BV-zoo) and the quotients $\mathcal{A} = \mathbb{F}/\sim$ of the free associative algebras taken for the target sets. The maps that take the sheaves M^n_{nC} of some other free associative algebras – in earnest, the sheaves of groups of walks, see §1.2 – over M^n to the cyclic-word quotients $\mathcal{A}^{(0|1)}$ of $\mathbb{Z}_2$-graded extensions of $\mathbb{F}$ now

[2]Noncommutative extensions of classical infinite-dimensional systems can acquire new components that are invisible in the commutative world: e.g., there appear – often, through nonlocalities – the terms that contain the commutants $a_i \circ a_j - a_j \circ a_i$.

[3]We note that the positive differential order calculus on infinite jet spaces lies far beyond the bare tensor calculus on usual commutative manifolds; for instance, compare [41] with [28] or contrast [1] vs [26] and [39] vs [19].

play the rôle of sections in the BV-bundle of (anti)fields and (anti)ghosts. — Models from theoretical physics motivate the study of precisely this construction over M^n, cf. Fig. 4 on p. 17 below.[4] In the frames of noncommutative picture, the cyclic words in the target set acquire the nature of paths which are thread through a cell-complex tiling of M^n, whereas the underlying manifold itself shows up in the formulae through the integration by parts, see Ch. 1. The point is that, even in this picture's graded-commutative reduction, all the path-like objects to-vary are first unrolled consecutively, forming one long path the components of which are then re-attached or modified by using the derivations ∂; only afterwards are the integrations by parts performed. This results in the sought-for regularisation (2a–2d) of the BV-geometry, cf. [15, §18].

A still wider approach to noncommutativity suggests that the manifolds – and derivative objects such as the fibre bundles – are determined as the spectra of noncommutative algebras, most commonly associative, provided that the algebras are 'smooth;' those algebras are viewed as the algebras of smooth functions on the objects which they determine. Nowadays, noncommutative geometry à la Connes [5] is a well-established domain that allows for consideration of much more general settings than ours. However, we keep the framework closer to the needs which one encounters in an intriguing class of path- and loop-based theoretical physics models ([24, 25], cf. [44, 11, 6]). Let us stay on the verge of maximal generality in favour of studying the language of closed strings of symbols – written around the circles and encoding paths in the granulated space M^n.

This paper consists of three parts. In Ch. 1 we introduce the static set-up of noncommutative infinite jet (super-)spaces. Based on the algorithmic construction of parity-odd Laplacian Δ and variational Schouten bracket $[\![\, ,\,]\!]$, the calculus of iterated variations of local functionals – i.e., kinematics – is developed in Ch. 2. Such BV-geometry of local functionals is then contrasted to the noncommutative Poisson formalism – that is, to the dynamics of variational multi-vectors, which we prove in Ch. 3 to be the paradigm of steps and stops, as far as the calculus of variations is concerned.

The text is structured as follows. The commutative but not associative algebra $\mathcal{A}$ of cyclic words written in the alphabet $\langle a^i\rangle$ of a free associative algebra is introduced in §1.1. The generators a^i themselves are viewed in §1.2 as words written in the alphabet $\langle \vec{\mathrm{x}}_i^{\pm 1}\rangle$ of edges in the adjacency graph for a cell-complex tiling of the substrate manifold M^n. The alphabets $\langle \vec{\mathrm{x}}_i^{\pm 1}\rangle$ and $\langle a^i\rangle$ provide the respective noncommutative analogues of base and fibre in a bundle; in §1.3 we build the jet space of maps $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$ from the sheaf of [unital extensions of] free associative algebras generated by $\langle \vec{\mathrm{x}}_i^{\pm 1}\rangle$ for a crystal tiling of M^n to [the unital extension of] the algebra $\mathcal{A}$ of cyclic words written in the alphabet $\langle a^i\rangle$. Various elements of the jet-space language are then recovered. In particular, we show in §1.3 why the Substitution Principle works for identities in total derivatives; the noncommutativity of set-up makes the reasoning particularly transparent.

[4] Let us recall that the notion of associative structures itself deserves the focused attention in mathematical physics literature (e.g., in the broad context of Yang–Baxter's equation). By construction, solutions of the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation *are* the regular generators of associative algebra structures [45, 9], cf. [35]. We now study the extent to which the differential calculus can be developed on the basis of that input data.

The second part begins with the definition of noncommutative analogue for the variational cotangent bundle over the infinite jet space $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$, see §2.1. The target algebra's alphabet $\langle a^i \rangle$ is doubled by using the canonical pairs $\langle a^i, a_i^\dagger \rangle$; sign convention (11) for the two ordered couplings of the virtual variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{a}^\dagger$ ensures the matching of signs in all the structures that are defined in what follows. In the meantime (see §2.3), the $\mathbb{Z}_2$-parity reversion $\Pi\colon a_i^\dagger \rightleftarrows b_i$ acts on the dual symbols $\boldsymbol{a}^\dagger$, producing the parity-odd slots $\boldsymbol{b}$. Now, the geometric approach of [19] to iterated variations works in the noncommutative set-up of *maps* $\boldsymbol{a} = \boldsymbol{s}(\boldsymbol{x}, \vec{\mathrm{x}}^{\pm 1})$ and *antimaps* $\boldsymbol{a}^\dagger = \boldsymbol{s}^\dagger(\boldsymbol{x}, \vec{\mathrm{x}}^{\pm 1})$ from the sheaf over M^n. Therefore, while giving the operational definitions of BV-Laplacian Δ in §2.6, we focus on the unlock-and-join reconfigurations of cyclic words. The variational Schouten bracket $[\![\,,\,]\!]$ is a derivative structure, that is, it is determined by the parity-odd operator Δ via its action on products, see (2a) above.[5] We then confirm that the variational Schouten bracket $[\![\,,\,]\!]$ is shifted-graded skew-symmetric and satisfies the Jacobi-identity. The two structures Δ and $[\![\,,\,]\!]$ endow the ring of local functionals with the structure of differential shifted-graded Lie algebra.

The third part of this text narrates on the noncommutative variational Poisson formalism. The notion of noncommutative variational multi-vectors is introduced in §3.1. We recall that not every grading-homogeneous integral functional over the infinite jet space of maps $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$, canonically extended in Ch. 2, would be a well defined variational multi-vector containing the respective number of parity-odd slots $\boldsymbol{b}$. Remark 2.2 on p. 17 is a key to that concept. Specifically, by viewing now the variational multi-vectors as maps that take the respective tuples of – possibly, exact – variational covectors to the senior horizonal cohomology space of cyclic word-valued integral functionals, we analyse in §3.2 the proper geometry of iterated variations that arise in the derived brackets encoding such maps. We then discover that the calculus of noncommutative variational multivectors is the paradigm of steps and stops. Finally, we arrive at the definition of Poisson brackets. In §3.3 we study the geometry of differential forms that stands behind the criterion — under which the variational noncommutative bi-vectors are Poisson, i.e., endow the space of noncommutative Hamiltonians with the variational Poisson brackets. In particular, in the course of showing that the Helmholtz lemma holds in the noncommutative case (see p. 36) we reveal a yet another mechanism for differentials to anticommute – besides the construction of top-degree volume forms $\mathrm{dvol}(\boldsymbol{x})$ on the oriented substrate manifols $M^n \ni \boldsymbol{x}$ and besides the sign convention on the couplings of dual variations, see (10) on p. 16.

[5] In geometric terms, the bracket $[\![\,,\,]\!]$ of cyclic word-valued functionals is encoded by the standard topological pair of pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ that links the cycles. In fact, this topological procedure underlies also each of the following structures and operations in the differential calculus under study:

- multiplication $\times$ of cyclic words and word-valued function(al)s,
- termwise action of derivations, including
- the commutation of vector fields, — and also
- evaluation of multi-vectors at the tuples of covectors: in particular,
- the Poisson bracket of Hamiltonian functionals.

Indeed, all of the above amounts to the detach-and-join trick $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$.

1. The nature of associative symbols

1.1. **The algebra $\mathcal{A}$ of cyclic words.** In this section we introduce the main object to consider in the future reasoning. Namely, by starting with a non-commutative free associative algebra, we define the commutative but not associative unital algebra $\mathcal{A}$ of cyclic words written in the free algebra's alphabet. Note that for the sake of clarity, neither of these two algebras is graded; however, in what follows we shall extend the alphabet by using symbols the $\mathbb{Z}_2$-valued parity of which is odd.

Throughout this text, the ground field $\Bbbk$ is the field $\mathbb{R}$ of real numbers.

Consider the free associative algebra $\mathsf{Free}\,(a^1, \ldots, a^m)$ with m generators $a^1, \ldots, a^m$; let $m < \infty$ for definition. Denote by $\circ$ the multiplication in that algebra. By definition, put

$$\mathfrak{t}\,(a^{i_1} \circ \ldots \circ a^{i_\lambda}) := a^{i_\lambda} \circ a^{i_1} \circ \ldots \circ a^{i_{\lambda-1}}; \tag{3}$$

otherwise speaking, the operator $\mathfrak{t}$ is the counterclockwise cyclic permutation of symbols in a homogeneous word of length λ.

Introduce the linear equivalence relation $\sim$ on $\mathsf{Free}\,(a^1, \ldots, a^m)$ by setting[6]

$$a \sim \mathfrak{t}(a),$$

where a is a homogeneous word as in (3), and then extending $\sim$ onto the algebra by linearity: $a \sim a'$ and $b \sim b'$ implies $a + b \sim a' + b'$. For instance, one has that[7]

$$a^1 + a^2 \circ a^3 + a^1 \circ a^2 \circ a^3 \sim a^1 + a^3 \circ a^2 + a^2 \circ a^3 \circ a^1.$$

Notice also that

$$a \sim \mathfrak{t}(a) \sim \ldots \sim \mathfrak{t}^{\lambda(a)-1}(a) \sim \frac{1}{\lambda(a)} \sum_{i=1}^{\lambda(a)} \mathfrak{t}^{i-1}(a)$$

for any word a of length $\lambda(a) > 0$; by convention, a word of zero length is an element of the ground field $\Bbbk$, see (6) below.

We denote by $\mathcal{A}$ the quotient $\mathsf{Free}\,(a^1, \ldots, a^m)/\sim$, that is, $\mathcal{A}$ is the vector space of (formal sums of) cyclic words such that each homogeneous component $a^{i_1} \circ \ldots \circ a^{i_\lambda}$ can be read starting from any letter a^{i_α} for $1 \leqslant \alpha \leqslant \lambda$. Therefore, let us denote by $(a) \in \mathcal{A}$ the equivalence class of an element $a \in \mathsf{Free}\,(a^1, \ldots, a^m)$ under cyclic permutations of symbols in all its homogeneous components (i. e., in all its "words" in proper sense).

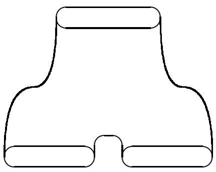
We now endow the vector space $\mathcal{A}$ of cyclic words with the algebra structure $\times$. Consider the equivalence classes (a_1) and (a_2) of two homogeneous elements $a_1, a_2 \in \mathsf{Free}\,(a^1, \ldots, a^m)$ of lengths $\lambda(a_1)$ and $\lambda(a_2)$ respectively. Let their product be

$$(a_1) \times (a_2) \stackrel{\text{def}}{=} \frac{1}{\max(1, \lambda(a_1)) \cdot \max(1, \lambda(a_2))} \Bigl(\sum_{i=1}^{\lambda(a_1)} \sum_{j=1}^{\lambda(a_2)} \mathfrak{t}^{i-1}\,(a_1) \circ \mathfrak{t}^{j-1}\,(a_2) \Bigr), \tag{4}$$

[6] It is readily seen that $a^{i_1} \circ \ldots \circ a^{i_\lambda} = \mathfrak{t}^{\lambda-1}\,(\mathfrak{t}\,(a^{i_1} \circ \ldots \circ a^{i_\lambda}))$ so that $a \sim a$ and $\mathfrak{t}(a) \sim a$, whence the transitive relation $\sim$ is reflexive and symmetric indeed.

[7] We emphasize that the cyclic invariance itself does *not* imply the commutativity: even though $a^i \sim a^i$ and $a^i \circ a^j \sim a^j \circ a^i$ one has that $a^i \circ a^j \circ a^k \nsim a^i \circ a^k \circ a^j$ unless some of the indexes coinside.

where the equivalence class in the right-hand side is normalized in such a way that the definition correlates with the commutative set-up (should it be recovered postfactum); now extend the product onto $\mathcal{A}$ by (bi-)linearity. The definition of operation $\times$ says that, each homogeneous string of symbols in the first co-multiple read, time after time, starting from every next letter, it is then pasted – time after time in its turn – in between every two consecutive letters occurring in each homogeneous string contained in the second co-multiple. Sure, this is the classical topological pair of pants $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$ in which every symbol in either of the factors has the right to be read first, see the figure.



Notice that not only the necklace (a_1) is unlocked at all possible multiplication signs $\circ$ and joined to (a_2) in between each pair of adjacent symbols in that word but, as one shifts the symbols in (a_2) around the circle, exactly the same is done with respect to the insertion of $\mathfrak{t}^{j-1}\,(a_2)$ into (a_1).

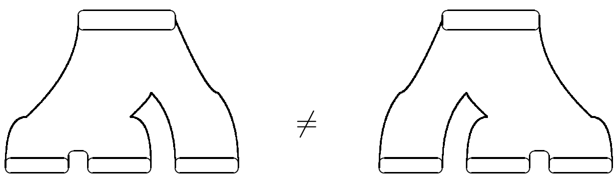
Corollary 1. Multiplication (4) on $\mathcal{A}$ is commutative.

However, it is readily seen that the symbols in homogeneous strings in (a_1) and (a_2) always stay next to each other in the nested product $\big((a_1) \times (a_2)\big) \times (a_3)$, whereas they are separated by the symbols from (a_3) in at least one homogeneous term in $(a_1) \times \big((a_2) \times (a_3)\big)$, provided that the alphabet contains at least two different letters and the length of the word a_3 is greater than one.

Proposition 2. If $m \geqslant 2$ so that the letters a^1 and a^2 are distinct in the alphabet, multiplication (4) on $\mathcal{A}$ is not associative:

$$\big((a_1) \times (a_2)\big) \times (a_3) \not\sim (a_1) \times \big((a_2) \times (a_3)\big), \tag{5}$$

see the figure below.



Obviously, the associativity equation for $\times$ can be satisfied incidentally, for a special choice of the three co-multiples.

Counterexample 1.1 ("abba")**.** Let $a_1 := a^{\mathbf{1}}$, $a_2 := a^{\mathbf{1}}$, and $a_3 := a^{\mathbf{2}}a^{\mathbf{2}}$. Then $(a_1) \times (a_2) = (a^{\mathbf{1}} \circ a^{\mathbf{1}})$ so that these two copies of the letter $a^{\mathbf{1}}$ always stay next to each other in any product of $(a_1) \times (a_2)$ with any other word. On the other hand (see Fig. 2),

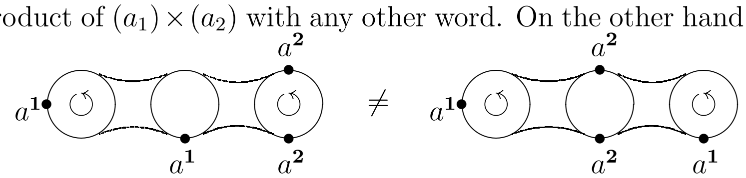


FIGURE 2. The letters $a^{\mathbf{1}}$ are (not) separated by the letters $a^{\mathbf{2}}$.

the word $(a_2) \times (a_3)$ is equal to $(a^{\mathbf{2}}a^{\mathbf{1}}a^{\mathbf{2}})$, whence the nested product $(a_1) \times \big((a_2) \times (a_3)\big)$ contains the term $\frac{1}{3}a^{\mathbf{1}}a^{\mathbf{2}}a^{\mathbf{1}}a^{\mathbf{2}}$, which is absent in the left-hand side of (5).

By interpreting the ground field $\Bbbk$ as the linear span of the zero-length word 1 and its equivalence class (1), we extend the commutative algebra of cyclic words to $\mathcal{A} \oplus \Bbbk \cdot (1)$, now endowed with the multiplication $\times$ such that, in agreement with the vector space structure of $\mathcal{A}$, formula (4) yields

$$(k) \times (a) \stackrel{\text{def}}{=} k \cdot (a) \tag{6}$$

for any $k \in \Bbbk$ and all cyclic words (a). Allowing for the slightest abuse of notation, we continue denoting by $\mathcal{A}$ the unital algebra of cyclic words that contains such zero-length but non-empty strings of symbols.

1.2. **The sheaves of algebras of walks.** In this section we motivate the construction of the algebra $\mathcal{A}$ that contains nonnegative-length cyclic words written in the alphabet a^1, …, a^m. By introducing several new elements into the picture now, in the next section we shall recover the notion of space of infinite jets of maps into the algebra $\mathcal{A}$.

