Scattering Equations and String Theory Amplitudes

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Abstract

Scattering equations for tree-level amplitudes are viewed in the context of string theory. To this end we are led to define a new dual model whose amplitudes coincide with string theory in both the small and large α' limit, computed algebraically on the surface of solutions to the scattering equations. Because it has support only on the scattering equations, it can be solved exactly, yielding a simple resummed model for α'-corrections to all orders. We use the same idea to generalize scattering equations to amplitudes with fermions and any mixture of scalars, gluons and fermions. In all cases checked we find exact agreement with known results.

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I. INTRODUCTION

In a series of remarkable papers, Cachazo, He and Yuan (CHY) have proposed that tree level scattering of massless particles in any dimension can be constructed from algebraic solutions of a set of kinematic scattering equations [1–3]. This idea had originally been supported by a number of highly non-trivial observations and checks, and also by explicit amplitude computations for a large number of external legs. A proof of this surprising construction has recently been provided for scalar amplitudes and gluon amplitudes by Dolan and Goddard in ref. [4] based on Britto-Cachazo-Feng-Witten (BCFW) [5] recursion. These authors have also shown how to generalize the construction to massive scalars, extending the specific construction for scalars of ref. [3] to any theory of scalars with only 3-point vertices, again in any dimension.

The whole setup of the scattering equation approach is eerily reminiscent of string theory, and indeed it was recognized early on [2] that these scattering equations coincide with the saddle point equations of the Gross-Mende limit [6]. But this also represents a conundrum: The Gross-Mende limit is that of high-energy scattering of strings corresponding to $\alpha' \to \infty$, not the opposite limit of $\alpha' \to 0$ where the field theory of pointlike particles emerges. Indeed, in the $\alpha' \to 0$ limit of string theory an entirely different formalism arises, even though, eventually, the same tree-level amplitudes come out. It is as if the scattering equation approach has managed to obtain a different limit of $\alpha' \to 0$, while retaining aspects of high-energy scattering of strings. In the twistor string frameworks [7–10], it has been demonstrated that one can naturally impose the scattering equations in an alternative path integral formulation. Many other indications of a close connection to string theory can be found. In [1] it was thus shown that the scattering equations are intimately related to the momentum kernel S [11, 12] between gauge and gravity theories (and hence between open and closed strings). Similarly, scattering equations manifestly operate with a basis of $(N-3)!$ amplitudes, in agreement with what is inferred from Bern-Carrasco-Johansson (BCJ) relations [13] and that follows directly from string theory [14, 15] (see [16–18] for applications to massless and massive amplitude). A more direct link between BCJ relations and scattering equations has also been proposed [3, 19, 20]. Finally, some algebraic relations arising in the string theory computation of disk amplitudes [21–24] have also found use in the scattering equation formalism. All of these examples indicate a close connection to string theory.

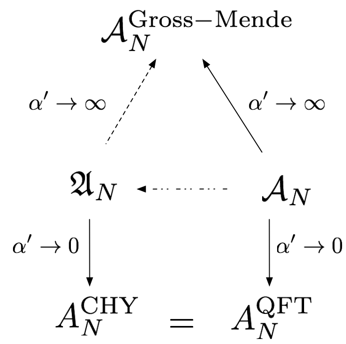
In this paper we suggest a new dual model that gives field theory amplitudes back in the $\alpha' \to 0$ limit and that, when $\alpha' \to \infty$, is governed by the Gross-Mende saddle point of high-energy string scattering:

$$\mathfrak{A}_N = \int \prod_{i=2}^{N-2} dz_i \prod_{1\le i<j\le N} |z_i - z_j|^{2\alpha' k_i\cdot k_j} \times \\ \times \frac{(z_1 - z_{N-1})^2 (z_{N-1} - z_N)^2 (z_N - z_1)^2}{\prod_{i=1}^{N} (z_i - z_{i+1})^2} \prod_{i\neq 1,N-1,N} \delta(S_i)\,, \quad \text{(I.1)}$$

where the integration is over the ordered set $z_1 < z_2 < \cdots < z_N$ and the three points z_1, z_{N-1} and z_N have been fixed by $SL(2,\mathbb{C})$ invariance. This expression differs from the CHY prescription by the Koba-Nielsen factor $\prod_{1\le i<j\le N-2} |z_i - z_j|^{2\alpha' k_i\cdot k_j}$, and differs from the usual string theory amplitude prescription by the delta function constraints $(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1) \prod_{i\neq 1,N-1,N} \delta(S_i) \prod_{i=2}^{N-2} (z_i - z_{i+1})^{-1}$.

At intermediate values of α', because of the delta function constraint, these tree-level amplitudes differ from the ones evaluated in the Ramond-Neveu-Schwarz (RNS) formalism or the pure spinor formulation [25]. The difference with the traditional string theory tree-level amplitude is discussed in section IV, where we show that the above prescription has a soft high-energy behavior similar to the one of the conventional string theory. Therefore the prescription retains some fundamental properties of stringy amplitudes. It would be interesting to relate the prescription given in this paper to an α'-extension of Berkovits' modified pure spinor prescription in the infinite tension limit [8]. We view it as a new dual model that could have been introduced long ago. Indeed, the approach by Fairlie et al. [26] (reviewed in [27]) by imposing on a scalar dual model a minimal area constraint is closely related to this, only missing the more general context and the new connection to the field theory limit $\alpha' \to 0$ that we provide here.

The connection with the usual quantum field theory limit of string theory and its high-energy limit is summarized in the following diagram showing that new amplitude $\mathfrak{A}_N$ interpolates between the CHY prescription and a high-energy limit with the Gross-Mende saddle point.



To show that the approach we suggest here also holds in a broader context than the original CHY prescription, we illustrate how the prescription in (I.1) can be extended to include fermions as in the superstring. We demonstrate explicitly that this produces correct amplitudes with fermions in a few simple cases. Also examples of mixed amplitudes with scalars, gluons and fermions will be considered and shown to agree with known results.

