Lectures on Regular and Irregular Holonomic D-modules

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Abstract

This is a survey paper based on lectures given by the authors at Ihes, February/March 2015. In a first part, we recall the main results on the tempered holomorphic solutions of D-modules in the language of indsheaves and, as an application, the Riemann-Hilbert correspondence for regular holonomic modules. In a second part, we present the enhanced version of the first part, treating along the same lines the irregular holonomic case.

Contents

Introduction

In these lectures, we assume the audience familiar with the language of sheaves and D-modules, in the derived sense.

Let X be a complex manifold. Denote by $\mathrm{Mod}(\mathscr{D}_X)$ the abelian category of left $\mathscr{D}_X$-modules, by $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$ the full subcategory of holonomic $\mathscr{D}_X$-modules and by $\mathrm{Perv}(\mathbb{C}_X)$ the abelian category of perverse sheaves with coefficients in $\mathbb{C}$. Consider the functor constructed in [Ka75]

$$\begin{aligned}&\mathcal{S}ol\colon \mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{Perv}(\mathbb{C}_X),\\ &\mathscr{M} \mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathcal{O}_X).\end{aligned}$$

(Note that at this time the notion of perverse sheaves was not explicit, but in his paper, the author proved that $\mathrm{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathcal{O}_X)$ is $\mathbb{C}$-constructible and satisfies the properties which are now called perversity.)

It is well-known that this functor is not faithful. For example, if $X = \mathbb{A}^1(\mathbb{C})$, the complex line with coordinate t, $P = t^2\partial_t - 1$ and $Q = t^2\partial_t + t$, then the two $\mathscr{D}_X$-modules $\mathscr{D}_X/\mathscr{D}_X \cdot P$ and $\mathscr{D}_X/\mathscr{D}_X \cdot Q$ will have the same sheaves of solutions.

A natural idea to overcome this difficulty is to replace the sheaf $\mathcal{O}_X$ with presheaves of holomorphic functions with various growths such as for example the sheaf $\mathcal{O}_X^{\mathrm{t}}$ of holomorphic functions with tempered growth. This presheaf is not a sheaf for the usual topology, but it becomes for a suitable Grothendieck topology, the subanalytic topology, and here we shall embed the category of subanalytic sheaves in that of indsheaves.

As we shall see, the indsheaf $\mathcal{O}_X^{\mathrm{t}}$ is not sufficient to obtain a Riemann-Hilbert correspondence, but it is a first step in this direction. To obtain a final result, it is necessary to add an extra variable and to work with an "enhanced" version of $\mathcal{O}_X^{\mathrm{t}}$ in order to describe "various growths" in a rigorous way.

In a first part, we shall recall the main results of the theory of indsheaves and subanalytic sheaves and we shall explain with some details the operations on D-modules and their tempered holomorphic solutions. As an application, we obtain the Riemann-Hilbert correspondence for regular holonomic D-modules as well as the fact that the De Rham functor commutes with integral transforms.

In a second part, we do the same for the sheaf of enhanced tempered solutions of (no more necessarily regular) holonomic D-modules. For that purpose, we first recall the main results of the theory of indsheaves on bordered spaces and its enhanced version, a generalization to indsheaves of a construction of Tamarkin [Ta08]. As an application, we study integral transforms with irregular kernels.

Bibliographical and historical comments. A first important step in a modern treatment of the Riemann-Hilbert correspondence is the book of Deligne [De70]. Then a detailed sketch of proof of the theorem establishing this correspondence (in the regular case) appeared in [Ka80] where the functor *Thom* of tempered cohomology was introduced, and a detailed proof appeared in [Ka84]. Many tools used in the proof of this result were first elaborated in [KK81]. A different proof to this correspondence appeared in [Me84]. The functorial operations on the functor *Thom*, as well as the dual notion, the Whitney tensor product $\overset{\mathrm{w}}{\otimes}$, are systematically studied in [KS96]. These two functors are in fact two particular applications of the theory of indsheaves appeared in [KS01]. This theory also contains that of the subanalytic sheaves, which is much easier and sufficient for most applications. (A direct construction of subanalytic sheaves may be found in [Pr08].)

In the early 2000, it became clear that the indsheaf $\mathcal{O}_X^{\mathrm{t}}$ of tempered holomorphic functions was an essential tool for the study of irregular holonomic modules and a toy model was studied in [KS03]. However, on $X = \mathbb{A}^1(\mathbb{C})$, the two $\mathscr{D}$-modules $\mathscr{M} = \mathscr{D}_X \cdot \exp(1/t)$ and $\mathscr{N} = \mathscr{D}_X \cdot \exp(2/t)$ have the same tempered holomorphic solutions which shows that $\mathcal{O}_X^{\mathrm{t}}$ is not precise enough to treat irregular holonomic $\mathscr{D}$-modules. This difficulty is overcome in [DK13] by adding an extra-variable in order to capture the growth at singular points. This is done, first by adapting to indsheaves a construction of Tamarkin [Ta08], leading to the notion of "enhanced indsheaves", then by defining the indsheaf of "enhanced tempered functions". Using fundamental results of Mochizuki [Mo09, Mo11] (see also Sabbah [Sa00] for preliminary results and see Kedlaya [Ke10, Ke11] for the analytic case), this leads to the solution of the Riemann-Hilbert correspondence for non necessarily regular holonomic D-modules.

Organization of the paper. In these Notes, Sections 1 and 2 are extracted from [KS96, KS01], Section 3 is extracted from [Ka84, Ka03, KS01, KS03] and Sections 4, 5 and 6 are extracted from [DK13] with the exception of Theorem 6.7.1 and subsection 6.10 which are extracted from [KS14].

1 Indsheaves

1.1 Ind-objects

References are made to [SGA4] or to [KS06] for an exposition on ind-objects.

Let $\mathscr{C}$ be a category (in a given universe). One denotes by $\mathscr{C}^\wedge$ the big category of functors from $\mathscr{C}^{\mathrm{op}}$ to **Set**. By the fully faithful functor $h^\wedge\colon \mathscr{C} \to \mathscr{C}^\wedge$, we regard $\mathscr{C}$ as a full subcategory of $\mathscr{C}^\wedge$.

An ind-object in $\mathscr{C}$ is an object $A \in \mathscr{C}^\wedge$ which is isomorphic to $\text{“}\varinjlim_{i\in I}\text{”}\, X_i$ where $X_i \in \mathscr{C}$ and I filtrant and small. Here, $\text{“}\varinjlim\text{”}$ is the inductive limit in $\mathscr{C}^\wedge$. One denotes by $\mathrm{Ind}(\mathscr{C})$ the full subcategory of $\mathscr{C}^\wedge$ consisting of ind-objects.

Theorem 1.1.1. *Let $\mathscr{C}$ be an abelian category.*

(i) *The category $\mathrm{Ind}(\mathscr{C})$ is abelian.*

(ii) *The natural functors $\iota\colon \mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ and $\mathrm{Ind}(\mathscr{C}) \to \mathscr{C}^\wedge$ are fully faithful.*

(iii) *The category $\mathrm{Ind}(\mathscr{C})$ admits exact small filtrant inductive limits, also denoted by $\text{“}\varinjlim\text{”}$ and the functor $\mathrm{Ind}(\mathscr{C}) \to \mathscr{C}^\wedge$ commutes with such limits.*

(iv) *Assume that $\mathscr{C}$ admits small projective limits. Then the category $\mathrm{Ind}(\mathscr{C})$ admits small projective limits, and the functor $\mathscr{C} \to \mathrm{Ind}(\mathscr{C})$ commutes with such limits.*

(v) *Assume that $\mathscr{C}$ admits small inductive limits, denoted by $\varinjlim$. Then the functor ι admits a left adjoint α. For $X = \text{“}\varinjlim_i\text{”}\, X_i$ with $X_i \in \mathscr{C}$ and I small and filtrant, $\alpha(X) \simeq \varinjlim_i X_i$.*

Note that for $X = \text{“}\varinjlim_i\text{”}\, X_i$ and $Y = \text{“}\varinjlim_j\text{”}\, Y_j \in \mathrm{Ind}(\mathscr{C})$ with $X_i, Y_j \in \mathscr{C}$, one has

$$\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(X,Y) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathscr{C}}(X_i, Y_j).$$

Example 1.1.2. Let $\mathbf{k}$ be a field. Denote by $\mathrm{Mod}(\mathbf{k})$ the category of $\mathbf{k}$-vector spaces and by $\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$ its full subcategory consisting of finite-dimensional vector spaces. Denote for short by $\mathrm{I}(\mathbf{k})$ the category of ind-objects of $\mathrm{Mod}(\mathbf{k})$. The functor $\alpha\colon \mathrm{I}(\mathbf{k}) \to \mathrm{Mod}(\mathbf{k})$ admits a left adjoint $\beta\colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{I}(\mathbf{k})$ defined as follows. For $V \in \mathrm{Mod}(\mathbf{k})$, set $\beta(V) = \text{“}\varinjlim\text{”}\, W$, where W ranges over the family of finite-dimensional vector subspaces of V. In other words, $\beta(V)$ is the functor

$$\begin{aligned}&\mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{Z}),\\ &M \mapsto \varinjlim_{W\subset V} \mathrm{Hom}_k(M,W), \quad W \text{ finite-dimensional.}\end{aligned}$$

Note that $\beta(V)(M) \simeq \mathrm{Hom}_{\mathbf{k}}(M,\mathbf{k}) \otimes V$.

If V is infinite-dimensional, $\beta(V)$ is not representable in $\mathrm{Mod}(\mathbf{k})$. Moreover, $\mathrm{Hom}_{\mathrm{I}(\mathbf{k})}(\mathbf{k}, V/\beta(V)) \simeq 0$.

Now, denote by $\mathrm{I}^f(\mathbf{k})$ the category of ind-objects of $\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$. There is an equivalence of categories

$$\alpha\colon \mathrm{I}^f(\mathbf{k}) \xrightarrow{\sim} \mathrm{Mod}(\mathbf{k}), \qquad \text{“}\varinjlim_i\text{” } V_i \mapsto \varinjlim_i V_i.$$

We get the *non* commutative diagram of categories

$$\begin{tikzcd} & \mathrm{I}^f(\mathbf{k}) \arrow[dl, "\sim"'] \arrow[d, "\widetilde{\iota}"] \arrow[dl, phantom, "NC"] \\ \mathrm{Mod}(\mathbf{k}) \arrow[r, "\iota"'] & \mathrm{I}(\mathbf{k}). \end{tikzcd} \tag{1.1.1}$$

Moreover, the functor $\widetilde{\iota}$ commutes with small inductive limits but the functor ι does not.

It is proved in [KS06] that the category $\mathrm{I}(\mathbf{k})$ does not have enough injectives.

Definition 1.1.3. An object $A \in \mathrm{Ind}(\mathscr{C})$ is quasi-injective if the functor $\mathrm{Hom}_{\mathrm{Ind}(\mathscr{C})}(\,\bullet\,, A)$ is exact on the category $\mathscr{C}$.

It is proved in loc. cit. that if $\mathscr{C}$ has enough injectives, then $\mathrm{Ind}(\mathscr{C})$ has enough quasi-injectives.

1.2 Indsheaves

We refer to [KS90] for all notions of sheaf theory used here. For simplicity, we denote by $\mathbf{k}$ a field, although most of the results would remain true when $\mathbf{k}$ is a commutative ring of finite global dimension.

A topological space is *good* if it is Hausdorff, locally compact, countable at infinity and has finite flabby dimension. Let M be such a space. One denotes by $\mathrm{Mod}(\mathbf{k}_M)$ the abelian category of sheaves of $\mathbf{k}$-modules on M and by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ its bounded derived category. Note that $\mathrm{Mod}(\mathbf{k}_M)$ has a finite homological dimension.

For a locally closed subset A of M, one denotes by $\mathbf{k}_A$ the constant sheaf on A with stalk $\mathbf{k}$ extended by 0 on $X \setminus A$. For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, one sets $F_A := F \otimes \mathbf{k}_A$.

We shall make use of the dualizing complex on M, denoted by ω_M, and the duality functors

$$\mathrm{D}'_M := \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \mathbf{k}_M), \quad \mathrm{D}_M := \mathrm{R}\mathscr{H}om\,(\,\bullet\,, \omega_M). \tag{1.2.1}$$

Recall that, when M is a real manifold, ω_M is isomorphic to the orientation sheaf shifted by the dimension.

One denotes by $\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$ the full subcategory of $\mathrm{Mod}(\mathbf{k}_M)$ consisting of sheaves with compact support. We set for short:

$$\mathrm{I}(\mathbf{k}_M) := \mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M))$$

and calls an object of this category an *indsheaf* on M.

When there is no risk of confusion, we shall simply write $\mathrm{I}\mathbf{k}_M$ instead of $\mathrm{I}(\mathbf{k}_M)$.

Theorem 1.2.1. *The prestack $\mathcal{I}(\mathbf{k}_M) : U \mapsto \mathrm{I}(\mathbf{k}_U)$, U open in M, is a stack.*

For $F = \text{“}\varinjlim_i\text{”}\, F_i \in \mathrm{I}(\mathbf{k}_M)$ and $G = \text{“}\varinjlim_j\text{”}\, G_j \in \mathrm{I}(k_M)$ with F_i, $G_j \in \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$, we set:

$$\begin{aligned} F \otimes G &= \text{“}\varinjlim_{i,j}\text{”}(F_i \otimes G_j), \\ \mathscr{I}hom\,(F, G) &= \varprojlim_i \text{“}\varinjlim_j\text{”}\, \mathscr{H}om\,(F_i, G_j). \end{aligned}$$

Note that for $F \in \mathrm{Mod}(\mathbf{k}_M)$ and $\{G_j\}_{j \in J}$ a small filtrant inductive system in $\mathrm{I}(k_M)$, we have

$$\mathscr{I}hom\,(F, \text{“}\varinjlim_j\text{”}\, G_j) \simeq \text{“}\varinjlim_j\text{”}\, \mathscr{I}hom\,(F, G_j).$$

Lemma 1.2.2. *The category $\mathrm{I}(\mathbf{k}_M)$ is a tensor category with $\otimes$ as a tensor product and $\mathbf{k}_M$ as a unit object.*

Note that $\mathscr{I}hom$ is the inner hom of the tensor category $\mathrm{I}(\mathbf{k}_M)$, i.e., we have

$$\mathrm{Hom}_{\mathrm{I}(\mathbf{k}_M)}(K_1 \otimes K_2, K_3) \simeq \mathrm{Hom}_{\mathrm{I}(\mathbf{k}_M)}\big(K_1, \mathscr{I}hom\,(K_2, K_3)\big).$$

We have two pairs (α_M, ι_M) and (β_M, α_M) of adjoint functors

$$\begin{tikzcd} \mathrm{Mod}(\mathbf{k}_M) \arrow[r, shift left=2, "\iota_M"] \arrow[r, leftarrow, "\alpha_M" description] \arrow[r, shift right=2, "\beta_M"'] & \mathrm{I}(\mathbf{k}_M). \end{tikzcd}$$

The functor ι_M is given by

$$\iota_M F = \text{“}\varinjlim_{U \subset\subset M}\text{”}\, F_U,\ U \text{ open relatively compact in } M.$$

The functor α_M is defined by

$$\alpha_M \colon \text{“}\varinjlim_i\text{”}\, F_i \mapsto \varinjlim_i F_i \ (I \text{ small and filtrant}).$$

For $F \in \mathrm{Mod}(\mathbf{k}_M)$, $\beta_M(F)$ is the functor

$$\beta_M(F) \colon G \mapsto \Gamma(M; H^0(\mathrm{D}'_M G) \otimes F), \quad (G \in \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)).$$

(This last formula is no more true if $\mathbf{k}$ is not a field.)

- ι_M is exact, fully faithful, and commutes with $\varprojlim$,
- α_M is exact and commutes with $\varprojlim$ and $\varinjlim$,
- β_M is exact, fully faithful and commutes with $\varinjlim$,
- α_M is left adjoint to ι_M,
- α_M is right adjoint to β_M,
- $\alpha_M \circ \iota_M \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_M)}$ and $\alpha_M \circ \beta_M \simeq \mathrm{id}_{\mathrm{Mod}(\mathbf{k}_M)}$.

Denote as usual by

$$\mathscr{H}om_{\mathrm{I}\mathbf{k}_M} \colon \mathrm{I}(\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{I}(\mathbf{k}_M) \to \mathrm{Mod}(\mathbf{k}_M)$$

the *hom* functor of the stack $\mathcal{I}(\mathbf{k}_M)$. Then

$$\mathscr{H}om_{\mathrm{I}\mathbf{k}_M} \simeq \alpha_M \circ \mathscr{I}hom\,.$$

Notation 1.2.3. As far as there is no risk of confusion, we shall not write the functor ι_M. Hence, we identify a sheaf F on M and its image by ι_M.

Example 1.2.4. Let $U \subset M$ be an open subset, $S \subset M$ a closed subset. Then

$$\beta_M(\mathbf{k}_U) \simeq \text{“}\varinjlim_{V}\text{”}\, \mathbf{k}_V,\ V \text{ open },V \subset\subset U,$$
$$\beta_M(\mathbf{k}_S) \simeq \text{“}\varinjlim_{V}\text{”}\, \mathbf{k}_{\overline{V}},\ V \text{ open },S \subset V.$$

Let $a \in M$ and consider the skyscraper sheaf $\mathbf{k}_{\{a\}}$. Then $\beta_M(\mathbf{k}_{\{a\}}) \to \mathbf{k}_{\{a\}}$ is an epimorphism in $\mathrm{I}(\mathbf{k}_M)$ and defining N_a by the exact sequence:

$$0 \to N_a \to \beta_M(\mathbf{k}_{\{a\}}) \to \mathbf{k}_{\{a\}} \to 0,$$

we get that $\mathrm{Hom}_{\mathbf{I}\mathbf{k}_M}(\mathbf{k}_U, N_a) \simeq 0$ for all open neighborhood U of a.

Let $f\colon M \to N$ be a continuous map.

