Graviton-Photon Scattering

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Abstract

We use that the gravitational Compton scattering factorizes on the Abelian QED amplitudes to evaluate various gravitational Compton processes. We examine both the QED and gravitational Compton scattering from a massive spin-1 system by the use of helicity amplitudes. In the case of gravitational Compton scattering we show how the massless limit can be used to evaluate the cross-section for graviton-photon scattering and discuss the difference between photon interactions and the zero mass spin-1 limit. We show that the forward scattering cross-section for graviton photo-production has a very peculiar behaviour, differing from the standard Thomson and Rutherford cross-sections for a Coulomb-like potential.

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1 Introduction

The treatment of electromagnetic interactions in quantum mechanics is well known and the discussion of electromagnetic effects via photon exchange is a staple of the graduate curriculum. In particular the photon exchange between charged particles can be shown to give rise to the Coulomb potential as well as to various higher order effects such as the spin-orbit and Darwin interactions [1]. The fact that the photon carries spin-1 means that the electromagnetic current is a four-vector and manipulations involving such vector quantities are familiar to most physicists. In a similar fashion, graviton exchange between a pair of masses can be shown to generate the gravitational potential as well as various higher order effects, but in this case the fact that the graviton is a spin-2 particle means that the gravitational "currents" are second rank tensors and the graviton propagator is a tensor of rank four. The resultant proliferation of indices is one reason why this quantum mechanical discussion of graviton exchange effects is not generally treated in introductory texts [2].

Recently, by using string-inspired methods, it has been demonstrated that the gravitational interaction factorizes in such a way that a gravitational amplitude can be written as the product of two more familiar vector amplitudes [3–7]. This factorization property, totally obscure at the level of the action, is a fundamental properties of gravity and has deep consequences at loop amplitude level, since many gravitational amplitudes can be constructed by an appropriate product of gauge theory integrand numerators [8]. This has triggered a lot of new results in extended supergravity [9–20], but quite remarkably these techniques can be applied as well to pure gravity [7, 21].

One remarkable property of amplitudes with emission of one or two gravitons is its factorization in terms of *Abelian* QED amplitudes [7, 22]. This factorization has the important consequence that the low-energy limit of the gravitational Compton amplitude for graviton photo-production is directly connected to the low-energy theorem for the QED Compton amplitudes [7].

In a previous paper [22] this property was used to evaluate processes such as graviton photo-production and gravitational Compton scattering for both spin-0 and spin-$\frac{1}{2}$ systems by simply evaluating the corresponding electromagnetic amplitude for Compton scattering. This permits the treatment of gravitational effects without long tedious computations, since they are now no more difficult than corresponding electromagnetic calculations. The simplicity optioned through the factorization have important consequences for the computations of long-range corrections to interaction potentials containing loops of intermediate photons- or gravitons [23–26]. In this paper we extend such considerations to electromagnetic and gravitational interactions of spin-1 systems. These calculations are useful not only as a generalization of our previous results but also, since the photon carries spin one, these methods can be used to consider the case of photon-graviton scattering, although there are subtleties in this case due to gauge invariance.

In all the cases under study, we show that the low-energy limit of the differential cross-section has an universal behaviour independent of the spin of the matter field on which photon or graviton is scattered. We demonstrate that this is a consequence of the well-known universal low-energy behaviour in quantum electrodynamics (QED) and the squaring relations between gravitational and electromagnetic processes.

The forward differential cross-section for the Compton scattering of photons on a massive target has the well-known constant behaviour of a Thomson cross-section

$$\lim_{\theta_L\to 0}\frac{d\sigma^{\rm Comp}_{lab,S}}{d\Omega}=\frac{\alpha^2}{2m^2}\,, \tag{1.1}$$

and the small-angle limit of gravitational Compton scattering of gravitons on a massive target has the expected behaviour due to a $1/r$ long-range potential of a Rutherford like cross-section

$$\lim_{\theta_L\to 0}\frac{d\sigma^{\rm g-Comp}_{lab,S}}{d\Omega}=\frac{16G^2m^2}{\theta_L^4}\,. \tag{1.2}$$

We explain in section 6 why this formula reproduces the small-angle limit of the classical cross-section for light bending in a Schwarzschild background.

The forward limit of the graviton photo-production cross-section has the rather unique behaviour

$$\lim_{\theta_L\to 0}\frac{d\sigma^{\rm photo}_{lab,S}}{d\Omega}=\frac{4G\alpha}{\theta_L^2}\,. \tag{1.3}$$

This limit is not only independent on the spin S but as well on the mass m of the target. The small-angle dependence is typical of an effective $1/r^2$ potential. We provide an explanation for this in section 6.

It may be very difficult to detect a single graviton [27] but photons are easily detected so it is would be interesting to be able to use the graviton photo-production process to provide an indirect detection of a graviton. The cross-section in Eq. (1.3) is suppressed by a power a Newton's constant G but, being independent of the mass m of the target, one can discriminate this effect when comparing to Compton scattering.

In the next section then we quickly review the electromagnetic interaction and derive the spin-1 couplings. In section 3, we analyze the Compton scattering of a spin-1 particle. The corresponding gravitational couplings are derived in section 4 and the graviton photo-production and gravitational Compton scattering reactions are calculated via both direct and factorization methods. In section 5 we discuss photon-graviton scattering and subtleties associated with gauge invariance. In section 6 we consider the forward small-angle limit of the various scattering cross-section derived in the previous section. We show that Compton, graviton photo-production and the gravitational Compton scattering have very different behaviour in each case. We summarize our findings in a brief concluding section.

2 Brief Review of Electromagnetism

In this section we present a quick review of the electromagnetic and gravitational interactions and the findings of our previous work. The electromagnetic interaction of a system may be found by using the minimal substitution $i\partial_\mu \rightarrow iD_\mu = i\partial_\mu - eA_\mu$ in the free particle Lagrangian, where A_μ is the photon field. In this way the Klein-Gordon Lagrangian

$$\mathcal{L}_0^{S=0} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \,, \tag{2.1}$$

which describes a free charged spin-less field, becomes

$$\mathcal{L}^{S=0} = (\partial_\mu - ieA_\mu)\phi^\dagger (\partial^\mu + ieA^\mu)\phi + m^2 \phi^\dagger \phi \,, \tag{2.2}$$

after this substitution. The corresponding interaction Lagrangian can then be identified as

$$\mathcal{L}_{int}^{S=0} = ieA_\mu \phi^\dagger \overleftrightarrow{\nabla}^\mu \phi + e^2 A^\mu A_\mu \phi^\dagger \phi \,, \tag{2.3}$$

where

$$C \overleftrightarrow{\nabla} D := C\vec{\nabla} D - (\vec{\nabla} D)C \,. \tag{2.4}$$

Similarly, for spin-$\frac{1}{2}$, the free Dirac Lagrangian

$$\mathcal{L}_0^{S=\frac{1}{2}} = \bar{\psi}(i \not{\nabla} - m)\psi \,, \tag{2.5}$$

becomes

$$\mathcal{L}^{S=\frac{1}{2}} = \bar{\psi}(i \not{\nabla} - e \not{A} - m)\psi\,, \tag{2.6}$$

and the interaction Lagrangian is found to be

$$\mathcal{L}_{int}^{S=\frac{1}{2}} = -e\bar{\psi} \not{A}\psi\,. \tag{2.7}$$

The resulting single-photon vertices are then

$$\langle p_f|V_{em}^{(1)\mu}|p_i\rangle_{S=0} = -ie(p_f + p_i)^\mu\,, \tag{2.8}$$

for spin-0 and

$$\langle p_f|V_{em}^{(1)\mu}|p_i\rangle_{S=\frac{1}{2}} = -ie\bar{u}(p_f)\gamma^\mu u(p_i)\,, \tag{2.9}$$

for spin-$\frac{1}{2}$, and in the case of spin 0 there exists also a two-photon (”seagull”) vertex

$$\langle p_f|V_{em}^{(2)\mu\nu}|p_i\rangle_{S=0} = 2ie^2\eta^{\mu\nu}\,. \tag{2.10}$$

The photon propagator in Feynman gauge is

$$D_f^{\alpha\beta}(q) = \frac{-i\eta^{\alpha\beta}}{q^2 + i\epsilon}\,. \tag{2.11}$$

The consequences of these Lagrangians were explored in ref. [22] and in the present paper we extend our considerations to the case of spin-1, for which the free Lagrangian has the Proca form

$$\mathcal{L}_0^{S=1} = -\frac{1}{2}B_{\mu\nu}^\dagger B^{\mu\nu} + m^2 B_\mu^\dagger B^\mu\,, \tag{2.12}$$

where B_μ is a spin one field subject to the constraint $\partial^\mu B_\mu = 0$ and $B_{\mu\nu}$ is the antisymmetric tensor

$$B^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu\,. \tag{2.13}$$

The minimal substitution then leads to the interaction Lagrangian

$$\mathcal{L}_{int}^{S=1} = ieA^\mu B^{\nu\dagger}\left(\eta_{\nu\alpha}\overleftrightarrow{\nabla}_\mu - \eta_{\alpha\mu}\overleftrightarrow{\nabla}_\nu\right)B^\alpha - e^2A^\mu A^\nu B^{\alpha\dagger}B^\beta(\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta})\,, \tag{2.14}$$

and the one, two photon vertices

$$\begin{aligned}
\langle p_f,\epsilon_B|V_{em}^{(1)\mu}|p_i,\epsilon_A\rangle_{S=1} &= -ie\epsilon_{B\beta}^*\left((p_f+p_i)^\mu\eta^{\alpha\beta} - \eta^{\beta\mu}p^{f\alpha} - \eta^{\alpha\mu}p^{i\beta}\right)\epsilon_{A\alpha}\,, \\
\langle p_f,\epsilon_B|V_{em}^{(2)\mu\nu}|p_i,\epsilon_A\rangle_{S=1} &= ie^2\epsilon_{B\beta}^*\left(2\eta^{\alpha\beta}\eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\beta\nu} - \eta^{\alpha\nu}\eta^{\beta\mu}\right)\epsilon_{A\alpha}\,.
\end{aligned} \tag{2.15}$$

However, Eq. (2.15) is *not* the correct result for a fundamental spin-1 particle such as the charged W boson. Because the W arises in a gauge theory, there exists an additional W-photon interaction, leading to which an "extra" contribution to the single photon vertex

$$\langle p_f, \epsilon_B | \delta V^{(1)\mu}_{em} | p_i, \epsilon_A \rangle_{S=1} = ie\epsilon^*_{B\beta} \left(\eta^{\alpha\mu}(p_i - p_f)^\beta - \eta^{\beta\mu}(p_i - p_f)^\alpha) \right) \epsilon_{A\alpha} \, . \tag{2.16}$$

The meaning of this term can be seen by using the mass-shell Proca constraints $p_i \cdot \epsilon_A = p_f \cdot \epsilon_B = 0$ to write the total on-shell single photon vertex as

$$\begin{aligned} \langle p_f, \epsilon_B | (V_{em} + \delta V_{em})^\mu | p_i, \epsilon_A \rangle_{S=1} &= -ie\epsilon^*_{B\beta} \left((p_f + p_i)^\mu \eta_{\alpha\beta} - 2\eta^{\alpha\mu}(p_i - p_f)^\beta \right. \\ &+ \left. 2\eta^{\beta\mu}(p_i - p_f)^\alpha \right) \epsilon_{A\alpha} \, , \end{aligned} \tag{2.17}$$

wherein we observe that the coefficient of the term $-\eta^{\alpha\mu}(p_i - p_f)^\beta + \eta^{\beta\mu}(p_i - p_f)^\alpha$ has been modified from unity to two. Since the rest frame spin operator can be identified via[1]

$$B^\dagger_i B_j - B^\dagger_j B_i = -i\epsilon_{ijk} \langle f | S_k | i \rangle \, , \tag{2.19}$$

the corresponding piece of the non relativistic interaction Lagrangian becomes

$$\mathcal{L}_{\rm int} = -g \frac{e}{2m} \langle f | \vec{S} | i \rangle \cdot \vec{\nabla} \times \vec{A} \, , \tag{2.20}$$

where g is the gyromagnetic ratio and we have included a factor $2m$ which accounts for the normalization condition of the spin one field. Thus the "extra" interaction required by a gauge theory changes the g-factor from its Belinfante value of unity [28] to its universal value of two, as originally proposed by Weinberg and more recently buttressed by a number of arguments [29, 30]. Use of $g = 2$ is required (as shown in [31]) in order to assure the validity of the factorization result of gravitional amplitudes in terms of QED amplitudes, as used below.

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[1] Equivalently, one can use the relativistic identity

$$\epsilon^*_{B\mu} q \cdot \epsilon_A - \epsilon_{A\mu} q \cdot \epsilon^*_B = \frac{1}{1 - \frac{q^2}{m^2}} \left(\frac{i}{m} \epsilon_{\mu\beta\gamma\delta} p_i^\beta q^\gamma S^\delta - \frac{1}{2m}(p_f + p_i)_\mu \epsilon^*_B \cdot q \epsilon_A \cdot q \right) \tag{2.18}$$

where $S^\delta = \frac{i}{2m} \epsilon^{\delta\sigma\tau\zeta} \epsilon^*_{B\sigma} \epsilon_{A\tau} (p_f + p_i)_\zeta$ is the spin four-vector.