Our paper is organized in the following way. First, in section II, we briefly review the scattering equations and their solution in the field theory limit. Next, we motivate the simple new dual model of scalars in (I.1). By imposing on the integrand the scattering equations, we obtain a simple scalar analog of the general framework of this paper: a model that reproduces the field theory limit on the surface of solutions to the scattering equations as $\alpha' \to 0$ and which reproduces the Gross-Mende solution in the limit of $\alpha' \to \infty$. Then in section III, we consider the case of amplitudes involving gauge fields, by first briefly recalling how to compute the corresponding gluon amplitudes in string theory. The expression for the string integrand is rather cumbersome, but it can be rearranged into a form identical to the Pfaffian prescription of refs. [1–3], up to additional terms that formally are suppressed as $1/\alpha'$. We show that all the additional pieces are proportional to the scattering equations after suitable integrations by parts manipulations familiar in string theory (see [28, eqs. (6.2.25)] and [21–24, 29, 30]). Therefore, on the surface of solutions to these equations they do not contribute, and the resulting modified integrand for our dual model in (I.1) yields the CHY amplitude prescription in the field theory limit $\alpha' \to 0$. In section IV we use this observation to show how to extend the scalar dual model prescription to include gauge fields. This elementary construction is particularly easily understood in the case of the four-point gluon amplitude. We also show how such manipulations extend to higher point amplitudes. Finally, in section V we discuss how to extend these considerations to compute amplitudes with external fermions on the basis of the scattering equations, and how mixed amplitudes with scalars, fermions and vectors can be computed as well. We end with an outlook for future work.

II. SCATTERING EQUATIONS AND A DUAL MODEL EXTENSION

For scalar theories, the prescription given by the CHY prescription for computing N-point scalar amplitudes reads

$$A_{N\,\mathrm{scalar}}^{\mathrm{CHY}} = \int \prod_{i\neq 1,N-1,N} \delta(S_i)\, \frac{(z_1 - z_{N-1})^2 (z_{N-1} - z_N)^2 (z_N - z_1)^2}{\prod_{i=1}^{N} (z_i - z_{i+1})^2} \prod_{i=2}^{N-2} dz_i\,, \tag{II.1}$$

where the legs are ordered canonically from 1 to N, and the notation is such that $z_{N+1} \equiv z_1$. Here S_i denotes the ith scattering equation

$$S_i = \sum_{j\neq i} \frac{k_i \cdot k_j}{z_i - z_j} \;=\; 0\,. \tag{II.2}$$

In the following we will fix the three points $z_1 = 0$, $z_{N-1} = 1$ and $z_N = \infty$. The CHY prescription given for computing N-point gauge theory amplitudes reads

$$A_{N\,\mathrm{gauge}}^{\mathrm{CHY}} = \int \mathrm{Pf}'\Psi_N(z_i) \prod_{i\neq 1,N-1,N} \delta(S_i)\, \frac{(z_1 - z_{N-1})^2 (z_{N-1} - z_N)^2 (z_N - z_1)^2}{\prod_{i=1}^{N} (z_i - z_{i+1})} \prod_{i=2}^{N-2} dz_i\,. \tag{II.3}$$

The function $\mathrm{Pf}'\Psi_N(z_i)$ is the reduced Pfaffian, $\mathrm{Pf}'(\Psi)_N(z_i)$, given by

$$\mathrm{Pf}'\Psi_N(z_i) = \frac{(-1)^{i+j}}{z_i - z_j} \mathrm{Pf}(\Psi_{ij}^{ij})\,, \tag{II.4}$$

where Ψ_{ij}^{ij} is the matrix obtained from Ψ by removing the rows and columns i and j (two rows and two columns removed). Gauge theory amplitudes are obtained with

$$\Psi_N(z_i) = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}\,, \tag{II.5}$$

where

$$A_{i,j} = \begin{cases} \dfrac{k_i \cdot k_j}{z_i - z_j} & i \neq j, \\ 0 & i = j, \end{cases} \quad B_{i,j} = \begin{cases} \dfrac{\epsilon_i \cdot \epsilon_j}{z_i - z_j} & i \neq j, \\ 0 & i = j, \end{cases} \quad C_{i,j} = \begin{cases} \dfrac{\epsilon_i \cdot k_j}{z_i - z_j} & i \neq j\,, \\ -\displaystyle\sum_{l\neq i} \frac{\epsilon_i \cdot k_l}{z_i - z_l} & i = j\,. \end{cases} \tag{II.6}$$

Let us now try to see this construction in the light of old-fashioned dual models with a dimensionful parameter α'. A simple dual model that yields the same massless scalar scattering amplitudes in the limit $\alpha' \to 0$ is the following:

$$\mathcal{A}_N = \left(\frac{g_o}{\sqrt{\alpha'}}\right)^{N-2} {\alpha'}^{N-3} \int \prod_{i=2}^{N-2} dz_i \, \frac{(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1)}{\prod_{i=1}^{N}(z_i - z_{i+1})} \prod_{1\leq i<j\leq N} |z_i - z_j|^{2\alpha' k_i\cdot k_j}, \tag{II.7}$$

where the integration is ordered along the real axis and g_o is the open string coupling constant.

Note how different the integration prescription is in the two cases. In the simple dual model defined above, we integrate in an ordered manner along the real line after having fixed again $z_1 = 0$, $z_{N-1} = 1$ and $z_N = \infty$. In the integral defining amplitudes based on scattering equations (II.1) the integral is saturated by the solutions to the delta function constraints. This means that singularities that normally carry the whole amplitude in the $\alpha' \to 0$ limit are harmless. Also the remaining part of the integrand is of course totally different, as there is no trace of α' in (II.1). Yet, remarkably, for all N the $\alpha' \to 0$ limit of (II.7) yields exactly the same answer as (II.1). This suggests that it may be advantageous to view (II.1) as the leading term of a more elaborate amplitude that depends on a parameter α'.

Based on this perhaps rather naïve argument, let us introduce a very simple new dual model defined by amplitudes (using the relation $g_o = g_{\text{Yang Mills}}\sqrt{2\alpha'}$ between the open string coupling constant and the Yang-Mills (YM) coupling constant in ten dimensions)

$$\mathfrak{A}_N = {g_{\text{YM}}^{N-2}}^2 \int \prod_{i=2}^{N-2} dz_i \prod_{1\leq i<j\leq N} |z_i - z_j|^{2\alpha' k_i\cdot k_j} \prod_{i\neq 1,N-1,N} \delta(S_i) \frac{(z_1 - z_{N-1})^2(z_{N-1} - z_N)^2(z_N - z_1)^2}{\prod_{i=1}^{N}(z_i - z_{i+1})^2}. \tag{II.8}$$

Note that, effectively, this simply amounts to taking the dual model expression and inserting the normalized delta function constraint[1]

$${\alpha'}^{3-N}(z_1 - z_{N-1})(z_{N-1} - z_N)(z_N - z_1) \prod_{i\neq 1,N-1,N} \delta(S_i) \prod_{i=1}^{N}(z_i - z_{i+1})^{-1}, \tag{II.9}$$

in the integrand. The overall powers of α' can be understood from the fact that it is natural from string theory to insert the delta function $\delta(\alpha' S_i) = {\alpha'}^{-1}\delta(S_i)$. Our claim is that

[1] The delta function constraint has to be understood to include signs as in [2]. In general, this can be given a precise interpretation in terms of contours in the complex plane via the global residue theorem [2, 4]. However in all cases we have considered (even in the case of complex solutions to the scattering equations) the naïve delta function constraint works as well, and of course the final result is real.

this prescription, applied to open string theory amplitudes, provides a constructive way to reproduce field theory amplitudes. In this expression one can set α' to zero in the integrand to recover the CHY prescription. The justification of this point is the subject of the next sections.