Let $G = \text{“}\varinjlim_{i}\text{”}\, G_i \in \mathrm{I}(k_N)$ with $G_i \in \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_N)$. One defines $f^{-1}G \in \mathrm{I}(\mathbf{k}_M)$ by the formula

$$f^{-1}G = \text{“}\varinjlim_{i}\text{”}\, f^{-1}G_i.$$

Let $F = \text{“}\varinjlim_{i}\text{”}\, F_i \in \mathrm{I}(\mathbf{k}_M)$ with $F_i \in \mathrm{Mod}^{\mathrm{c}}(k_M)$. One defines $f_*F \in \mathrm{I}(\mathbf{k}_N)$ by the formula:

$$f_*(\text{“}\varinjlim_{i}\text{”}\, F_i) \quad = \quad \varprojlim_{K} \text{“}\varinjlim_{i}\text{”}\, f_*(F_{iK}) \ (K \text{ compact in } M).$$

The two functors f_* and f^{-1} commute with both the functors ι_M and α_M and that is the reason why we keep the same notations as for usual sheaves.

Recall that for a usual sheaf F, its proper direct image is defined by

$$f_!F = \varinjlim_{U \subset\subset M} f_*F_U.$$

Hence, one defines the proper direct image of $F = \text{“}\varinjlim_{i}\text{”}\, F_i \in \mathrm{I}(\mathbf{k}_M)$ with $F_i \in \mathrm{Mod}^{\mathrm{c}}(k_M)$ by

$$f_{!!}(\text{“}\varinjlim_{i}\text{”}\, F_i) \quad = \quad \text{“}\varinjlim_{i}\text{”}\, f_*(F_i).$$

However, $f_{!!}\circ\iota_M \neq \iota_M\circ f_!$ in general. That is why we have used a different notation.

The category $\mathrm{I}(\mathbf{k}_M)$ does not have enough injectives even for $M = \mathrm{pt}$ (see [KS06, Proposition 15.1.2]). In particular, it is not a Grothendieck category. One can however construct the derived functors and the six operations for indsheaves. The functor f^{-1} has a right adjoint $\mathrm{R}f_*$. The functor $\mathrm{R}f_{!!}$ admits a left adjoint, denoted by $f^{\,!}$.

Hence we have functors

$$
\begin{aligned}
\iota_M &: \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M),\\
\alpha_M &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M),\\
\beta_M &: \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M),\\
\otimes &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)\times\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M),\\
\mathrm{R}\mathcal{I}hom &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}}\times\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathrm{D}^{+}(\mathrm{I}\mathbf{k}_M),\\
\mathrm{R}\mathcal{H}om_{\mathrm{I}\mathbf{k}_M} &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}}\times\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathrm{D}^{+}(\mathbf{k}_M),\\
\mathrm{R}f_* &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N),\\
f^{-1} &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M),\\
\mathrm{R}f_{!!} &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N),\\
f^{\,!} &: \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M).
\end{aligned}
$$

We may summarize the commutativity of the various functors we have introduced in the table below. Here, "$\circ$" means that the functors commute, and "$\times$" they do not. Moreover, $\varinjlim$ are taken over small filtrant categories.

(1.2.2)

	$\otimes$	f^{-1}	f_*	$f_{!!}$	$f^{\,!}$	$\varinjlim$	$\varprojlim$
ι	$\circ$	$\circ$	$\circ$	$\times$	$\circ$	$\times$	$\circ$
α	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\circ$	$\circ$
β	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\circ$	$\times$

Note that the pairs $(f^{-1}, \mathrm{R}f_*)$ and $(\mathrm{R}f_{!!}, f^{\,!})$ are pairs of adjoint functors, which implies their commutation with $\varinjlim$ or $\varprojlim$, as usual. Finally, note that the functor $f^{\,!}$ commutes with filtrant inductive limits (after taking the cohomology).

1.3 Ring action

We do not recall here the notion of a ring object $\mathcal{B}$ or a $\mathcal{B}$-module in a tensor category $\mathcal{S}$ (see [KS01, § 5.4]). (In the sequel, we shall consider the tensor category $\mathrm{I}(\mathbf{k}_M)$, see Lemma 1.2.2.) For such a ring object $\mathcal{B}$ in $\mathcal{S}$, we denote by $\mathrm{Mod}(\mathcal{B})$ the abelian category of $\mathcal{B}$-modules in $\mathcal{S}$ and by $\mathsf{D}^{\mathrm{b}}(\mathcal{B})$ its derived category.

We shall encounter the following situation. Let $\mathcal{A}$ be a sheaf of $\mathbf{k}$-algebras on M. Consider an object $\mathscr{M}$ of $\mathrm{I}(\mathbf{k}_M)$ together with a morphism of sheaves of $\mathbf{k}$-algebras

$$\mathcal{A} \to \mathscr{E}nd_{\mathcal{I}(\mathbf{k}_M)}(\mathscr{M}).$$

In this case one says that $\mathscr{M}$ is an $\mathcal{A}$-module in $\mathrm{I}(\mathbf{k}_M)$. One denotes by

- $\mathrm{I}(\mathcal{A})$ the abelian category of $\mathcal{A}$-modules in $\mathrm{I}(\mathbf{k}_M)$,
- $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A}) := \mathsf{D}^{\mathrm{b}}(\mathrm{I}(\mathcal{A}))$ its bounded derived category. We use similar notations with D^{b} replaced with D^{+}, D^{-} and D.

One shall not confuse the category $\mathrm{I}(\mathcal{A})$ with the category $\mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathcal{A}))$ of ind-objects of the category of sheaves of $\mathcal{A}$-modules with compact support, and we shall not confuse their derived categories.

If $\mathcal{A}$ is a sheaf of $\mathbf{k}$-algebras as above, then $\beta_M\mathcal{A}$ is a ring-object in the tensor category $\mathrm{I}(\mathbf{k}_M)$. Since

$$\mathrm{Hom}_{\mathbf{k}_M}(\mathcal{A}, \mathscr{H}om_{\mathrm{I}\mathbf{k}_M}(\mathscr{M}, \mathscr{M})) \simeq \mathrm{Hom}_{\mathrm{I}(\mathbf{k}_M)}(\beta_M\mathcal{A}, \mathscr{I}hom\,(\mathscr{M}, \mathscr{M})),$$

we get equivalences of categories

$$\mathrm{Mod}(\beta_M\mathcal{A}) \simeq \mathrm{I}(\mathcal{A}), \quad \mathsf{D}^{\mathrm{b}}(\beta_M\mathcal{A}) \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A}).$$

Remark 1.3.1. Our notations differ from that of [KS01, § 5.4, § 5.5].
- For a ring object $\mathscr{B}$ in $\mathrm{I}(\mathbf{k}_M)$, $\mathrm{Mod}(\mathscr{B})$ in our notation was denoted by $\mathrm{I}(\mathscr{B})$ in [KS01].
- For a sheaf of rings $\mathscr{A}$, $\mathrm{I}(\mathcal{A})$ in our notation was denoted by $\mathrm{I}(\beta\mathcal{A})$ and $\mathrm{Ind}(\mathrm{Mod}^{\mathrm{c}}(\mathcal{A}))$ in our notation was denoted by $\mathrm{I}(\mathcal{A})$ in [KS01].

See [KS01, Exe. 3.4, Def. 4.1.2, Def. 5.4.4, Exe. 5.3].

We have the quasi-commutative diagram

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{A}) \arrow[r, shift left, "\beta_M"] \arrow[d] & \mathrm{I}(\mathcal{A}) \arrow[l, shift left, "\alpha_M"] \arrow[d] \\ \mathrm{Mod}(\mathbf{k}_M) \arrow[r, shift left, "\beta_M"] & \mathrm{I}(\mathbf{k}_M). \arrow[l, shift left, "\alpha_M"] \end{tikzcd} \tag{1.3.1}$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathcal{A}^{\mathrm{op}})$ and $K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A})$ one gets the objects, functorially in $\mathscr{M}$, $\mathscr{N}$, K:

$$\mathrm{R}\mathscr{H}om_{\mathcal{A}}(\mathscr{M}, K) \in \mathrm{D}^{+}(\mathrm{I}\mathbf{k}_M), \quad \mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} K \in \mathrm{D}^{-}(\mathrm{I}\mathbf{k}_M).$$

They are characterized by

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}(\mathrm{I}\mathbf{k}_M)}\big(L, \mathrm{R}\mathscr{H}om_{\mathcal{A}}(\mathscr{M}, K)\big) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}\big(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbf{k}_M}(L, K)\big),\\
&\mathrm{Hom}_{\mathsf{D}(\mathrm{I}\mathbf{k}_M)}\big(\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} K, L\big) \simeq \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}\big(\mathscr{N}, \mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbf{k}_M}(K, L)\big)
\end{aligned}$$

for any $L \in \mathsf{D}(\mathrm{I}\mathbf{k}_M)$.

Proposition 1.3.2. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$, $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathcal{A}^{\mathrm{op}})$ and $\mathscr{K} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A})$. There are natural isomorphisms:*

$$\begin{aligned}
\mathrm{R}\mathscr{H}om_{\mathcal{A}}(\mathscr{M}, \mathscr{K}) &\simeq \mathrm{R}\mathscr{I}hom_{\beta_M\mathcal{A}}(\beta_M\mathscr{M}, \mathscr{K}) \quad \textit{in } \mathrm{D}^{+}(\mathrm{I}\mathbf{k}_M),\\
\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} \mathscr{K} &\simeq \beta_M\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\beta_M\mathcal{A}} \mathscr{K} \quad \textit{in } \mathrm{D}^{-}(\mathrm{I}\mathbf{k}_M).
\end{aligned}$$

Proof. Let $L \in \mathsf{D}^{+}(\mathrm{I}\mathbf{k}_M)$. We have the sequence of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(\mathrm{I}\mathbf{k}_M)}(L, \mathrm{R}\mathscr{H}om_{\mathcal{A}}(\mathscr{M}, \mathscr{K})) &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbf{k}_M}(L, \mathscr{K}))\\
&\simeq \mathrm{Hom}_{\mathrm{D}(\beta_M\mathcal{A})}(\beta_M\mathscr{M}, \mathrm{R}\mathscr{I}hom(L, \mathscr{K}))\\
&\simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{I}\mathbf{k}_M)}(L, \mathrm{R}\mathscr{I}hom_{\beta_M\mathcal{A}}(\beta_M\mathscr{M}, \mathscr{K})).
\end{aligned}$$

The second formula is proved similarly. Q.E.D.

Notation 1.3.3. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A})$, $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A}^{\mathrm{op}})$ and $\mathscr{K} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A})$, we shall use the notations $\mathrm{R}\mathscr{I}hom_{\beta\mathcal{A}}(\mathscr{M}, \mathscr{K})$ and $\mathscr{N} \overset{\mathrm{L}}{\otimes}_{\beta\mathcal{A}} \mathscr{K}$, objects of $\mathrm{D}(\mathrm{I}\mathbf{k}_M)$.

Let us briefly recall a few basic formulas.

We consider the following situation: $f\colon M \to N$ is a continuous map of good topological spaces and $\mathscr{R}$ is a sheaf of $\mathbf{k}$-algebras on N.

In the sequel, $\mathsf{D}^{\dagger}$ is D, D^{b}, D^{+} or D^{-}.

Theorem 1.3.4. (a) *The functor $f^{-1}\colon \mathrm{I}(\mathbf{k}_N) \to \mathrm{I}(\mathbf{k}_M)$ induces a functor $f^{-1} : \mathsf{D}^{\dagger}(\mathrm{I}\mathscr{R}) \to \mathsf{D}^{\dagger}(\mathrm{I}f^{-1}\mathscr{R})$,*

(b) *The functor $f_*\colon \mathrm{I}(\mathbf{k}_M) \to \mathrm{I}(\mathbf{k}_N)$ induces a functor $\mathrm{R}f_*\colon \mathsf{D}^{\dagger}(\mathrm{I}f^{-1}\mathscr{R}) \to \mathsf{D}^{\dagger}(\mathrm{I}\mathscr{R})$,*

(c) *The functor* $f_{!!}\colon \mathrm{I}(\mathbf{k}_M) \to \mathrm{I}(\mathbf{k}_N)$ *induces a functor* $\mathrm{R}f_{!!}\colon \mathsf{D}^{\dagger}(\mathrm{I}f^{-1}\mathscr{R}) \to \mathsf{D}^{\dagger}(\mathrm{I}\mathscr{R})$,

(d) *the functor* $\mathrm{R}f_{!!}\colon \mathsf{D}^{+}(\mathrm{I}f^{-1}\mathscr{R}) \to \mathsf{D}^{+}(\mathrm{I}\mathscr{R})$ *admits a a right adjoint, denoted by* $f^{\,!}$.

Theorem 1.3.5. (a) *For* $G \in \mathsf{D}^{-}(\mathrm{I}\mathscr{R})$ *and* $F \in \mathsf{D}^{+}(\mathrm{I}f^{-1}\mathscr{R})$, *one has the isomorphism*

$$\mathrm{R}\mathscr{I}hom_{\beta_N\mathscr{R}}(G, \mathrm{R}f_*F) \;\simeq\; \mathrm{R}f_*\mathrm{R}\mathscr{I}hom_{f^{-1}\beta_N\mathscr{R}}(f^{-1}G, F).$$

(b) *For* $G \in \mathsf{D}^{+}(\mathrm{I}\mathscr{R})$ *and* $F \in \mathsf{D}^{-}(\mathrm{I}f^{-1}\mathscr{R})$, *one has the isomorphism*

$$\mathrm{R}\mathscr{I}hom_{\beta_N\mathscr{R}}(\mathrm{R}f_{!!}F, G) \;\simeq\; \mathrm{R}f_*\mathrm{R}\mathscr{I}hom_{f^{-1}\beta_N\mathscr{R}}(F, f^{\,!}G).$$

(c) (Projection formula.)*For* $F \in \mathsf{D}^{-}(\mathrm{I}f^{-1}\mathscr{R})$ *and* $G \in \mathsf{D}^{-}(\mathrm{I}\mathscr{R}^{\mathrm{op}})$, *one has the isomorphism*

$$G \overset{\mathrm{L}}{\otimes}_{\beta_N\mathscr{R}} \mathrm{R}f_{!!}F \simeq \mathrm{R}f_{!!}(f^{-1}G \overset{\mathrm{L}}{\otimes}_{f^{-1}\beta_N\mathscr{R}} F).$$

(d) (Base change formula.) *Consider the Cartesian square of good topological spaces*

(1.3.2)
$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & N' \arrow[d, "g"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

There are natural isomorphisms of functors from $\mathsf{D}^{\dagger}(\mathrm{I}f^{-1}\mathscr{R})$ *to* $\mathsf{D}^{\dagger}(\mathrm{I}g^{-1}\mathscr{R})$:

(1.3.3) $$\mathrm{R}f'_{!!}g'^{-1} \simeq g^{-1}\mathrm{R}f_{!!},$$

(1.3.4) $$\mathrm{R}f'_{*}g'^{\,!} \simeq g^{\,!}\mathrm{R}f_{*}.$$

Note that Theorem 1.3.6 below has no counterpart in classical sheaf theory.

Theorem 1.3.6. *Let* $\mathscr{A}$ *be a sheaf of* $\mathbf{k}_M$*-algebras, let* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, *let* $\mathscr{K} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathcal{A}^{\mathrm{op}})$ *and let* $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathcal{A})$. *Then one has the isomorphism:*

(1.3.5) $$\mathrm{R}\mathscr{I}hom(F, \mathscr{K}) \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} \mathscr{L} \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom(F, \mathscr{K} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}} \mathscr{L}).$$

Thanks to Proposition 1.3.2, isomorphism (1.3.5) may also be formulated as

$$\mathrm{R}\mathscr{I}hom(F,\mathscr{K})\overset{\mathrm{L}}{\otimes}_{\beta_M\mathcal{A}}\beta_M\mathscr{L}\xrightarrow{\sim}\mathrm{R}\mathscr{I}hom(F,\mathscr{K}\overset{\mathrm{L}}{\otimes}_{\beta_M\mathcal{A}}\beta_M\mathscr{L}). \tag{1.3.6}$$

Also note that (1.3.5) is no more true if we relax the hypothesis that $F\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$.

1.4 Sheaves on the subanalytic site

Recall first that, for real analytic manifolds M, N and a closed subanalytic subset S of M, we say that a map $f\colon S\to N$ is subanalytic if its graph is subanalytic in $M\times N$. One denotes by $\mathscr{A}_S^{\mathbb{R}}$ the sheaf of continuous $\mathbb{R}$-valued subanalytic maps on S. A *subanalytic space* $(M,\mathscr{A}_M^{\mathbb{R}})$, or simply M for short, is an $\mathbb{R}$-ringed space locally isomorphic to $(S,\mathscr{A}_S^{\mathbb{R}})$ for a closed subanalytic subset S of a real analytic manifold.

We can define the notion of subanalytic subsets of a subanalytic space, as well as $\mathbb{R}$-constructible sheaves on a subanalytic space.

Definition 1.4.1. Let M be a subanalytic space, Op_M the category of its open subsets, the morphisms being the inclusion. One denotes by $\mathrm{Op}_{M_{\mathrm{sa}}}$ the full subcategory of Op_M consisting of subanalytic relatively compact open subsets. The site M_{sa} is obtained by deciding that a family $\{U_i\}_{i\in I}$ of subobjects of $U\in\mathrm{Op}_{M_{\mathrm{sa}}}$ is a covering of U if there exists a finite subset $J\subset I$ such that $\bigcup_{j\in J}U_j=U$.

Let us denote by

$$\rho_M\colon M\to M_{\mathrm{sa}} \tag{1.4.1}$$

the natural morphism of sites and, as usual, by $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$ the Grothendieck category of sheaves of $\mathbf{k}$-modules on M_{sa}. Hence, (ρ_M^{-1},ρ_{M*}) is a pair of adjoint functors.