3 Compton Scattering

The vertices given in the previous section can now be used to evaluate the Compton scattering amplitude for a spin-1 system having charge e and mass m by summing the contributions of the three diagrams shown in Figure 1, yielding

$$\begin{aligned}
\mathrm{Amp}^{\mathrm{Comp}}_{S=1} &= e^2\Big\{2\epsilon_A\cdot\epsilon_B^*\left[\frac{\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\epsilon_i\cdot\epsilon_f^*\right] \\
&- g\left[\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot\epsilon_B^*\left(\frac{\epsilon_i\cdot p_i}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f}{p_i\cdot k_f}\right)-\epsilon_A\cdot[\epsilon_i,k_i]\cdot\epsilon_B^*\left(\frac{\epsilon_f\cdot p_f}{p_i\cdot k_i}-\frac{\epsilon_f^*\cdot p_i}{p_i\cdot k_f}\right)\right] \\
&- g^2\left[\frac{1}{2p_i\cdot k_i}\epsilon_A\cdot[\epsilon_i,k_i]\cdot[\epsilon_f^*,k_f]\cdot\epsilon_B^*-\frac{1}{2p_i\cdot k_f}\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot[\epsilon_i,k_i]\epsilon_B^*\right] \\
&- \frac{(g-2)^2}{m^2}\left[\frac{1}{2p_i\cdot k_i}\epsilon_A\cdot[\epsilon_i,k_i]\cdot p_i\epsilon_B^*\cdot[\epsilon_f^*,k_f]\cdot p_f\right. \\
&- \left.\frac{1}{2p_i\cdot k_f}\epsilon_A\cdot[\epsilon_f^*,k_f]\cdot p_i\epsilon_B^*\cdot[\epsilon_i,k_i]\cdot p_i\right]\Big\}\,, \qquad (3.1)
\end{aligned}$$

with the momentum conservation condition $p_i + k_i = p_f + k_f$ and where we have defined

$$S\cdot[Q,R]\cdot T := S\cdot QT\cdot R - S\cdot RT\cdot Q.$$

We can verify the gauge invariance of the above form by noting that this amplitude can be written in the equivalent form

$$\begin{aligned}
\mathrm{Amp}^{\mathrm{Comp}}_{S=1} &= \frac{e^2}{p_i\cdot k_ip_i\cdot k_f}\Big\{2\epsilon_B^*\cdot\epsilon_A(p_i\cdot F_i\cdot F_f\cdot p_i) \\
&+ g\left[(\epsilon_B^*\cdot F_f\cdot\epsilon_A)(p_i\cdot F_i\cdot p_f)+(\epsilon_B^*\cdot F_i\cdot\epsilon_A)(p_i\cdot F_f\cdot p_f)\right] \\
&- \frac{g^2}{2}\left[p_i\cdot k_f(\epsilon_B^*\cdot F_f\cdot F_i\cdot\epsilon_A)-p_i\cdot k_i(\epsilon_B^*\cdot F_i\cdot F_f\cdot\epsilon_A)\right] \\
&- \frac{(g-2)^2}{2m^2}\left[(\epsilon_B^*\cdot F_f\cdot p_f)(p_i\cdot F_i\cdot\epsilon_A)-(\epsilon_B^*\cdot F_i\cdot p_i)(p_i\cdot F_f\cdot\epsilon_A)\right]\Big\}\,, \qquad (3.2)
\end{aligned}$$

where $F_i^{\mu\nu} = \epsilon_i^\mu k_i^\nu - \epsilon_i^\nu k_i^\mu$ and $F_f^{\mu\nu} = \epsilon_f^{*\mu} k_f^\nu - \epsilon_f^{*\nu} k_f^\mu$. Since $F_{i,f}$ are obviously invariant under the substitutions $\epsilon_{i,f} \to \epsilon_{i,f} + \lambda k_{i,f},\ i = 1, 2$, it is clear that Eq. (3.1) satisfies the gauge invariance strictures

$$\epsilon_f^{*\mu} k_i^\nu \mathrm{Amp}^{\mathrm{Comp}}_{\mu\nu,S=1} = k_f^\mu \epsilon_i^\nu \mathrm{Amp}^{\mathrm{Comp}}_{\mu\nu,S=1} = 0\,. \qquad (3.3)$$

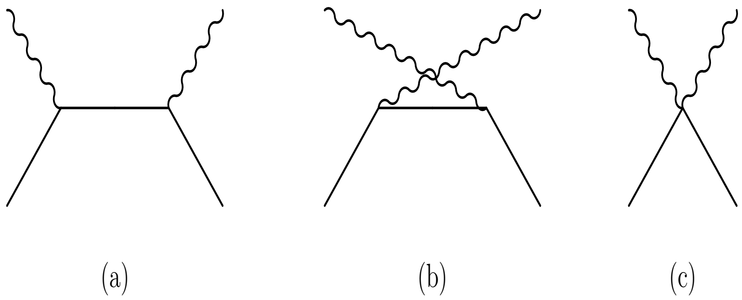

Figure 1: Diagrams relevant to Compton scattering.

Henceforth in this manuscript we shall assume the g-factor of the spin-1 system to have its "natural" value $g = 2$, since it is in this case that the high-energy properties of the scattering are well controlled and factorization methods of gravity amplitudes are valid [29, 30].

In order to make the transition to gravity in the next section, it is useful to utilize the helicity formalism [32], whereby we evaluate the matrix elements of the Compton amplitude between initial and final spin-1 and photon states having definite helicity, where helicity is defined as the projection of the particle spin along the momentum direction. We shall work initially in the center of mass frame. For a photon incident with four-momentum $k_{i\mu} = p_{\rm CM}(1, \hat{z})$ we choose the polarization vectors

$$\epsilon_i^{\lambda_i} = -\frac{\lambda_i}{\sqrt{2}}(\hat{x} + i\lambda_i\hat{y}), \qquad \lambda_i = \pm\,, \tag{3.4}$$

while for an outgoing photon with $k_{f\mu} = p_{\rm CM}(1, \cos\theta_{\rm CM}\hat{z} + \sin\theta_{\rm CM}\hat{x})$ we use polarizations

$$\epsilon_f^{\lambda_f} = -\frac{\lambda_f}{\sqrt{2}}(\cos\theta_{\rm CM}\hat{x} + i\lambda_f\hat{y} - \sin\theta_{\rm CM}\hat{z}), \quad \lambda_f = \pm\,. \tag{3.5}$$

We can define corresponding helicity states for the spin-1 system. In this case the initial and final four-momenta are $p_i = (E_{\rm CM}, -p_{\rm CM}\hat{z})$ and $p_f = (E_{\rm CM}, -p_{\rm CM}(\cos\theta_{\rm CM}\hat{z} + \sin\theta_{\rm CM}\hat{x}))$ and there are transverse polarization four-vectors

$$\begin{aligned}
\epsilon^{\pm}_{A\mu} &= (0, \mp\frac{-\hat{x} \pm i\hat{y}}{\sqrt{2}})\,, \\
\epsilon^{\pm}_{B\mu} &= (0, \mp\frac{-\cos\theta_{\rm CM}\hat{x} \pm i\hat{y} + \sin\theta_{\rm CM}\hat{z}}{\sqrt{2}})\,,
\end{aligned} \tag{3.6}$$

while the longitudinal mode has polarization four-vectors

$$\begin{aligned}
\epsilon^{0}_{A\mu} &= \frac{1}{m}(p_{\rm CM}, -E_{\rm CM}\hat{z})\,, \\
\epsilon^{0}_{B\mu} &= \frac{1}{m}(p_{\rm CM}, -E_{\rm CM}(\cos\theta_{\rm CM}\hat{z} + \sin\theta_{\rm CM}\hat{x}))\,,
\end{aligned} \tag{3.7}$$

In terms of the usual invariant kinematic variables

$$s = (p_i + k_i)^2, \quad t = (k_i - k_f)^2, \quad u = (p_i - k_f)^2\,,$$

we identify

$$\begin{aligned}
p_{\rm CM} &= \frac{s - m^2}{2\sqrt{s}}\,, \\
E_{\rm CM} &= \frac{s + m^2}{2\sqrt{s}}\,, \\
\cos\frac{1}{2}\theta_{\rm CM} &= \frac{((s - m^2)^2 + st)^{\frac{1}{2}}}{s - m^2} = \frac{(m^4 - su)^{\frac{1}{2}}}{s - m^2}\,, \\
\sin\frac{1}{2}\theta_{\rm CM} &= \frac{(-st)^{\frac{1}{2}}}{(s - m^2)}\,.
\end{aligned} \tag{3.8}$$

The invariant cross-section for unpolarized Compton scattering is then given by

$$\frac{d\sigma^{\rm Comp}_{S=1}}{dt} = \frac{1}{16\pi(s - m^2)^2}\frac{1}{3}\sum_{a,b=-,0,+}\frac{1}{2}\sum_{c,d=-,+}|B^1(ab;cd)|^2\,. \tag{3.9}$$

where

$$B^1(ab;cd) = \langle p_f, b; k_f, d|{\rm Amp}^{\rm Comp}_{S=1}|p_i, a; k_i, c\rangle\,, \tag{3.10}$$

is the Compton amplitude for scattering of a photon with four-momentum k_i, helicity a from a spin-1 target having four-momentum p_i, helicity c to a photon with four-momentum k_f, helicity d and target with four-momentum

p_f, helicity b and the sum in Eq. (3.9). The helicity amplitudes can now be calculated straightforwardly. There exist $3^2 \times 2^2 = 36$ such amplitudes but, since helicity reverses under spatial inversion, parity invariance of the electromagnetic interaction requires that[2]

$$|B^1(ab;cd)| = |B^1(-a-b;-c-d)|\,.$$

Also, since helicity is unchanged under time reversal, but initial and final states are interchanged, time reversal invariance of the electromagnetic interaction requires that

$$|B^1(ab;cd)| = |B^1(ba;dc)|\,.$$

Consequently there exist only twelve *independent* helicity amplitudes. Using Eq. (3.1) we can calculate the various helicity amplitudes in the center of mass frame and then write these results in terms of invariants using Eq. (3.8), yielding

$$\begin{aligned}
|B^1(++;++)| &= |B^1(--;--)| = 2e^2\frac{((s-m^2)^2+m^2t)^2}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(++;--)| &= |B^1(--;++)| = 2e^2\frac{(m^4-su)^2}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(+-;+-)| &= |B^1(-+;-+)| = 2e^2\frac{m^4t^2}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(+-;-+)| &= |B^1(-+;+-)| = 2e^2\frac{s^2t^2}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(++;+-)| &= |B^1(--;-+)| = |B^1(++;-+)| = |B^1(--;+-)|\,, \\
&= 2e^2\frac{m^2t(m^4-su)}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(+-;++)| &= |B^1(-+;--)| = |B^1(-+;++)| = |B^1(+-;--)|\,, \\
&= 2e^2\frac{m^2t(m^4-su)}{(s-m^2)^3(u-m^2)}\,. \qquad (3.11)
\end{aligned}$$

[2]Note that we require only that the magnitudes of the helicity amplitudes related by parity and/or time reversal be the same. There could exist unobservable phases.

and

$$
\begin{aligned}
|B^1(0+;++)| &= |B^1(0-;--)| = |B^1(+0;++)| = |B^1(-0;--)|\,, \\
&= 2e^2\frac{\sqrt{2}m(tm^2+(s-m^2)^2)\sqrt{-t(m^4-su)}}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(0+;+-)| &= |B^1(0-;-+)| = |B^1(+0;-+)| = |B^1(-0;+-)|\,, \\
&= 2e^2\frac{\sqrt{2}mst\sqrt{-t(m^4-su)}}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(0+;-+)| &= |B^1(0-;+-)| = |B^1(+0;+-)| = |B^1(-0;-+)|\,, \\
&= 2e^2\frac{\sqrt{2}m^3t\sqrt{-t(m^4-su)}}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(0+;--)| &= |B^1(0-;++)| = |B^1(+0;--)| = |B^1(-0;++)|\,, \\
&= 2e^2\frac{\sqrt{2}m(-t(m^4-su))^{\frac{3}{2}}}{(s-m^2)^3t(u-m^2)}\,, \\
|B^1(00;++)| &= |B^1(00;--)| = 2e^2\frac{(2tm^2+(s-m^2)^2)(m^4-su)}{(s-m^2)^3(u-m^2)}\,, \\
|B^1(00;+-)| &= |B^1(00;-+)| = 2e^2\frac{(m^2t((s-m^2)^2+2st)}{(s-m^2)^3(u-m^2)}\,. \qquad (3.12)
\end{aligned}
$$

Substitution into Eq. (3.9) then yields the invariant cross-section for unpolarized Compton scattering from a spin-1 target

$$
\frac{d\sigma^{\rm Comp}_{S=1}}{dt} = \frac{e^4}{12\pi(s-m^2)^4(u-m^2)^2}[(m^4-su+t^2)(3(m^4-su)+t^2)+t^2(t-m^2)(t-3m^2)]\,, \qquad (3.13)
$$

which can be compared with the corresponding results for unpolarized Compton scattering from spin-0 and spin-$\frac{1}{2}$ targets found in ref. [22]—

$$
\begin{aligned}
\frac{d\sigma^{\rm Comp}_{S=0}}{dt} &= \frac{e^4}{4\pi(s-m^2)^4(u-m^2)^2}\left[(m^4-su)^2+m^4t^2\right]\,, \\
\frac{d\sigma^{\rm Comp}_{S=\frac{1}{2}}}{dt} &= \frac{e^4\left[(m^4-su)(2(m^4-su)+t^2)+m^2t^2(2m^2-t)\right]}{8\pi(s-m^2)^4(u-m^2)^2}\,. \qquad (3.14)
\end{aligned}
$$

Usually such results are written in the *laboratory* frame, wherein the target is at rest, by use of the relations

$$\begin{aligned} s - m^2 &= 2m\omega_i, \quad u - m^2 = -2m\omega_f\,, \\ m^4 - su &= 4m^2\omega_i\omega_f \cos^2\frac{\theta_L}{2}, \quad m^2 t = -4m^2\omega_i\omega_f \sin^2\frac{\theta_L}{2}\,, \end{aligned} \tag{3.15}$$

and

$$\frac{dt}{d\Omega} = \frac{d}{2\pi d\cos\theta_L}\left(-\frac{2\omega_i^2(1-\cos\theta_L)}{1+\frac{\omega_i}{m}(1-\cos\theta_L)}\right) = \frac{\omega_f^2}{\pi}\,. \tag{3.16}$$

Introducing the fine structure constant $\alpha = e^2/4\pi$, we find then

$$\begin{aligned} \frac{d\sigma^{\rm Comp}_{lab,S=1}}{d\Omega} &= \frac{\alpha^2}{m^2}\frac{\omega_f^4}{\omega_i^4}\left[(\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2})^2\right. \\ &+ \left.\frac{16\omega_i^2}{3m^2}\sin^4\frac{\theta_L}{2}(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2}) + \frac{32\omega_i^4}{3m^4}\sin^8\frac{\theta_L}{2}\right]\,, \\ \frac{d\sigma^{\rm Comp}_{lab,S=\frac{1}{2}}}{d\Omega} &= \frac{\alpha^2}{m^2}\frac{\omega_f^3}{\omega_i^3}\left((\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2}) + 2\frac{\omega_i^2}{m^2}\sin^4\frac{\theta_L}{2}\right)\,, \\ \frac{d\sigma^{\rm Comp}_{lab,S=0}}{d\Omega} &= \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}\left(\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2}\right)\,. \end{aligned} \tag{3.17}$$