Massive scalar amplitudes can be dealt with easily, as they simply correspond to replacing

$$\prod_{i=1}^{N}(z_i - z_{i+1})^{-1} \;\rightarrow \prod_{i=1}^{N}(z_i - z_{i+1})^{-1-\alpha' m^2}\,, \tag{II.10}$$

in the integrand of (II.7). By differentiation of the integrand with respect to z_i we obtain the massive scattering equation proposed and proven to be correct in ref. [4]. The fact that scattering equations arise from differentiation with respect to the z_i of external legs in the integrand will play a crucial role in what follows.

In contrast to a more conventional dual model such as (II.7), the new integral (II.8) has a totally smooth and finite limit $\alpha' \to 0$, where it of course coincides with scalar field theory. So has anything been achieved in making such a trivial extension? A hint that this may be so is that in the opposite limit $\alpha' \to \infty$, the amplitudes of (II.8) and (II.7), are both fixed by the same Gross-Mende saddle point of high-energy string scattering. So this simple extension (II.8) retains all the nice properties of (II.1) when $\alpha' = 0$, and yields stringy amplitudes in the opposite limit of $\alpha' \to \infty$. In between these two limits we obviously have no immediate way to interpret the amplitudes (II.8), but these amplitudes are all trivially computable due to the δ-function constraint in the measure.

What could be the meaning of the dimensional parameter α' here? It would be tempting to view it as an inverse string tension. However, such a point of view is not tenable. This becomes clear already in the case of four-particle scattering, which has almost no resemblance at finite α' to the corresponding Veneziano amplitude of (II.7). There is not an infinite series of poles in the amplitude that, rather, is more like that of ordinary field theory with a trivial exponential damping factor. Indeed, because the limit $\alpha' \to 0$ meets no singularity, amplitudes with either small or large momenta can be found immediately at any value of α'. At $\alpha' = 0$ the scattering amplitudes of (II.8) are just those of field theory, up to arbitrarily high energies. The extension of (II.1) to the new dual model (II.8) looks much like dualized (color-ordered) scalar field theory regularized with an ultraviolet cutoff $1/\sqrt{\alpha'}$.

At this point, the dual model (II.8) cannot be viewed as anything else but a curiosity. If there is to be any substance in it, and insight to be gained, we must see if a slightly more sophisticated line of approach can yield new results. We therefore turn to ordinary string theory, and explore the extent to which similar considerations can be extended to massless gauge boson scattering.

III. SCATTERING EQUATIONS AND GAUGE FIELDS

In this section we explore in some detail the properties of the prescription (II.8). It is well known that the requirement of multilinearity in external polarization vectors conveniently can be implemented in terms of auxiliary fermionic integrations in the string integrand. These real Grassmann variables, when integrated out, produce a Pfaffian. This suggests that the Pfaffian prescription of the previous section may be viewed as a remnant of the string theory integrand, now only evaluated on the solutions to the scattering equations. As we shall see, this is indeed the case. But instead of computing the resulting Pfaffian directly, it is convenient to split it up into its separate components, in this way illuminating which pieces give rise to the Pfaffian of the previous section, and which do not.

A. Multi-Pfaffian Structure of N-point Open-String Integrand.

We first provide a new way to decompose the string theory integrand for the scattering of N gluons in the open superstring as a sum of Pfaffians. This will include terms in the integrand of increasing powers of $1/\alpha'$ as N grows, but of course the full integral starts with terms of order $1/\alpha'$ only. These terms of higher powers of $1/\alpha'$ in the integrand can indeed be re-cast into terms that carry no explicit factor of α' by means of integrations by parts. Such rewritings show that these terms do not contribute on the surface of solutions to the scattering equations.

In the RNS formalism, the vertex operators come in various ghost pictures with respect to the superconformal ghost ($\beta = \partial\xi\, e^{-\varphi}, \gamma = e^{\varphi}$) [31]. The -1 ghost picture of the unintegrated vertex operator for the emission of a gauge boson is then given by

$$U^{(-1)} = g_o T^a : e^{-\varphi} \epsilon \cdot \psi\, e^{ik\cdot X} : , \tag{III.1}$$

while these in the 0 ghost picture read

$$U^{(0)} = g_o\sqrt{\frac{2}{\alpha'}}T^a : (i\partial X^\mu + 2\alpha'(k\cdot\psi)(\epsilon\cdot\psi))e^{ik\cdot X} : . \quad \text{(III.2)}$$

The corresponding integrated vertex operators are given by

$$\begin{aligned} V^{(-1)} &= \int dz : U^{(-1)} : , \\ V^{(0)} &= \int dz : U^{(0)} : . \end{aligned} \quad \text{(III.3)}$$

The normalization of the operator-product expansion (OPE) on the boundary of the disk is such that

$$\begin{aligned} X^\mu(z)X^\nu(0) &\simeq -\alpha' \log|z|^2 , \\ \psi^\mu(z)\psi^\nu(0) &\simeq \frac{\eta^{\mu\nu}}{z} , \\ e^{q_1\varphi(z)}e^{q_2\varphi(0)} &\simeq \frac{1}{z^{q_1 q_2}} . \end{aligned} \quad \text{(III.4)}$$

At tree-level, to saturate the +2 background superghost charge, one should set two vertex operators in the −1 ghost-picture, the rest can be chosen in the 0 ghost-picture. These two operators chosen in the −1 ghost picture, for instance V_1 and V_2, determine which lines and columns of the matrix one should remove to get the correct reduced Pfaffian of equation (II.4). The n-gluon open-string amplitude $\mathcal{A}_N$ reads:

$$\mathcal{A}_N = \frac{1}{\alpha' g_o^2}\langle cU^{(-1)}(z_1)cU^{(-1)}(z_{N-1})cU^{(0)}(z_N)\int\prod_{i=2}^{N-2} dz_i U^{(0)}(z_2)\cdots U^{(0)}(z_{N-2})\rangle . \quad \text{(III.5)}$$

where g_o is the open-string coupling constant. A Pfaffian comes out of this integral simply because of the Grassmann integral over a product of fermionic fields.

