Differential symmetry breaking operators. II. Rankin–Cohen operators for symmetric pairs

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DIFFERENTIAL SYMMETRY BREAKING OPERATORS.
II. RANKIN–COHEN OPERATORS FOR SYMMETRIC PAIRS

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Abstract. Rankin–Cohen brackets are symmetry breaking operators for the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})$. We address a general problem to find explicit formulæ for such intertwining operators in the setting of multiplicity-free branching laws for reductive symmetric pairs.

For this purpose we use a new method (F-method) developed in [KP14-1] and based on the *algebraic Fourier transform for generalized Verma modules.* The method characterizes symmetry breaking operators by means of certain systems of partial differential equations of second order.

We discover explicit formulæ of new differential symmetry breaking operators for all the six different complex geometries arising from semisimple symmetric pairs of split rank one, and reveal an intrinsic reason why the coefficients of orthogonal polynomials appear in these operators (Rankin–Cohen type) in the three geometries and why normal derivatives are symmetry breaking operators in the other three cases. Further, we analyze a new phenomenon that the multiplicities in the branching laws of Verma modules may jump up at singular parameters.

Key words and phrases: *branching laws, Rankin–Cohen brackets, F-method, symmetric pair, invariant theory, Verma modules, Hermitian symmetric spaces, Jacobi polynomial.*

Contents

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1. Introduction

What kind of differential operators do preserve modularity? R. A. Rankin [Ra56] and H. Cohen [C75] introduced a family of differential operators transforming a given pair of modular forms into another modular form of a higher weight. Let f_1 and f_2 be holomorphic modular forms for a given arithmetic subgroup of $SL(2,\mathbb{R})$ of weight k_1 and k_2, respectively. The bidifferential operators, referred to as the *Rankin–Cohen brackets* of degree a and defined by

$$\mathcal{RC}^{k_3}_{k_1,k_2}(f_1,f_2)(z) := \sum_{\ell=0}^{a}(-1)^{\ell}\binom{k_1+a-1}{\ell}\binom{k_2+a-1}{a-\ell} f_1^{(a-\ell)}(z) f_2^{(\ell)}(z), \tag{1.1}$$

where $f^{(n)}(z) = \frac{d^n f}{dz^n}(z)$, yield holomorphic modular forms of weight $k_3 = k_1 + k_2 + 2a$ ($a = 0, 1, 2, \cdots$). (In the usual notation, these operators are written as $RC^{a}_{k_1,k_2}$.)

The Rankin–Cohen bidifferential operators have attracted considerable attention in recent years particularly because of their applications to various areas including

- theory of modular and quasimodular forms (special values of L-functions, the Ramanujan and Chazy differential equations, van der Pol and Niebur equalities) [CL11, MR09, Z94],
- covariant quantization [BTY07, CMZ97, CM04, OS00, DP07, P08, UU96],
- ring structures on representations spaces [DP07, Z94].

Existing methods for the $SL(2,\mathbb{R})$-case. A prototype of the Rankin–Cohen brackets was already found by P. Gordan and S. Guldenfinger [Go1887, Gu1886] in the 19th century by using recursion relations for invariant binary forms and the Cayley processes. For explicit constructions of the equivariant bidifferential operators (1.1), several different methods have been developed:

- Recurrence relations [C75, El06, HT92, P12, Z94].
- Taylor expansions of Jacobi forms [EZ85, IKO12, Ku75].
- Reproducing kernels for Hilbert spaces [PZ04, UU96, Zh10].
- Dual pair correspondence [B06, EI98].

In the first part of our work [KP14-1] we proposed yet another method (*F-method*) to find differential symmetry breaking operators in a more general setting of branching laws for infinite-dimensional representations, based on the algebraic Fourier transform of generalized Verma modules. Even in the $SL(2,\mathbb{R})$-case, the method is original and simple, and yields missing operators for singular parameters (k_1, k_2, k_3), see Corollary 9.3 for the complete classification. Moreover, the F-method leads us to discover new families of covariant differential operators for six different complex geometries beyond the $SL(2,\mathbb{R})$ case (see Table 1.1).

Branching laws for symmetric pairs. By *branching law* we mean the decomposition of an irreducible representation π of a group G when restricted to a given subgroup G'. An important and fruitful source of examples is provided by pairs of groups (G, G') such that G' is the fixed point group of an involutive automorphism τ of G, called *symmetric pairs.*

The decomposition of the tensor product of two representations is a special case of branching laws with respect to symmetric pairs (G, G'). Indeed, if $G = G_1 \times G_1$ and τ is an involutive automorphism of G given by $\tau(x, y) = (y, x)$, then $G' \simeq G_1$ and the restriction of the outer tensor product $\pi_1 \boxtimes \pi_2$ to the subgroup G' is nothing but the tensor product $\pi_1 \otimes \pi_2$ of two representations π_1 and π_2 of G_1. The Littlewood–Richardson rule for finite-dimensional representations is another classical example of branching laws with respect to the symmetric pair $(GL(p+q,\mathbb{C}), GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$. Our approach relies on recent progress in the theory of branching laws of infinite-dimensional representations for symmetric pairs even beyond completely reducible cases (see Section 9 for instance).

Rankin–Cohen operators as intertwining operators. From the view point of representation theory the Rankin–Cohen operators are intertwiners in the branching law for the tensor product of two holomorphic discrete series representations π_{k_1} and π_{k_2} of $SL(2,\mathbb{R})$. More precisely, the discrete series representation $\pi_{k_1+k_2+2a}$ ($a \in \mathbb{N}$) occurs in the following branching law [Mo80, Re79]:

$$\pi_{k_1} \otimes \pi_{k_2} \simeq \sum_{a\in\mathbb{N}}^{\oplus} \pi_{k_1+k_2+2a}, \tag{1.2}$$

and the operator (1.1) gives an explicit intertwining operator from $\pi_{k_1} \otimes \pi_{k_2}$ to the irreducible summand $\pi_{k_1+k_2+2a}$.

In our work [KP14-1] we developed a new method to find explicit intertwining operators for irreducible components of branching laws in a broader setting of symmetric pairs. Such operators are unique up to scalars if the representation π is a highest weight module of scalar type (or equivalently π is realized in the space of holomorphic sections of a homogeneous holomorphic line bundle over a bounded symmetric domain) and (G, G') is any symmetric pair, by the multiplicity-free theorems ([K08, K12]).

The subject of this paper is to study concrete examples where the F-method turns out to be surprisingly efficient.

Let $\mathcal{V}_X \to X$ be a homogeneous vector bundle of a Lie group G and $\mathcal{W}_Y \to Y$ a homogeneous vector bundle of G'. Then we have a natural representation π of G on the space $\Gamma(X, \mathcal{V}_X)$ of sections on X, and similarly that of G' on $\Gamma(Y, \mathcal{W}_Y)$. Assume G' is a subgroup of G. We address the following question:

Question 1. *Find explicit G'-intertwining operators from $\Gamma(X, \mathcal{V}_X)$ to $\Gamma(Y, \mathcal{W}_Y)$.*

To illustrate the nature of such operators we also refer to them as *continuous symmetry breaking operators.* They are said to be *differential symmetry breaking operators* if the operators are differential operators.

The F-method proposed in [KP14-1] provides necessary tools to give an answer to Question 1 for all symmetric pairs (G, G') of split rank one inducing a holomorphic embedding $Y \hookrightarrow X$ (see Table 2.1). We remark that the split rank one condition does not force the rank of G/G' to be equal to one (see Table 1.1 (1), (5) below).

Normal derivatives and Jacobi–type differential operators. In representation theory, taking normal derivatives with respect to an equivariant embedding $Y \hookrightarrow X$ is a standard tool to find abstract branching laws for representations that are realized on X (see [JV79] for instance).

However, we should like to emphasize that the common belief "normal derivatives with respect to $Y \hookrightarrow X$ are intertwining operators in the branching laws" is not true. Actually, it already fails for the tensor product of two holomorphic discrete series of $SL(2, \mathbb{R})$ where the Rankin–Cohen brackets are not normal derivatives with respect to the diagonal embedding $Y \hookrightarrow Y \times Y$ with Y being the Poincaré upper half plane.

We discuss when normal derivatives become intertwiners in the following six complex geometries arising from real symmetric pairs of split rank one:

(1)	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathbb{P}^n\mathbb{C} \times \mathbb{P}^n\mathbb{C}$	(4)	$\mathrm{Gr}_{p-1}(\mathbb{C}^{p+q}) \hookrightarrow \mathrm{Gr}_p(\mathbb{C}^{p+q})$
(2)	$\mathrm{LGr}(\mathbb{C}^{2n-2}) \times \mathrm{LGr}(\mathbb{C}^2) \hookrightarrow \mathrm{LGr}(\mathbb{C}^{2n})$	(5)	$\mathbb{P}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{2n}\mathbb{C}$
(3)	$\mathrm{Q}^n\mathbb{C} \hookrightarrow \mathrm{Q}^{n+1}\mathbb{C}$	(6)	$\mathrm{IGr}_{n-1}(\mathbb{C}^{2n-2}) \hookrightarrow \mathrm{IGr}_n(\mathbb{C}^{2n})$

TABLE 1.1. Equivariant embeddings of flag varieties

Here $\mathrm{Gr}_p(\mathbb{C}^n)$ is the Grassmanian of p-planes in $\mathbb{C}^n$, $\mathrm{Q}^m\mathbb{C} := \{z \in \mathbb{P}^{m+1}\mathbb{C} : z_0^2 + \cdots + z_{m+1}^2 = 0\}$ is the complex quadric, and $\mathrm{IGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, \, Q|_V \equiv 0\}$

is the Grassmanian of isotropic subspaces of $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form Q, and $\mathrm{LGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, \omega|_{V\times V} \equiv 0\}$ is the Grassmanian of Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form ω.

For $Y \hookrightarrow X$ as in Table 1.1 and any equivariant line bundle $\mathcal{L}_\lambda \to X$ with sufficiently positive λ we give a necessary and sufficient condition for normal derivatives to become intertwiners:

Theorem A. (1) *Any continuous G'-homomorphism from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W})$ is given by normal derivatives with respect to the equivariant embedding $Y \hookrightarrow X$ if the embedding $Y \hookrightarrow X$ is of type (4), (5) or (6) in Table 1.1.*

(2) *None of normal derivatives of positive order is a G'-homomorphism if the embedding $Y \hookrightarrow X$ is of type (1), (2) and (3) in Table 1.1.*

See Theorem 5.3 for the precise formulation of the first statement. For the three geometries (1), (2), and (3) in Table 1.1, we construct explicitly all the continuous G'-homomorphisms which are actually holomorphic differential operators (differential symmetry breaking operators). For this, let $P_\ell^{\alpha,\beta}(x)$ be the Jacobi polynomial, and $\widetilde{C}_\ell^\alpha(x)$ the normalized Gegenbauer polynomial (see Appendix 11.3). We inflate them into polynomials of two variables by

$$P_\ell^{\alpha,\beta}(x,y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right) \quad \text{and} \quad \widetilde{C}_\ell^\alpha(x,y) := x^{\frac{\ell}{2}} \widetilde{C}_\ell^\alpha\left(\frac{y}{\sqrt{x}}\right).$$

In what follows, $\mathcal{L}_\lambda$ stands for a homogeneous holomorphic line bundle, and $\mathcal{W}_\lambda^a$ a homogeneous vector bundle with typical fiber $S^a(\mathbb{C}^m)$ ($m = n$ in (1); $= n-1$ in (2); m=1 in (3)) with parameter λ (see Lemma 5.5 for details). Then we prove:

Theorem B. (1) *For the symmetric pair $(U(n,1)\times U(n,1), U(n,1))$ the differential operator*

$$P_a^{\lambda'-1,\,-\lambda'-\lambda''-2a+1}\left(\sum_{i=1}^n v_i \frac{\partial}{\partial z_i}, \sum_{j=1}^n v_j \frac{\partial}{\partial z_j}\right)$$

is an intertwining operator from $\mathcal{O}(Y, \mathcal{L}_{(\lambda_1',\lambda_2')}) \widehat{\otimes} \mathcal{O}(Y, \mathcal{L}_{(\lambda_1'',\lambda_2'')})$ to $\mathcal{O}(Y, \mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')})$, for any $\lambda_1', \lambda_1'', \lambda_2', \lambda_2'' \in \mathbb{Z}$, and $a \in \mathbb{N}$. Here we set $\lambda' = \lambda_1' - \lambda_2'$ and $\lambda'' = \lambda_1'' - \lambda_2''$.

(2) *For the symmetric pair $(Sp(n,\mathbb{R}), Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R}))$ the differential operator*

$$C_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_i v_j \frac{\partial^2}{\partial z_{ij}\partial z_{nn}}, \sum_{1\le j\le n-1} v_j \frac{\partial}{\partial z_{jn}}\right)$$

is an intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_\lambda^a)$, for any $\lambda \in \mathbb{Z}$, and $a \in \mathbb{N}$.

(3) *For the symmetric pair $(SO(n,2), SO(n-1,2))$ the differential operator*

$$\widetilde{C}_a^{\lambda-\frac{n-1}{2}}\left(-\Delta^z_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right)$$

is an intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ *to* $\mathcal{O}(Y, \mathcal{L}_{\lambda+a})$, *for any* $\lambda \in \mathbb{Z}$ *and* $a \in \mathbb{N}$.

See Theorems 8.1, 7.1, and 6.3 for the precise statements, respectively. By the localness theorem [KP14-1, Theorem 5.3], any continuous G'-homomorphisms are differential operators. Then we prove that the above operators exhaust all continuous symmetry breaking operators in (2) and (3), and for generic parameter (λ', λ'') in (1), see (8.7) for the exact condition on the parameter. The first statement of Theorem B corresponds to the decomposition of the tensor product, and gives rise to the classical Rankin–Cohen brackets in the case where $n = 1$. An analogous formula for Theorem B (3) was recently found in a completely different way by A. Juhl [J09] in the setting of conformally equivariant differential operators with respect to the embedding of Riemannian manifolds $S^{n-1} \hookrightarrow S^n$.

The proof of Theorem B is built on the F-method, which establishes in the present setting a bijection between the space

$$\mathrm{Hom}_{G'}(\mathcal{O}(X, \mathcal{L}_\lambda), \mathcal{O}(Y, \mathcal{W}_\lambda^a))$$

of symmetry breaking operators and the space of polynomial solutions to a certain ordinary differential equation, namely

$$\begin{aligned} &\mathrm{Sol}_{\mathrm{Jacobi}}(\lambda' - 1, -\lambda' - \lambda'' - 2a + 1, a) \cap \mathrm{Pol}_a[s] \\ &\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - 1, a) \cap \mathrm{Pol}_a[s]_{\mathrm{even}} \\ &\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - \frac{n-1}{2}, a) \cap \mathrm{Pol}_a[s]_{\mathrm{even}}, \end{aligned}$$

for the geometries (1), (2), and (3) in Table 1.1, respectively. Here $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_a[s]$ and $\mathrm{Sol}_{\mathrm{Gegen}} \cap \mathrm{Pol}_a[s]$ denote the space of polynomial solutions of degree at most a to the Jacobi differential equation (11.4) and to the Gegenbauer differential equation (11.14), respectively. (The subscript "even" stands for a parity condition (6.12).)

Surprisingly, the dimension of the space of symmetry breaking operators for the tensor product case (1) jumps up at some singular parameters. We illustrate this phenomenon by the the following result in the $\mathfrak{sl}_2$-case:

Theorem C (Theorem 9.1)**.** *The following three conditions on the parameters* $(\lambda', \lambda'', \lambda''') \in \mathbb{Z}^3$ *are equivalent:*

(i) $\dim_{\mathbb{C}} \mathrm{Hom}_{SL(2,\mathbb{R})}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) = 2$.

(ii) $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'''), \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda') \otimes \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'')) = 2$, *where* $\mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)$ *is the Verma module* $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{-\lambda}$ *of* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

(iii) $\lambda', \lambda'' \le 0, 2 \le \lambda''', \lambda' + \lambda'' \equiv \lambda'''$ *mod* 2, $-(\lambda' + \lambda'') \ge \lambda''' - 2 \ge |\lambda' - \lambda''|$.

We also prove that the analytic continuations of the Rankin–Cohen bidifferential operators $\mathcal{RC}_{\lambda', \lambda''}^{\lambda'''}$ vanish exactly at these singular parameters $(\lambda', \lambda'', \lambda''')$ in this case. Moreover, we construct explicitly *three* symmetry breaking operators in this case, and prove that any two of the three are linearly independent. Furthermore we show that each of these three symmetry breaking operators factors into two natural intertwining operators as follows:

$$\begin{tikzcd}
& \mathcal{O}(\mathcal{L}_{2-\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}) \arrow[dr, "\mathcal{RC}^{\lambda'''}_{2-\lambda',\lambda''}"] & \\
\mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{\lambda''}) \arrow[ur, "\left(\frac{\partial}{\partial z_1}\right)^{1-\lambda'}\otimes \mathrm{id}"] \arrow[r, "\mathrm{id}\otimes\left(\frac{\partial}{\partial z_2}\right)^{1-\lambda''}"] \arrow[dr, "\mathcal{RC}^{2-\lambda'''}_{\lambda',\lambda''}"'] & \mathcal{O}(\mathcal{L}_{\lambda'})\widehat{\otimes}\mathcal{O}(\mathcal{L}_{2-\lambda''}) \arrow[r, "\mathcal{RC}^{\lambda'''}_{\lambda',2-\lambda''}"] & \mathcal{O}(\mathcal{L}_{\lambda'''}), \\
& \mathcal{O}(\mathcal{L}_{2-\lambda'''}) \arrow[ur, "\left(\frac{d}{dz}\right)^{\lambda'''-1}"'] &
\end{tikzcd}$$

whereas the linear relation among the three is explicitly given by using Kummer's connection formula for Gauss hypergeometric functions via the F-method.

In Section 10 we briefly discuss some new applications of the explicit formulæ of differential symmetry breaking operators. Namely, we describe an explicit construction of the discrete spectrum of complementary series representations of $O(n+1,1)$ when restricted to $O(n,1)$ by means of the differential operator given in Theorem B (3).

In Appendix (Section 11) we collect some results on classical ordinary differential equations with focus on singular parameters for which there exist two linearly independent polynomial solutions which correspond, via the F-method, to the failure of multiplicity-one results in the branching laws.

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Notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

2. Geometric setting: Hermitian symmetric spaces

In this section we describe the geometric setting in which Question 1 will be answered.

2.1. **Complex submanifolds in Hermitian symmetric spaces.** Let G be a connected real reductive Lie group, θ a Cartan involution, and G/K the associated Riemannian symmetric space. We write $\mathfrak{c}(\mathfrak{k})$ for the center of the complexified Lie algebra $\mathfrak{k} := \mathrm{Lie}(K) \otimes_{\mathbb{R}} \mathbb{C} \equiv \mathfrak{k}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. We suppose that G/K is a Hermitian symmetric space. This means that there exists a characteristic element $H_o \in \mathfrak{c}(\mathfrak{k})$ such that we have an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{n}_+ + \mathfrak{n}_-$$

of $\mathrm{ad}(H_o)$ with eigenvalues 0, 1, and -1, respectively. We note that $\mathfrak{c}(\mathfrak{k})$ is one-dimensional if G is simple.

Let $G_{\mathbb{C}}$ be a complex reductive Lie group with Lie algebra $\mathfrak{g}$, and $P_{\mathbb{C}}$ the maximal parabolic subgroup having Lie algebra $\mathfrak{p} := \mathfrak{k} + \mathfrak{n}_+$ with abelian nilradical $\mathfrak{n}_+$. The complex structure of the homogeneous space G/K is induced from the Borel embedding

$$G/K \subset G_{\mathbb{C}}/K_{\mathbb{C}} \exp \mathfrak{n}_+ = G_{\mathbb{C}}/P_{\mathbb{C}}.$$

Let G' be a θ-stable, connected reductive subgroup of G. We set $K' := K \cap G'$ and assume

$$H_o \in \mathfrak{k}'. \tag{2.1}$$

Then the homogeneous space G'/K' carries a G'-invariant complex structure such that the embedding $G'/K' \hookrightarrow G/K$ is holomorphic by the following diagram:

$$\begin{tikzcd} Y = G'/K' \arrow[r, hook] \arrow[d, phantom, "\bigcap" description, "\text{open}"'] & G/K = X \arrow[d, phantom, "\bigcap" description, "\text{open}"] \\ G'_{\mathbb{C}}/P'_{\mathbb{C}} \arrow[r, hook] & G_{\mathbb{C}}/P_{\mathbb{C}}, \end{tikzcd} \tag{2.2}$$

where $G'_{\mathbb{C}}$ and $P'_{\mathbb{C}} = K'_{\mathbb{C}} \exp \mathfrak{n}'_+$ are the connected complex subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}' := \mathrm{Lie}(G') \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathfrak{p}' := \mathfrak{k}' + \mathfrak{n}'_+ \equiv (\mathfrak{k} \cap \mathfrak{g}') + (\mathfrak{n}_+ \cap \mathfrak{g}')$, respectively.

Given a finite-dimensional representation of K on a complex vector space V, we extend it to a holomorphic representation of $P_{\mathbb{C}}$ by letting the unipotent subgroup $\exp(\mathfrak{n}_+)$ act trivially, and form a holomorphic vector bundle $\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}} = G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$ over $G_{\mathbb{C}}/P_{\mathbb{C}}$. The restriction to the open set G/K defines a G-equivariant holomorphic vector bundle $\mathcal{V} := G \times_K V$. We then have a natural representation of G on the vector space $\mathcal{O}(G/K, \mathcal{V})$ of global holomorphic sections.

Likewise, given a finite-dimensional representation W of K', we form the G'-equivariant holomorphic vector bundle $\mathcal{W} = G' \times_{K'} W$ and consider the representation of G' on $\mathcal{O}(G'/K', \mathcal{W})$.

Let $V^\vee$ and $W^\vee$ be the contragredient representations of V and W, respectively, and we define $\mathfrak{g}$- and $\mathfrak{g}'$-modules (*generalized Verma modules*) by

$$\begin{aligned} \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) &:= U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee, \\ \mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee) &:= U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} W^\vee, \end{aligned}$$

where $U(\mathfrak{g})$ and $U(\mathfrak{g}')$ denote the universal enveloping algebras of the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, respectively. We endow the spaces $\mathcal{O}(G/K, \mathcal{V})$ and $\mathcal{O}(G'/K', \mathcal{W})$

with the Fréchet topology of uniform convergence on compact sets, and denote by $\mathrm{Hom}_{G'}(\,\cdot\,,\,\cdot\,)$ the space of continuous symmetry breaking operators (*i.e.* continuous G'-homomorphisms), and by $\mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}}, \mathcal{W}_{G'_{\mathbb{C}}/P'_{\mathbb{C}}})$ the space of $G'_{\mathbb{C}}$-equivariant holomorphic differential operators with respect to the holomorphic map $G'_{\mathbb{C}}/P'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/P_{\mathbb{C}}$ (see [KP14-1, Definition 2.1] for the definition of differential operators between vector bundles with different base spaces). Then the localness theorem [KP14-1, Theorem 5.3] and the duality theorem (*op. cit.*, Theorem 2.12) assert:

Theorem 2.1. *We have the following natural isomorphisms:*

$$\begin{aligned} \mathrm{Hom}_{G'}(\mathcal{O}(G/K,\mathcal{V}), \mathcal{O}(G'/K',\mathcal{W})) &\simeq \mathrm{Diff}^{\mathrm{hol}}_{G'_{\mathbb{C}}}(\mathcal{V}_{G_{\mathbb{C}}/P_{\mathbb{C}}}, \mathcal{W}_{G'_{\mathbb{C}}/P'_{\mathbb{C}}}) \\ &\simeq \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}^{\mathfrak{g}'}_{\mathfrak{p}'}(W^{\vee}), \mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(V^{\vee})). \end{aligned}$$

2.2. **Semisimple symmetric pairs of holomorphic type and split rank.** Let τ be an involutive automorphism of a semisimple Lie group G. Without loss of generality we may and do assume that τ commutes with the Cartan involution θ of G. We define a θ-stable subgroup by

$$G^{\tau} := \{g \in G : \tau g = g\}.$$

Then the homogeneous space G/G^{τ} carries a G-invariant pseudo-Riemannian structure g induced from the Killing form of $\mathfrak{g}(\mathbb{R}) = \mathrm{Lie}(G)$, and becomes an affine symmetric space with respect to the Levi-Civita connection. We use the same letters τ and θ to denote the differentials and also their complex linear extensions. We set $\mathfrak{g}(\mathbb{R})^{\tau} := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau Y = Y\}$, the Lie algebra of G^{τ}. The pair $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^{\tau})$ is said to be a semisimple symmetric pair. It is *irreducible* if $\mathfrak{g}(\mathbb{R})$ is simple or is a direct sum of two copies of a simple Lie algebra $\mathfrak{g}'(\mathbb{R})$ with $\mathfrak{g}(\mathbb{R})^{\tau} \simeq \mathfrak{g}'(\mathbb{R})$. Then any semisimple symmetric pair is isomorphic to a direct sum of irreducible ones.

Definition 2.2. Geometrically, the *split rank* of the semisimple symmetric space G/G^{τ} is the dimension of a maximal flat, totally geodesic submanifold B in G/G^{τ} such that the restriction of g to B is positive definite. Algebraically, it is the dimension of a maximal abelian subspace of $\mathfrak{g}(\mathbb{R})^{-\tau,-\theta} := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau Y = \theta Y = -Y\}$. The dimension is independent of the choice of the data, and the geometric and algebraic definitions coincide. We denote it by $\mathrm{rank}_{\mathbb{R}}\, G/G^{\tau}$.

The automorphism $\tau\theta$ is also an involution because $\tau\theta = \theta\tau$. Since

$$\mathfrak{g}(\mathbb{R})^{\tau\theta,-\theta} := \{Y \in \mathfrak{g}(\mathbb{R}) : \tau\theta Y = Y, \theta Y = -Y\}$$

coincides with $\mathfrak{g}(\mathbb{R})^{-\tau,-\theta}$, we have $\mathrm{rank}_{\mathbb{R}} G/G^{\tau} = \mathrm{rank}_{\mathbb{R}} G^{\tau\theta}$, the split rank of the reductive Lie group $G^{\tau\theta}$.

Suppose now that G/K is a Hermitian symmetric space with a characteristic element H_o as in Section 2.1.

	$\mathfrak{g}(\mathbb{R})$	$\mathfrak{g}(\mathbb{R})^{\tau}$	$\mathfrak{g}(\mathbb{R})^{\tau\theta}$
1	$\mathfrak{su}(n,1)\oplus\mathfrak{su}(n,1)$	$\mathfrak{su}(n,1)$	$\mathfrak{su}(n,1)$
2	$\mathfrak{sp}(n+1,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})\oplus\mathfrak{sp}(1,\mathbb{R})$	$\mathfrak{u}(1,n)$
3	$\mathfrak{so}(n,2)$	$\mathfrak{so}(n-1,2)$	$\mathfrak{so}(n-1)\oplus\mathfrak{so}(1,2)$
4	$\mathfrak{su}(p,q)$	$\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(p-1,q))$	$\mathfrak{s}(\mathfrak{u}(1,q)\oplus\mathfrak{u}(p-1))$
5	$\mathfrak{so}(2,2n)$	$\mathfrak{u}(1,n)$	$\mathfrak{u}(1,n)$
6	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(2)\oplus\mathfrak{so}^*(2n-2)$	$\mathfrak{u}(1,n-1)$

TABLE 2.1. Split rank one irreducible symmetric pairs of holomorphic type

Definition 2.3. An irreducible symmetric pair $(\mathfrak{g}(\mathbb{R}),\mathfrak{g}(\mathbb{R})^{\tau})$ (or (G,G^{τ})) is said to be of *holomorphic type* (with respect to the complex structure on G/K defined by the characteristic element H_o) if $\tau(H_o)=H_o$, namely $H_o\in\mathfrak{k}^{\tau}$.