We observe that the nonrelativistic laboratory cross-section has an identical form for *any* spin

$$\left.\frac{d\sigma^{\rm Comp}_{lab,S}}{d\Omega}\right|^{NR} = \frac{\alpha^2}{2m^2}\left[(\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})(1+\mathcal{O}(\frac{\omega_i}{m}))\right]\,, \tag{3.18}$$

which follows from the universal form of the Compton amplitude for scattering from a spin S target in the low-energy ($\omega \ll m$) limit, which in turn arises from the universal form of the Compton amplitude for scattering from a spin S target in the low-energy limit—

$$\langle S, M_f; \epsilon_f | {\rm Amp}_S^{\rm Comp} | S, M_i; \epsilon_i\rangle_{\omega\ll m} = 2e^2\epsilon_f^*\cdot\epsilon_i\delta_{M_i,M_f} + \dots\,, \tag{3.19}$$

and obtains in an effective field theory approach to Compton scattering [33].[3]

[3]That the seagull contribution dominates the non relativistic cross-section is clear from the feature that

$${\rm Amp}_{\rm Born} \sim 2e^2\frac{\epsilon_f^*\cdot p\epsilon_i\cdot p}{p\cdot k} \sim \frac{\omega}{m}\times{\rm Amp}_{\rm seagull} = 2e^2\epsilon_f^*\cdot\epsilon_i. \tag{3.20}$$

4 Gravitational Interactions

In the previous section we discussed the treatment the familiar electromagnetic interaction, using Compton scattering on a spin-1 target as an example. In this section we show how the gravitational interaction can be evaluated via methods parallel to those used in the electromagnetic case. An important difference is that while in the electromagnetic case we have the simple interaction Lagrangian

$$\mathcal{L}_{int} = -eA_\mu J^\mu\,, \tag{4.1}$$

where J^μ is the electromagnetic current matrix element, for gravity we have

$$\mathcal{L}_{int} = -\frac{\kappa}{2} h^{\mu\nu} T^{\mu\nu}\,. \tag{4.2}$$

Here the field tensor $h_{\mu\nu}$ is defined in terms of the metric via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \tag{4.3}$$

where κ is defined in terms of Newton's constant via $\kappa^2 = 32\pi G$. The Einstein-Hilbert action is

$$\mathcal{S}_{\rm Einstein-Hilbert} = \int d^4x\,\sqrt{-g}\,\frac{2}{\kappa^2}\,\mathcal{R}\,, \tag{4.4}$$

where

$$\sqrt{-g} = \sqrt{-{\rm det}g} = \exp\frac{1}{2}{\rm trlog}g = 1 + \frac{1}{2}\eta^{\mu\nu}h_{\mu\nu} + \ldots\,, \tag{4.5}$$

is the square root of the determinant of the metric and $\mathcal{R} := R^\lambda{}_{\mu\lambda\nu} g^{\mu\nu}$ is the Ricci scalar curvature obtained by contracting the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ with the metric. The energy-momentum tensor is defined in terms of the matter Lagrangian via

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{\rm mat}}{\delta g^{\mu\nu}}\,. \tag{4.6}$$

The prescription Eq. (4.6) yields the forms

$$T^{S=0}_{\mu\nu} = \partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi - g_{\mu\nu}(\partial_\lambda\phi^\dagger\partial^\lambda\phi - m^2\phi^\dagger\phi)\,, \tag{4.7}$$

for a scalar field and

$$T^{S=\frac{1}{2}}_{\mu\nu} = \bar\psi[\frac{1}{4}\gamma_\mu i\overleftrightarrow{\nabla}_\nu + \frac{1}{4}\gamma_\nu i\overleftrightarrow{\nabla}_\mu - g_{\mu\nu}(\frac{i}{2}\overleftrightarrow{\not\nabla} - m)]\psi\,, \tag{4.8}$$

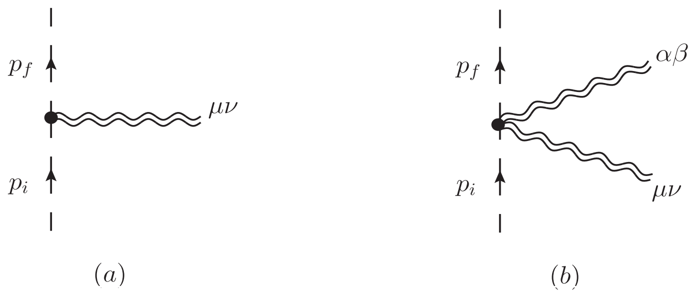

Figure 2: (a) The one-graviton and (b) two-graviton emission vertices from either a scalar, spinor or vector particle.

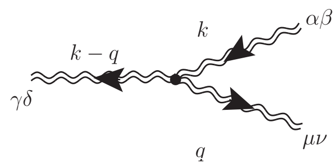

Figure 3: The three graviton vertex

for spin-$\frac{1}{2}$, where we have defined

$$\bar{\psi}i\overleftrightarrow{\nabla}_\mu\psi := \bar{\psi}i\nabla_\mu\psi - (i\nabla_\mu\bar{\psi})\psi\,. \tag{4.9}$$

The one graviton emission vertices of figure 2(a) can now be identified as

$$\langle p_f|V^{(1)\mu\nu}_{grav}|p_i\rangle_{S=0} = -i\frac{\kappa}{2}\left(p_f^\mu p_i^\nu + p_f^\nu p_i^\mu - \eta^{\mu\nu}(p_f\cdot p_i - m^2)\right)\,, \tag{4.10}$$

for spin-0,

$$\langle p_f|V^{(1)\mu\nu}_{grav}|p_i\rangle_{S=\frac{1}{2}} = -i\frac{\kappa}{2}\bar{u}(p_f)[\frac{1}{4}\gamma^\mu(p_f+p_i)^\nu + \frac{1}{4}\gamma^\nu(p_f+p_i)^\mu]u(p_i)\,, \tag{4.11}$$

for spin-$\frac{1}{2}$, and

$$\begin{aligned}\langle p_f,\epsilon_B|V^{(1)\mu\nu}_{grav}|p_i,\epsilon_A\rangle_{S=1} &= -i\frac{\kappa}{2}\left\{\epsilon_B^*\cdot\epsilon_A(p_i^\mu p_f^\nu + p_i^\nu p_f^\mu) - \epsilon_B^*\cdot p_i(p_f^\mu\epsilon_A^\nu + \epsilon_A^\mu p_f^\nu)\right.\\ &- \epsilon_A\cdot p_f(p_i^\nu\epsilon_B^{*\mu} + p_i^\mu\epsilon_B^{*\nu}) + (p_f\cdot p_i - m^2)(\epsilon_A^\mu\epsilon_B^{*\nu} + \epsilon_A^\nu\epsilon_B^{*\mu})\\ &- \left.\eta^{\mu\nu}\left[(p_i\cdot p_f - m^2)\epsilon_B^*\cdot\epsilon_A - \epsilon_B^*\cdot p_i\epsilon_A\cdot p_f\right]\right\}\,,\end{aligned} \tag{4.12}$$

for spin-1. There also exist two-graviton (seagull) vertices shown in figure 2(b), which can be found by expanding the stress-energy tensor to second order in $h_{\mu\nu}$. In the case of spin-0

$$\begin{aligned}\langle p_f|V^{(2)\mu\nu,\alpha\beta}_{grav}|p_i\rangle_{S=0} &= i\kappa^2\left[I^{\mu\nu,}{}_{\rho\xi}I^{\xi}{}_{\zeta,\alpha\beta}(p_f^\zeta p_i^\rho + p_f^\rho p_i^\zeta) - \frac{1}{2}\left(\eta^{\mu\nu}I^{\rho\zeta,\alpha\beta} + \eta^{\alpha\beta}I^{\rho\zeta,\mu\nu}p_f^\rho p_i^\zeta\right)\right.\\ &- \left.\frac{1}{2}\left(I^{\mu\nu;\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\right)(p_f\cdot p_i - m^2)\right]\,,\end{aligned} \tag{4.13}$$

where

$$I_{\alpha\beta,\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}). \tag{4.14}$$

For spin-$\frac{1}{2}$

$$\begin{aligned}\langle p_f|V^{(2)\mu\nu,\alpha\beta}_{grav}|p_i\rangle_{S=\frac{1}{2}} &= i\kappa^2\,\bar{u}(p_f)\left\{\frac{1}{16}[\eta^{\mu\nu}(\gamma^\alpha(p_f+p_i)^\beta + \gamma^\beta(p_f+p_i)^\alpha)\right.\\ &+ \eta^{\alpha\beta}(\gamma^\mu(p_f+p_i)^\nu + \gamma^\nu(p_f+p_i)^\mu)]\\ &+ \frac{3}{16}(p_f+p_i)_\epsilon\gamma_\xi(I^{\xi\phi,\mu\nu}I_\phi{}^{\epsilon,\alpha\beta} + I^{\xi\phi,\alpha\beta}I_\phi{}^{\epsilon,\mu\nu})\\ &+ \left.\frac{i}{16}\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(I^{\mu\nu,\eta}{}_\zeta I^{\alpha\beta,\sigma\zeta}p_{f\rho} - I^{\alpha\beta,\eta}{}_\zeta I^{\mu\nu,\sigma\zeta}p_{i\rho})\right\}u(p_i)\,,\end{aligned} \tag{4.15}$$

while for spin-1

$$
\begin{aligned}
&\langle p_f,\epsilon_B;k_f|V^{(2)\,\mu\nu,\rho\sigma}_{grav}|p_i,\epsilon_A;k_i\rangle_{S=1} = -i\frac{\kappa^2}{4}\,\{[p_{i\beta}p_{f\alpha} \\
-\ & \eta_{\alpha\beta}(p_i\cdot p_f-m^2)](\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho}-\eta_{\mu\nu}\eta_{\rho\sigma}) \\
+\ & \eta_{\mu\rho}[\eta_{\alpha\beta}(p_{i\nu}p_{f\sigma}+p_{i\sigma}p_{f\nu})-\eta_{\alpha\nu}p_{i\beta}p_{f\sigma}-\eta_{\beta\nu}p_{i\sigma}p_{f\alpha} \\
-\ & \eta_{\beta\sigma}p_{i\nu}p_{f\alpha}-\eta_{\alpha\sigma}p_{i\beta}p_{f\nu}+(p_i\cdot p_f-m^2)(\eta_{\alpha\nu}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\nu})] \\
+\ & \eta_{\mu\sigma}[\eta_{\alpha\beta}(p_{i\nu}p_{f\rho}+p_{i\rho}p_{f\nu})-\eta_{\alpha\nu}p_{i\beta}p_{f\rho}-\eta_{\beta\nu}p_{i\rho}p_{f\alpha} \\
-\ & \eta_{\beta\rho}p_{i\nu}p_{f\alpha}-\eta_{\alpha\rho}p_{i\beta}p_{f\nu}+(p_i\cdot p_f-m^2)\eta_{\alpha\nu}\eta_{\beta\rho}+\eta_{\alpha\rho}\eta_{\beta\nu})] \\
+\ & \eta_{\nu\rho}[\eta_{\alpha\beta}(p_{i\mu}p_{f\sigma}+p_{i\sigma}p_{f\mu})-\eta_{\alpha\mu}p_{i\beta}p_{f\sigma}-\eta_{\beta\mu}p_{i\sigma}p_{f\alpha} \\
-\ & \eta_{\beta\sigma}p_{i\mu}p_{f\alpha}-\eta_{\alpha\sigma}p_{i\beta}p_{f\mu}+(p_i\cdot p_f-m^2)(\eta_{\alpha\mu}\eta_{\beta\sigma}+\eta_{\alpha\sigma}\eta_{\beta\mu})] \\
+\ & \eta_{\nu\sigma}[\eta_{\alpha\beta}(p_{i\mu}p_{f\rho}+p_{i\rho}p_{f\mu})-\eta_{\alpha\mu}p_{i\beta}p_{f\rho}-\eta_{\beta\mu}p_{i\rho}p_{f\alpha} \\
-\ & \eta_{\beta\rho}p_{i\mu}p_{f\alpha}-\eta_{\alpha\rho}p_{i\beta}p_{f\mu}+(p_i\cdot p_f-m^2)(\eta_{\alpha\mu}\eta_{\beta\rho}+\eta_{\alpha\rho}\eta_{\beta\mu})] \\
-\ & \eta_{\mu\nu}[\eta_{\alpha\beta}(p_{i\rho}p_{f\sigma}+p_{i\sigma}p_{f\rho})-\eta_{\alpha\rho}p_{i\beta}p_{f\sigma}-\eta_{\beta\rho}p_{i\sigma}p_{f\alpha} \\
-\ & \eta_{\beta\sigma}p_{i\rho}p_{f\alpha}-\eta_{\alpha\sigma}p_{i\beta}p_{f\rho}+(p_i\cdot p_f-m^2)(\eta_{\alpha\rho}\eta_{\beta\sigma}+\eta_{\beta\rho}\eta_{\alpha\sigma})] \\
-\ & \eta_{\rho\sigma}[\eta_{\alpha\beta}(p_{i\mu}p_{f\nu}+p_{i\nu}p_{f\mu})-\eta_{\alpha\mu}p_{i\beta}p_{f\nu}-\eta_{\beta\mu}p_{i\nu}p_{f\alpha} \\
-\ & \eta_{\beta\nu}p_{i\mu}p_{f\alpha}-\eta_{\alpha\nu}p_{i\beta}p_{f\mu}+(p_i\cdot p_f-m^2)(\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\beta\mu}\eta_{\alpha\nu})] \\
+\ & (\eta_{\alpha\rho}p_{i\mu}-\eta_{\alpha\mu}p_{i\rho})(\eta_{\beta\sigma}p_{f\nu}-\eta_{\beta\mu}p_{f\sigma}) \\
+\ & (\eta_{\alpha\sigma}p_{i\nu}-\eta_{\alpha\nu}p_{i\sigma})\eta_{\beta\rho}p_{f\mu}-\eta_{\beta\mu}p_{f\rho}) \\
+\ & (\eta_{\alpha\sigma}p_{i\mu}-\eta_{\alpha\mu}p_{i\sigma})(\eta_{\beta\rho}p_{f\nu}-\eta_{\beta\nu}p_{f\rho}) \\
+\ & (\eta_{\alpha\rho}p_{i\nu}-\eta_{\alpha\nu}p_{i\rho})(\eta_{\beta\sigma}p_{f\mu}-\eta_{\beta\mu}p_{f\sigma})\}\,\epsilon^\alpha_A(\epsilon^\beta_B)^*\,. \qquad (4.16)
\end{aligned}
$$