Focusing first on the purely fermionic part of the correlator (III.5), it involves a product of $2N-2$ fermionic fields, among which $N-3$ are bilinears:

$$\langle(\epsilon_1\cdot\psi(z_1))(\epsilon_2\cdot\psi(z_2))\prod_{i=3}^{N} : (k_i\cdot\psi(z_i))(\epsilon_i\cdot\psi) :\rangle . \quad \text{(III.6)}$$

The integral

$$\int[d\psi]\epsilon_1\cdot\psi(z_1))(\epsilon_2\cdot\psi(z_2))\prod_{i=3}^{N} : (k_i\cdot\psi(z_i))(\epsilon_i\cdot\psi) : \exp\left(-1/2\int\psi\bar\partial\psi\right) , \quad \text{(III.7)}$$

can therefore be written in terms of the following $(2N-2)\times(2N-2)$ matrix:

$$M' = \begin{pmatrix} A & -C'^{\mathrm{T}} \\ C' & B \end{pmatrix} , \qquad \text{(III.8)}$$

composed of the block matrices A, B given in (II.6) and C' for which we have

$$C'_{i,i} = 0\,, \quad C'_{ij} = \frac{\epsilon_i \cdot k_j}{z_i - z_j}\,, \qquad i = 1, 2, ..., N\,, \quad j = 3, 4, ..., N\,, \quad j \neq i\,. \qquad \text{(III.9)}$$

These matrices are of sizes $(N-2)\times(N-2)$, $N \times N$ and $N \times (n-2)$, respectively, because the vertex operators corresponding to particles 1 and 2 do not have corresponding $k_i \cdot \psi$.

This is not yet the Pfaffian of eq. (II.4) because the matrix C' has 0's on the diagonal since the self contraction $\langle : (k_i \cdot \psi(z_i))(\epsilon_i \cdot \psi(z_i)) : \rangle$ vanishes. This self contraction must be replaced by the bosonic contraction of a ∂X field with the plane-wave factor just as in ref. [28, eq. (6.2.25)] (see also [7, 21]),

$$: (\epsilon_i \cdot \partial X(z_i)) e^{i \sum_l k_l X(z_l)} : \sim \left(-2\alpha' \sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l} \right) : e^{i \sum_{l \neq i} k_l X(z_l)} : + O(z_i - z_l)\,, \qquad \text{(III.10)}$$

providing the correct factor to add to the diagonal of the matrix C'

$$C_{i,i} = -\sum_l \frac{\epsilon_i \cdot k_l}{z_i - z_l}\,, \quad C_{ij} = C'_{ij}\,, \quad j \neq i\,, \qquad \text{(III.11)}$$

and thus matching the Pfaffian of the matrix Ψ^{12}_{12}. After including the superghost correlator $\langle e^{\varphi_1} e^{\varphi_2} \rangle = z_{12}^{-1}$, we end up with $\mathrm{Pf}'\Psi$ defined in (II.4).

In the approach of ref. [7] there are here no other contractions to perform because the ∂X field by construction is taken to be a momentum P field frozen by the scattering equations. However, here the story is different as we are here dealing with actual string theory. The ∂X fields do have nonvanishing OPEs with other ∂X fields. This is also the mechanism that prevents unwanted tachyon poles from appearing in the string theory amplitudes.

In order to derive these remaining terms, one can simply recursively apply Wick's theorem. In the first step, one finds OPEs only between ∂X's and the plane-wave factor; this gives the Pfaffian in eq. (II.4). In the second step, one performs all possible contractions between only two ∂X's, the rest as before; this yields a sum of Pfaffians where two more sets of lines and rows have been crossed out, with a corresponding $\langle \partial X(z) \partial X(w) \rangle \sim (z-w)^{-2}$ propagator in front of it (this induces a weighing $1/\alpha'$ compared to the term of the first step). By iterating the process, one finally deduces that the chiral kinematic correlator is expressed as a sum of Pfaffians and the full answer is

$$\mathcal{A}_N = \left(\frac{g_o}{\sqrt{\alpha'}}\right)^{N-2} \alpha'^{N-3} \int \prod_{i=2}^{N-2} dz_i \prod_{1\le i<j\le N} |z_{ij}|^{2\alpha' k_i\cdot k_j} \times (z_1 - z_{N-1})(z_{N-1}-z_N)(z_N - z_1)\times$$
$$\left(\mathrm{Pf}'(\Psi) + \sum_{k=1}^{\lfloor \frac{N}{2}\rfloor} \frac{1}{(2\alpha')^k} \sum_{\substack{\text{distinct pairs}\\ (i_3,i_4),\dots,(i_{2k-1},i_{2k})}} \prod_{p=3}^{2k-1} \frac{(\epsilon_{i_p}\cdot\epsilon_{i_{p+1}})}{(z_{i_p i_{p+1}})^2} \mathrm{Pf}\,'(\Psi^{i_3 i_4\dots i_{2k}}_{i_3 i_4\dots i_{2k}})\right), \tag{III.12}$$

where $z_{ij} = z_i - z_j$ and a global normalization factor has been set to 1 and where $\mathrm{Pf}'(\Psi^{i_3i_4\dots i_{2k}}_{i_3i_4\dots i_{2k}})$ stands for $\frac{1}{z_{12}}\mathrm{Pf}(\Psi^{12i_3i_4\dots i_{2k}}_{12i_3i_4\dots i_{2k}})$.

IV. FROM STRING THEORY TO SCATTERING EQUATIONS

In the previous section we have identified which piece of string theory gives rise to the Pfaffian of eq. (II.4), and which yields additional terms. We will now show that the additional terms, through partial integrations, can be put in a form that makes them proportional to the scattering equations, causing them to vanish with the alternative integration measure that imposes scattering equations as a delta function constraint. In this form the full expression can be integrated over these two different measures, both yielding the correct field theory result when taking the $\alpha' \to 0$ limit. Some simple examples will illustrate this.