If (G,G^{τ}) is of holomorphic type, then G^{τ}/K^{τ} carries a G^{τ}-invariant complex structure such that the embedding $G^{\tau}/K^{\tau}\hookrightarrow G/K$ is holomorphic.

Among irreducible symmetric pairs $(\mathfrak{g}(\mathbb{R}),\mathfrak{g}(\mathbb{R})^{\tau})$ of holomorphic type Table 2.1 gives the infinitesimal classification of those of split rank one.

The pairs of flag varieties (see (2.2)) associated with the six pairs (G,G^{τ}) in Table 2.1 correspond to the six complex parabolic geometries given in Table 1.1.

3. F-METHOD IN HOLOMORPHIC SETTING

In this section we reformulate the recipe of the F-method ([KP14-1, Section 4]) in the holomorphic setting, that is, in the setting of Section 2.1 where G'/K' is a complex submanifold of the Hermitian symmetric space G/K.

3.1. F-method for Hermitian symmetric spaces.

The algebraic Fourier transform on a vector space E is an isomorphism of the Weyl algebras of holomorphic differential operators with polynomial coefficients on a complex vector spaces E and its dual space $E^{\vee}$:

$$\mathcal{D}(E)\to\mathcal{D}(E^{\vee}),\qquad T\mapsto\widehat{T}$$

induced by

$$\widehat{\frac{\partial}{\partial z_j}} := -\zeta_j,\quad \widehat{z_j} := \frac{\partial}{\partial\zeta_j},\quad 1\le j\le n=\dim E, \tag{3.1}$$

where $(z_1,\dots,z_n)$ are coordinates on E and $(\zeta_1,\dots,\zeta_n)$ are the dual coordinates on $E^{\vee}$.

Let $G_{\mathbb{C}}$ be a connected complex reductive Lie group with Lie algebra $\mathfrak{g}$ and $P_{\mathbb{C}}=K_{\mathbb{C}}N_{+,\mathbb{C}}$ be a parabolic subgroup with Lie algebra $\mathfrak{p}=\mathfrak{k}+\mathfrak{n}_{+}$. Let λ be a holomorphic representation of $K_{\mathbb{C}}$ on V. We extend it to $P_{\mathbb{C}}$ by letting $N_{+,\mathbb{C}} = \exp(\mathfrak{n}_+)$ act trivially, and form a $G_{\mathbb{C}}$-equivariant holomorphic vector bundle $\mathcal{V} = G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$ over $G_{\mathbb{C}}/P_{\mathbb{C}}$. Let $\mathbb{C}_{2\rho}$ be the holomorphic character defined by $p \mapsto \det(\mathrm{Ad}(p) : \mathfrak{p} \to \mathfrak{p})$, and define a twist of the contragredient representation $(\lambda^\vee, V^\vee)$ of $P_{\mathbb{C}}$ by $\lambda^* := \lambda^\vee \otimes \mathbb{C}_{2\rho}$. We set $\mathcal{V}^* \equiv \mathcal{V}^\vee_{2\rho} := G_{\mathbb{C}} \times_{P_{\mathbb{C}}} (V^\vee \otimes \mathbb{C}_{2\rho})$, which is isomorphic to the tensor bundle of the dual bundle $\mathcal{V}^\vee$ and the canonical line bundle of $G_{\mathbb{C}}/P_{\mathbb{C}}$. We shall apply the algebraic Fourier transform to the infinitesimal representation $d\pi_{\lambda^*}$ of $\mathfrak{g}$ on $\mathcal{O}(G_{\mathbb{C}}/P_{\mathbb{C}}, \mathcal{V}^*)$ as follows.

We recall that the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+$ induces a diffeomorphism

$$\mathfrak{n}_- \times K_{\mathbb{C}} \times \mathfrak{n}_+ \to G_{\mathbb{C}}, \quad (X, \ell, Y) \mapsto (\exp X)\ell(\exp Y),$$

into an open dense subset, denoted by $G_{\mathbb{C}}^{\mathrm{reg}}$, of $G_{\mathbb{C}}$. Let $p_\pm : G_{\mathbb{C}}^{\mathrm{reg}} \to \mathfrak{n}_\pm, p_o : G_{\mathbb{C}}^{\mathrm{reg}} \to K_{\mathbb{C}}$, be the projections characterized by the identity

$$\exp(p_-(g))p_o(g)\exp(p_+(g)) = g.$$

Furthermore, we introduce the following maps:

$$\alpha : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{k}, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_o\left(e^{tY} e^Z\right), \tag{3.2}$$

$$\beta : \mathfrak{g} \times \mathfrak{n}_- \to \mathfrak{n}_-, \qquad (Y, Z) \mapsto \left.\frac{d}{dt}\right|_{t=0} p_-\left(e^{tY} e^Z\right). \tag{3.3}$$

For $F \in \mathcal{O}(\mathfrak{n}_-, V^\vee) \simeq \mathcal{O}(\mathfrak{n}_-) \otimes V^\vee$, we set $f : G_{\mathbb{C}}^{\mathrm{reg}} \longrightarrow V^\vee$ by $f(\exp Zp) = \lambda^*(p)^{-1}F(Z)$ for $Z \in \mathfrak{n}_-$ and $p \in P_{\mathbb{C}}$. Then the infinitesimal action of $\mathfrak{g}$ on $\mathcal{O}(\mathfrak{n}_-, V^\vee)$ is given by

$$\begin{aligned} (d\pi_{\lambda^*}(Y)F)(Z) &= \left.\frac{d}{dt}\right|_{t=0} f(e^{-tY}e^Z) \\ &= \lambda^*(\alpha(Y, Z))F(Z) - (\beta(Y, \cdot)F)(Z) \quad \text{for } Y \in \mathfrak{g}, \end{aligned} \tag{3.4}$$

where we use the same letter λ^* to denote the infinitesimal action of $\mathfrak{p}$ on $V^\vee$. This action yields a Lie algebra homomorphism

$$d\pi_{\lambda^*} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee). \tag{3.5}$$

In turn, we get another Lie algebra homomorphism by the algebraic Fourier transform on the Weyl algebra $\mathcal{D}(\mathfrak{n}_-)$:

$$\widehat{d\pi_{\lambda^*}} : \mathfrak{g} \to \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee), \tag{3.6}$$

where we identify $\mathfrak{n}_-^\vee$ with $\mathfrak{n}_+$ by a $\mathfrak{g}$-invariant non-degenerate bilinear form on $\mathfrak{g}$ (*e.g.* the Killing form).

Theorem 3.1 (F-method for Hermitian symmetric spaces)**.** *Suppose we are in the setting of Section 2.1.*

(1) *We have the following commutative diagram of three isomorphisms:*

(3.7)

$$\begin{tikzcd}
& \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} & \\
\mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \arrow[ur, "F_c\otimes\mathrm{id}"] \arrow[rr, "\sim", "D_{X\to Y}"'] & & \mathrm{Hom}_{G'}(\mathcal{O}(X,\mathcal{V}), \mathcal{O}(Y,\mathcal{W})). \arrow[ul, "\mathrm{Symb}\otimes\mathrm{id}"']
\end{tikzcd}$$

(2) *Let $\mathfrak{b}(\mathfrak{k}')$ be a Borel subalgebra of $\mathfrak{k}'$, and assume that W is the irreducible representation of K' with lowest weight $-\chi$. Then we have the following isomorphism:*

$$\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} \xrightarrow{\sim} \{P \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee : P \,\text{satisfies}\,(3.8)\,\text{and}\,(3.9)\}$$

$$ZP = \chi(Z)P, \qquad \text{for all } Z \in \mathfrak{b}(\mathfrak{k}'). \tag{3.8}$$

$$\widehat{d\pi_{\lambda^*}}(C)P = 0, \qquad \text{for all } C \in \mathfrak{n}'_+. \tag{3.9}$$

Proof. 1) The first statement follows from Theorem 2.1 and [KP14-1, Corollary 4.3].

2) Via the linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$, we have an isomorphism

$$\begin{aligned} & \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)} \\ \simeq\ & \{\psi \in \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W) : \psi \text{ satisfies } (3.10) \text{ and } (3.11)\}, \end{aligned}$$

$$\nu(\ell) \circ \mathrm{Ad}_\sharp(\ell)\psi \circ \lambda(\ell^{-1}) = \psi \qquad \text{for all}\, \ell \in K', \tag{3.10}$$

$$(\widehat{d\pi_{\lambda^*}}(C) \otimes \mathrm{id}_W)\psi = 0 \qquad \text{for all}\, C \in \mathfrak{n}'_+, \tag{3.11}$$

where $\mathrm{Ad}_\sharp(\ell) : \mathrm{Pol}(\mathfrak{n}_+) \to \mathrm{Pol}(\mathfrak{n}_+), \varphi \mapsto \varphi \circ \mathrm{Ad}(\ell)^{-1}$.

On the other hand, if $-\chi$ is the lowest weight of the irreducible representation W of K', we have an isomorphism

$$\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi, \tag{3.12}$$

where

$$(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi := \{P \in \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee : P \text{ satisfies } (3.8)\}.$$

Therefore, Theorem 3.1 (2) is deduced from Theorem 3.1 (1) and from the following natural isomorphism:

$$\{\psi \,\text{satisfying}\,(3.10)\,\text{and}\,(3.11)\} \xrightarrow{\sim} \{P \,\text{satisfying}\,(3.8)\,\text{and}\,(3.9)\}.$$

See also [*op. cit.*, Lemma 4.6]. ☐

The F-method (see [*op.cit.*, Section 4.4]) in this setting consists of the following five steps:

Step 0. Fix a finite-dimensional representation (λ, V) of the maximal compact subgroup K. Form a G-equivariant holomorphic vector bundle $\mathcal{V}_X \equiv \mathcal{V} = G \times_K V$ on $X = G/K$.

Step 1. Extend λ to a representation of the Lie algebra $\mathfrak{p} = \mathfrak{k} + \mathfrak{n}_+$ by letting $\mathfrak{n}_+$ act trivially, and define another representation $\lambda^* := \lambda^\vee \otimes \mathbb{C}_{2\rho}$ of $\mathfrak{p}$ on $V^\vee$. Compute $d\pi_{\lambda^*}$ and $\widehat{d\pi_{\lambda^*}}$.

Step 2. Find a finite-dimensional representation (ν, W) of the Lie group K' such that

$$\operatorname{Hom}_{\mathfrak{g}'}(\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\},$$

or equivalently,

$$\operatorname{Hom}_{\mathfrak{k}'}(W^\vee, \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \neq \{0\}.$$

Form a G'-equivariant holomorphic vector bundle $\mathcal{W}_Y \equiv \mathcal{W} = G' \times_{K'} W$ on $Y = G'/K'$. According to the duality theorem [KP14-1, Theorem 2.12] the space of differential symmetry breaking operators $\operatorname{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ is then non-trivial.

Step 3. Write down the condition on $\operatorname{Hom}_{K'}(V, \operatorname{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}$, namely, the space of $\psi \in \operatorname{Pol}(\mathfrak{n}_+) \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$ satisfying (3.10) and (3.11) or equivalently $P \in \operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee$ satisfying (3.8) and (3.9).

Step 4. Use the invariant theory and give a simple description of

$$\operatorname{Hom}_{K'}(V, \operatorname{Pol}(\mathfrak{n}_+) \otimes W) \simeq \left(\operatorname{Pol}(\mathfrak{n}_+) \otimes V^\vee\right)_\chi, \quad \psi \leftrightarrow P$$

by means of "regular functions $g(s)$ on a slice" S for generic $K'_{\mathbb{C}}$-orbits on $\mathfrak{n}_+$. Induce differential equations for $g(s)$ on S from (3.11) (or equivalently (3.9)). Concrete computations are based on the technique of the T-saturation of differential operators, see Section 3.2. Solve the differential equations of $g(s)$.

Step 5. Transfer a solution g obtained in Step 4 into a polynomial solution ψ to (3.10) and (3.11). In the diagram (3.7), $(\operatorname{Symb} \otimes \operatorname{id})^{-1}(\psi)$ gives the desired differential symmetry breaking operator in the coordinates $\mathfrak{n}_-$ of X. As a byproduct, obtain an explicit K'-type $W^\vee$ annihilated by $\mathfrak{n}'_+$ in $\operatorname{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(V^\vee)$ (sometimes referred to as *singular vectors*) as $(F_c \otimes \operatorname{id})^{-1}(\psi)$.

We shall give some comments on Steps 3 and 4 in Sections 3.3 and 3.2 respectively. For Step 2, there are two approaches: one is to use (abstract) branching laws for the restriction of $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ to the subalgebra $\mathfrak{g}'$ (*e.g.* Fact 4.2) or the restriction of $\mathcal{O}(G/K, \mathcal{V})$ to the subgroup G' (*e.g.* Fact 4.3). The other one is to apply the F-method and reduce it to a question of solving differential equations of second order. The former approach works well for generic parameters. We shall see that the latter approach is efficient for singular parameters in our setting (Theorems 6.1, 7.1 and 8.1, see also [KØSS13]).

3.2. **T-saturation of differential operators.** In order to implement Step 4, our idea is to introduce saturated differential operators as follows. For simplicity consider the case when $\dim_{\mathbb{C}} V = 1$. Then $\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)$ is identified with a subspace of $\mathrm{Pol}(\mathfrak{n}_+)$ via the isomorphism (3.12). Let $\mathbb{C}(\mathfrak{n}_+)$ denote the field of rational functions on $\mathfrak{n}_+$. Suppose that we have a morphism $T : \mathbb{C}[S] \longrightarrow \mathbb{C}(\mathfrak{n}_+)$ such that T induces an isomorphism

$$T : \Gamma(S) \overset{\sim}{\to} \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)$$

for some algebraic variety S ("slice" of a generic $K'_{\mathbb{C}}$-orbit on $\mathfrak{n}_+$), and for some appropriate function space $\Gamma(S)$ (*e.g.* $\Gamma(S) = \mathrm{Pol}_a[t]_{\mathrm{even}}$, see (6.12)). In the special case where V and W are the trivial one-dimensional representations of K and K', respectively, we may take $S = \mathfrak{n}_+ // K'_{\mathbb{C}}$ (geometric quotient) and T is the natural morphism $\mathbb{C}[S] \overset{\sim}{\longrightarrow} \mathbb{C}[\mathfrak{n}_+]^{K'_{\mathbb{C}}}$.

Definition 3.2. A differential operator R on $\mathfrak{n}_+$ with rational coefficients is T-*saturated* if there exists an operator D such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}[S] \arrow[r, "T"] \arrow[d, "D"'] & \mathbb{C}(\mathfrak{n}_+) \arrow[d, "R"] \\
\mathbb{C}[S] \arrow[r, "T"] & \mathbb{C}(\mathfrak{n}_+).
\end{tikzcd}$$

Such an operator D is unique (if exists), and we denote it by $T^\sharp R$. Then we have

$$T^\sharp(R_1 \cdot R_2) = T^\sharp(R_1) T^\sharp(R_2) \tag{3.13}$$

whenever it makes sense.

Proposition 3.3. *Let* $C_1, \cdots, C_k$ *be a basis of* $\mathfrak{n}'_+$. *Suppose there exist non-zero* $Q_j \in \mathbb{C}(\mathfrak{n}_+)$ *such that* $Q_j \widehat{d\pi_{\lambda^*}}(C_j)$ *is* T*-saturated* $(1 \le j \le k)$ *and set* $D_j := T^\sharp(Q_j \widehat{d\pi_{\lambda^*}}(C_j))$. *Then* T *induces a bijection*

$$\begin{aligned}
& \{g \in \Gamma[S] : D_j g = 0,\ (1 \le j \le k)\} \\
\overset{\sim}{\to}\ & \{\psi \in \mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) : \psi \,\text{satisfies}\, (3.10) \,\text{and}\, (3.11)\} \\
\simeq\ & \left\{P \in \left(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee\right)_\chi : P \,\text{satisfies}\, (3.9)\right\}.
\end{aligned}$$

We shall use this idea in Sections 6-8 where S is one-dimensional and D_j are ordinary differential operators. We note that $D_j g = 0$ $(1 \le \forall j \le k)$ is equivalent to a single equation $D_i g = 0$ if K' acts irreducibly on $\mathfrak{n}'_+$.

3.3. **Complement for the F-method in vector-valued cases and highest weight varieties.** If the target $\mathcal{W}_Y$ is no longer a line bundle but a vector bundle, *i.e.*, if W is an arbitrary finite-dimensional, irreducible $\mathfrak{k}'$-module, we recall two supplementary ingredients of Step 3 in the recipe by reducing (3.10) to a simpler algebraic question on polynomial rings, so that we can focus on the crucial part consisting of a system of differential equations of second order (3.11). This construction is based on the notion of *highest weight variety* of the fiber W and is summarized in the following two lemmas (see [KP14-1, Lemmas 4.6 and 4.7].

We fix a Borel subalgebra $\mathfrak{b}(\mathfrak{k}')$ of $\mathfrak{k}'$. Let $\chi : \mathfrak{b}(\mathfrak{k}') \to \mathbb{C}$ be a character. For a $\mathfrak{k}'$-module U, we set

$$U_\chi := \{u \in U \,:\, Zu = \chi(Z)u \quad \text{for any } Z \in \mathfrak{b}(\mathfrak{k}')\}.$$

Suppose that W is the irreducible representation of $\mathfrak{k}'$ with lowest weight $-\chi$. Then the contragredient representation $W^\vee$ has a highest weight χ. We fix a non-zero highest weight vector $w^\vee \in (W^\vee)_\chi$. Then the contraction map

$$U \otimes W \to U, \quad \psi \mapsto \langle \psi, w^\vee \rangle,$$

induces a bijection between the following two subspaces:

$$(U \otimes W)^{\mathfrak{k}'} \xrightarrow{\sim} U_\chi, \tag{3.14}$$

if U is completely reducible as a $\mathfrak{k}'$-module. By using the isomorphism (3.14), we reformulate Step 3 of the recipe for the F-method as follows:

Lemma 3.4. *Assume that W is an irreducible representation of the parabolic subalgebra $\mathfrak{p}'$. Let $-\chi$ be the lowest weight of W as a $\mathfrak{k}'$-module. Then we have a natural injective homomorphism*

$$\mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \hookrightarrow \left\{Q \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi : \widehat{d\pi_\mu}(C)Q = 0 \quad \text{for all } C \in \mathfrak{n}'_+\right\},$$

which is bijective if K' is connected.

See [KP14-1, Lemma 4.6] for the proof.

Since any non-zero vector in $W^\vee$ is cyclic, the next lemma explains how to recover $D_{X\to Y}(\varphi)$ from Q given in Lemma 3.4.

We assume, for simplicity, that the $\mathfrak{k}$-module (λ, V) lifts to $K_\mathbb{C}$, the $\mathfrak{k}'$-module (ν, W) lifts to $K'_\mathbb{C}$, and use the same letters to denote their liftings.

Lemma 3.5. *For any $\varphi \in \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_\mathfrak{p}^\mathfrak{g}(V^\vee))$, $k \in K'_\mathbb{C}$ and $w^\vee \in W^\vee$,*

$$\langle D_{X\to Y}(\varphi), \nu^\vee(k)w^\vee\rangle = (\mathrm{Ad}(k) \otimes \lambda^\vee(k))\,\langle D_{X\to Y}(\varphi), w^\vee\rangle\,. \tag{3.15}$$

See [KP14-1, Lemma 4.7] for the proof.

4. Branching laws and Hermitian symmetric spaces

The existence, respectively the uniqueness (up to scaling) of differential symmetry breaking operators from $\mathcal{V}_X$ to $\mathcal{W}_Y$ are subject to the conditions

$$\dim \mathrm{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \geq 1, \text{ respectively } \leq 1. \tag{4.1}$$

So we need to find the geometric settings (i.e. the pair $Y \subset X$ of generalized flag varieties and two homogeneous vector bundles $\mathcal{V}_X \to X$ and $\mathcal{W}_Y \to Y$) satisfying (4.1). This is the main ingredient of Step 2 in the recipe of the F-method, and thanks to [KP14-1, Theorem 2.9], the existence and uniqueness are equivalent to the following question concerning (abstract) branching laws: Given a $\mathfrak{p}$-module V, find all finite-dimensional $\mathfrak{p}'$-modules W such that $\dim \mathrm{Hom}_{\mathfrak{p}'}(W^\vee, \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) = 1$, and equivalently,

$$\dim \mathrm{Hom}_{\mathfrak{g}'}(\mathrm{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) = 1. \tag{4.2}$$

This section briefly reviews what is known on this question (see Fact 4.2).

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $\mathfrak{j}$ a Cartan subalgebra of $\mathfrak{g}$. We fix a positive root system $\Delta^+ \equiv \Delta^+(\mathfrak{g}, \mathfrak{j})$, write ρ for half the sum of positive roots, $\alpha^\vee$ for the coroot for $\alpha \in \Delta$, and $\mathfrak{g}_\alpha$ for the root space. Define a Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ with nilradical $\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$.

The BGG category $\mathcal{O}$ is defined as the full subcategory of $\mathfrak{g}$-modules whose objects are finitely generated, $\mathfrak{j}$-semisimple and locally $\mathfrak{n}$-finite [BGG76].

As in the previous sections, fix a standard parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ such that the Levi factor $\mathfrak{l}$ contains $\mathfrak{j}$. We set $\Delta^+(\mathfrak{l}) := \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{j})$. The parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ is defined as the full subcategory of $\mathcal{O}$ whose objects are locally $\mathfrak{l}$-finite.

We define

$$\Lambda^+(\mathfrak{l}) := \{\lambda \in \mathfrak{j}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \text{ for any } \alpha \in \Delta^+(\mathfrak{l})\},$$

the set of linear forms λ on $\mathfrak{j}$ whose restrictions to $\mathfrak{j} \cap [\mathfrak{l}, \mathfrak{l}]$ are dominant integral. We write V_λ for the finite-dimensional simple $\mathfrak{l}$-module with highest weight λ, regard it as a $\mathfrak{p}$-module by letting $\mathfrak{n}_+$ act trivially, and consider the generalized Verma module

$$\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda.$$

Then $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$ and any simple object in $\mathcal{O}^{\mathfrak{p}}$ is the quotient of some generalized Verma module. If

$$\langle \lambda, \alpha^\vee \rangle = 0 \qquad \text{for all} \qquad \alpha \in \Delta(\mathfrak{l}), \tag{4.3}$$

then V_λ is one-dimensional, to be denoted also by $\mathbb{C}_\lambda$. In this case we say $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of *scalar type*.

Let $\tau \in \mathrm{Aut}(\mathfrak{g})$ be an involutive automorphism of the Lie algebra $\mathfrak{g}$. We write

$$\mathfrak{g}^{\pm\tau} := \{v \in \mathfrak{g} : \tau v = \pm v\}$$

for the ±1 eigenspaces of τ, respectively. We say that $(\mathfrak{g}, \mathfrak{g}')$ is a symmetric pair if $\mathfrak{g}' = \mathfrak{g}^\tau$ for some τ.

For a general choice of τ and $\mathfrak{p}$, the space considered in (4.2) may be reduced to zero for all $\mathfrak{p}'$-modules W. Suppose $V \equiv V_\lambda$ with $\lambda \in \Lambda^+(\mathfrak{k})$ generic. Then a necessary and sufficient condition for the existence of W such that the left-hand side of (4.2) is non-zero is given by the geometric requirement on the generalized flag variety $G_{\mathbb{C}}/P_{\mathbb{C}}$, namely, the set $G_{\mathbb{C}}^\tau P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$, see [K12, Proposition 3.8].

Consider now the case where the nilradical $\mathfrak{n}_+$ of $\mathfrak{p}$ is abelian. Then, the following result holds :

Fact 4.1 ([K12])**.** *If the nilradical* $\mathfrak{n}_+$ *of* $\mathfrak{p}$ *is abelian, then for any symmetric pair* $(\mathfrak{g}, \mathfrak{g}^\tau)$ *the restriction of a generalized Verma module of scalar type* $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)|_{\iota(\mathfrak{g}^\tau)}$ *is multiplicity-free for any embedding* $\iota : \mathfrak{g}^\tau \to \mathfrak{g}$ *such that* $\iota(G_{\mathbb{C}}^\tau)P_{\mathbb{C}}$ *is closed in* $G_{\mathbb{C}}$ *and for any sufficiently positive* λ.

A combinatorial description of the branching law is given as follows. Suppose that $\mathfrak{p}$ is $\mathfrak{g}^\tau$-compatible (see [KP14-1, Definition 4.5]). Then the involution τ stabilizes $\mathfrak{k}$ and $\mathfrak{n}_+$, respectively, the nilradical $\mathfrak{n}_+$ decomposes into a direct sum of eigenspaces $\mathfrak{n}_+ = \mathfrak{n}_+^\tau + \mathfrak{n}_+^{-\tau}$ and $G_{\mathbb{C}}^\tau P_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$. Fix a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{k}$ such that $\mathfrak{j}^\tau := \mathfrak{j} \cap \mathfrak{g}^\tau$ is a Cartan subalgebra of $\mathfrak{k}^\tau$.

We define $\theta \in \operatorname{End}(\mathfrak{g})$ by $\theta|_{\mathfrak{k}} = \mathrm{id}$ and $\theta|_{\mathfrak{n}_+ + \mathfrak{n}_-} = -\mathrm{id}$. Then θ is an involution commuting with τ. Moreover it is an automorphism if $\mathfrak{n}_+$ is abelian. The reductive subalgebra $\mathfrak{g}^{\tau\theta} = \mathfrak{k}^\tau + \mathfrak{n}_-^{-\tau} + \mathfrak{n}_+^{-\tau}$ decomposes into simple or abelian ideals $\bigoplus_i \mathfrak{g}_i^{\tau\theta}$, and we write the decomposition of $\mathfrak{n}_+^{-\tau}$ as $\mathfrak{n}_+^{-\tau} = \bigoplus_i \mathfrak{n}_{+,i}^{-\tau}$ correspondingly. Each $\mathfrak{n}_{+,i}^{-\tau}$ is a $\mathfrak{j}^\tau$-module, and we denote by $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$ the set of weights of $\mathfrak{n}_{+,i}^{-\tau}$ with respect to $\mathfrak{j}^\tau$. The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu_1^{(i)}, \cdots, \nu_{k_i}^{(i)}\}$ in $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$ inductively as follows:

1) $\nu_1^{(i)}$ is the highest root of $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$.
2) $\nu_{j+1}^{(i)}$ is the highest root among the elements in $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$ that are strongly orthogonal to $\nu_1^{(i)}, \cdots, \nu_j^{(i)}$ $(1 \le j \le k_i - 1)$.

We define the following subset of $\mathbb{N}^k$ $(k = \sum k_i)$ by

$$A^+ := \prod_i A_i, \quad A_i := \{(a_j^{(i)})_{1 \le j \le k_i} \in \mathbb{N}^{k_i} : a_1^{(i)} \ge \cdots \ge a_{k_i}^{(i)} \ge 0\}. \tag{4.4}$$

Introduce the following positivity condition:

$$\langle \lambda - \rho_{\mathfrak{g}}, \alpha \rangle > 0 \quad \text{for any } \alpha \in \Delta(\mathfrak{n}_+, \mathfrak{j}). \tag{4.5}$$

Fact 4.2 ([K08])**.** *Suppose* $\mathfrak{p}$ *is* $\mathfrak{g}^\tau$*-compatible, and* λ *satisfies* (4.3) *and* (4.5)*. Then the generalized Verma module* $\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)$ *decomposes into a multiplicity-free direct sum* *of irreducible $\mathfrak{g}^\tau$-modules :*

$$\operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(-\lambda)|_{\mathfrak{g}^\tau} \simeq \bigoplus_{(a_j^{(i)})\in A^+} \operatorname{ind}_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}\Big(-\lambda|_{\mathfrak{j}^\tau} - \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}\Big). \tag{4.6}$$

In particular, for a simple $\mathfrak{p}^\tau$-module W (namely, a simple $\mathfrak{k}^\tau$-module with trivial action of $\mathfrak{n}^\tau$),

$$\dim \operatorname{Hom}_{\mathfrak{g}^\tau}(\operatorname{ind}_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(W^\vee), \operatorname{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_{-\lambda})) = 1$$

if and only if the highest weight of the $\mathfrak{k}^\tau$-module W is of the form $\lambda|_{\mathfrak{j}^\tau} + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}$ for some $(a_j^{(i)}) \in A^+$.