Note that

$$\begin{cases}\text{a presheaf } F \text{ on } M_{\mathrm{sa}} \text{ is a sheaf if and only if } F(\emptyset)=0 \text{ and for any}\\ \text{pair } (U_1,U_2) \text{ in } \mathrm{Op}_{M_{\mathrm{sa}}}\text{, the sequence below is exact:}\\ \qquad 0\to F(U_1\cup U_2)\to F(U_1)\oplus F(U_2)\to F(U_1\cap U_2).\end{cases} \tag{1.4.2}$$

The functor ρ_M^{-1} also admits a left adjoint, denoted by $\rho_{M!}$. For $F\in\mathrm{Mod}(\mathbf{k}_M)$, $\rho_{M!}F$ is the sheaf associated to the presheaf $U\mapsto F(\overline{U})$, $U\in$

$\mathrm{Op}_{M_{\mathrm{sa}}}$. Hence we have the two pairs of adjoint functors (ρ_M^{-1}, ρ_{M*}) and $(\rho_{M!}, \rho_M^{-1})$

$$\mathrm{Mod}(\mathbf{k}_M) \underset{\rho_{M!}}{\overset{\rho_{M*}}{\rightleftarrows}} \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

The functor $\rho_{M!}$ is exact.

One denotes by "$\varinjlim$" the inductive limit in the category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Inductive limits do not commute with the functor ρ_{M*}.

Remark 1.4.2. It would be possible to develop the theory of subanalytic sheaves and in particular the six operations (see [Pr08]). However, in these Notes, we prefer to embed the category of subanalytic sheaves into that of indsheaves, as we shall do now.

Denote by $\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)$ the small abelian category of $\mathbb{R}$-constructible sheaves (see [KS90] for an exposition) and denote by $\mathbb{R}\text{-}\mathrm{C}^c(\mathbf{k}_M)$ the full subcategory consisting of sheaves with compact support. Recall that $\mathsf{D}^{\mathrm{b}}(\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)) \simeq \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}\mathrm{c}}(\mathbf{k}_M)$. Set

$$\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) = \mathrm{Ind}(\mathbb{R}\text{-}\mathrm{C}^c(\mathbf{k}_M)).$$

The fully faithful functor $\mathbb{R}\text{-}\mathrm{C}^c(\mathbf{k}_M) \to \mathrm{Mod}^{\mathrm{c}}(\mathbf{k}_M)$ induces a fully faithful functor $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \to \mathrm{I}(\mathbf{k}_M)$, by which we regard $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M)$ as a full subcategory of $\mathrm{I}(\mathbf{k}_M)$.

We say that an indsheaf on M is a *subanalytic* indsheaf if it is isomorphic to an object of $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M)$.

We have a quasi-commutative diagram of categories in which all arrows are exact and fully faithful:

(1.4.3)
$$\begin{tikzcd} \mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M) \arrow[r, "\iota_M^{rc}"] \arrow[d] & \mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \arrow[d] \\ \mathrm{Mod}(\mathbf{k}_M) \arrow[r, "\iota_M"] & \mathrm{I}(\mathbf{k}_M). \end{tikzcd}$$

Proposition 1.4.3. *The restriction of the functor ρ_{M*} to the category $\mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M)$ is exact and fully faithful.*

We have a natural functor

$$\lambda_M \colon \mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \to \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}), \quad \text{“}\varinjlim_i\text{”}\, F_i \mapsto \text{“}\varinjlim_i\text{”}\, \rho_{M*}F_i, \tag{1.4.4}$$

where the first “$\varinjlim$” is taken in the category $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M)$ and the second one is taken in the category $\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

Theorem 1.4.4. *The functor λ_M in (1.4.4) is an equivalence.*

In other words, subanalytic indsheaves are usual sheaves on the subanalytic site. By this result, the embedding $\mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \hookrightarrow \mathrm{I}(\mathbf{k}_M)$ gives an exact and fully faithful functor

$$\widetilde{\iota}_M \colon \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \to \mathrm{I}(\mathbf{k}_M).$$

Note that for $G \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$, one has

$$\widetilde{\iota}_M G \simeq \text{“}\varinjlim_{\rho_{M*}F \to G}\text{”}\, F, \text{ where } F \in \mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M).$$

Also note that

$$\widetilde{\iota}_M \mathscr{H}om\,(F,G) \simeq \mathscr{I}hom\,(F,\widetilde{\iota}_M G) \text{ for } F \in \mathbb{R}\text{-}\mathrm{C}(\mathbf{k}_M),\ G \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}).$$

We have the following diagrams, where the one in the left is *non* commutative and the one in the right is commutative (see Diagram 1.1.1 for the case $M = \mathrm{pt}$):

$$\begin{tikzcd}
\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \arrow[dr, "\widetilde{\iota}_M"] & \\
\mathrm{Mod}(\mathbf{k}_M) \arrow[u, "\rho_{M*}"] \arrow[r, "\iota_M"'] \arrow[ur, phantom, "NC"] & \mathrm{I}(\mathbf{k}_M),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \arrow[r, "\sim"] \arrow[dr, "\widetilde{\iota}_M"'] & \mathrm{I}_{\mathbb{R}-\mathrm{c}}(\mathbf{k}_M) \arrow[d] \\
& \mathrm{I}(\mathbf{k}_M).
\end{tikzcd} \tag{1.4.5}$$

The functors ι_M and $\widetilde{\iota}_M$ are exact but ρ_{M*} is not right exact in general.

Lemma 1.4.5. *The two diagrams below commute:*

$$\begin{tikzcd}
\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \arrow[d, "\rho_M^{-1}"'] \arrow[dr, "\widetilde{\iota}_M"] & \\
\mathrm{Mod}(\mathbf{k}_M) & \mathrm{I}(\mathbf{k}_M), \arrow[l, "\alpha_M"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}}) \arrow[dr, "\widetilde{\iota}_M"] & \\
\mathrm{Mod}(\mathbf{k}_M) \arrow[u, "\rho_{M!}"] \arrow[r, "\beta_M"'] & \mathrm{I}(\mathbf{k}_M).
\end{tikzcd} \tag{1.4.6}$$

Proof. (i) Let us prove the commutation of the diagram on the left. Since all functors in the diagram commute with inductive limits, we are reduced to prove the isomorphism $\rho_M^{-1}\rho_{M*}F \simeq \alpha_M\widetilde{\iota}_M\rho_{M*}F$ for $F \in \mathbb{R}\text{-C}^c(\mathbf{k}_M)$ and the result is clear in this case.
(ii) Let us prove the commutation of the diagram on the right. Again all functors in the diagram commute with inductive limits. We shall first prove that

(1.4.7) the functor β_M factors as $\beta_M = \widetilde{\iota}_M \circ \lambda_M$ for a functor $\lambda_M \colon \mathrm{Mod}(\mathbf{k}_M) \to \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$.

First consider the case of $F = \mathbf{k}_U$ for U open and relatively compact in M. In this case,

$$\beta_M\mathbf{k}_U \simeq \text{“}\varinjlim_{V\subset\subset U}\text{”}\,\mathbf{k}_V,\ V \text{ open in } M$$

and we may assume that V is subanalytic. Hence $\beta_M\mathbf{k}_U$ is a subanalytic indsheaf. Since any $F \in \mathrm{Mod}(\mathbf{k}_M)$ is obtained by taking direct sums and cokernels of sheaves of the type $\mathbf{k}_U$ and the subcategory of subanalytic indsheaves is stable by these operations, $\beta_M F$ is a subanalytic indsheaf for any $F \in \mathrm{Mod}(\mathbf{k}_M)$ and we get (1.4.7). It remains to prove that $\lambda_M \simeq \rho_{M!}$. Let $F \in \mathrm{Mod}(\mathbf{k}_M)$ and $G \in \mathrm{Mod}(\mathbf{k}_{M_{\mathrm{sa}}})$. Using (i) and the fact that $\widetilde{\iota}_M$ is fully faithful, we have

$$\begin{aligned}
\mathrm{Hom}\,(\rho_{M!}F, G) &\simeq \mathrm{Hom}\,(F, \rho_M^{-1}G) \simeq \mathrm{Hom}\,(F, \alpha_M\widetilde{\iota}_M G)\\
&\simeq \mathrm{Hom}\,(\beta_M F, \widetilde{\iota}_M G) \simeq \mathrm{Hom}\,(\widetilde{\iota}_M\lambda_M F, \widetilde{\iota}_M G)\\
&\simeq \mathrm{Hom}\,(\lambda_M F, G).
\end{aligned}$$

Q.E.D.

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$ the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ consisting of objects with subanalytic indsheaves as cohomologies. By [KS01, Th 7.1], we have:

Theorem 1.4.6. *The functor $\widetilde{\iota}_M$ induces an equivalence of triangulated categories*

(1.4.8) $$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{\mathrm{sa}}}) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M).$$

Proposition 1.4.7. *Let M be a subanalytic space.*

(i) *Let* $K, L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$. *Then* $K \otimes L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$.

(ii) *Let* $K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$ *and let* $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbf{k}_M)$. *Then* $\mathrm{R}\mathscr{I}hom\,(F, K) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$.

Proposition 1.4.8. *Let* $f\colon M \to N$ *be a morphism of subanalytic spaces.*

(i) *For* $L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_N)$, *we have* $f^{-1}L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$ *and* $f^{\,!}\,L \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$.

(ii) *For* $K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$, *we have* $\mathrm{R}f_{!!}K \in \mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_N)$.

The next result will be of a constant use.

Proposition 1.4.9. *A morphism* $u\colon K \to L$ *in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$ *is an isomorphism if and only if, for any relatively compact subanalytic open subset* U *of* M *and any* $n \in \mathbb{Z}$, u *induces an isomorphism* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)}(\mathbf{k}_U[n], K) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)}(\mathbf{k}_U[n], L)$.

1.5 Some classical sheaves on the subanalytic site

In this subsection, M is a real analytic manifold and we take $\mathbb{C}$ as the base field $\mathbf{k}$.

1.5.1 Tempered and Whitney functions and distributions

As usual, we denote by $\mathcal{C}^{\infty}_M$ (resp. $\mathcal{C}^{\omega}_M$) the sheaf of $\mathbb{C}$-valued functions of class C^{∞} (resp. real analytic) and by $\mathcal{D}b_M$ (resp. $\mathscr{B}_M$) the sheaf of Schwartz's distributions (resp. Sato's hyperfunctions). We also use the notation $\mathscr{A}_M = \mathcal{C}^{\omega}_M$. We denote by $\mathscr{D}_M$ the sheaf of finite-order differential operators with real analytic coeffiecients. References for D-modules are made to [Ka03].

Definition 1.5.1. Let U be an open subset of M and $f \in \mathcal{C}^{\infty}_M(U)$. One says that f has *polynomial growth* at $p \in M$ if f satisfies the following condition: for a local coordinate system $(x_1, \dots, x_n)$ around p, there exist a sufficiently small compact neighborhood K of p and a positive integer N such that

$$\sup_{x \in K \cap U} \big(\mathrm{dist}(x, K \setminus U)\big)^N |f(x)| < \infty\,. \tag{1.5.1}$$

Here we understand that the left-hand side of (1.5.1) is 0 if $K \cap U = \emptyset$ or $K \setminus U = \emptyset$. It is obvious that f has polynomial growth at any point of U. We say that f is tempered at p if all its derivatives have polynomial growth at p. We say that f is tempered if it is tempered at any point of M.

An important property of subanalytic subsets is given by the lemma below. (See Lojasiewicz [Lo59] and also [Ma66] for a detailed study of its consequences.)

Lemma 1.5.2. *Let U and V be two relatively compact open subanalytic subsets of $\mathbb{R}^n$. Then there exist a positive integer N and $C > 0$ such that*

$$\operatorname{dist}\big(x, \mathbb{R}^n \setminus (U \cup V)\big)^N \le C\big(\operatorname{dist}(x, \mathbb{R}^n \setminus U) + \operatorname{dist}(x, \mathbb{R}^n \setminus V)\big).$$

For an open subanalytic subset U in M, denote by $\mathcal{C}_M^{\infty,\mathrm{t}}(U)$ the subspace of $\mathcal{C}_M^{\infty}(U)$ consisting of tempered functions.

Denote by $\mathcal{D}b_M^{\mathrm{t}}(U)$ the space of tempered distributions on U, the image of the restriction map $\Gamma(M; \mathcal{D}b_M) \to \Gamma(U; \mathcal{D}b_M)$. Using Lemma 1.5.2 and (1.4.2) one proves:

- the presheaf $U \mapsto \mathcal{C}_M^{\infty,\mathrm{t}}(U)$ is a sheaf on M_{sa},
- the presheaf $U \mapsto \mathcal{D}b_M^{\mathrm{t}}(U)$ is a sheaf on M_{sa}.

One denotes them by $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{t}}$ and $\mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{t}}$.

For a closed subanalytic subset S in M, denote by $\mathrm{I}_{M,S}^{\infty}$ the space of C^{∞}-functions defined on M which vanish up to infinite order on S. In [KS96], one introduced the sheaf:

$$\mathbb{C}_U \overset{\mathrm{w}}{\otimes} \mathcal{C}_M^{\infty} \quad := \quad V \longmapsto \mathrm{I}_{V, V\setminus U}^{\infty}$$

and showed that it uniquely extends to an exact functor

$$\bullet \overset{\mathrm{w}}{\otimes} \mathscr{C}_M^{\infty}, \quad \mathrm{Mod}_{\mathbb{R}\text{-c}}(\mathbb{C}_M) \to \mathrm{Mod}(\mathbb{C}_M).$$

One denotes by $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{w}}$ the sheaf on M_{sa} given by

$$\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{w}}(U) = \Gamma(M; H^0(\mathrm{D}'_M \mathbf{k}_U) \overset{\mathrm{w}}{\otimes} \mathscr{C}_M^{\infty}), U \in \mathrm{Op}_{M_{\mathrm{sa}}}.$$

If $\mathrm{D}'_M \mathbb{C}_U \simeq \mathbb{C}_{\overline{U}}$, then $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{w}}(U) \simeq \mathcal{C}_M^{\infty}(M)/\mathrm{I}_{M,\overline{U}}^{\infty}$ is the space of Whitney functions on $\overline{U}$. It is thus natural to call $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{w}}$ the sheaf of Whitney $\mathcal{C}_M^{\infty}$-functions on M_{sa}.

Note that the sheaf $\rho_{M*}\mathscr{D}_M$ does not operate on the sheaves $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{t}}$, $\mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{w}}$, $\mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{t}}$ but $\rho_{M!}\mathscr{D}_M$ does.

Notation 1.5.3. Recall the exact and fully faithful functor $\widetilde{\iota}_M \colon \mathrm{Mod}(\mathbb{C}_{M_{\mathrm{sa}}}) \to \mathrm{Mod}(\mathrm{I}\mathbb{C}_{\mathrm{M}})$. We denote by $\mathcal{C}_M^{\infty,\mathrm{t}}$ and $\mathcal{D}b_M^{\mathrm{t}}$ the indsheaves $\widetilde{\iota}_M \mathcal{C}_{M_{\mathrm{sa}}}^{\infty,\mathrm{t}}$ and $\widetilde{\iota}_M \mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{t}}$ and calls them the *indsheaves of tempered* C^∞*-functions and tempered distributions,* respectively.

We have the sequence of monomorphisms

$$
\begin{tikzcd}
\mathcal{C}_M^{\infty,\mathrm{w}} \arrow[r, tail] & \mathcal{C}_M^{\infty,\mathrm{t}} \arrow[r, tail] \arrow[d, tail] & \mathcal{C}_M^{\infty} \arrow[d, tail] \\
 & \mathcal{D}b_M^{\mathrm{t}} \arrow[r, tail] & \mathcal{D}b_M .
\end{tikzcd}
$$

Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)$. One has the isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)$:

$$
\begin{aligned}
\rho_M^{-1} \mathrm{R}\mathscr{H}om\,(\mathrm{R}\rho_{M*}F, \mathcal{D}b_{M_{\mathrm{sa}}}^{\mathrm{t}}) &\simeq \mathrm{R}\mathscr{H}om_{\,\mathrm{I}\mathbb{C}_{\mathrm{M}}}(F, \mathcal{D}b_M^{\mathrm{t}}) \\
&\simeq \mathit{Thom}(F, \mathcal{D}b_M),
\end{aligned} \tag{1.5.2}
$$

where the functor

$$
\mathit{Thom}(\,\bullet\,, \mathcal{D}b_M) \colon \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_M)^{\mathrm{op}} \to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_M)
$$

was defined in [Ka80, Ka84] as the main tool for the proof of the Riemann-Hilbert correspondence for regular holonomic D-modules.

1.5.2 Operations on tempered distributions

Let us describe without detailed proofs the behaviour of the indsheaf of tempered distributions with respect to direct or inverse images (see [KS01]). In [KS96] these operations are treated in the language of the functor *Thom* introduced in [Ka84], but we prefer to use the essentially equivalent language of indsheaves.

For a real analytic manifold M and for a morphism of real analytic manifolds $f \colon M \to N$, we denote by

- $\dim M$ the dimension of M,
- $\mathscr{A}_M^{(\dim M)}$ the sheaf of real analytic forms of top degree,
- Θ_M the sheaf of real analytic vector fields,
- or_M the orientation sheaf,

- $\mathscr{V}_M := \mathscr{A}_M^{(\dim M)} \otimes \mathrm{or}_M$ the sheaf of real analytic densities on M,
- $\mathcal{D}b_M^{\mathrm{t}\vee} := \mathscr{V}_M \overset{\mathrm{L}}{\otimes}_{\mathscr{A}_M} \mathcal{D}b_M^{\mathrm{t}}$ the indsheaf of tempered distributions densities,
- $\mathscr{D}_{M\to N} = \mathscr{A}_M \otimes_{f^{-1}\mathscr{A}_N} f^{-1}\mathscr{D}_N$, the transfer bimodule. Recall that the left $\mathscr{D}_M$-module structure of $\mathscr{D}_{M\to N}$ is deduced from the action of Θ_M. For $v \in \Theta_M$, denoting by $\sum_i a_i \otimes w_i$ its image in $\mathscr{A}_M \otimes_{f^{-1}\mathscr{A}_N} f^{-1}\Theta_N$, the action of v on $\mathscr{D}_{M\to N}$ is given by $v(a\otimes P) = v(a)\otimes P + \sum_i aa_i \otimes w_i P$.