Let us for simplicity focus first on the four-gluon amplitude. As explained in the previous section, it takes the form

$$\mathcal{A}_4(1,2,3,4) = \left(\frac{g_o}{\sqrt{\alpha'}}\right)^2 \alpha' \int_0^1 \left(\mathrm{Pf}'(\Psi) + \frac{(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)}{2\alpha' z_2^2}\right) z_2^{2\alpha' k_1\cdot k_2}(1-z_2)^{2\alpha' k_2\cdot k_3}\, dz_2\,, \tag{IV.1}$$

where as usual $z_1 = 0$, $z_3 = 1$, and $z_4 = \infty$. The additional piece proportional to $1/\alpha'$ is crucial in the string theory context, as it removes a tachyon pole and allows the limit $\alpha' \to 0$ to be taken, yielding the field theory answer.

One notices that the term

$$\delta\mathcal{A}_4 = \int_0^1 dz_2 \frac{1}{z_2^2} \exp\left(2\alpha' k_1\cdot k_2 \log(z_2) + 2\alpha' k_2\cdot k_3 \log(1-z_2)\right), \tag{IV.2}$$

can be integrated by part to give

$$\begin{aligned}\delta\mathcal{A}_4 &= -\int_0^1 dz_2\,\partial_{z_2}\left(\frac{1}{z_2}\right)\exp\left(2\alpha' k_1\cdot k_2\log(z_2)+2\alpha' k_2\cdot k_3\log(1-z_2)\right)\\ &= \int_0^1 dz_2\,\frac{1}{z_2}\,\partial_{z_2}\Big(\exp\left(2\alpha' k_1\cdot k_2\log(z_2)+2\alpha' k_2\cdot k_3\alpha'\log(1-z_2)\right)\Big)\,. \end{aligned}\tag{IV.3}$$

By analytic continuation we can choose a kinematic region where the boundary terms vanish. Eq. (IV.3) can be rewritten as

$$\delta\mathcal{A}_4 = \alpha'\int_0^1 dz_2\,\frac{1}{z_2}\Big(\frac{k_1\cdot k_2}{z_2}+\frac{k_2\cdot k_3}{1-z_2}\Big)\Big(\exp\left(2\alpha' k_1\cdot k_2\log(z_2)+2\alpha' k_2\cdot k_3\alpha'\log(1-z_2)\right)\Big)\,,\tag{IV.4}$$

where we recognize the four-point scattering equation

$$S_2 = \frac{k_1\cdot k_2}{z_2}+\frac{k_2\cdot k_3}{1-z_2}\ .\tag{IV.5}$$

From this we can write new dual model prescription for gauge field amplitudes by evaluating the string integrand on the solution of the scattering equation by inserting the delta function factor given in (II.9). Since the $1/\alpha'$ term is proportional to the scattering equation in (IV.1) we have (using the relation between the open-string coupling constant the Yang-Mills coupling constant in ten dimensions $g_o = g_{\rm YM}\sqrt{\alpha'}$)

$$\mathfrak{A}_4(1,2,3,4) = g_{\rm YM}^2\int_0^1 \mathrm{Pf}'(\Psi)\, z_2^{2\alpha' k_1\cdot k_2-1}(1-z_2)^{2\alpha' k_2\cdot k_3-1}\,\delta(S_2)\,dz_2\,.\tag{IV.6}$$

Another ordering of the external legs will yield another scattering equation. The various ordered amplitudes are of course related by the action of the momentum kernel [12].

We see that in string theory we can trade the explicit $1/\alpha'$ term by an integration over a term proportional to the scattering equation. In string theory this term of course gives a contribution.

The same phenomenon occurs for amplitudes with higher N. It gets increasingly tedious to carry out the sequence of partial integrations, but the origin of the mechanism seems to be closely related to a similar situation in string-based rules, proven in Appendix B of ref. [32] (see also ref. [21]). In this procedure, the last step is always a single integration by part on a variable that has been isolated, which, when the partial derivative hits the Koba-Nielsen factor, brings down a scattering equation in the integrand, just as in this four-point example, leading the following form for the new dual model amplitude prescription

$$\mathfrak{A}_N(1,2,3,\ldots,N) = g_{\rm YM}^{N-2}\int\prod_{i=2}^{N-2}dz_i\prod_{1\leq i<j\leq N}(z_i-z_j)^{2\alpha' k_i\cdot k_j}\times$$
$$\frac{(z_1-z_{N-1})^2(z_{N-1}-z_N)^2(z_N-z_1)^2}{\prod_{i=1}^N(z_i-z_{i+1})}\times {\rm Pf}'(\Psi)\prod_{i\neq 1,N-1,N}\delta(S_i)\,. \quad \text{(IV.7)}$$

After having done these partial integrations, the new integrand now has the property that it corresponds to the CHY integrand at first order in $1/\alpha'$. As we already emphasized, this is natural from the point of view of the ambitwistor string models [7, 9].

Once again, the reason for this is because we have shown that the higher order term in $1/\alpha'$, after IBP reduction, is exactly killed by the scattering equation constraint. Although we calculate the Pfaffian according to standard conformal field theory rules, the integrations by part of the $1/\alpha'$-terms are only a valid operation in the string theory integrand. This is why one can set α' to zero in the integrand to recover the CHY prescription, without meeting any singularities. This is very different from the usual infinite tension limit of string theory where one needs to scale the variables of integrations to reach the pinching limits of the string integrand (see [33] for a recent discussion).

In the Gross-Mende $\alpha'\to\infty$ limit, the $1/\alpha'$ correction for the string amplitudes in (III.12) vanish. Consequently, the string theory amplitude and the new dual model prescription in (IV.7) have the $\alpha'\to\infty$ Gross-Mende saddle point, but with different prefactors compared to the usual high-energy limit of the string theory amplitudes.

V. AMPLITUDES WITH FERMIONS AND MIXED AMPLITUDES

A. The Four-Fermion Amplitude

In this section we show the generality of the delta function measure (II.9) by calculating a few tree level amplitudes directly from string theory integrands. As a first example, we check how fermion amplitudes can come out from our prescription. In the case of the fermion four-point amplitude one has [31, 34]

$$\mathcal{A}_4 = \left(\frac{g_o}{\sqrt{\alpha'}}\right)^2\alpha'\int_0^1 dz_2 z_2^{-2\alpha' t-1}(1-z_2)^{-2\alpha' s-1}\left[(1-z_2)(\bar v_1\gamma^\mu u_2)(\bar v_3\gamma_\mu u_4)-z_2(\bar v_1\gamma^\mu u_4)(\bar v_3\gamma_\mu u_2)\right]\,, \quad \text{(V.1)}$$

where $\bar{v}_i$ and u_i are the incoming and outgoing fermion wave functions. As is well known, this string theory integral can be done in terms of two beta functions. In the field theory limit $\alpha' \to 0$ it of course yields the correct answer corresponding to the two channels s and t.