Notice that when τ is a Cartan involution, G^τ is compact and $\mathfrak{g}^\tau = \mathfrak{p}^\tau$. In this case, the formula (4.6) is due to L. K. Hua [H63] (classical case), B. Kostant (unpublished), and W. Schmid [Sch69]. In general G^τ is non-compact, and we need to consider infinite-dimensional irreducible representations of G^τ when we consider the branching law $G \downarrow G^\tau$.

In remaining Sections 5, 6, 7 and 8 we construct a family of equivariant differential operators for all symmetric pairs $(\mathfrak{g}, \mathfrak{g}^\tau)$ with G^τ non-compact and $k = 1$ (in particular, $\Delta(\mathfrak{n}_{+,i}^{-\tau}, \mathfrak{j}^\tau)$ is empty for all but one i).

In conclusion, we recall the corresponding branching laws in the category of unitary representations, which are the dual of the formulæ in Fact 4.2. We denote by $\mathcal{H}^2(M, \mathcal{V})$ the Hilbert space of square integrable holomorphic sections of the Hermitian vector bundle $\mathcal{V}$ over a Hermitian manifold M. If the positivity condition (4.5) holds, then $\mathcal{H}^2(G/K, \mathcal{L}_\lambda) \neq \{0\}$, and G acts unitarily and irreducibly on it.

Given $\underline{a} = (a_j^{(i)}) \in A^+$ ($\subset \mathbb{N}^k$), we write $\mathcal{W}_\lambda^{\underline{a}}$ for the G^τ-equivariant holomorphic vector bundle over G^τ/K^τ associated to the irreducible representation $\mathcal{W}_\lambda^{\underline{a}}$ of $\mathfrak{k}^\tau$ with highest weight $\lambda|_{\mathfrak{j}^\tau} + \sum_i \sum_{j=1}^{k_i} a_j^{(i)} \nu_j^{(i)}$.

Fact 4.3 ([K08])**.** *If the positivity condition* (4.5) *is satisfied, then $\mathcal{H}^2(G^\tau/K^\tau, \mathcal{W}_\lambda^{\underline{a}})$ is non-zero and G^τ acts on it irreducibly and unitarily for any $\underline{a} \in A^+$. Moreover, the branching law for the restriction $G \downarrow G^\tau$ is given by*

$$\mathcal{H}^2(G/K, \mathcal{L}_\lambda) \simeq \sum_{\underline{a}\in A^+}^{\oplus} \mathcal{H}^2(G^\tau/K^\tau, \mathcal{W}_\lambda^{\underline{a}}) \quad \text{(Hilbert direct sum)}. \tag{4.7}$$

5. Normal derivatives versus intertwining operators

Let G'/K' be a subsymmetric space of the Hermitian symmetric space G/K as in Section 2.1. Consider the Taylor expansion of any holomorphic function (section) on G/K with respect to the normal direction. Then the coefficients give rise to holomorphic sections of a family of vector bundles over the submanifold G'/K'. This idea was used earlier by Jakobsen and Vergne [JV79], and by the first author [K08] for filtered modules to find *abstract* branching laws.

However, it should be noted that normal derivatives do not always give rise to symmetry breaking operators. In this section we clarify the reason in the general setting, and then give a classification of all irreducible symmetric pairs $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ of split rank one for which it happens.

5.1. **Normal derivatives and the Borel embedding.** Suppose $E = E' \oplus E''$ is a direct sum of complex vector spaces. Let $\mathcal{V}_E := E \times V$ and $\mathcal{W}_{E'} := E' \times W$ be direct product vector bundles over E and E', respectively. Clearly, we have isomorphisms $\mathcal{O}(E, \mathcal{V}_E) \simeq \mathcal{O}(E) \otimes V$, and $\mathcal{O}(E', \mathcal{W}_{E'}) \simeq \mathcal{O}(E') \otimes W$.

Take coordinates $y = (y_1, \cdots, y_p)$ in E' and $z = (z_1, \cdots, z_n)$ in E''. The subspace E' is given by the condition $z = 0$ in $E = \{(y, z) : y \in E', z \in E''\}$. A holomorphic differential operator $\widetilde{T} : \mathcal{O}(E) \otimes V \longrightarrow \mathcal{O}(E') \otimes W$, $f(y, z) \mapsto (\widetilde{T} f)(y)$ is said to be a *normal derivative with respect to the decomposition* $E = E' \oplus E''$ if it is of the form

$$\left(\widetilde{T} f\right)(y) = \sum_{\alpha \in \mathbb{N}^q} T_\alpha(y) \left(\left. \frac{\partial^{|\alpha|} f(y, z)}{\partial z^\alpha} \right|_{z=0} \right), \tag{5.1}$$

for some $T_\alpha \in \mathcal{O}(E') \otimes \operatorname{Hom}_{\mathbb{C}}(V, W)$.

We write $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_E, \mathcal{W}_{E'})$ for the space of (holomorphic) normal derivatives. This notion depends on the direct sum decomposition $E = E' \oplus E''$.

Since the commutative groups $E \supset E'$ act on the direct product bundles $\mathcal{V}_E$ and $\mathcal{W}_{E'}$, respectively, we can consider symmetry breaking operators in this abelian setting, namely, E'-equivariant normal derivatives, which amount to the condition that $T_\alpha(y)$ in (5.1) is a differential operator with constant coefficients for every $\alpha \in \mathbb{N}^q$. We denote $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_E, \mathcal{W}_{E'})$ the subspace of $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_E, \mathcal{W}_{E'})$ consisting of those operators.

Thus we have seen the following:

Lemma 5.1. *There is a natural isomorphism:*

$$\operatorname{Hom}_{\mathbb{C}}(V, W) \otimes S(E'') \xrightarrow{\sim} \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_E, \mathcal{W}_{E'}).$$

Suppose we are in the setting of Section 2.1. We apply the concept of normal derivatives to the subsymmetric space G'/K' in the Hermitian symmetric space G/K. Let $\mathcal{V}$ be a homogeneous vector bundle over $X = G/K$ associated with a finite-dimensional representation V of K. Similarly, let $\mathcal{W}$ be a homogeneous vector bundle over the subsymmetric space $Y = G'/K'$ associated with a finite-dimensional representation W of K'.

By using the Killing form, we take a complementary subspace $\mathfrak{g}''$ of $\mathfrak{g}'$ in $\mathfrak{g}$ so that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$ is a direct sum of G'-modules. We set $\mathfrak{n}''_- := \mathfrak{n}_- \cap \mathfrak{g}''$. Since the characteristic element $H_o \in \mathfrak{g}'$ (see (2.1)), we have a direct sum decomposition of K'-modules:

$$\mathfrak{n}_- = \mathfrak{n}'_- \oplus \mathfrak{n}''_-. \tag{5.2}$$

Accordingly, we can consider the space $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$ of holomorphic normal derivatives with respect to (5.2).

We write $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$ and $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$ for the images of $\mathcal{N}\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$ and $\mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-})$, respectively, under the natural injective map:

$$\mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_{\mathfrak{n}_-}, \mathcal{W}_{\mathfrak{n}'_-}) \hookrightarrow \mathrm{Diff}^{\mathrm{hol}}(\mathcal{V}_X, \mathcal{W}_Y)$$

induced by the following map:

$$\begin{tikzcd}
\mathcal{O}(\mathfrak{n}_-, V) \arrow[r] \arrow[d, hook, "\text{restriction}"'] & \mathcal{O}(\mathfrak{n}'_-, W) \arrow[d, hook, "\text{restriction}"] \\
\mathcal{O}(G/K, \mathcal{V}) \arrow[r, dotted] & \mathcal{O}(G'/K', \mathcal{W}).
\end{tikzcd} \tag{5.3}$$

Since the trivialization of the vector bundle $G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V$

$$\begin{tikzcd}
\mathfrak{n}_- \times V \arrow[r, hook] \arrow[d] & G_{\mathbb{C}} \times_{P_{\mathbb{C}}} V \arrow[d] & \mathcal{V}_X \arrow[l, hook'] \arrow[d] \\
\mathfrak{n}_- \arrow[r, hook] & G_{\mathbb{C}}/P_{\mathbb{C}} & X = G/K \arrow[l, hook']
\end{tikzcd}$$

is $K_{\mathbb{C}}$-equivariant, Lemma 5.1 implies:

Proposition 5.2. *There is a natural isomorphism:*

$$\mathrm{Hom}_{K'}(V, S(\mathfrak{n}''_-) \otimes W) \xrightarrow{\sim} \mathcal{N}\mathrm{Diff}^{\mathrm{const}}_{K'}(\mathcal{V}_X, \mathcal{W}_Y).$$

We study whether or not the following two subspaces

- $\mathcal{N}\mathrm{Diff}_{K'}(\mathcal{V}_X, \mathcal{W}_Y)$ of K'-equivariant normal derivatives and
- $\mathrm{Hom}_{G'}(\mathcal{O}(\mathcal{V}_X), \mathcal{O}(\mathcal{W}_Y))$ of symmetry breaking operators

coincide in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}(\mathcal{V}_X), \mathcal{O}(\mathcal{W}_Y))$. Owing to Theorem 3.1 and Proposition 5.2, it reduces to an algebraic problem to compare

- $\mathrm{Hom}_{K'}(V, S(\mathfrak{n}''_-) \otimes W)$ and
- $\mathrm{Hom}_{K'}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}$

in $\mathrm{Hom}_{\mathbb{C}}(V, \mathrm{Pol}(\mathfrak{n}_+) \otimes W) \simeq \mathrm{Hom}_{\mathbb{C}}(V, S(\mathfrak{n}_-) \otimes W)$. We shall see in the next subsection that they actually coincide for the three families of symmetric pairs out of the six listed in Table 2.1.

5.2. **When are normal derivatives intertwining operators?** Let $\dim V = 1$, and we write as before $\mathcal{L}_\lambda$ for the homogeneous line bundle over $X = G/K$ associated to the character $\mathbb{C}_\lambda$ of K.

Theorem 5.3. *Suppose $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ is a split rank one irreducible symmetric pair of holomorphic type (see Definition 2.3). Then, the following three conditions on the pair $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ are equivalent:*

(i) *For any λ satisfying the positivity condition* (4.5) *and for any irreducible K^τ-module W, all continuous G^τ-homomorphisms*

$$\mathcal{O}(X, \mathcal{L}_\lambda) \longrightarrow \mathcal{O}(Y, \mathcal{W}),$$

are given by normal derivatives with respect to the decomposition $\mathfrak{n}_- = \mathfrak{n}_-^\tau \oplus \mathfrak{n}_-^{-\tau}$.

(ii) *For some λ satisfying* (4.5) *and for some irreducible K^τ-module W, there exists a non-trivial G^τ-intertwining operator*

$$\mathcal{O}(X, \mathcal{L}_\lambda) \longrightarrow \mathcal{O}(Y, \mathcal{W})$$

which is given by normal derivatives of positive order.

(iii) *The symmetric pair $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ is isomorphic to one of $(\mathfrak{su}(p,q), \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(p-1,q)))$, $(\mathfrak{so}(2,2n), \mathfrak{u}(1,n))$ or $(\mathfrak{so}^*(2n), \mathfrak{so}(2) \oplus \mathfrak{so}^*(2n-2))$.*

Notice that the geometric nature of embeddings $Y \hookrightarrow X$ mentioned in the condition (iii) corresponds to the following inclusions of flag varieties:

$$\begin{array}{rcl} \mathrm{Gr}_{p-1}(\mathbb{C}^{p+q}) & \hookrightarrow & \mathrm{Gr}_p(\mathbb{C}^{p+q}); \\ \mathbb{P}^n\mathbb{C} & \hookrightarrow & Q^{2n}\mathbb{C}; \\ \mathrm{IGr}_{n-1}(\mathbb{C}^{2n-2}) & \hookrightarrow & \mathrm{IGr}_n(\mathbb{C}^{2n}), \end{array}$$

where $\mathrm{Gr}_p(\mathbb{C}^k) := \{V \subset \mathbb{C}^k : \dim V = p\}$ is the complex Grassmanian, $Q^m\mathbb{C} := \{z \in \mathbb{P}^{m+1}\mathbb{C} : z_0^2 + \cdots + z_{m+1}^2 = 0\}$ is the complex quadric and $\mathrm{IGr}_n(\mathbb{C}^{2n}) := \{V \subset \mathbb{C}^{2n} : \dim V = n, Q\big|_V \equiv 0\}$ is the isotropic Grassmanian for $\mathbb{C}^{2n}$ equipped with a non-degenerate quadratic form Q.

5.3. **Outline of the proof of Theorem 5.3.** The implication (i)⇒(ii) is obvious. On the other hand, for split rank one symmetric spaces there are three other cases (i.e., (1), (2) and (3) in Table 2.1) where the G^τ-intertwining operators are not given by normal derivatives. In Sections 6, 7 and 8 we construct them explicitly. This will conclude the implication (ii)⇒(iii). For the rest of this section we shall give a proof for the implication (iii)⇒(i).

Consider a homomorphism: $T : W^\vee \longrightarrow S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee$. We regard $S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee$ as a subspace of $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ on which the Lie algebra $\mathfrak{g}$ acts by $\widehat{d\pi_{\lambda^*}}$, see (3.6). If T is a K^τ-homomorphism, the differential operator $\widetilde{T} : \mathcal{O}(G/K, \mathcal{V}_X) \to \mathcal{O}(G^\tau/K^\tau, \mathcal{W}_Y)$

is K^τ-equivariant. The following statement gives a sufficient condition for $\widetilde{T}$ to be G^τ-equivariant.

Proposition 5.4. *The normal derivative* $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$ *induces a* G^τ*-equivariant differential operator from* $\mathcal{V}_X$ *to* $\mathcal{W}_Y$ *if and only if* T *is a* K^τ*-homomorphism and* $T(W^\vee)$ *is contained in* $(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}$.

Proof. The proof is a direct consequence of the F-method. Indeed, by Theorem 3.1, $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y) \subset \mathrm{Diff}^{\mathrm{const}}(\mathfrak{n}_-) \otimes \mathrm{Hom}_{\mathbb{C}}(V, W)$ is a G^τ-equivariant differential operator if and only if $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{p}^\tau)}$ where

$$\begin{aligned} & (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{p}^\tau)} \\ = \; & (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{k}^\tau)} \cap (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}. \end{aligned}$$

Furthermore, by Theorem 3.1, for $\widetilde{T} \in \mathcal{N}\mathrm{Diff}^{\mathrm{const}}(\mathcal{V}_X, \mathcal{W}_Y)$, we have $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) \in (\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathrm{Hom}(V, W))^{\widehat{d\pi_{\lambda^*}}(\mathfrak{k}^\tau)}$ if and only if $T \in \mathrm{Hom}_{\mathfrak{k}^\tau}(W^\vee, S(\mathfrak{n}_-^{-\tau}) \otimes V^\vee)$, as $(\mathrm{Symb} \otimes \mathrm{id})(\widetilde{T}) = (F_c \otimes \mathrm{id})(T)$. Hence the statement is proved. ☐

Lemma 5.5. *Suppose* $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ *is a split rank one irreducible symmetric pair of holomorphic type and* λ *satisfying* (4.3) *and* (4.5). *For* $a \in \mathbb{N}$ *we define a* K^τ*-module:*

$$W_\lambda^a := S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}_\lambda. \tag{5.4}$$

1. *The module* W_λ^a *is irreducible for any* $a \in \mathbb{N}$.
2. *If for an irreducible* K^τ*-module* W *there exists a non-zero continuous* G^τ*-homomorphism* $\mathcal{O}(G/K, \mathcal{L}_\lambda) \to \mathcal{O}(G^\tau/K^\tau, \mathcal{W})$, *then the module* W *is isomorphic to* W_λ^a *for some* $a \in \mathbb{N}$.
3. *Assume that*

$$\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}), S^{a_1}(\mathfrak{n}_-^\tau) \otimes S^{a-a_1}(\mathfrak{n}_-^{-\tau})) = \{0\} \quad \text{for any } 1 \le a_1 \le a. \tag{5.5}$$

Then, the normal derivative $\widetilde{T}$ *corresponding to the natural inclusion* $T : (W_\lambda^a)^\vee \to S(\mathfrak{n}_-^{-\tau}) \otimes (\mathbb{C}_\lambda)^\vee$ *is a* G^τ*-equivariant differential operator.*

Proof. If $\mathrm{rank}_{\mathbb{R}} G/G^\tau = 1$, then the non-compact part of $\mathfrak{g}(\mathbb{R})^{\tau\theta}$ is isomorphic to $\mathfrak{su}(1, n)$ for some n. Thus the first statement follows from the observation that $S^a(\mathbb{C}^n)$ is an irreducible $\mathfrak{gl}_n(\mathbb{C})$-module for any $a \in \mathbb{N}$ because the action of $\mathfrak{k}^\tau$ on $\mathfrak{n}_+^{-\tau}$ corresponds to the natural action of $\mathfrak{gl}_n(\mathbb{C})$ on $\mathbb{C}^n$.

The second statement is due to the localness theorem [KP14-1, Theorem 5.3] for $k = \mathrm{rank}_{\mathbb{R}} G/G^\tau = 1$.

To show the third statement, observe that we have the following natural inclusions $A \subset B \supset C$, where

$$A := \mathrm{Pol}^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_\lambda^\vee,\; B := \mathrm{Pol}^a(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee,\; C := (\mathrm{Pol}^a(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda^\vee)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+^\tau)}.$$

Therefore

$$\mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, A) \hookrightarrow \mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B) \hookleftarrow \mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, C).$$

By Proposition 5.2 and Theorem 3.1, we have

$$\mathcal{N}\mathrm{Diff}^{\mathrm{const}}_{K^\tau}(\mathcal{V}_X, \mathcal{W}_Y) \hookrightarrow \mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B) \hookleftarrow \mathrm{Hom}_{G'}(\mathcal{O}(X,\mathcal{V}), \mathcal{O}(Y,\mathcal{W})).$$

Since $\mathrm{Pol}^a(\mathfrak{n}_+) \simeq \bigoplus_{a_1=0}^{a} \mathrm{Pol}^{a_1}(\mathfrak{n}_+^\tau) \otimes \mathrm{Pol}^{a-a_1}(\mathfrak{n}_+^{-\tau})$, the assumption (5.5) implies that $\mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, A) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{k}^\tau}((W^a_\lambda)^\vee, B)$, and therefore the first inclusion is an isomorphism. Moreover, since A is isomorphic to the irreducible $\mathfrak{k}^\tau$-module $(W^a_\lambda)^\vee$, the first term is one-dimensional by Schur's lemma. The last one is also one-dimensional according to the multiplicity-one decomposition given in Fact 4.2. Therefore, all the three terms coincide.

Hence the canonical isomorphism $T : (W^a_\lambda)^\vee \to S(\mathfrak{n}_-^{-\tau}) \otimes (\mathbb{C}_\lambda)^\vee$ satisfies the assumption of Proposition 5.4. Thus Lemma follows. □

Remark 5.6. The highest weight vectors of the generalized Verma module $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)$ with respect to $\mathfrak{p}^\tau$ have a particularly simple form if the condition (5.5) is satisfied. In fact, by Poincaré–Birkhoff–Witt theorem $\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)$ is isomorphic, as a $\mathfrak{k}$-module, to $S(\mathfrak{n}_-) \otimes \mathbb{C}^\vee_\lambda$, when $\mathfrak{n}_-$ is abelian. Under the assumption (5.5) we thus have

$$\left(\mathrm{ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}^\vee_\lambda)\right)^{\mathfrak{n}_+^\tau} \simeq \bigoplus_{a=0}^{\infty} S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}^\vee_\lambda.$$

This formula is an algebraic explanation of the fact that G^τ-equivariant operators are given by normal derivatives in this setting.

In order to conclude the proof of Theorem 5.3 we have to show that in all cases mentioned in (iii) the condition (5.5) is fulfilled. It will be done in the next subsection.

5.4. **An application of the classical branching rules.**

In what follows, we shall verify the condition (5.5) for the last three cases (4), (5) and (6) in Table 2.1 by using some classical branching rules of irreducible representations of $\mathfrak{gl}_m(\mathbb{C})$.

Denote by $F(\mathfrak{gl}_m(\mathbb{C}), \mu)$ the finite dimensional irreducible $\mathfrak{gl}_m(\mathbb{C})$-module with highest weight μ. For example, the natural representation of the Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ on $\mathbb{C}^m$ corresponds to $F(\mathfrak{gl}_m(\mathbb{C}), (1,0,\dots,0))$ and its contragredient representation on $(\mathbb{C}^m)^\vee$ to $F(\mathfrak{gl}_m(\mathbb{C}), (0,0,\dots,0,-1))$, while the action of $\mathfrak{gl}_m(\mathbb{C})$ on the space of symmetric matrices $\mathrm{Sym}(m,\mathbb{C}) \simeq S^2(\mathbb{C}^m)$ given by $C \mapsto XC\,{}^tX$ for $X \in \mathfrak{gl}_m(\mathbb{C})$ and $C \in Sym(m,\mathbb{C})$ corresponds to $F(\mathfrak{gl}_m(\mathbb{C}), (2,0,\dots,0))$. More generally, the action of $\mathfrak{gl}_m(\mathbb{C})$ on the space of i-th symmetric tensors is no longer irreducible and decomposes as follows:

$$\begin{aligned} S^i\left(\mathrm{Sym}(m,\mathbb{C})\right) &\simeq S^i\left(S^2(\mathbb{C}^m)\right) \\ &\simeq \bigoplus_{\substack{i_1\geq\cdots\geq i_m\geq 0\\ i_1+\cdots+i_m=i}} F(\mathfrak{gl}_m(\mathbb{C}),\,(2i_1,2i_2,\ldots,2i_m)). \end{aligned} \tag{5.6}$$

In turn, classical Pieri's rule gives the following irreducible decomposition for the tensor product of such modules:

$$S^i\left(S^2(\mathbb{C}^m)\right)\otimes S^k\left(\mathbb{C}^m\right) \simeq \bigoplus_{\substack{i_1\geq\cdots\geq i_m\geq 0,\\ i_1+\cdots+i_m=i}} \bigoplus_{\substack{\ell_1\geq 2i_1\geq\cdots\geq\ell_m\geq 2i_m,\\ \sum_{r=1}^m(\ell_r-2i_r)=k}} F(\mathfrak{gl}_m(\mathbb{C}),\,(\ell_1,\ldots,\ell_m)).$$

Remark 5.7. The summand of the form $F(\mathfrak{gl}_m(\mathbb{C}),\,(\ell,0,\ldots,0))$ occurs in the right-hand side if and only if $i_2=\cdots=i_m=0$, hence $i_1=i$ and $\ell-2i=k$. This remark will be used in Section 7.

Example 5.8. *Let* $G=U(p,q)$, $G^\tau=U(1)\times U(p-1,q)$ *and* $\mathfrak{k}^\tau=\mathfrak{k}^\tau(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{gl}_1(\mathbb{C})\oplus\mathfrak{gl}_{p-1}(\mathbb{C})\oplus\mathfrak{gl}_q(\mathbb{C})$. *Then, the decomposition* $\mathfrak{n}_-=\mathfrak{n}_-^\tau\oplus\mathfrak{n}_-^{-\tau}$ *as a* $\mathfrak{k}^\tau$*-module amounts to*

$$(\mathbb{C}^p)^\vee\boxtimes\mathbb{C}^q\simeq(\mathbb{C}\boxtimes(\mathbb{C}^{p-1})^\vee\boxtimes\mathbb{C}^q)\oplus(\mathbb{C}_{-1}\boxtimes\mathbb{C}\boxtimes\mathbb{C}^q),$$

where $\boxtimes$ *stands for the outer tensor product representation. Therefore, for* $a=a_1+a_2$,

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}),S^{a_1}(\mathfrak{n}_-^\tau)\otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\;&\mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a},\mathbb{C}_{-a_2})\otimes\mathrm{Hom}_{\mathfrak{gl}_{p-1}(\mathbb{C})}(\mathbb{C},S^{a_1}((\mathbb{C}^{p-1})^\vee))\otimes\mathrm{Hom}_{\mathfrak{gl}_q(\mathbb{C})}(S^a(\mathbb{C}^q),S^{a_2}(\mathbb{C}^q)) \end{aligned}$$

is not reduced to zero if and only if $a_1=0$ *and* $a_2=a$. *Thus, the condition* (5.5) *is satisfied.*

Example 5.9. *Let* $G=SO(2,2n)$, $G^\tau=U(1,n)$ *and* $\mathfrak{k}^\tau=\mathfrak{gl}_1(\mathbb{C})\oplus\mathfrak{gl}_n(\mathbb{C})$. *Then the decomposition* $\mathfrak{n}_-=\mathfrak{n}_-^\tau\oplus\mathfrak{n}_-^{-\tau}$ *as a* $\mathfrak{k}^\tau$*-module amounts to*

$$\mathbb{C}_{-1}\boxtimes\mathbb{C}^{2n}\simeq(\mathbb{C}_{-1}\boxtimes\mathbb{C}^n)\oplus(\mathbb{C}_{-1}\boxtimes(\mathbb{C}^n)^\vee).$$

Therefore, for $a=a_1+a_2$, *we have*

$$\begin{aligned} &\mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}),S^{a_1}(\mathfrak{n}_-^\tau)\otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\;&\mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a},\mathbb{C}_{-a_1-a_2})\otimes\mathrm{Hom}_{\mathfrak{gl}_n(\mathbb{C})}(S^a((\mathbb{C}^n)^\vee),S^{a_1}(\mathbb{C}^n)\otimes S^{a_2}((\mathbb{C}^n)^\vee)) \\ \simeq\;&\bigoplus_{b=0}^{\min(a_1,a_2)}\mathrm{Hom}_{\mathfrak{gl}_n(\mathbb{C})}(F(\mathfrak{gl}_n(\mathbb{C}),(0,\cdots,0,-a)),F(\mathfrak{gl}_n(\mathbb{C}),(a_1-b,0,\cdots,0,-a_2+b))), \end{aligned}$$

where the second isomorphism follows from Pieri's rule. Thus, the left-hand side is not reduced to zero if and only if $a_1=0$ *and* $a_2=a$. *Hence, the condition* (5.5) *is satisfied.*

Example 5.10. *Let* $G = SO^*(2n)$, $G^\tau = SO^*(2n-2) \times SO(2)$ *and* $\mathfrak{k}^\tau = \mathfrak{gl}_{n-1}(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$. *In this case, the decomposition* $\mathfrak{n}_- = \mathfrak{n}_-^\tau \oplus \mathfrak{n}_-^{-\tau}$ *as a* $\mathfrak{k}^\tau$*-module amounts to*

$$(\mathrm{Alt}(\mathbb{C}^{n-1})^\vee \boxtimes \mathbf{1}) \oplus ((\mathbb{C}^{n-1})^\vee \boxtimes \mathbb{C}_{-1}).$$

Therefore, for $a = a_1 + a_2$

$$\begin{aligned} & \mathrm{Hom}_{\mathfrak{k}^\tau}(S^a(\mathfrak{n}_-^{-\tau}), S^{a_1}(\mathfrak{n}_-^\tau) \otimes S^{a_2}(\mathfrak{n}_-^{-\tau})) \\ \simeq\ & \mathrm{Hom}_{\mathfrak{gl}_{n-1}(\mathbb{C})}(S^a((\mathbb{C}^{n-1})^\vee), S^{a_1}(\mathrm{Alt}(\mathbb{C}^{n-1})^\vee) \otimes S^{a_2}((\mathbb{C}^{n-1})^\vee)) \otimes \mathrm{Hom}_{\mathfrak{gl}_1(\mathbb{C})}(\mathbb{C}_{-a}, \mathbb{C}_{-a_2}). \end{aligned}$$

In view of the $\mathfrak{gl}_1(\mathbb{C})$*-action on the right-hand side, it is non-zero only if* $a_2 = a$ *(and therefore* $a_1 = 0$*). Thus the condition* (5.5) *is satisfied.*

Hence we have verified the assumption (5.5) for all the three symmetric pairs $(\mathfrak{g}(\mathbb{R}), \mathfrak{g}(\mathbb{R})^\tau)$ corresponding to the three complex geometries (4), (5) and (6) in Table 1.1, and have proved the implication (iii) ⇒ (i) in Theorem 5.3 by Lemma 5.5 (3).