Finally, we require the triple graviton vertex of figure 3

$$\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) &= -\frac{i\kappa}{2}\left\{(I_{\alpha\beta,\gamma\delta}-\frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta})\left[k^\mu k^\nu+(k-q)^\mu(k-q)^\nu+q^\mu q^\nu-\frac{3}{2}\eta^{\mu\nu}q^2\right]\right.\\
&+ 2q_\lambda q_\sigma\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}-I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta}-I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\right]\\
&+ [q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta})+q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta})\\
&- q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})-\eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}+\eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})]\\
&+ [2q^\lambda(I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}(k-q)^\mu+I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}(k-q)^\nu\\
&- I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}k^\mu-I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}k^\nu)\\
&+ q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu+I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta})+\eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta}+I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})]\\
&+ [(k^2+(k-q)^2)\left(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu+I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu-\frac{1}{2}\eta^{\mu\nu}(I_{\alpha\beta,\gamma\delta}-\frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta})\right)\\
&- \left.(k^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+(k-q)^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})]\right\}\,. \qquad (4.17)
\end{aligned}$$

We work in harmonic (de Donder) gauge which satisfies, in lowest order,

$$\partial^\mu h_{\mu\nu}=\frac{1}{2}\partial_\nu h\,, \qquad (4.18)$$

with

$$h=\mathrm{tr}h_{\mu\nu}\,, \qquad (4.19)$$

and in which the graviton propagator has the form

$$D_{\alpha\beta;\gamma\delta}(q)=\frac{i}{q^2+i\epsilon}\frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta}+\eta_{\alpha\delta}\eta_{\beta\gamma}-\eta_{\alpha\beta}\eta_{\gamma\delta})\,. \qquad (4.20)$$

Then just as the (massless) photon is described in terms of a spin-1 polarization vector ϵ_μ which can have projection (helicity) either plus- or minus-1 along the momentum direction, the (massless) graviton is a spin two particle which can have the projection (helicity) either plus- or minus-2 along the momentum direction. Since $h_{\mu\nu}$ is a symmetric tensor, it can be described in terms of a direct product of unit spin polarization vectors—

$$\begin{aligned}
\text{helicity} &= +2: \quad h^{(2)}_{\mu\nu}=\epsilon^+_\mu\epsilon^+_\nu\,,\\
\text{helicity} &= -2: \quad h^{(-2)}_{\mu\nu}=\epsilon^-_\mu\epsilon^-_\nu\,, \qquad (4.21)
\end{aligned}$$

and just as in electromagnetism, there is a gauge condition—in this case Eq. (4.18)—which must be satisfied. Note that the helicity states given in Eq. (4.21) are consistent with the gauge requirement, since

$$\eta^{\mu\nu}\epsilon^+_\mu\epsilon^+_\nu=\eta^{\mu\nu}\epsilon^-_\mu\epsilon^-_\nu=0, \quad \text{and} \quad k^\mu\epsilon^\pm_\mu=0\,. \qquad (4.22)$$

With this background we can now examine reactions involving gravitons, as discussed below.

4.1 Graviton Photo-production

We first use the above results to discuss the problem of graviton photo-production on a target of spin-1—$\gamma + S \rightarrow g + S$—for which the four diagrams we need are shown in Figure 4. The electromagnetic and gravitational vertices needed for the Born terms and photon pole diagrams—Figures 4a, 4b, and 4d—have been given above. For the photon pole diagram we require the graviton-photon coupling, which is found from the electromagnetic energy-momentum tensor [34]

$$T_{\mu\nu} = -F_{\mu\alpha}F_{\nu}^{\alpha} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\,, \tag{4.23}$$

and yields the photon-graviton vertex[4]

$$\begin{aligned}
\langle k_f,\epsilon_f|V_{grav}^{(\gamma)\mu\nu}|k_i,\epsilon_i\rangle \;&=\; i\frac{\kappa}{2}\left[\epsilon_f^*\cdot\epsilon_i(k_i^\mu k_f^\nu + k_i^\nu k_f^\mu) - \epsilon_f^*\cdot k_i(k_f^\mu\epsilon_i^\nu + \epsilon_i^\mu k_f^\nu)\right.\\
&-\; \epsilon_i\cdot k_f(k_i^\nu\epsilon_f^{*\mu} + k_i^\mu\epsilon_f^{*\nu}) + k_f\cdot k_i(\epsilon_i^\mu\epsilon_f^{*\nu} + \epsilon_i^\nu\epsilon_f^{*\mu})\\
&-\; \eta^{\mu\nu}\left[k_f\cdot k_i\epsilon_f^*\cdot\epsilon_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f\right]\Big\}\,.
\end{aligned} \tag{4.24}$$

Finally, we need the seagull vertex which arises from the feature that the energy-momentum tensor depends on p_i, p_f and therefore yields a contact interaction when the minimal substitution is made, yielding the spin-1 seagull amplitude shown in Figure 4c.

$$\begin{aligned}
&\langle p_f,\epsilon_B;k_f,\epsilon_f\epsilon_f|T|p_i,\epsilon_A;k_i,\epsilon_i\rangle_{seagull} = \frac{i}{2}\kappa e\left[\epsilon_f^*\cdot(p_f+p_i)\epsilon_f^*\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_A\right.\\
-\;& \epsilon_B^*\cdot\epsilon_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_A - \epsilon_B^*\cdot p_i\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot\epsilon_A - \epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_B^*\\
-\;& \left.\epsilon_A\cdot p_f\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot\epsilon_B^* - \epsilon_f^*\cdot\epsilon_A\epsilon_i\cdot(p_f+p_i)\epsilon_f^*\cdot\epsilon_B^*\right]\,.
\end{aligned} \tag{4.25}$$

The individual contributions from the four diagrams in Figure 4 are given in Appendix A and have a rather complex form. However, when added together we find a *much* simpler result—the full graviton photo-production amplitude

[4]Note that this form agrees with the previously derived form for the massive graviton-spin-1 energy-momentum tensor—Eq. (4.12)—in the $m \rightarrow 0$ limit.

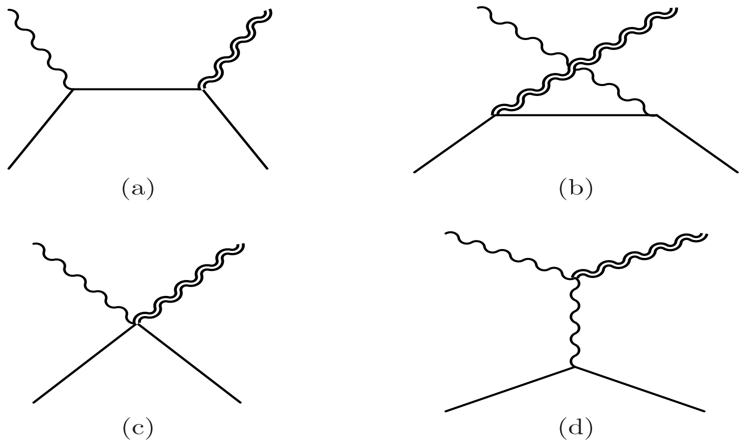

Figure 4: Diagrams relevant to graviton photo-production.

is found to be proportional to the already calculated Compton amplitude for spin-1—Eq. (3.1)—times a universal factor. That is,

$$\langle p_f; k_f, \epsilon_f \epsilon_f | T | p_i; k_i, \epsilon_i \rangle = H \times \left(\epsilon^*_{f\alpha} \epsilon_{i\beta} T^{\alpha\beta}_{Compton}(S=1) \right) , \tag{4.26}$$

where

$$H = \frac{\kappa}{2e} \frac{p_f \cdot F_f \cdot p_i}{k_i \cdot k_f} = \frac{\kappa}{2e} \frac{\epsilon^*_f \cdot p_f k_f \cdot p_i - \epsilon^*_f \cdot p_i k_f \cdot p_f}{k_i \cdot k_f} , \tag{4.27}$$

and $\epsilon^*_{f\alpha} \epsilon_{i\beta} T^{\alpha\beta}_{Compton}(S)$ is the Compton scattering amplitude for particles of spin S calculated in the previous section. The gravitational and electromagnetic gauge invariance of Eq. (4.26) is obvious, since it follows directly from the gauge invariance already shown for the Compton amplitude together with the explicit gauge invariance of the factor H. The validity of Eq. (4.26) allows the calculation of the cross-section by helicity methods since the graviton photo-production helicity amplitudes are given by

$$C^1(ab; cd) = H \times B^1(ab; cd) , \tag{4.28}$$

where $B^1(ab; cd)$ are the Compton helicity amplitudes found in the previous section. We can then evaluate the invariant photo-production cross-section using

$$\frac{d\sigma^{\rm photo}_{S=1}}{dt} = \frac{1}{16\pi(s-m^2)^2} \frac{1}{3} \sum_{a=-,0,+} \frac{1}{2} \sum_{c=-,+} |C^1(ab; cd)|^2 , \tag{4.29}$$

yielding

$$\frac{d\sigma^{\rm photo}_{S=1}}{dt} = -\frac{e^2\kappa^2(m^4-su)}{96\pi t(s-m^2)^4(u-m^2)^2}\big[(m^4-su+t^2)(3(m^4-su)+t^2) + t^2(t-m^2)(t-3m^2)\big]\,. \qquad (4.30)$$

Since

$$|H| = \frac{\kappa}{e}\left(\frac{m^4-su}{-2t}\right)^{\frac{1}{2}}\,, \qquad (4.31)$$

the laboratory value of the factor H is

$$|H_{lab}|^2 = \frac{\kappa^2 m^2}{2e^2}\frac{\cos^2\frac{1}{2}\theta_L}{\sin^2\frac{1}{2}\theta_L}\,, \qquad (4.32)$$

the corresponding laboratory cross-section is

$$\begin{aligned}\frac{d\sigma^{\rm photo}_{lab,S=1}}{d\Omega} &= |H_{lab}|^2\frac{d\sigma^{\rm Comp}_{lab,S=1}}{dt}\\ &= G\alpha\frac{\omega_f^4}{\omega_i^4}\cos^2\frac{\theta_L}{2}\left[(\text{ctn}^2\frac{\theta_L}{2}\cos^2\frac{\theta_L}{2}+\sin^2\frac{\theta_L}{2})(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2})^2\right.\\ &+ \left.\frac{16\omega_i^2}{3m^2}\sin^2\frac{\theta_L}{2}(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2})+\frac{32\omega_i^4}{3m^4}\sin^6\frac{\theta_L}{2}\right]\,. \qquad (4.33)\end{aligned}$$

The factor $|H_{lab}|^2$ can be thought of as "dressing" the photon into a graviton. We see that just as in Compton scattering the low-energy laboratory cross-section has a universal form, which is valid for a target of arbitrary spin

$$\frac{d\sigma^{\rm photo}_{lab,S}}{d\Omega} = G\alpha\cos^2\frac{\theta_L}{2}(\text{ctn}^2\frac{\theta_L}{2}\cos^2\frac{\theta_L}{2}+\sin^2\frac{\theta_L}{2})(1+\mathcal{O}(\frac{\omega_i}{m}))\,. \qquad (4.34)$$

In this case the universality can be understood from the feature that at low-energy the leading contribution to the graviton photo-production amplitude comes *not* from the seagull, as in Compton scattering, but rather from the photon pole term,

$$\text{Amp}_{\gamma-pole}\underset{\omega\ll m}{\longrightarrow}\kappa\frac{\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot k_i}{2k_f\cdot k_i}\times k_i^\mu\langle p_f;S,M_f|J_\mu|p_i;S,M_i\rangle\,. \qquad (4.35)$$

The leading piece of the electromagnetic current has the universal low-energy structure

$$\langle p_f; S, M_f|J_\mu|p_i; S, M_i\rangle = \frac{e}{2m}(p_f + p_i)_\mu \delta_{M_f,M_i}(1 + \mathcal{O}(\frac{p_f - p_i}{m}))\,, \qquad (4.36)$$

where we have divided by the factor $2m$ to account for the normalization of the target particle. Since $k_i \cdot (p_f + p_i) \underset{\omega\to 0}{\longrightarrow} 2m\omega$, we find the universal low-energy amplitude

$$\text{Amp}^{NR}_{\gamma-pole} = \kappa e\omega \frac{\epsilon^*_f \cdot \epsilon_i \epsilon^*_f \cdot k_i}{2k_f \cdot k_i}\,, \qquad (4.37)$$

whereby the resulting helicity amplitudes have the form

$$\text{Amp}^{NR}_{\gamma-pole} = \frac{\kappa e}{2\sqrt{2}} \begin{cases} \frac{1}{2}\sin\theta_L \left(\frac{1+\cos\theta_L}{1-\cos\theta_L}\right) = \frac{\cos\frac{\theta_L}{2}}{\sin\frac{\theta_L}{2}}\cos^2\frac{\theta_L}{2} & ++ = --\,, \\ \frac{1}{2}\sin\theta_L \left(\frac{1-\cos\theta_L}{1-\cos\theta_L}\right) = \frac{\cos\frac{\theta_L}{2}}{\sin\frac{\theta}{2}}\sin^2\frac{\theta_L}{2} & +- = -+\,. \end{cases} \qquad (4.38)$$

Squaring and averaging, summing over initial, final spins we find

$$\frac{d\sigma^{\text{photo}}_{lab,S}}{d\Omega} \underset{\omega\to 0}{\longrightarrow} G\alpha \cos^2\frac{\theta_L}{2}(\text{ctn}^2\frac{\theta_L}{2}\cos^2\frac{\theta_L}{2} + \sin^2\frac{\theta_L}{2})\,. \qquad (4.39)$$

as found above—*cf.* Eq. (4.34).

The power of the factorization theorem is obvious and, as we shall see in the next section, allows the straightforward evaluation of even more complex reactions such as gravitational Compton scattering.