Let M^n be an oriented smooth real manifold of positive dimension n. Suppose now that a tiling of the manifold M^n is given, that is, M^n is realised by $M^n = \cup_{\alpha\in\mathcal{I}}\overline{\Delta}_\alpha$ via the complex of cells Δ_α of dimension n, see Fig. 3a. We remark that the choice of a tiling

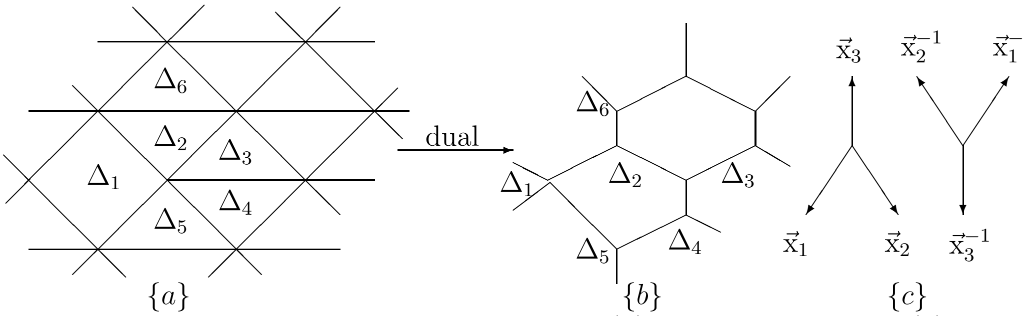


FIGURE 3. A fragment of cell-complex tiling (a), its adjacency graph (b), and the alphabet of a crystal tiling (c).

can be not unique for a given manifold M^n. Construct the tiling's adjacensy graph: each cell Δ_α represented by the vertex in the dual picture (see Fig. 3b), two vertices are connected by the edge iff the respective cells in the tiling are adjacent through a common face of lower dimension.[8]

Over the substrate manifold M^n, let us construct now the sheaf of algebras of walks along the edges in the adjacency graph, see [36] for the basic definition.[9] The sheaf is glued from the algebras of walks defined over every subset $U \subseteq M^n$ open with respect to a chosen topology on that manifold (e. g., the standard Euclidean one – however, cf. Remark 1.1 below). By definition, if $U \cap \Delta_\alpha \neq \varnothing$ for some $\alpha \in \mathcal{I}$, the respective

[8]The *discrete* adjacency table, *finite* for every vertex Δ_α in the dual complex, is the main profit that one gains by taking the tiling of space, however tiny be the diameter of each cell with respect to a given distance function on M^n.

[9]The *algebra* of walks – that is, the vector space of formal sums of paths that can also be multiplied by using the concatenation, – but not the *group* of walks in which every element could always be singled out and inverted or dealt with separately in other respects, is considered here for the sake of beauty, strangeness, and charm.

algebra over U is formed by all the walks along the adjacency graph's edges that connect those vertices Δ_α such that $U \cap \Delta_\alpha \neq \varnothing$. (In particular, whenever $U \subseteq \Delta_\alpha$ for some $\alpha \in \mathcal{I}$, then all walks amount to the null path $\mathbf{1}$ that does not leave the vertex Δ_α in the adjacency graph.) Obviously, the algebra of walks over a union $U_i \cup U_j \subseteq M^n$ of open sets in M^n consists of (the formal sums of) walks along the union of two sets of edges that interconnect the adjacent cells having non-empty intersections with U_i or U_j. Conversely, the algebra of walks over the open intersection $U_i \cap U_j$ is built by using the intersection of the sets of edges for U_i and U_j alone.

Remark 1.1. As an alternative to the ever-present Euclidean topology induced on M^n, non-Hausdorff topologies on the sheaf's substrate manifold can be determined by the cell complex itself. Namely, let its vertices, edges, faces and so on up to the n-dimensional cells Δ_α be proclaimed open. By this argument, the cells Δ_α of higher dimension acquire the status of things that have no parts; for all the points of every such cell are indistinguishable indeed.

The sheaf structure is then set equal to $\varnothing$ over the empty subset of M^n. By definition, the structure over all the lower-dimensional components of the complex such as the vertices, edges, or faces (i.e., for all $U \subseteq M^n$ open such that $U \cap \Delta_\alpha = \varnothing$ for all $\alpha \in \mathcal{I}$) is set equal to $\Bbbk \cdot \mathbf{1}$. Consequently, a scalar is the only type of data which the substrate manifold does carry whenever it is shrunk to The One Point. Finally, for all subsets U open in M^n and such that $U \cap \Delta_\alpha \neq \varnothing$ for some $\alpha \in \mathcal{I}$, the sheaf structure is set equal to the unital algebra of walks along the adjacency graph's edges interconnecting the respective vertices Δ_α of the dual graph. Note that under the shrinking of such open domains $U \subseteq M^n$ to non-empty open parts of the lower-dimensional skeleton of the cell complex, the unital algebra of walks is canonically projected onto its null-path component $\Bbbk \cdot \mathbf{1}$.

Note further that two (or say, three) different tilings of the substrate manifold M^n determine the two (resp., three) topologies on it; all of them would not be equivalent to the standard Euclidean topology.

Without any extra assumptions made about the tiling, the cells' adjacency table and the portrait of edges in the dual graph are *local*. Indeed, a *quasi*crystal structure of the cell complex realisation of M^n could contain defects. Consequently, the larger an open domain $U \subseteq M^n$ is, the larger can be the alphabet of edges which are used to encode paths as words. On the other hand, such robust sheaf structure is uniform with respect to the presence or absence of any defects in the (quasi)crystal tiling $\{\Delta_\alpha\}$; this would be convenient if there are some defects indeed.

For the sake of clarity, we shall assume from now on that the substrate manifold's tiling is globally regular, so that the crystal structure $\{\Delta_\alpha\}$ is formed by (in)finite replication of a finite union of cells. The edge alphabet is then completely determined by N adjacency relations within the generating set of cells in that finite union. For instance, consider the honeycomb triangular tiling of the plane, see Fig. 3c. This regularity assumption makes the alphabet $\vec{\mathbf{x}}^{\pm 1}$ finite even if the tiling of the (non)compact manifold M^n is infinite. The price that one has to pay is that the coding of edges can no longer be referred to any specific cell, hence a presence of defects is no longer possible.

Be that as it may, over M^n let us construct the (almost) *constant* sheaf of unital extensions $\Bbbk \cdot \mathbf{1} \oplus \mathsf{Free}_{\Bbbk}\bigl(\vec{x}_1^{\pm 1}, \ldots, \vec{x}_N^{\pm 1}\bigr)$ of free algebras generated by the symbols $\vec{x}_i^{+1}$ and

$\vec{x}_i^{-1}$ that denote the edges passed in the adjacency graph in either of the two directions.[10] The idea that the sheaf be almost constant is expressed as follows.[11] Whenever one shrinks an open subset $U_i \subseteq M^n$ to a smaller open set $U_j \subseteq U_i$ such that $U_j \cap \Delta_\alpha \neq \varnothing$ for some $\alpha \in \mathcal{I}$, the restriction map acting on the algebra is the identity map. Over any non-empty set U open in M^n but such that $U \cap \Delta_\alpha = \varnothing$ for all $\alpha \in \mathcal{I}$ (see Remark 1.1 above), the sheaf structure is the unital component $\Bbbk \cdot \mathbf{1}$ and the restriction map is the canonical projection. For the empty subset of M^n, the sheaf structure is empty by definition.

This sheaf over M^n will be denoted by M^n_{nC}; it remembers the topology on the substrate manifold and it carries the finite alphabet $\vec{\mathbf{x}}^{\pm 1}$ of the N edges that interconnect the cells in (the replicas of) a fundamental domain in the tiling.

We now start building a noncommutative analogue of the variational cotangent bundle or Batalin–Vilkovisky bundle over the space-time. Recalling from §1.1 the construction of the algebra $\mathcal{A}$ of cyclic words, we notice that it always suffices to define maps to the generators a^i of the free associative algebra; the maps are then extended onto (the quotient of) the target space $\mathsf{Free}\,(a^1, \ldots, a^m)$ by using both the multiplicative and additive structures ($\circ$ and $+$, respectively). Indeed, consider the map $\boldsymbol{s}\colon M^n_{\text{nC}} \to \mathsf{Free}\,(a^1, \ldots, a^m)$ which, in a chart $U \subseteq M^n$ containing a point $\boldsymbol{x}$ of the substrate manifold M^n, is described by the formulas[12]

$$a^i = s^i(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}), \qquad 1 \leqslant i \leqslant m. \tag{7}$$

Otherwise speaking, each component s^i of such "section" is a word (or a formal sum of words) in the alphabet $\vec{\mathbf{x}}^{\pm 1} = \{\vec{x}_j^{\pm 1},\ 1 \leqslant j \leqslant N\}$, each word taken with a smooth coefficient from $C^\infty(M^n)$. By construction, the value of a homogeneous word written in the alphabet $\boldsymbol{a} = \{a^i,\ 1 \leqslant i \leqslant m\}$ is the associative product of the map's values at the word's consecutive letters. For instance, we postulate that

$$(a^i \circ a^j)\big|_{\boldsymbol{s}}(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}) = s^i(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}) \circ s^j(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1});$$

the multiplication $\circ$ in the left-hand side is thus inherited from the associative multiplication in the sheaves (that is, from the bare concatenation of paths that run along the chosen adjacency graph).

The construction of maps that take the sheaf M^n_{nC} to the quotient $\mathcal{A}$ is immediate because the equivalence relation $\sim$ was introduced in §1.1 without reference to the evaluation of letters a^i via (7); however, we now refer to Remark 2.6 on p. 19 below.

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[10] A possibility to walk every edge, hence every path *backwards* – along the respective reverses $\vec{x}_i^{\mp 1}$, reading the words right to left, – is a forerunner of the introduction of canonical conjugate symbols $a_j^\dagger$, which are responsible for the dual, parity-odd part of the picture. This will be discussed in §2.1 and §2.3, see Fig. 4 on p. 17 in particular.

[11] We remark that by intention do we consider the sheaves over M^n but not the spaces of maps taking that commutative manifold to a given (non)commutative algebra. In view of what has been said before, such target spaces themselves could depend on the point $\boldsymbol{x}$ as it runs through the domains Δ_α.

[12] Actually, formula (7) is a compact notation: its right-hand side evaluates at $\boldsymbol{x} \in M^n$ the infinitely many coefficients of $\vec{x}_i^{\pm 1}$, $\vec{x}_i^{\pm 1} \circ \vec{x}_j^{\pm 1}$, $\vec{x}_i^{\pm 1} \circ \vec{x}_j^{\pm 1} \circ \vec{x}_k^{\pm 1}$, etc.; those auxiliary objects are not even given their own names in this transcription.

Remark 1.2. Evaluation (7) of a word a from $\mathcal{A}$ paves the way (weighted by elements of $C^\infty(M^n)$) along the edges $\vec{\mathrm{x}}_i^{\pm 1}$ of the graph which we started with. However, the cyclic invariance of the word a does *not* generally imply that this path is closed.[13] For a given map (7), not every cyclic word may have its proper meaning[14] (Still the converse is true: every path is encoded by the respective word and every closed path is described by the equivalence class of cyclic words.) ; moreover, not every word written in the alphabet $\vec{\mathbf{x}}_i^{\pm 1}$ encodes some path connecting cells in the tiling.

Remark 1.3 ($\mathbf{1}(\boldsymbol{x}) \in C^\infty(M^n)$). As soon as the unital algebra $\mathcal{A}$ of cyclic words is placed over the "points" of M^n_{nC} – in earnest, over usual points $\boldsymbol{x} \in M^n$ of the substrate manifold – the zero-length words in $\mathcal{A}$ are specified pointwise over M^n by elements of the ring $C^\infty(M^n)$ that plays now the rôle of the ground field $\Bbbk$. This blow-up $\Bbbk \hookrightarrow C^\infty(M^n)$ is standard in differential calculus on (jet) bundles in the commutative case (cf. [37, 28, 26, 18]).

Remark 1.4 (Positive proper length). Obviously, the case when $a^i = s^i(\boldsymbol{x})$ for some i is somewhat special: the algebra $\mathcal{A}$ of nonnegative-length cyclic words was unital by construction, but the assignment above would convert the generator a^i to the multiple of the neutral element at every $\boldsymbol{x}$ in a chart. To exclude this unfavourable situation from the study, let us technically assume that the lexicographic length of all the word(s) in each component s^i is strictly positive.[15]

1.3. The geometry of jet space $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$.

In this section we outline the standard construction of infinite jet space $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$ for maps from the substrate manifold M^n through the sheaf M^n_{nC} to the quotient $\mathcal{A}$ of the free associative algebra. We emphasize that this construction (local with respect to $\boldsymbol{x} \in U \subseteq M^n$) refers only to the smooth structure on the domain set $U \subseteq M^n$ and to the vector-space organisation of objects over it.

The construction which we revealed in footnote 12 yields the (infinite sets of) jet coordinates $a^i \equiv a^i_\varnothing, a^i_{x^j}, a^i_{x^j x^k}, \ldots, a_\sigma$ for $|\sigma| \geqslant 0$ over a chart $U \subseteq M^n$ with local coordinates $\boldsymbol{x} = (x^1, \ldots, x^n)$. Let us denote by $[\boldsymbol{a}]$ the differential dependence on letters a^i, $a^i_{x^j}$, …, a_σ up to some arbitrarily high but always finite order $|\sigma| < \infty$. The construction of the algebra $\mathcal{F}(M^n_{\mathrm{nC}} \to \mathcal{A})$ of cyclic-word valued functions on $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$ is standard: namely, it is the inductive limit of filtered algebras ([37, 18]). Likewise, the *total derivatives* $\frac{\mathrm{d}}{\mathrm{d}x^i}$, which we denote synonymically by D_{x^i} for $1 \leqslant i \leqslant n$ making no further distinction between $\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\sigma$ and $D^\sigma_{\boldsymbol{x}}$, are introduced by using the restrictions of elements $f \in \mathcal{F}(M^n_{\mathrm{nC}} \to \mathcal{A})$ to graphs of (7), i.e.,

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x^i}(f)\Bigg|_{\mathrm{jet}_\infty(\boldsymbol{a}=\boldsymbol{s}(\,\cdot\,,\vec{\mathrm{x}}^{\pm 1}))}(\boldsymbol{x}_0) \stackrel{\mathrm{def}}{=} \frac{\partial}{\partial x^i}\bigg|_{\boldsymbol{x}_0}\Bigl(f\big|_{\mathrm{jet}_\infty(\boldsymbol{a}=\boldsymbol{s}(\,\cdot\,,\vec{\mathrm{x}}^{\pm 1}))}\Bigr). \tag{8}$$

[13] Actually, this circumstance refers to a distinction between all *sections* of a given bundle and all *solutions* of a given equations for sections of that bundle.

[14] Alternatively, it could require some effort to endow a given cyclic word with a meaning by contracting the graph between the path's loose ends.

[15] Moreover, one should even require that the *walk* s^i along the edges $\vec{\mathrm{x}}_i^{\pm 1}$ of the graph be more than a null path $\mathbf{1}$, for it could be that the walk is contractable: e. g., $s^i = \vec{\mathrm{x}}_j \circ \vec{\mathrm{x}}_j^{-1} = \mathbf{1}$.

This determines the usual coordinate expressions,

$$\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m}\sum_{|\sigma|\geqslant 0} a^j_{\sigma\cup\{i\}}\cdot\frac{\overrightarrow{\partial}}{\partial a^j_\sigma} \quad\text{and}\quad \frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + \sum_{j=1}^{m}\sum_{|\sigma|\geqslant 0}\frac{\overleftarrow{\partial}}{\partial a^j_\sigma}\cdot a^j_{\sigma\cup\{i\}}$$

for $1\leqslant i\leqslant n$. When subjected to close scrutiny, both the operators $\overleftarrow{D}_{x^i}$ and $\overrightarrow{D}_{x^i}$ show up, first, through the substrate part $\mathbf{1}\cdot\partial/\partial x^i$ plus the m sums –formally, infinite– of the cyclic words such that the derivations $\partial/\partial a^j_\sigma$ sit in their locks. The Leibniz-rule action of each term in such operator on another cyclic word that contains the respective jet letters a^j_σ is again a topological pair of pants $\mathbb{S}^1\times\mathbb{S}^1\to\mathbb{S}^1$.

Similarly, one could now think of the *variational covectors* $(\boldsymbol{p}\circ\delta\boldsymbol{a}) = \bigl(p_\alpha(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1},[a])\circ\delta a^\alpha\bigr)$ on $J^\infty(M^n_{\mathrm{nC}}\to\mathcal{A})$ as of (the formal sums of) necklaces equipped with the extra earrings δa^α, by which those cyclic words are handled.

We emphasize that, unlike it is the case studied in §1.1 –the cyclic words in $\mathcal{A}$ do not carry any marked point,– the earrings $\partial/\partial\boldsymbol{a}_\sigma$ and $\delta\boldsymbol{a}$ are the only places where the (co)vectors can be unlocked. Let us establish an immediate implication of this principle; very helpful, it remains valid in the purely commutative set-up.