6. SYMMETRY BREAKING OPERATORS FOR THE RESTRICTION $SO(n,2) \downarrow SO(n-1,2)$

Let $n \geq 3$. In what follows, we realize the indefinite orthogonal group $SO(n,2)$ in a slightly non-standard way, namely, use a non-degenerate quadratic form on $\mathbb{C}^{n+2}$ defined by

$$\widetilde{Q}(w) := w_0^2 + \cdots + w_n^2 - w_{n+1}^2 \quad \text{for } w = (w_0, \cdots, w_{n+1}) \in \mathbb{C}^{n+2},$$

and restrict it to a certain real form $E(\mathbb{R})$ (see (6.3) below) of $\mathbb{C}^{n+2}$. (The restriction to the standard real form $\mathbb{R}^{n+2}$ yields conformally covariant differential operators corresponding to another pair of real forms $(SO(n+1,1), SO(n,1))$, see Remark 6.13.)

Let $G_{\mathbb{C}}$ be the complex special orthogonal group $SO(\mathbb{C}^{n+2}, \widetilde{Q})$ with respect to the quadratic form $\widetilde{Q}$. Then $G_{\mathbb{C}}$ acts transitively on the isotropic cone

$$\Xi_{\mathbb{C}} := \{w \in \mathbb{C}^{n+2} \setminus \{0\} : \widetilde{Q}(w) = 0\},$$

and also on the complex quadric

$$\mathrm{Q}^n\mathbb{C} := \Xi_{\mathbb{C}}/\mathbb{C}^* \subset \mathbb{P}^{n+1}\mathbb{C}$$

by $w \to g \cdot [w] := [gw]$ for $w \in \mathbb{C}^{n+1} \setminus \{0\}$. Let $w_o = {}^t(1, 0, \cdots, 0, 1) \in \Xi_{\mathbb{C}}$, and $P_{\mathbb{C}}$ be the stabilizer of the base point $[w_o] = [1 : 0 : \cdots : 0 : 1] \in Q^n\mathbb{C}$, which is a maximal parabolic subgroup of $G_{\mathbb{C}}$. Then we have an isomorphism $Q^n\mathbb{C} \simeq G_{\mathbb{C}}/P_{\mathbb{C}}$. We define an embedding

$$\iota : \mathbb{C}^n \to \Xi_{\mathbb{C}}, \quad z \mapsto {}^t(1 - Q_n(z), 2z_1, \cdots, 2z_n, 1 + Q_n(z)), \tag{6.1}$$

where $Q_n(z) := \sum_{j=1}^n z_j^2$ for $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. Then we get coordinates on $Q^n\mathbb{C}$ by

$$\mathbb{C}^n \hookrightarrow \mathrm{Q}^n\mathbb{C}, \quad z \mapsto [\iota(z)] \tag{6.2}$$

which define the open Bruhat cell (see (6.7) below).

The quadratic form $\widetilde{Q}$ is of signature $(n, 2)$ when restricted to the real vector space

$$E(\mathbb{R}) := \sqrt{-1}\mathbb{R}e_0 + \sum_{j=1}^{n+1} \mathbb{R}e_j, \tag{6.3}$$

where $\{e_j : 0 \le j \le n+1\}$ is the standard basis in $\mathbb{C}^{n+2}$. Thus we have an isomorphism:

$$SO(\mathbb{C}^{n+2}, \widetilde{Q}) \cap GL_{\mathbb{R}}(E(\mathbb{R})) \simeq SO(n, 2).$$

Let G be its identity component $SO_o(n, 2)$. Then the G-orbit through the base point $[w_o]$ in $Q^n\mathbb{C}$ is still contained in $\mathbb{C}^n$, and is identified with the Lie ball $X := \{z \in \mathbb{C}^n : |z\,{}^t z|^2 + 1 - 2\overline{z}\,{}^t z > 0, |z\,{}^t z| < 1\} \simeq G/K$ which is the bounded Hermitian symmetric domain of type IV in the É. Cartan classification.

Let τ be the involution of $GL(n+1, \mathbb{C})$ by conjugation by $\mathrm{diag}(1, \dots, 1, -1, 1)$. It leaves G invariant, and we denote by G' the identity component of the fixed point group G^τ. The group $G' = SO_o(n-1, 2)$ acts on the subsymmetric domain

$$Y := X \cap \{z_n = 0\}.$$

Then $Y \simeq G'/K' = SO_o(n-1, 2)/SO(n-1) \times SO(2)$ a subsymmetric space of X of complex codimension one.

We take $H_o := E_{0,n+1} + E_{n+1,0}$.Then H_o is a characteristic element as in Section 2.1. For $\lambda \in \mathbb{Z}$ we define a character of $\mathfrak{c}(\mathfrak{k})$ by $tH_o \mapsto \lambda t$, and lift it to a character $\mathbb{C}_\lambda$ of K. Let $\mathcal{L}_\lambda$ be the G-equivariant holomorphic line bundle $G \times_K \mathbb{C}_\lambda$. The holomorphic line bundle $\mathcal{L}_\lambda \to X$ is trivialized by using the open Bruhat cell, and the representation of G on $\mathcal{O}(X, \mathcal{L}_\lambda)$ is identified with the multiplier representation $\pi_\lambda \equiv \pi_\lambda^G$ of the same group on $\mathcal{O}(X)$ given by

$$F(z) \mapsto (\pi_\lambda(g)F)(z) = J(g^{-1}, z)^{-\lambda} F(g^{-1} \cdot z), \tag{6.4}$$

where we define a map $J : G \times X \to \mathbb{C}^*$ by

$$J(g, z) := \frac{1}{2} {}^t w_o g \iota(z), \quad \text{for } g \in G \text{ and } z \in X.$$

Since $H_o \in \mathfrak{k}'$ (see (2.1)), we can also define a G'-equivariant holomorphic line bundle $\mathcal{L}_\nu = G' \times_{K'} \mathbb{C}_\nu$ over $Y = G'/K'$ for $\nu \in \mathbb{Z}$.

Let $\widetilde{G}$ be the universal covering group of $G = SO_o(n, 2)$. Then for any $\lambda \in \mathbb{C}$ one can define a $\widetilde{G}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda = \widetilde{G} \times_{\widetilde{K}} \mathbb{C}_\lambda$ over $X = G/K \simeq \widetilde{G}/\widetilde{K}$, and a representation of the same group on $\mathcal{O}(X, \mathcal{L}_\lambda)$. Similarly, for $\nu \in \mathbb{C}$, the universal covering group $\widetilde{G'}$ of $G' = SO_o(n-1, 2)$ acts on $\mathcal{O}(Y, \mathcal{L}_\nu)$.

Here is a complete classification of symmetry breaking operators from $\mathcal{O}(X,\mathcal{L}_\lambda)$ to $\mathcal{O}(Y,\mathcal{L}_\nu)$ with respect to the symmetric pair $\widetilde{G} \supset \widetilde{G}'$:

Theorem 6.1. *Let $n \geq 3$ and $\widetilde{G}'$ be the universal covering group of $SO_o(n-1,2)$. Suppose $\lambda,\nu \in \mathbb{C}$. Then the following three conditions on the parameters $(\lambda,\nu) \in \mathbb{C}^2$ are equivalent:*

(i) $\mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda),\mathcal{O}(Y,\mathcal{L}_\nu)) \neq \{0\}$.
(ii) $\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda),\mathcal{O}(Y,\mathcal{L}_\nu)) = 1$.
(iii) $\nu - \lambda \in \mathbb{N}$.

Remark 6.2. The equivalence (i)$\Leftrightarrow$(ii) in Theorem 6.1 is not true for singular parameters (λ,ν) in the case of $n = 2$. This situation will be treated carefully in Section 9. In fact, the symmetric pair $(SO_o(2,2), SO_o(2,1))$ is locally isomorphic to the pair $(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}), \Delta(SL(2,\mathbb{R}))$ modulo the center. We note that $n = 2$ in Theorem 6.1 corresponds to $\lambda' = \lambda''$ in Theorem 9.1.

Let $\widetilde{C}_\ell^\alpha(x)$ be the renormalized Gegenbauer polynomial (see Appendix 11.3). We inflate it to a polynomial of two variables x and y:

$$\begin{aligned} \widetilde{C}_\ell^\alpha(x,y) &:= x^{\frac{\ell}{2}} \widetilde{C}_\ell^\alpha\left(\frac{y}{\sqrt{x}}\right) \\ &= \sum_{k=0}^{[\frac{\ell}{2}]} (-1)^k \frac{\Gamma(\ell-k+\alpha)}{\Gamma\left(\alpha+\left[\frac{\ell+1}{2}\right]\right)\Gamma(k+1)\Gamma(\ell-2k+1)} (2y)^{\ell-2k} x^k. \end{aligned} \tag{6.5}$$

For instance, $\widetilde{C}_0^\alpha(x,y) = 1$, $\widetilde{C}_1^\alpha(x,y) = 2y$, $\widetilde{C}_2^\alpha(x,y) = 2(\alpha+1)y^2 - x$, etc. Notice that $\widetilde{C}_\ell^\alpha(x^2,y)$ is a homogeneous polynomial of x and y of degree ℓ.

Theorem 6.3. *Retain the setting of Theorem 6.1. Let $a := \nu - \lambda \in \mathbb{N}$. Then the differential operator from $\mathcal{O}(X)$ to $\mathcal{O}(Y)$ defined by*

$$D_{X\to Y,a} := \widetilde{C}_a^{\lambda-\frac{n-1}{2}}\left(-\Delta^z_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right) \tag{6.6}$$

intertwines the restriction $\pi_\lambda^{\widetilde{G}}\big|_{\widetilde{G}'}$ with $\pi_{\lambda+a}^{\widetilde{G}'}$ (see (6.4)). Here $\Delta^z_{\mathbb{C}^m} := \sum_{k=1}^m \frac{\partial^2}{\partial z_k^2}$ denotes the holomorphic Laplacian on $\mathbb{C}^m$ in the coordinates $(z_1,\cdots,z_m)$.

It follows from Theorems 6.1 and 6.3 that any symmetry breaking operator from $\mathcal{O}(X,\mathcal{L}_\lambda)$ to $\mathcal{O}(Y,\mathcal{L}_{\lambda+a})$ is proportional to $D_{X\to Y,a}$ for any $\lambda \in \mathbb{C}$ and $a \in \mathbb{N}$.

Remark 6.4. If $\lambda \in \mathbb{R}$ and $\lambda > n-1$, then $\mathcal{H}^2(X,\mathcal{L}_\lambda) := \mathcal{O}(X,\mathcal{L}_\lambda) \cap L^2(X,\mathcal{L}_\lambda)$ is a non-zero Hilbert space on which $\widetilde{G}$ acts unitarily and irreducibly, giving a holomorphic discrete series representation of $\widetilde{G}$ modulo the center. By [KP14-1, Theorem 5.13] the same statement as Theorems 6.1 and 6.3 remains true for symmetry breaking operators between the unitary representations $\mathcal{H}^2(X, \mathcal{L}_\lambda)$ and $\mathcal{H}^2(Y, \mathcal{L}_{\lambda+a})$.

In order to prove Theorems 6.1 and 6.3 we apply the F-method (see Section 3.1). The Lie algebra $\mathfrak{g} = \mathfrak{so}(\mathbb{C}^{n+2}, \widetilde{Q})$ has a direct sum decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+$$

of $-1, 0$, and 1 eigenspaces of $\mathrm{ad}(H_o)$, respectively. Then the maximal parabolic subgroup $P_{\mathbb{C}}$ has a Levi decomposition $P_{\mathbb{C}} = K_{\mathbb{C}} N_{+,\mathbb{C}}$, where $N_{+,\mathbb{C}} = \exp \mathfrak{n}_+$.

As Step 1 of the F-method we define a standard basis of $\mathfrak{n}_+ \simeq \mathbb{C}^n$ by

$$C_j \ := \ E_{j,0} - E_{j,n+1} - E_{0,j} - E_{n+1,j} \quad (1 \le j \le n),$$

and similarly a standard basis of $\mathfrak{n}_- \simeq \mathbb{C}^n$ by

$$\overline{C}_j \ := \ E_{j,0} + E_{j,n+1} - E_{0,j} + E_{n+1,j} \quad (1 \le j \le n).$$

Then the decomposition $\mathfrak{n}_+ = \mathfrak{n}_+^{\tau} \oplus \mathfrak{n}_+{-\tau}$ is given by

$$\mathfrak{n}_+ = \sum_{j=1}^{n-1} \mathbb{C} C_j \oplus \mathbb{C} C_n.$$

Let $Z = \sum_{i=1}^n z_i \overline{C}_i \in \mathfrak{n}_-$ and $Y = \sum_{j=1}^n y_j C_j \in \mathfrak{n}_+$. By a simple computation we have

$$\exp(Z) \cdot w_o = \iota(z) \in \mathbb{C}^{n+2}, \tag{6.7}$$

the open Bruhat cell is given by (6.2). Moreover, by using

$$\exp(tY) \exp(Z) w_o = \iota(z) - 2t \begin{pmatrix} (y,z) \\ Q(z)y \\ (y,z) \end{pmatrix} + o(t),$$

we obtain formulæ of the maps (3.2) and (3.3), as

$$\begin{aligned} \alpha(Y, Z) &= -2(z,y) H_o \mod \mathfrak{so}(n, \mathbb{C}); \\ \beta(Y, Z) &= 2(z,y) E_z - Q_n(z) \sum_{j=1}^n y_j \frac{\partial}{\partial z_j}, \end{aligned}$$

where we regard $\beta(Y, \cdot)$ as a holomorphic vector field on $\mathfrak{n}_-$ and recall that $E_z := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$, $Q_n(z) = z_1^2 + \cdots + z_n^2$ and $(z, y) = z_1 y_1 + \cdots + z_n y_n$.

Then the infinitesimal action $d\pi_{\lambda^*}(C_j)$ with

$$\lambda^* = \lambda^\vee \otimes \mathbb{C}_{2\rho} = -\lambda + n,$$

is given by

$$d\pi_{\lambda^*}(C_j) = 2(\lambda - n) z_j - 2 z_j E_z + Q_n(z) \frac{\partial}{\partial z_j}. \tag{6.8}$$

Lemma 6.5. *For $C \in \mathbb{C}^n \simeq \mathfrak{n}_+$ and $\zeta \in \mathbb{C}^n \simeq \mathfrak{n}_-$ one has,*

$$\widehat{d\pi_{\lambda^*}}(C_j) = 2\lambda \frac{\partial}{\partial \zeta_j} + 2E_\zeta \frac{\partial}{\partial \zeta_j} - \zeta_j \Delta^\zeta_{\mathbb{C}^n}, \qquad 1 \le j \le n,$$

where $E_\zeta := \sum_{i=1}^n \zeta_i \frac{\partial}{\partial \zeta_i}$ and $\Delta^\zeta_{\mathbb{C}^n} = \frac{\partial^2}{\partial \zeta_1^2} + \cdots \frac{\partial^2}{\partial \zeta_n^2}$.

Proof. According to Definition 3.1 we have $\widehat{z_j} = \frac{\partial}{\partial \zeta_j}$ and hence $\widehat{E_z} = -E_\zeta - n$. On the other hand, using the commutation relations of the Weyl algebra (see *e.g.* [KP14-1, (3.2)]) we get

$$\Delta^\zeta_{\mathbb{C}^n} \zeta_j = \zeta_j \Delta^\zeta_{\mathbb{C}^n} + 2\frac{\partial}{\partial \zeta_j}, \qquad \frac{\partial}{\partial \zeta_j} E_\zeta = E_\zeta \frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j}.$$

Thus the above formula for the algebraic Fourier transform $\widehat{d\pi_{\lambda^*}}(C_j)$ of the differential operator (6.8) follows. ☐

For Step 2 we apply Lemma 5.5 (2) and get the following.

Proposition 6.6. *Assume $\lambda > n-1$. If*

$$\mathrm{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_\lambda), \mathcal{O}(G'/K', \mathcal{W})) \neq \{0\}$$

for an irreducible representation W of K', then W must be one-dimensional and of the form

$$W^a_\lambda := S^a(\mathfrak{n}_-^{-\tau}) \otimes \mathbb{C}_\lambda \simeq \mathrm{Pol}^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_\lambda \tag{6.9}$$

for some $a \in \mathbb{N}$.

We denote by ν the action of K' on W^a_λ. In our setting where $\dim V = \dim W^a_\lambda = 1$ we write $\zeta = (\zeta', \zeta_n) \in \mathbb{C}^n$ with $\zeta' = (\zeta_1, \dots, \zeta_{n-1}) \in \mathbb{C}^{n-1}$, and identify an element of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes W^a_\lambda)$ with a polynomial $\psi(\zeta)$ of n variables. Then, for Step 3, the condition (3.10) implies that $\psi(\zeta)$ is homogeneous of degree a and the condition (3.11) amounts to the system of differential equations:

$$\widehat{d\pi_{\lambda^*}}(C_j)\psi = \left(2\lambda \frac{\partial}{\partial \zeta_j} + 2E_\zeta \frac{\partial}{\partial \zeta_j} - \zeta_j \Delta^\zeta_{\mathbb{C}^n}\right)\psi = 0, \qquad 1 \le j \le n-1$$

by Lemma 6.5.

To be prepared for Step 4, observe that the $K'_{\mathbb{C}}$-action on $\mathfrak{n}_- = \mathfrak{n}_-^\tau \oplus \mathfrak{n}_-^{-\tau}$ is identified with the action of $SO(n-1, \mathbb{C}) \times SO(2, \mathbb{C})$ on $\mathbb{C}^n$ given as

$$\mathbb{C}^n \boxtimes \mathbb{C}_{-1} \simeq (\mathbb{C}^{n-1} \boxtimes \mathbb{C}_{-1}) \oplus (\mathbb{C} \boxtimes \mathbb{C}_{-1}).$$

Then generic $K'_{\mathbb{C}}$-orbits are of codimension one in $\mathfrak{n}_-$, and the $K'_{\mathbb{C}}$-orbit space in $\{\zeta \in \mathbb{C}^n : Q_{n-1}(\zeta') \neq 0\}$ has coordinates $\dfrac{\zeta_n^2}{Q_{n-1}(\zeta')}$.

For $a \in \mathbb{N}$, we introduce an operator T_a by

$$(T_a g)(\zeta) := Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right), \tag{6.10}$$

for $g \in \mathbb{C}[t]$. We note that $T_a g$ is a (multi-valued) meromorphic function of $\zeta_1, \dots, \zeta_n$.

We set

$$\mathrm{Pol}_a[t] := \mathbb{C}\text{-span}\left\langle t^{a-i} : 0 \le i \le a\right\rangle, \tag{6.11}$$

$$\mathrm{Pol}_a[t]_{\mathrm{even}} := \mathbb{C}\text{-span}\left\langle t^{a-2j} : 0 \le j \le \left[\frac{a}{2}\right]\right\rangle. \tag{6.12}$$

Then $(T_a g)(\zeta)$ is a homogeneous polynomial of degree a if $g \in \mathrm{Pol}_a[t]_{\mathrm{even}}$.

Remark 6.7. In this section we have assumed $n \ge 3$, and therefore $Q_{n-1}(\zeta')^{\frac{1}{2}} = (\zeta_1^2 + \cdots + \zeta_{n-1}^2)^{\frac{1}{2}}$ is not a polynomial and the parity condition in (6.12) is necessary. However, for $n = 2$, $T_a g$ is a polynomial for $g \in \mathrm{Pol}_a[t]$ as we can take a branch as $Q_1(\zeta')^{\frac{1}{2}} = \zeta_1$.

The first half of Step 4 is summarized in the following lemma:

Lemma 6.8. *For $n \ge 3$ we have,*

$$\mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \simeq \begin{cases} \{0\} & \text{if } \nu - \lambda \notin \mathbb{N}, \\ T_{\nu-\lambda}(\mathrm{Pol}_{\nu-\lambda}[t]_{\mathrm{even}}) & \text{if } \nu - \lambda \in \mathbb{N}. \end{cases}$$

Proof. As modules of $\mathfrak{k}' = \mathfrak{so}(n-1, \mathbb{C}) \oplus \mathfrak{so}(2, \mathbb{C})$, we have the following isomorphisms:

$$\mathrm{Pol}(\mathfrak{n}_+) \simeq S(\mathfrak{n}_-) \simeq \bigoplus_{a_1, a_2 \in \mathbb{N}} S^{a_1}(\mathfrak{n}_-^{\tau}) \otimes S^{a_2}(\mathfrak{n}_-^{-\tau}) \simeq \bigoplus_{a=0}^{\infty} \bigoplus_{a_1=0}^{a} S^{a_1}(\mathbb{C}^{n-1}) \boxtimes \mathbb{C}_{-a}.$$

Therefore

$$\mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \simeq \bigoplus_{a=0}^{\infty} \bigoplus_{a_1=0}^{a} \left(S^{a_1}(\mathbb{C}^{n-1})\right)^{SO(n-1,\mathbb{C})} \boxtimes \left(\mathbb{C}_{\nu-a-\lambda}\right)^{SO(2,\mathbb{C})}.$$

The right-hand side is non-zero only when $\nu - \lambda \in \mathbb{N}$. In this case the summand is non-trivial only when $a = \nu - \lambda$. On the other hand, since $n \ge 3$, we have

$$S^{a_1}(\mathbb{C}^{n-1})^{SO(n-1,\mathbb{C})} \simeq \begin{cases} \mathbb{C} Q_{n-1}(\zeta')^{\frac{a_1}{2}} & \text{if } a_1 \in 2\mathbb{N}, \\ 0 & \text{if } a_1 \notin 2\mathbb{N}. \end{cases}$$

Hence the lemma follows. ☐

To implement the second part of Step 4 we apply Proposition 3.3 to the map (6.10). For this we collect some formulæ for saturated differential operators that we shall use later.

Lemma 6.9. *For every* $0 \leq j \leq n-1$ *one has:*

$$T_a^\sharp\left(\zeta_j E_{\zeta'} - Q_{n-1}(\zeta')\frac{\partial}{\partial \zeta_j}\right) = 0, \tag{6.13}$$

$$T_a^\sharp\left((a-1)\zeta_n - E_{\zeta}\frac{\partial}{\partial \zeta_j}\right) = 0. \tag{6.14}$$

Proof. The proof of both statements is straightforward from the definition of T_a. ☐

Lemma 6.10. *Let* T_a *be the operator defined in* (6.10). *We write* $\zeta' = (\zeta_1, \cdots, \zeta_{n-1})$ *and* $\vartheta_t := t\frac{d}{dt}$. *One then has:*

(1) $T_a^\sharp(E_{\zeta'}) = a - \vartheta_t$.
(2) $T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\frac{\partial}{\partial \zeta_j}\right) = a - \vartheta_t$, $(1 \leq j \leq n-1)$.
(3) $T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}E_\zeta\frac{\partial}{\partial \zeta_j}\right) = (a-1)(a-\vartheta_t)$, $(1 \leq j \leq n-1)$.
(4) $T_a^\sharp(\zeta_n^2\Delta^\zeta_{\mathbb{C}^{n-1}}) = t^2(\vartheta_t - a)(\vartheta_t - n - a + 3)$.
(5) $T_a^\sharp(Q_{n-1}(\zeta')\Delta^\zeta_{\mathbb{C}^{n-1}}) = (\vartheta_t - a)(\vartheta_t - n - a + 3)$.
(6) $T_a^\sharp(Q_{n-1}(\zeta')\frac{\partial^2}{\partial \zeta_n^2}) = t^{-2}(\vartheta_t^2 - \vartheta_t)$.
(7) $T_a^\sharp(\zeta_n\frac{\partial}{\partial \zeta_n}) = \vartheta_t$.
(8) $T_a^\sharp(\zeta_n^2\frac{\partial^2}{\partial \zeta_n^2}) = \vartheta_t^2 - \vartheta_t$.

Proof. Notice first that the identity (1) is equivalent to (2) according to (6.13) and that the identity (3) may be deduced from (1) or (2) by (6.14). Furthermore, identities (4) and (5) on the one hand and (6) and (8) on the other are equivalent according to the definition of the T-saturation as $t = \frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}$.

Thus, it would be enough to show the identities (1), (4), (7) and (8). We give a proof for the first statement, and the remaining cases can be treated in a similar way. Let $1 \leq j \leq n-1$. Then

$$\begin{aligned}
\left(T_a^\sharp(E_{\zeta'})g\right)(t) &= \sum_{j=1}^{n-1}\zeta_j\frac{\partial}{\partial \zeta_j}\left(Q_{n-1}(\zeta')^{\frac{a}{2}}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\right)\\
&= aQ_{n-1}(\zeta')^{\frac{a}{2}-1}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\sum_{j=1}^{n-1}\zeta_j^2 - Q_{n-1}(\zeta')^{\frac{a}{2}}g'\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\sum_{j=1}^{n-1}\frac{\zeta_j^2\zeta_n}{\sqrt{Q_{n-1}^3(\zeta')}}\\
&= aQ_{n-1}(\zeta')^{\frac{a}{2}}g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right) - \frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}Q_{n-1}(\zeta')^{\frac{a}{2}}g'\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)\\
&= \left(a - t\frac{d}{dt}\right)g(t).
\end{aligned}$$

☐

For the second half of Step 4 we apply the idea of T-saturated differential operators (see Definition 3.2). Although the differential operator $\widehat{d\pi_{\lambda^*}}(C_j)$ itself is not T_a-saturated, we shall see that $Q_j\widehat{d\pi_{\lambda^*}}(C_j)$ is T_a-saturated if we set $Q_j = \zeta_j^{-1}Q_{n-1}(\zeta')$. In the following lemma, we note that the right-hand side is independent of j.

Lemma 6.11. *The T_a-saturation of the differential operators $\widehat{d\pi_{\lambda^*}}(C_j)$ with $C_j \in \mathfrak{n}_+^\tau$ is given for any $1 \le j \le n-1$ by*

$$T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\widehat{d\pi_{\lambda^*}}(C_j)\right) = \frac{-1}{t^2}\left((1+t^2)\vartheta_t^2 - (1-(2\lambda-n+1)t^2)\vartheta_t - a(a+2\lambda-n+1)t^2\right).$$

Proof. Suppose $1 \le j \le n-1$. Applying (2), (3) and (5), (6) of Lemma 6.10, respectively, we have following identities:

$$\begin{aligned} T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\frac{\partial}{\partial\zeta_j}\right) &= a - \theta_t, \\ T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}E_\zeta\frac{\partial}{\partial\zeta_j}\right) &= (a-1)(a-\vartheta_t), \\ T_a^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_j}\zeta_j\Delta_{\mathbb{C}^n}^\zeta\right) &= T_a^\sharp\left(Q_{n-1}(\zeta')\left(\Delta_{\mathbb{C}^{n-1}}^\zeta + \frac{\partial^2}{\partial\zeta_n^2}\right)\right) \\ &= (\vartheta_t - a)(\vartheta_t - n + 3 - a) + t^{-2}(\vartheta_t^2 - \vartheta_t). \end{aligned}$$

We recall from Lemma 6.5 that $\widehat{d\pi_{\lambda^*}}(C_j) = 2\lambda\frac{\partial}{\partial\zeta_j} + 2E_\zeta\frac{\partial}{\partial\zeta_j} - \zeta_j\Delta_{\mathbb{C}^n}^\zeta$. Summing up these terms we get the lemma. ☐

Proposition 6.12. *Let $a \in \mathbb{N}$, and T_a be as in* (6.10). *The polynomial $\psi(\zeta) = (T_a g)(\zeta)$ of n variables satisfies the system of partial differential equations* (3.11) *if and only if $g(t)$ satisfies the following single ordinary differential equation:*

$$\left((1-s^2)\vartheta_s^2 - (1+(2\lambda-n+1)s^2)\vartheta_s + a(a+2\lambda-n+1)s^2\right)g(-\sqrt{-1}s) = 0, \tag{6.15}$$

or equivalently, $g(t)$ is proportional to the normalized Gegenbauer polynomial $\widetilde{C}_a^{\lambda-\frac{n-1}{2}}(\sqrt{-1}t)$. (For the Gegenbauer polynomial, see Section 11.3.)