4.2 Gravitational Compton Scattering

In the previous section we observed some of the power of the factorization theorem in the context of graviton photo-production on a spin-1 target in that we only needed to calculate the simpler Compton scattering process rather than to consider the full gravitational interaction. In this section we tackle a more challenging example, that of gravitational Compton scattering—$g + S \rightarrow g + S$—from a spin-1 target, for which there exist the four diagrams shown in Figure 5.

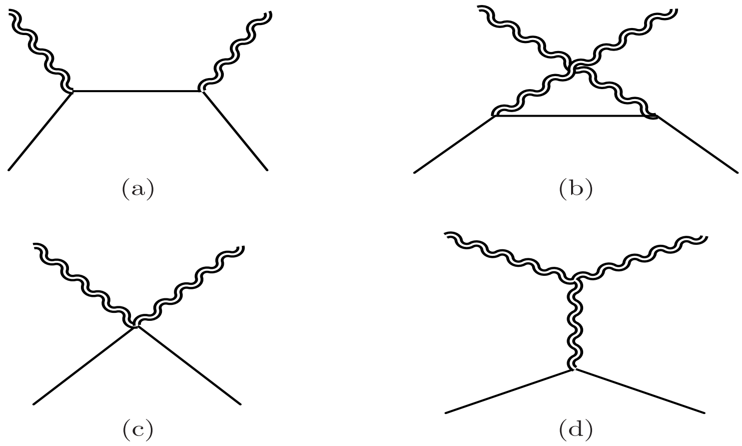

Figure 5: Diagrams relevant for gravitational Compton scattering.

The contributions from the four individual diagrams can now be calculated and are quoted in Appendix A. Each of the four diagrams has a rather complex form. However, when added together the result simplifies enormously. Defining the kinematic factor

$$Y = \frac{\kappa^2}{8e^4} \frac{p_i \cdot k_i p_i \cdot k_f}{k_i \cdot k_f} = \frac{\kappa^4}{16e^4} \frac{(s - m^2)(u - m^2)}{t}\,, \tag{4.40}$$

the sum of the four diagrams is found to be given by

$$\begin{aligned} & \langle p_f, \epsilon_B; k_f, \epsilon_f \epsilon_f | \mathrm{Amp}_{grav} | p_i, \epsilon_A; k_i, \epsilon_i \epsilon_i \rangle_{S=1} \\ = \; & Y \times \langle p_f, \epsilon_B; k_i, \epsilon_f | \mathrm{Amp}_{em} | p_i, \epsilon_A; k_i, \epsilon_i \rangle_{S=1} \times \langle p_f; k_i, \epsilon_f | \mathrm{Amp}_{em} | p_i; k_i, \epsilon_i \rangle_{S=0}\,, \end{aligned} \tag{4.41}$$

where

$$\langle p_f; k_i, \epsilon_f | \mathrm{Amp}_{em} | p_i; k_i, \epsilon_i \rangle_{S=0} = 2e^2 \left[\frac{\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f}{p_i \cdot k_i} - \frac{\epsilon_i \cdot p_f \epsilon_f^* \cdot p_i}{p_i \cdot k_f} - \epsilon_f^* \cdot \epsilon_i \right], \tag{4.42}$$

is the Compton amplitude for a spinless target.

In ref. [22] the identity Eq. (4.41) was verified for simpler cases of spin-0 and spin-$\frac{1}{2}$. This relation is a consequence of the general relations between gravity and gauge theory tree-level amplitudes derived from string theory as explained in [26]. Here we have shown its validity for the much more complex case of spin-1 scattering. The corresponding cross-section can be calculated by helicity methods using the identity

$$D^1(ab;cd) = Y \times B^1(ab;cd) \times A^0(cd)\,, \tag{4.43}$$

where $B^1(ab;cd)$ is the spin-1 Compton helicity amplitude calculated in section 2 while

$$\begin{aligned} A^0(++) &= 2e^2 \frac{m^4 - su}{(s-m^2)(u-m^2)}\,, \\ A^0(+-) &= 2e^2 \frac{-m^2 t}{(s-m^2)(u-m^2)}\,, \end{aligned} \tag{4.44}$$

are the helicity amplitudes for spin zero Compton scattering. Using Eq. (4.41) the invariant cross-section for unpolarized spin-1 gravitational Compton scattering

$$\frac{d\sigma^{\rm g-Comp}_{S=1}}{dt} = \frac{1}{16\pi(s-m^2)^2}\frac{1}{3}\sum_{a=-,0,+}\frac{1}{2}\sum_{c=-,+}|D^1(ab;cd)|^2\,, \tag{4.45}$$

is found to be

$$\begin{aligned} \frac{d\sigma^{\rm g-Comp}_{S=1}}{dt} &= \frac{\kappa^4}{768\pi(s-m^2)^4(u-m^2)^2t^2}\left[(m^4-su)^2(3(m^4-su)+t^2)(m^4-su+t^2)\right) \\ &+ \left. m^4t^4(3m^2-t)(m^2-t)\right]\,. \end{aligned} \tag{4.46}$$

This form can be compared with the corresponding unpolarized gravitational Compton cross-sections found in ref. [22]

$$\begin{aligned} \frac{d\sigma^{\rm g-Comp}_{S=\frac{1}{2}}}{dt} &= \frac{\kappa^4}{512\pi}\frac{((m^4-su)^3(2(m^4-su)+t^2)+m^6t^4(2m^2-t))}{t^2(s-m^2)^4(u-m^2)^2}\,, \\ \frac{d\sigma^{\rm g-Comp}_{S=0}}{d\Omega} &= \frac{\kappa^4}{256\pi^2(s-m^2)^4(u-m^2)^2t^2}\left[(m^4-su)^4+m^8t^4\right]\,. \end{aligned} \tag{4.47}$$

The corresponding laboratory frame cross-sections are

$$\begin{aligned}
\frac{d\sigma^{\rm g-Comp}_{lab,S=1}}{d\Omega} &= G^2m^2\frac{\omega_f^4}{\omega_i^4}\Big[(\mathrm{ctn}^4\frac{\theta_L}{2}\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2})^2 \\
&+ \frac{16}{3}\frac{\omega_i^2}{m^2}(\cos^6\frac{\theta_L}{2}+\sin^6\frac{\theta_L}{2})(1+2\frac{\omega_i}{m}\sin^2\frac{\theta_L}{2}) \\
&+ \frac{16}{3}\frac{\omega_i^4}{m^4}\sin^2\frac{\theta_L}{2}(\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})\Big]\;, \\
\frac{d\sigma^{\rm g-Comp}_{lab,S=\frac{1}{2}}}{d\Omega} &= G^2m^2\frac{\omega_f^3}{\omega_i^3}\Big((\mathrm{ctn}^4\frac{\theta_L}{2}\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})+2\frac{\omega_i}{m}(\mathrm{ctn}^2\frac{\theta_L}{2}\cos^6\frac{\theta_L}{2}+\sin^6\frac{\theta_L}{2}) \\
&+ 2\frac{\omega_i^2}{m^2}(\cos^6\frac{\theta_L}{2}+\sin^6\frac{\theta_L}{2})\Big)\;, \\
\frac{d\sigma^{\rm g-Comp}_{lab,S=0}}{d\Omega} &= G^2m^2\frac{\omega_f^2}{\omega_i^2}\Big(\mathrm{ctn}^4\frac{\theta_L}{2}\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2}\Big)\;. \qquad (4.48)
\end{aligned}$$

We observe that the low-energy laboratory cross-section has the universal form for any spin

$$\frac{d\sigma^{\rm g-Comp}_{lab,S}}{d\Omega} = G^2m^2\Big[\mathrm{ctn}^4\frac{\theta_L}{2}\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2}+\mathcal{O}(\frac{\omega_i}{m})\Big]\;. \qquad (4.49)$$

It is interesting to note that the "dressing" factor for the leading (++) helicity Compton amplitude—

$$|Y||A^{++}| = \frac{\kappa^2}{2e^2}\frac{m^4-su}{-t} \xrightarrow{\rm lab} \frac{\kappa^2m^2}{2e^2}\frac{\cos^2\frac{\theta_l}{2}}{\sin^2\frac{\theta}{2}}\;, \qquad (4.50)$$

—is simply the square of the photo-production dressing factor H, as might intuitively be expected since now *both* photons must be dressed in going from the Compton to the gravitational Compton cross-section.[5] In this case the universality of the non relativistic cross-section follows from the leading

[5]In the case of $+-$ helicity the "dressing" factor is

$$|Y||A^{+-}| = \frac{\kappa^2}{2e^2}m^2\,. \qquad (4.51)$$

so that the non-leading contributions will have different dressing factors.

contribution arising from the graviton pole term

$$\mathrm{Amp}_{g-pole} \underset{\omega\ll m}{\longrightarrow} \frac{\kappa}{4k_f\cdot k_i}(\epsilon_f^*\cdot\epsilon_i)^2(k_f^\mu k_f^\nu + k_i^\mu k_i^\nu)\frac{\kappa}{2}\langle p_f;S,M_f|T_{\mu\nu}|p_i;S,M_i\rangle\,. \tag{4.52}$$

Here the matrix element of the energy-momentum tensor has the universal low-energy structure

$$\frac{\kappa}{2}\langle p_f;S,M_f|T_{\mu\nu}|p_i;S,M_i\rangle = \frac{\kappa}{4m}(p_{f\mu}p_{i\nu}+p_{f\nu}p_{i\mu})\delta_{M_f,M_i}\Big(1+\mathcal{O}(\frac{p_f-p_i}{m})\Big)\,, \tag{4.53}$$

where we have divided by the factor $2m$ to account for the normalization of the target particle. We find then the universal form for the leading graviton pole amplitude

$$\mathrm{Amp}_{g-pole} \underset{\mathrm{non-rel}}{\longrightarrow} \frac{\kappa^2}{8mk_f\cdot k_i}(\epsilon_f^*\cdot\epsilon_i)^2(p_i\cdot k_f p_f\cdot k_f + p_i\cdot k_i p_f\cdot k_i)\delta_{M_f,M_i}\,. \tag{4.54}$$

Since $p\cdot k \underset{\omega\ll m}{\longrightarrow} m\omega$ the corresponding helicity amplitudes become

$$\mathrm{Amp}^{NR}_{g-pole} = 4\pi G m \begin{cases} \frac{(1+\cos\theta_L)^2}{2(1-\cos\theta_L)} = \frac{\cos^4\frac{\theta_L}{2}}{\sin^2\frac{\theta_L}{2}} & ++ = --\,,\\ \frac{(1-\cos\theta_L)^2}{2(1-\cos\theta_L)} = \frac{\sin^4\frac{\theta_L}{2}}{\sin^2\frac{\theta_L}{2}} & +- = -+\,. \end{cases} \tag{4.55}$$

Squaring and averaging, summing over initial, final spins we find

$$\frac{d\sigma^{\mathrm{g-Comp}}_{lab,S}}{d\Omega} \underset{\omega\to 0}{\longrightarrow} G^2m^2(\mathrm{ctn}^4\frac{\theta_L}{2}\cos^4\frac{\theta_L}{2}+\sin^4\frac{\theta_L}{2})\,, \tag{4.56}$$

as found in Eq. (4.49) above.

5 Graviton-Photon Scattering

In the previous sections we have generalized the results of reference [22] to the case of a massive spin-1 target. Here we show how these techniques can be used to calculated the cross-section for photon-graviton scattering. In the Compton scattering calculation we assumed that the spin-1 target had charge e. However, the photon couplings to the graviton are identical to those of a graviton coupled to a charged spin-1 system in the massless limit, and one might assume then that, since the results of the gravitational Compton scattering are independent of charge, the graviton-photon cross-section can be calculated by simply taking the $m \to 0$ limit of the graviton-spin-1 cross-section. Of course, the laboratory cross-section no longer makes sense since the photon cannot be brought to rest, but the invariant cross-section is finite in this limit—

$$\frac{d\sigma_{S=1}^{\mathrm{g-Comp}}}{dt} \underset{m\to 0}{\longrightarrow} \frac{4\pi G^2(3s^2u^2 - 4t^2su + t^4)}{3s^2t^2}\,, \tag{5.1}$$

and it might be naively assumed that Eq. (5.1) is the graviton-photon scattering cross-section. However, this is *not* the case and the resolution of this problem involves some interesting physics.

We begin by noting that in the massless limit the only non vanishing helicity amplitudes are

$$\begin{aligned} D^1(++;++)_{m=0} = D^1(--;--)_{m=0} &= 8\pi G\frac{s^2}{t}\,, \\ D^1(--;++)_{m=0} = D^1(++;--)_{m=0} &= 8\pi G\frac{u^2}{t}\,, \\ D^1(00;++)_{m=0} = D^1(00;--)_{m=0} &= 8\pi G\frac{su}{t}\,, \end{aligned} \tag{5.2}$$

which lead to the cross-section

$$\begin{aligned} \frac{d\sigma_{S=1}^{\mathrm{g-Comp}}}{dt} &= \frac{1}{16\pi s^2}\frac{1}{3}\sum_{a=+,0,-}\frac{1}{2}\sum_{c=+,-}|D^1(ab;cd)|^2 \\ &= \frac{1}{16\pi s^2}\frac{1}{3\cdot 2}(8\pi G)^2 \times 2 \times \left[\frac{s^4}{t^2} + \frac{u^4}{t^2} + \frac{s^2u^2}{t^2}\right] \\ &= \frac{4\pi}{3}G^2\frac{s^4+u^4+s^2u^2}{s^2t^2}\,, \end{aligned} \tag{5.3}$$

in agreement with Eq. (5.1). However, this result demonstrates the problem. We know that in Coulomb gauge the photon has only two transverse degrees of freedom, having positive and negative helicity—there exists *no* longitudinal degree of freedom. Thus the correct photon-graviton cross-section is actually

$$\begin{aligned} \frac{d\sigma_{g\gamma}}{dt} &= \frac{1}{16\pi s^2}\frac{1}{3}\sum_{a=+,-}\frac{1}{2}\sum_{c=+,-}|D^1(ab;cd)|^2 \\ &= \frac{1}{16\pi s^2}\frac{1}{2\cdot 2}(8\pi G)^2 \times 2 \times \left[\frac{s^4}{t^2} + \frac{u^4}{t^2}\right] = 2\pi G^2\frac{s^4+u^4}{s^2t^2}\,, \end{aligned} \tag{5.4}$$

which agrees with the value calculated via conventional methods by Skobelev [35]. Alternatively, since in the center of mass frame

$$\frac{dt}{d\Omega} = \frac{\omega_{\rm CM}}{\pi}, \tag{5.5}$$

we can write the center of mass graviton-photon cross-section in the form

$$\frac{d\sigma_{\rm CM}}{d\Omega} = 2G^2\omega_{\rm CM}^2\left(\frac{1+\cos^8\frac{\theta_{\rm CM}}{2}}{\sin^4\frac{\theta_{\rm CM}}{2}}\right), \tag{5.6}$$

again in agreement with the value given by Skobelev [35].