Proposition 1.5.4. *Let M and N be two real analytic manifolds. There exists a natural morphism*

$$\mathcal{D}b_M^{\mathrm{t}} \boxtimes \mathcal{D}b_N^{\mathrm{t}} \to \mathcal{D}b_{M\times N}^{\mathrm{t}} \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}(\mathscr{D}_M \boxtimes \mathscr{D}_N)). \tag{1.5.3}$$

The next result is a reformulation of a theorem of [Ka84].

Theorem 1.5.5. *Let $f\colon M \to N$ be a morphism of real analytic manifolds. There exists a natural isomorphism*

$$\mathcal{D}b_M^{\mathrm{t}\vee} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M} \mathscr{D}_{M\to N} \xrightarrow{\sim} f^{!}\mathcal{D}b_N^{\mathrm{t}\vee} \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}f^{-1}\mathscr{D}_N^{\mathrm{op}}). \tag{1.5.4}$$

Sketch of proof. (i) First, we construct the morphism in (1.5.4). By adjunction it is enough to construct a morphism

$$\mathrm{R}f_{!!}(\mathcal{D}b_M^{\mathrm{t}\vee} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M} \mathscr{D}_{M\to N}) \to \mathcal{D}b_N^{\mathrm{t}\vee}. \tag{1.5.5}$$

Denote by $\mathrm{Sp}_\bullet(\mathscr{M})$ the Spencer complex of a coherent $\mathscr{D}_M$-module $\mathscr{M}$. There is a quasi-isomorphism $\mathrm{Sp}_\bullet(\mathscr{M}) \to \mathscr{M}$. Denoting by Θ_M the sheaf of real analytic vector fields on M, we have $\mathrm{Sp}_k(\mathscr{M}) = \mathscr{D}_M \otimes_{\mathscr{A}_M} \bigwedge^k \Theta_M \otimes_{\mathscr{A}_M} \mathscr{M}$. Then $\mathrm{Sp}_\bullet(\mathscr{D}_{M\to N})$ gives a resolution of $\mathscr{D}_{M\to N}$ as a $(\mathscr{D}_M, f^{-1}\mathscr{D}_N)$-bimodule locally free over $\mathscr{D}_M$. Note that $\mathcal{D}b_M^{\mathrm{t}} \otimes_{\mathscr{D}_M} \mathrm{Sp}_k(\mathscr{D}_{M\to N})$ is acyclic with respect to the functor $f_{!!}$ for any k. Hence, in order to construct morphism (1.5.5), it is enough to construct a morphism of complexes in $\mathrm{I}(\mathscr{D}_X)$

$$f_{!!}\big(\mathcal{D}b_M^{\mathrm{t}\vee} \otimes_{\mathscr{D}_M} \mathrm{Sp}_\bullet(\mathscr{D}_{M\to N})\big) \to \mathcal{D}b_N^{\mathrm{t}\vee}. \tag{1.5.6}$$

Set for short

$$\mathscr{K}_\bullet = \mathcal{D}b_M^{\mathrm{t}\vee} \otimes_{\mathscr{D}_M} \mathrm{Sp}_\bullet(\mathscr{D}_{M\to N}) \simeq \mathcal{D}b_M^{\mathrm{t}\vee} \otimes_{\mathscr{A}_M} \overset{\bullet}{\bigwedge} \Theta_M \otimes_{f^{-1}\mathscr{A}_N} f^{-1}\mathscr{D}_N.$$

Then we have $f_{!!}(\mathscr{K}_0) = f_{!!}(\mathcal{D}b_M^{\mathrm{t}\vee}) \otimes_{\mathscr{A}_N} \mathscr{D}_N$. The integration of distributions gives a morphism

$$\int_f : f_{!!}(\mathcal{D}b_M^{\mathrm{t}\vee}) \to \mathcal{D}b_N^{\mathrm{t}\vee}. \tag{1.5.7}$$

Since $\mathcal{D}b_N^{\mathrm{t}\vee}$ is a right $\mathscr{D}_N$-module, we obtain the morphism $u\colon f_{!!}(\mathscr{K}_0) \to \mathcal{D}b_N^{\mathrm{t}\vee}$. By an explicit calculation, one checks that the composition

$$f_{!!}(\mathscr{K}_1) \xrightarrow{d_1} f_{!!}\mathscr{K}_0 \xrightarrow{u} \mathcal{D}b_N^{\mathrm{t}\vee}$$

vanishes. This defines morphism (1.5.5) and hence the morphism in (1.5.4).

(ii) One can treat separately the case of a closed embedding and a submersion.

(a) If $f\colon M \to N$ is a closed embedding, the result follows from the isomorphism

$$\Gamma_M(\mathcal{D}b_N^{\mathrm{t}\vee}) \simeq \mathcal{D}b_M^{\mathrm{t}\vee} \otimes_{\mathscr{D}_M} \mathscr{D}_{M\to N}.$$

(b) When f is a submersion, one reduces to the case where $M = N \times \mathbb{R}$ and f is the projection. Let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$ such that f is proper on $\mathrm{Supp}(F)$ and let us apply the functor $\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F, \bullet)$ to the morphism (1.5.4). Using $\mathrm{R}\mathscr{H}om\,(F, \bullet) \simeq \alpha_M \circ \mathrm{R}\mathscr{I}hom\,(F, \bullet)$, we get the morphism

$$\begin{aligned} \mathrm{R}f_*\big(\mathrm{R}\mathscr{H}om\,(F, \mathcal{D}b_M^{\mathrm{t}\vee}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M} \mathscr{D}_{M\to N}\big) &\to \mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F, f^{\,!}\mathcal{D}b_N^{\mathrm{t}\vee}) \\ &\simeq \mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F, \mathcal{D}b_N^{\mathrm{t}\vee}). \end{aligned} \tag{1.5.8}$$

By Proposition 1.4.9, it remains to prove that (1.5.8) is an isomorphism.

One then reduces to the case where $F = \mathbb{C}_Z$ for a closed subanalytic subset Z of $N\times\mathbb{R}$ proper over N. Then, by using the structure of subanalytic sets, one reduces to the case where $f^{-1}(x) \cap Z$ is a closed interval for each $x \in f(Z)$. Finally, one proves that the sequence below is exact.

$$0 \longrightarrow f_!\Gamma_Z\mathcal{D}b_M \xrightarrow{\partial_t} f_!\Gamma_Z\mathcal{D}b_M \xrightarrow{\int_{\mathbb{R}}(\cdot)dt} \Gamma_{f(Z)}\mathcal{D}b_N \to 0.$$

Q.E.D.

One often needs to compactify the real analytic manifold M. In order to check that the construction does not depend on the compactification, the next lemma is useful.

Lemma 1.5.6. *Consider a morphism $f\colon M \to N$ of real analytic manifolds and let $V \subset N$ be a subanalytic open subset. Set $U = f^{-1}V$ and assume that f induces an isomorphism of real analytic manifolds $U \xrightarrow{\sim} V$. Then*

$$\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{D}b^{\mathrm{t}}_M) \simeq f^{\,!}\,\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_V, \mathcal{D}b^{\mathrm{t}}_N). \tag{1.5.9}$$

Proof. By Theorem 1.5.5, we have

$$\begin{aligned} f^{\,!}\,\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_V, \mathcal{D}b^{\mathrm{t}}_N) &\simeq \mathrm{R}\mathscr{I}hom\,(f^{-1}\mathbb{C}_V, f^{\,!}\,\mathcal{D}b^{\mathrm{t}}_N) \\ &\simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{D}b^{\mathrm{t}\vee}_M \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M} \mathscr{D}_{M\to N}). \end{aligned}$$

Since we have a monomorphism $\mathscr{D}_M \rightarrowtail \mathscr{D}_{M\to N}$ whose cokernel has its support contained in $X \setminus U$, we have an isomorphism

$$\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{D}b^{\mathrm{t}\vee}_M) \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{D}b^{\mathrm{t}\vee}_M \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_M} \mathscr{D}_{M\to N}).$$

Q.E.D.

Remark 1.5.7. By choosing $N = \mathrm{pt}$ and $F = \mathbb{C}_U$ for U open subanalytic, we obtain that $\mathrm{RHom}\,(\mathrm{R}f_!\mathbb{C}_U, \mathbb{C}) \simeq \mathrm{R}\Gamma(U; \omega_M)$ is isomorphic to the De Rham complex with coefficients in $\mathcal{D}b^{\mathrm{t}}_M(U)$. This is a vast generalization of a well-known theorem of Grothendieck [Gr66] which asserts that the cohomology of the complementary of an algebraic hypersurface S may be calculated as the De Rham complex with coefficients the meromorphic functions with poles on S. This result has been generalized to the semi-analytic setting by Poly [Po74].

1.5.3 Whitney and tempered holomorphic functions

Let X be a complex manifold. We denote by X^c the complex conjugate manifold to X and by $X_{\mathbb{R}}$ the underlying real analytic manifold.

We define the following indsheaves

$$\begin{aligned} \mathcal{O}^{\omega}_X &:= \beta_X \mathcal{O}_X, \\ \mathcal{O}^{\mathrm{w}}_X &:= \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c}}(\mathcal{O}_{X^c}, \mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathbb{R}}}) \simeq \Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X^c}} \mathcal{C}^{\infty,\mathrm{w}}_{X_{\mathbb{R}}}[-d_X], \\ \mathcal{O}^{\mathrm{t}}_X &:= \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c}}(\mathcal{O}_{X^c}, \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}) \simeq \Omega_{X^c} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X^c}} \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}[-d_X]. \end{aligned}$$

These are objects of $\mathsf{D}^{\mathrm{b}}_{\mathrm{I}\mathbb{R}\text{-c}}(\mathrm{I}\mathscr{D}_X)$. Hence $\mathcal{O}^{\mathrm{t}}_X$ is isomorphic to the Dolbeault complex with coefficients in $\mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}$:

$$0 \longrightarrow \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}} \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{t}\,(0,1)}_{X_{\mathbb{R}}} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathcal{D}b^{\mathrm{t}\,(0,d_X)}_{X_{\mathbb{R}}} \longrightarrow 0,$$

where $\mathcal{D}b^{\mathrm{t}\,(0,p)}_{X_{\mathbb{R}}} = \Omega^p_{X^c} \otimes_{\mathcal{O}_{X^c}} \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}$.

One calls $\mathcal{O}^{\mathrm{w}}_X$ and $\mathcal{O}^{\mathrm{t}}_X$ the *indsheaves of Whitney and tempered holomorphic functions*, respectively. We have the morphisms in the category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$:

$$\mathcal{O}^{\omega}_X \to \mathcal{O}^{\mathrm{w}}_X \to \mathcal{O}^{\mathrm{t}}_X \to \mathcal{O}_X.$$

One proves the isomorphism

$$(1.5.10) \qquad \mathcal{O}^{\mathrm{t}}_X \;\simeq\; \mathrm{R}\mathscr{H}om_{\mathscr{D}_{X^c}}(\mathcal{O}_{X^c}, \mathcal{C}^{\infty,\mathrm{t}}_{X_{\mathbb{R}}}) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X).$$

Note that the object $\mathcal{O}^{\mathrm{t}}_X$ is not concentrated in degree zero if $d_X > 1$. Indeed, with the subanalytic topology, only finite coverings are allowed. If one considers for example the open subset $U \subset \mathbb{C}^n$, the difference of an open ball of radius $R > 0$ and a closed ball of radius $0 < r < R$, then the Dolbeault complex will not be exact after any finite covering.

Example 1.5.8. (i) Let Z be a closed complex analytic subset of the complex manifold X. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:

$$\begin{aligned}
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_Z, \mathcal{O}^{\omega}_X) \simeq (\mathcal{O}_X)_Z && \text{(restriction)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_Z, \mathcal{O}^{\mathrm{w}}_X) \simeq \widehat{\mathcal{O}_X|}_Z && \text{(formal completion)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathbb{C}_Z, \mathcal{O}^{\mathrm{t}}_X) \simeq \mathrm{R}\Gamma_{[Z]}(\mathcal{O}_X) && \text{(algebraic cohomology)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathbb{C}_Z, \mathcal{O}_X) \simeq \mathrm{R}\Gamma_Z(\mathcal{O}_X) && \text{(local cohomology)}.
\end{aligned}$$

(ii) Let M be a real analytic manifold such that X is a complexification of M. We have the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_M)$:

$$\begin{aligned}
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}^{\omega}_X)|_M \simeq \mathscr{A}_M && \text{(real analytic functions)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}^{\mathrm{w}}_X)|_M \simeq \mathcal{C}^{\infty}_M && \text{(}C^{\infty}\text{ functions)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}^{\mathrm{t}}_X)|_M \simeq \mathcal{D}b_M && \text{(distributions)},\\
&\mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathrm{D}'_X\mathbb{C}_M, \mathcal{O}_X)|_M \simeq \mathscr{B}_M && \text{(hyperfunctions)}.
\end{aligned}$$

There is a kind of duality between the indsheaves $\mathcal{O}^{\mathrm{w}}_X$ and $\mathcal{O}^{\mathrm{t}}_X$, but we shall not develop this point here (see [KS96, Th. 6.1]).

2 Tempered solutions of D-modules

2.1 D-modules

References are made to [Ka03] for the theory of $\mathscr{D}$-modules.

Let $(X, \mathcal{O}_X)$ be a *complex* manifold and denote as usual by d_X its complex dimension, by Ω_X the invertible sheaf of differential forms of top degree and by $\mathscr{D}_X$ the sheaf of algebras of finite-order differential operators.

Denote by $\mathrm{Mod}(\mathscr{D}_X)$ the category of left $\mathscr{D}_X$-modules, and by $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ its bounded derived category.

Notation 2.1.1. According to Proposition 1.3.2, for $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, we have the functors

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \bullet) \quad &: \quad \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X) \to \mathsf{D}^{+}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \\ \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M} \quad &: \quad \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X^{\mathrm{op}}) \to \mathsf{D}^{-}(\mathrm{I}\mathbb{C}_{\mathrm{X}}). \end{aligned}$$

We also have the functor

$$\bullet \overset{\mathrm{D}}{\otimes} \bullet \quad : \quad \mathsf{D}^{-}(\mathrm{I}\mathscr{D}_X) \times \mathsf{D}^{-}(\mathrm{I}\mathscr{D}_X) \to \mathsf{D}^{-}(\mathrm{I}\mathscr{D}_X)$$

constructed as follows. The $(\mathscr{D}_X, \mathscr{D}_X \otimes \mathscr{D}_X)$-bimodule structure on $\mathscr{D}_X \otimes_{\mathcal{O}_X} \mathscr{D}_X$ gives

$$\mathscr{M} \otimes_{\mathcal{O}_X} \mathscr{N} \simeq (\mathscr{D}_X \otimes_{\mathcal{O}_X} \mathscr{D}_X) \otimes_{\mathscr{D}_X \otimes \mathscr{D}_X} (\mathscr{M} \otimes \mathscr{N})$$

the structure of a $\mathscr{D}_X$-module.

There are similar constructions with right $\mathscr{D}_X$-modules.

Let $f : X \to Y$ be a morphism of complex manifolds. One denotes by

- $\overset{\mathrm{D}}{\boxtimes}$ the (derived) operation of external tensor product for $\mathscr{D}$-modules,
- $\mathscr{D}_{X \to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathscr{D}_Y$, the transfer $(\mathscr{D}_X, \mathscr{D}_Y)$-bimodule,
- $\mathscr{D}_{Y \leftarrow X} = f^{-1}\mathscr{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$, the opposite transfer $(\mathscr{D}_Y, \mathscr{D}_X)$-bimodule,
- $\mathrm{D}f^*$, $\mathrm{D}f_!$ and $\mathrm{D}f_*$ the (derived) operations of inverse image, proper direct images and direct images for $\mathscr{D}$-modules,
- $\mathbb{D}_X \mathscr{M} = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})[d_X]$, the dual of $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$,

- $\mathrm{r}\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X^{\mathrm{op}})$ the equivalence of categories given by $\mathscr{M}^{\mathrm{r}} = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{O}_X} \mathscr{M}$.

Note that

$$\mathrm{D}f^*\mathscr{O}_Y \simeq \mathscr{O}_X, \quad \mathrm{D}f^*\Omega_Y \simeq \Omega_X.$$

Recall that to a coherent $\mathscr{D}_X$-module $\mathscr{M}$ one associates its characteristic variety $\mathrm{char}(\mathscr{M})$, a closed conic involutive subset of the cotangent bundle T^*X.

If $\mathrm{char}(\mathscr{M})$ is Lagrangian, $\mathscr{M}$ is called *holonomic*. It is immediately checked that the full subcategory $\mathrm{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$ of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of holonomic $\mathscr{D}$-modules is a thick abelian subcategory.

A $\mathscr{D}_X$-module $\mathscr{M}$ is *quasi-good* if, for any relatively compact open subset $U \subset X$, $\mathscr{M}|_U$ is a sum of coherent $(\mathscr{O}_X|_U)$-submodules. A $\mathscr{D}_X$-module $\mathscr{M}$ is *good* if it is quasi-good and coherent. The subcategories of $\mathrm{Mod}(\mathscr{D}_X)$ consisting of quasi-good (resp. good) $\mathscr{D}_X$-modules are abelian and thick. Therefore, one has the triangulated categories

- $\mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is coherent for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is holonomic for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is quasi-good for all } j \in \mathbb{Z}\}$,
- $\mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X) = \{\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)\,;\, H^j(\mathscr{M}) \text{ is good for all } j \in \mathbb{Z}\}$.

Note that the properties of being quasi-good are stable by inverse image and tensor product, as well as by direct image by maps proper on the support of the module. The property of being good is stable by duality.

Let $f\colon X \to Y$ be a morphism of complex manifolds. One associates the maps

$$\begin{tikzcd}
T^*X \arrow[drr, "\pi_X"'] & & X \times_Y T^*Y \arrow[ll, "f_d"'] \arrow[rr, "f_\pi"] \arrow[d] & & T^*Y \arrow[d, "\pi_Y"] \\
& & X \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

One says that f is non-characteristic for $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ if the map f_d is proper (hence, finite) on $f_\pi^{-1}\big(\mathrm{char}(\mathscr{N})\big)$.