Theorem 3 (The Substitution Principle)**.** *Suppose that a tuple of identities*

$$\boldsymbol{I}\bigl((\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}),[\boldsymbol{a}],\bigl[\boldsymbol{p}_1(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})\bigr],\ldots,\bigl[\boldsymbol{p}_k(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})\bigr]\bigr)\equiv 0$$

holds on $J^\infty(M^n_{\mathrm{nC}}\to\mathcal{A})$ *for every* k*-tuple of noncommutative variational* (*co*)*vectors the coefficients* $\boldsymbol{p}_{i,\alpha}(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})$ *of which can depend only on points* $\boldsymbol{x}\in M^n$ *and letters from the edge alphabet* $\vec{\mathrm{x}}^{\pm1}$*. Then the identities*

$$\boldsymbol{I}\bigl((\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}),[\boldsymbol{a}],\bigl[\boldsymbol{p}_1\bigl((\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}),[\boldsymbol{a}]\bigr)\bigr],\ldots,\bigl[\boldsymbol{p}_k\bigl((\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}),[\boldsymbol{a}]\bigr)\bigr]\bigr)\equiv 0$$

in total derivatives (8) *with respect to* $\boldsymbol{p}_i$ *are valid on* $J^\infty(M^n_{\mathrm{nC}}\to\mathcal{A})$ *for all* (*local*) (*co*)*vectors* $\boldsymbol{p}_i$ *depending not only on* $\boldsymbol{x}$ *and* $\vec{\mathrm{x}}^{\pm1}$ *but also endowed with arbitrary, finite differential order dependence on the jet variables* $\boldsymbol{a}_\sigma$*,* $|\sigma|<\infty$*.*

Remark 1.5. At this moment it is legitimate to view the variational (co)vectors $\boldsymbol{p}_i = (p_{i,\alpha}\circ\delta a^\alpha)$ as bare collections of their indexed open-word components $p_{i,\alpha}$ that are already built into the identities $\boldsymbol{I}$. The geometric mechanism telling *how* the variational (co)vectors got there will be revealed gradually in what follows (see footnote 33 on p. 33).

Corollary 4. If, under the assumptions of Theorem 3, the identities $\boldsymbol{I}\bigl((\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}),[\boldsymbol{a}],[\boldsymbol{p}_i]\bigr)\equiv 0$ in total derivatives with respect to $\boldsymbol{p}_1$, …, $\boldsymbol{p}_k$ hold on $J^\infty(M^n_{\mathrm{nC}}\to\mathcal{A})$ for every k-tuple of *exact* variational covectors $\boldsymbol{p}_i = (\delta\mathcal{H}_i/\delta\boldsymbol{a}\circ\delta\boldsymbol{a})$ which are obtained by variation of arbitrary linear integral functionals $\mathcal{H}\in\bar{H}^n(M^n_{\mathrm{nC}}\to\mathcal{A})$, then these identities hold for *all* covectors $\boldsymbol{p}_i$, i.e., not necessarily exact.

Indeed, it is always possible to represent locally an $(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})$-dependent cyclic word $\sum_{\alpha=1}^m\bigl(p_{i,\alpha}(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})\circ\delta a^\alpha\bigr)$ as the variation $\delta\mathcal{H}$ of the functional $\sum_{j=1}^m\int\bigl(p_{i,\alpha}(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})\circ a^\alpha\,\mathrm{dvol}(\boldsymbol{x})\bigr)$ and then apply Theorem 3.

Proof of Theorem 3. For the sake of brevity, let each variational noncommutative *co*vector $\boldsymbol{p}_i$ consist of just one word written in the alphabet of $J^\infty(M^n_{\mathrm{nC}}\to\mathcal{A})$. The crucial idea is that the position of the locks $\delta\boldsymbol{a}$ is fixed on the circles which carry the words $\boldsymbol{p}_i$.

This means that, whenever one declares an arbitrary differential dependence of $\boldsymbol{p}_i$ on $\boldsymbol{a}$, the words $\boldsymbol{I}$ in principle lengthen but still, in the course of multiplications $\times$ within the identities, each $\boldsymbol{p}_i$ is never torn in between any consecutive pair of letters $\boldsymbol{a}$. Namely, during the evaluation of $\boldsymbol{I}$ at the words $\boldsymbol{p}_i$ those are unlocked, the letters and the words' overall coefficients depending on $\boldsymbol{x}$ are then stretched to open strings (ordered counterclockwise). These strings are pasted into $\boldsymbol{I}$ without splitting, i.e., the adjacent letters of $\boldsymbol{p}_i$ never become separated by any other symbols.[16]

Total derivatives (8) then work according to their definition: under a restriction of $\boldsymbol{I}$ (hence of all $\boldsymbol{p}_i$) to the jet of a mapping $\boldsymbol{a} = \boldsymbol{s}(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm})$, each symbol a^j is replaced with the respective sum of open strings $s^j(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})$ so that derivations (8) in $\boldsymbol{a}_\sigma$ occurring anywhere (either in $\boldsymbol{p}_i$ or in $\boldsymbol{I}$ if the identities explicitly depend on $[\boldsymbol{a}]$) then reduce to the derivations $\partial/\partial x^i$ of real-valued functions defined at $\boldsymbol{x} \in U \subseteq M^n$. By the initial assumption of the theorem, its assertion is valid for all strings written in the basic alphabet $(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})$ that replace[17] the entries $\boldsymbol{p}_i$ in $\boldsymbol{I}$. Hence we conclude that the identities $\boldsymbol{I} \equiv 0$ hold on $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A})$ for the full set of arguments of the (co)vectors. □

Remark 1.6. The proof remains literally valid in the case of (evolutionary) vector fields instead of variational *co*vectors. This would be important for the description of variational noncommutative symplectic structures. However, the proof reveals *why* this noncommutative phrasing of the Substitution Principle does *not* hold for arbitrary cyclic words $\boldsymbol{p}_i\big((\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}), [\boldsymbol{a}]\big)$ of unspecified nature.

[16]This scenario is realised irrespectively of presence or absence of letters $\boldsymbol{a}$'s on the necklaces $\boldsymbol{p}_i$, which is in contrast with formula (4).

[17]One does not even have to postulate that the mappings $\boldsymbol{a} = \boldsymbol{s}(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})$ inserted in the explicit dependence of $\boldsymbol{I}$ on $[\boldsymbol{a}]$ coincide with the mappings now standing for $\boldsymbol{a}$ in the implicit dependence $\big[\boldsymbol{p}_i\big((\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}), [\boldsymbol{a}]\big)\big]$.

2. Differential graded Lie algebra of noncommutative local functionals

2.1. The variational symplectic dual. We now extend the alphabet $a^1, \ldots, a^m$ of the associative algebra $\mathsf{Free}_{\Bbbk}(a^1, \ldots, a^m)$ which we started with. Namely, we introduce the new symbols $a_1^\dagger, \ldots, a_m^\dagger$ that *ought to be* the canonical conjugates of the respective variables $a^1, \ldots, a^m$; let us explain what this means by viewing their construction from the four different perspectives (e.g., by putting these new symbols in context of Schwinger–Dyson's equation in Batalin–Vilkovisky formalism — or by tracking their origin in the (non)commutative variational Poisson geometry).

First, let us consider the free associative algebra standing alone, that is, *before* the evaluation of generators by (7) under a given map $\boldsymbol{s}\colon M^n_{\text{nC}} \to \mathsf{Free}\,(a^1, \ldots, a^m)$. In this set-up, there still remain two ways to understand the nature of new generators $a_i^\dagger$, namely, the coarse and fine. The former is to proclaim that the vector space $V^\dagger := \text{span}_{\Bbbk}(a_1^\dagger, \ldots, a_m^\dagger)$ is dual to the linear span $V := \text{span}_{\Bbbk}(a^1, \ldots, a^m)$ under the $\Bbbk$-valued coupling; by construction, the elements $a_i^\dagger$ specify the basis dual to that of a^i in V. The new letters are then incorporated into the set of generators of (the unital extention of) the associative algebra $\Bbbk \cdot \mathbf{1} \oplus \mathsf{Free}_{\Bbbk}(a^1, \ldots, a^m; a_1^\dagger, \ldots, a_m^\dagger)$. This definition is sufficient (which is explained in Chapter 3) to make the noncommutative variational Poisson formalism work.

The fine approach is as follows; although less is required, it is still enough to construct the (non)commutative Batalin–Vilkovisky geometry. Suppose that the generators a^i of the free associative algebra are subjected to a virtual shift $\delta\boldsymbol{a} = \delta a^i \cdot \vec{e}_i$, where the m vectors $\vec{e}_i$ constitute the adapted[18] basis in $T_{\boldsymbol{a}}V$, each of them pointing along the respective generator in the vector space $V = \text{span}_{\Bbbk}(a^1, \ldots, a^m)$. Likewise, consider the adapted basis $\vec{e}^{\,\dagger,i}$ in the tangent space $T_{\boldsymbol{a}^\dagger}V^\dagger$ at the point $\boldsymbol{a}^\dagger$ of the vector space $V^\dagger = \text{span}_{\Bbbk}(a_1^\dagger, \ldots, a_m^\dagger)$. We require that the frame $\vec{e}^{\,\dagger,i}$ be $\Bbbk$-dual to the frame $\vec{e}_i$, $1 \leqslant i \leqslant m$, so that the virtual variation $\delta\boldsymbol{a}^\dagger = \delta a_i^\dagger \cdot \vec{e}^{\,\dagger,i}$ is canonical conjugate to the diagonal deformation $\delta\boldsymbol{a} = \delta a^i \cdot \vec{e}_i$, see (10) and (11) below.

Remark 2.1. In the second approach, we do not proclaim that the new symbols $a_i^\dagger$ are the duals of the old generators a^i (or their inverses, or *re*verses, cf. (9)). In other words, we do not use the isomorphism between the vector space $V^\dagger = \text{span}_{\Bbbk}(a_1^\dagger, \ldots, a_m^\dagger)$ and the vector space $T_{\boldsymbol{a}^\dagger}V^\dagger$ tangent to it at a point. Note that the left-hand side of the isomorphism $V^\dagger \simeq T_{\boldsymbol{a}^\dagger}V^\dagger$ exploits the *global* vector-space organisation of $V^\dagger$ whereas the right-hand side refers to its *local* portrait near the point $\boldsymbol{a}^\dagger$. This is what the Batalin–Vilkovisky and Poisson formalisms really need.

Next, let us recall that the algebra generated now by the symbols $a^1, \ldots, a^m; a_1^\dagger, \ldots, a_m^\dagger$ is the target space for maps from the (locally constant) sheaf M^n_{nC} that provides

[18] In other words, only the *diagonal* deformations of the associative algebra generators are now allowed. This should be expected; for in the commutative BV-geometry, the variables a^i and $b_i = \Pi(a_i^\dagger)$, see below, describe the conjugate field-antifield or ghost-antighost pairs that stem from the different generations of Noether's identities between the Euler–Lagrange equations of motion. Hence by construction, the variables a^i or b_i at different values of the index i are fibre coordinates in different vector bundles, merged later to their Whitney sum (see [18, §§2, 6, 11] or [23] and references therein).

the alphabet $\vec{\mathrm{x}}^{\pm1}$ in a chart $U \subseteq M^n$ containing a point $\boldsymbol{x}$. We now discuss the three admissible scenarios of extending the mappings $\boldsymbol{s}$ in (7) to the new set-up.[19] The guiding principle that we keep in mind is that the Schwinger–Dyson condition, which is imposed in the Batalin–Vilkovisky picture but which is devised to constrain the objects already defined, makes the Feynman path integrals of local functionals, the geometry of which will be defined later in this text, effectively independent of any actual values of the symbols $a_i^\dagger$ (or their parity-odd descendants b_i and their derivatives with respect to $\boldsymbol{x}$). That is, the objects $a_i^\dagger$ as elements of the target set for maps from M^n_{nC} could acquire *whatever* values; indeed, no physics depends on the mapping's part that hits them. If so, leaving the symbols $a_i^\dagger$ unspecified would be the first option. To reduce the number of essences, we could let the mapping $M^n_{\mathrm{nC}} \to \big(\Bbbk\cdot\mathbf{1}\oplus \mathsf{Free}_{\Bbbk}(a^1,\ldots,a^m;a_1^\dagger,\ldots,a_m^\dagger)\big)/\sim$ be *not onto* but hit only the (unital) half generated by the symbols a^i.

However, we are also free to assign the values $\boldsymbol{a} = \boldsymbol{s}(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})$ and $\boldsymbol{a}^\dagger = \boldsymbol{s}^\dagger(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})$ in a way we choose. Hence the third option is to set the components of $\boldsymbol{s}^\dagger$ equal by definition to the sum of formal *re*verses for each nonzero, homogeneous words in $\boldsymbol{s}$,

$$a_i^\dagger := s_i^\dagger(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}) = \sum_J \frac{1}{f^{i,J}(\boldsymbol{x})}\,\vec{\mathrm{x}}_{j_\lambda}^{-\alpha(\lambda)}\circ\ldots\circ\vec{\mathrm{x}}_{j_1}^{-\alpha(1)} \tag{9}$$

for

$$a^i = s^i(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1}) = \sum_J f^{i,J}(\boldsymbol{x})\,\vec{\mathrm{x}}_{j_1}^{\alpha(1)}\circ\ldots\circ\vec{\mathrm{x}}_{j_\lambda}^{\alpha(\lambda)}, \qquad f^{i,J}\not\equiv 0,$$

where, at every point $\boldsymbol{x}\in U\subseteq M^n$, the sum is taken over the indexes J such that the coefficients $f^{i,J}$do not vanish.[20]

Example 2.1. If

$$a^i = \sum\nolimits_{k\in\mathbb{Z}}(\mathrm{loop})^k, \qquad \text{then} \qquad a_i^\dagger = \sum\nolimits_{k\in\mathbb{Z}}(\mathrm{loop})^{-k},$$

that is, all the reiterations of a closed path are walked backwards.

Convention (9) means that, whenever each component s^i of the map $\boldsymbol{s}$ is just a single word, the respective dual $a_i^\dagger$ becomes the weighted *re*verse – and true *in*verse – of the path $a^i(\boldsymbol{x},\vec{\mathrm{x}}^{\pm1})$.

2.2. **Elementary (non)commutative variations.** The precedence $\vec{e}_1 \prec \ldots \prec \vec{e}_m \prec \vec{e}^{\dagger,1} \prec \ldots \prec \vec{e}^{\dagger,m}$ of the basic vectors for virtual shifts endows the Cartesian sum $T_{\boldsymbol{a}}\,\mathrm{span}(a^1,\ldots,a^m)\mathbin{\widehat{\oplus}}T_{\boldsymbol{a}^\dagger}\,\mathrm{span}(a_1^\dagger,\ldots,a_m^\dagger)$ of the *dual* spaces with an orientation; it fixes the signs in all the structures of (non)commutative symplectic geometry. The signs show up through the two couplings $T_{\boldsymbol{a}}V\times T_{\boldsymbol{a}^\dagger}V^\dagger\to\Bbbk$ and $T_{\boldsymbol{a}^\dagger}V^\dagger\times T_{\boldsymbol{a}}V\to\Bbbk$

[19]The *fourth* scenario is specific to the (non)commutative variational Poisson formalism, in the frames of which the symbols $\boldsymbol{a}^\dagger$ play the rôles of placeholders for the variational covectors that are not exact; but still, the isomorphism $V^\dagger\simeq T_{\boldsymbol{a}^\dagger}V^\dagger$ is explicitly used in the assignment $\boldsymbol{a}^\dagger := \boldsymbol{p}$ (we shall discuss this in Chapter 3).

[20]In view of what has been said before, the fact that the extension $\boldsymbol{s}^\dagger$ remains undefined at the zero locus of all these coefficients makes no harm.

(which we denote by $\langle\,,\,\rangle$ in both cases, making no confusion; for the sequential order is essential). Namely, we have that

$$\big\langle \vec{e}_i, \vec{e}^{\,\dagger,j}\big\rangle = \boldsymbol{\delta}_i{}^j \quad\text{and}\quad \big\langle \vec{e}^{\,\dagger,j}, \vec{e}_i\big\rangle = -\boldsymbol{\delta}_i{}^j, \tag{10}$$

where $\boldsymbol{\delta}_i{}^j$ is the Kronecker symbol that equals unit iff $i=j$ and which is set equal to zero otherwise, see [19, §2.2].