So what has gone wrong here? Ordinarily in the massless limit of a spin-1 system, the longitudinal mode decouples because the zero helicity spin-1 polarization vector becomes

$$\epsilon^0_\mu \underset{m\to 0}{\longrightarrow} \frac{1}{m}(p,(p+\frac{m^2}{2p}+\ldots)\hat{z}) = \frac{1}{m}p_\mu + (0,\frac{m}{2p}\hat{z}) + \ldots \tag{5.7}$$

However, the term proportional to p_μ vanishes when contracted with a conserved current by gauge invariance while the term in $\frac{m}{2p}$ vanishes in the massless limit. That the spin-1 Compton scattering amplitude becomes gauge invariant for the spin-1 particles in the massless limit can be seen from the fact that the Compton amplitude can be written as

$$\begin{aligned}{\rm Amp}^{\rm Comp}_{S=1} \underset{m\to 0}{\longrightarrow} & \frac{e^2}{p_i\cdot q_i p_i\cdot q_f}\left[{\rm Tr}(F_iF_fF_AF_B)+{\rm Tr}(F_iF_AF_fF_B)+{\rm Tr}(F_iF_AF_BF_f)\right.\\ & \left.-\frac{1}{4}\left({\rm Tr}(F_iF_f){\rm Tr}(F_AF_B)+{\rm Tr}(F_iF_A){\rm Tr}(F_fF_B)+{\rm Tr}(F_iF_B){\rm Tr}(F_fF_A)\right)\right],\end{aligned} \tag{5.8}$$

which can be checked by a bit of algebra. Equivalently, one can check that the massless spin-1 amplitude vanishes if one replaces either $\epsilon_{A\mu}$ by $p_{i\mu}$ or $\epsilon_{B\mu}$ by $p_{f\mu}$. However, what happens when we have *two* longitudinal spin-1 particles is that the product of longitudinal polarization vectors is proportional to $1/m^2$, while the correction term to the four-momentum p_μ is $\mathcal{O}(m^2)$ so that the product is non-vanishing in the massless limit. That is why the multipole $D(00;++)_{m=0} = D(00;--)_{m=0}$ is non vanishing. One can deal with this problem by simply omitting the longitudinal degree of freedom explicitly, as we did above, but this seems a rather crude way to proceed. Shouldn't this behaviour arise naturally?

The problem here is that as long as the mass of the spin-1 particle remains finite everything is fine. However, when the spin-1 particle become massless the theory becomes undefined. This can be seen from the neutral spin-1 (Proca) Lagrangian, which has the form

$$\mathcal{L}^1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu \,. \tag{5.9}$$

The classical equation of motion then becomes

$$\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0 \,. \tag{5.10}$$

Taking the divergence of Eq. (5.10) we find

$$m^2 \partial^\nu A_\nu = 0 \,, \tag{5.11}$$

which yields the constraint $m^2\partial^\nu A_\nu = 0$. Then provided that $m^2 \neq 0$ we have the stricture $\partial^\nu A_\nu = 0$, which is the condition that changes the number of degrees of freedom from four to three, as required for a spin-1 particle. However, in the massless limit, this is no longer the case. Another way to see this is to integrate by parts, whereby Eq. (5.9) can be written in the form

$$\mathcal{L}^1_{m=0} = \frac{1}{2}A_\mu \mathcal{O}^{\mu\nu} A_\nu \,, \quad \text{with} \quad \mathcal{O}^{\mu\nu} = \eta^{\mu\nu}\Box - \partial^\mu\partial^\nu \,. \tag{5.12}$$

In particle physics the photon propagator is given by the inverse of this operator—$\mathcal{O}^{-1}_{\mu\nu}$—which is defined via $\mathcal{O}^{\mu\nu}\mathcal{O}^{-1}_{\nu\alpha} = \delta^\mu_\alpha$ [1]. However, the operator $\mathcal{O}^{\mu\nu}$ does not have an inverse, since it has a zero eigenvalue, as can be seen by operating on a quantity of the form $\partial_\nu\Lambda(x)$ where $\Lambda(x)$ is an arbitrary scalar function. The solution to this problem is well known. The Lagrangian must be altered by adding a gauge fixing term

$$\mathcal{L}^1_{m=0} \longrightarrow -\frac{1}{4}F_{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2 \,, \tag{5.13}$$

where λ is an arbitrary constant. We now have $\mathcal{O}^{\mu\nu} = \eta^{\mu\nu}\Box - (1-\lambda)\partial^\mu\partial^\nu$ which does possess an inverse—$\mathcal{O}^{-1}_{\mu\nu} = \frac{1}{\Box}\left(\eta_{\mu\nu} - \frac{1-\lambda}{\lambda}\frac{\partial_\mu\partial_\nu}{\Box}\right)$. It is this gauge fixing term, which is required in the massless limit, and which eliminates the longitudinal degree of freedom. This degree of freedom acts like simple scalar field (spin-0 particle) and must be subtracted from the massless limit of the spin-1 result. Indeed, from ref. [22] we see that the massless limit of the $++$ graviton scattering from a spin-0 target becomes

$$D^0(++) = (2e^2)^2 \times Y = 8\pi G \frac{su}{t}\,, \tag{5.14}$$

while the $+-$ helicity amplitude vanishes. This scalar amplitude is identical the amplitude $D_1(00;++)$ and eliminates the longitudinal degree of freedom when it is subtracted from the massless spin-1 limit.

An alternative way to obtain this result is to use the Stueckelberg form of the spin-1 Lagrangian, which involves coupling a new spin-0 field B [36]

$$\mathcal{L}_S = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}(A_\mu + \frac{1}{m}\partial_\mu B)(A^\mu + \frac{1}{m}\partial^\mu B) - \frac{1}{2}(\partial_\mu A^\mu + mB)(\partial_\nu A^\nu + mB)\,. \tag{5.15}$$

As long as $m \neq 0$ the fields A_μ and B are coupled. However, if we take the massless limit Eq. (5.15) becomes

$$\mathcal{L}_S \underset{m\to 0}{\longrightarrow} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu A^\mu \partial_\nu A^\nu + \frac{1}{2}\partial_\mu B \partial^\mu B\,, \tag{5.16}$$

and represents the sum of two independent massless fields—a spin-1 component A_μ with the Lagrangian (in Feynman gauge $\lambda = 1$)

$$\mathcal{L}_S^1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu A^\mu \partial_\nu A^\nu = -\frac{1}{2}A^\mu \Box A_\mu\,, \tag{5.17}$$

for which we *do* have an inverse and an independent spin zero component having the Lagrangian

$$\mathcal{L}_S^0 = \frac{1}{2}\partial_\mu B \partial^\mu B\,. \tag{5.18}$$

It is the scattering due to the spin-1 component which is physical and leads to the graviton-photon scattering amplitude, while the spin zero component is *unphysical* and generates the longitudinal component of the massless limit of the graviton-spin-1 scattering.

As a final comment we note that the graviton-graviton scattering amplitude can be obtained by dressing the product of two massless spin-1 Compton amplitudes [4]—

$$\begin{aligned} & \langle p_f, \epsilon_B\epsilon_B; k_f, \epsilon_f\epsilon_f | \mathrm{Amp}_{grav}^{tot} | p_i\epsilon_A\epsilon_A; k_i, \epsilon_i\epsilon_i\rangle_{m=0,S=2} \\ = \;& Y \times \langle p_f, \epsilon_B; k_f, \epsilon_f | \mathrm{Amp}_{em}^{\mathrm{Comp}} | p_i, \epsilon_A; k_i\epsilon_i\rangle_{m=0,S=1} \\ \times \;& \langle p_f, \epsilon_B; k_f, \epsilon_f | \mathrm{Amp}_{em}^{\mathrm{Comp}} | p_i, \epsilon_A; k_i\epsilon_i\rangle_{m=0,S=1}\,. \end{aligned} \tag{5.19}$$

Then for the helicity amplitudes we have

$$E^2(++;++)_{m=0} = Y(B^1(++;++)_{m=0})^2\,, \tag{5.20}$$

where $E^2(++;++)$ is the graviton-graviton $++;++$ helicity amplitude while $B^1(++;++)$ is the corresponding spin-1 Compton helicity amplitude. Thus we find

$$E^2(++;++)_{m=0} = \frac{\kappa^2}{16e^4}\frac{su}{t} \times (2e^2\frac{s}{u})^2 = 8\pi G\frac{s^3}{ut}\,, \tag{5.21}$$

which agrees with the result calculated via conventional methods [37]. In this case there exist non-zero helicity amplitudes related by crossing symmetry. However, we defer detailed discussion of this result to a future communication.

6 The forward cross-section

The forward limit, *i.e.*, $\theta_L \to 0$, of the laboratory frame, Compton cross-sections evaluated in section 3 has a universal structure independent of the spin S of the massive target

$$\lim_{\theta_L\to 0} \frac{d\sigma^{\rm Comp}_{lab,S}}{d\Omega} = \frac{\alpha^2}{2m^2}\,, \tag{6.1}$$

reproducing the Thomson scattering cross-section.

For graviton photo-production, the small angle limit is very different, since the forward scattering cross-section is divergent—the small angle limit of the graviton photo-production of section 4.1 is given by

$$\lim_{\theta_L\to 0} \frac{d\sigma^{\rm photo}_{lab,S}}{d\Omega} = \frac{4G\alpha}{\theta_L^2}\,, \tag{6.2}$$

and arises from the photon pole in figure 4(d). Notice that this behaviour differs from the familiar $1/\theta^4$ small-angle Rutherford cross-section for scattering in a Coulomb-like potential. This divergence of the forward cross-section indicates that a long range force is involved but with an effective $1/r^2$ potential. This effective potential arising from the γ-pole in figure 4(d), is the Fourier transform with respect to the momentum transfer $q = k_f - k_i$ of the low-energy limit given in Eq. (4.37). Because of the linear dependence in the momenta in the numerator one obtains

$$\int \frac{d^3\vec{q}}{(2\pi)^3}\, e^{i\vec{q}\cdot\vec{r}}\, \frac{1}{|\vec{q}|} = \frac{1}{2\pi^2 r^2}\,, \tag{6.3}$$

and that leads to the peculiar forward scattering behaviour of the cross-section. Another contrasting feature of graviton photo-production is the independence of the forward cross-section on the mass m of the target.

The small angle limit of the gravitational Compton cross-section derived in section 4.2 is given by

$$\lim_{\theta_L\to 0} \frac{d\sigma^{\rm g-Comp}_{lab,S}}{d\Omega} = \frac{16G^2m^2}{\theta_L^4}\,. \tag{6.4}$$

The limit is, of course, independent of the spin S of the matter field. Finally, the photon-graviton cross-section derived in section 5, has the forward scattering dependence

$$\lim_{\theta_{\rm CM}\to 0} \frac{d\sigma_{\rm CM}}{d\Omega} = \frac{32G^2\omega_{\rm CM}^2}{\theta_{\rm CM}^4}\,. \tag{6.5}$$

The behaviours in Eq. (6.4) and (6.5) are due to the graviton pole in figure 5(d), and are typical of the small-angle behaviour of Rutherford scattering in a Coulomb potential.

The classical bending of the geodesic for a massless particle in a Schwarzschild metric produced by a point-like mass m is given by $b = 4Gm/\theta + O(1)$ [38], where b is the classical impact parameter. The associated classical cross-section is

$$\frac{d\sigma^{\rm classical}}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \simeq \frac{16G^2m^2}{\theta^4} + O(\theta^{-3})\,, \tag{6.6}$$

matching the expression in Eq. (6.4). The diagram in figure 5(d) describes the gravitational interaction between a massive particle of spin S and a graviton. In the forward scattering limit the remaining diagrams of figure 5 have vanishing contributions. Since this limit is independent of the spin of the particles interacting gravitationally, the expression in Eq. (6.4) describes the forward gravitational scattering cross-section of *any* massless particle on the target of mass m and explains the match with the classical formula given above.

Eq. (6.5) can be interpreted in a similar way, as the bending of a geodesic in a geometry curved by the energy density with an effective Schwarzschild radius of $\sqrt{2}\,G\omega_{\rm CM}$ determined by the center-of-mass energy [39]. However, the effect is fantastically small since the cross-section in Eq. (6.5) is of order $\ell_P^4/(\lambda^2\,\theta_{\rm CM}^4)$ where $\ell_P^2 = \hbar G/c^3 \sim 1.62\,10^{-35}\,{\rm m}$ is the Planck length, and λ the wavelength of the photon.

7 Conclusion

In ref. [22] it was demonstrated that the gravitational interactions of a charged spin-0 or spin-$\frac{1}{2}$ particle are greatly simplified by use of the recently discovered factorization theorem, which asserts that the gravitational amplitudes must be identical to corresponding electromagnetic amplitudes multiplied by universal kinematic factors. In the present work we demonstrated that the same simplification applies when the target particle carries spin-1. Specifically, we evaluated the graviton photo-production and graviton Compton scattering amplitudes explicitly using direct and factorized techniques and showed that they are identical. However, the factorization methods are *enormously* simpler and allow the use of familiar electromagnetic calculational methods, eliminating the need for the use of less familiar and more cumbersome tensor quantities. We also studied the massless limit of the spin-1 system and showed how the use of factorization permits a relatively simple calculation of graviton-photon scattering. Finally, we discussed a subtlety in this graviton-photon calculation having to do with the feature that the spin-1 system must change from three to two degrees of freedom when $m \to 0$ and studied why the zero mass limit of the spin-1 gravitational Compton scattering amplitude does not correspond to that for photon scattering. We noted that graviton-graviton scattering is also simply obtained by taking the product of Compton amplitudes dressed by the appropriate kinematic factor.

We discussed the main feature of the forward cross-section for each process studied in this paper. Both the Compton and the gravitational Compton scattering have the expected behaviour, while graviton photo-production has a different shape that could in principle lead to an interesting new experimental signature of a graviton scattering on matter. An extension of the present discussion at loop order and implications for the photo-production of gravitons from stars [40, 41] will be given elsewhere.