Proof. The statement follows from Lemma 6.11 after the change of variable $t = -\sqrt{-1}s$. ☐

We have carried out the crucial part of the F-method. Let us complete the proof of Theorems 6.1 and 6.3.

Proof of Theorems 6.1 and 6.3. By the general result of the F-method (see Theorem 3.1), the symbol map of differential operators gives an isomorphism

$$\operatorname{Hom}_{\widetilde{G}'}(\mathcal{O}(X,\mathcal{L}_\lambda),\mathcal{O}(Y,\mathcal{L}_\nu)) \overset{\text{Symb}}{\underset{\sim}{\to}} \operatorname{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \operatorname{Pol}(\mathfrak{n}_+)\otimes\mathbb{C}_\nu)^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}_+')}.$$

By Lemma 6.8, the right-hand side is reduced to zero if $\nu - \lambda \notin \mathbb{N}$. From now on, we assume $a := \nu - \lambda \in \mathbb{N}$, and identify the right-hand side with a subspace of $\mathrm{Pol}(\mathfrak{n}_+)$. Then it follows from Lemma 6.8 and Proposition (6.12) that the bijections

$$\begin{array}{ccccc} \mathrm{Pol}_a[s]_{\mathrm{even}} & \overset{T_a}{\xrightarrow{\sim}} & \mathrm{Pol}_a[t]_{\mathrm{even}} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu) \\ h(s) & \mapsto & g(t) = h(\sqrt{-1}t) & \mapsto & Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\dfrac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right) \end{array}$$

induces an isomorphism

$$\mathrm{Sol}_{\mathrm{Gegen}}\left(\lambda - \frac{n-1}{2}, a\right) \cap \mathrm{Pol}_a[s]_{\mathrm{even}} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{k}'}(\mathbb{C}_\lambda, \mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}_\nu)^{\widehat{d\pi_{\lambda^*}(\mathfrak{n}'_+)}}.$$

Since the left-hand side is always one-dimensional (see Theorem 11.4 in Appendix), the first statement follows.

Furthermore, since $\mathrm{Sol}_{\mathrm{Gegen}}\left(\lambda - \frac{n-1}{2}, a\right) \cap \mathrm{Pol}_a[s]_{\mathrm{even}}$ is spanned by $\widetilde{C}_a^{\lambda - \frac{n-1}{2}}(s)$ by Theorem (11.4) (2), the space $\mathrm{Hom}_{\widetilde{G'}}(\mathcal{O}(X, \mathcal{L}_\lambda), \mathcal{O}(Y, \mathcal{L}_\nu))$ is spanned by

$$\mathrm{Symb}^{-1} \circ T_a \widetilde{C}_a^{\lambda - \frac{n-1}{2}}(\sqrt{-1}t) = (-1)^{-\frac{a}{2}} \widetilde{C}_a^{\lambda - \frac{n-1}{2}}\left(-\Delta^z_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right).$$

Hence Theorems 6.1 and 6.3 are proved. ☐

Remark 6.13. Theorem 6.3 is a "holomorphic version" of the conformally covariant operator considered by A. Juhl [J09] in the setting $S^{n-1} \hookrightarrow S^n$, with equivariant actions of the pair of groups $SO(n,1) \subset SO(n+1,1)$, respectively. Our proof based on the F-method is much shorter than the original proof in [J09, Chapter 6] that relies on combinatorial argument using recurrence relations of the coefficients of differential operators. The F-method gives a conceptual explanation for the appearance of Gegenbauer polynomials in Theorem 6.3. The relationship of symmetry breaking operators between real flag varieties (*e.g.* [J09, KØSS13]) and the holomorphic setting is illustrated by an SL_2-example in [KKP15].

7. Symmetry breaking operators for the restriction $Sp(n, \mathbb{R}) \downarrow Sp(n-1, \mathbb{R}) \times Sp(1, \mathbb{R})$

Let $n \geq 2$. In what follows, we realize the real symplectic group $G = Sp(n, \mathbb{R})$ as a subgroup of the indefinite unitary group $U(n,n)$, so that we can directly apply the computation of $d\pi_{\lambda^*}(C)$ $(C \in \mathfrak{n}_+)$ in [KP14-1, Example 3.7].

Let $G_{\mathbb{C}}$ be the complex symplectic group $Sp(n, \mathbb{C})$ which preserves the standard symplectic form ω defined on $\mathbb{C}^{2n}$ by

$$\omega(u, v) := {}^t u J_n v, \quad \text{for } u, v \in \mathbb{C}^{2n},$$

where $J_n := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Let $E(\mathbb{R}) := \left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} : z \in \mathbb{C}^n \right\}$ be a totally real vector subspace of $\mathbb{C}^{2n}$, and we set

$$G := GL_{\mathbb{R}}(E(\mathbb{R})) \cap Sp(n, \mathbb{C}) \simeq Sp(n, \mathbb{R}).$$

Then the Lie algebra $\mathfrak{g}(\mathbb{R}) \simeq \mathfrak{sp}(n, \mathbb{R})$ of G is given by

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{gl}_{\mathbb{R}}(E(\mathbb{R})) \cap \mathfrak{sp}(n, \mathbb{C}) = \left\{ \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} : A = -{}^t\overline{A}, B \in \operatorname{Sym}(n, \mathbb{C}) \right\},$$

where we recall that $\operatorname{Sym}(n, \mathbb{C})$ is the space of complex symmetric matrices.

Let $H_n := \{Z \in \operatorname{Sym}(n, \mathbb{C}) : \|Z\|_{\mathrm{op}} < 1\}$ be the bounded symmetric domain of type CI in the É. Cartan classification, where $\|Z\|_{\mathrm{op}}$ denotes the operator norm of $Z \in \operatorname{End}(\mathbb{C}^n)$. The Lie group $G = Sp(n, \mathbb{R})$ acts biholomorphically on H_n by

$$g \cdot Z = (aZ + b)(cZ + d)^{-1} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \ Z \in H_n.$$

The isotropy subgroup K of G at the origin 0 is identified with $U(n)$ by the isomorphism:

$$K \overset{\sim}{\to} U(n), \quad \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \mapsto A.$$

We write $\widetilde{G}$ for the universal covering of G, and $\widetilde{K}$ for the connected subgroup with Lie algebra $\mathfrak{k}(\mathbb{R})$.

Let G' be the subgroup of $G = Sp(n, \mathbb{R})$ that preserves the direct sum decomposition $E(\mathbb{R}) \simeq \mathbb{R}^{2n} = \mathbb{R}^{2n-2} \oplus \mathbb{R}^2$ in the standard coordinates. Then G' is isomorphic to the connected group $Sp(n-1, \mathbb{R}) \times Sp(1, \mathbb{R})$. The pair (G, G') is a symmetric pair as G' is the fixed point subgroup of an involution τ of G defined by

$$\tau(g) = \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & I_{n-1,1} \end{pmatrix} g \begin{pmatrix} I_{n-1,1} & 0 \\ 0 & I_{n-1,1} \end{pmatrix},$$

where $I_{n-1,1} = \operatorname{diag}(1, \cdots, 1, -1)$.

We set $X := H_n \simeq G/K$ and $Y := X \cap \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \operatorname{Sym}(n-1, \mathbb{C}), d \in \mathbb{C} \right\} \simeq H_{n-1} \times H_1 \simeq G'/K'$. The symmetric pair (G, G') is of holomorphic type, and the embedding of the complex manifold $Y \hookrightarrow X$ is G'-equivariant.

Let $\mathfrak{j}$ be the standard Cartan subalgebra $\sum_{i=1}^n \mathbb{C}(E_{ii} - E_{n+i,n+i})$ of $\mathfrak{k}$, and $\{e_1, \cdots, e_n\}$ the standard basis. Then $\mathfrak{j}$ is a Cartan subalgebra of $\mathfrak{g}$ and we choose $\Delta^+(\mathfrak{k}, \mathfrak{j}) = \{e_i - e_j : 1 \le i < j \le n\}$ and $\Delta(\mathfrak{n}_+, \mathfrak{j}) = \{-(e_i + e_j) : 1 \le i \le j \le n\}$ so that $\rho_{\mathfrak{g}} = (-1, -2, \cdots, -n)$.

Then we have the following decomposition of the Lie algebra

$$\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C}) = \mathfrak{n}_- + \mathfrak{k} + \mathfrak{n}_+, \quad \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \mapsto (B, A, C)$$

with $B = {}^tB$ and $C = {}^tC$. Here we have chosen a realization of $\mathfrak{n}_+$ in the *lower* triangular matrices. Accordingly, we adopt the following notation for characters of $\mathfrak{k} \simeq \mathfrak{gl}_n(\mathbb{C})$: for $\lambda \in \mathbb{C}$ the character $\mathbb{C}_\lambda$ of $\mathfrak{k}$ is defined by:

$$\mathfrak{k} \longrightarrow \mathbb{C}, \quad \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \mapsto -\lambda \operatorname{Trace} A.$$

Its restriction to $\mathfrak{j}$ is given by $(-\lambda, \cdots, -\lambda) \in \mathfrak{j}^\vee \simeq \mathbb{C}^n$.

For $\lambda \in \mathbb{C}$, the character $\mathbb{C}_\lambda$ lifts to $\widetilde{K}$ and defines a $\widetilde{G}$-equivariant holomorphic line bundle $\mathcal{L}_\lambda$ over $X = \widetilde{G}/\widetilde{K} \simeq G/K$. It descends to a G-equivariant bundle if $\lambda \in \mathbb{Z}$. In our parametrization, $\mathcal{L}_{n+1}$ is the canonical line bundle of $X = G/K$, namely, $\mathbb{C}_{2\rho} = \mathbb{C}_{n+1}$.

We shall construct differential symmetry breaking operators from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_Y)$ where $\mathcal{W}_Y$ is a G'-equivariant holomorphic vector bundle over Y. Unlike in the previous section, we have to deal with vector bundles rather than line bundles because, by Proposition 7.4 below, there exists a non-trivial G'-intertwining operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_Y)$ only if $\dim W > 1$ for generic λ except for the case when $\mathcal{W}_Y = \mathcal{L}_\lambda|_Y$ or $n = 2$.

More precisely, such an irreducible representation W of $\mathfrak{k}' \simeq \mathfrak{gl}_{n-1}(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ must be isomorphic to

$$W_\lambda^a = F(\mathfrak{gl}_{n-1}(\mathbb{C}), (-\lambda, \cdots, -\lambda, -\lambda - a)) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), (-\lambda - a)e_n), \tag{7.1}$$

for some $a \in \mathbb{N}$. This is a representation of $K' = GL(n-1,\mathbb{C}) \times GL(1,\mathbb{C})$ on the space $\operatorname{Pol}^a[v_1, \cdots, v_{n-1}]$ of homogeneous polynomials of degree a on $\mathbb{C}^{n-1}$ twisted by the one-dimensional representation $(\det_{n-1})^{-\lambda}(\det_1)^{-\lambda-a}$ of K' where $\det_k A$ denotes the determinant of $A \in M(k,\mathbb{C})$.

In order to give a concrete model for the natural action of G on $\mathcal{O}(X, \mathcal{V})$ consider an irreducible representation ν of $U(m)$ with highest weight $(\nu_1, \cdots, \nu_m)$ acting on a finite-dimensional complex vector space W. We extend it into a holomorphic representation denoted by the same letter ν of $GL(m,\mathbb{C})$ on W. Then the holomorphic vector bundle $\mathcal{W} = Sp(m,\mathbb{R}) \times_{U(m)} W$ over H_m is trivialized using the open Bruhat cell, and the regular representation of $Sp(m,\mathbb{R})$ on $\mathcal{O}(H_m, \mathcal{W})$ is identified with the multiplier representation of the same group on $\mathcal{O}(H_m) \otimes W$ given by

$$\left(\pi_{(\nu_1,\cdots,\nu_m)}^{Sp(m,\mathbb{R})}(g)F\right)(Z) = \nu\left({}^t(cZ+d)\right) F\left((aZ+b)(cZ+d)^{-1}\right),$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m,\mathbb{R})$, $Z \in H_m$. For $\lambda \in \mathbb{Z}$, the one-dimensional representation $\mathbb{C}_\lambda$ of K has a highest weight $(-\lambda, \cdots, -\lambda)$ and we shall simply write $\pi_\lambda^{Sp(m,\mathbb{R})}$ for the representation $\pi_{(-\lambda,\cdots,-\lambda)}^{Sp(m,\mathbb{R})}$ of $Sp(m,\mathbb{R})$ on $\mathcal{O}(H_m)$ given by

$$\left(\pi_\lambda^{Sp(m,\mathbb{R})}(g)F\right)(Z) = \det(cZ+d)^{-\lambda} F\left((aZ+b)(cZ+d)^{-1}\right),$$

for $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m,\mathbb{R})$, $Z \in H_m$. For $\lambda \in \mathbb{C}$, it gives a representation of $\widetilde{Sp(m,\mathbb{R})}$ on the same space $\mathcal{O}(H_m)$. Similarly, for $a \in \mathbb{N}$, we denote by $\pi_{\lambda,a}^{Sp(m,\mathbb{R})}$ the representation $\pi_{(0,\cdots,0,-a)+(-\lambda,\cdots,-\lambda)}^{Sp(m,\mathbb{R})}$ of the same group on $\mathcal{O}(H_m) \otimes \mathrm{Pol}^a[v_1, \cdots, v_m]$.

The representation W_λ^a may be realized on the space $\mathrm{Pol}^a[v_1, \cdots, v_{n-1}]$ where $(v_1, \cdots, v_{n-1})$ are the standard coordinates on $\mathfrak{n}_-^{-\tau} \simeq \mathbb{C}^{n-1}$. Hence, the differential symmetry breaking operators can be thought of as elements of $\mathbb{C}\left[\frac{\partial}{\partial z_{ij}}\right] \otimes \mathrm{Pol}^a[v_1, \ldots, v_{n-1}]$, where z_{ij} $(1 \le i, j \le n)$ are the standard coordinates on $\mathfrak{n}_- \simeq \mathrm{Sym}(n,\mathbb{C})$.

Theorem 7.1. *Let $n \ge 2$. Suppose $\lambda \in \mathbb{C}$ and $a \in \mathbb{N}$.*

(1) *The vector space*

$$\mathrm{Hom}_{\widetilde{Sp(n-1,\mathbb{R})\times Sp(1,\mathbb{R})}}(\mathcal{O}(H_n, \mathcal{L}_\lambda), \mathcal{O}(H_{n-1} \times H_1, \mathcal{W}_\lambda^a))$$

is one-dimensional.

(2) *The vector-valued differential operator from $\mathcal{O}(X)$ to $\mathcal{O}(Y) \otimes W$ defined by* (7.2)

$$D_{X\to Y,a} := \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_i v_j \frac{\partial^2}{\partial z_{ij}\partial z_{nn}}, \sum_{1\le j\le n-1} v_j \frac{\partial}{\partial z_{jn}}\right) \in \mathbb{C}\left[\frac{\partial}{\partial z_{ij}}\right] \otimes \mathrm{Pol}^a[v_1, \cdots, v_{n-1}]$$

intertwines the restriction $\pi_\lambda^{Sp(n,\mathbb{R})}\Big|_{Sp(n-1,\mathbb{R})\times Sp(1,\mathbb{R})}$ and $\pi_{\lambda,a}^{Sp(n-1,\mathbb{R})} \boxtimes \pi_{\lambda+a}^{Sp(1,\mathbb{R})}$. Here the polynomial $\widetilde{C}_a^{\lambda-1}(x,y)$ is the inflated normalized Gegenbauer polynomial defined in (6.5).

It follows from Theorem 7.1 that any symmetry breaking operator from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W}_\lambda^a)$ is proportional to $D_{X\to Y,a}$.

Remark 7.2. If $\lambda > n$ then $\mathcal{H}^2(X, \mathcal{L}_\lambda) := \mathcal{O}(X, \mathcal{L}_\lambda) \cap L^2(X, \mathcal{L}_\lambda)$ is a non-zero Hilbert space on which G acts unitarily and irreducibly. Then, $\mathcal{H}^2(Y, \mathcal{W}_\lambda^a) := \mathcal{O}(Y, \mathcal{W}_\lambda^a) \cap L^2(Y, \mathcal{W}_\lambda^a) \ne \{0\}$ for any $a \in \mathbb{N}$, and the same statements as in Theorem 7.1 remain true for symmetry breaking operators between the representation spaces $\mathcal{H}^2(X, \mathcal{L}_\lambda)$ and $\mathcal{H}^2(Y, \mathcal{W}_\lambda^a)$.

In order to prove Theorem 7.1 we apply the F-method. Its Step 1 is given by

Lemma 7.3. *For* $\lambda \in \mathbb{C}$*, we set* $\lambda^* = \lambda^\vee \otimes \mathbb{C}_{2\rho} = -\lambda + n + 1$*. For* $C \in Sym(n, \mathbb{C}) \simeq \mathfrak{n}_+$ *and* $Z \in Sym(n, \mathbb{C}) \simeq \mathfrak{n}_-$ *we have*

$$\begin{aligned} d\pi_{\lambda^*}(C) &= (-\lambda + n + 1)\operatorname{Trace}(CZ) + \sum_{i\le j}\sum_{k,\ell} C_{k\ell} z_{ik} z_{j\ell} \frac{\partial}{\partial z_{ij}}, \\ \widehat{d\pi_{\lambda^*}}(C) &= -\lambda \sum_{i\le j} C_{ij} \frac{\partial}{\partial \zeta_{ij}} - \frac{1}{2}\left(\sum_{i\le k, j\le \ell} C_{k\ell}\zeta_{ij} \frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} + \sum_{i\ge k, j\ge \ell} C_{k\ell}\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} \right). \end{aligned}$$

Proof. We embed the group $Sp(n, \mathbb{R})$ into $U(n, n)$ and apply the results of [KP14-1, Example 3.7] with $p = q = n$. Thus, the first statement follows from the formula (3.4).

We consider a bilinear form

$$\mathfrak{n}_+ \times \mathfrak{n}_- \to \mathbb{C}, \qquad (C, Z) \mapsto \operatorname{Trace}(C\,{}^tZ),$$

where $\mathfrak{n}_+ \simeq \mathrm{Sym}(n, \mathbb{C}) \simeq \mathfrak{n}_-$. Recall that ζ_{ij} with $1 \le i \le j \le n$ are the coordinates on $\mathfrak{n}_+ \simeq \mathrm{Sym}(n, \mathbb{C})$. However, it is convenient for the computations below to allow us to use $\frac{\partial}{\partial\zeta_{ij}}$ $(i > j)$ for the same meaning with $\frac{\partial}{\partial\zeta_{ji}}$. Then

$$\widehat{z_{ij}} = \frac{1}{2}(1 + \delta_{ij})\frac{\partial}{\partial\zeta_{ij}}, \quad \widehat{\frac{\partial}{\partial z_{ij}}} = (\delta_{ij} - 2)\zeta_{ij}.$$

Thus the algebraic Fourier transform of the first term of $d\pi_{\lambda^*}(C)$ amounts to

$$(\operatorname{Trace}(CZ))^{\widehat{}} = \frac{1}{2}\sum_{i,j} C_{ij}(1 + \delta_{ij})\frac{\partial}{\partial\zeta_{ij}} = \sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}},$$

whereas that of the second term of $d\pi_{\lambda^*}(C)$ amounts to

$$\begin{aligned} &\widehat{\left(\sum_{i\le j}\sum_{k,\ell} C_{k\ell} z_{ik} z_{j\ell}\frac{\partial}{\partial z_{ij}} \right)} = -(n+1)\sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}} - \frac{1}{4}\sum_{i,j,k,l} C_{kl}(1+\delta_{ik})(1+\delta_{jl})\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} \\ &= -(n+1)\sum_{i\le j} C_{ij}\frac{\partial}{\partial\zeta_{ij}} - \frac{1}{2}\left(\sum_{i\le k, j\le\ell} C_{k\ell}\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} + \sum_{i\ge k, j\ge\ell} C_{k\ell}\zeta_{ij}\frac{\partial^2}{\partial\zeta_{ik}\partial\zeta_{j\ell}} \right). \end{aligned}$$

Hence the formula for $\widehat{d\pi_{\lambda^*}}(C)$ follows. □

The condition (4.5) amounts to $\langle(-\lambda+1, \cdots, -\lambda+n), -(e_i + e_j)\rangle > 0$ for any $1 \le i \le j \le n$, namely $\lambda > n$.

For the Step 2 we apply Lemma 5.5.

Proposition 7.4. *Assume* $\lambda > n$*. If*

$$\operatorname{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_\lambda), \mathcal{O}(G'/K', \mathcal{W})) \ne \{0\}$$

for an irreducible representation W *of* K'*, then* W *is of the form*

$$W = W_\lambda^a = S^a(\mathfrak{n}_+^{-\tau}) \otimes (-\lambda \operatorname{Trace}_n),$$

for some $a \in \mathbb{N}$ *see* (7.1).

From now on, we aim to construct (differential) symmetry breaking operators from $\mathcal{O}(X, \mathcal{L}_\lambda)$ to $\mathcal{O}(Y, \mathcal{W})$ in the case $W = W^a_\lambda$.

Define a Borel subalgebra $\mathfrak{b}(\mathfrak{k}')$ corresponding to the positive root system $\Delta^+(\mathfrak{k}', \mathfrak{j}) := \Delta^+(\mathfrak{k}, \mathfrak{j}) \cap \Delta(\mathfrak{k}', \mathfrak{j})$.

For Step 3 we apply Lemma 3.4 and we get:

Lemma 7.5. *Let* W^a_λ *be the irreducible* $\mathfrak{k}'$*-module defined in* (7.1).

(1) *The highest weight of* $(W^a_\lambda)^\vee$ *is given by*

$$\chi = (a, 0, \dots, 0; a) + (\lambda, \dots, \lambda; \lambda).$$

(2) *For the* $\mathfrak{k}$*-module* $\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}^\vee_\lambda$*, the* χ*-weight space for* $\mathfrak{b}(\mathfrak{k}')$ *is given by:*

$$(\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}^\vee_\lambda)_\chi \simeq \bigoplus_{2j+k=a} \mathbb{C}\zeta_{11}^j \zeta_{1n}^k \zeta_{nn}^j, \tag{7.3}$$

where we identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes \mathbb{C}^\vee_\lambda$ *with* $\mathrm{Pol}(\mathfrak{n}_+)$ *as vector spaces.*

Proof. The statement (1) is clear from the definition of W^a_λ given in (7.1). Notice that in our convention $\Delta(\mathfrak{n}_-)$ is given as $\Delta(\mathfrak{n}_-) = \{e_i + e_j : 1 \le i \le j \le n\}$. Thus $\mathfrak{n}_-$ decomposes into irreducible representations of $\mathfrak{k}'$ as

$$\begin{aligned} \mathfrak{n}_- &\simeq (\mathrm{Sym}(n-1), \mathbb{C}) \boxtimes \mathbb{C}) \oplus (\mathbb{C} \boxtimes \mathbb{C}_2) \oplus (\mathbb{C}^{n-1} \boxtimes \mathbb{C}_1) \\ &\simeq (F(\mathfrak{gl}_{n-1}, 2e_1) \boxtimes F(\mathfrak{gl}_1, 0)) \oplus (F(\mathfrak{gl}_{n-1}, 0) \boxtimes F(\mathfrak{gl}_1, 2e_n)) \\ &\quad \oplus (F(\mathfrak{gl}_{n-1}, e_1) \boxtimes F(\mathfrak{gl}_1, e_n)). \end{aligned} \tag{7.4}$$

Accordingly we get an isomorphism of $\mathfrak{k}'$-modules:

$$\mathrm{Pol}(\mathfrak{n}_+) \simeq S(\mathfrak{n}_-) \simeq \bigoplus_{i,j,k} \left(S^i(\mathrm{Sym}(n-1), \mathbb{C})) \otimes S^k(\mathbb{C}^{n-1})\right) \boxtimes \mathbb{C}_{2j+k}. \tag{7.5}$$

Since ζ_{11}, ζ_{nn} and ζ_{1n} are highest weight vectors in the $\mathfrak{k}'$-module $\mathfrak{n}_-$ with respect to $\Delta^+(\mathfrak{k}')$ (see (7.4)), so is any monomial $\zeta_{11}^i \zeta_{nn}^j \zeta_{1n}^k$ in the $\mathfrak{k}'$-module $S(\mathfrak{n}_-) \simeq \mathrm{Pol}(\mathfrak{n}_+)$ of weight $(2i+k)e_1 + (k+2j)e_n$.

According to the irreducible decomposition (7.5) and Remark 5.7, it follows that the right-hand side of (7.3) exhausts all highest weight vectors in $\mathrm{Pol}(\mathfrak{n}_+)$ of weight $a(e_1 + e_n)$. Thus, taking into account the $\mathfrak{k}'$-action on $\mathbb{C}^\vee_\lambda \simeq \lambda\,\mathrm{Trace}_n$, we get Lemma. □

As Step 4, we reduce the system of differential equations (3.11), *i.e.* $\widehat{d\pi_{\lambda^*}}(C)\psi = 0$, to an ordinary differential equation. For this, we identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ with the space of polynomials in ζ on $\mathfrak{n}_+ \simeq \mathrm{Sym}(n, \mathbb{C})$. For a polynomial $g(t) \in \mathrm{Pol}_a[t]_{\mathrm{even}}$ (see (6.12)) we set

$$(T_a g)(\zeta) := (\sqrt{2\zeta_{11}\zeta_{nn}})^a g\left(\frac{\zeta_{1n}}{\sqrt{2\zeta_{11}\zeta_{nn}}}\right).$$

Proposition 7.6. *Let* χ *be as in Lemma 7.5 (1).*

(1) $T_a : \mathrm{Pol}_a[t]_{\mathrm{even}} \overset{\sim}{\to} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi$.

(2) *The map* T_a *induces an isomorphism*

$$\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - 1, a) \cap \mathrm{Pol}_a[t]_{\mathrm{even}} \overset{\sim}{\to} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi^{\widehat{d\pi_{\lambda^*}}(\mathfrak{n}'_+)}.$$

(3) *Any polynomial* $\psi(\zeta) \equiv \psi(\zeta_{ij})$ *in the right-hand side of (7.3) is given by*

$$\psi(\zeta) = (T_a g)(\zeta) := (\sqrt{2\zeta_{11}\zeta_{nn}})^a g\left(\frac{\zeta_{1n}}{\sqrt{2\zeta_{11}\zeta_{nn}}}\right), \tag{7.6}$$

for some $g(t) \in \mathrm{Pol}_a[t]_{\mathrm{even}}$.

(4) *The polynomial* $\psi(\zeta)$ *on* $\mathrm{Sym}(n, \mathbb{C})$ *satisfies the system of partial differential equations* $\widehat{d\pi_{\lambda^*}}(C)\psi = 0$ *for any* $C \in \mathfrak{n}'_+$ *if and only if* $g(t)$ *satisfies the Gegenbauer differential equation*

$$\left((1 - t^2)\vartheta_t^2 - (1 + 2(\lambda - 1)t^2)\vartheta_t + a(a + 2(\lambda - 1))t^2\right) g(t) = 0, \tag{7.7}$$

where we denote $\vartheta_t = t\frac{d}{dt}$ *as before.*

Proof. The first two statements follow from Theorem 3.1, Proposition 3.3 and Lemma 3.4. The third statement is clear from (7.3). The proof of the last assertion is similar to the one of Lemma 6.11 and uses the following identities for T_a-saturated differential operators:

$$T_a^\sharp \vartheta_{\zeta_{11}} = T_a^\sharp \vartheta_{\zeta_{nn}} = \frac{1}{2}(a - \vartheta_t), \quad T_a^\sharp \vartheta_{\zeta_{1n}} = \vartheta_t,$$

where $\vartheta_{\zeta_{ij}} = \zeta_{ij}\frac{\partial}{\partial \zeta_{ij}}$. ☐

We are ready to complete the proof of Theorem 7.1.