Note that the virtual deformations $\delta\boldsymbol{a} = \delta a^i(\boldsymbol{x})\cdot\vec{e}_i(\boldsymbol{x})$ and $\delta\boldsymbol{a}^\dagger = \delta a^\dagger_j(\boldsymbol{x})\cdot\vec{e}^{\,\dagger,j}(\boldsymbol{x})$ can be dependent on $\boldsymbol{x}\in M^n$ — and they should even be such. We let the shifts be *normalised* at all $\boldsymbol{x}\in\operatorname{supp}(\delta a^i)\subseteq M^n$ by the constraint

$$\delta a^i(\boldsymbol{x})\cdot\delta a^\dagger_i(\boldsymbol{x})\equiv 1. \qquad \text{(no summation!)}$$

This is why the couplings of virtual deformations are invisible in the ready-to-use formulae. Indeed, it is enough to know the signs

$$\langle \delta a^i(\boldsymbol{x})\cdot\vec{e}_i(\boldsymbol{x}),\ \delta a^\dagger_i(\boldsymbol{y})\cdot\vec{e}^{\,\dagger,i}(\boldsymbol{y})\rangle\Big|_{\boldsymbol{x}=\boldsymbol{y}} = +1 \tag{11a}$$

and

$$\langle \delta a^\dagger_i(\boldsymbol{y})\cdot\vec{e}^{\,\dagger,i}(\boldsymbol{y}),\ \delta a^i(\boldsymbol{x})\cdot\vec{e}_i(\boldsymbol{x})\rangle\Big|_{\boldsymbol{x}=\boldsymbol{y}} = -1, \tag{11b}$$

at all the internal points $\boldsymbol{x}$ of the support $\operatorname{supp}(\delta a^i)$, see [22] and [19] for illustrations.[21]

2.3. **Parity-odd neighbours $\boldsymbol{b}=\Pi(\boldsymbol{a}^\dagger)$.** From now on, let the set-up be $\mathbb{Z}_2$-graded by the function $|\cdot|$ that takes values in $\mathbb{Z}$ and determines the parity $(-)^{|\cdot|}$. In the cyclic world, the concept of $\mathbb{Z}_2$-grading works as follows:

$$\mathfrak{t}\,(\gamma_1\circ\ldots\circ\gamma_\lambda) = (-)^{|\gamma_1\circ\ldots\circ\gamma_{\lambda-1}|\cdot|\gamma_\lambda|}\gamma_\lambda\circ\gamma_1\circ\ldots\circ\gamma_{\lambda-1}. \tag{12}$$

That is, all the cyclic words containing parity-odd letters are equipped with the observation point $\boldsymbol{\infty}$ which is located between the last and first symbols with respect to the cyclic order. Whenever a graded letter γ_λ standing last in a closed string of symbols overtakes its predecessors, thus becoming the first in a row, it monitors the rest of the word from the point $\boldsymbol{\infty}$ and contributes to the exponent of $(-)$ with the product of gradings.

All the objects which have been considered in the preceding sections were parity-even, of proper grading 0. Let us relay the parity of symbols $a^\dagger_i$ by postulating that the new parity-odd variables carry the grading $+1$ (or *minus* one, or any other (un)conventional odd integer number). To keep track of the reversed parity, let us denote[22] these generators by $\boldsymbol{b}=(b_1,\ldots,b_m)$ so that $\Pi\colon a^\dagger_i \rightleftarrows b_i$. Likewise, we denote by $\mathcal{A}^{(0|1)}$ the graded commutative unital non-associative algebra of cyclic words written in the alphabet $\mathbf{1}$, $a^1,\ldots,a^m,b_1,\ldots,b_m$.

[21]The usefullness of carrying the coefficients $\delta\boldsymbol{a}(\,\cdot\,)$ and $\delta\boldsymbol{a}^\dagger(\,\cdot\,)$ all way long is revealed in the geometry of iterated variations; let us also remember that we shall not always indicate the dependence of frames $\vec{e}_i(\,\cdot\,)$ and $\vec{e}^{\,\dagger,i}(\,\cdot\,)$ on points of substrate manifold M^n. However, the fact that such dependence is not impossible is crucial for the consistency of the formalism.

[22]Note that the new rule of arithmetic (12) does not modify our earliest convention (9) for the evaluation of symbols — as soon as a calculation governed by such graded arithmetic rule is over. Note also that the parity reversion Π does not modify the topology of spaces, whence conventions (11) remain valid for the virtual variations $\delta\boldsymbol{b}=\delta b_i(\boldsymbol{x})\cdot\vec{e}^{\,\dagger,i}(\boldsymbol{x})$.

Remark 2.2 ("$(abab) = 0\,?$"). The idea that cyclic words acquire and accumulate the extra sign factors, whenever a parity-odd symbol overtakes the rest of the word by passing through the circle's observation point $\boldsymbol{\infty}$, creates the following subtlety.

Set $m = 1$ for definition and, omitting the symbols $\circ$ of associative multiplication, first consider the cyclic word $(abaab)$. The identical, parity-odd letters b contained in it can be distinguished nevertheless: one of them is *followed* by aa but preceded only by a, whereas the other is *preceded* by aa and followed by just a single copy of letter a; we have that $(abaab) \sim -(aabab)$.

On the other hand, the cyclic word $(abab)$ does not contain any mechanism to distinguish between the two parity-odd entries b, yet $(\underline{ab}ab) \sim -(ab\underline{ab})$ by construction. In fact, this word is synonymic to zero in the algebra of cyclic words which are written in the parity-extended alphabet. It will also be readily seen that both the Batalin–Vilkovisky Laplacian of such words – or the Schouten bracket taken for such words with any other cyclic-word functional – vanish identically.

Let us be aware of the existence of this class of synonyms for zero; the calculus of iterated variations which we develop in Ch. 2 is indifferent to these synonyms' presence. (Arising in the Batalin–Vilkovisky formalism in retrospect, the Schwinger–Dyson condition neutralises the idea of evaluating such dual symbols by a use of the extentions $\boldsymbol{s}^\dagger(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1})$ for the initially defined maps $\boldsymbol{s}\colon M^n_{\text{nC}} \to \mathcal{A}$.) However, when the time comes in Ch. 3 to view integral functionals as well-defined totally antisymmetric maps of k-tuples of variational covectors – but not as maps from the sheaf M^n_{nC} to the algebra of cyclic words written using $\vec{\mathbf{x}}^{\pm 1}$ and weighted at every $\boldsymbol{x} \in M^n$, – then we restrict ourself to the study of spaces of (non)commutative variational *multivectors* only, regarding the parity-odd slots $\boldsymbol{b}$ as those covectors' placeholders.

Remark 2.3 (Comparison with the BV-geometry). We extended the alphabet of (the quotient of the unital extension for) the free associative algebra that serves as the target space for maps from the (locally constant) sheaf M^n_{nC} over the substrate manifold M^n. Let us summarise this picture in Fig. 4(a), in which one easily recognises the noncommutative generalisation of the classical Batalin–Vilkovisky geometry (see Fig. 4(b)).

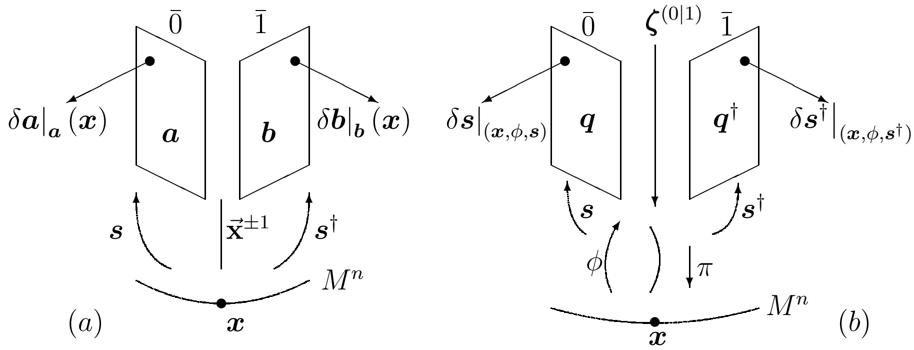

FIGURE 4. The elementaty displacements $\vec{\mathbf{x}}^{\pm 1}$ in a tiling of M^n versus the gauge connection fields ϕ over the space-time M^n; the canonical duality of diagonal variations for the opposite-parity halves of the alphabet versus the opposite-parity field-antifield and ghost-antighost pairs.

The rôle of physical fields ϕ as sections of their bundle π is now played by the primitive displacements $\vec{\mathbf{x}}^{\pm 1}$ in granulated space, cf. [24, §3.1]. The target algebra generated by the symbols a^i and b_i was known to us before as the Whitney sum of parity-even and odd components in the Batalin–Vilkovisky superbundle $\boldsymbol{\zeta}^{(0|1)}$, pulled back – by the projection π – over the total space of the bundle of physical fields. The symbols $\boldsymbol{a}$ and $\boldsymbol{b} = \Pi(\boldsymbol{a}^\dagger)$ of opposite parities form the noncommutative analogue of the BV-zoo $\boldsymbol{q}$, $\boldsymbol{q}^\dagger$ inhabited by the (anti)fields and (anti)ghosts. The rôle of the BV-bundle's sections is granted to the two maps $\boldsymbol{s}$ and $\boldsymbol{s}^\dagger$; the latter, see (9) above, has been studied in the context of Schwinger–Dyson's equation.[23]

2.4. **The ring of noncommutative local functionals.** Let us proceed from *functions* on the space $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ of jets of maps (9) to the notion of *functionals* that take such mappings $(\boldsymbol{s}, \boldsymbol{s}^\dagger)$ to formal cyclic words[24] written in the alphabet $\vec{\mathbf{x}}^{\pm 1}$ of edges in the adjacency graph for a given crystal tiling of the substrate manifold M^n.

Remark 2.4. On the infinite jet space $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ for maps from the sheaf over $M^n \ni \boldsymbol{x}$ to the quotient $\mathcal{A}^{(0|1)}$ of $\mathbb{Z}_2$-graded associative algebra, one could use the full alphabet $\boldsymbol{x}$, $\vec{\mathbf{x}}^{\pm 1}$, $\boldsymbol{a}_\sigma$, $\boldsymbol{b}_\tau$ when writing the cyclic words (those, in turn, can be equipped with an extra structure $\text{dvol}(\boldsymbol{x})$, see Remark 2.5 below). Every such object is the sum of its homogeneous components, each weighted by the coefficients that (can) depend on points $\boldsymbol{x}$ of the substrate manifold M^n. For the sake of definition, let us assume that every such coefficient is C^∞-smooth on M^n; their asymptotic behaviour must also be specified in advance so that the integration by parts makes sense.

Specifically, if the manifold M^n is closed, then there is nothing to discuss: the empty boundary carries no boundary terms. However, should there be one, $\partial M^n \neq \varnothing$, or should the manifold M^n be non-compact (e.g., let $M^n = \mathbb{R}^n$ with the standard Euclidean topology), then we postulate that the coefficients decay rapidly towards the boundary ∂M^n or spatial infinity, respectively.

Likewise, we suppose that the supports $\operatorname{supp} \delta a^i$ of the $C^\infty(M^n)$-smooth infinitesimal variations $\delta a^i(\cdot) \cdot \vec{e}_i(\cdot) \colon M^n \to T_{\boldsymbol{a}} \operatorname{span}(a^1, \ldots, a^m)$ are compact.

Remark 2.5. The *volume element* $\text{dvol}(\boldsymbol{x})$ on M^n in the construction of jet space $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ is an important ingredient in the notion of integral functionals. We suppose that a volume element $\text{dvol}(\boldsymbol{x})$ is given at all points $\boldsymbol{x} \in M^n$ (possibly, in a way that depends on the tiling at hand). Also, we technically assume that the volume element $\text{dvol}(\boldsymbol{x})$ may not depend on a choice of the mappings $(\boldsymbol{s}, \boldsymbol{s}^\dagger)$ — that is, in a sense, on a configuration of noncommutative "fields" over the granulation M^n_{nC} of the physical space M^n.

[23]We recall from [19] that the normalised variations $\delta\boldsymbol{s}$ and $\delta\boldsymbol{s}^\dagger$ were the dual components in sections of the tangent bundle $\mathbb{T}\boldsymbol{\zeta}^{(0|1)}$; the vectors $\delta\boldsymbol{s}\,(\boldsymbol{x}, \phi(\boldsymbol{x}), \boldsymbol{s}(\boldsymbol{x}, \phi(\boldsymbol{x})))$ and $\delta\boldsymbol{s}^\dagger\,\bigl(\boldsymbol{x}, \phi(\boldsymbol{x}), \boldsymbol{s}^\dagger(\boldsymbol{x}, \phi(\boldsymbol{x}))\bigr)$ were attached at points of graphs of sections for the BV-superbundle induced over π. The construction of these test shifts was laborious indeed in the graded-commutative world. On the other hand, the noncommutative target spaces contain nothing else but the basic letters $\boldsymbol{a}$ and $\boldsymbol{b}$ that undergo the virtual deformations, so that the picture is simplified considerably.

[24]Such cyclic words are *formal* because (i) they could encode no realisable paths along the edges of the graph and (ii), although "cyclic" by construction, each homogeneous component of such words could not encode a *closed* walk, even if it did specify *some* walk along the edges.

One could think that the volume element $\mathrm{dvol}(\cdot)$ is placed in the locks of cyclic words; this idea is practical because, whenever any such word is unlocked, it is converted at once into a singular linear integral operator; the volume element then disappears, giving way to the attachment points' congruence mechanism through the locality of couplings (10) in (11).

From now on we focus on the class of *integral* functionals such as $F = \int f\bigl(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, [\boldsymbol{a}], [\boldsymbol{b}]\bigr) \circ \mathrm{dvol}(\boldsymbol{x})$, where the cyclic word $f \circ \mathrm{dvol}(\boldsymbol{x})$ marks an equivalence class modulo integrations by parts (no boundary terms!). By definition, the value of such integral functional at a given mapping $(\boldsymbol{s}, \boldsymbol{s}^\dagger)$ is

$$F(\boldsymbol{s}, \boldsymbol{s}^\dagger) \stackrel{\text{def}}{=} \int_{M^n} f\bigl(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, \mathrm{jet}_\infty(\boldsymbol{s}), \mathrm{jet}_\infty(\boldsymbol{s}^\dagger)\bigr) \circ \mathrm{dvol}(\boldsymbol{x}) \in \mathcal{X}(\vec{\mathbf{x}}^{\pm 1}); \tag{13}$$

the integral makes sense due to our earlier assumptions on the global choice of alphabet $\vec{\mathbf{x}}^{\pm 1}$ on the entire M^n (that is, the tiling $M^n = \bigcup_\alpha \overline{\Delta}_\alpha$ is not *quasi*crystal) and on the class of functional coefficients depending on $\boldsymbol{x}$, so that the (im)proper integral converges. The vector space of integral functionals will be denoted by $\bar{H}^n\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, to keep track of the target algebra of cyclic words (even though neither the letters a^i nor b_j show up in the functionals' values that belong to the space $\mathcal{X}(\vec{\mathbf{x}}^{\pm 1})$ of cyclic words written in the edge alphabet $\vec{\mathbf{x}}^{\pm 1}$).

Integral functionals $F_1, \ldots, F_\ell \in \bar{H}^n\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ are the building blocks in the *local* functionals such as $F_1 \times \ldots \times F_\ell \in \bar{H}^{n^{\otimes \ell}}\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$.

Definition 1. Let $F_1 = \int f_1\bigl(\boldsymbol{x}_1, [\boldsymbol{a}], [\boldsymbol{b}]\bigr) \circ \mathrm{dvol}(\boldsymbol{x}_1)$ and $F_2 = \int f_2\bigl(\boldsymbol{x}_2, [\boldsymbol{a}], [\boldsymbol{b}]\bigr) \circ \mathrm{dvol}(\boldsymbol{x}_2)$ be two linear integral functionals the densities of which do not depend explicitly on letters from the edge alphabet $\vec{\mathbf{x}}^{\pm 1}$. The *product*

$$F_1 \times F_2 = \iint f_1\big|_{(\boldsymbol{x}_1, [\boldsymbol{a}], [\boldsymbol{b}])} \times f_2\big|_{(\boldsymbol{x}_2, [\boldsymbol{a}], [\boldsymbol{b}])} \circ \mathrm{dvol}(\boldsymbol{x}_1) \cdot \mathrm{dvol}(\boldsymbol{x}_2) \in \bar{H}^{n^{\otimes 2}}\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$$

is the horizontal cohomology class of linear integral functionals over $\bigl(M^{n^{\otimes 2}}, \mathrm{dvol}(\,\cdot\,)^{\otimes 2}\bigr)$ such that their densities are equivalent to the product $f_1 \times f_2$ in $\mathcal{A}^{(0|1)}$.

Setting $\bar{H}^{n^{\otimes 0}}\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ equal to $\Bbbk \cdot (\mathbf{1})$ by definition, we extend the bi-linear operation $\times$ recursively from pairs of integral functionals to the multiplication of products of any nonnegative numbers of functionals. Because the operation $\times$ is not associative, there are the respective Catalan number ways to arrange the multiplications in $F_1 \times \ldots \times F_\ell$ by inserting the $\ell - 1$ balanced pairs of parentheses. We let the *default ordering* be lexicographic: $(\cdots(F_1 \times F_2) \times \ldots \times F_{\ell-1}) \times F_\ell$.

Denote by $\overline{\mathfrak{M}}^n\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ the $\mathbb{Z}_2$-graded commutative non-associative unital ring $\bigoplus_{\ell \geqslant 0} \bar{H}^{n^{\otimes \ell}}\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ of local functionals in the noncommutative set-up under study.

The evaluation of products $F_1 \times \ldots \times F_\ell$ of functionals at a given mapping $(\boldsymbol{s}, \boldsymbol{s}^\dagger)$ goes as follows; without loss of generality suppose $\ell = 2$. First, double $(\boldsymbol{s}, \boldsymbol{s}^\dagger) \mapsto (\boldsymbol{s}, \boldsymbol{s}^\dagger)^{\otimes 2}$ for the $\ell = 2$ copies of the substrate manifold M^n, and then integrate over $M^{n^{\otimes 2}}$ in the element of $\bar{H}^{n^{\otimes 2}}\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$.