The classical de Rham and solution functors are defined by

$$\begin{aligned} \mathcal{DR}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), & \mathscr{M} &\mapsto \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \\ \mathcal{S}ol_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} &\to \mathsf{D}^{\mathrm{b}}(\mathbb{C}_X), & \mathscr{M} &\mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X). \end{aligned}$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has

$$\mathcal{S}ol_X(\mathscr{M}) \simeq \mathcal{DR}_X(\mathbb{D}_X\mathscr{M})[-d_X]. \tag{2.1.1}$$

Theorem 2.1.2 (Projection formulas [Ka03, Th. 4.2.8, Th 4.40])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}})$. There are natural isomorphisms*:

$$\mathrm{D}f_!(\mathrm{D}f^*\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathscr{L} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M}, \tag{2.1.2}$$

$$\mathrm{R}f_!(\mathrm{D}f^*\mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}) \simeq \mathscr{L} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathrm{D}f_!\mathscr{M}. \tag{2.1.3}$$

In particular, there is an isomorphism (*commutation of the De Rham functor and direct images*)

$$\mathrm{R}f_!(\mathcal{DR}_X(\mathscr{M})) \simeq \mathcal{DR}_Y(\mathrm{D}f_!\mathscr{M}). \tag{2.1.4}$$

Theorem 2.1.3 (Commutativity with duality [Ka03, Sc86])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{M})$*. Then $\mathrm{D}f_!\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_Y)$ and $\mathbb{D}_Y(\mathrm{D}f_!\mathscr{M}) \simeq \mathrm{D}f_!\mathbb{D}_X\mathscr{M}$.*

(ii) *Let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_Y)$. Then $\mathrm{D}f^*\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$. Moreover, if $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ and f is non-characteristic for $\mathscr{N}$, then $\mathrm{D}f^*\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathbb{D}_X(\mathrm{D}f^*\mathscr{N}) \simeq \mathrm{D}f^*\mathbb{D}_Y\mathscr{N}$.*

Corollary 2.1.4. *Let $f\colon X \to Y$ be a morphism of complex manifolds.*

(i) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ and assume that f is proper on* $\mathrm{Supp}(\mathscr{M})$*. Then we have the isomorphism for $\mathscr{N} \in \mathrm{D}(\mathscr{D}_Y)$:*

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{D}f^*\mathscr{N})\,[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathrm{D}f_*\mathscr{M}, \mathscr{N})\,[d_Y]. \tag{2.1.5}$$

In particular, with the same hypotheses, we have the isomorphism (*commutation of the Sol functor and direct images*)

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}_X)\,[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathrm{D}f_*\mathscr{M}, \mathcal{O}_Y)\,[d_Y]. \tag{2.1.6}$$

(ii) *Let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_Y)$ and assume that f is non-characteristic for $\mathscr{N}$. Then we have the isomorphism for $\mathscr{M} \in \mathrm{D}(\mathscr{D}_X)$:*

$$\text{(2.1.7)}\quad \mathrm{R}f_*\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathrm{D}f^*\mathscr{N}, \mathscr{M})[d_X] \simeq \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{N}, \mathrm{D}f_*\mathscr{M})[d_Y].$$

A *transversal Cartesian diagram* is a commutative diagram

$$\text{(2.1.8)}\qquad \begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $X' \simeq X \times_Y Y'$ and such that the map of tangent spaces

$$T_{g'(x)}X \oplus T_{f'(x)}Y' \to T_{f(g'(x))}Y$$

is surjective for any $x \in X'$.

Proposition 2.1.5 (Base change formula)**.** *Consider the transversal Cartesian diagram (2.1.8). Then, for any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$ such that* $\mathrm{supp}(\mathscr{M})$ *is proper over Y, we have the isomorphism*

$$\mathrm{D}g^*\,\mathrm{D}f_*\mathscr{M} \simeq \mathrm{D}f'_*\,\mathrm{D}g'^*\mathscr{M}.$$

2.2 Tempered De Rham and Sol functors

Setting $\Omega^{\mathrm{t}}_X := \Omega_X \otimes_{\mathscr{O}_X} \mathcal{O}^{\mathrm{t}}_X$, the tempered de Rham and solution functors are given by

$$\begin{aligned} \mathcal{DR}^{\mathrm{t}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) &\to \mathsf{D}^-(\mathrm{I}\mathbb{C}_X), & \mathscr{M} &\mapsto \Omega^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \\ \mathcal{S}ol^{\mathrm{t}}_X \colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} &\to \mathsf{D}^+(\mathrm{I}\mathbb{C}_X), & \mathscr{M} &\mapsto \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}^{\mathrm{t}}_X). \end{aligned}$$

One has

$$\mathcal{S}ol_X \simeq \alpha_X \mathcal{S}ol^{\mathrm{t}}_X, \quad \mathcal{DR}_X \simeq \alpha_X \mathcal{DR}^{\mathrm{t}}_X.$$

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, one has

$$\text{(2.2.1)}\qquad \mathcal{S}ol^{\mathrm{t}}_X(\mathscr{M}) \simeq \mathcal{DR}^{\mathrm{t}}_X(\mathbb{D}_X\mathscr{M})[-d_X].$$

The next result is a reformulation of a theorem of [Ka84] (see also [KS01, Th. 7.4.1])

Theorem 2.2.1. *Let $f\colon X \to Y$ be a morphism of complex manifolds. There is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}f^{-1}\mathscr{D}_Y^{\mathrm{op}})$:*

$$\Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\,[d_X] \xrightarrow{\sim} f^{\,!}\,\Omega_Y^{\mathrm{t}}\,[d_Y]. \tag{2.2.2}$$

Proof. Consider the isomorphism (1.5.4) with $M = X_{\mathbb{R}}$ and $N = Y_{\mathbb{R}}$ and apply $\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{Y^c}} \mathscr{O}_{Y^c}$. We get the result since

$$\begin{aligned}
\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X^c}} & \mathscr{D}_{X\times X^c\to Y\times Y^c} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{Y^c}} \mathscr{O}_{Y^c} \\
&\simeq \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X^c}} \mathscr{D}_{X\times X^c\to Y\times Y^c} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{Y^c}} \mathscr{D}_{Y^c\to \mathrm{pt}} \\
&\simeq \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X^c}} \mathscr{D}_{X\times X^c\to Y\times Y^c} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{Y\times Y^c}} \mathscr{D}_{Y\times Y^c\to Y} \\
&\simeq \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X\times X^c}} \mathscr{D}_{X\times X^c\to Y} \\
&\simeq \bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_{X^c}} \mathscr{O}_{X^c}.
\end{aligned}$$

Q.E.D.

Note that this isomorphism (2.2.2) is equivalent to the isomorphism

$$\mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{O}_X^{\mathrm{t}}\,[d_X] \xrightarrow{\sim} f^{\,!}\,\mathscr{O}_Y^{\mathrm{t}}\,[d_Y]. \tag{2.2.3}$$

Corollary 2.2.2. *Let $f\colon X \to Y$ be a morphism of complex manifolds and let $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$. Then (2.2.2) induces the isomorphism*

$$\mathcal{DR}_X^{\mathrm{t}}(\mathrm{D}f^*\mathscr{N})\,[d_X] \simeq f^{\,!}\,\mathcal{DR}_Y^{\mathrm{t}}(\mathscr{N})\,[d_Y] \ \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X). \tag{2.2.4}$$

Proof. Apply $\bullet \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{N}$ to isomorphism (2.2.2). Q.E.D.

Corollary 2.2.3. *For any complex manifold X, we have*

$$\mathcal{DR}_X^{\mathrm{t}}(\mathscr{O}_X) \simeq \mathbb{C}_X[d_X].$$

Corollary 2.2.4. *Let $f\colon X \to Y$ be a morphism of complex manifolds. There is a natural morphism*

$$f^{-1}\Omega_Y^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X} \longrightarrow \Omega_X^{\mathrm{t}} \ \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X^{\mathrm{op}}). \tag{2.2.5}$$

Proof. (i) Assume that f is a closed embedding. We have

$$\begin{aligned} f^{-1}\Omega^{\mathrm{t}}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X} &\simeq f^{\,!}\mathrm{R}f_{!!}(f^{-1}\Omega^{\mathrm{t}}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X}) \\ &\simeq f^{\,!}(\Omega^{\mathrm{t}}_Y \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathrm{R}f_!\mathscr{D}_{Y\leftarrow X}) \\ &\simeq f^{\,!}\Omega^{\mathrm{t}}_Y \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathscr{D}_{Y\leftarrow X} \\ &\simeq \Omega^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X}\,[d_X - d_Y] \\ &\simeq \Omega^{\mathrm{t}}_X. \end{aligned}$$

(ii) Assume that f is submersive. We have

$$\begin{aligned} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{D}_{Y\leftarrow X}, \Omega^{\mathrm{t}}_X) &\simeq \Omega^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{D}_{Y\leftarrow X}, \mathscr{D}_X) \\ &\simeq \Omega^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}\,[d_Y - d_X] \\ &\simeq f^{\,!}\Omega^{\mathrm{t}}_Y\,[2d_Y - 2d_X] \simeq f^{-1}\Omega^{\mathrm{t}}_Y. \end{aligned}$$

Then use

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{D}_{Y\leftarrow X}, \Omega^{\mathrm{t}}_X) \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} \mathscr{D}_{Y\leftarrow X} \to \Omega^{\mathrm{t}}_X.$$

Q.E.D.

Note that morphism (2.2.5) is equivalent to the morphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$

$$\mathscr{D}_{X\to Y} \overset{\mathrm{L}}{\otimes}_{f^{-1}\mathscr{D}_Y} f^{-1}\mathcal{O}^{\mathrm{t}}_Y \longrightarrow \mathcal{O}^{\mathrm{t}}_X.$$

The next result is a kind of Grauert theorem for tempered holomorphic functions. It will be generalised to D-modules in Corollary 2.2.6. Its proof uses difficult results of functional analysis.

Theorem 2.2.5 ([KS96, Th. 7.3])**.** (Tempered Grauert theorem.) *Let* $f\colon X \to Y$ *be a morphism of complex manifolds, let* $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_X)$ *and assume that* f *is proper on* $\mathrm{Supp}(\mathscr{F})$*. Then there is a natural isomorphism*

$$\mathrm{R}f_{!!}(\mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathscr{F}) \simeq \mathcal{O}^{\mathrm{t}}_Y \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_Y} \mathrm{R}f_!\mathscr{F}. \tag{2.2.6}$$

Sketch of proof. It is enough to prove that for any $G \in \mathbb{R}\text{-C}(\mathbb{C}_Y)$, we have

$$\mathrm{R}\mathscr{H}om\,(f^{-1}G, \mathcal{O}_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathscr{F}) \simeq \mathrm{R}\mathscr{H}om\,(G, \mathcal{O}_Y^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_Y} \mathrm{R}f_!\mathscr{F}). \tag{2.2.7}$$

Since $\mathscr{F}$ and $\mathrm{R}f_!\mathscr{F}$ are coherent, (2.2.7) is equivalent to

$$\mathrm{R}\mathscr{H}om\,(f^{-1}G, \mathcal{O}_X^{\mathrm{t}}) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathscr{F} \simeq \mathrm{R}\mathscr{H}om\,(G, \mathcal{O}_Y^{\mathrm{t}}) \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_Y} \mathrm{R}f_!\mathscr{F}. \tag{2.2.8}$$

Such a formula is proved in [KS96, Th. 7.3]. Q.E.D.

Corollary 2.2.6 ([KS01, Th. 7.4.6])**.** *Let $f\colon X \to Y$ be a morphism of complex manifolds. Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and assume that f is proper on* $\operatorname{supp}\mathscr{M}$. *There is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$*

$$\mathcal{DR}_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}) \xrightarrow{\sim} \mathrm{R}f_*\mathcal{DR}_X^{\mathrm{t}}(\mathscr{M}). \tag{2.2.9}$$

Proof. Applying the functor $\mathrm{R}f_{!!}(\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})$ to the morphism (2.2.5) we obtain the morphism in (2.2.9). To check it is an isomorphism, we reduce to the case where $\mathscr{M} = \mathscr{D}_X \otimes_{\mathcal{O}_X} \mathscr{F}$ with $\mathscr{F}$ a coherent $\mathcal{O}_X$-module such that f is proper on $\operatorname{Supp}(\mathscr{F})$. Then we apply Theorem 2.2.5. Q.E.D.

Corollary 2.2.7. *Let f and $\mathscr{M}$ be as in Corollary 2.2.6. Then we have the isomorphism*

$$\mathrm{D}f_*(\mathcal{O}_X^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathcal{O}_Y^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_*\mathscr{M} \ \textit{in}\ \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_Y). \tag{2.2.10}$$

Proof. We have

$$\begin{aligned}
\Omega_Y^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M} &\simeq \mathcal{O}_Y^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} (\mathscr{D}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M}) \\
&\simeq \mathcal{O}_Y^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} \mathrm{D}f_!(\mathrm{D}f^*\mathscr{D}_Y \overset{\mathrm{D}}{\otimes} \mathscr{M}) \\
&\simeq \mathcal{DR}_Y^{\mathrm{t}}(\mathrm{D}f_!(\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})),
\end{aligned}$$

where the second isomorphism follows from the projection formula (2.1.2). Applying Corollary 2.2.6, we obtain

$$\Omega_Y^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M} \simeq \mathrm{R}f_*(\Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M})).$$

On the other-hand, we have

$$\Omega^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{D}_{X\to Y} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \;\simeq\; (\Omega^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{D}_{X\to Y}.$$

Therefore,

$$\Omega^{\mathrm{t}}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D}f_!\mathscr{M} \;\simeq\; \mathrm{D}f_*(\Omega^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M}).$$

To conclude, use the equivalence of categories $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y^{\mathrm{op}}) \simeq \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$ given by $\mathscr{M}^{\mathrm{r}} = \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathscr{M}$. Q.E.D.

Remark 2.2.8. If one replaces (2.2.2) with its non-tempered version, then the formula is no more true, contrarily to isomorphism (2.2.9) which remains true by Theorem 2.1.2.

2.3 Localization along a hypersurface

In order to prove Theorem 3.3.2 below, we need some lemmas.

If $S \subset X$ is a closed hypersurface, denote by $\mathcal{O}_X(*S)$ the sheaf of meromorphic functions with poles at S. It is a regular holonomic $\mathscr{D}_X$-module, and it is a flat $\mathcal{O}_X$-module. For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$, set

$$\mathscr{M}(*S) = \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X(*S).$$

Lemma 2.3.1. *Let S be a closed complex hypersurface in X. There are isomorphisms*

$$\begin{aligned} \mathcal{O}^{\mathrm{t}}_X(*S) &\simeq \mathrm{R}\mathscr{I}\!hom(\mathbb{C}_{X\setminus S}, \mathcal{O}^{\mathrm{t}}_X) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X), \\ \mathcal{O}_X(*S) &\simeq \mathrm{R}\mathscr{H}\!om_{\mathrm{I}\mathbb{C}_X}(\mathbb{C}_{X\setminus S}, \mathcal{O}^{\mathrm{t}}_X) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X). \end{aligned} \tag{2.3.1}$$

Proof. (i) The second isomorphism follows from the first one by applying the functor α_X.
(ii) By taking the Dolbeault resolution of $\mathcal{O}^{\mathrm{t}}_X$ we are reduced to prove a similar result with $\mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}$ instead of $\mathcal{O}^{\mathrm{t}}_X$. More precisely, consider a real analytic manifold M a real analytic map $f\colon M \to \mathbb{C}$. Set $S = \{f = 0\}$ and denote by $j\colon (M \setminus S) \hookrightarrow M$ the open embedding. Define the sheaf $\mathscr{A}_M[1/f]$ as the inductive limit of the sequence of embeddings $\mathscr{A}_M \xrightarrow{f} \mathscr{A}_M \xrightarrow{f} \cdots$.

Equivalently, $\mathscr{A}_M[1/f]$ is the subsheaf of $j_*j^{-1}\mathscr{A}_M$ consisting of sections u such that there locally exists an integer m with $f^m \cdot u \in \mathscr{A}_M$. Set

$$\mathcal{D}b^{\mathrm{t}}_M[1/f] := \mathcal{D}b^{\mathrm{t}}_M \otimes_{\mathscr{A}_M} (\mathscr{A}_M[1/f]).$$

(Note that $\mathcal{D}b^{\mathrm{t}}_M[1/f]$ is isomorphic to the inductive limit of the sequence of morphisms $\mathcal{D}b^{\mathrm{t}}_M \xrightarrow{f} \mathcal{D}b^{\mathrm{t}}_M \xrightarrow{f} \cdots$.) It is enough to prove the isomorphism

$$\mathcal{D}b^{\mathrm{t}}_M[1/f] \simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{M\setminus S}, \mathcal{D}b^{\mathrm{t}}_M), \tag{2.3.2}$$

or, equivalently, the isomorphism for any open relatively compact subanalytic subset U of M

$$\Gamma(U; \mathcal{D}b^{\mathrm{t}}_{M_{\mathrm{sa}}}[1/f]) \;\simeq\; \Gamma(U \setminus S; \mathcal{D}b^{\mathrm{t}}_{M_{\mathrm{sa}}}). \tag{2.3.3}$$

This follows from the fact that $f\colon \Gamma(U \setminus S; \mathcal{D}b^{\mathrm{t}}_{M_{\mathrm{sa}}}) \to \Gamma(U \setminus S; \mathcal{D}b^{\mathrm{t}}_{M_{\mathrm{sa}}})$ is bijective. (See also Lojasiewicz [Lo59].) Q.E.D.