But this integral also defines the correct field theory limit if we instead integrate over the delta function measure given by the scattering equations as provided by the additional measure factor (II.9),

$$\mathfrak{A}_4 = g_{\rm YM}^2 \int_0^1 dz_2\, \delta\left(S_2\right) z_2^{-2\alpha' t-2}(1-z_2)^{-2\alpha' s-2} \times \\ \left((1-z_2)(\bar{v}_1\gamma^\mu u_2)(\bar{v}_3\gamma_\mu u_4) - z_2(\bar{v}_1\gamma^\mu u_4)(\bar{v}_3\gamma_\mu u_2)\right)\,, \tag{V.2}$$

where S_2 is the scattering equation in k_2. Explicitly, we get in the limit $\alpha' \to 0$,

$$A_4 = g_{\rm YM}^2 \left[\frac{1}{s}(\bar{v}_1\gamma^\mu u_2)(\bar{v}_3\gamma_\mu u_4) - \frac{1}{t}(\bar{v}_1\gamma^\mu u_4)(\bar{v}_3\gamma_\mu u_2)\right]\,, \tag{V.3}$$

which is the correct field theory answer.

B. The Two-Fermion Two-Gluon Amplitude

As another example of how this procedure works, one can similarly work out the expression for the two-fermion two-gluon amplitude. For the corresponding string theory integrand see, *e.g.*, refs. [31, 34]. This amplitude has also been considered in the ambitwistor framework of ref. [9], but here we explain how to derive the result starting from ordinary string theory.

We have explicitly verified in this case that the delta function measure (II.9) yields exactly the tree level amplitude in the limit $\alpha' \to 0$. In this case it follows in essentially one line, as there are no cancellations between tachyonic terms in the amplitudes. It indeed seems that we can directly take superstring integrands for amplitudes including fermions and integrate over a measure that localizes exactly on the scattering equations.

C. The Five-Point Mixed Scalar-Gluon Amplitude

To give further credence to the procedure, let us finally consider a five-point case involving mixed external states of four scalars and a gluon. Because of the combination of scalars and a gluon, the string theory integrand of this amplitude contains two tachyonic terms canceling each other in the integral, and we again first make this cancellation manifest by means of a single partial integration. We borrow the expression for the string theory integrand of the amplitude from ref. [29] (the explicit prefactor K_a in front of the integral can be found in that paper, but we do not need it for the arguments here),

$$\begin{aligned}\mathcal{A}_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_a \int \Big(\prod_{k=4}^{5} dz_k\Big)\Big(\prod_{i<j} z_{ij}^{\alpha' s_{ij}}\Big)\Big(\frac{1}{z_{35}}\Big(\frac{(\zeta_5\cdot k_4)}{z_{45}}\frac{\alpha' s_{12} z_{34}}{z_{24}z_{13}z_{14}z_{23}}\Big) \\ + \frac{(\zeta_5\cdot k_1)}{z_{15}z_{24}}\Big(\frac{(1-\alpha' s_{24})}{z_{24}z_{13}} + \frac{\alpha' s_{24}}{z_{14}z_{23}}\Big) + \frac{(\zeta_5\cdot k_2)}{z_{14}z_{25}}\Big(\frac{(1-\alpha' s_{14})}{z_{14}z_{23}} + \frac{\alpha' s_{14}}{z_{13}z_{24}}\Big)\Big)\,,\end{aligned} \tag{V.4}$$

where $s_{ij} = 2k_i \cdot k_j$, ζ_i and k_i are the polarizations and momenta. Using the integration-by-parts relation in z_4 for the terms with ζ_5 dotted with k_1 and k_2 we can rewrite these explicit $1/\alpha'$ terms exactly as in the pure gluon case. This replaces that term by the scattering equation in leg 4, *e.g.* $(1-\alpha' s_{14}) \to (\mathrm{IBP}(S)_4 z_{14} - \alpha' s_{14})$. Using the prescription (II.9), we get

$$\begin{aligned}\mathfrak{A}_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_a \int \Big(\prod_{k=4}^{5} dz_k\Big)\Big(\prod_{i<j} z_{ij}^{\alpha' s_{ij}}\Big)\delta(S_4)\,\delta(S_5)\,\frac{z_{12}^2 z_{23}^2 z_{31}^2}{\prod_{1\le i\le 5}(z_i - z_{i+1})} \\ \Big(\frac{1}{z_{35}}\Big(\frac{(\zeta_5\cdot k_4)}{z_{45}}\frac{\alpha' s_{12} z_{34}}{z_{24}z_{13}z_{14}z_{23}}\Big) + \frac{(\zeta_5\cdot k_1)}{z_{15}z_{24}}\Big(\frac{\alpha' z_{24} S_4 - \alpha' s_{24}}{z_{24}z_{13}} + \frac{\alpha' s_{24}}{z_{14}z_{23}}\Big) \\ + \frac{(\zeta_5\cdot k_2)}{z_{14}z_{25}}\Big(\frac{\alpha' z_{14} S_4 - \alpha' s_{14}}{z_{14}z_{23}} + \frac{\alpha' s_{14}}{z_{13}z_{24}}\Big)\Big)\,,\end{aligned} \tag{V.5}$$

where the delta function measure now has been adapted to the situation where legs $(1,2,3)$ are fixed as $(-\infty,0,1)$ following the convention used in in [29]. We see that the delta function effectively removes the $1/\alpha'$ term after having canceled the tachyon pole explicitly by use of the partial integration that introduces the scattering equation in leg 4, $S_4 = 0$. After some algebra we arrive in the limit of $(\alpha' \to 0)$ at

$$A_5(\phi_1,\phi_2,\phi_3,\phi_4,g_5) = K_{ft}\Big((\zeta_5\cdot k_1)\Big(\frac{1}{s_{23}} - \frac{s_{34}}{s_{23}s_{15}}\Big) + (\zeta_5\cdot k_2)\Big(\frac{1}{s_{23}}\Big) + (\zeta_5\cdot k_4)\Big(\frac{s_{12}}{s_{23}s_{45}}\Big)\Big)\,, \tag{V.6}$$

which is the correct result. Here K_{ft} denotes the prefactor of the amplitude in the limit $(\alpha' \to 0)$. There thus seem to be no additional problems associated with mixed amplitudes. We therefore expect that any generic amplitude involving gluons, scalars and fermions in any combination can be computed in the same manner, imposing the same delta function measure after having manifestly canceled all tachyon poles (if present) through integrations by parts.