Remark 2.6. It is readily seen that, generally speaking,

$$\big(F_1 \overset{\mathcal{A}^{(0|1)}}{\times} F_2\big)\,(\boldsymbol{s},\boldsymbol{s}^\dagger) \neq F_1(\boldsymbol{s},\boldsymbol{s}^\dagger) \overset{\mathcal{X}(\vec{\mathbf{x}}^{\pm 1})}{\times} F_2(\boldsymbol{s},\boldsymbol{s}^\dagger).$$

Namely, the first-step multiplication $f_1 \times f_2$ of the two densities, still referred to the respective copies of base manifold M^n, followed by the object's evaluation at $(\boldsymbol{s},\boldsymbol{s}^\dagger)^{\otimes 2}$ for a given mapping $M^n_{\text{nC}} \to \mathsf{Free}\,(a^1, \ldots, a^m; b_1, \ldots, b_m)$, does yield the cyclic word written in the alphabet $\vec{\mathbf{x}}^{\pm 1}$ but the definition of multiplicative structure *in* $\mathcal{A}^{(0|1)}$ makes that word not necessarily equal to the values' product in the commutative non-associative unital algebra $\mathcal{X}(\vec{\mathbf{x}}^{\pm 1})$ of such words. Indeed, the multiplication $\times$ in $\mathcal{A}^{(0|1)}$ unlocks the cyclic words in between the letters $\boldsymbol{a}$ or $\boldsymbol{b}$ that will later be evaluated at $(\boldsymbol{s},\boldsymbol{s}^\dagger)(\boldsymbol{x},\vec{\mathbf{x}}^{\pm 1})$ whereas the multiplication $\times$ in $\mathcal{X}(\vec{\mathbf{x}}^{\pm 1})$ unlocks the cyclic words, already evaluated at a given mapping $(\boldsymbol{s},\boldsymbol{s}^\dagger)(\boldsymbol{x},\vec{\mathbf{x}}^{\pm 1})$, in between *every* two consecutive symbols from the edge alphabet.[25]

Also, note that the multiplication $\times$ in $\mathcal{A}^{(0|1)}$ is $\mathbb{Z}_2$-graded commutative — whereas that grading is lost in the course of functionals' evaluation at the mappings $(\boldsymbol{s},\boldsymbol{s}^\dagger)$.

In the remaining part of this chapter we reveal the structure of differential (shifted-) graded Lie algebra on the $\mathbb{Z}_2$-graded commutative non-associative unital ring $\overline{\mathfrak{M}}^n\big(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\big)$ of local functionals.

2.5. **Elements of the geometric theory of variations.** For consistency, let us outline the key ideas in the geometry of iterated variations; we note however that an attempt to describe and motivate every detail of the picture would inevitably mean a verbatim reproduction of the text [19]. Yet at the same time, we recall that the concept of writing words in a given alphabet — associative but without reference to the commutativity — does contribute to the consistency of formalism in the (graded-)commutative set-up; the bonus we get there, it is the full matching of signs in formulae, whenever the sequential order in which the (co)vectors combine in couplings (10) is never broken.

The promised key points are as follows.

- The unlinking of a cyclic word, together with an intention to paste the open string of symbols contained in it into another word as an uninterrupted fragment, converts the (procedure of) insertion of that string into a singular linear integral operator.
- Such operators are singular because the restriction to the diagonal over substrate points in M^n is ensured by the ordered couplings (10) which are *not defined* off the diagonal $\boldsymbol{x} = \boldsymbol{y}$ in (11).
- The definitions of the Batalin–Vilkovisky Laplacian Δ and variational Schouten bracket $[\![\,,\,]\!]$ are operational, that is, every such definition is an algorithm for the on-the-diagonal reconfiguration of the couplings.

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[25]Otherwise speaking, the arithmetic of local functionals' values does indicate the existence of elementary paths a^i and $a_i^\dagger$ that one can neither single out as closed walks nor still grind into the smallest possible displacements $\vec{\mathrm{x}}_j^{\pm 1}$ in the granulated space M^n_{nC}.

- The right-to-left order in which the operators are accumulated and the lexicographic, left-to-right direction in which they act, variations occur, and differentials are taken are essential; the ordering may not be violated — lest the formalism becomes inconsistent and formulae contradict each other.
- The objects that are usually viewed in the calculus of variations as differential forms are either the volume element $\mathrm{dvol}(\boldsymbol{x})$ on the substrate manifold M^n or the dual bases $\vec{e}_i$, $\vec{e}^{\,\dagger,i}$ in the tangent spaces attached at the point $(\boldsymbol{a},\boldsymbol{b})$ of the target algebra (this is what its alphabet was doubled for). In both cases, the *orientation* uniquely determines the signs of the couplings by ordering the tangent vectors. This explains why such differential 1-forms anticommute.
- In the course of virtual variation of the symbols a^i_σ and $b_{j,\tau}$ by using[26]

$$(\delta a^i)\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{x}}\right)^{\sigma}(\boldsymbol{x})\cdot\vec{e}_i(\boldsymbol{x})\quad\text{and}\quad(\delta b_j)\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{x}}\right)^{\tau}(\boldsymbol{x})\cdot\vec{e}^{\,\dagger,j}(\boldsymbol{x}),$$

the responses of integral functionals are always expanded with respect to respective dual bases $\vec{e}^{\,\dagger,i}$ and $\vec{e}_i$. For instance, we obtain the singular linear integral operators

$$\overrightarrow{\delta\boldsymbol{a}}(\cdot)=\int_{M^n}\mathrm{d}\boldsymbol{y}\sum_{i=1}^{m}\sum_{|\sigma|\geqslant 0}(\delta a^i)\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{y}}\right)^{\sigma}(\boldsymbol{y})\cdot\big\langle\vec{e}_i(\boldsymbol{y}),\vec{e}^{\,\dagger,i}(\cdot)\big\rangle\,\frac{\overrightarrow{\partial}}{\partial a^i_\sigma}$$

and

$$\overrightarrow{\delta\boldsymbol{b}}(\cdot)=\int_{M^n}\mathrm{d}\boldsymbol{z}\sum_{j=1}^{m}\sum_{|\tau|\geqslant 0}(\delta b_j)\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{z}}\right)^{\tau}(\boldsymbol{z})\cdot\big\langle(-\vec{e}^{\,\dagger,j})(\boldsymbol{z}),\vec{e}_i(\cdot)\big\rangle\,\frac{\overrightarrow{\partial}}{\partial b_{j,\tau}}\,.$$

This convention will be illustrated in the sequel.
- Given by its own singular integral operator, each variation brings a new copy of the integration domain M^n into the picture. In consequence, all the intermediate objects that emerge in the course of calculations do retain a kind of memory of the way how they were obtained from the input data.[27] That is, no calculation can be interrupted along the way.
- In every calculation, the integrations by parts are performed last, prior only to the reconfigurations of couplings and their evaluation by using (11). For instance, the derivative $(\overleftarrow{\partial}/\partial\boldsymbol{y})^\sigma$ in the formula above channels through $\vec{e}_i(\boldsymbol{y})$ and $\vec{e}^{\,\dagger,i}(\boldsymbol{x})$ on the diagonal $\boldsymbol{y}=\boldsymbol{x}$ which is the only place where the coupling is defined; the derivative thus becomes $(-\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\sigma$ that falls on (a derivative of) the argument's density at $\boldsymbol{x}\in M^n$.

  This principle makes the variations (graded-)permutable.

[26]It is readily seen that the congruence of multi-indexes σ in $(\partial/\partial x)^\sigma$ and a^i_σ (as well as in the partial derivative $\vec{\partial}/\partial a^i_\sigma$, see below) refers to the definition of vector as an equivalence class of curves passing through a point.

[27]In the (graded-)commutative language of bundles this means that their products $\boldsymbol{\zeta}^{(0|1)}\times\mathbb{T}\boldsymbol{\zeta}^{(0|1)}\times\ldots\times\mathbb{T}\boldsymbol{\zeta}^{(0|1)}$, standing over $M^n\times M^n\times\ldots\times M^n$, are taken, but not their Whitney sums $\boldsymbol{\zeta}^{(0|1)}\times_{M^n}\mathbb{T}\boldsymbol{\zeta}^{(0|1)}\times_{M^n}\ldots\times_{M^n}\mathbb{T}\boldsymbol{\zeta}^{(0|1)}$ are fibred over a single copy of the base manifold M^n.

- By construction, *iterated* variations of a functional never spread from it to the fragments of other functionals in any composite object during multiple integrations by parts.

We refer to [19, 20, 22] for more details and illustrations of these guiding principles.[28]

2.6. **How the Batalin–Vilkovisky Laplacian determines the Schouten bracket.** We are now ready to outline the construction of parity-odd Batalin–Vilkovisky Laplacian Δ. On the space of local functionals over the jet space $J^\infty(M^n_{\text{nC}} \to A^{(0|1)})$ of maps, it is the parent structure for the noncommutative variational Schouten bracket $[\![\,,\,]\!]$. We establish the main properties of these structures, recalling further the relations between them.

Definition 2. The Batalin–Vilkovisky Laplacian is the reconfiguration −shown in Fig. 5− of (co)vector couplings in the second variation $\overrightarrow{\delta\boldsymbol{a}}(\overrightarrow{\delta\boldsymbol{b}}(\cdot))$ of a local functional on the jet space $J^\infty(M^n \to \mathcal{A}^{(0|1)})$ of maps.

$$\begin{array}{ccc} \langle{}^{(1)}\male \quad {}^{(2)}\female\rangle & & \langle{}^{(1)}\male \qquad {}^{(3)}\female\rangle \\ & \longmapsto & \\ \qquad\qquad \langle{}^{(3)}\female \quad {}^{(4)}\male\rangle & & \qquad \langle{}^{(2)}\female \qquad {}^{(4)}\male\rangle \end{array}$$

FIGURE 5. The on-the-diagonal reconfiguration of couplings is the operational definition of BV-Laplacian Δ.

First, let us consider an *integral* functional $F \in \bar{H}^n(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$. Let $\delta a^{i_1}(\boldsymbol{y}_1) \cdot \vec{e}_{i_1}(\boldsymbol{y}_1)$ and $\delta b_{i_2}(\boldsymbol{y}_2) \cdot \vec{e}^{\,\dagger,i_2}(\boldsymbol{y}_2)$ be a pair of test shifts of the parity-even and odd letters in the target alphabet; assume normalization (11). Construct the second variation

$$\begin{aligned} \overrightarrow{\delta\boldsymbol{a}}(\overrightarrow{\delta\boldsymbol{b}}(F)) = \iint\limits_{M^n} \mathrm{d}\boldsymbol{y}_1\, \mathrm{d}\boldsymbol{y}_2 \int \mathrm{dvol}(\boldsymbol{x}) \cdot \\ \cdot \Big\{ (\delta a^{i_1}) \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{y}_1}\right)^{\sigma_1} (\boldsymbol{y}_1) \cdot \Big\langle \vec{e}_{i_1}(\boldsymbol{y}_1) \Big| \vec{e}^{\,\dagger,i_1}(\boldsymbol{x}) \Big\rangle \frac{\overrightarrow{\partial}}{\partial a^{i_1}_{\sigma_1}} \circ \\ \circ (\delta b_{i_2}) \left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{y}_2}\right)^{\sigma_2} (\boldsymbol{y}_2) \cdot \Big\langle (-\vec{e}^{\,\dagger,i_2})(\boldsymbol{y}_2) \Big| \vec{e}_{i_2}(\boldsymbol{x}) \Big\rangle \frac{\overrightarrow{\partial}}{\partial b_{i_2,\sigma_2}} f(\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]) \Big\}. \end{aligned}$$

[28] Simple as they may look, these geometric rules substantiate $\gtrsim 10^4$ man-hours of research on BV-quantisation of gauge fields; the estimate is this: in the past 30 years, each member of the communities of $\gtrsim 30$ researchers in circa 30 countries invested annually $\gtrsim 30$ hours into the (manual regularisation of) calculations related to the Batalin–Vilkovisky geometry [2, 3].

At the end of a reasoning (of which the object ΔF could be only a small piece), the integrations by parts carry the derivatives off the virtual test shifts, which yields

$$\iint\limits_{M^n} \mathrm{d}\boldsymbol{y}_1\,\mathrm{d}\boldsymbol{y}_2 \int \mathrm{dvol}(\boldsymbol{x}) \cdot \Bigl\{ \delta a^{i_1}(\boldsymbol{y}_1) \cdot \bigl\langle \vec{e}_{i_1}(\boldsymbol{y}_1) \big| \vec{e}^{\dagger,i_1}(\boldsymbol{x}) \bigr\rangle \cdot \\ \cdot\, \delta b_{i_2}(\boldsymbol{y}_2) \cdot \bigl\langle (-\vec{e}^{\dagger,i_2})(\boldsymbol{y}_2) \big| \vec{e}_{i_2}(\boldsymbol{x}) \bigr\rangle \left(-\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma_1 \cup \sigma_2} \frac{\vec{\partial}^{\,2}}{\partial a^{i_1}_{\sigma_1} \partial b_{i_2,\sigma_2}} f(\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]) \Bigr\}.$$

Finally, the two pairs of couplings are reconfigured according to the scenario in Fig. 5, which gives the action of operator

$$\iint\limits_{M^n} \mathrm{d}\boldsymbol{y}_1\,\mathrm{d}\boldsymbol{y}_2 \left\{ \begin{matrix} \langle \delta a^{i_1}(\boldsymbol{y}_1) \vec{e}_{i_1}(\boldsymbol{y}_1)| & & |\delta b_{i_2}(\boldsymbol{y}_2) \cdot (-\vec{e}^{\dagger,i_2})(\boldsymbol{y}_2)\rangle & \\ & \langle \vec{e}^{\dagger,i_1}(\boldsymbol{x})| & & |\vec{e}_{i_2}(\boldsymbol{x})\rangle \end{matrix} \right\}$$

on the basic (co)vectors at $\boldsymbol{x} \in M^n$. The couplings wright the diagonal $i_1 = i_2$ in the summation over the indexes. Normalization (11) and the couplings' values (10) make each line in the formula above equal to -1; their product equals unit.

Corollary 5. In particular, this gives us the integrand of ΔF whenever this object *is* the endpoint of a reasoning; namely, we obtain

$$\sum_{i=1}^{m} \sum_{\substack{|\sigma_1| \geqslant 0 \\ |\sigma_2| \geqslant 0}} \left(-\frac{\vec{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}} \right)^{\sigma_1 \cup \sigma_2} \left(\frac{\vec{\partial}^{\,2}}{\partial a^{i}_{\sigma_1} \partial b_{i,\sigma_2}} f \right) (\boldsymbol{x}, [\boldsymbol{a}], [\boldsymbol{b}]).$$

We emphasize that, should the object ΔF itself be a constituent element of a larger expression, other partial derivatives $\vec{\partial}/\partial a^{j_1}_{\tau_1}$ or $\vec{\partial}/\partial b_{j_2,\tau_2}$ could accumulate at the given density f of the functional F, whereas all the powers of minus the total derivatives would then gather outside those higher-order partial derivatives.

Let F, G, and H be homogeneous integral functionals on $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$, of respective gradings $|F|$, $|G|$, and $|H|$, cf. Definition 1 on p. 19.

Definition 3 ($\Delta(F \times G)$)**.** Whenever applied to the product $F \times G$ of two integral functionals, the Batalin–Vilkovisky Laplacian Δ, which was defined above as reconfiguration (cf. Fig. 5) of the (co)vector couplings, is the parent structure for the (non)commutative variational *Schouten bracket* $[\![\,,\,]\!]$, or *antibracket*,

$$\Delta(F \times G) \stackrel{\text{def}}{=} \Delta(F) \times G + (-)^{|F|} [\![F, G]\!] + (-)^{|F|} F \times \Delta G. \tag{14}$$

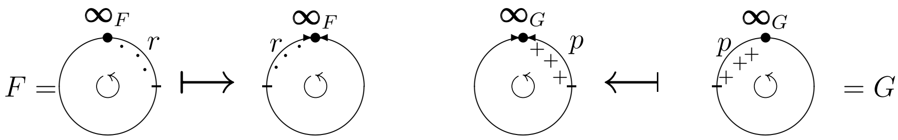
In other words, the bracket $[\![\,,\,]\!]$ measures the deviation for Δ from being a graded derivation.

Corollary 6. The (non)commutative variational Schouten bracket $[\![\,,\,]\!]$ itself is a (shifted-) graded derivation of the product $\times$ in the algebra of local functionals,

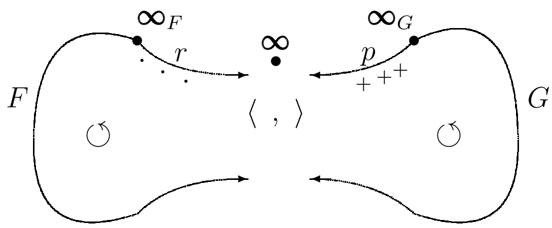
$$[\![F, G \times H]\!] = [\![F, G]\!] \times H + (-)^{(|F|-1)\cdot|G|} G \times [\![F, H]\!]. \tag{15}$$

The proof refers to a check of definitions.

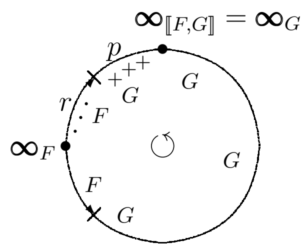
Remark 2.7 (the geometric realization of $[\![\,,\,]\!]$). The geometric construction of every term in the noncommutative variational Schouten bracket goes as follows. Without loss of generality, suppose that the given integral functionals F and G each consist of just a single cyclic word (otherwise, proceed by linearity).