Proof of Theorem 7.1. By the general result of the F-method (see Theorem 2.1) and owing to Proposition 3.3 and Lemma 3.4, we have the following isomorphism

$$\mathrm{Sol}_{\mathrm{Gegen}}(\lambda - 1, a) \cap \mathrm{Pol}_a[t]_{even} \simeq \mathrm{Hom}_{\widetilde{G'}}(\mathcal{O}(X, \mathcal{L}_\lambda), \mathcal{O}(Y, \mathcal{W}_\lambda^a)).$$

Hence, the uniqueness of the G'-intertwining operator amounts to the fact that the Gegenbauer differential equation has a unique polynomial solution up to a scalar multiple (see Theorem 11.4 (2) in Appendix).

Let us prove that $D_{X \to Y, a}$ defined in (7.2) belongs to $\mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}_\lambda^a)$. Using the F-method we have proved that if $D \in \mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}_\lambda^a)$ and $w^\vee$ is a highest weight vector in $(W_\lambda^a)^\vee$, then $\langle D, w^\vee \rangle$ is of the form $(\mathrm{Symb}^{-1} \otimes \mathrm{id})T_a g$, where $g(t)$ is a polynomial satisfying (7.7). Hence $g(t)$ is, up to a scalar multiple, the Gegenbauer polynomial $\widetilde{C}_a^{\lambda-1}(t)$. In turn, $(T_a g)(\zeta) = \widetilde{C}_a^{\lambda-1}(2\zeta_{11}\zeta_{nn}, \zeta_{1n})$ up to a scalar.

Thus, in order to show $D_{X\to Y,a} \in \mathrm{Diff}_{G'}(\mathcal{L}_\lambda, \mathcal{W}_\lambda^a)$ it is sufficient to verify for all $\ell \in K'_{\mathbb{C}}$:

$$(\mathrm{Symb} \otimes \mathrm{id})\langle D_{X\to Y,a}, \nu^\vee(\ell^{-1})w^\vee\rangle = (\mathrm{Ad}_\sharp(\ell^{-1}) \otimes \lambda^\vee(\ell^{-1}))(T_a g), \tag{7.8}$$

by Lemma 3.5 and by the observation that every non-zero $w^\vee \in W^\vee$ is cyclic. The left-hand side of (7.8) amounts to

$$\begin{aligned} &\left\langle \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2v_i v_j \zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1} v_j\zeta_{jn}\right), \nu^\vee(\ell^{-1})w^\vee\right\rangle \\ = \;& (\det \ell)^{-\lambda}\left\langle \widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2(\ell v)_i(\ell v)_j\zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1}(\ell v)_j\zeta_{jn}\right), w^\vee\right\rangle, \end{aligned}$$

where $v = {}^t(v_1, \dots, v_{n-1})$ stands for the column vector. Since $\langle Q(v), w^\vee\rangle$ gives the coefficients of v_1^a in the polynomial $Q(v)$, it is equal to

$$\begin{aligned} &(\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2\ell_{i1}\ell_{j1}\zeta_{ij}\zeta_{nn}, \sum_{1\le j\le n-1}\ell_{j1}\zeta_{jn}\right) \\ = \;& (\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2({}^t\ell\zeta\ell)_{11}\zeta_{nn}, \sum_{1\le j\le n-1}({}^t\ell\zeta)_{1n}\right). \end{aligned}$$

On the other hand, the action of $\mathrm{Ad}(\ell^{-1})$ on $\mathrm{Pol}(\mathfrak{n}_+)$ is generated by

$$\zeta_{ij} \mapsto ({}^t\ell\zeta\ell)_{ij}, \quad \zeta_{in} \mapsto ({}^t\ell\zeta)_{in}.$$

Hence, the right-hand side of (7.8) amounts to

$$(\det \ell)^{-\lambda}\widetilde{C}_a^{\lambda-1}\left(\sum_{1\le i,j\le n-1} 2({}^t\ell\zeta\ell)_{11}\zeta_{nn}, \sum_{1\le j\le n-1}({}^t\ell\zeta)_{1n}\right),$$

whence the equality (7.8).

For the existence, we know that $\mathrm{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_\lambda), \mathcal{O}(G'/K', \mathcal{W}_\lambda^a)) \neq \{0\}$for $\lambda > n$ by Theorem 2.1 and the branching law given by Fact 4.2. In this case, it is given by the differential operator (7.2) by the F-method. The same formula defines a non-zero differential operator which depends holomorphically on $\lambda \in \mathbb{C}$. Since the actions of $\widetilde{G}$ on $\mathcal{O}(G/K, \mathcal{L}_\lambda)$ and that of $\widetilde{G}'$ on $\mathcal{O}(G'/K', \mathcal{W}_\lambda^a)$ can be realized on H_n and $H_{n-1} \times H_1$, respectively, by operators depending holomorphically on $\lambda \in \mathbb{C}$, the differential operator (7.2) respects the $\widetilde{G}'$ for all $\lambda \in \mathbb{C}$ by holomorphic continuation. ☐

8. Symmetry breaking operators for the tensor product representations of $U(n,1)$

In this section we discuss a higher dimensional generalization of the Rankin–Cohen bidifferential operators by considering the symmetric pair $(G' \times G', G')$ with $G' = U(n,1)$. First we fix some notations. Let $U(n,1)$ be the Lie group of all matrices preserving the standard Hermitian form of signature $(n,1)$ on $\mathbb{C}^{n+1}$ given by $I_{n,1} = \operatorname{diag}(1,\cdots,1,-1) \in GL(n+1,\mathbb{C})$.

Let D be the unit ball $\{Z \in \mathbb{C}^n : \|Z\| < 1\}$, where $\|Z\|^2 := \sum_{j=1}^n |z_j|^2$ for $Z = (z_1,\cdots,z_n)$. It is the Hermitian symmetric domain of type AIII in $\mathbb{C}^n$ in É. Cartan classification. Then the Lie group $U(n,1)$ acts biholomorphically on D by

$$g\cdot Z = (aZ+b)(cZ+d)^{-1} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1),\ Z \in D,$$

and the isotropy subgroup at the origin is isomorphic to $U(n)\times U(1)$. Since $cZ+d \in GL(1,\mathbb{C})$, we identify $cZ+d$ as a non-zero complex number and write $\frac{aZ+b}{cZ+d}$ instead of $(aZ+b)(cZ+d)^{-1}$ from now on.

We adapt the same convention as in [KP14-1, Example 3.7] with $p = n$ and $q = 1$. In particular, we use the decomposition of the Lie algebra

$$\operatorname{Lie}(U(n,1)) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathfrak{gl}_{n+1}(\mathbb{C}) = \mathfrak{n}'_- + \mathfrak{k}' + \mathfrak{n}'_+, \quad \begin{pmatrix} A & B \\ C & d \end{pmatrix} \mapsto (B,(A,d),C).$$

Given a representation $\nu = \nu_1 \boxtimes \nu_2$ of $U(n)\times U(1)$ on a finite-dimensional complex vector space W, we extend it to a holomorphic representation, denoted by the same letter $\nu = \nu_1 \boxtimes \nu_2$, of $GL(n,\mathbb{C})\times GL(1,\mathbb{C})$ on W. Then the holomorphic vector bundle $\mathcal{W} = U(n,1)\times_{U(n)\times U(1)} W$ over D is trivialized by using the open Bruhat cell $\mathfrak{n}'_- \simeq \mathbb{C}^n$, and the regular representation of $U(n,1)$ on $\mathcal{O}(D,\mathcal{W})$ is identified with the multiplier representation π_W of the same group on $\mathcal{O}(D)\otimes W$ given by

$$(\pi_W(g)F)(Z) := \nu_1\left(a - \frac{(aZ+b)c}{cZ+d}\right)^{-1} \nu_2(cZ+d)^{-1} F\left(\frac{aZ+b}{cZ+d}\right), \tag{8.1}$$

for $F \in \mathcal{O}(D)\otimes W, g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(n,1)$ and $Z \in D$. We note that $cZ+d \neq 0$.

For $\lambda_1, \lambda_2 \in \mathbb{C}$, the map

$$\mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C}) \to \mathbb{C}, (A,d) \mapsto -\lambda_1 \operatorname{Trace} A - \lambda_2 d \tag{8.2}$$

is a one-dimensional representation of the Lie algebra $\mathfrak{k}'$, which we denote by $\mathbb{C}_{(\lambda_1,\lambda_2)}$. The negative signature in (8.2) is chosen according to our realization of $\mathfrak{n}_+$ in the lower triangular matrices. For integral values of λ_1 and λ_2 the character $\mathbb{C}_{(\lambda_1,\lambda_2)}$ lifts to $U(n)\times U(1)$. The restriction of the one-dimensional representation (8.2) to the Cartan subalgebra $\bigoplus_{i=1}^{n+1} \mathbb{C}E_{ii}$ is given by $(-\lambda_1,\cdots,-\lambda_1;-\lambda_2)$ in the dual basis $\{e_1,\cdots,e_{n+1}\}$.

For $\lambda_1,\lambda_2 \in \mathbb{Z}$, we form a $U(n,1)$-equivariant holomorphic line bundle $\mathcal{L}_{\lambda_1,\lambda_2} = U(n,1)\times_{U(n)\times U(1)}\mathbb{C}_{(\lambda_1,\lambda_2)}$ over D. By (8.1), the representation of $U(n,1)$ on $\mathcal{O}(D,\mathcal{L}_{\lambda_1,\lambda_2})$

is identified with the multiplier representation, denoted simply by $\pi_{\lambda_1,\lambda_2}$, of $U(n,1)$ on $\mathcal{O}(D)$ given by

$$(\pi_{\lambda_1,\lambda_2}(g)F)(Z) = (cZ+d)^{-\lambda_1+\lambda_2}(\det g)^{-\lambda_1}F\left(\frac{aZ+b}{cZ+d}\right).$$

In our normalization, the canonical bundle of D is given by $\mathcal{L}_{(1,-n)}$ associated with $\mathbb{C}_{2\rho} = \operatorname{Trace}(\operatorname{ad}(\cdot) : \mathfrak{n}_+ \to \mathfrak{n}_+) \simeq \mathbb{C}_{(1,-n)}$ with the notation of (8.2), and the dualizing bundle of $\mathcal{L}_{\lambda_1,\lambda_2}$ is given as

$$\mathcal{L}^*_{\lambda_1,\lambda_2} = \mathcal{L}^\vee_{\lambda_1,\lambda_2} \otimes \mathbb{C}_{2\rho} \simeq \mathcal{L}_{-\lambda_1+1,-\lambda_2-n}, \tag{8.3}$$

associated with

$$\mathbb{C}^*_{(\lambda_1,\lambda_2)} = \mathbb{C}_{(-\lambda_1,-\lambda_2)} \otimes \mathbb{C}_{2\rho} \simeq \mathbb{C}_{(-\lambda_1+1,-\lambda_2-n)}.$$

Now we consider the setting of symmetry breaking operators for the tensor product representations. We set $X := D \times D$ and $Y := \Delta(D)$. Thus, we have the following diagram:

$$\begin{array}{ccccccc} X = D\times D & \subset & \mathbb{C}^n \times \mathbb{C}^n & \simeq & \mathfrak{n}_- & \subset & \mathbb{P}^n\mathbb{C}\times\mathbb{P}^n\mathbb{C} \\ \cup & & \cup & & \cup & & \cup \\ Y = \Delta(D) & \subset & \Delta(\mathbb{C}^n) & \simeq & \mathfrak{n}'_- & \subset & \Delta(\mathbb{P}^n\mathbb{C}). \end{array}$$

We also set

$$G := U(n,1)\times U(n,1),$$

and let τ be the involution of G acting by $\tau : (g,h) \mapsto (h,g)$. Then the fixed point subgroup G^τ is isomorphic to $\Delta(U(n,1))$. Its identity component G' coincides with G^τ which is already connected. We consider the symmetric pair of holomorphic type (G,G').

According to the branching law in Fact 4.3, for $(\lambda_1',\lambda_2',\lambda_1'',\lambda_2'') \in \mathbb{Z}^4$ with $\lambda_1'-\lambda_2' > n$ and $\lambda_1''-\lambda_2'' > n$, there exists a non-trivial G'-intertwining operator $D_{X\to Y}(\varphi)$ from $\mathcal{O}(X, \mathcal{L}_{(\lambda_1',\lambda_2')} \boxtimes \mathcal{L}_{(\lambda_1'',\lambda_2'')})$ to $\mathcal{O}(Y,\mathcal{W}_Y)$ if and only the irreducible representation W of $U(n)\times U(1)$ has the highest weight $(-\lambda_1,\cdots,-\lambda_1,-\lambda_1-a;-\lambda_2+a)$ for some $a \in \mathbb{N}$. We denote it by $W^a_{(\lambda_1,\lambda_2)}$ and realize on the space $\operatorname{Pol}^a[v_1,\cdots,v_n]$ of homogeneous polynomials of degree a where $(v_1,\ldots,v_n)$ are the standard coordinates on $\mathfrak{n}_-^{-\tau} \simeq \mathbb{C}^n$. Then the vector-valued differential symmetry breaking operators can be thought of as elements of

$$\mathbb{C}\left[\frac{\partial}{\partial z_1'},\ldots,\frac{\partial}{\partial z_n'},\frac{\partial}{\partial z_1''},\ldots,\frac{\partial}{\partial z_n''}\right] \otimes \operatorname{Pol}^a[v_1,\ldots,v_n], \tag{8.4}$$

where z_i', z_j'' $(1 \le i,j \le n)$ are the standard coordinates on $\mathfrak{n}_- \simeq \mathbb{C}^n \oplus \mathbb{C}^n$.

Let $P_\ell^{\alpha,\beta}(t)$ be the Jacobi polynomial defined by

$$P_\ell^{\alpha,\beta}(t) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)}\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{\Gamma(\alpha+\beta+\ell+m+1)}{\ell!\,\Gamma(\alpha+m+1)}\left(\frac{t-1}{2}\right)^m, \tag{8.5}$$

see Appendix 11.2 for more details. We inflate it to a homogeneous polynomial of two variables x and y by

$$P_\ell^{\alpha,\beta}(x,y) := y^\ell P_\ell^{\alpha,\beta}\left(2\frac{x}{y}+1\right). \tag{8.6}$$

For instance, $P_0^{\alpha,\beta}(x,y) = 1$, $P_1^{\alpha,\beta}(x,y) = (2+\alpha+\beta)x + (\alpha+1)y$, etc.

We write $\widetilde{U(n,1)}$ for the universal covering of the group $U(n,1)$. Then we can define a $\widetilde{U(n,1)}$-equivariant holomorphic line bundle $\mathcal{L}_{(\lambda_1,\lambda_2)}$ over D for all $\lambda_1, \lambda_2 \in \mathbb{C}$, as well as a representation of $\widetilde{U(n,1)}$ on $\mathcal{O}(D, \mathcal{L}_{(\lambda_1,\lambda_2)})$.

We denote by $\widehat{\otimes}$ the completion of the tensor product of two nuclear spaces.

Theorem 8.1. *Suppose that* $\lambda_1', \lambda_2', \lambda_1'', \lambda_2'' \in \mathbb{C}$ *and* $a \in \mathbb{N}$. *We set* $\lambda' := \lambda_1' - \lambda_2'$ *and* $\lambda'' := \lambda_1'' - \lambda_2''$.

(1) *The dimension of the vector space*

$$\mathrm{Hom}_{\widetilde{U(n,1)}}\left(\mathcal{O}(D,\mathcal{L}_{(\lambda_1',\lambda_2')})\,\widehat{\otimes}\,\mathcal{O}(D,\mathcal{L}_{(\lambda_1'',\lambda_2'')}), \mathcal{O}(D,\mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')})\right)$$

is either one or two. It is equal to two if and only if

$$\lambda', \lambda'' \in \{-1,-2,\cdots\} \quad \text{and} \quad a \geq \lambda' + \lambda'' + 2a - 1 \geq |\lambda' - \lambda''|. \tag{8.7}$$

(2) *The vector-valued differential operator from* $\mathcal{O}(D\times D)$ *to* $\mathcal{O}(D)\otimes\mathrm{Pol}^a[v_1,\cdots,v_n]$ *defined by*

$$D_{X\to Y,a} := P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}\left(\sum_{i=1}^n v_i\frac{\partial}{\partial z_i}, \sum_{j=1}^n v_j\frac{\partial}{\partial z_j}\right) \tag{8.8}$$

intertwines $\pi_{\lambda_1',\lambda_2'} \boxtimes \pi_{\lambda_1'',\lambda_2''}\big|_{G'}$ *and* π_W, *where* $W \simeq W^a_{\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2''}$.

(3) *If the triple* (λ', λ'', a) *satisfies* (8.7), *then* $D_{X\to Y,a} = 0$. *Otherwise, any symmetry breaking operator is proportional to* $D_{X\to Y,a}$.

Remark 8.2.

(1) The representation theoretic interpretation of the condition (8.7) will be clarified in Section 9 in the case $n = 1$, where we construct three symmetry breaking operators for singular parameters satisfying (8.7) and discuss their linear relations.

(2) The fiber of the vector bundle $\mathcal{W}^a_{(\lambda_1,\lambda_2)}$ is isomorphic to the space $S^a(\mathbb{C}^n)$ of symmetric tensors of degree a. It is a line bundle if and only if $a = 0$ or $n = 1$. In the case $n = 1$, the formula (8.8) reduces to the classical Rankin–Cohen bidifferential operators (see (1.1)) with an appropriate choice of spectral parameters, namely, for $a := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$, the following identity holds:

$$\mathcal{RC}^{\lambda'''}_{\lambda',\lambda''} = (-1)^a P_a^{\lambda'-1,1-\lambda'''}\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right)\Big|_{z_1=z_2=z}. \tag{8.9}$$

Remark 8.3.
(1) If $\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2 \in \mathbb{Z}$ and $a \in \mathbb{N}$, then the linear groups G and G' act equivariantly on the two bundles $\mathcal{L}_{(\lambda'_1,\lambda'_2)} \boxtimes \mathcal{L}_{(\lambda''_1,\lambda''_2)} \to D \times D$ and $\mathcal{W}^a_{(\lambda_1,\lambda_2)} \to D$, respectively.
(2) If $\lambda', \lambda'' > n$, then analogous statements as in Theorem 8.1 remain true for continuous G'-homomorphisms between the Hilbert spaces $\mathcal{H}^2\left(X, \mathcal{L}_{(\lambda'_1,\lambda'_2)} \otimes \mathcal{L}_{(\lambda''_1,\lambda''_2)})\right)$ and $\mathcal{H}^2\left(Y, \mathcal{W}^a_{(\lambda'_1+\lambda''_1,\lambda'_2+\lambda''_2)}\right)$.
(3) Similar statements hold for continuous G'-homomorphisms between the Casselman–Wallach globalizations by the localness theorem [KP14-1, Theorem 5.3].

In order to prove Theorem 8.1, we apply again the F-method. Its Step 1 is given by

Lemma 8.4. *For* $(\lambda'_1, \lambda'_2) \in \mathbb{C}^2$, *we set* $(\mu'_1, \mu'_2) := (-\lambda'_1 + 1, -\lambda'_2 - n)$ *and likewise we define* (μ''_1, μ''_2) *from* $(\lambda''_1, \lambda''_2)$. *Let* $C := C' + C'' = (c'_1, \dots, c'_n) + (c''_1, \dots, c''_n) \in \mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$. *Then*

$$\begin{aligned} d\pi_{\mu'_1,\mu'_2}(C') \oplus d\pi_{\mu''_1,\mu''_2}(C'') &= \sum_{i=1}^n c'_i z'_i (E_{z'} - \lambda' + n + 1) + \sum_{j=1}^n c''_j z''_j (E_{z''} - \lambda'' + n + 1), \\ \widehat{d\pi}_{\mu'_1,\mu'_2}(C') \oplus \widehat{d\pi}_{\mu''_1,\mu''_2}(C'') &= -\left(\lambda' \sum_{i=1}^n c'_i \frac{\partial}{\partial \zeta'_i} + \sum_{i,j=1}^n c'_i \zeta'_j \frac{\partial^2}{\partial \zeta'_i \partial \zeta'_j}\right) \\ &\quad -\left(\lambda'' \sum_{j=1}^n c''_j \frac{\partial}{\partial \zeta''_j} + \sum_{i,j=1}^n c''_i \zeta''_j \frac{\partial^2}{\partial \zeta''_i \partial \zeta''_j}\right). \end{aligned}$$

For the Step 2 we apply Lemma 5.5.

Proposition 8.5. *Assume* $\lambda' = \lambda'_1 - \lambda'_2 > n$ *and* $\lambda'' = \lambda''_1 - \lambda''_2 > n$. *If*

$$\operatorname{Hom}_{G'}(\mathcal{O}(G/K, \mathcal{L}_{(\lambda'_1,\lambda'_2)} \otimes \mathcal{L}_{(\lambda''_1,\lambda''_2)}, \mathcal{O}(G'/K', \mathcal{W})) \neq \{0\}$$

for an irreducible representation W *of* K', *then* W *is of the form*

$$\begin{aligned} W &= W^a_{(\lambda'_1+\lambda''_1,\lambda'_2+\lambda''_2)} = S^a(\mathfrak{n}_+^{-\tau}) \otimes \mathbb{C}_{(\lambda'_1+\lambda''_1,\lambda'_2+\lambda''_2)} \\ &\simeq (S^a((\mathbb{C}^n)^\vee) \otimes (-\lambda_1 \operatorname{Trace}_n)) \boxtimes F(\mathfrak{gl}_1, (-\lambda_2 + a)e_{n+1}) \end{aligned} \tag{8.10}$$

for some $a \in \mathbb{N}$.

For Step 3 we apply Lemma 3.4 and we get:

Lemma 8.6. *Suppose* $\lambda'_1, \lambda'_2, \lambda''_1, \lambda''_2 \in \mathbb{C}$ *and* $a \in \mathbb{N}$. *Let* V *be the one-dimensional representation* $\mathbb{C}_{(\lambda'_1,\lambda'_2)} \boxtimes \mathbb{C}_{(\lambda''_1,\lambda''_2)}$ *of* $\mathfrak{k}$, *and* W *the irreducible representation of* $\mathfrak{k}' \simeq \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ *defined in* (8.10).

(1) *The highest weight of the contragredient representation* $W^\vee$ *with respect to the standard Borel subalgebra* $\mathfrak{b}(\mathfrak{k}')$ *of* $\mathfrak{k}'$ *is given by*

$$\chi = (a, 0, \cdots, 0; -a) + (\lambda'_1 + \lambda''_1, \cdots, \lambda'_1 + \lambda'_1; \lambda'_2 + \lambda''_2).$$

(2) *We regard the* $\mathfrak{k}$*-module* $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ *as a* $\mathfrak{b}(\mathfrak{k}')$*-module. Then the* χ*-weight space is given by*

$$(\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi \simeq \bigoplus_{i+j=a} \mathbb{C}(\zeta_1')^i(\zeta_1'')^j, \tag{8.11}$$

where we identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ *with* $\mathrm{Pol}(\mathfrak{n}_+)$ *as vector spaces.*

Proof. 1) Since the highest weight of W is given by

$$(-\lambda_1' - \lambda_1'', \cdots, -\lambda_1' - \lambda_1''; -\lambda_2' - \lambda_2'') + (0, \cdots, 0, -a; a),$$

see (7.1), the first statement is clear.

2) The Lie algebra $\mathfrak{k}' \simeq \mathfrak{gl}_n(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ acts on $\mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$ as the direct sum of two copies of irreducible representations

$$F(\mathfrak{gl}_n(\mathbb{C}), (0, \cdots, 0; -1)) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), 1),$$

and thus one has the following irreducible decomposition

$$\begin{aligned} \mathrm{Pol}(\mathfrak{n}_+) &\simeq \bigoplus_{i,j} \mathrm{Pol}^i(\mathbb{C}^n) \otimes \mathrm{Pol}^j(\mathbb{C}^n) \\ &\simeq \bigoplus_{i,j} (F(\mathfrak{gl}_n(\mathbb{C}), (i, 0, \cdots, 0)) \otimes F(\mathfrak{gl}_n(\mathbb{C}), (j, 0, \cdots, 0))) \boxtimes F(\mathfrak{gl}_1(\mathbb{C}), -(i+j)) \\ &\simeq \bigoplus_{i,j} \bigoplus_{\underline{s}} F(\mathfrak{gl}_n(\mathbb{C}), (s_1, s_2, 0, \cdots, 0)) \otimes F(\mathfrak{gl}_1(\mathbb{C}), -(i+j)), \end{aligned}$$

where the sum in the last line is taken over all $\underline{s} = (s_1, s_2, 0, \cdots, 0) \in \mathbb{N}^n$ satisfying $s_1 \geq s_2 \geq 0$, and $i + j \geq s_1 \geq \max(i, j)$ and $s_1 + s_2 = i + j$. In particular, the weight χ occurs a highest weight in $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$, or equivalently, the one-dimensional $\mathfrak{b}(\mathfrak{k}')$-module $(a, 0, \cdots, 0; -a)$ occurs in $\mathrm{Pol}(\mathfrak{n}_+)$, if and only if $i + j = a$ and $s_2 = 0$. In this case the weight vectors are the monomials $(\zeta_1')^i(\zeta_1'')^j$. Lemma follows. □

As Step 4, we reduce the system of differential equations (3.9) to an ordinary differential equation. For this, we recall from (6.11) that $\mathrm{Pol}_a[t]$ is the space of polynomials in one variable t of degree at most a. We identify $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ with the space of polynomials in (ζ', ζ'') on $\mathfrak{n}_+ \simeq \mathbb{C}^n \oplus \mathbb{C}^n$. For $g \in \mathrm{Pol}_a[t]$ we set

$$(T_a g)(\zeta', \zeta'') := (\zeta_1'')^a g\left(\frac{\zeta_1'}{\zeta_1''}\right).$$

Proposition 8.7. *Let* χ *be the character of* $\mathfrak{b}(\mathfrak{k}')$ *given in Lemma 8.6.*

(1) *The map* T_a *induces an isomorphism* $T_a : \mathrm{Pol}_a[t] \xrightarrow{\sim} (\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee)_\chi$.

(2) *The polynomial* $T_a g$ *satisfies the system of partial differential equations* (3.9) *if and only if the polynomial* $g(t)$ *solves the single ordinary differential equation*

$$\left((t + t^2)\frac{d^2}{dt^2} + (\lambda' - (\lambda'' - 2a + 2)t)\frac{d}{dt} + a(\lambda'' + a - 1)\right) g(t) = 0. \tag{8.12}$$

For the proof of Proposition 8.7 we use the following identities for T_a-saturated operators whose verification is similar to the one for Lemma 6.10.

Lemma 8.8. *One has:*

(1) $T_a^\sharp\left(\zeta_1''\frac{\partial}{\partial\zeta_1'}\right)=\frac{d}{dt}$.
(2) $T_a^\sharp\left(\zeta_1'\zeta_1''\frac{\partial^2}{\partial(\zeta_1')^2}\right)=t\frac{d^2}{dt^2}$.
(3) $T_a^\sharp\left(\zeta_1''\frac{\partial}{\partial\zeta_1''}\right)=a-t\frac{d}{dt}$.
(4) $T_a^\sharp\left((\zeta_1'')^2\frac{\partial^2}{\partial(\zeta_1'')^2}\right)=a(a-1)-2(a-1)t\frac{d}{dt}+t^2\frac{d^2}{dt^2}$.