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Appendix A

Here we give the detailed contributions from each of the four diagrams contributing to graviton photo-production and to gravitational Compton scattering. In the case of graviton photo-production—Figure 4—we have the four pieces

Graviton Photo-production: spin-1

$$\begin{aligned}
\mathrm{Born-a:}\qquad \mathrm{Amp_a(S=1)} &= \frac{\kappa e}{p_i\cdot k_i}\Big\{\epsilon_i\cdot p_i\left[\epsilon_B^*\cdot\epsilon_A\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_f-\epsilon_B^*\cdot k_f\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_A\right.\\
&- \epsilon_A\cdot p_f\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_B^*+p_f\cdot k_f\epsilon_f^*\cdot\epsilon_A\epsilon_f^*\cdot\epsilon_B^*\big]\\
&+ \epsilon_A\cdot\epsilon_i\left[\epsilon_B^*\cdot k_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_f-\epsilon_B^*\cdot k_f\epsilon_f^*\cdot p_f\epsilon_f^*\cdot k_i-p_f\cdot k_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_B^*\right.\\
&+ p_f\cdot k_f\epsilon_f^*\cdot k_i\epsilon_f^*\cdot\epsilon_B^*\big]\\
&- \epsilon_A\cdot k_i\left[\epsilon_B^*\cdot\epsilon_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_f-\epsilon_B^*\cdot k_f\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_i-\epsilon_i\cdot p_f\epsilon_f^*\cdot p_f\epsilon_f^*\cdot\epsilon_B^*\right.\\
&+ p_f\cdot k_f\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot\epsilon_B^*\big]\\
&- \epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_fp_i\cdot k_i\Big\}\ .
\end{aligned}\tag{7.1}$$

$$\begin{aligned}
\mathrm{Born-b:}\qquad \mathrm{Amp_b(S=1)} &= -\frac{\kappa e}{p_i\cdot k_f}\Big\{\epsilon_i\cdot p_f\left[\epsilon_A\cdot\epsilon_B^*\epsilon_f^*\cdot p_i\epsilon_f^*\cdot p_i-\epsilon_B^*\cdot p_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_A\right.\\
&+ \epsilon_A\cdot k_f\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_B^*-p_i\cdot k_f\epsilon_f^*\cdot\epsilon_A\epsilon_f^*\cdot\epsilon_B^*\big]\\
&+ \epsilon_B^*\cdot k_i\left[\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot p_i-\epsilon_i\cdot p_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_A+\epsilon_A\cdot k_f\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_i\right.\\
&- p_i\cdot k_f\epsilon_f^*\cdot\epsilon_A\epsilon_f^*\cdot\epsilon_i\big]\\
&+ \epsilon_i\cdot\epsilon_B^*\left[\epsilon_A\cdot k_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot p_i-p_i\cdot k_i\epsilon_f^*\cdot p_i\epsilon_f^*\cdot\epsilon_A+\epsilon_A\cdot k_f\epsilon_f^*\cdot p_i\epsilon_f^*\cdot k_i\right.\\
&- p_i\cdot k_f\epsilon_f^*\cdot\epsilon_A\epsilon_f^*\cdot k_i\big]\\
&- \epsilon_A\cdot\epsilon_f^*\epsilon_f^*\cdot p_i\epsilon_B^*\cdot\epsilon_ip_i\cdot k_f\Big\}\ .
\end{aligned}\tag{7.2}$$

$$\text{Seagull}-\text{c}: \qquad \text{Amp}_c(S=1) = \kappa e\left[\epsilon_f^*\cdot\epsilon_i(\epsilon_B^*\cdot\epsilon_A\epsilon_f^*\cdot(p_f+p_i) - \epsilon_A\cdot p_f\epsilon_B^*\cdot\epsilon_f^* - \epsilon_B^*\cdot p_i\epsilon_A\cdot\epsilon_f^*)\right.$$
$$\left. - \quad \epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_i - \epsilon_A\cdot\epsilon_f^*\epsilon_B^*\cdot\epsilon_i\epsilon_f^*\cdot p_f + \epsilon_f^*\cdot\epsilon_A\epsilon_f^*\cdot\epsilon_B^*\epsilon_i\cdot(p_f+p_i)\right] , \qquad (7.3)$$

and finally, the photon pole contribution

$$\begin{aligned}
\gamma-\text{pole}-\text{d}: \qquad \text{Amp}_\text{d}(\text{S}=1) &= -\frac{e\kappa}{2k_f\cdot k_i} \\
\times &\quad \left\{\epsilon_B^*\cdot\epsilon_A\left[\epsilon_f^*\cdot(p_f+p_i)(k_f\cdot k_i\epsilon_f^*\cdot\epsilon_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f)\right.\right. \\
+ &\quad \left.\epsilon_f^*\cdot k_i(\epsilon_f^*\cdot\epsilon_i k_i\cdot(p_i+p_f) - \epsilon_f^*\cdot k_i\epsilon_i\cdot(p_f+p_i))\right] \\
- &\quad 2\epsilon_B^*\cdot p_i\left[\epsilon_f^*\cdot\epsilon_A(k_f\cdot k_i\epsilon_f^*\cdot\epsilon_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f)\right. \\
+ &\quad \left.\epsilon_f^*\cdot k_i(\epsilon_f^*\cdot\epsilon_i\epsilon_A\cdot k_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot\epsilon_A)\right] \\
- &\quad 2\epsilon_A\cdot p_f\left[\epsilon_f^*\cdot\epsilon_B^*(k_f\cdot k_i\epsilon_f^*\cdot\epsilon_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot k_f)\right. \\
+ &\quad \left.\left.\epsilon_f^*\cdot k_i(\epsilon_f^*\cdot\epsilon_i\epsilon_B^*\cdot k_i - \epsilon_f^*\cdot k_i\epsilon_i\cdot\epsilon_B^*)\right]\right\} . \qquad (7.4)
\end{aligned}$$

In the case of gravitational Compton scattering—Figure 5—we have the four contributions

Gravitational Compton Scattering: spin-1

$$\begin{aligned}
\text{Born}-\text{a}: \qquad \text{Amp}_a(S=1) &= \kappa^2\frac{1}{2p_i\cdot k_i}[(\epsilon_i\cdot p_i)^2(\epsilon_f^*\cdot p_f)^2\epsilon_A\cdot\epsilon_B^* \\
- &\quad (\epsilon_f^*\cdot p_f)^2\epsilon_i\cdot p_i(\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_i + \epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot p_i) \\
- &\quad (\epsilon_i\cdot p_i)^2\epsilon_f^*\cdot p_f(\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot p_f + \epsilon_B^*\cdot k_f\epsilon_A\cdot\epsilon_f^*) \\
+ &\quad \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f\epsilon_i\cdot p_f\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_f^* + \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f\epsilon_f^*\cdot p_i\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot k_f \\
+ &\quad (\epsilon_f^*\cdot p_f)^2\epsilon_B^*\cdot\epsilon_i\epsilon_A\cdot\epsilon_i p_i\cdot k_i + (\epsilon_i\cdot p_i)^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_f^* p_f\cdot k_f \\
+ &\quad \epsilon_i\cdot p_i\epsilon_f^*\cdot p_f(\epsilon_A\cdot k_i\epsilon_B^*\cdot k_f\epsilon_i\cdot\epsilon_f^* + \epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i p_i\cdot p_f) \\
- &\quad \epsilon_i\cdot p_i\epsilon_f^*\cdot p_i\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i p_f\cdot k_f - \epsilon_f^*\cdot p_f\epsilon_i\cdot p_f\epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^* p_i\cdot k_i \\
- &\quad \epsilon_i\cdot p_i\epsilon_A\cdot k_i\epsilon_B^*\cdot\epsilon_f^*\epsilon_f^*\cdot\epsilon_i p_f\cdot k_f - \epsilon_f^*\cdot p_f\epsilon_B^*\cdot k_f\epsilon_A\cdot\epsilon_i\epsilon_i\cdot\epsilon_f^* p_i\cdot k_i \\
+ &\quad \epsilon_A\cdot\epsilon_i\epsilon_B^*\cdot\epsilon_f^* p_i\cdot k_i p_f\cdot k_f\epsilon_i\cdot\epsilon_f^* - m^2\epsilon_B^*\cdot\epsilon_f^*\epsilon_A\cdot\epsilon_i\epsilon_f^*\cdot p_f\epsilon_i\cdot p_i] . \\
&\qquad (7.5)
\end{aligned}$$

$$
\begin{aligned}
\text{Born} - \text{b}: \qquad & \text{Amp}_b(S=1) = -\kappa^2 \frac{1}{2p_i \cdot k_f}[(\epsilon_f^* \cdot p_i)^2(\epsilon_i \cdot p_f)^2 \epsilon_A \cdot \epsilon_B^* \\
+ \quad & (\epsilon_i \cdot p_f)^2 \epsilon_f^* \cdot p_i(\epsilon_A \cdot k_f \epsilon_B^* \cdot \epsilon_f^* - \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot p_i) \\
+ \quad & (\epsilon_f^* \cdot p_i)^2 \epsilon_i \cdot p_f(\epsilon_B^* \cdot k_i \epsilon_A \cdot \epsilon_i - \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot p_f) \\
- \quad & \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f \epsilon_f^* \cdot p_f \epsilon_A \cdot k_f \epsilon_B^* \cdot \epsilon_i - \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f \epsilon_i \cdot p_i \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot k_i \\
- \quad & (\epsilon_i \cdot p_f)^2 \epsilon_B^* \cdot \epsilon_f^* \epsilon_A \cdot \epsilon_f^* p_i \cdot k_f - (\epsilon_f^* \cdot p_i)^2 \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot \epsilon_i p_f \cdot k_i \\
+ \quad & \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f(\epsilon_A \cdot k_f \epsilon_B^* \cdot k_i \epsilon_i \cdot \epsilon_f^* + \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot \epsilon_f^* p_i \cdot p_f) \\
+ \quad & \epsilon_f^* \cdot p_i \epsilon_i \cdot p_i \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot \epsilon_f^* p_f \cdot k_i + \epsilon_i \cdot p_f \epsilon_f^* \cdot p_f \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot \epsilon_i p_i \cdot k_f \\
- \quad & \epsilon_f^* \cdot p_i \epsilon_A \cdot k_f \epsilon_B^* \cdot \epsilon_i \epsilon_i \cdot \epsilon_f^* p_f \cdot k_i - \epsilon_i \cdot p_f \epsilon_B^* \cdot k_i \epsilon_A \cdot \epsilon_f^* \epsilon_f^* \cdot \epsilon_i p_i \cdot k_f \\
+ \quad & \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot \epsilon_i p_i \cdot k_f p_f \cdot k_i \epsilon_i \cdot \epsilon_f^* - m^2 \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot \epsilon_f^* \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i]\,.
\end{aligned} \tag{7.6}
$$

$$
\begin{aligned}
\text{Seagull} - \text{c}: \qquad & \text{Amp}_c(S=1) = -\frac{\kappa^2}{4}[(\epsilon_i \cdot \epsilon_f^*)^2(m^2 - p_i \cdot p_f)\epsilon_A \cdot \epsilon_B^* \\
+ \quad & \epsilon_A \cdot p_f \epsilon_B^* \cdot p_i(\epsilon_i \cdot \epsilon_f^*)^2 + \epsilon_i \cdot p_i \epsilon_f^* \cdot p_f(2\epsilon_i \cdot \epsilon_f^* \epsilon_A \cdot \epsilon_B^* - 2\epsilon_A \cdot \epsilon_2 \epsilon_B^* \cdot \epsilon_1) \\
+ \quad & \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i(2\epsilon_i \cdot \epsilon_f^* \epsilon_A \cdot \epsilon_B^* - 2\epsilon_A \cdot \epsilon_i \epsilon_B^* \cdot \epsilon_f^*) \\
+ \quad & 2\epsilon_i \cdot p_i \epsilon_1 \cdot p_f \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot \epsilon_f^* + 2\epsilon_f^* \cdot p_f \epsilon_f^* \cdot p_i \epsilon_A \cdot \epsilon_i \epsilon_B^* \cdot \epsilon_i \\
- \quad & 2\epsilon_i \cdot p_i \epsilon_i \cdot \epsilon_f^* \epsilon_A \cdot p_f \epsilon_B^* \cdot \epsilon_f^* - 2\epsilon_f^* \cdot p_f \epsilon_i \cdot \epsilon_f^* \epsilon_A \cdot \epsilon_i \epsilon_f^* \cdot p_i \\
- \quad & 2\epsilon_i \cdot p_f \epsilon_i \cdot \epsilon_f^* \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot p_i - 2\epsilon_f^* \cdot p_i \epsilon_i \cdot \epsilon_f^* \epsilon_B^* \cdot \epsilon_i \epsilon_A \cdot p_f \\
- \quad & 2(m^2 - p_f \cdot p_i)\epsilon_i \cdot \epsilon_f^*(\epsilon_A \cdot \epsilon_i \epsilon_B^* \cdot \epsilon_f^* + \epsilon_A \cdot \epsilon_f^* \epsilon_B^* \cdot \epsilon_i)]\,,
\end{aligned} \tag{7.7}
$$