First, rotate the necklace F counterclockwise until $r \geqslant 0$ parity-odd symbols would have passed through the lock $\boldsymbol{\infty}_F$; when a parity-even or the next, $(r+1)$th parity-odd symbol reaches $\boldsymbol{\infty}_F$, open that lock. Likewise, rotate the ring G clockwise and, as soon as $p \geqslant 0$ parity-odd symbols would have passed through $\boldsymbol{\infty}_G$, unlock G at a symbol the parity of which is opposite to that of the letter at which F was unlocked.



Second, place the loose ends of the two open words next to each other, preserving the orientation of the strings of symbols. Now integrate by parts, throwing the derivatives off the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ by letting them fall on the chains of letters from the words where the variations emerged from. Next, couple the variations $\delta\boldsymbol{a}$ and $\delta\boldsymbol{b}$ by using (11) and join the facing ends of the two strings, forming the new cyclic word that carries the orientation and sign factor from $\langle\, ,\, \rangle$.



Finally, rotate the letters around the new word counterclockwise so that the old location of $\boldsymbol{\infty}_G$ in between the symbols of G reaches the new linking $\boldsymbol{\infty}_{[\![F,G]\!]}$ of strings, nearest to $\boldsymbol{\infty}_G$ in the positive direction. The terminal configuration is displayed above; it carries $|F|+|G|-1$ parity-odd symbols, it preserves the orientation of both the input words F and G, and it carries the sign factor determined by the ordered coupling of (co)vectors.

Corollary 7. For a given homogeneous integral functional $F \in \bar{H}^n(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ of grading $|F|$, the operator $[\![F, \cdot\,]\!]$ proceeds over letters of its cyclic-word argument by the graded Leibniz rule; the operator's proper grading $|[\![F, \cdot\,]\!]|$ is $|F| = 1$.

Proposition 8. The (non)commutative variational Schouten bracket of homogeneous integral functionals is shifted-graded skew-symmetric:

$$[\![G, F]\!] = -(-)^{(|F|-1)\cdot(|G|-1)}[\![F, G]\!]$$

for $F, G \in \bar{H}^n(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$.

Theorem 9. *Let F, G, and H be homogeneous integral functionals on $J^\infty(M^n_{nC} \to \mathcal{A}^{(0|1)})$ so that their gradings are $|F|$, $|G|$, and $|H|$, respectively. Then the following three equivalent statements are valid*:

(i) *The noncommutative variational Schouten bracket satisfies the shifted-graded Jacobi identity*

$$(-)^{(|F|-1)\cdot(|H|-1)}[\![F, [\![G, H]\!]]\!] + (-)^{(|F|-1)\cdot(|G|-1)}[\![G, [\![H, F]\!]]\!] + \\ + (-)^{(|G|-1)\cdot(|H|-1)}[\![H, [\![F, G]\!]]\!] = 0.$$

(ii) *The Jacobi identity for the bracket $[\![\,,\,]\!]$ is the graded Leibniz rule for the operator $[\![F, \cdot\,]\!]$ acting on $[\![G, H]\!]$, namely,*

$$[\![F, [\![G, H]\!]]\!] = [\![[\![F, G]\!], H]\!] + (-)^{(|F|-1)\cdot(|G|-1)}[\![G, [\![F, H]\!]]\!]. \tag{16}$$

(iii) *The shifted-graded commutator of operators $[\![F, \cdot\,]\!]$ and $[\![G, \cdot\,]\!]$ is equal to the operator $[\![[\![F, G]\!] \cdot\,]\!]$, that is,*

$$[\![F, [\![G, \cdot\,]\!]]\!](H) - (-)^{(|F|-1)\cdot(|G|-1)}[\![G, [\![F, \cdot\,]\!]]\!](H) = [\![[\![F, G]\!], \cdot\,]\!](H). \tag{17}$$

Corollary 10. Let F and G be two noncommutative local functionals; suppose F is homogeneous. The Batalin–Vilkovisky Laplacian Δ satisfies the relation

$$\Delta\big([\![F, G]\!]\big) = [\![\Delta F, G]\!] + (-)^{|F|-1}[\![F, \Delta G]\!]. \tag{18}$$

In other words, the operator Δ is a graded derivation of the noncommutative variational Schouten bracket $[\![\,,\,]\!]$.

Corollary 11. The Batalin–Vilkovisky Laplacian Δ is a differential on the space of local functionals over $J^\infty(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$,

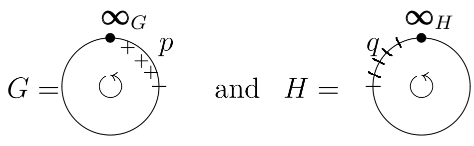
$$\Delta^2 = 0.$$

The proof of Corollaries 10 and 11 is straightforward, see [19] for the scheme of that reasoning.

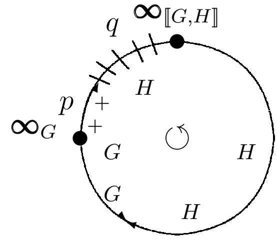
Proof of Theorem 9. Consider the consecutive action on the operators $[\![F \cdot\,]\!]$ and $[\![G \cdot\,]\!]$ of gradings $|F| = 1$ and $|G| = 1$, respectively, on an integral functional H. Each operator proceeds over letters in every cyclic word of H by the graded Leibniz rule. It is readily seen that by taking the shifted-graded *difference* of the two applications, as it stands in the left-hand side of (17), we cancel all the terms in which the strings of symbols from F and G are pasted into H not hitting each other (that is, rather staying next to each other or becoming separated by the argument's own letters). Therefore, both sides of (17) contain the *second* variation of F and G but only the *first* variation of H.

Note further that all the integrals by parts always fall only on letters that belong to (what remains of) the functional which is varied, see section 2.5. Consequently, both sides of (17) contain the same configurations of powers of total derivatives that fall on the letters from the second, second, and first variations of F, G, and H, respectively. This shows that it is sufficient to inspect the matching of signs — as they occur in the left- and right-hand side of (17) — in front of the insertions of symbols from F into G, and *vice versa*. Without loss of generality, let us suppose that each of the functionals F and G consist of just a single cyclic word.

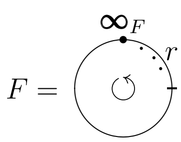
Every term in $[\![G,\,\cdot\,]\!](H)$ is obtained from the cyclic words



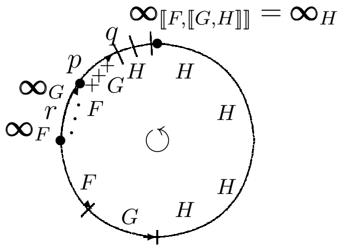
as follows (see Remark 2.7). First, the ring G is rotated counterclockwise, transporting p odd symbols through ∞_G, which gives the sign $(-)^{p\cdot(|G|-1)}$ and then G is unlocked at ∞_G. At the same time, H is rotated clockwise and unlocked as soon as q odd letters would have passed the lock ∞_H. Contracting one pair of variations $\delta\boldsymbol{a}, \delta\boldsymbol{b})$ destroys one parity-odd symbol in either G or H. Now, the word obtained from G by erasing one letter in it is pasted, orientation preserved, into the similarly obtained fragments of H. The loose ends of the two strings are joined, making a new circle. Finally, the q letters of H are pushed counterclockwise — so many of them that the old ∞_H coincides with $\infty_{[\![G,H]\!]}$, placed at the moment of linking at the concatenation of strings' loose ends nearest to ∞_H in positive direction. The sign factor which is gained when the lock of H is restored on its proper place equals $(-)^{q\cdot(|G|-1)}$; the *minus one* in the exponent counts the parity-odd letter destroyed by the coupling. The resulting necklace – a term in $[\![G,H]\!]$ – looks like this:



The total sign accumulated up to this moment is $(-)^{p\cdot(|G|-1)} \cdot (-)^{q\cdot(|G|-1)}$. Now the operator $[\![F,\,\cdot\,]\!]$ approaches that ring from the left. Arguing as above, we rotate the cyclic word



counterclockwise, letting r parity-odd symbols pass through ∞_F (this yields $(-)^{r\cdot(|F|-1)}$). Having unlocked that ring at ∞_F, we carry this term in $[\![F,\,\cdot\,]\!]$ of grading $|F|-1$ along the $p+q$ parity-odd symbols in the pre-fabricated linking of G and H. By the time the loose ends of $[\![F,\cdot]\!]$ reach the former location of ∞_G in G, the sign factor $(-)^{(p+q)\cdot(|F|-1)}$ is accumulated, and the configuration is this:

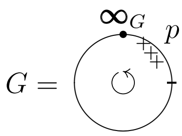


By having realised the scenario which the left-hand side of (17) provides, we obtained the overall sign

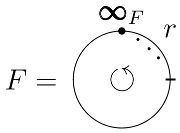
$$(-)^{r\cdot(|F|-1)}\cdot(-)^{p\cdot(|G|-1)}\cdot(-)^{q\cdot(|G|-1)}\cdot(-)^{(p+q)\cdot(|F|-1)} = (-)^{r\cdot(|F|-1)}\cdot(-)^{(p+q)\cdot(|F|+|G|-2)}. \quad (19)$$

Moreover, it is clear now what the extra sign contribution to the formula above would there be, should the insertion of the unlocked F start later – with respect to the cyclic order – than the starting point ∞_G of the turned-and-unlocked cyclic word G.

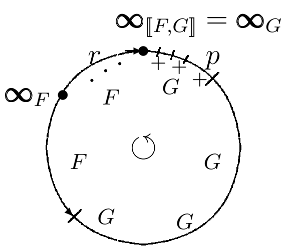
On the other hand, let us calculate the overall sign factor of the very same geometric configuration in the right-hand side of (17). So, we first produce the respective term in $[\![F,G]\!]$. Let us recall from the above that the word



is unlocked straight after ∞_G, but



is first rotated counterclockwise by r parity-odd slots; this yields the sign $(-)^{r\cdot(|F|-1)}$ and gives the word



It contains $|F|+|G|-1$ parity-odd letters; let us use it in the action of $[\![[\![F,G]\!],\cdot]\!]$ on H. By rotating the word to-paste counterclockwise by p parity-odd symbols, we gain the sign $(-)^{p\cdot(|F|+|G|-2)}$; proceeding by the Leibniz rule over q parity-odd letters in H, we obtain another sign factor $(-)^{q\cdot(|F|+|G|-2)}$. In total, the overall sign that occurs in the right-hand side of (17) for the configuration that we knew before is

$$(-)^{r\cdot(|F|-1)}\cdot(-)^{p\cdot(|F|+|G|-2)}\cdot(-)^{q\cdot(|F|+|G|-2)}.$$

This is exactly (19).

To process the configurations in which the symbols from G are pasted in between the letters of F, and those are already installed in H, let us first swap F and G in the right-hand side of (17). By Proposition 8, it becomes

$$-(-)^{(|F|-1)\cdot(|G|-1)}[\![[\![G,F]\!],\,\cdot\,]\!](H).$$

Second, multiply both sides of (17) by the sign factor $-(-)^{(|F|-1)\cdot(|G|-1)}$; this gives

$$-(-)^{(|F|-1)\cdot(|G|-1)}[\![F,[\![G,\,\cdot\,]\!]]\!](H)+[\![G,[\![F,\,\cdot\,]\!]]\!](H) \quad \text{versus} \quad [\![[\![G,F]\!],\,\cdot\,]\!](H).$$

Finally, relabel $F\rightleftarrows G$; by having thus recovered both sides of (17) in its authentic form, we convert the configurations to-consider into those which we did cope with. The proof is complete. □

Remark 2.8. We conclude that the proof of all these assertions about the Batalin–Vilkovisky Laplacian and variational Schouten bracket remains literally valid in the graded-commutative set-up. Indeed, when the proof is over, it suffices to let $N:=0$ and proclaim that the letters a^i_σ and $b_{j,\tau}$ are graded-permutable; the proof itself does not require that assumption.

Likewise, by shrinking the substrate manifold M^n to a point, so that $n=0$ and $N=0$, we recover the standard properties of the parity-odd differential $\Delta_0=\overrightarrow{\partial}^2/\partial a^i\partial b_i$ and parity-odd Poisson bracket in the (formal non)commutative geometry of symplectic supermanifolds of superdimension $(m|m)$. The locality of couplings (10) still in force, our reasoning explains why the differentials of *two* Hamiltonians are referred to *the same* point when their Poisson bracket is constructed.

3. Noncommutative variational Poisson formalism

The noncommutative variational cotangent superspace, which we built in Ch. 1 for spaces of maps $M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}$, and the calculus of local functionals on jet spaces $J^\infty(M^n_{\text{nC}} \to A^{(0|1)})$, see Ch. 2, refer to the canonical *symplectic* structure encoded by (10). Let us now introduce a more narrow class of variational noncommutative geometries in which the *Poisson* structures are defined.

3.1. **Noncommutative variational multivectors.** Let us recall that the notion of space of integral functionals $\bar{H}^n(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ was based in Ch. 2 on an obvious analytic idea to integrate the maps $\boldsymbol{s}\colon M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}$ over $\text{dvol}(\boldsymbol{x})$ on the substrate manifold M^n; the integrals take every such mapping to the cyclic word(s) written in the edge alphabet $\vec{\mathbf{x}}^{\pm 1}$ (see (13) on p. 19). When the $\mathbb{Z}_2$-valued parity function was introduced, the parity-odd symbols $\boldsymbol{b}$ and extension $\boldsymbol{s}^\dagger$ of $\boldsymbol{s}$ to maps landing in $\mathcal{A}^{(0|1)}$ were felt as the objects that make everything go much better as soon as one gets rid of them; we refer to Remark 2.2 in particular.

Taking this into account, let us describe a very different geometric approach to the use of $\mathbb{Z}_2$-parity graded noncommutative integral functionals. Namely, we propose to view the parity-odd symbols $\boldsymbol{b}$ and their derivatives as *placeholders* for (non)commutative variational covectors; such placeholders appear in the fully skew-symmetric poly-linear maps for the space $\bar{H}(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ of purely even Hamiltonian functionals. By making this construction precise, which forces us to narrow the class of graded-homogeneous functionals under study, we resolve the difficulty which is known from Remark 2.2.

The key idea is that – unlike it is the case for cyclic-word integral functionals of generic nature – the (non)commutative variational *multivectors* are organised in precisely the same way with respect to each parity-odd entry $\boldsymbol{b}$, as long as the shifts $\mathfrak{t}$ around the circle and integrations by parts are allowed.

Let $P \in \bar{H}^n(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ be a homogeneous functional of grading $|P| =: k \geqslant 0$. If $k = 0$, none of the cyclic words in P contains any parity-odd symbols b^i_τ. If $k = 1$, then there is the noncommutative linear total differential operator A (that is, an operator which is polynomial in the total derivatives and the coefficients of which are operators of left and right multiplication by functions of $\boldsymbol{x}$ or by parity-even symbols $\vec{\mathbf{x}}^{\pm 1}$ or a^i_σ from the alphabet on $J^\infty(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$) such that

$$P = \bigl(A(\boldsymbol{b})\bigr).$$

Clearly, there remains nothing more to do; for the key idea above is already realized.

Suppose now $k = 2$; pick *one* parity-odd letter in every cyclic word of P and throw all the derivations off every such letter by using a suitable number of integrations by parts; then, if necessary, transport the letters around the circle so that those $b^i_\varnothing$ stand immediately after the observation point $\boldsymbol{\infty}$ in the positive, counterclockwise direction. This brings P to the normal shape

$$P \cong \tfrac{1}{2}\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr) \ ; \tag{20}$$

by construction, A is the arising $(m \times m)$-size matrix linear noncommutative total differential operator of one argument.

Arguing as above and picking *some* parity-odd letter in every word of a given integral functional P of grading k, we transform it to the sum of cyclic words, each starting with $b^j_\varnothing$ for $1 \leqslant j \leqslant m$,

$$P \cong \frac{1}{k!}\bigl(\boldsymbol{b} \circ A(\underbrace{\boldsymbol{b}, \ldots, \boldsymbol{b}}_{k-1 \text{ slots}})\bigr), \tag{21}$$

where the noncommutative total differential operator A is poly-linear in its $k-1$ arguments.[29]

To make the construction of operator A independent of our initial choice of *some* parity-odd entries, let us analyse the properties such an operator must have. We now consider the case $k = 2$ because it will be essential in what follows. Through the chain of integrations by parts and by carrying the parity-odd letters around the circle,

$$P = \tfrac{1}{2}(\boldsymbol{b} \circ A(\boldsymbol{b})) \cong \tfrac{1}{2}\left((\boldsymbol{b})\overleftarrow{A}^{\dagger} \circ \boldsymbol{b}\right) \sim -\tfrac{1}{2}\left(\boldsymbol{b} \circ (\boldsymbol{b})\overleftarrow{A}^{\dagger}\right) \stackrel{\text{def}}{=} -\tfrac{1}{2}\left(\boldsymbol{b} \circ A^{\dagger}(\boldsymbol{b})\right), \tag{22}$$

we define the *adjoint operator* $A^\dagger$ that acts on its argument in the left-to-right direction.[30] The starting objects P and the resulting functional are identically the same if we require that

$$A = -A^\dagger. \tag{23}$$

For example, let $n = 1$, $m = 1$ and consider $P = \frac{1}{2}(b \circ b_x)$ with $A = \vec{\mathrm{d}}/\mathrm{d}x$, see [40].