In the sequel, we set for a closed complex analytic hypersurface S

$$\mathcal{O}^{\mathrm{t}}_X(*S) := \mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathcal{O}_X(*S) \simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{M\setminus S}, \mathcal{O}^{\mathrm{t}}_X). \tag{2.3.4}$$

Lemma 2.3.2. *Let S be a closed complex hypersurface in X. There are isomorphisms*

$$\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathrm{t}}_X(*S) \xrightarrow{\sim} \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}_X(*S) \simeq \mathrm{R}\mathscr{H}om\,(\mathbb{C}_{X\setminus S}, \mathbb{C}_X)\,[d_X]. \tag{2.3.5}$$

Proof. It follows from Lemma 2.3.1 that

$$\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathrm{t}}_X(*S) \;\simeq\; \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{X\setminus S}, \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathrm{t}}_X).$$

Then the result follows from the isomorphisms

$$\Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathrm{t}}_X \;\simeq\; \Omega_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}_X \simeq \mathbb{C}_X\,[d_X].$$

Q.E.D.

3 Regular holonomic D-modules

3.1 Regular normal form for holonomic modules

For the notion of regular holonomic D-modules, refer e.g. to [Ka03, §5.2] and [KK81].

Definition 3.1.1. A holonomic $\mathscr{D}_X$-module $\mathscr{M}$ is *regular* if, denoting by Λ its characteristic variety in T^*X and by $\mathscr{I}_\Lambda$ the ideal of $\mathrm{gr}(\mathscr{D}_X)$ of functions vanishing on Λ, there exists locally a good filtration on $\mathscr{M}$ such that $\mathscr{I}_\Lambda \cdot \mathrm{gr}(\mathscr{M}) = 0$.

One can prove that the full subcategory $\mathrm{Mod}_{\mathrm{rh}}(\mathscr{D}_X)$ of $\mathrm{Mod}_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of regular holonomic $\mathscr{D}_X$-modules is a thick abelian subcategory, stable by duality. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ he full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ whose objects have holonomic cohomologies. Then $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ is triangulated.

For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, denote by $\mathrm{SingSupp}(\mathscr{M})$ the set of $x \in X$ such that $\mathscr{M}$ is not a coherent $\mathscr{O}_X$-module on a neighborhood of x.

Definition 3.1.2. Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has *regular normal form* along D if locally on D, for a local coordinate system $(z_1, \dots, z_n)$ on X such that $D = \{z_1 \cdots z_r = 0\}$, $\mathscr{M} \simeq \mathscr{D}_X/\mathscr{I}_\lambda$ for $\lambda = (\lambda_1, \dots, \lambda_r) \in (\mathbb{C}\setminus\mathbb{Z}_{\geq 0})^r$. Here, $\mathscr{I}_\lambda$ is the left ideal generated by the operators $(z_i\partial_i - \lambda_i)$ and ∂_j for $i \in I := \{1, \dots, r\}$, $j \in \{r+1, \dots, n\}$.

One shall be aware that the property of being of normal form is not stable by duality. Note that, for $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{C}^m$, $\mathscr{D}_X/\mathscr{I}_\lambda \simeq (\mathscr{D}_X/\mathscr{I}_\lambda)(*D)$ if and only if $\lambda_i \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ for any $i \in \{1, \dots, r\}$.

Of course, if a holonomic $\mathscr{D}_X$-module has regular normal form, then it is regular holonomic.

Lemma 3.1.3. *Let $\mathscr{L}$ be a holonomic module with regular normal form along D. Then $\mathcal{S}ol_X(\mathscr{L}) \otimes \mathbb{C}_{X\setminus D} \xrightarrow{\sim} \mathcal{S}ol_X(\mathscr{L})$.*

Proof. It is enough to prove that $\mathcal{S}ol_X(\mathscr{L})|_D \simeq 0$. In a local coordinate system $(z_1, \dots, z_n)$, set $Z_i = \{z_i = 0\}$. Setting $P_i = z_i\partial_i - \lambda_i$ with $\lambda_i \in \mathbb{C}\setminus\mathbb{Z}_{\geq 0}$, it is enough to check that P_i induces an isomorphism $P_i \colon \mathscr{O}_X|_{Z_i} \xrightarrow{\sim} \mathscr{O}_X|_{Z_i}$, which is clear. Q.E.D.

Lemma 3.1.4. *Let $P_X(\mathscr{M})$ be a statement concerning a complex manifold X and a regular holonomic object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i\in I} U_i$ be an open covering. Then $P_X(\mathscr{M})$ is true if and only if $P_{U_i}(\mathscr{M}|_{U_i})$ is true for any $i \in I$.*

(b) *If $P_X(\mathscr{M})$ is true, then $P_X(\mathscr{M}[n])$ is true for any $n \in \mathbb{Z}$.*

(c) *Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ be a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. If $P_X(\mathscr{M}')$ and $P_X(\mathscr{M}'')$ are true, then $P_X(\mathscr{M})$ is true.*

(d) *Let $\mathscr{M}$ and $\mathscr{M}'$ be regular holonomic $\mathscr{D}_X$-modules. If $P_X(\mathscr{M} \oplus \mathscr{M}')$ is true, then $P_X(\mathscr{M})$ is true.*

(e) *Let $f\colon X \to Y$ be a projective morphism and let $\mathscr{M}$ be a good regular holonomic $\mathscr{D}_X$-module. If $P_X(\mathscr{M})$ is true, then $P_Y(\mathrm{D}f_*\mathscr{M})$ is true.*

(f) *If $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module with a regular normal form along a normal crossing divisor of X, then $P_X(\mathscr{M})$ is true.*

If conditions (a)–(f) *are satisfied, then $P_X(\mathscr{M})$ is true for any complex manifold X and any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$.*

Sketch of proof. (i) If D is a normal crossing hypersurface of X and $\mathscr{M}$ is a regular holonomic $\mathscr{D}_X$-module satisfying
• $\mathscr{M} \simeq \mathscr{M}(*D)$,
• $\mathrm{SingSupp}(\mathscr{M}) \subset D$,
then, locally on X, there exists a filtration

$$\mathscr{M} = \mathscr{M}_0 \supset \mathscr{M}_1 \supset \cdots \supset \mathscr{M}_l \supset \mathscr{M}_{l+1} = 0$$

such that $\mathscr{M}_j/\mathscr{M}_{j+1}$ has regular normal form. It follows that in this case, $P_X(\mathscr{M})$ is true.
(ii) Let us take a closed complex analytic subset Z of X such that the support of $\mathscr{M}$ is contained in Z. We argue by induction on the dimension m of Z. There exists a morphism $f\colon W \to Z$ such that
(1) W is non singular with dimension m,
(2) f is projective,
(3) there exists a closed complex analytic subset S of Z with dimension $< m$

such that

(3-i) $f^{-1}(Z \setminus S) \to Z \setminus S$ is an isomorphism,

(3-ii) $D := f^{-1}S$ is a normal crossing hypersurface of W,

(3-iii) $H^{m-d_X}\mathrm{D}g^*\mathscr{M}$ has no singularities outside D, where g is the composition $W \xrightarrow{f} Z \hookrightarrow X$.

We have

$$\big(\mathrm{D}g^*\mathscr{M}\big)(*D) \simeq \big(\big(H^{m-d_X}\mathrm{D}g^*\mathscr{M}\big)(*D)\big)[d_X - m].$$

Then by step (i), $P_W\big((\mathrm{D}g^*\mathscr{M})(*D)\big)$ is true. Hence $P_X\big(\mathrm{D}g_*\big((\mathrm{D}g^*\mathscr{M})(*D)\big)\big)$ is true. Let us consider a distinguished triangle

$$\mathscr{M} \longrightarrow \mathrm{D}g_*\big((\mathrm{D}g^*\mathscr{M})(*D)\big)[m - d_X] \longrightarrow \mathscr{N} \xrightarrow{1} .$$

Since $\mathrm{Supp}(\mathscr{N}) \subset S$, $P_X(\mathscr{N})$ is true by the induction hypothesis. Hence $P_X(\mathscr{M})$ is true. Q.E.D.

Remark 3.1.5. In fact, we could remove condition (d) in the regular case. We keep it by analogy with the irregular case.

3.2 Real blow-up

A classical tool in the study of differential equations is the real blow-up and we shall use this construction in the proof of Theorems 3.3.2, 6.7.1 and in the definition of normal form given in § 6.5.

Recall that $\mathbb{C}^\times$ denotes $\mathbb{C} \setminus \{0\}$ and $\mathbb{R}_{>0}$ the multiplicative group of positive real numbers. Consider the action of $\mathbb{R}_{>0}$ on $\mathbb{C}^\times \times \mathbb{R}$:

$$\mathbb{R}_{>0} \times (\mathbb{C}^\times \times \mathbb{R}) \to \mathbb{C}^\times \times \mathbb{R}, \quad (a, (z,t)) \mapsto (az, a^{-1}t) \tag{3.2.1}$$

and set

$$\widetilde{\mathbb{C}}^{\mathrm{tot}} = (\mathbb{C}^\times \times \mathbb{R})/\mathbb{R}_{>0}, \, \widetilde{\mathbb{C}}^{\geq 0} = (\mathbb{C}^\times \times \mathbb{R}_{\geq 0})/\mathbb{R}_{>0}, \, \widetilde{\mathbb{C}}^{>0} = (\mathbb{C}^\times \times \mathbb{R}_{>0})/\mathbb{R}_{>0}.$$

One denotes by ϖ^{tot} the map:

$$\varpi^{\mathrm{tot}} \colon \widetilde{\mathbb{C}}^{\mathrm{tot}} \to \mathbb{C}, \quad (z,t) \mapsto tz. \tag{3.2.2}$$

Then we have

$$\widetilde{\mathbb{C}}^{\mathrm{tot}} \supset \widetilde{\mathbb{C}}^{\geq 0} \supset \widetilde{\mathbb{C}}^{>0} \xrightarrow{\sim} \mathbb{C}^\times.$$

Let $X = \mathbb{C}^n \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$ and let D be the divisor $\{z_1 \cdots z_r = 0\}$. Set

$$\widetilde{X}^{\mathrm{tot}} = (\widetilde{\mathbb{C}}^{\mathrm{tot}})^r \times \mathbb{C}^{n-r},\ \widetilde{X}^{>0} = (\widetilde{\mathbb{C}}^{>0})^r \times \mathbb{C}^{n-r},\ \widetilde{X} = (\widetilde{\mathbb{C}}^{\geq 0})^r \times \mathbb{C}^{n-r}.$$

Then $\widetilde{X}$ is the closure of $\widetilde{X}^{>0}$ in $\widetilde{X}^{\mathrm{tot}}$.

The map ϖ^{tot} in (3.2.2) defines the map

$$\varpi\colon \widetilde{X} \to X.$$

The map ϖ is proper and induces an isomorphism

$$\varpi|_{\widetilde{X}^{>0}}\colon \widetilde{X}^{>0} = \varpi^{-1}(X \setminus D) \xrightarrow{\sim} X \setminus D.$$

We call $\widetilde{X}$ the *real blow up* along D.

Remark 3.2.1. The real manifold $\widetilde{X}$ (with boundary) as well as the map $\varpi\colon \widetilde{X} \to X$ may be intrinsically defined for a complex manifold X and a normal crossing divisor D, but $\widetilde{X}^{\mathrm{tot}}$ is only intrinsically defined as a germ of a manifold in a neighborhood of $\widetilde{X}$.

We set

$$\begin{aligned} \mathcal{D}b^{\mathrm{t}}_{\widetilde{X}} &:= \mathscr{I}hom(\mathbb{C}_{\widetilde{X}^{>0}}, \mathcal{D}b^{\mathrm{t}}_{\widetilde{X}^{\mathrm{tot}}})|_{\widetilde{X}} \\ &\simeq \varpi^! \mathscr{I}hom(\mathbb{C}_{X\setminus D}, \mathcal{D}b^{\mathrm{t}}_{X_{\mathbb{R}}}), \end{aligned} \tag{3.2.3}$$

where the last isomorphism follows from Lemma 1.5.6. Now we set

$$\begin{aligned} \mathcal{O}^{\mathrm{t}}_{\widetilde{X}} &:= \mathrm{R}\mathscr{H}om_{\varpi^{-1}\mathscr{D}_{X^c}}(\varpi^{-1}\mathcal{O}_{X^c}, \mathcal{D}b^{\mathrm{t}}_{\widetilde{X}}), \\ \mathcal{A}_{\widetilde{X}} &:= \alpha_{\widetilde{X}} \mathcal{O}^{\mathrm{t}}_{\widetilde{X}}, \\ \mathscr{D}^{\mathcal{A}}_{\widetilde{X}} &:= \mathcal{A}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathscr{D}_X. \end{aligned} \tag{3.2.4}$$

Note that $\mathcal{A}_{\widetilde{X}}$ and $\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}$ are concentrated in degree 0, and hence they are sheaves of $\mathbb{C}$-algebras on $\widetilde{X}$. Also note that:

$$\mathcal{D}b^{\mathrm{t}}_{\widetilde{X}} \text{ is an object of } \mathrm{I}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}} \otimes \varpi^{-1}\mathscr{D}_{X^c}), \tag{3.2.5}$$

$$\mathcal{O}^{\mathrm{t}}_{\widetilde{X}} \text{ is an object of } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}). \tag{3.2.6}$$

By using (3.2.3), we get the isomorphism

$$\mathcal{O}^{\mathrm{t}}_{\widetilde{X}} \simeq \varpi^! \mathcal{O}^{\mathrm{t}}_X(*D) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\varpi^{-1}\mathscr{D}_X). \tag{3.2.7}$$

Recall that the map ϖ is proper, and hence $\mathrm{R}\varpi_{!!} \simeq \mathrm{R}\varpi_*$.

Lemma 3.2.2. *Let $\mathscr{F} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}})$ and assume that $\mathscr{F} \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{X\setminus D}, \mathscr{F})$. Then $\mathrm{R}\varpi_{!!}\varpi^{!}\mathscr{F} \xrightarrow{\sim} \mathscr{F}$.*

Proof. One has

$$\begin{aligned}
\mathrm{R}\varpi_{!!}\varpi^{!}\mathscr{F} &\simeq \mathrm{R}\varpi_{*}\varpi^{!}\,\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{X\setminus D}, \mathscr{F}) \\
&\simeq \mathrm{R}\varpi_{*}\,\mathrm{R}\mathscr{I}hom\,(\varpi^{-1}\mathbb{C}_{X\setminus D}, \varpi^{!}\mathscr{F}) \\
&\simeq \mathrm{R}\mathscr{I}hom\,(\mathrm{R}\varpi_{!!}\varpi^{-1}\mathbb{C}_{X\setminus D}, \mathscr{F}) \simeq \mathscr{F}.
\end{aligned}$$

Q.E.D.

As a corollary, we obtain the isomorphism

$$\mathrm{R}\varpi_{*}\mathcal{O}^{\mathrm{t}}_{\widetilde{X}} \simeq \mathcal{O}^{\mathrm{t}}_{X}(*D) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{X}). \tag{3.2.8}$$

For $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}})$, we set

$$\mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{N}) = \Omega^{\mathrm{t}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}} \mathscr{N}, \tag{3.2.9}$$

$$\mathcal{S}ol^{\mathrm{t}}_{\widetilde{X}}(\mathscr{N}) = \mathrm{R}\mathscr{H}om_{\,\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathscr{N}, \mathcal{O}^{\mathrm{t}}_{\widetilde{X}}). \tag{3.2.10}$$

Here $\Omega^{\mathrm{t}}_{\widetilde{X}} = \varpi^{-1}\Omega_X \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{O}_X} \mathcal{O}^{\mathrm{t}}_{\widetilde{X}}$. It is an object of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}((\mathscr{D}^{\mathcal{A}}_{\widetilde{X}})^{\mathrm{op}}))$.

For a $\mathscr{D}_X$-module $\mathscr{M}$ we set:

$$\mathscr{M}^{\mathcal{A}} := \mathscr{D}^{\mathcal{A}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathscr{D}_X} \varpi^{-1}\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}). \tag{3.2.11}$$

Lemma 3.2.3. *For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, we have*

$$\varpi^{!}\,\mathcal{DR}^{\mathrm{t}}_{X}(\mathscr{M}(*D)) \simeq \mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}), \tag{3.2.12}$$

$$\mathrm{R}\varpi_{*}\mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \simeq \mathcal{DR}^{\mathrm{t}}_{X}(\mathscr{M}(*D)). \tag{3.2.13}$$

Proof. By (3.2.7), we have

$$\begin{aligned}
\varpi^{!}\,\mathcal{DR}^{\mathrm{t}}_{X}(\mathscr{M}(*D)) &\simeq \varpi^{!}\left(\Omega^{\mathrm{t}}_{X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}(*D)\right) \\
&\simeq \varpi^{!}\left(\Omega^{\mathrm{t}}_{X}(*D) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}\right) \\
&\simeq (\varpi^{!}\,\Omega^{\mathrm{t}}_{X}(*D)) \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathscr{D}_X} \varpi^{-1}\mathscr{M} \\
&\simeq \Omega^{\mathrm{t}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}} \mathscr{D}^{\mathcal{A}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathscr{D}_X} \varpi^{-1}\mathscr{M} \\
&\simeq \Omega^{\mathrm{t}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}} \mathscr{M}^{\mathcal{A}} \simeq \mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}).
\end{aligned}$$

Hence we obtain the first isomorphism.

Since

$$\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}(*D)) \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom\big(\mathbb{C}_{X\setminus D}, \mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}(*D))\big),$$

the second isomorphism follows from Lemma 3.2.2. Q.E.D.

Proposition 3.2.4. *Let $\mathscr{L}$ be a holonomic $\mathscr{D}_X$-module with regular normal form along D. Then, locally on $\widetilde{X}$,*

$$\mathscr{L}^{\mathcal{A}} \simeq \mathcal{A}_{\widetilde{X}} \simeq \mathcal{O}^{\mathcal{A}}_X \text{ in } \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}).$$

Proof. Let us keep the notations of Definition 3.1.2. We may assume that $\mathscr{L} = \mathscr{D}_X/\mathscr{I}_\lambda$. Since $z^\lambda := \prod_{i\in I} z_i^{\lambda_i}$ is a locally invertible section of $\mathcal{A}_{\widetilde{X}}$, the result follows from

$$(z_i\partial_i - \lambda_i)z^\lambda = z^\lambda z_i\partial_i.$$

Q.E.D.

3.3 Regular Riemann-Hilbert correspondence

We shall first prove the regularity theorem for regular holonomic D-modules, namely, any solution of such a D-module is tempered.