VI. CONCLUSION

We have provided an natural interpolation between the CHY prescription for tree level amplitudes in field theory and the Gross-Mende limit of string theory. We have introduced a new kind of dual model defined as the string theory localized on the surface of the solutions to the scattering equations. We have shown how this can be used to derive new amplitudes, those with external fermions, on the basis of merging string theory with the scattering equations. Numerous other examples can be derived similarly: mixed amplitudes with gluons and fermions, scalars and fermions, and so on. We have provided some examples, and argued that the general prescription is to rewrite the string integrand by manifestly canceling tachyon poles and then evaluate the string integrand on the solutions to the scattering equations. It would be very interesting to relate the prescription given in this paper to an α'-extension of Berkovits' prescription given in [8].

From this prescription no further calculations are necessary since one can use the form of the string integrand with the $1/\alpha'$ expansion of Pfaffian, perform the partial integration to remove the second order poles, and evaluate it on the scattering equations.

Another very important question concerns closed string. The whole CHY construction, and the subsequent ambitwistor/pure spinor models are intrinsically closed-string like models. The way in which scalar, gauge, or gravity interactions are implemented at the integrand level is indeed highly reminiscent of the string theory realization of these interactions, by the left-right moving mixing [35]. It is an interesting question how the prescription used here transcribe into closed-string language. The reason this is nontrivial is the absence of chirality in the closed string, where one sector is holomorphic while the other is antiholomorphic. This is very different from the CHY prescription, where both sectors of the theory possess the same chirality as in ambitwistor models.

An obvious question is what happens at the genuine quantum level, *i.e.* at loop order. Tree level amplitudes correspond to vertex operators on the sphere. Using again string theory as the guide, one would be led to consider the corresponding scattering equations associated with the N external momenta but integrated over correlation functions on higher genus surfaces. Integrations will remain even after imposing the scattering equations. It would be interesting to see if they reproduce the result of field theory loop computations in the $\alpha' \to 0$ limit.

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Appendix A: Appendix: Further details on integration by parts.

In Section 5 we did not want to clutter the text with more explicit details of higher-point issues with respect to the needed integration by parts. In this Appendix we provide a few details of what happens at five points.

To illustrate in this slightly more complicated case how to do the integration by parts, we consider the term originating from the $\partial X(z_i)\partial X(z_j)$ contractions in the ghost picture changing formalism. As in the four-point case we choose to remove lines 1 and 2 in the matrix of the Pfaffian.

Explicitly, we have the following two types of terms that are of type $\frac{1}{\alpha'}$

$$\begin{aligned}\sim \ldots - \frac{\epsilon_1\cdot k_3\epsilon_2\cdot\epsilon_3\epsilon_4\cdot\epsilon_5}{\alpha' z_{12}z_{13}z_{23}z_{45}^2} + \frac{\epsilon_1\cdot\epsilon_3\epsilon_2\cdot k_3\epsilon_4\cdot\epsilon_5}{\alpha' z_{12}z_{13}z_{23}z_{45}^2} - \frac{\epsilon_1\cdot k_4\epsilon_2\cdot\epsilon_4\epsilon_3\cdot\epsilon_5}{\alpha' z_{12}z_{14}z_{24}z_{35}^2} \\ + \frac{\epsilon_1\cdot\epsilon_4\epsilon_2\cdot k_4\epsilon_3\cdot\epsilon_5}{\alpha' z_{12}z_{14}z_{24}z_{35}^2} - \frac{\epsilon_1\cdot k_5\epsilon_2\cdot\epsilon_5\epsilon_3\cdot\epsilon_4}{\alpha' z_{12}z_{15}z_{25}z_{34}^2} + \frac{\epsilon_1\cdot\epsilon_5\epsilon_2\cdot k_5\epsilon_3\cdot\epsilon_4}{\alpha' z_{12}z_{15}z_{25}z_{34}^2}\,,\end{aligned} \tag{A.1}$$

and

$$\begin{aligned}&\sim \ldots - \frac{\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4\left(\frac{\epsilon_5\cdot k_1}{z_{15}}+\frac{\epsilon_5\cdot k_2}{z_{25}}+\frac{\epsilon_5\cdot k_3}{z_{35}}+\frac{\epsilon_5\cdot k_4}{z_{45}}\right)}{\alpha' z_{12}^2 z_{34}^2} \\ &- \frac{\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_5\left(\frac{\epsilon_4\cdot k_1}{z_{14}}+\frac{\epsilon_4\cdot k_2}{z_{24}}+\frac{\epsilon_4\cdot k_3}{z_{34}}-\frac{\epsilon_4\cdot k_5}{z_{45}}\right)}{\alpha' z_{12}^2 z_{35}^2} + \frac{\epsilon_1\cdot\epsilon_2\epsilon_4\cdot\epsilon_5\left(-\frac{\epsilon_3\cdot k_1}{z_{13}}-\frac{\epsilon_3\cdot k_2}{z_{23}}+\frac{\epsilon_3\cdot k_4}{z_{34}}+\frac{\epsilon_3\cdot k_5}{z_{35}}\right)}{\alpha z_{12}^2 z_{45}^2}\,.\end{aligned} \tag{A.2}$$

We will now show that in all cases we can find integration-by-part relations that are equivalent to inserting the scattering equations.

- In the first equation (A.1) we will in terms 1 - 2 use the relation involving z_4, while for the terms 3 - 6 we will instead use the integration-by-part relation in z_3.

- In the second equation (A.2) for the first term we will use the relation in the variable z_3, except for the next-to-last term where we will use the one for z_4. For the second and third terms here we will use those in z_1, except for the first terms where we use those in z_3 and z_4.

By this prescription we have absorbed all the $\frac{1}{\alpha'}$ terms of the five-point amplitude. Again we observe that at the solution to the scattering equations the reduced Pfaffian will be unchanged, since all we have done is to turn them into terms proportional to the scattering equations.

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[1] F. Cachazo, S. He and E. Y. Yuan, "Scattering Equations and Klt Orthogonality," Phys. Rev. D **90** (2014) 065001 [arXiv:1306.6575 [hep-th]].

[2] F. Cachazo, S. He and E. Y. Yuan, "Scattering of Massless Particles in Arbitrary Dimension," arXiv:1307.2199 [hep-th].

[3] F. Cachazo, S. He and E. Y. Yuan, "Scattering of Massless Particles: Scalars, Gluons and Gravitons," JHEP **1407** (2014) 033 [arXiv:1309.0885 [hep-th]].