The requirements which the poly-linear operator A of $k-1$ arguments must satisfy are imposed for all $k \geqslant 3$ in the exactly same way as in (22).

In what follows, we shall consider only the grading-homogeneous functionals on $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ for which the poly-linear operators A are well defined, so that normalisation (21) can be attained by starting from any parity-odd entry in every cyclic word of the functional at hand.

Definition 4. Homogeneous integral functionals $P \in \bar{H}^n(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ of grading $k \geqslant 0$ and such that either $k \leqslant 1$ or normalisation (21) is well defined are called *noncommutative variational k-vectors.*

Let us denote by $\bar{H}^n_k(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}) \subsetneq \bar{H}^n(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ the vector space of noncommutative variational k-vectors on $J^\infty(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$.

Note that by Remark 2.2, the subspaces $\bar{H}^n_k(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ do *not* exhaust the homogeneous components of grading k in $\bar{H}^n(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ for $k \geqslant 2$.

Remark 3.1. We claim that the vector space $\bigoplus_{k\geqslant 0} \bar{H}^n_k(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$ of all noncommutative variational multivectors is closed under $[\![\,,\,]\!]$, which endows it with the structure of Gerstenhaber algebra with respect to the noncommutative variational Schouten bracket.

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[29]Of course, the notation for A acting on the m-tuples $\boldsymbol{b}$ is symbolic; in reality, every cyclic word of P carries k parity-odd entries $b^{i_1}_\varnothing$, $b^{i_2}_{\sigma^{i_2}_2}$, $\ldots$, $b^{i_k}_{\sigma^{i_k}_k}$, where $1 \leqslant i_\alpha \leqslant m$ and the multi-indexes are word-dependent. It is often the case that $|\sigma^i_\alpha| \neq |\sigma^j_\alpha|$ for $i \neq j$ at some α; for instance, recall the differential order of entries in the matrix operator for the second Poisson structure of the renowned Boussinesq hierarchy.

[30]Note that the *left* multiplications in A become the right multiplications in $\overleftarrow{A}^{\dagger}$, and *vice versa.* At the same time, the total derivative operators are reshaped by $(\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\sigma\circ \mapsto \circ(-\overleftarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\sigma \mapsto (-\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\sigma\circ$, e.g., the adjoint to $(aa\circ)\overrightarrow{D}_x(\,\cdot\,)(\circ a)$ is $(-\overrightarrow{D}_x) \circ ((a\circ)(\,\cdot\,)(\circ aa))$. Thirdly, the operator's matrix is transposed: $(A^\dagger)^{ij} = (A^{ji})^\dagger$.

Definition 4 is constructive but implicit. It is instructive to see why the Schouten bracket $[\![F, G]\!]$ of a k-vector F and ℓ-vector G is a $(k + \ell - 1)$-vector: this fact relies on a very distinguished structure – of the local variational differential operators $[\![F, \cdot\,]\!]$ or $[\![\,\cdot\,, G]\!]$ – which normalization (21) provides for the geometric model of $[\![\,,\,]\!]$ in Remark 2.7.

Remark 3.2. The price that one pays for the (non)commutative variational multivectors' realisation – uniform with respect to every parity-odd entry $\boldsymbol{b}$ under integration by parts and cyclic shifts – is precisely having that legal possibility to integrate by parts. Yet we remember from §2.5 that all such integration is postponed until the ultimate end of every object's construction in the frames of the geometry of iterated variations. Therefore, the variational calculus of (non)commutative variational multivectors is *step-by-step* indeed; every intermediate object is let to exist as a well-defined notion.

For instance, Poisson bi-vectors $\boldsymbol{\mathcal{P}}$ first take the Hamiltonians F to the respective one-vectors X_F, which are also known to us under the name of Hamiltonian evolution equations (e. g., of (non)commutative Korteveg–de Vries type). In turn, the well-defined one-vector X_F acts by the Schouten bracket $[\![X_F, \cdot\,]\!]$ on a given 0-vector H, which defines the Poisson bracket $\{F, G\}_{\boldsymbol{\mathcal{P}}}$, see §3.3 below.

Notice that no multiplication of copies of the substrate manifold M^n can be seen from this way of reasoning; in fact, the on-the-diagonal restriction in the last phase of construction of the Schouten bracket becomes the immediate next to the first step. This is why the Poisson framework for (non)commutative variational multivectors was not capable of providing the intrinsic self-regularisation of the Batalin–Vilkovisky formalism with generic local functionals.

3.2. **Derived brackets.** Let $P \in \bar{H}^n_k(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ be a noncommutative variational k-vector. Consider k integral functionals $H_1, \ldots, H_k \in \bar{H}^n_0(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ of grading zero (that is, a k-tuple of 0-vectors).

Definition 5. The k-linear bracket $\{\,\cdot\,, \ldots, \cdot\,\}_P \colon (\bar{H}^n_0 \times \ldots \times \bar{H}^n_0)(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}) \to \bar{H}^n_0(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)})$ is defined by the noncommutative variational k-vector P as the derived bracket,[31]

$$\{H_1, \ldots, H_k\}_P \stackrel{\text{def}}{=} (-)^k \, \underline{[\![\ldots \underline{[\![P, H_1]\!]}, \ldots, H_k]\!]}. \tag{24}$$

The nested Schouten brackets are underlined in order to emphasize that each of them produces an *object*, i. e., the noncommutative variational multivector with one parity-odd entry less than the two arguments had together. In consequence, the integrations by parts are legitimate at every such step. This makes the Poisson formalism on the jet spaces a science of steps and stops.

Example 3.1. If $k = 1$ and the noncommutative variational one-vector is the cyclic word $P = (A(\boldsymbol{b}))$ for some total differential operator A (i. e., for a linear operator that

[31]We refer to [31] for a review of the concept of derived brackets in the geometry of usual manifolds. The algebraic classification of N-ary brackets is obtained in [43]; by analysing the jet-bundle geometry in this context, in the paper [16] we developed the notion of Wronskian determinants for functions in many variables. In particular, we proved that every such structure W on the space of functions encodes a differential $\mathbf{d}_W^2 = 0$.

is polynomial in the total derivatives), then

$$\{H_1\}_P = -[\![P, H_1]\!] = (A(\delta H_1/\delta \boldsymbol{a})).$$

Likewise, if $k = 2$ and, after a suitable number of integrations by parts, the noncommutative variational bi-vector is represented by the cyclic word(s) $P = \frac{1}{2}(\boldsymbol{b} \circ A(\boldsymbol{b}))$, then it is readily seen that[32]

$$\{H_1, H_2\}_P = \underline{[\![\underline{[\![H_1, P]\!]}, H_2]\!]} \cong \left(A\left(\frac{\delta H_1}{\delta \boldsymbol{a}}\right) \circ \frac{\delta H_2}{\delta \boldsymbol{a}}\right) \sim \left(\frac{\delta H_2}{\delta a^i} \circ A^{ij}\left(\frac{\delta H_1}{\delta a^j}\right)\right). \qquad (25)$$

Let us comment on every step in this construction. First, the variational one-vector X_{H_1} is produced from P and H_1; consider

$$[\![H_1, \tfrac{1}{2}(\boldsymbol{b} \circ A(\boldsymbol{b}))]\!] = \left(\frac{\delta H_1}{\delta \boldsymbol{a}} \circ \frac{1}{2} \sum_{|\tau|} \left(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^{\tau} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{b}_\tau} (\boldsymbol{b} \circ A(\boldsymbol{b}))\right).$$

When $P = \frac{1}{2}\bigl(b \circ A(\boldsymbol{b})\bigr)$ is varied with respect to $\boldsymbol{b}$, the partial derivatives $\overrightarrow{\partial}/\partial b^j_\tau$ reach the first occurence $\boldsymbol{b}_\varnothing$ with $\tau = \varnothing$ at once; before they reach the argument $\boldsymbol{b}$ of skew-adjoint operator A, let us integrate by parts: $\frac{1}{2}(\boldsymbol{b} \circ A(\underline{\boldsymbol{b}})) \cong \frac{1}{2}(-A(\boldsymbol{b}) \circ \underline{\boldsymbol{b}}) \sim \frac{1}{2}(\underline{\boldsymbol{b}} \circ A(\boldsymbol{b}))$. This shows that due to the particular structure of bi-vectors – if compared with generic functionals of grading two, – the second term doubles and absorbs $\frac{1}{2}$. We get the one-vector $(\delta H_1/\delta \boldsymbol{a} \circ A(\boldsymbol{b}))$; integrating by parts once again and using (23), we obtain the object

$$X_{H_1} = \left(-A\left(\frac{\delta H_1}{\delta \boldsymbol{a}}\right) \circ \boldsymbol{b}\right).$$

Now the construction of the outer Schouten bracket in (25) is elementary.

Lemma 12. Derived bracket (24) is totally antisymmetric under permutations of its arguments:

$$\{H_{\omega(1)}, \ldots, H_{\omega(k)}\}_P = (-)^{\omega} \{H_1, \ldots, H_k\}_P$$

for any $\omega \in S_k$ and any $H_1, \ldots, H_k \in \bar{H}^n_0(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)})$.

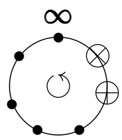
Remark 3.3. The total skew-symmetry of object (24) produced in k separate steps – with integration by parts and full stop after each step – does not follow from the Jacobi identity for $[\![\,,\,]\!]$. Rather, this is a manifestation of the noncommutative variational k-vectors' intrinsic property to be structurally identical with respect to every two graded entries $\boldsymbol{b}$.

Sketch of the proof. It suffices to show that the derived bracket $\{\cdot, \ldots, \cdot\}_P$ changes its sign under a swap of two consecutive arguments H_i and H_{i+1} :

$$\ldots \underline{[\![\underline{[\![Q, H_i]\!]}, H_{i+1}]\!]} \ldots \cong - \ldots \underline{[\![\underline{[\![Q, H_{i+1}]\!]}, H_i]\!]} \ldots .$$

Consider the noncommutative variational multivector's necklace Q and mark, by using $\otimes$ and $\oplus$, two parity-odd entries $\boldsymbol{b}$ (e.g., the two *consecutive* ones for the sake of clarity), see the figure,

[32]The first equality tells us that the bracket $\{\cdot, \cdot\}_P$ which the bi-vector P determines is a bracket *between* its arguments indeed.



This object's inner Schouten bracket with H_i does basically the following: normalisation (21) throws all the derivatives off the entry $\otimes$ and implants $\delta H_i/\delta\boldsymbol{a}$ in its stead (the normalisation does exactly the same with every other entry $\boldsymbol{b}$ by the definition of multivector, but let us focus on the term such that the variation $\delta H_i/\delta\boldsymbol{a}$ hits $\otimes$). Now reshape this output by making $\oplus$ free of derivatives falling on it. Note that this session of integrations by parts again amounts to bringing the multivector to normalized shape (21), – only the neighbouring entry $\otimes$ is occupied now by $\delta H_i/\delta\boldsymbol{a}$, not by $\boldsymbol{b}$. The outer Schouten bracket installs $\delta H_{i+1}/\delta\boldsymbol{a}$ for $\oplus$ (or for any other parity-odd entry; we consider just one term, for definition).

On the other hand, consider the very same scenario of putting $\delta H_i/\delta\boldsymbol{a}$ for $\otimes$ and $\delta H_{i+1}/\delta\boldsymbol{a}$ for $\oplus$, done in the reverse order. To reach $\oplus$ first in the construction of (now, inner) Schouten bracket, the derivation $\overleftarrow{\partial}/\partial\boldsymbol{b}$ has to overtake $\otimes$ currently occupied by the parity-odd placeholder $\boldsymbol{b}$; this overtake yields the sought-for minus sign. The variation $\delta H_{i+1}/\delta\boldsymbol{a}$ pasted for $\oplus$, we cast all the derivatives off the still-unused slot $\otimes$, leave $\delta H_i/\delta\boldsymbol{a}$ there, and integrate by parts back, to isolate $\delta H_{i+1}/\delta\boldsymbol{a}$ in the socket $\oplus$. It is readily seen that the two algorithms produce the identical portraits of letters and derivatives, yet those two differ by the sign factor. □

Remark 3.4. Continuing this line of reasoning, we conclude that for a given noncommutative variational k-vector P, the value $\{H_1, \ldots, H_k\}_P$ of derived bracket (24) at k arguments $H_1, \ldots, H_k \in \bar{H}^n_0(M^n_{\text{nC}} \to A^{(0|1)})$ is equivalent, up to integration by parts, to the 0-vector

$$(-)^{\frac{k(k-1)}{2}} \cdot \frac{1}{k!} \sum_{\omega\in S_k} (-)^{\omega} \left(\frac{\delta H_{\omega(1)}}{\delta\boldsymbol{a}} \cdot A\left(\frac{\delta H_{\omega(2)}}{\delta\boldsymbol{a}}, \ldots, \frac{\delta H_{\omega(k)}}{\delta\boldsymbol{a}} \right)\right) \cong \{H_1, \ldots, H_k\}_P, \quad (26)$$

where the alternating sum runs through the entire permutation group S_k; note that it is the parity-even arguments H_i but not the slots for them which are shuffled.

Observation (26) allow us to extend the *mapping* P from the geometry of exact (non)commutative variational covectors $\delta H_i/\delta\boldsymbol{a}$,

$$P\left(\delta H_1/\delta\boldsymbol{a}, \ldots, \delta H_k/\delta\boldsymbol{a}\right) \stackrel{\text{def}}{=} \{H_1, \ldots, H_k\}_P,$$

to k-tuples of arbitrary variational covectors $\boldsymbol{p}_i = (p_{i,\alpha} \circ \delta a^\alpha)$. The case $k = 1$ with $P(\boldsymbol{p}_1) := (A(\boldsymbol{p}_1))$ is elementary; for $k \geqslant 2$, we put[33]

$$P(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k) := (-)^{\frac{k(k-1)}{2}} \cdot \frac{1}{k!} \sum_{\omega\in S_k} (-)^{\omega} \left(\boldsymbol{p}_{\omega(1)} \circ A(\boldsymbol{p}_{\omega(2)}, \ldots, \boldsymbol{p}_{\omega(k)})\right). \quad (27)$$

[33]The variations $\delta\boldsymbol{a}$ serve as the earrings by which the open-word components $p_{i,\alpha}$ of $\boldsymbol{p}_i$ are hooked and dragged into the cyclic word of P. We emphasize that the isomorphism $V^\dagger \simeq T_{\boldsymbol{a}^\dagger}V^\dagger$ *is* used here to convert the placeholders $\boldsymbol{b}$ for $\boldsymbol{p}_i$ into the virtual offsets $\sum_{\alpha=1}^{m} 1\cdot\vec{e}^{\,\dagger,\alpha}$. The absorption of each argument $\boldsymbol{p}_i$ then goes closely to the lines of geometric construction of the Schouten bracket, see Remark 2.7 on p. 23. The various details of this construction – e. g., earrings, hooks, and placeholders – are left to the reader.

However, generic variational covectors, not necessarily exact, will not be studied in particular in what follows – rather, the converse can be assumed in view of the Substitution Principle (see §1.3).

Remark 3.5. Attempts to define the (non)commutative variational Schouten bracket of multivectors via a recursive procedure that involves the use of the two arguments' values at test covectors are sometimes practised in the literature; see [29] and references therein for the hydrography of underwater stones and for the analysis of other difficulties which arise on that way.

3.3. **Noncommutative variational Poisson structures.** We now analyse the construction of noncommutative variational Poisson brackets, recalling and re-proving several important facts from the general theory — here, under the coarse assumption of cyclic invariance (e.g., the Helmholtz lemma reveals a yet another mechanism for the differentials to anticommute).

Remark 3.6. Although the formalism is based on the noncommutative variational *symplectic* geometry from Ch. 2, the presence of differential operators A in the definition of the Poisson bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ as derived with respect to a given Poisson bi-vector $\boldsymbol{\mathcal{P}}$, see (24), usually makes such brackets degenerate. Their Casimirs, forming the zeroth Poisson cohomology group with respect to $\partial_{\boldsymbol{\mathcal{P}}_1}$, starts the Magri scheme for systems possessing the bi-Hamiltonian structures $(\boldsymbol{\mathcal{P}}_1, \boldsymbol{\mathcal{P}}_2)$, see [18, §9.2] and [8, 7].

3.3.1. *The definition of Poisson bracket.* Consider a noncommutative variational bi-vector $\boldsymbol{\mathcal{P}}$ and let H_1, H_2, $H_3 \in \bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$ be any three noncommutative variational 0-vectors.