Theorem 3.3.1. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. Then there are isomorphisms:*

$$\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}) \xrightarrow{\sim} \mathcal{DR}_X(\mathscr{M}) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}), \tag{3.3.1}$$

$$\mathcal{S}ol^{\mathrm{t}}_X(\mathscr{M}) \xrightarrow{\sim} \mathcal{S}ol_X(\mathscr{M}) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}). \tag{3.3.2}$$

Proof. (i) Note that, thanks to (2.2.1), the isomorphism in (3.3.2) is equivalent to the isomorphism in (3.3.1) for $\mathbb{D}_X\mathscr{M}$. We shall only prove (3.3.1).

(ii) We shall apply Lemma 3.1.4. Denote by $P_X(\mathscr{M})$ the statement which asserts that the morphism in (3.3.1) is an isomorphism.

(a-d) of this lemma are clearly satisfied.

(e) follows from isomorphism (2.2.9) in Corollary 2.2.6 and its non-tempered version, isomorphism (2.1.4) in Theorem 2.1.2.

(f) Let us check property (f). Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module with regular normal form along a normal crossing divisor D.

We want to prove the isomorphism $\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}) \xrightarrow{\sim} \alpha_X \mathcal{DR}^{\mathrm{t}}_X(\mathscr{M})$. Since $\mathrm{R}\varpi_* \mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \xrightarrow{\sim} \mathcal{DR}^{\mathrm{t}}_X(\mathscr{M})$ by Lemma 3.2.3 and since $\mathrm{R}\varpi_*$ commutes with α, we are reduced to prove the isomorphism

$$\mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \xrightarrow{\sim} \alpha_{\widetilde{X}} \mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}).$$

This is a local problem on $\widetilde{X}$ and we may apply Proposition 3.2.4. Hence it is enough to show

$$\mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathcal{O}^{\mathcal{A}}_X) \xrightarrow{\sim} \alpha_{\widetilde{X}} \mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathcal{O}^{\mathcal{A}}_X),$$

which follows from

$$\mathcal{DR}^{\mathrm{t}}_{\widetilde{X}}(\mathcal{O}^{\mathcal{A}}_X) \simeq \mathbb{C}_{\widetilde{X}}[d_X].$$

This completes the proof of property (f). Q.E.D.

The following theorem is a generalized form of the Riemann-Hilbert correspondence for regular holonomic D-modules (see Remark 3.3.3).

Theorem 3.3.2. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. Then, there is an isomorphism functorial in $\mathscr{M}$*

$$\mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M} \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol^{\mathrm{t}}_X(\mathscr{M}), \mathcal{O}^{\mathrm{t}}_X) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X). \tag{3.3.3}$$

Proof. (i) The morphism in (3.3.3) is obtained by adjunction from the composition of the morphisms

$$\mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M} \overset{\mathrm{L}}{\otimes} \mathrm{R}\mathscr{I}hom_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}^{\mathrm{t}}_X) \;\to\; \mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{L}}{\otimes}_{\beta\mathcal{O}_X} \mathcal{O}^{\mathrm{t}}_X \to \mathcal{O}^{\mathrm{t}}_X. \tag{3.3.4}$$

(ii) We shall apply Lemma 3.1.4. Denote by $P_X(\mathscr{M})$ the statement which asserts that the morphism in (3.3.3) is an isomorphism.
(a-d) Properties (a)–(d) of this lemma are clearly satisfied.
(e) By Corollary 2.2.7, we have

$$\mathcal{O}^{\mathrm{t}}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D}f_*\mathscr{M} \simeq \mathrm{D}f_*(\mathcal{O}^{\mathrm{t}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M}). \tag{3.3.5}$$

On the other hand we have

$$\mathcal{S}ol^{\mathrm{t}}_Y(\mathrm{D}f_*\mathscr{M}) \simeq \mathrm{R}f_{!!}\,\mathcal{S}ol^{\mathrm{t}}_X(\mathscr{M})[d_X - d_Y]$$

by (2.2.1), (2.2.9) and Theorem 2.1.3 (i). Hence we have

$$\begin{aligned}&\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}),\mathcal{O}_Y^{\mathrm{t}})\\&\qquad\simeq \mathrm{R}\mathscr{I}hom\,(\mathrm{R}f_{!!}\,\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M})[d_X-d_Y],\mathcal{O}_Y^{\mathrm{t}})\\&\qquad\simeq \mathrm{R}f_*\,\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M})[d_X-d_Y],f^!\,\mathcal{O}_Y^{\mathrm{t}}).\end{aligned}$$

By (2.2.3), we have

$$f^!\,\mathcal{O}_Y^{\mathrm{t}}\simeq \mathscr{D}_{Y\leftarrow X}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathcal{O}_X^{\mathrm{t}}\,[d_X-d_Y].$$

Hence we have

$$\begin{aligned}&\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}),\mathcal{O}_Y^{\mathrm{t}})\\&\qquad\simeq \mathrm{R}f_*\,\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}),\mathscr{D}_{Y\leftarrow X}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathcal{O}_X^{\mathrm{t}})\\&\qquad\simeq \mathrm{R}f_*\big(\mathscr{D}_{Y\leftarrow X}\overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X}\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}),\mathcal{O}_X^{\mathrm{t}})\big)\\&\qquad\simeq \mathrm{D}f_*\,\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}),\mathcal{O}_X^{\mathrm{t}}).\end{aligned}$$

Combining with (3.3.5), we finally obtain

$$\begin{aligned}\mathcal{O}_Y^{\mathrm{t}}\overset{\mathrm{D}}{\otimes}\mathrm{D}f_*\mathscr{M}&\simeq \mathrm{D}f_*(\mathcal{O}_X^{\mathrm{t}}\overset{\mathrm{D}}{\otimes}\mathscr{M})\\&\simeq \mathrm{D}f_*\,\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}),\mathcal{O}_X^{\mathrm{t}})\\&\simeq \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_Y^{\mathrm{t}}(\mathrm{D}f_*\mathscr{M}),\mathcal{O}_Y^{\mathrm{t}}).\end{aligned}$$

Here the second isomorphism follows from $P_X(\mathscr{M})$.

(f) Let us check property (f) for (3.3.3). Hence, we assume that $\mathscr{M}$ has regular normal form along D.

By Lemmas 3.1.3 and 2.3.1 we have

$$\begin{aligned}\mathrm{R}\mathscr{I}hom\,\big(\mathcal{S}ol_X(\mathscr{M}),\mathcal{O}_X^{\mathrm{t}}\big)&\simeq \mathrm{R}\mathscr{I}hom\,\big(\mathcal{S}ol_X(\mathscr{M})\otimes\mathbb{C}_{X\setminus D},\mathcal{O}_X^{\mathrm{t}}\big)\\&\simeq \mathrm{R}\mathscr{I}hom\,\big(\mathcal{S}ol_X(\mathscr{M}),\mathrm{R}\mathscr{I}hom\,(\mathbb{C}_{X\setminus D},\mathcal{O}_X^{\mathrm{t}})\big)\\&\simeq \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X(\mathscr{M}),\mathrm{R}\varpi_*\mathcal{O}_{\widetilde{X}}^{\mathrm{t}})\\&\simeq \mathrm{R}\varpi_*\,\mathrm{R}\mathscr{I}hom\,(\varpi^{-1}\mathcal{S}ol_X(\mathscr{M}),\mathcal{O}_{\widetilde{X}}^{\mathrm{t}})\\&\simeq \mathrm{R}\varpi_*\,\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}),\mathcal{O}_{\widetilde{X}}^{\mathrm{t}}).\end{aligned}$$

Here the last isomorphism follows from

$$\mathbb{C}_{\widetilde{X}>0}\overset{\mathrm{L}}{\otimes}\varpi^{-1}\mathcal{S}ol_X(\mathscr{M})\simeq \mathbb{C}_{\widetilde{X}>0}\overset{\mathrm{L}}{\otimes}\mathcal{S}ol_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}).$$

On the other-hand, we have

$$\begin{aligned}\mathcal{O}_X^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathscr{M} \simeq \mathcal{O}_X^{\mathrm{t}}(*D) \overset{\mathrm{D}}{\otimes} \mathscr{M} &\simeq (\mathrm{R}\varpi_* \mathcal{O}_{\widetilde{X}}^{\mathrm{t}}) \overset{\mathrm{D}}{\otimes} \mathscr{M} \\ &\simeq \mathrm{R}\varpi_*(\mathcal{O}_{\widetilde{X}}^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\varpi^{-1}\mathcal{O}_X} \varpi^{-1}\mathscr{M}) \\ &\simeq \mathrm{R}\varpi_*(\mathcal{O}_{\widetilde{X}}^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_{\widetilde{X}}} \mathscr{M}^{\mathcal{A}}).\end{aligned}$$

Hence it is enough to show that

$$\mathcal{O}_{\widetilde{X}}^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_{\widetilde{X}}} \mathscr{M}^{\mathcal{A}} \to \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}), \mathcal{O}_{\widetilde{X}}^{\mathrm{t}}) \tag{3.3.6}$$

is an isomorphism. Note that this morphism is obtained from a similar morphism to (3.3.4) by adjunction. By Proposition 3.2.4, $\mathscr{M}^{\mathcal{A}}$ is locally isomorphic to $\mathcal{A}_{\widetilde{X}}$. Then $\mathcal{S}ol_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \simeq \mathbb{C}_{\widetilde{X}}$, and it is obvious that (3.3.6) is an isomorphism.

Q.E.D.

Remark 3.3.3. Isomorphism (3.3.1) already appeared in [Ka84]. Isomorphism (3.3.3) (with a different formulation) is essentially due to Björk [Bj93, Th. 7.9.11].

Applying the functor α_X to the isomorphism (3.3.3), we get the Riemann-Hilbert correspondence for regular holonomic D-modules:

Corollary 3.3.4 ([Ka80])**.** *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$. There is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$:*

$$\mathscr{M} \simeq \mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(\mathcal{S}ol_X(\mathscr{M}), \mathcal{O}_X^{\mathrm{t}}). \tag{3.3.7}$$

Corollary 3.3.5. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ and let $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$. Then isomorphism (3.3.3) induces the isomorphism*

$$\mathcal{DR}^{\mathrm{t}}(\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}), \mathcal{DR}^{\mathrm{t}}(\mathscr{L})). \tag{3.3.8}$$

Proof. We have

$$\begin{aligned}\mathcal{DR}^{\mathrm{t}}(\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) &= \Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \\ &\simeq (\Omega_X^{\mathrm{t}} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{L} \\ &\simeq \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}), \Omega_X^{\mathrm{t}}) \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{L} \\ &\simeq \mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{M}), \Omega_X^{\mathrm{t}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{L}).\end{aligned}$$

Here, the last isomorphism follows from Theorem 1.3.6, using the fact that $\mathcal{S}ol^{\mathrm{t}}(\mathscr{M}) \xrightarrow{\sim} \mathcal{S}ol(\mathscr{M})$. Q.E.D.

As an application of isomorphism (3.3.2), we get:

Corollary 3.3.6. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_X)$ and let $F \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathbb{C}_X)$. Then we have the natural isomorphism*

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om_{\mathrm{I}\mathbb{C}_X}(F, \mathcal{O}^{\mathrm{t}}_X)) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathrm{R}\mathscr{H}om(F, \mathcal{O}_X)).$$

Let M be a real analytic manifold and X a complexification of M. Choosing for F the object $\mathrm{D}'_X\mathbb{C}_M$ we get the isomorphism between the complexes of distribution solutions and hyperfunction solutions of $\mathscr{M}$:

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{D}b_M) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M).$$

Remark 3.3.7. Of course, isomorphism (3.3.3) is no more true if one replaces $\mathcal{O}^{\mathrm{t}}_X$ with $\mathcal{O}_X$. For example, choosing $\mathscr{M} = \mathcal{O}_X(*Y)$ for Y a closed hypersurface, the left-hand is the sheaf of meromorphic functions with poles on Y and the right-hand side the sheaf of holomorphic functions with possibly essential singularities on Y.

3.4 Integral transforms with regular kernels

Consider morphisms of complex manifolds

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Notation 3.4.1. (i) For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_S)$ one sets

$$\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L} := \mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}). \tag{3.4.1}$$

(ii) For $L \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_S)$, $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$ and $G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_Y)$ one sets

$$L \circ G := \mathrm{R}f_{!!}(L \otimes g^{-1}G),$$
$$\Phi_L(G) = L \circ G, \quad \Psi_L(F) = \mathrm{R}g_*\mathrm{R}\mathscr{I}hom(L, f^!F).$$

Note that we have a pair of adjoint functors

$$\Phi_L\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}}) \rightleftarrows \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})\colon \Psi_{\mathrm{L}} \tag{3.4.2}$$

Theorem 3.4.2. *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, let $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_S)$ and set $L := \mathcal{S}ol_S(\mathscr{L})$. Assume that $f^{-1}\operatorname{supp}(\mathscr{M}) \cap \operatorname{supp}(\mathscr{L})$ is proper over Y and that $\mathscr{L}$ is good. Then there is a natural isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$:*

$$\Psi_L\big(\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M})\big)\,[d_X - d_S] \simeq \mathcal{DR}^{\mathrm{t}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}). \tag{3.4.3}$$

Note that any regular holonomic $\mathscr{D}$-module is good.

Proof. Applying Corollaries 2.2.2, 2.2.6 and 3.3.5, we get:

$$\begin{aligned}
\mathcal{DR}^{\mathrm{t}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}) &= \mathcal{DR}^{\mathrm{t}}_Y(\mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}))\\
&\simeq \mathrm{R}g_*\mathcal{DR}^{\mathrm{t}}_S(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L})\\
&\simeq \mathrm{R}g_*\mathrm{R}\mathscr{I}hom\,(\mathcal{S}ol^{\mathrm{t}}_S(\mathscr{L}), \mathcal{DR}^{\mathrm{t}}_S(\mathrm{D}f^*\mathscr{M}))\\
&\simeq \mathrm{R}g_*\mathrm{R}\mathscr{I}hom\,(L, f^!\,\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}))\,[d_X - d_S]\\
&= \Psi_L(\mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}))\,[d_X - d_S].
\end{aligned}$$

Q.E.D.

By applying the functor $\mathrm{RHom}\,(G;\,\bullet\,)$ with $G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$ to both sides of (3.4.3), one gets

Corollary 3.4.3 ([KS01, Th. 7.4.13])**.** *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, let $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathscr{D}_S)$ and let $L := \mathcal{S}ol_S(\mathscr{L})$. Assume that $f^{-1}\operatorname{supp}(\mathscr{M}) \cap \operatorname{supp}(\mathscr{L})$ is proper over Y and that $\mathscr{L}$ is good. Let $G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$. Then one has the isomorphism*

$$\begin{aligned}
&\mathrm{RHom}_{\mathrm{I}\mathbb{C}_{\mathrm{X}}}(L \circ G, \mathcal{DR}^{\mathrm{t}}_X(\mathscr{M}))[d_X - d_S]\\
&\qquad\qquad \simeq \mathrm{RHom}_{\mathrm{I}\mathbb{C}_{\mathrm{Y}}}(G, \mathcal{DR}^{\mathrm{t}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L})).
\end{aligned} \tag{3.4.4}$$

Note that a similar formula holds when replacing $\mathcal{O}^{\mathrm{t}}_X$ and $\mathcal{O}^{\mathrm{t}}_Y$ with their non tempered versions $\mathscr{O}_X$ and $\mathscr{O}_Y$ (and indsheaves with usual sheaves), but the hypotheses are different. Essentially, $\mathscr{M}$ has to be coherent, f non characteristic for $\mathscr{M}$ and $\mathrm{D}f^*\mathscr{M}$ has to be transversal to the holonomic module $\mathscr{L}$. On the other hand, we do not need the regularity assumption on $\mathscr{L}$. See [DS96] for such a non tempered formula (in a more particular setting).

However, if one removes the hypothesis that the holonomic module $\mathscr{L}$ is regular in Theorem 3.4.2, formula (3.4.3) do not hold anymore and we have to replace $\mathcal{O}_X^{\mathrm{t}}$ with its enhanced version, as we shall see in the next sections.

3.5 Irregular $\mathscr{D}$-modules: an example

In this subsection we recall an example treated in [KS03] which emphasizes the role of the sheaf $\mathcal{O}_X^{\mathrm{t}}$ in the study of irregular holonomic D-modules.

Let $X = \mathbb{C}$ endowed with the holomorphic coordinate z. Define

$$U = X \setminus \{0\}, \quad j\colon U \hookrightarrow X \text{ the open embedding.}$$

Consider the differential operator $P = z^2\partial_z + 1$ and the $\mathscr{D}_X$-module $\mathscr{L} := \mathscr{D}_X \exp(1/z) \simeq \mathscr{D}_X/\mathscr{D}_X \cdot P$.

Notice first that $\mathcal{O}_X^{\mathrm{t}}$ is concentrated in degree 0 (since $\dim X = 1$) and it is a sub-indsheaf of $\mathcal{O}_X$. Therefore the morphism $H^0(\mathcal{S}ol_X^{\mathrm{t}}(\mathscr{L})) \to H^0(\mathcal{S}ol_X(\mathscr{L})) \simeq \mathbb{C}_U$ is a monomorphism. It follows that for $V \subset X \setminus \{0\}$ a connected open subset, $\Gamma(V; H^0\mathcal{S}ol^{\mathrm{t}}(\mathscr{M})) \neq 0$ if and only if $V \subset U$ and $\exp(1/z)|_V$ is tempered.