[4] L. Dolan and P. Goddard, "Proof of the Formula of Cachazo, He and Yuan for Yang-Mills Tree Amplitudes in Arbitrary Dimension," JHEP **1405** (2014) 010 [arXiv:1311.5200 [hep-th]].

[5] R. Britto, F. Cachazo, B. Feng and E. Witten, "Direct Proof of tree-level Recursion Relation in Yang-Mills Theory," Phys. Rev. Lett. **94** (2005) 181602 [hep-th/0501052].

[6] D. J. Gross and P. F. Mende, "String Theory Beyond the Planck Scale," Nucl. Phys. B **303** (1988) 407.

[7] L. Mason and D. Skinner, "Ambitwistor strings and the scattering equations," JHEP **1407** (2014) 048 [arXiv:1311.2564 [hep-th]].

[8] N. Berkovits, "Infinite Tension Limit of the Pure Spinor Superstring," JHEP **1403** (2014) 017 [arXiv:1311.4156 [hep-th]].

[9] T. Adamo, E. Casali and D. Skinner, "Ambitwistor strings and the scattering equations at one loop," JHEP **1404** (2014) 104 [arXiv:1312.3828 [hep-th]].

[10] H. Gomez and E. Y. Yuan, "N-point tree-level scattering amplitude in the new Berkovits' string," JHEP **1404** (2014) 046 [arXiv:1312.5485 [hep-th]].

[11] N. E. J. Bjerrum-Bohr, P. H. Damgaard, B. Feng and T. Sondergaard, "Gravity and Yang-

Mills Amplitude Relations," Phys. Rev. D **82** (2010) 107702 [arXiv:1005.4367 [hep-th]]; "Proof of Gravity and Yang-Mills Amplitude Relations," JHEP **1009** (2010) 067 [arXiv:1007.3111 [hep-th]].

[12] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, "The Momentum Kernel of Gauge and Gravity Theories," JHEP **1101** (2011) 001 [arXiv:1010.3933 [hep-th]].

[13] Z. Bern, J. J. M. Carrasco and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. D **78** (2008) 085011 [arXiv:0805.3993 [hep-ph]].

[14] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "Minimal Basis for Gauge Theory Amplitudes," Phys. Rev. Lett. **103** (2009) 161602 [arXiv:0907.1425 [hep-th]].

[15] S. Stieberger, "Open & Closed Vs. Pure Open String Disk Amplitudes," arXiv:0907.2211 [hep-th].

[16] S. G. Naculich, "Scattering equations and virtuous kinematic numerators and dual-trace functions," arXiv:1404.7141 [hep-th].

[17] S. G. Naculich, "Scattering equations and BCJ relations for gauge and gravitational amplitudes with massive scalar particles," JHEP **1409** (2014) 029 [arXiv:1407.7836 [hep-th]].

[18] B. U. W. Schwab, "Subleading Soft Factor for String Disk Amplitudes," JHEP 08 (2014) 062, arXiv:1406.4172 [hep-th].

[19] S. Litsey and J. Stankowicz, "Kinematic Numerators and a Double-Copy Formula for N = 4 Super-Yang-Mills Residues," Phys. Rev. D **90** (2014) 025013 [arXiv:1309.7681 [hep-th]].

[20] R. Monteiro and D. O'Connell, "The Kinematic Algebras from the Scattering Equations," JHEP **1403** (2014) 110 [arXiv:1311.1151 [hep-th]].

[21] C. R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation," Nucl. Phys. B **873** (2013) 419 [arXiv:1106.2645 [hep-th]].

[22] C. R. Mafra, O. Schlotterer and S. Stieberger, "Complete N-Point Superstring Disk Amplitude II. Amplitude and Hypergeometric Function Structure," Nucl. Phys. B **873** (2013) 461 [arXiv:1106.2646 [hep-th]].

[23] J. Broedel, O. Schlotterer and S. Stieberger, "Polylogarithms, Multiple Zeta Values and Superstring Amplitudes," Fortsch. Phys. **61** (2013) 812 [arXiv:1304.7267 [hep-th]].

[24] L. A. Barreiro and R. Medina, "RNS Derivation of N-Point Disk Amplitudes from the Revisited S-Matrix Approach," Nucl. Phys. **B886**, 870 (2014), arXiv:1310.5942 [hep-th].

[25] N. Berkovits, "Relating the RNS and Pure Spinor Formalisms for the Superstring," JHEP

0108 (2001) 026 [hep-th/0104247].

[26] D.E. Roberts, Mathematical Structure of Dual Amplitude s, PhD thesis (unpublished), Durham University Library (1972), chapter IV; D. B. Fairlie and D. E. Roberts, "Dual Models Without Tachyons - A New Approach," PRINT-72-2440.

[27] D. B. Fairlie, "A Coding of Real Null Four-Momenta into World-Sheet Co-Ordinates," Adv. Math. Phys. **2009** (2009) 284689 [arXiv:0805.2263 [hep-th]].

[28] J. Polchinski, "String theory. Vol. 1: An introduction to the bosonic string," Cambridge, UK: Univ. Pr. (1998) 402 p

[29] S. Stieberger and T. R. Taylor, "Supersymmetry Relations and MHV Amplitudes in Superstring Theory," Nucl. Phys. B **793** (2008) 83 [arXiv:0708.0574 [hep-th]].

[30] S. Stieberger and T. R. Taylor, "Closed String Amplitudes as Single-Valued Open String Amplitudes," Nucl. Phys. B **881**, 269 (2014) [arXiv:1401.1218 [hep-th]].

[31] D. Friedan, E. J. Martinec and S. H. Shenker, "Conformal Invariance, Supersymmetry and String Theory," Nucl. Phys. B **271** (1986) 93.

[32] Z. Bern and D. A. Kosower, "Color Decomposition of One Loop Amplitudes in Gauge Theories," Nucl. Phys. B **362** (1991) 389.

[33] P. Tourkine, "Tropical Amplitudes," arXiv:1309.3551 [hep-th].

[34] J. Cohn, D. Friedan, Z. a. Qiu and S. H. Shenker, "Covariant Quantization of Supersymmetric String Theories: the Spinor Field of the Ramond-Neveu-Schwarz Model," Nucl. Phys. B **278** (1986) 577.

[35] A. Ochirov and P. Tourkine, "Bcj Duality and Double Copy in the Closed String Sector," JHEP **1405** (2014) 136 [arXiv:1312.1326 [hep-th]].