Definition 6. Bi-linear, skew-symmetric derived bracket (25),

$$\{H_i, H_j\}_{\boldsymbol{\mathcal{P}}} = \underline{\underline{[\![\underline{[\![H_i, \boldsymbol{\mathcal{P}}]\!]}, H_j]\!]}}, \qquad 1 \leqslant i < j \leqslant 3,$$

is called the noncommutative variational *Poisson bracket* if it satisfies Jacobi's identity,

$$\{\{H_1, H_2\}_{\boldsymbol{\mathcal{P}}}, H_3\}_{\boldsymbol{\mathcal{P}}} + \{\{H_2, H_3\}_{\boldsymbol{\mathcal{P}}}, H_1\}_{\boldsymbol{\mathcal{P}}} + \{\{H_3, H_1\}_{\boldsymbol{\mathcal{P}}}, H_2\}_{\boldsymbol{\mathcal{P}}} \cong 0 \tag{28}$$

for all H_1, H_2, $H_3 \in \bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, which are then called the *Hamiltonians*.

If identity (28) holds, the noncommutative variational bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr)$ is called *Poisson*; the skew-adjoint noncommutative linear operator A in total derivatives is then called a *Hamiltonian operator*, and the noncommutative variational one-vectors $X_{H_i} \stackrel{\text{def}}{=} [\![\boldsymbol{\mathcal{P}}, H_i]\!]$ are the *Hamiltonian one-vectors* (or *one-vector fields*) specified by their Hamiltonians H_i and the Poisson bi-vector $\boldsymbol{\mathcal{P}}$.

Criterion 13. *A noncommutative variational bi-vector $\boldsymbol{\mathcal{P}}$ is Poisson (i.e., the derived bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ satisfies Jacobi's identity (28)) if the bi-vector $\boldsymbol{\mathcal{P}}$ satisfies the classical master-equation*

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong 0 \in \bar{H}^n_3\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr). \tag{29}$$

The bi-vector $\boldsymbol{\mathcal{P}}$ is Poisson only if the value of $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]$ at any triple H_1, H_2, H_3 of Hamiltonians is cohomologically trivial:

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!](H_1, H_2, H_3) \cong 0 \in \bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr).$$

The assertion is aimed to emphasize that the Poisson bi-vectors are the primary objects, whereas the Poisson brackets are the derived structures.[34]

Lemma 14. If a noncommutative variational k-vector $\boldsymbol{\mathcal{Q}}$ represents the class of zero in $\bar{H}^n_k\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, then, $\boldsymbol{\mathcal{Q}}$ viewed as the map $\bigl(\bar{H}^n_0 \times \ldots \times \bar{H}^n_0\bigr)\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr) \to \bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, its value $\boldsymbol{\mathcal{Q}}\bigl(\delta H_1/\delta \boldsymbol{a}, \ldots, \delta H_k/\delta \boldsymbol{a}\bigr) = \{H_1, \ldots, H_k\}_{\boldsymbol{\mathcal{Q}}}$ is cohomologically trivial for every k-tuple of the arguments $H_1, \ldots, H_k \in \bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$.

Sketch of the proof. Indeed, whenever the cyclic word $\boldsymbol{\mathcal{Q}} = \mathrm{d}_h\boldsymbol{\mathcal{R}}(\boldsymbol{b}, \ldots, \boldsymbol{b})$ carrying k parity-odd entries $\boldsymbol{b}$ is exact with respect to the lift d_h of the de Rham differential for M^n onto $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, so is every term – in the sum over the $|S_k|$ ways to permute the arguments $H_1, \ldots, H_k$ by using $\omega \in S_k$ – obtained by pasting whatever open string $\delta H_{\omega(i)}/\delta a^j$ of parity-even symbols instead of the ith copy of the symbol b_j. □

3.3.2. *Noncommutative differential forms.* To approach the proof of Criterion 13, let us have a glimpse of the classical set of structures that appear on the infinite jet spaces $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}\bigr)$ – in particular, in the context of Vinogradov's $\mathcal{C}$-spectral sequence [42].

By definition, now put

$$\vec{\partial}^{(\boldsymbol{a})}_{\varphi(\boldsymbol{x},\vec{\mathbf{x}}^{\pm 1},[\boldsymbol{a}])} = \sum_{i=1}^{m} \sum_{|\sigma|\geqslant 0} \Bigl((\varphi^i)\Bigl(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\sigma}\Bigr)\bigl(\boldsymbol{x}, \vec{\mathbf{x}}^{\pm 1}, [\boldsymbol{a}]\bigr) \circ \frac{\overrightarrow{\partial}}{\partial a^i_\sigma}.$$

It is readily seen that these *evolutionary derivations* commute with the total derivatives on $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}\bigr)$:

$$\bigl[\vec{\partial}^{(\boldsymbol{a})}_{\varphi}, \vec{\mathrm{d}}/\mathrm{d}x^k\bigr] = 0 \qquad \text{for all } k = 1, \ldots, n.$$

Consequently, for any operator A in total derivatives we have that

$$\vec{\partial}^{(\boldsymbol{a})}_{\varphi}\bigl(A(\boldsymbol{p})\bigr) = \bigl(\vec{\partial}^{(\boldsymbol{a})}_{\varphi}(A)\bigr)(\boldsymbol{p}) + A\bigl(\vec{\partial}^{(\boldsymbol{a})}_{\varphi}(\boldsymbol{p})\bigr).$$

Next, define the *linearization* $\ell^{(\boldsymbol{a})}_{\boldsymbol{p}}$ of an object $\boldsymbol{p}$ over $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}\bigr)$ by setting

$$(\varphi)\overleftarrow{\ell}^{(\boldsymbol{a})}_{\boldsymbol{p}} = \overrightarrow{\partial}^{(\boldsymbol{a})}_{\varphi}(\boldsymbol{p})$$

whenever the right-hand side is well defined.

Thirdly, for each value of the index i running from 1 to m and for every multi-index σ let us introduce the symbol $\mathrm{d}a^i_\sigma$. Now define the *Cartan differential* $\mathrm{d}_{\mathcal{C}}\colon a^i_\sigma \mapsto \mathrm{d}a^i_\sigma$,

[34] The gap between necessity and sufficience is the conjecture that, whenever the value $\boldsymbol{\mathcal{Q}}(\delta H_1/\delta\boldsymbol{a}, \ldots, \delta H_k/\delta\boldsymbol{a})$ of a (non)commutative variational k-vector $\boldsymbol{\mathcal{Q}}$ at every k-tuple of exact variational covectors $\delta H_i/\delta\boldsymbol{a}$ is cohomologically trivial in $\bar{H}^n_0\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$, the k-vector $\boldsymbol{\mathcal{Q}}$ itself is cohomologically trivial in $\bar{H}^n_k\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$. This conjecture proven, Lemma 14 would convert into an equivalence.

It is quite paradoxical that, to the best of our knowledge, no proof of this claim has been obtained yet — even in the graded-commutative set-up; some think it is too obvious to be proved and others think it is too broad to be true, unless extra assumptions on the topology of fibre bundle π over M^n are incorporated in the precise phrasing of that claim for the structures over $J^\infty(\pi)$.

We expect however that there *is* a proof and that it is particularly transparent in the cyclic-word setting $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}^{(0|1)}\bigr)$. The Substitution Principle working first (see Theorem 3 on p. 12), a suitable homotopy then restores the k-linear horizontal $(n-1)$-form $\boldsymbol{\mathcal{R}}(\boldsymbol{b}, \ldots, \boldsymbol{b})$ such that the explicit construction $\boldsymbol{\mathcal{Q}} = \mathrm{d}_h\boldsymbol{\mathcal{R}}$ showing $\boldsymbol{\mathcal{Q}}$ is trivial is uniform with respect to all the k-tuples $H_1, \ldots, H_k$.

$\mathrm{d}a^i_\sigma \mapsto 0$, also setting its action equal to zero on $\boldsymbol{x}$ and $\vec{\mathbf{x}}^{\pm 1}$ and postulating that $\mathrm{d}_{\mathcal{C}}$ is a derivation. By construction, let the differential $\mathrm{d}_{\mathcal{C}}$ be correlated with other structures on $J^\infty\bigl(M^n_{\text{nC}} \to \mathcal{A}\bigr)$ in the standard way: e.g., set $\vec{D}_{x^k}(\mathrm{d}a^i_\sigma) = \mathrm{d}a^i_{\sigma\cup\{k\}}$.

Let us explain what it means that the symbols $\mathrm{d}a^i_\sigma$ and $\mathrm{d}a^j_\tau$ "anticommute." The key idea is that the precedence-succedence relation of such symbols in a given cyclic word manifests that circle's orientation, which is provided by construction.

Consider a cyclic word that carries *one* symbol $\mathrm{d}a^i_\sigma$; the word thus acquires a marked point. The derivation $\mathrm{d}_{\mathcal{C}}$ acts on (the rest of) the word by starting at $\mathrm{d}a^i_\sigma$ and processing the letters a^j_τ by going in the positive direction. We say that all the symbols $\mathrm{d}a^j_\tau$, newly produced by $\mathrm{d}_{\mathcal{C}}$ from such a^j_τ are *succedent* with respect to the mark $\mathrm{d}a^i_\sigma$; in turn, the old symbol $\mathrm{d}a^i_\sigma$ is *precedent* for each new object $\mathrm{d}a^j_\tau$. To change this precedence-succedence relation $\mathrm{d}a^i_\sigma \prec \mathrm{d}a^j_\tau$ but still let the circle's orientation stay intact, the object $\mathrm{d}a^j_\tau$ is proclaimed the new marked point — so that $\mathrm{d}a^i_\sigma$ now *succeeds* it with respect to the positive order of letters written along the oriented circle. By definition, such involution of the relative order $\prec$ of the two symbols, $\mathrm{d}a^i_\sigma$ and $\mathrm{d}a^j_\tau$, produces the factor -1 in front of the cyclic word that carries both of them. Clearly, $\mathrm{d}^2_{\mathcal{C}} = 0$.

Lemma 15 (Helmholtz)**.** The linearization $\vec{\ell}^{(\boldsymbol{a})}_{\delta H/\delta \boldsymbol{a}}$ of an element in the image of variational derivative $\delta/\delta\boldsymbol{a}$ is self-adjoint:

$$\vec{\ell}^{(\boldsymbol{a})}_{\delta H/\delta \boldsymbol{a}} = \vec{\ell}^{(\boldsymbol{a})\,\dagger}_{\delta H/\delta \boldsymbol{a}}. \tag{30}$$

Note that this half of Helmholtz' *criterion* does not refer to the topology of the set-up.

Proof. Let H be a noncommutative variational 0-vector. Up to an integration by parts, we have that $\mathrm{d}_{\mathcal{C}}H \cong \bigl(\mathrm{d}\boldsymbol{a} \circ \delta H/\delta\boldsymbol{a}\bigr)$. By the above,

$$0 = \mathrm{d}^2_{\mathcal{C}}(H) \cong \bigl(\mathrm{d}\boldsymbol{a} \circ \overrightarrow{\ell}^{(\boldsymbol{a})}_{\delta H/\delta\boldsymbol{a}}(\underline{\mathrm{d}\boldsymbol{a}})\bigr) \cong \bigl((\mathrm{d}\boldsymbol{a})\overleftarrow{\ell}^{(\boldsymbol{a})\,\dagger}_{\delta H/\delta\boldsymbol{a}} \circ \underline{\mathrm{d}\boldsymbol{a}}\bigr) \sim -\bigl(\underline{\mathrm{d}\boldsymbol{a}} \circ \overrightarrow{\ell}^{(\boldsymbol{a})\,\dagger}_{\delta H/\delta\boldsymbol{a}}(\mathrm{d}\boldsymbol{a})\bigr),$$

whence (30). □

3.3.3. *Proof of Criterion 13.* First, let us recall the renowned cancellation mechanism in the left-hand side of Jacobi's identity (28). By definition, put $\boldsymbol{p}_i = \delta H_i/\delta\boldsymbol{a}$ for the three Hamiltonians. Integrating by parts in the inner and outer Poisson brackets in (28) and using formula (25), we get

$$\begin{aligned}
&\vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_1)}\bigl(\boldsymbol{p}_2 \circ A(\boldsymbol{p}_3)\bigr) + \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_2)}\bigl(\boldsymbol{p}_3 \circ A(\boldsymbol{p}_1)\bigr) + \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_3)}\bigl(\boldsymbol{p}_1 \circ A(\boldsymbol{p}_2)\bigr)\\
&\quad = \bigl(\vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_2) \circ A(\boldsymbol{p}_3)\bigr) + \bigl(\boldsymbol{p}_2 \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_1)}(A)(\boldsymbol{p}_3)\bigr) - \bigl(A(\boldsymbol{p}_2) \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_3)\bigr)\\
&\qquad + \bigl(\vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_2)}(\boldsymbol{p}_3) \circ A(\boldsymbol{p}_1)\bigr) + \bigl(\boldsymbol{p}_3 \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_2)}(A)(\boldsymbol{p}_1)\bigr) - \bigl(A(\boldsymbol{p}_3) \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_2)}(\boldsymbol{p}_1)\bigr)\\
&\qquad + \bigl(\vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_3)}(\boldsymbol{p}_1) \circ A(\boldsymbol{p}_2)\bigr) + \bigl(\boldsymbol{p}_1 \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_3)}(A)(\boldsymbol{p}_2)\bigr) - \bigl(A(\boldsymbol{p}_1) \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_3)}(\boldsymbol{p}_2)\bigr).
\end{aligned} \tag{31}$$

Applying Lemma 15 to the variational covectors $\boldsymbol{p}_i = \delta H_i/\delta\boldsymbol{a}$ as follows,

$$\begin{aligned}
\bigl(\vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_1)}(\boldsymbol{p}_2) \circ A(\boldsymbol{p}_3)\bigr) &\stackrel{\text{def}}{=} \bigl(\vec{\ell}^{(\boldsymbol{a})}_{\boldsymbol{p}_2}(A(\boldsymbol{p}_1)) \circ A(\boldsymbol{p}_3)\bigr) = \bigl(\vec{\ell}^{(\boldsymbol{a})\,\dagger}_{\boldsymbol{p}_2}(A(\boldsymbol{p}_1)) \circ A(\boldsymbol{p}_3)\bigr)\\
&\cong \bigl(A(\boldsymbol{p}_1) \circ \vec{\ell}^{(\boldsymbol{a})}_{\boldsymbol{p}_2}\bigl(A(\boldsymbol{p}_3)\bigr)\bigr) \stackrel{\text{def}}{=} \bigl(A(\boldsymbol{p}_1) \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\boldsymbol{p}_3)}(\boldsymbol{p}_2)\bigr),
\end{aligned}$$

we conclude that it is only the second column which survives the cancellation in (31). The left-hand side of Jacobi's identity thus equals

$$\Bigl(\tfrac{\delta H_1}{\delta \boldsymbol{a}} \circ \vec{\partial}^{(\boldsymbol{a})}_{A(\delta H_3/\delta \boldsymbol{a})}(A)\bigl(\tfrac{\delta H_2}{\delta \boldsymbol{a}}\bigr)\Bigr) + \text{cyclic permutations.} \tag{32}$$

On the other hand, consider the bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr)$ and construct

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong \Bigl(\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr)\Bigl(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{a}_\sigma} \circ \Bigl(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\sigma}\bigl(A(\boldsymbol{b})\bigr)\Bigr)\Bigr);$$

the right-hand side contains, for every multi-index σ, the derivation that pastes its coefficient for each a^i_σ occurring in the coefficients of operator A within $\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr)$.

The only thing which the evaluation of $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]$ at H_1, H_2, and H_3 does,

$$[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]\bigl(\delta H_1/\delta \boldsymbol{a}, \delta H_2/\delta \boldsymbol{a}, \delta H_3/\delta \boldsymbol{a}\bigr) = (-)^3\,\underline{[\![\underline{[\![\underline{[\![\underline{[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]}, H_1]\!]}, H_2]\!]}, H_3]\!]},$$

is the speading of variational derivatives $\delta H_i/\delta \boldsymbol{a}$ over the three slots $\boldsymbol{b}$ in the tri-vector $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]$. In view of the evaluation's total skew-symmetry (see Lemma 12), it is enough to sum up over the cyclic (hence, even) permutations in the group S_3, and then double. This yields the three terms

$$\Bigl(\tfrac{\delta H_1}{\delta \boldsymbol{a}} \circ \bigl((A)\overleftarrow{\partial}^{(\boldsymbol{a})}_{A(\delta H_3/\delta \boldsymbol{a})}\bigr)\bigl(\tfrac{\delta H_2}{\delta \boldsymbol{a}}\bigr)\Bigr) + \text{cyclic permutations.} \tag{33}$$

Uniting the two parts of the reasoning, we conclude that the left-hand side (32) of Jacobi's identity (28) for the bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ and the value of tri-vector $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]$ at the same Hamiltonians H_1, H_2, and H_3 as in (28) are equal, hence simultaneously (non)trivial, as elements of the cohomology group $\bar{H}^n_0\bigl(M^n_{\mathrm{nC}} \to \mathcal{A}^{(0|1)}\bigr)$. □

Example 3.2. Every noncommutative variational bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\bigl(\boldsymbol{b} \circ A(\boldsymbol{b})\bigr)$ such that the coefficients of skew-adjoint linear total differential operator A do not depend on any symbol a^i_σ –in particular, the operator A has constant coefficients– is Poisson.

Referring to the conjecture in footnote 34 on p. 35 and setting $\boldsymbol{\mathcal{Q}} = [\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!]$ there, one could now argue that the bracket $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ is Poisson if and only if the classical master-equation $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong 0$ holds for $\boldsymbol{\mathcal{P}}$.

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