and finally the (lengthy) graviton pole contribution is

$$
\begin{aligned}
\mathrm{g-pole-d}: \qquad & \mathrm{Amp}_d(S=1) = -\frac{\kappa^2}{16k_i\cdot k_f}\{\epsilon^*_B\cdot\epsilon_A[(\epsilon_i\cdot\epsilon^*_f)^2(4k_i\cdot p_ip_f\cdot k_i+4k_f\cdot p_ik_f\cdot p_f \\
- & \ 2(p_i\cdot k_ip_f\cdot k_f+p_f\cdot k_ip_i\cdot k_f)+6p_i\cdot p_fk_i\cdot k_f) \\
+ & \ 4((\epsilon_i\cdot k_f)^2\epsilon^*_f\cdot p_f\epsilon^*_f\cdot p_i+(\epsilon^*_f\cdot k_i)^2\epsilon_i\cdot p_i\epsilon_i\cdot p_f \\
+ & \ \epsilon_i\cdot k_f\epsilon^*_f\cdot k_i(\epsilon_i\cdot p_i\epsilon^*_f\cdot p_f+\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i)) \\
- & \ 4\epsilon_i\cdot\epsilon^*_f(\epsilon_i\cdot k_f(\epsilon^*_f\cdot p_ip_f\cdot k_f+\epsilon^*_f\cdot p_fk_f\cdot p_i) \\
+ & \ \epsilon^*_f\cdot k_i(\epsilon_i\cdot p_ip_f\cdot k_i+\epsilon_i\cdot p_fp_i\cdot k_i)) \\
- & \ 4k_i\cdot k_f\epsilon_i\cdot\epsilon^*_f(\epsilon_i\cdot p_i\epsilon^*_f\cdot p_f+\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i)-4p_i\cdot p_f\epsilon_i\cdot\epsilon^*_f\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i] \\
- & \ (p_i\cdot p_f\epsilon^*_B\cdot\epsilon_A-\epsilon^*_B\cdot p_i\epsilon_A\cdot p_f)[10(\epsilon_i\cdot\epsilon^*_f)^2k_i\cdot k_f+4\epsilon_i\cdot\epsilon^*_f\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i \\
- & \ 4(\epsilon_i\cdot\epsilon^*_f)^2k_i\cdot k_f-8\epsilon_i\cdot\epsilon^*_f\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i] \\
+ & \ (p_i\cdot p_f-m^2)[(\epsilon_i\cdot\epsilon^*_f)^2(4\epsilon_A\cdot k_i\epsilon^*_B\cdot k_i+4\epsilon_A\cdot k_f\epsilon^*_B\cdot k_f \\
- & \ 2(\epsilon_A\cdot k_i\epsilon^*_B\cdot k_f+\epsilon_A\cdot k_f\epsilon^*_B\cdot k_i)+6\epsilon^*_B\cdot\epsilon_Ak_i\cdot k_f) \\
+ & \ 4[(\epsilon_i\cdot k_f)^2\epsilon_A\cdot\epsilon^*_f\epsilon^*_B\cdot\epsilon^*_f+(\epsilon^*_f\cdot k_i)^2\epsilon_A\cdot\epsilon_i\epsilon^*_B\cdot\epsilon_i \\
+ & \ \epsilon_i\cdot k_f\epsilon^*_f\cdot k_f(\epsilon_A\cdot\epsilon_i\epsilon^*_B\cdot\epsilon^*_f+\epsilon_A\cdot\epsilon^*_f\epsilon^*_B\cdot\epsilon_i)] \\
- & \ 4\epsilon_i\cdot\epsilon^*_f[\epsilon_i\cdot k_f(\epsilon_A\cdot\epsilon^*_f\epsilon^*_B\cdot k_f+\epsilon^*_B\cdot\epsilon^*_f\epsilon_A\cdot k_f) \\
+ & \ \epsilon^*_f\cdot k_i(\epsilon_A\cdot\epsilon_i\epsilon^*_B\cdot k_i+\epsilon^*_B\cdot\epsilon_i\epsilon_A\cdot k_i) \\
+ & \ k_i\cdot k_f(\epsilon_A\cdot\epsilon_i\epsilon^*_B\cdot\epsilon^*_f+\epsilon^*_B\cdot\epsilon_i\epsilon_A\cdot\epsilon^*_f)+\epsilon_A\cdot\epsilon^*_B\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i]] \\
- & \ 2\epsilon_A\cdot p_f[(\epsilon^*_f\cdot\epsilon_i)^2[2\epsilon^*_B\cdot k_ip_i\cdot k_i+2\epsilon^*_B\cdot k_fp_i\cdot k_f+3\epsilon^*_B\cdot p_ik_i\cdot k_f \\
- & \ (\epsilon^*_B\cdot k_ip_i\cdot k_f+\epsilon^*_B\cdot k_fp_i\cdot k_i)] \\
+ & \ 2(\epsilon_i\cdot k_f)^2\epsilon^*_B\cdot\epsilon^*_f\epsilon^*_f\cdot p_i+2(\epsilon^*_f\cdot k_i)^2\epsilon^*_B\cdot\epsilon_i\epsilon_i\cdot p_i \\
+ & \ 2\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i(\epsilon^*_B\cdot\epsilon_i\epsilon^*_f\cdot p_i+\epsilon_i\cdot p_i\epsilon^*_B\cdot\epsilon^*_f) \\
- & \ 2\epsilon_i\cdot\epsilon^*_f[\epsilon_i\cdot k_f(\epsilon^*_B\cdot\epsilon^*_fp_i\cdot k_f+\epsilon^*_f\cdot p_i\epsilon^*_B\cdot k_f) \\
+ & \ \epsilon^*_f\cdot k_i(\epsilon^*_B\cdot\epsilon_ip_i\cdot k_i+\epsilon^*_B\cdot k_i\epsilon_i\cdot p_i)] \\
- & \ 2k_i\cdot k_f\epsilon_i\cdot\epsilon^*_f(\epsilon^*_B\cdot\epsilon_i\epsilon^*_f\cdot p_i+\epsilon^*_B\cdot\epsilon^*_f\epsilon_i\cdot p_i)-2\epsilon^*_B\cdot p_i\epsilon_i\cdot\epsilon^*_f\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i] \\
- & \ 2\epsilon^*_B\cdot p_i[(\epsilon^*_f\cdot\epsilon_i)^2[2\epsilon_A\cdot k_ip_f\cdot k_i+2\epsilon_A\cdot k_fp_f\cdot k_f+3\epsilon_A\cdot p_fk_i\cdot k_f \\
- & \ (\epsilon_A\cdot k_ip_f\cdot k_f+\epsilon_A\cdot k_fp_f\cdot k_f)] \\
+ & \ 2(\epsilon_i\cdot k_f)^2\epsilon_A\cdot\epsilon^*_f\epsilon^*_f\cdot p_f+2(\epsilon^*_f\cdot k_i)^2\epsilon_A\cdot\epsilon_i\epsilon_i\cdot p_f \\
+ & \ 2\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i(\epsilon_A\cdot\epsilon_i\epsilon^*_f\cdot p_f+\epsilon_i\cdot p_f\epsilon_A\cdot\epsilon^*_f \\
- & \ 2\epsilon_i\cdot\epsilon^*_f[\epsilon_i\cdot k_f(\epsilon_A\cdot\epsilon^*_fp_f\cdot k_f+\epsilon^*_f\cdot p_f\epsilon_A\cdot k_f) \\
+ & \ \epsilon^*_f\cdot k_i(\epsilon_A\cdot\epsilon_ip_f\cdot k_i+\epsilon_A\cdot k_i\epsilon_i\cdot p_f)] \\
- & \ 2k_i\cdot k_f\epsilon_i\cdot\epsilon^*_f(\epsilon_A\cdot\epsilon_i\epsilon^*_f\cdot p_f+\epsilon_A\cdot\epsilon^*_f\epsilon_i\cdot p_f)-2\epsilon_A\cdot p_f\epsilon_i\cdot\epsilon^*_f\epsilon_i\cdot k_f\epsilon^*_f\cdot k_i]\}\,.
\end{aligned}
\tag{7.8}
$$

References

[1] See, *e.g.*, B.R. Holstein, **Topics in Advanced Quantum Mechanics**, Dover, New York (2014).

[2] An exception is M.D. Scadron, **Advanced Quantum Theory and its Applications through Feynman Diagrams**, Springer-Verlag, New York (1979).

[3] H. Kawai, D. C. Lewellen and S. H. H. Tye, "A Relation Between Tree Amplitudes of Closed and Open Strings," Nucl. Phys. B **269** (1986) 1.

[4] S.Y. Choi, J.S. Shim, and H.S. Song, "Factorization and Polarization in Linearized Gravity," Phys. Rev. **D51**, 2751-69 (1995).

[5] Z. Bern, "Perturbative Quantum Gravity and Its Relation to Gauge Theory," Living Rev. Rel. **5** (2002) 5 [gr-qc/0206071].

[6] S. Stieberger, "Open & Closed Vs. Pure Open String Disk Amplitudes," arXiv:0907.2211 [hep-th].

[7] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, "The Momentum Kernel of Gauge and Gravity Theories," JHEP **1101** (2011) 001 [arXiv:1010.3933 [hep-th]].

[8] Z. Bern, J. J. M. Carrasco and H. Johansson, "Perturbative Quantum Gravity as a Double Copy of Gauge Theory," Phys. Rev. Lett. **105** (2010) 061602 [arXiv:1004.0476 [hep-th]].

[9] M. Chiodaroli, M. Günaydin, H. Johansson and R. Roiban, "Scattering Amplitudes in $\mathcal{N} = 2$ Maxwell-Einstein and Yang-Mills/Einstein Supergravity," arXiv:1408.0764 [hep-th].

[10] A. Ochirov and P. Tourkine, "Bcj Duality and Double Copy in the Closed String Sector," JHEP **1405** (2014) 136 [arXiv:1312.1326 [hep-th]].

[11] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "The Ultraviolet Behavior of $\mathcal{N} = 8$ Supergravity at Four Loops," Phys. Rev. Lett. **103** (2009) 081301 [arXiv:0905.2326 [hep-th]].

[12] Z. Bern, T. Dennen, Y. t. Huang and M. Kiermaier, "Gravity as the Square of Gauge Theory," Phys. Rev. D **82** (2010) 065003 [arXiv:1004.0693 [hep-th]].

[13] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes," Phys. Rev. D **85** (2012) 105014 [arXiv:1201.5366 [hep-th]].

[14] Z. Bern, C. Boucher-Veronneau and H. Johansson, "N ¿= 4 Supergravity Amplitudes from Gauge Theory at One Loop," Phys. Rev. D **84** (2011) 105035 [arXiv:1107.1935 [hep-th]].

[15] J. J. M. Carrasco, M. Chiodaroli, M. Günaydin and R. Roiban, "One-Loop Four-Point Amplitudes in Pure and Matter-Coupled N ¡= 4 Supergravity," JHEP **1303** (2013) 056 [arXiv:1212.1146 [hep-th]].

[16] Z. Bern, S. Davies, T. Dennen and Y. t. Huang, "Absence of Three-Loop Four-Point Divergences in $\mathcal{N} = 4$ Supergravity," Phys. Rev. Lett. **108** (2012) 201301 [arXiv:1202.3423 [hep-th]].

[17] Z. Bern, S. Davies, T. Dennen, Y. t. Huang and J. Nohle, "Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops," arXiv:1303.6605 [hep-th].

[18] Z. Bern, S. Davies and T. Dennen, "The Ultraviolet Structure of Half-Maximal Supergravity with Matter Multiplets at Two and Three Loops," Phys. Rev. D **88** (2013) 065007 [arXiv:1305.4876 [hep-th]].

[19] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov and V. A. Smirnov, "Ultraviolet Properties of $\mathcal{N} = 4$ Supergravity at Four Loops," Phys. Rev. Lett. **111** (2013) 23, 231302 [arXiv:1309.2498 [hep-th]].

[20] Z. Bern, S. Davies and T. Dennen, "Enhanced Ultraviolet Cancellations in $\mathcal{N} = 5$ Supergravity at Four Loop," arXiv:1409.3089 [hep-th].

[21] H. Johansson and A. Ochirov, "Pure Gravities via Color-Kinematics Duality for Fundamental Matter," arXiv:1407.4772 [hep-th].

[22] B.R. Holstein, "Graviton Physics", Am. J. Phys. **74**, 1002-11 (2006).

[23] B. R. Holstein and A. Ross, "Spin Effects in Long Range Electromagnetic Scattering," arXiv:0802.0715 [hep-ph].

[24] B. R. Holstein and A. Ross, "Spin Effects in Long Range Gravitational Scattering," arXiv:0802.0716 [hep-ph].

[25] B. R. Holstein and A. Ross, "Long Distance Effects in Mixed Electromagnetic-Gravitational Scattering," arXiv:0802.0717 [hep-ph].

[26] N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, "On-Shell Techniques and Universal Results in Quantum Gravity," JHEP **1402** (2014) 111 [arXiv:1309.0804 [hep-th], arXiv:1309.0804].

[27] F. Dyson, "Is a Graviton Detectable?", Int.J.Mod.Phys. **A28** (2013) 1330041.

[28] F.J. Belinfante, "Intrinsic Magnetic Moment of Elementary Particles of Spin 3/2," Phys. Rev. **92**, 997-1001 (1953).

[29] S. Weinberg, in **Lectures on Elementary Paricles and Quantum Field Theory**, Proc. Summer Institute, Brandeis Univ. (1970), ed. S. Deser, MIT Press, Cambridge, MA (1970), Vol. 1.;

[30] B.R. Holstein, "How Large is the 'Natural' Magnetic Moment," Am. J. Phys. **74**, 1104-11 (2006).

[31] B.R. Holstein, "Factorization in Graviton Scattering and the "Natural" Value of the g-factor," Phys. Rev. **D74**, 085002 (2006).

[32] M. Jacob and G.C. Wick, "On the General Theory of Collisions for Particles with Spin," Ann. Phys. (NY) **7**, 404-28 (1959).

[33] B.R. Holstein, "Blue Skies and Effective Interactions," Am. J. Phys. **67**, 422-27 (1999).

[34] J.D. Jackson, **Classical Electrodynamics**, Wiley, New York (1970).

[35] V.V. Skobelev, "Graviton-Photon Interaction", Sov. Phys. J. **18**, 62-65 (1975).

[36] See, *e.g.*, H.Ruegg and M. Ruiz-Altaba, "The Stueckelberg Field," Int. J. Mod. Phys. **A19**, 3265-3348 (2004).

[37] J.F. Donoghue and T. Torma, “Infrared Behavior of Graviton-Graviton Scattering,” Phys. Rev. **D60**, 024003 (1999).

[38] L. D. Landau and E. M. Lifschits, “The Classical Theory of Fields : Course of Theoretical Physics, Volume 2”, Butterworth-Heinemann; 4 edition (December 31, 1975)

[39] D. Amati, M. Ciafaloni and G. Veneziano, “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions,” Int. J. Mod. Phys. A **3** (1988) 1615.

[40] W. K. De Logi and A. R. Mickelson, “Electrogravitational Conversion Cross-Sections in Static Electromagnetic Fields,” Phys. Rev. D **16** (1977) 2915.

[41] G. Papini and S. R. Valluri, “Gravitons in Minkowski Space-Time. Interactions and Results of Astrophysical Interest,” Phys. Rept. **33** (1977) 51.