Proof of Proposition 8.7. The general condition (3.9) of the F-method amounts to the following differential equation:

$$\left(\lambda'\frac{\partial}{\partial\zeta_i'}+\zeta_i'\frac{\partial^2}{\partial(\zeta_i')^2}+\lambda''\frac{\partial}{\partial\zeta_i''}+\zeta_i''\frac{\partial^2}{\partial(\zeta_i'')^2}\right)\psi(\zeta',\zeta'')=0, \tag{8.13}$$

for $C_i=(\underbrace{0,\dots,0}_{i-1},1,0,\cdots,0)+(\underbrace{0,\dots,0}_{i-1},1,0,\cdots,0)\in\Delta(\mathfrak{n}_+)\simeq\mathfrak{n}_+'\simeq\mathbb{C}^n$ $(1\le i\le n)$. Applying this to $\psi=T_a g$, and using Lemma 8.8, we obtain the differential equation (8.12) for $g(t)$. ☐

We give a proof of Theorem 8.1 below. Note that the proof requires some general argument on the Jacobi polynomials, which is summarized in Appendix, namely, Section 11.2. We naturally quote necessary facts from the section, although they are discussed later.

Proof of Theorem 8.1. We set

$$h(s):=g\left(\frac{s-1}{2}\right).$$

Then $g(t)\in\mathrm{Pol}_a[t]$ if and only if $h(s)\in\mathrm{Pol}_a[s]$, and $g(t)$ satisfies (8.12) if and only if $h(s)$ satisfies

$$\left((1-s^2)\frac{d^2}{ds^2}+(\beta-\alpha-(\alpha+\beta+2)s)\frac{d}{ds}+a(a+\alpha+\beta+1)\right)h(s)=0, \tag{8.14}$$

where $\alpha:=\lambda'-1$ and $\beta:=-\lambda'-\lambda''-2a+1$. Thus, combining with Theorem 3.1, we have shown the following bijection

$$\begin{aligned}&\mathrm{Hom}_{\widetilde{U(n,1)}}\left(\mathcal{O}(D,\mathcal{L}_{(\lambda_1',\lambda_2')})\widehat{\otimes}\mathcal{O}(D,\mathcal{L}_{(\lambda_1'',\lambda_2'')}),\mathcal{O}(D,\mathcal{W}^a_{(\lambda_1'+\lambda_1'',\lambda_2'+\lambda_2'')})\right)\\ \simeq\ &\mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1,-\lambda'-\lambda''-2a+1,a)\cap\mathrm{Pol}_a[s],\end{aligned} \tag{8.15}$$

where $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha,\beta,\ell)\cap\mathrm{Pol}_a[s]$ denotes the space of polynomials of degree at most a satisfying the Jacobi differential equation (11.4).

By the bijection (8.15) the first statement is reduced to Theorem 11.2 in Appendix on the dimension of polynomial solutions to the Jacobi differential equation.

Since the Jacobi polynomial $P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}(s)$ belongs to the right-hand side of (8.15), it follows from Theorem 3.1 (2) and Lemma 3.5 that $D_{X\to Y,a}$ is a symmetry breaking operator. The last statement follows from the fact that Jacobi polynomial $P_a^{\lambda'-1,-\lambda'-\lambda''-2a+1}(t)$ is identically zero as a polynomial of t if and only if the triple (λ',λ'',a) satisfies (8.7), by Theorem 11.2 (1) in Appendix. ☐

Remark 8.9. In all the three cases we have reduced a system of partial differential equations to a single ordinary differential equation in Step 4 of the F-method. The latter equation has regular singularities at $t = \pm 1$ and ∞. We describe the corresponding singularities via the map T_a as follows:

(1) The singularities of the differential equation (6.15) correspond to the varieties given by $\zeta_n = 0$ and $Q_{n-1}(\zeta') = 0$.
(2) The singularities of the differential equation (7.7) correspond to the varieties given by $\zeta_{1n} = 0$ and $\det\begin{vmatrix}\zeta_{11} & \zeta_{1n}\\ \zeta_{1n} & \zeta_{nn}\end{vmatrix} = 0$.
(3) The singularities of the differential equation (8.14) correspond to the varieties given by $\zeta_1' = 0$ and $\zeta_1' = \pm\zeta_1''$.

9. Higher multiplicity phenomenon for singular parameter

It is well-known that the branching law for the tensor product of two holomorphic discrete series representations of $SL(2,\mathbb{R})\,(\simeq SU(1,1))$ is multiplicity free. More generally, the branching laws for holomorphic discrete series representations of scalar type in the setting of reductive symmetric pairs remain multiplicity free for positive parameters [K08], as well as their counterpart for generalized Verma modules for *generic* parameters [K12]. However, we discover that such multiplicity one results may fail for singular parameters. In this section, we examine why and how it happens in the example of $SL(2,\mathbb{R})$. We shall see that the F-method reduces it to the question of finding polynomial solutions to the Gauss hypergeometric equation with all the parameters being negative integers. We give a complete answer to this question in Appendix.

9.1. Multiplicity two results for singular parameters.

9.1. Multiplicity two results for singular parameters. From now on, we consider the setting of the previous section for $n = 1$, and let $G = SU(1,1)$ rather than $U(1,1)$.

For $\lambda \in \mathbb{Z}$, we write $\mathcal{L}_\lambda$ for the G-equivariant holomorphic line bundle over the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, where $\lambda = \lambda_1 - \lambda_2$ in the notations of the previous section. Using the Bruhat decomposition, we trivialize the line bindle $\mathcal{L}_\lambda$ and identify the regular representation of G on $\mathcal{O}(D, \mathcal{L}_\lambda)$ with the following multiplier representation on $\mathcal{O}(D)$:

$$(\pi_\lambda(g)F)(z) = (cz+d)^{-\lambda} F\left(\frac{az+b}{cz+d}\right), \quad \text{for} \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } F \in \mathcal{O}(D).$$

For $\lambda \in \mathbb{C}$, we extend π_λ to a representation of the universal covering group $\widetilde{G} = \widetilde{SU(1,1)}$.

We write $\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\nu)$ for the Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\nu$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In our parametrization, if $\lambda = 1 - k$ ($k \in \mathbb{N}$), then the k-dimensional irreducible representation occurs as a subrepresentation of $(\pi_\lambda, \mathcal{O}(D))$ and as a quotient of $\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)$.

We consider symmetry breaking operators from the tensor product representation $\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''})$ to $\mathcal{O}(\mathcal{L}_{\lambda'''})$, where $\widehat{\otimes}$ denotes the completion of the tensor product of two nuclear spaces. As we saw in (1.1), the Rankin–Cohen bidifferential operator $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''}$ is an example of such an operator when $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ (see also Example 9.9 below).

For $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$, we set

$$\begin{aligned} H(\lambda', \lambda'', \lambda''') &:= \operatorname{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ &= \operatorname{Diff}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ &\simeq \operatorname{Hom}_{\mathfrak{g}}(\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'''), \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda') \otimes \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda'')), \end{aligned}$$

where the second equality and the third isomorphism follow from Theorem 2.1. The general theory (see Fact 4.2) shows that $H(\lambda', \lambda'', \lambda''')$ is generically equal to 0 or 1. Here is a precise dimension formula:

Theorem 9.1. *The vector space $H(\lambda', \lambda'', \lambda''')$ is finite dimensional for any $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$. More precisely,*

(1) $\dim_{\mathbb{C}} H(\lambda', \lambda'', \lambda''') \in \{0, 1, 2\}$.
(2) $H(\lambda', \lambda'', \lambda''') \neq \{0\}$ *if and only if*

$$\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}. \tag{9.1}$$

(3) *Suppose* (9.1) *is satisfied. Then the following three conditions are equivalent:*
  (i) $\dim_{\mathbb{C}} H(\lambda', \lambda'', \lambda''') = 2$.
  (ii)

$$\lambda', \lambda'', \lambda''' \in \mathbb{Z}, \quad 2 \geq \lambda' + \lambda'' + \lambda''', \quad \text{and} \quad \lambda''' \geq |\lambda' - \lambda''| + 2. \tag{9.2}$$

  (iii) $\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = 0$.

Next, let us give an explicit basis of $H(\lambda', \lambda'', \lambda''')$. For this consider the polynomials of one variable $\widetilde{g}_j$ $(j = 1, 2, 3)$ which will be defined in Lemma 11.3 with

$$\alpha = \lambda' - 1,\ \beta = 1 - \lambda''', \quad \text{and} \quad \ell = \frac{1}{2}(-\lambda' - \lambda'' + \lambda''').$$

We inflate $\widetilde{g}_j$ into homogeneous polynomials of degree ℓ of two variables by

$$G_j(x, y) := (-y)^{\ell} \widetilde{g}_j \left(1 + \frac{2x}{y}\right),$$

and set

$$D_j := \operatorname{Rest}_{z_1 = z_2 = z} \circ\, G_j \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}\right),$$

for $j = 1, 2, 3$.

Theorem 9.2. *Suppose the conditions* (9.1) *and* (9.2) *hold.*

(1) *The operators D_j $(j = 1, 2, 3)$ are G-homomorphisms from $\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''})$ to $\mathcal{O}(\mathcal{L}_{\lambda'''})$.*

(2) *$1 - \lambda', 1 - \lambda''$ and $1 - \lambda''' \in \mathbb{N}_+$, and the operators D_j $(j = 1, 2, 3)$ factorize into two natural intertwining operators as follows:*

$$\begin{aligned} D_1 &= \mathcal{RC}^{\lambda'''}_{2-\lambda', \lambda''} \circ \left(\left(\frac{\partial}{\partial z_1} \right)^{1-\lambda'} \otimes \mathrm{id} \right), \\ D_2 &= \mathcal{RC}^{\lambda'''}_{\lambda', 2-\lambda''} \circ \left(\mathrm{id} \otimes \left(\frac{\partial}{\partial z_2} \right)^{1-\lambda''} \right), \\ D_3 &= \left(\frac{d}{dz} \right)^{\lambda''' - 1} \circ \mathcal{RC}^{2-\lambda'''}_{\lambda', \lambda''}. \end{aligned}$$

(3) *The following linear relation holds:*

$$D_1 - D_2 + (-1)^{\lambda'} D_3 = 0.$$

The factorizations in Theorem 9.2 are illustrated by the following diagram:
(9.3)

$$\begin{tikzcd}
 & \mathcal{O}(\mathcal{L}_{2-\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}) \arrow[dr, "\mathcal{RC}^{\lambda'''}_{2-\lambda', \lambda''}"] & \\
\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}) \arrow[ur, "\left(\frac{\partial}{\partial z_1}\right)^{1-\lambda'} \otimes \mathrm{id}"] \arrow[r, "\mathrm{id} \otimes \left(\frac{\partial}{\partial z_2}\right)^{1-\lambda''}"] \arrow[dr, "\mathcal{RC}^{2-\lambda'''}_{\lambda', \lambda''}"'] & \mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{2-\lambda''}) \arrow[r, "\mathcal{RC}^{\lambda'''}_{\lambda', 2-\lambda''}"] & \mathcal{O}(\mathcal{L}_{\lambda'''}), \\
 & \mathcal{O}(\mathcal{L}_{2-\lambda'''}) \arrow[ur, "\left(\frac{d}{dz}\right)^{\lambda'''-1}"'] &
\end{tikzcd}$$

To summarize we consider the following three cases.

Case 0. $\lambda''' - \lambda' - \lambda'' \notin 2\mathbb{N}$.
Case 1. $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ but the condition (9.2) is not fulfilled.
Case 2. $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$ and the condition (9.2) is satisfied.

Corollary 9.3.

$$H(\lambda', \lambda'', \lambda''') = \begin{cases} \{0\} & \text{Case}\,0, \\ \mathbb{C} \cdot \mathcal{RC}_{\lambda', \lambda''}^{\lambda'''} & \text{Case}\,1, \\ \mathbb{C}\langle D_1, D_2 \rangle = \mathbb{C}\langle D_1, D_3 \rangle = \mathbb{C}\langle D_2, D_3 \rangle & \text{Case}\,2. \end{cases}$$

The rest of this section is devoted to the proof of Theorems 9.1 and 9.2.

9.2. **Application of the F-method.** For $\alpha, \beta \in \mathbb{C}$, and $\ell \in \mathbb{N}$, we denote by $\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$ the space of polynomials $g(t)$ of degree at most ℓ satisfying the Jacobi differential equation (see Appendix 11.2):

$$(1 - t^2) g''(t) + (\beta - \alpha - (\alpha + \beta + 2)t) g'(t) + \ell(\ell + \alpha + \beta + 1) g(t) = 0.$$

Lemma 9.4. *Suppose* $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$. *Then,*

(1) $H(\lambda', \lambda'', \lambda''') = \{0\}$ *if* $\lambda''' - \lambda' - \lambda'' \notin 2\mathbb{N}$.

(2) *Suppose* $\lambda''' - \lambda' - \lambda'' \in 2\mathbb{N}$. *Then the F-method gives a bijection*

$$H(\lambda', \lambda'', \lambda''') \xrightarrow{\sim} \mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t],$$

with $\alpha = \lambda' - 1$, $\beta = 1 - \lambda'''$, *and* $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$.

Proof. By Step 3 of the F-method, the symbol map induces a bijection between $H(\lambda', \lambda'', \lambda''')$ and the space of polynomials $\psi(\zeta_1, \zeta_2)$ of two variables satisfying the following two conditions

- $\psi(\zeta_1, \zeta_2)$ is homogeneous of degree $\frac{1}{2}(\lambda''' - \lambda' - \lambda'')$,
- $\left(\lambda' \frac{\partial}{\partial \zeta_1} + \zeta_1 \frac{\partial^2}{\partial \zeta_1^2}\right) \psi = \left(\lambda'' \frac{\partial}{\partial \zeta_2} + \zeta_2 \frac{\partial^2}{\partial \zeta_2^2}\right) \psi = 0$,

corresponding to (3.10) and (3.11), respectively. Hence the first statement follows.

The second statement follows from Step 4 of the F-method, namely, Proposition 8.7 with $n = 1$ shows that there is a correspondence between $\psi(\zeta_1, \zeta_2)$ and $g(t) \in \mathrm{Sol}_{\mathrm{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$ with α, β and ℓ as above given by

$$\psi(\zeta_1, \zeta_2) = \zeta_2^\ell \, g\left(\frac{2\zeta_1}{\zeta_2} + 1\right).$$

$\square$

We consider the transformation $(\lambda', \lambda'', \lambda''') \mapsto (\alpha, \beta, \ell)$ given by

$$\alpha := \lambda' - 1, \quad \beta := 1 - \lambda''', \quad \ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda''). \tag{9.4}$$

For $\ell \in \mathbb{N}$, we define a finite set by

$$\Lambda_\ell := \{(\alpha, \beta) \in \mathbb{Z}^2 : \alpha + \ell \geq 0, \beta + \ell \geq 0, \alpha + \beta \leq -(\ell + 1)\}. \tag{9.5}$$

We note that $\Lambda_\ell \in (-\mathbb{N}_+) \times (-\mathbb{N}_+)$ and $\# \Lambda_\ell = \frac{1}{2}\ell(\ell + 1)$.

Lemma 9.5. *Suppose* α, β, ℓ *are given by* (9.4). *Then* $\ell \in \mathbb{N}$ *and* $(\alpha, \beta) \in \Lambda_\ell$ *if and only if* $(\lambda', \lambda'', \lambda''') \in \mathbb{C}^3$ *satisfies the following two conditions:*

$$\lambda', \lambda'', \lambda''' \in \mathbb{Z}, \quad \lambda' + \lambda'' \equiv \lambda''' \operatorname{mod} 2, \tag{9.6}$$

$$-(\lambda' + \lambda'') \geq \lambda''' - 2 \geq |\lambda' - \lambda''|. \tag{9.7}$$

Since the proof is elementary and follows from the definition, we omit it. Note that the conditions (9.6) and (9.7) imply that

$$\lambda' \leq 0, \quad \lambda'' \geq 0, \quad \text{and} \quad 2 \leq \lambda''',$$

which are equivalent to $\alpha \leq -1$, $\alpha + \beta + 2\ell \geq 0$, and $\beta \leq -1$, respectively.

Proof of Theorem 9.1. By Lemma 9.4, the proof is reduced to the computation of the dimension of $\mathrm{Sol}_{\mathsf{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t]$.

1) Since the Jacobi differential equation is of second order, the space of its polynomial solutions is at most two-dimensional.

2) If $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$, then Theorem 11.1 (1) shows that $\dim \mathrm{Sol}_{\mathsf{Jacobi}}(\alpha, \beta, \ell) \cap \mathrm{Pol}_\ell[t] \geq 1$ for any $\alpha, \beta \in \mathbb{C}$.

3) The equivalence follows from Theorem 11.2 (1) in light of Lemma 9.5. $\square$

9.3. Factorization of symmetry breaking operators.

We have seen in Theorem 9.1 that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) = 2,$$

when $(\lambda', \lambda'', \lambda''')$ satisfies (9.6) and (9.7). In this subsection, we show that the other three symmetry breaking operators in the diagram (9.3) are unique up to scalars. To be precise, we prove the following.

Proposition 9.6. *Suppose* $(\lambda', \lambda'', \lambda''')$ *satisfies* (9.6) *and* (9.7). *Then*

$$\begin{aligned} & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{2-\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ = \; & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{2-\lambda''}), \mathcal{O}(\mathcal{L}_{\lambda'''})) \\ = \; & \dim_{\mathbb{C}} \mathrm{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_{\lambda'}) \widehat{\otimes} \mathcal{O}(\mathcal{L}_{\lambda''}), \mathcal{O}(\mathcal{L}_{2-\lambda'''})) = 1. \end{aligned}$$

Proof. The transformation $(\lambda', \lambda'', \lambda''') \mapsto (\alpha, \beta, \ell)$ given by (9.4) yields

$$\begin{aligned} (2 - \lambda', \lambda'', \lambda''') &\mapsto (-\alpha, \beta, \alpha + \ell), \\ (\lambda', 2 - \lambda'', \lambda''') &\mapsto (\alpha, \beta, -\alpha - \beta - \ell - 1), \\ (\lambda', \lambda'', 2 - \lambda''') &\mapsto (\alpha, -\beta, \beta + \ell). \end{aligned}$$

Moreover, if $(\alpha, \beta) \in \Lambda_\ell$ for some $\ell \in \mathbb{N}$, then

(1) $\alpha + \ell \in \mathbb{N}$ and $(-\alpha, \beta) \notin \Lambda_{\alpha+\ell}$,
(2) $-\alpha - \beta - \ell - 1 \in \mathbb{N}$ and $(\alpha, \beta) \notin \Lambda_{-\alpha-\beta-\ell-1}$,
(3) $\beta + \ell \in \mathbb{N}$ and $(\alpha, -\beta) \notin \Lambda_{\beta+\ell}$.

Then the proposition follows from Lemma 9.4 (2) and Theorem 11.2 (1). ☐

9.4. **Differential intertwining operators for SL_2.** Obviously, both the F-method and the localness theorem hold in the case when $G = G'$, for which symmetry breaking operators are usual intertwining operators, and have been extensively studied. Lemma 9.7 below is well-known, but we illustrate its proof by using the F-method. The operators $\left(\frac{d}{dz}\right)^k$ are used for the factorization of D_j $(j = 1, 2, 3)$ in Theorem 9.2.

For $(\lambda, \nu) \in \mathbb{C}^2$, we set

$$\begin{aligned} H(\lambda, \nu) &:= \operatorname{Hom}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_\lambda), \mathcal{O}(\mathcal{L}_\nu)) \\ &= \operatorname{Diff}_{\widetilde{G}}(\mathcal{O}(\mathcal{L}_\lambda), \mathcal{O}(\mathcal{L}_\nu)) \\ &\simeq \operatorname{Hom}_{\mathfrak{g}}(\operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\nu), \operatorname{ind}_{\mathfrak{b}}^{\mathfrak{g}}(-\lambda)). \end{aligned}$$

Lemma 9.7.

(1) $\dim_{\mathbb{C}} H(\lambda, \nu) \leq 1$, *and the equality holds if and only if* $\lambda = \nu$ *or* $(\lambda, \nu) = (1 - k, 1 + k)$ *for some* $k \in \mathbb{N}$.
(2) *If* $(\lambda, \nu) = (1 - k, 1 + k)$ *for some* $k \in \mathbb{N}$, *then*

$$H(\lambda, \nu) = \mathbb{C}\left(\frac{d}{dz}\right)^k.$$

Proof. By the F-method, we have the following bijection between $H(\lambda, \nu)$ and the space of polynomials $g(t)$ of one variable satisfying the following two conditions

- $g(t)$ is a monomial of degree $\frac{1}{2}(\nu - \lambda)$, *i.e.* $g(t) = C\,t^{\frac{\nu-\lambda}{2}}$ for some $C \in \mathbb{C}$,
- $\left(\lambda \frac{d}{dt} + \frac{d^2}{dt^2}\right) g(t) = 0$,

according to (3.10) and (3.11).

The first condition forces $\nu - \lambda$ to be in $2\mathbb{N}$ in order to have $H(\lambda, \nu)$ not reduced to zero, whereas the second one implies $(\nu - \lambda)(\lambda + \nu - 2) = 0$. Hence either $\lambda = \nu$ or $(\lambda, \nu) = (1 + k, 1 - k)$ for some $k \in \mathbb{N}$. In the latter case, $g(t) = Ct^k$ for some $k \in \mathbb{N}$, which yields $\left(\frac{d}{dz}\right)^k$ as a $\widetilde{G}$-intertwining operator from $\mathcal{O}(\mathcal{L}_\lambda)$ to $\mathcal{O}(\mathcal{L}_\nu)$. ☐

9.5. **Construction of homogeneous polynomials by inflation.** In order to analyze symmetry breaking operators in the setting when the Rankin–Cohen bidifferential operators $\mathcal{RC}_{\lambda', \lambda''}^{\lambda'''}$ vanish identically, we introduce the following notation.

For a polynomial $g(s)$ of degree *at most* ℓ, we set a polynomial of two variables

$$(\mathcal{I}_\ell g)(x, y) = (-y)^\ell g\left(-\frac{x}{y}\right).$$

The proof of factorization of symmetry breaking operators will be reduced to the following elementary factorization of homogeneous polynomials $(\mathcal{I}_\ell g)(x,y)$. The following observation follows immediately from the definition.

Lemma 9.8.

(1) *Suppose* $g_1(s)$ *is of the form* $g_1(s) = s^m h_1(s)$ *for some polynomial* $h_1(s)$ *of degree* $\ell - m$*, then*

$$(\mathcal{I}_\ell g_1)(x,y) = (-x)^m (\mathcal{I}_{\ell-m} h_1)(x,y).$$

(2) *Suppose* $g_2(s)$ *is a polynomial of degree* $\ell - m$*, then*

$$(\mathcal{I}_\ell g_2)(x,y) = (-y)^m (\mathcal{I}_{\ell-m} g_2)(x,y).$$

(3) *Suppose* $g_3(s)$ *is a polynomial of the form* $g_3(s) = (1-s)^m h_3(s)$ *for some polynomial* $h_3(s)$ *of degree* $\ell - m$*, then*

$$(\mathcal{I}_\ell g_3)(x,y) = (-1)^m (x+y)^m (\mathcal{I}_{\ell-m} h_3)(x,y).$$

Suppose $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$. Then it follows from the proof of Lemma 9.4 that the inverse of the following bijection

$$H(\lambda',\lambda'',\lambda''') \xrightarrow{\sim} \mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1, 1-\lambda''', \ell) \cap \mathrm{Pol}_\ell[t], \quad D \mapsto g \tag{9.8}$$

is given (up to multiplication by $(-1)^\ell$) by

$$D = \mathrm{Rest}_{z_1=z_2=z} \circ (\mathcal{I}_\ell g(1-2s)) \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right).$$

Example 9.9. *The Rankin–Cohen bidifferential operator* (1.1) *is given for* $(\lambda',\lambda'',\lambda''') \in \mathbb{C}^3$ *with* $\ell := \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$ *by*

$$\mathcal{RC}_{\lambda',\lambda''}^{\lambda'''} = \mathrm{Rest}_{z_1=z_2=z} \circ \left(\mathcal{I}_\ell P_\ell^{\lambda'-1, 1-\lambda'''}(1-2s) \right) \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \right). \tag{9.9}$$

Proof of Theorem 9.2. 1) Since $\widetilde{g}_j \in \mathrm{Sol}_{\mathrm{Jacobi}}(\lambda'-1, 1-\lambda''', \ell) \cap \mathrm{Pol}_\ell[t]$ with $\ell = \frac{1}{2}(\lambda''' - \lambda' - \lambda'') \in \mathbb{N}$ by Theorem 11.2 in the Appendix, we have $D_j \in H(\lambda',\lambda'',\lambda''')$ by (9.8).

2) Combining Lemmas 9.8 and 11.3 we have the following identities of the homogeneous polynomials $G_j(x,y)$:

$$\begin{aligned} G_1(x,y) &= (-x)^{-\alpha} \left(\mathcal{I}_{\alpha+\ell} P_{\alpha+\ell}^{-\alpha,\beta}(1-2s) \right)(x,y), \\ G_2(x,y) &= (-y)^{-\beta} \left(\mathcal{I}_{\beta+\ell} P_{-\alpha-\beta-\ell-1}^{\alpha,\beta}(1-2s) \right)(x,y), \\ G_3(x,y) &= (-x-y)^{-\beta} \left(\mathcal{I}_{\beta+\ell} P_{\beta+\ell}^{\alpha,-\beta}(1-2s) \right)(x,y). \end{aligned}$$

The first two identities yield the factorization of D_1 and D_2, and the last one yields the factorization of G_3 in light of the formula:

$$\mathrm{Rest}_{z_1=z_2=z} \circ \left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right)^j = \left(\frac{d}{dz} \right)^j \circ \mathrm{Rest}_{z_1=z_2=z}, \quad \text{for all } j \in \mathbb{N}.$$

3) The identity is reduced to the linear relations among the polynomials $\widetilde{g}_j(s)$ ($j = 1,2,3$) (see Lemma 11.3) which are obtained by Kummer's connection formula for the Gauss hypergeometric function at the regular singularities $s = 0$ ($\widetilde{g}_1(s)$ and $\widetilde{g}_2(s)$) and $s = 1$ ($\widetilde{g}_3(s)$). Hence Theorem 9.2 is proved. ☐

10. An application of differential symmetry breaking operators

10.1. Remark on the discrete spectrum of the branching rule for complementary series for $O(n+1,1) \downarrow O(n,1)$. B. Kostant proved in [Kos69] the existence of the "long" complementary series representations of $SO(n,1)$ and $SU(n,1)$. In general, branching problems for the complementary series are more involved than the ones for principal series representations because the Mackey machinery does not apply.

In this section we explain briefly how the differential operators $D_{X\to Y,a}$ ($a \in \mathbb{N}$) given in Theorem 6.3 explicitly characterize discrete summands in the branching laws of the complementary series representations of $O(n+1,1)$ when restricted to the subgroup $O(n,1)$.

For this we first observe that $G'_{\mathbb{C}}$-equivariant holomorphic differential operators $D_{X\to Y,a}$ associated to the embedding of complex flag varieties $G_{\mathbb{C}}/P_{\mathbb{C}} \hookleftarrow G'_{\mathbb{C}}/P'_{\mathbb{C}}$ induce $G_{\mathbb{R}}$-equivariant differential operators associated to the embedding of the real flag varieties $G_{\mathbb{R}}/P_{\mathbb{R}} \hookleftarrow G'_{\mathbb{R}}/P'_{\mathbb{R}}$ for any pair $(G_{\mathbb{R}}, G'_{\mathbb{R}})$ of real forms of $(G_{\mathbb{C}}, G'_{\mathbb{C}})$ as far as $(P_{\mathbb{C}}, P'_{\mathbb{C}})$ have real forms $(P_{\mathbb{R}}, P'_{\mathbb{R}})$ in $(G_{\mathbb{R}}, G'_{\mathbb{R}})$.

In particular, for the pair $(G, G') = (SO_o(n,2), SO_o(n-1,2))$ and $(G_{\mathbb{R}}, G'_{\mathbb{R}}) := (SO_o(n+1,1), SO_o(n,1))$ whose complexifications are the same, we see that G-equivariant holomorphic differential operators $D_{X\to Y,a} : \mathcal{O}(G/K, \mathcal{L}_\lambda) \to \mathcal{O}(G'/K', \mathcal{L}_{\lambda+a})$ induce a $G'_{\mathbb{R}}$-equivariant differential operators

$$D_{X_{\mathbb{R}}\to Y_{\mathbb{R}},a} : C^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{L}_\lambda) \to C^\infty(G'_{\mathbb{R}}/P'_{\mathbb{R}}, \mathcal{L}_{\lambda+a}), \tag{10.1}$$

for two spherical principal series representations of $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$, owing to [KP14-1, Theorem 5.3 (2)] (extension theorem). In our parametrization, for $0 < \lambda < n$, there is a complementary series $\mathcal{H}_\lambda$ that contains $C^\infty(G_{\mathbb{R}}/P_{\mathbb{R}}, \mathcal{L}_\lambda)$ as a dense subset.

We define a family of Hilbert spaces $L^2(\mathbb{R}^n)_s$ with parameter $s \in \mathbb{R}$ by

$$L^2(\mathbb{R}^n)_s := L^2(\mathbb{R}^n, (\xi_1^2 + \cdots + \xi_n^2)^{\frac{s}{2}} d\xi_1 \cdots d\xi_n).$$

Then, for $0 < \lambda < n$, the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ on the N-picture gives a unitary isomorphism

$$\mathcal{F}_{\mathbb{R}^n} : \mathcal{H}_{n-\lambda} \xrightarrow{\sim} L^2(\mathbb{R}^n)_{2\lambda-n}.$$

Correspondingly to the explicit formula

$$D_{X_{\mathbb{R}}\to Y_{\mathbb{R}},a} = \widetilde{C}_a^{\lambda-\frac{n-1}{2}}\left(-\Delta_{\mathbb{C}^{n-1}}, \frac{\partial}{\partial z_n}\right)$$

that was established in Theorem 6.3, we see that the multiplication of the inflated Gegenbauer polynomial $\widetilde{C}_a^{\lambda-\frac{n-1}{2}}(|\xi|^2, \xi_n)$ (see (6.5)) yields an explicit construction of discrete summands of the branching law for the restriction of complementary series as follows:

Proposition 10.1. *Suppose* $a \in \mathbb{N}$ *and* $0 < \lambda < \frac{n-1}{2} - a$. *For* $\xi = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1}$, *we set* $|\xi| := (\xi_1^2 + \cdots + \xi_{n-1}^2)^{\frac{1}{2}}$. *Then,*

$$L^2(\mathbb{R}^{n-1})_{2(\lambda+a)-n-1} \hookrightarrow L^2(\mathbb{R}^n)_{2\lambda-n}, \quad v(\xi) \mapsto C_a^{\lambda-\frac{n-1}{2}}\left(|\xi|^2, \xi_n\right) v(\xi)$$

is an isometric and $G'_{\mathbb{R}}$*-intertwining map from the complementary series of* $G'_{\mathbb{R}} = SO_o(n,1)$ *to that of* $G_{\mathbb{R}} = SO_o(n+1,1)$.

See [KS13, Chapter 15] for the proof that (10.1) implies the proposition in the case $a \in 2\mathbb{N}$ (with both $G_{\mathbb{R}}$ and $G'_{\mathbb{R}}$ replaced by disconnected groups $O(n+1,1)$ and $O(n,1)$, respectively).

11. Appendix: Jacobi polynomials and Gegenbauer polynomials

11.1. Polynomial solutions to the hypergeometric differential equation. In this subsection we discuss polynomial solutions to the Gauss hypergeometric differential equation

$$\left(z(1-z)\frac{d^2}{dz^2} - (c - (a+b+1)z)\frac{d}{dz} - ab\right) u(z) = 0. \tag{11.1}$$

For $c \notin -\mathbb{N}$, the hypergeometric series

$$ {}_2F_1(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j j!} z^j \tag{11.2}$$

is a non-zero solution to (11.1). It is easy to see from (11.2) that ${}_2F_1(a,b;c;z)$ is a polynomial if and only if $a \in -\mathbb{N}$ or $b \in -\mathbb{N}$.

Furthermore, we may ask if there exist two linearly independent polynomial solutions to (11.1). In fact, this never happens when $c \notin -\mathbb{N}$. More precisely, we have the following:

Theorem 11.1. *Suppose* $a, b, c \in \mathbb{C}$.

(1) *The following two conditions are equivalent.*
  (i) *There exists a non-zero polynomial solution to* (11.1).
  (ii) $a \in -\mathbb{N}$ *or* $b \in -\mathbb{N}$.
(2) *The following two conditions are equivalent.*
  (iii) *There exist two linearly independent polynomial solutions to* (11.1).
  (iv) $a, b, c \in -\mathbb{N}$ *and either* (iv-a) *or* (iv-b) *holds:*
    (iv-a) $a \geq c > b$,

(iv-b) $b \geq c > a$.

In this case the two linearly independent polynomial solutions are of degree $-a$ *and* $-b$.

Proof. (1) We have already discussed the case where $c \notin -\mathbb{N}$. Suppose now that $c \in -\mathbb{N}$. Since $1 - c > 0$, we have linearly independent solutions to (11.1) near $z = 0$ as follows

$$\begin{aligned} h_1(z) &= z^{1-c}{}_2F_1(a-c+1, b-c+1; 2-c; z), \\ h_2(z) &= g(z) + (\mathrm{Res}_{\gamma=c}\,{}_2F_1(a,b;\gamma;z)) \log z, \end{aligned}$$

where $g(z)$ is a holomorphic function near $z = 0$ satisfying $g(0) = 1$. We divide the proof into two cases depending on whether $\mathrm{Res}_{\gamma=c}\,{}_2F_1(a,b;\gamma;z) = 0$ or not.

Case 1. Assume $\mathrm{Res}_{\gamma=c}\,{}_2F_1(a,b;\gamma;z) = 0$. In view of the residue formula

$$\mathrm{Res}_{\gamma=c}\,{}_2F_1(a,b;\gamma;z) = \frac{(-1)^c (a)_{1-c}(b)_{1-c}}{(-c)!(1-c)!} z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z)$$

this expression vanishes if and only if $(a)_{1-c}(b)_{1-c} = 0$, namely

$$-\mathbb{N} \ni a \geq c \quad \text{or} \quad -\mathbb{N} \ni b \geq c.$$

In this case ${}_2F_1(a,b;\gamma;z)$ is holomorphic in γ near $\gamma = c$, and

$$\lim_{\gamma \to c} {}_2F_1(a,b;\gamma;z) = \sum_{j=0}^{L} \frac{(a)_j (b)_j}{(c)_j j!} z^j,$$

where $L = -a$ or $-b$, is a polynomial solution to (11.1).

Case 2. Assume $\mathrm{Res}_{\gamma=c}\,{}_2F_1(a,b;\gamma;z) \neq 0$. Since the logarithmic term does not vanish, there exists a non-zero polynomial solution to (11.1) if and only if $h_1(z)$ is a polynomial, or equivalently,

$$a - c + 1 \in -\mathbb{N} \quad \text{or} \quad b - c + 1 \in -\mathbb{N},$$

namely,

$$-\mathbb{N} \ni a < c \quad \text{or} \quad -\mathbb{N} \ni b < c.$$

Combining Case 1 and Case 2, we conclude the equivalence of (i) and (ii) in (1) for $c \in -\mathbb{N}$.

(2) We recall that the differential equation (11.1) has regular singularities at $z = 0, 1$, and ∞, and its characteristic exponents are indicated in the Riemann scheme

$$P \begin{Bmatrix} z = 0 & 1 & \infty & \\ 0 & 0 & a & ; z \\ 1 - c & c - a - b & b & \end{Bmatrix}.$$

(iii)⇒(iv). Suppose (iii) holds. Since the space of local solutions to (11.1) is two dimensional, any solution must be a polynomial. This forces the characteristic exponents to satisfy the following conditions:

$$1 - c, c - a - b \in \mathbb{N}, \quad \text{and} \quad a, b \in \mathbb{N}.$$

Furthermore, the condition (iii) shows that there is no local solution which involves a non-zero logarithmic term near each regular singularity point, which in particular implies that the two characteristic exponents at $z = 0, 1$ or ∞ cannot coincide. Hence we get

$$1 - c \neq 0,\ c - a - b \neq 0, \quad \text{and} \quad a \neq b.$$

Thus we have shown that the condition (iii) implies

$$a, b, c \in -\mathbb{N}. \tag{11.3}$$

From now we assume $c \in -\mathbb{N}$. As in the proof of (1), the condition (iii) implies that $\mathrm{Res}_{\gamma=c}\, {}_2F_1(a, b; \gamma; z) = 0$, and $h_1(z)$ is a polynomial. The latter conditions amount to

$$\begin{aligned} -\mathbb{N} \ni a \geq c \quad &\text{or} \quad -\mathbb{N} \ni b \geq c, \\ -\mathbb{N} \ni a < c \quad &\text{or} \quad -\mathbb{N} \ni b < c, \end{aligned}$$

respectively. Equivalently, we have either $a \geq c > b$ or $b \geq c > a$ under the condition that $a, b, c \in -\mathbb{N}$ (see (11.3)). Hence the implication (iii)⇒(iv) is proved.

(iv)→(iii). Conversely, suppose (iv) holds. Then as we saw in the proof of (1), $h_1(z)$ and

$$\lim_{\gamma \to c} {}_2F_1(a, b; \gamma; z) = \sum_{j=0}^{\min(-a,-b)} \frac{(a)_j (b)_j}{(c)_j j!} z^j$$

are both polynomial solutions to (11.1), corresponding to the characteristic exponents $1-c$ and 0, respectively. Thus they are linearly independent, and we have completed the proof of the equivalence of (iii) and (iv). □

11.2. **Jacobi polynomials.** In this subsection, we discuss polynomial solutions to the Jacobi differential equation with emphasis on singular parameters where the corresponding Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ vanishes. In particular, we give a criterion for the space of polynomial solutions to be two-dimensional, and find its explicit basis.

First we quickly review the classical facts on Jacobi polynomials. Suppose $\alpha, \beta \in \mathbb{C}$ and $\ell \in \mathbb{N}$. The Jacobi differential equation

$$\left((1 - t^2)\frac{d^2}{dt^2} + (\beta - \alpha - (\alpha + \beta + 2)t)\frac{d}{dt} + \ell(\ell + \alpha + \beta + 1)\right) y = 0 \tag{11.4}$$

is a particular case of the Gauss hypergeometric equation (11.1), and has at least one non-zero polynomial solution by Theorem 11.1 (1).

The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is the normalized polynomial solution to (11.4) that is subject to the Rodrigues formula

$$(1-t)^\alpha(1+t)^\beta P_\ell^{\alpha,\beta}(t) = \frac{(-1)^\ell}{2^\ell \ell!}\left(\frac{d}{dt}\right)^\ell \left((1-t)^{\ell+\alpha}(1+t)^{\ell+\beta}\right),$$

from which we have

$$P_\ell^{\beta,\alpha}(-t) = (-1)^\ell P_\ell^{\alpha,\beta}(t). \tag{11.5}$$

The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is generically non-zero (see Theorem 11.2 below for a precise condition) and is a polynomial of degree ℓ satisfying $P_\ell^{\alpha,\beta}(1) = \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\ell!}$. Explicitly, for $\alpha \notin -\mathbb{N}_+$,

$$\begin{aligned} P_\ell^{\alpha,\beta}(t) &= \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+1)\ell!}\,{}_2F_1\left(-\ell, \alpha+\beta+\ell+1; \alpha+1; \frac{1-t}{2}\right) \\ &= \frac{\Gamma(\alpha+\ell+1)}{\Gamma(\alpha+\beta+\ell+1)}\sum_{m=0}^{\ell}\binom{\ell}{m}\frac{\Gamma(\alpha+\beta+\ell+m+1)}{\Gamma(\alpha+m+1)\ell!}\left(\frac{t-1}{2}\right)^m. \end{aligned} \tag{11.6}$$

Here are the first three Jacobi polynomials.

- $P_0^{\alpha,\beta}(t) = 1$.
- $P_1^{\alpha,\beta}(t) = \frac{1}{2}(\alpha-\beta+(2+\alpha+\beta)t)$.
- $P_2^{\alpha,\beta}(t) = \frac{1}{2}(1+\alpha)(2+\alpha)+\frac{1}{2}(2+\alpha)(3+\alpha+\beta)(t-1)+\frac{1}{8}(3+\alpha+\beta)(4+\alpha+\beta)(t-1)^2$.

If $\alpha > -1$ and $\beta > -1$, then the Jacobi polynomials $P_\ell^{\alpha,\beta}(t)$ ($\ell \in \mathbb{N}$) form an orthogonal basis in $L^2([-1,1],(1-t)^\alpha(1+t)^\beta dt)$.

When $\alpha = \beta$ these polynomials yield Gegenbauer polynomials (see the next section for more details), and they further reduce to Legendre polynomials in the case when $\alpha = \beta = 0$.

Theorem 11.2. *Suppose $\ell \in \mathbb{N}$. We recall from* (9.5) *that $\Lambda_\ell \subset (-\mathbb{N})^2$ is a finite set of the cardinality $\frac{1}{2}\ell(\ell+1)$.*

(1) *The following three conditions on $(\alpha,\beta) \in \mathbb{C}^2$ are equivalent:*
   (i) *The Jacobi polynomial $P_\ell^{\alpha,\beta}(t)$ is equal to zero as a polynomial of t.*
   (ii) *There exist two linearly independent polynomial solutions to* (11.4) *of degree less than or equal to ℓ, namely,*

$$\dim_{\mathbb{C}}(\mathrm{Sol}_{\mathrm{Jacobi}}(\alpha,\beta,\ell) \cap \mathrm{Pol}_\ell[t]) = 2.$$

   (iii) $(\alpha,\beta) \in \Lambda_\ell$.

(2) *If one of (therefore any of) the equivalent conditions (i)-(iii) is satisfied, then*

$$\lim_{\varepsilon\to 0}{}_2F_1(-\ell, \alpha+\beta+1; \alpha+\varepsilon+1; z) \tag{11.7}$$

exists and is a polynomial in z*, which we denote by* ${}_2F_1(-\ell, \alpha+\beta+1; \alpha+1; z)$*. Then any two of the following three polynomials*

$$g_1(z) := z^{-\alpha}{}_2F_1(-\alpha-\ell, \beta+\ell+1; 1-\alpha; z), \tag{11.8}$$

$$g_2(z) := {}_2F_1(-\ell, \alpha+\beta+\ell+1; \alpha+1; z), \tag{11.9}$$

$$g_3(z) := (1-z)^{-\beta}{}_2F_1(-\beta-\ell, \alpha+\ell+1; 1-\beta; 1-z), \tag{11.10}$$

with $z = \frac{1}{2}(1-t)$ *are linearly independent polynomial solutions to* (11.4) *of degree* ℓ*,* $-(\alpha+\beta+\ell+1)$*, and* ℓ*, respectively. In particular, any polynomial solution is of degree at most* ℓ*.*

Proof. (1). (i)⇔(iii). By the expression

$$P_\ell^{\alpha,\beta}(t) = \sum_{j=0}^{\ell} \frac{(\alpha+j+1)_{\ell-j}(\alpha+\beta+\ell+1)_j}{j!(\ell-j)!}\left(\frac{t-1}{2}\right)^j,$$

one has $P_\ell^{\alpha,\beta}(t) \equiv 0$ as a polynomial of t if and only if

$$\underbrace{(\alpha+j+1)\cdots(\alpha+\ell)}_{\ell-j}\underbrace{(\alpha+\beta+\ell+1)\cdots(\alpha+\beta+\ell+j)}_{j} = 0, \text{for all}\, j\ (0 \le j \le \ell). \tag{11.11}$$

The condition (11.11) implies $\alpha \in \{-1, \cdots, -\ell\}$ by taking $j = 0$. Conversely, if $\alpha \in \{-1, \cdots, -\ell\}$, then $(\alpha+j+1)\cdots(\alpha+\ell) = 0$ for all j $(0 \le j \le \ell)$, and therefore (11.11) is equivalent to $(\alpha+\beta+\ell+1)\cdots(\alpha+\beta+\ell+j) = 0$ with $j = 1-\alpha$, namely, $\alpha+\beta+\ell+1 \le 0 \le \beta+\ell+1$. Hence the equivalence of (i) and (iii) is proved.

(ii)⇔(iii). We recall from Theorem 11.1 that if the condition (iii), or equivalently (iv), is satisfied, then there are two linearly independent polynomial solutions to (11.1) of degrees $-a$ and $-b$, respectively. Applying Theorem 11.1 (2) with

$$a = -\ell, \quad b = \alpha+\beta+\ell+1, \quad \text{and} \quad c = 1+\alpha,$$

we see that the condition on the degree of polynomials in (ii) corresponds to the condition $-a \ge -b$, which excludes (iv-b) in Theorem 11.1, and therefore, the condition (ii) is equivalent to

$$-\ell, \alpha+\beta+\ell+1, 1+\alpha \in -\mathbb{N}, \quad \alpha+\beta+\ell+1 \ge 1+\alpha > -\ell,$$

which is nothing but $(\alpha, \beta) \in \Lambda_\ell$.

(2). Suppose $(\alpha, \beta) \in \Lambda_\ell$ for some $\ell \in \mathbb{N}$.

Since $-\alpha-\ell \in -\mathbb{N}$ and $\beta+\ell+1, 1-\alpha \notin -\mathbb{N}$, the polynomial $g_1(z)$ is of degree $-\alpha+(\alpha+\ell) = \ell$.

Secondly, the expression $-(\alpha+\beta+\ell+1)$ defines a non-negative integer smaller than $-\ell$ and we have:

$${}_2F_1(-\ell, \alpha+\beta+1; \alpha+\varepsilon+1; z) = \sum_{j=0}^{-(\alpha+\beta+\ell+1)} \frac{(-\ell)_j(\alpha+\beta+\ell+1)_j}{(\alpha+\varepsilon+1)_j j!} z^j.$$

Since $\alpha + j \le -(\beta + \ell + 1) < 0$ for all j with $0 \le j \le -(\alpha + \beta + \ell + 1)$, the denominator in each summand does not vanish at $\varepsilon = 0$, and therefore, $g_2(z)$ is well-defined and is a polynomial of degree $-(\alpha + \beta + \ell + 1)$.

Thirdly, since $-\beta - \ell \in -\mathbb{N}$ and $\alpha + \ell + 1, 1 - \beta \in \mathbb{N}_+$, the function ${}_2F_1(-\beta - \ell, \alpha + \ell + 1; 1 - \beta; 1 - z)$ is a polynomial of homogeneous degree $\ell + \beta$, and thus $g_3(z)$ is a polynomial of degree ℓ.

Moreover, since $g_j(z)$ $(j = 1, 2, 3)$ are local solutions to

$$\left(z(1-z)\frac{d^2}{dz^2} - ((\alpha+1) - (\alpha+\beta+2)z)\frac{d}{dz} + \ell(\alpha+\beta+\ell+1)\right)u(z) = 0 \tag{11.12}$$

near zero depending meromorphically on parameters $(\alpha, \beta) \in \mathbb{C}^2$, and since they do not admit poles at any point of Λ_ℓ, they are actually solutions to (11.12). Since $g_1(0) = 0$ and $g_2(0) = 1$, these functions are linearly independent.

Finally, we apply Kummer's connection formula (see [EMOT53, 2.9 (4.3)])

$$\begin{aligned} & (1-z)^{c-a-b}{}_2F_1(c-a, c-b; c-a-b+1; 1-z) \\ = \; & \frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} z^{1-c}{}_2F_1(a+1-c, b+1-c; 2-c; z) \\ + \; & \frac{\Gamma(1-c)\Gamma(c-a-b+1)}{\Gamma(1-a)\Gamma(1-b)} {}_2F_1(a, b; c; z) \end{aligned}$$

with

$$a = -\ell, \quad b = \alpha + \beta + \ell + 1, \quad c = 1 + \alpha + \varepsilon,$$

and taking the limit $\varepsilon \to 0$, we obtain

$$g_3(z) = (-1)^{\alpha+\beta+\ell}\frac{(-\beta)!(\beta+\ell)!}{(-\alpha)!(\alpha+\ell)!}g_1(z) + \frac{(-\alpha-1)!(-\beta)!}{l!(-\alpha-\beta-\ell-1)!}g_2(z). \tag{11.13}$$

Since the scalars of this linear combination are non-zero, both pairs $\{g_1(z), g_3(z)\}$ and $\{g_2(z), g_3(z)\}$ are linearly independent. □

To end this subsection, we express $g_j(z)$ $(j = 1, 2, 3)$ in terms of the Jacobi polynomials. As a byproduct, we also give an identity among the Jacobi polynomials when $(\alpha, \beta) \in \Lambda_\ell$, or equivalently, when $P_\ell^{\alpha,\beta}(t) \equiv 0$ (Theorem 11.2).

Lemma 11.3. *Suppose* $(\alpha,\beta)\in\Lambda_\ell$. *Then,*

$$\begin{aligned}
(1)\quad \widetilde{g}_1(z) &:= \binom{\ell}{-\alpha}\cdot g_1(z) = z^{-\alpha}P^{-\alpha,\beta}_{\ell+\alpha}(1-2z);\\
\widetilde{g}_2(z) &:= (-1)^{-\ell-\alpha-\beta-1}\binom{-\alpha-1}{\ell+\beta}\cdot g_2(z) = P^{\alpha,\beta}_{-\ell-\alpha-\beta-1}(1-2z);\\
\widetilde{g}_3(z) &:= (-1)^{\beta+\ell}\binom{\ell}{-\beta}\cdot g_3(z) = (1-z)^{-\beta}P^{\alpha,-\beta}_{\ell+\beta}(1-2z).
\end{aligned}$$

$$\begin{aligned}
(2)\quad & (-1)^{\alpha}\widetilde{g}_3(z) = \widetilde{g}_1(z) - \widetilde{g}_2(z), \quad \text{namely,}\\
& P^{\alpha,\beta}_{-\ell-\alpha-\beta-1}(t) = (-1)^{\alpha+1}\left(\frac{1+t}{2}\right)^{-\beta}P^{\alpha,-\beta}_{\beta+\ell}(t) + \left(\frac{1-t}{2}\right)^{-\alpha}P^{-\alpha,\beta}_{\alpha+\ell}(t).
\end{aligned}$$

Proof. 1) The first and third formulæ follow from the equation (11.6) and the identity $\Gamma(\lambda)\Gamma(1-\lambda)=\frac{\lambda}{\sin\pi\lambda}$. The second one is more subtle because $g_2(z)$ is defined as the limit of the Gauss hypergeometric function in a specific direction (see (11.7)). Taking this into account, we deduce the second formula from (11.6).

2) The second identity follows directly from the first statement and (11.13). □

11.3. **Gegenbauer Polynomials.** Let $\vartheta_t := t\frac{t}{dt}$. For $\alpha\in\mathbb{C}$ and $\ell\in\mathbb{N}$, the Gegenbauer differential equation

$$\left((1-t^2)\frac{d^2}{dt^2} - (2\alpha+1)t\frac{d}{dt} + \ell(\ell+2\alpha)\right)y = 0$$

or, equivalently,

$$\left((1-t^2)\vartheta_t^2 - (1+2\alpha t^2)\vartheta_t + \ell(\ell+2\alpha)t^2\right)y = 0 \tag{11.14}$$

is a particular case of the Jacobi differential equation (11.4) where (α,β) are set to be $(\alpha-\frac{1}{2},\alpha-\frac{1}{2})$, and has at least one non-zero polynomial solution owing to Theorem 11.1 (1). The Gegenbauer (or ultraspherical) polynomial $C^{\alpha}_{\ell}(t)$ is a solution to (11.14) given by the following formula:

$$\begin{aligned}
C^{\alpha}_{\ell}(t) &= \frac{\Gamma(\ell+2\alpha)}{\Gamma(2\alpha)\Gamma(\ell+1)}\,{}_2F_1\left(-\ell,\ell+2\alpha;\alpha+\frac{1}{2};\frac{1-t}{2}\right)\\
&= \sum_{k=0}^{[\frac{\ell}{2}]}(-1)^k\frac{\Gamma(\ell-k+\alpha)}{\Gamma(\alpha)\Gamma(k+1)\Gamma(\ell-2k+1)}(2t)^{\ell-2k}.
\end{aligned}$$

It is a specialization of the Jacobi polynomial

$$C^{\alpha}_{\ell}(t) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\ell+2\alpha)}{\Gamma(2\alpha)\Gamma(\ell+\alpha+\frac{1}{2})}P^{\alpha-\frac{1}{2},\alpha-\frac{1}{2}}_{\ell}(t). \tag{11.15}$$

The Gegenbauer polynomial $C_\ell^\alpha(t)$ is a polynomial of degree ℓ. Here are the first five Gegenbauer polynomials.

- $C_0^\alpha(t) = 1$.
- $C_1^\alpha(t) = 2\alpha t$.
- $C_2^\alpha(t) = -\alpha(1 - 2(\alpha+1)t^2)$.
- $C_3^\alpha(t) = -2\alpha(\alpha+1)(t - \frac{2}{3}(\alpha+2)t^3)$.
- $C_4^\alpha(t) = \frac{1}{2}\alpha(\alpha+1)(1 - 4(\alpha+2)t^2 + \frac{4}{3}(\alpha+2)(\alpha+3)t^4)$.

We note that $C_\ell^\alpha(t) \equiv 0$ if $\ell \geq 1$ and $\alpha = 0, -1, -2, \cdots, -\left[\frac{\ell-1}{2}\right]$. Slightly differently from the usual notation in the literature, we renormalize the Gegenbauer polynomial by

$$\widetilde{C}_\ell^\alpha(t) := \frac{\Gamma(\alpha)}{\Gamma\left(\alpha + \left[\frac{\ell+1}{2}\right]\right)} C_\ell^\alpha(t). \tag{11.16}$$

Then $\widetilde{C}_\ell^\alpha(t)$ is a non-zero solution to (11.14) for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$.

As in the case of the Jacobi differential equation, there are some exceptional parameters (α, ℓ) for which the Gegenbauer differential equation (11.14) has two linearly independent polynomial solutions. For this we denote by

$$\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}[t]$$

the space of polynomial solutions to (11.14), and consider its subspace $\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ where $\mathrm{Pol}_\ell[t]_{\mathrm{even}} = \mathbb{C}\text{-span}\left\langle t^{\ell-2j} : 0 \leq j \leq \left[\frac{\ell}{2}\right]\right\rangle$. Then we have the following:

Theorem 11.4. *(1) Suppose* $\ell \in \mathbb{N}$ *and* $\alpha \in \mathbb{C}$. *Then*

$$\dim_{\mathbb{C}}(\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}[t]) = 2$$

if and only if (α, ℓ) *satisfies*

$$\alpha \in \mathbb{Z} + \frac{1}{2} \quad \text{and} \quad 1 - 2\ell \leq 2\alpha \leq -\ell. \tag{11.17}$$

(2) For any $\ell \in \mathbb{N}$ *and any* $\alpha \in \mathbb{C}$, *the space* $\mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ *is one-dimensional, and is spanned by* $\widetilde{C}_\ell^\alpha(t)$.

Proof. (1) The first statement follows immediately from Theorem 11.2 by replacing (α, β) with $(\alpha - \frac{1}{2}, \alpha - \frac{1}{2})$.

(2) Clearly, $\widetilde{C}_\ell^\alpha(t) \in \mathrm{Sol}_{\mathrm{Gegen}}(\alpha, \ell) \cap \mathrm{Pol}_\ell[t]_{\mathrm{even}}$ for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Hence it suffices to show that another solution (see Theorem 11.2 and (11.7))

$${}_2F_1\left(-\ell, 2\alpha + \ell; \alpha + \frac{1}{2}; \frac{1-t}{2}\right) \notin \mathrm{Pol}_\ell[t]_{\mathrm{even}}$$

when α satisfies (11.17). Indeed ${}_2F_1\left(-\ell, 2\alpha + \ell; \alpha + \frac{1}{2}; \frac{1-t}{2}\right)$ is a polynomial in t whose top term is a non-zero multiple of $t^{-(2\alpha+\ell)}$, but $-(2\alpha + \ell) \not\equiv \ell \mod 2$ because $\alpha \in \mathbb{Z} + \frac{1}{2}$. Hence Theorem is proved. $\square$

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