Denote by $\overline{B}_\varepsilon$ the closed ball with center $(\varepsilon, 0)$ and radius ε and set

$$U_\varepsilon = X \setminus \overline{B}_\varepsilon = \{z \in \mathbb{C} \setminus \{0\}; \mathrm{Re}(1/z) < 1/2\varepsilon\}.$$

One proves that $\exp(1/z)$ is tempered (in a neighborhood of 0) on an open subanalytic subset $V \subset X \setminus \{0\}$ if and only if $\mathrm{Re}(1/z)$ is bounded on V, that is, if and only if $V \subset U_\varepsilon$ for some $\varepsilon > 0$. We get the isomorphism

$$\mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \otimes \mathbb{C}_U \simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{U_\varepsilon}. \tag{3.5.1}$$

Since $\mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \simeq \mathcal{DR}^{\mathrm{t}}(\mathbb{D}\mathscr{L})$ and $\mathbb{D}\mathscr{L} \simeq \mathbb{D}\mathscr{L}(*\{0\})$, we get that

$$\mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{S}ol^{\mathrm{t}}(\mathscr{L})) \simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \otimes \mathbb{C}_U).$$

Therefore,

$$\begin{aligned}
&\mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \simeq \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{U_\varepsilon}),\\
&H^0(\mathcal{S}ol^{\mathrm{t}}(\mathscr{L})) \simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathbb{C}_{U_\varepsilon},\\
&H^1(\mathcal{S}ol^{\mathrm{t}}(\mathscr{L})) \simeq \text{“}\varinjlim_{\varepsilon>0}\text{”}\, \mathscr{E}xt^1(\mathbb{C}_U, \mathbb{C}_{U_\varepsilon}) \simeq \mathbb{C}_{\{0\}},\\
&\mathcal{S}ol(\mathscr{L}) \simeq \alpha_X \mathcal{S}ol^{\mathrm{t}}(\mathscr{L}) \simeq \mathrm{R}\mathscr{H}om\,(\mathbb{C}_U, \mathbb{C}_U),\\
&H^0(\mathcal{S}ol(\mathscr{L})) \simeq \mathbb{C}_U, \quad H^1(\mathcal{S}ol(\mathscr{L})) \simeq \mathbb{C}_{\{0\}}.
\end{aligned}$$

The functor $\mathcal{S}ol^{\mathrm{t}}$ is not fully faithful since the $\mathscr{D}_X$-modules $\mathscr{M} := \mathscr{D}_X \exp(1/z)$ and $\mathscr{N} := \mathscr{D}_X \exp(2/z)$ have the same indsheaves of tempered holomorphic solutions although they are not isomorphic.

However, $\mathcal{S}ol^{\mathrm{t}}_X(\mathscr{D}_X \exp(1/z)) \not\simeq \mathcal{S}ol^{\mathrm{t}}_X(\mathscr{D}_X \exp(1/z^m))$ for any $m > 1$.

Hence, the functor $\mathcal{S}ol^{\mathrm{t}}$ is sensitive enough to distinguish $m \in \mathbb{Z}_{>0}$ in $\mathscr{D}_X \exp(z^{-m})$ but is not sensitive enough to distinguish $c \in \mathbb{R}_{>0}$ in $\mathscr{D}_X \exp(cz^{-1})$.

In order to capture c, we need to work in the framework of enhanced indsheaves, which we are going to explain in the next sections.

4 Indsheaves on bordered spaces

4.1 Bordered spaces

Definition 4.1.1. The category of *bordered spaces* is the category whose objects are pairs $(M, \widehat{M})$ with $M \subset \widehat{M}$ an open embedding of good topological spaces. Morphisms $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ are continuous maps $f\colon M \to N$ such that

$$\overline{\Gamma}_f \to \widehat{M} \text{ is proper.} \tag{4.1.1}$$

Here $\Gamma_f \subset M \times N$ is the graph of f an $\overline{\Gamma}_f$ is its closure in $\widehat{M} \times \widehat{N}$.

The composition of $(L, \widehat{L}) \xrightarrow{g} (M, \widehat{M}) \xrightarrow{f} (N, \widehat{N})$ is given by $f \circ g\colon L \to N$ (see Lemma 4.1.2 below), and the identity $\mathrm{id}_{(M,\widehat{M})}$ is given by id_M.

Lemma 4.1.2. *Let $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ and $g\colon (L, \widehat{L}) \to (M, \widehat{M})$ be morphisms of bordered spaces. Then the composition $f \circ g$ is a morphism of bordered spaces.*

One shall identify a space M and the bordered space (M, M). Then, by using the identifications $M = (M, M)$ and $\widehat{M} = (\widehat{M}, \widehat{M})$, there are natural morphisms of bordered spaces

$$M \to (M, \widehat{M}) \to \widehat{M}.$$

Note however that $(M, \widehat{M}) \to M$ is a morphism of bordered spaces if and only if M is a closed subset of $\widehat{M}$.

We can easily see that the category of bordered spaces admits products:

$$(M, \widehat{M}) \times (N, \widehat{N}) \simeq (M \times N, \widehat{M} \times \widehat{N}). \tag{4.1.2}$$

Let $(M, \widehat{M})$ a bordered space. Denote by $i\colon \widehat{M} \setminus M \to \widehat{M}$ the closed embedding. Identifying $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}\setminus M})$ with its essential image in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}})$ by the fully faithful functor $\mathrm{R}i_! \simeq \mathrm{R}i_*$, the restriction functor $F \mapsto F|_M$ induces an equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}})/\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{\widehat{M}\setminus M}) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M).$$

This is no longer true for indsheaves. Therefore one sets

$$\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) := \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})/\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M}).$$

where $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M})$ is identified with its essential image in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$ by $\mathrm{R}i_{!!} \simeq \mathrm{R}i_*$, as for usual sheaves.

Recall that if $\mathscr{T}$ is a triangulated category and $\mathscr{I}$ a subcategory, one denotes by ${}^\perp\mathscr{I}$ and $\mathscr{I}^\perp$ the left and right orthogonal to $\mathscr{I}$ in $\mathscr{T}$, respectively:

$$\begin{aligned} {}^\perp\mathscr{I} &:= \{A \in \mathscr{T}\,;\, \mathrm{Hom}_{\mathscr{C}}(A, B) = 0 \text{ for any } B \in \mathscr{I}\}\,, \\ \mathscr{I}^\perp &:= \{A \in \mathscr{T}\,;\, \mathrm{Hom}_{\mathscr{C}}(B, A) = 0 \text{ for any } B \in \mathscr{I}\}\,. \end{aligned}$$

Proposition 4.1.3. *Let $(M, \widehat{M})$ be a bordered space. Then we have*

$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M}) &= \{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}}); \mathbf{k}_M \otimes F \simeq 0\} \\ &= \{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}});\, \mathrm{R}\mathscr{I}hom\,(\mathbf{k}_M, F) \simeq 0\}, \\ {}^\perp\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M}) &= \{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}}); \mathbf{k}_M \otimes F \xrightarrow{\sim} F\}, \\ \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M})^\perp &= \{F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}}); F \xrightarrow{\sim} \mathrm{R}\mathscr{I}hom\,(\mathbf{k}_M, F)\}. \end{aligned}$$

Moreover, there are equivalences

$$\begin{aligned}
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) &\xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M})^{\perp}, \quad F \mapsto \mathrm{R}\mathscr{I}hom\,(\mathbf{k}_M, F),\\
\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) &\xrightarrow{\sim} {}^{\perp}\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}\setminus M}), \quad F \mapsto \mathbf{k}_M \otimes F,
\end{aligned}$$

with quasi-inverse induced by the quotient functor.

Corollary 4.1.4. *For* $F, G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$ *one has*

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})})}(F, G) &\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})}(\mathbf{k}_M \otimes F, G)\\
&\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})}(F, \mathrm{R}\mathscr{I}hom\,(\mathbf{k}_M, G)).
\end{aligned}$$

The functors $\otimes$ and $\mathrm{R}\mathscr{I}hom$ in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$ induce well defined functors (we keep the same notations)

$$\begin{aligned}
\otimes\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) &\to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}),\\
\mathrm{R}\mathscr{I}hom : \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) &\to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}).
\end{aligned}$$

4.2 Operations

Let $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ be a morphism of bordered spaces, and recall that Γ_f denotes the graph of the associated map $f\colon M \to N$. Since Γ_f is closed in $M \times N$, it is locally closed in $\widehat{M} \times \widehat{N}$. One can then consider the sheaf $\mathbf{k}_{\Gamma_f}$ on $\widehat{M} \times \widehat{N}$. Let $q_1\colon \widehat{M} \times \widehat{N} \to \widehat{M}$ and $q_2\colon \widehat{M} \times \widehat{N} \to \widehat{N}$ be the projections.

Definition 4.2.1. Let $f\colon (M, \widehat{M}) \to (N, \widehat{N})$ be a morphism of bordered spaces. For $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$ and $G \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{N}})$, we set

$$\begin{aligned}
\mathrm{R}f_{!!}F &= \mathrm{R}q_{2!!}(\mathbf{k}_{\Gamma_f} \otimes q_1^{-1}F),\\
\mathrm{R}f_{*}F &= \mathrm{R}q_{2*}\mathrm{R}\mathscr{I}hom\,(\mathbf{k}_{\Gamma_f}, q_1^{!}F),\\
f^{-1}G &= \mathrm{R}q_{1!!}(\mathbf{k}_{\Gamma_f} \otimes q_2^{-1}G),\\
f^{!}G &= \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom\,(\mathbf{k}_{\Gamma_f}, q_2^{!}G).
\end{aligned}$$

Remark 4.2.2. Considering a continuous map $f\colon M \to N$ as a morphism of bordered spaces with $M = \widehat{M}$ and $N = \widehat{N}$, the above functors are isomorphic to the usual external operations for indsheaves.

Lemma 4.2.3. *The above definition induces well-defined functors*

$$\mathrm{R}f_{!!}, \mathrm{R}f_* \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(N,\widehat{N})}),$$
$$f^{-1}, f^{\,!} \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(N,\widehat{N})}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}).$$

Lemma 4.2.4. *Let $j_M \colon (M, \widehat{M}) \to \widehat{M}$ be the morphism given by the open embedding $M \subset \widehat{M}$. Then*

(i) *The functors $j_M^{-1} \simeq j_M^{\,!} \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})})$ are isomorphic to the quotient functor.*

(ii) *For $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$ one has the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\widehat{M}})$*

$$\mathrm{R}j_{M\,!!}j_M^{-1}F \simeq \mathbf{k}_M \otimes F, \quad \mathrm{R}j_{M*}j_M^{\,!}F \simeq \mathrm{R}\mathscr{I}hom\,(\mathbf{k}_M, F).$$

(iii) *The functors $\otimes$ and $\mathrm{R}\mathscr{I}hom$ commute with $j_M^{-1} \simeq j_M^{\,!}$.*

(iv) *The functor $\otimes$ commutes with $\mathrm{R}j_{M\,!!}$ and the functor $\mathrm{R}\mathscr{I}hom$ commutes with $\mathrm{R}j_{M*}$.*

The operations for indsheaves on bordered spaces satisfy similar properties as for usual spaces.

Lemma 4.2.5. *Let $f \colon (M, \widehat{M}) \to (N, \widehat{N})$ and $g \colon (L, \widehat{L}) \to (M, \widehat{M})$ be morphisms of bordered spaces.*

(i) *The functor $\mathrm{R}f_{!!}$ is left adjoint to $f^{\,!}$.*

(ii) *The functor f^{-1} is left adjoint to $\mathrm{R}f_*$.*

(iii) *One has $\mathrm{R}(f \circ g)_{!!} \simeq \mathrm{R}f_{!!} \circ \mathrm{R}g_{!!}$, $\mathrm{R}(f \circ g)_* \simeq \mathrm{R}f_* \circ \mathrm{R}g_*$, $(f \circ g)^{-1} \simeq g^{-1} \circ f^{-1}$ and $(f \circ g)^{\,!} \simeq g^{\,!} \circ f^{\,!}$.*

Corollary 4.2.6. *If $f \colon (M, \widehat{M}) \to (N, \widehat{N})$ is an isomorphism of bordered spaces, then $\mathrm{R}f_* \simeq \mathrm{R}f_{!!}$ and $f^{-1} \simeq f^{\,!}$. Moreover, $\mathrm{R}f_*$ and f^{-1} are quasi-inverse to each other.*

Most of the formulas for indsheaves on usual spaces extend to bordered spaces.

Proposition 4.2.7. *Let* $f\colon (M,\widehat{M}) \to (N,\widehat{N})$ *be a morphism of bordered spaces. For* $F \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})})$ *and* $G, G_1, G_2 \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(N,\widehat{N})})$, *one has isomorphisms*

$$\begin{aligned}
\mathrm{R}f_{!!}(f^{-1}G \otimes F) &\simeq G \otimes \mathrm{R}f_{!!}F,\\
f^{-1}(G_1 \otimes G_2) &\simeq f^{-1}G_1 \otimes f^{-1}G_2,\\
\mathrm{R}\mathscr{I}hom\,(G, \mathrm{R}f_*F) &\simeq \mathrm{R}f_*\,\mathrm{R}\mathscr{I}hom\,(f^{-1}G, F),\\
\mathrm{R}\mathscr{I}hom\,(\mathrm{R}f_{!!}F, G) &\simeq \mathrm{R}f_*\,\mathrm{R}\mathscr{I}hom\,(F, f^{\,!}G),\\
f^{\,!}\,\mathrm{R}\mathscr{I}hom\,(G_1, G_2) &\simeq \mathrm{R}\mathscr{I}hom\,(f^{-1}G_1, f^{\,!}G_2),
\end{aligned}$$

and a morphism

$$f^{-1}\,\mathrm{R}\mathscr{I}hom\,(G_1, G_2) \to \mathrm{R}\mathscr{I}hom\,(f^{-1}G_1, f^{-1}G_2).$$

Lemma 4.2.8. *Consider a Cartesian diagram in the category of bordered spaces*

$$\begin{tikzcd}
(M',\widehat{M}') \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "\Box"] & (N',\widehat{N}') \arrow[d, "g"] \\
(M,\widehat{M}) \arrow[r, "f"] & (N,\widehat{N}).
\end{tikzcd}$$

Then there are isomorphisms of functors $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M',\widehat{M}')}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(N,\widehat{N})})$

$$g^{-1}\mathrm{R}f_{!!} \simeq \mathrm{R}f'_{!!}g'^{-1}, \qquad g^{\,!}\,\mathrm{R}f_* \simeq \mathrm{R}f'_*g'^{!}.$$

The notion of proper morphisms of topological spaces is extended to the case of bordered spaces as follows.

Definition 4.2.9. The morphism of bordered spaces $f\colon (M,\widehat{M}) \to (N,\widehat{N})$ is *proper* if the following two conditions hold:

(a) $f\colon M \to N$ is proper,

(b) the projection $\overline{\Gamma}_f \to \widehat{N}$ is proper.

Lemma 4.2.10. *The map* $f\colon (M,\widehat{M}) \to (N,\widehat{N})$ *is proper if and only if the following two conditions hold:*

(a) $\overline{\Gamma}_f \times_{\widehat{N}} N \subset \Gamma_f$.

(b) *the projection* $\overline{\Gamma}_f \to \widehat{N}$ *is proper.*

Proposition 4.2.11. *Assume that* $f\colon (M,\widehat{M}) \to (N,\widehat{N})$ *is proper. Then* $\mathrm{R}f_{!!} \xrightarrow{\sim} \mathrm{R}f_*$ *as functors* $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(M,\widehat{M})}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{(N,\widehat{N})})$.

5 Enhanced indsheaves

In this section, extracted from [DK13], one extends some constructions of Tamarkin [Ta08] to indsheaves on bordered spaces. We refer to [GS12] for a detailed exposition and some complements to Tamarkin's paper.

5.1 Tamarkin's construction

Let M be a smooth manifold and denote by T^*M its cotangent bundle. Given $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, its microsupport $SS(F) \subset T^*M$ (see [KS90]) describes the codirections of non propagation for the cohomology of F. It is a closed conic co-isotropic subset of T^*M.

In order to treat co-isotropic subsets of T^*M which are not necessarily conic, Tamarkin adds a real variable $t \in \mathbb{R}$. Denoting by (t, t^*) the symplectic coordinates of $T^*\mathbb{R}$, consider the full subcategory $\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) \subset \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$ whose objects K satisfy $SS(K) \subset \{t^* \leq 0\}$. There are equivalences

$$^{\perp}\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) \simeq \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})/\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) \simeq \mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}})^{\perp}$$

between the quotient category and the left and right orthogonal categories.

Let us recall the description of the first equivalence.

For $K, L \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})$, consider the convolution functor with respect to the t variable

$$K \overset{+}{\otimes} L := \mathrm{R}\mu_!(q_1^{-1}K \otimes q_2^{-1}L),$$

where $\mu(x, t_1, t_2) = (x, t_1 + t_2)$, $q_1(x, t_1, t_2) = (x, t_1)$ and $q_2(x, t_1, t_2) = (x, t_2)$.

One sets

$$\mathbf{k}_{\{t\geq 0\}} = \mathbf{k}_{\{(x,t)\in M\times\mathbb{R}\,;\,t\in\mathbb{R},\ t\geq 0\}}, \tag{5.1.1}$$

and we use similar notation for $\mathbf{k}_{\{t=0\}}$, $\mathbf{k}_{\{t>0\}}$, $\mathbf{k}_{\{t\leq 0\}}$, $\mathbf{k}_{\{t<0\}}$, $\mathbf{k}_{\{t\neq 0\}}$ and $\mathbf{k}_{\{t=a\}}$, etc. These are sheaves on $M \times \mathbb{R}$.

Note that $\mathbf{k}_{\{t=0\}} \overset{+}{\otimes} K \simeq K$. Then

$$\begin{aligned}
\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) &= \{K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}}); \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \xrightarrow{\sim} 0\},\\
^{\perp}\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) &= \{K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}}); \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \xrightarrow{\sim} K\},
\end{aligned}$$

and one has an equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})/\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}) \xrightarrow{\sim} {}^{\perp}\mathsf{D}^{\mathrm{b}}_{t^*\leq 0}(\mathbf{k}_{M\times\mathbb{R}}), \quad K \mapsto \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K.$$

5.2 Convolution products

Consider the 2-point compactification of the real line $\overline{\mathbb{R}}:=\mathbb{R}\sqcup\{\pm\infty\}$. Denote by $\mathbb{P}^1(\mathbb{R}) = \mathbb{R}\sqcup\{\infty\}$ the real projective line. Then $\overline{\mathbb{R}}$ has a structure of subanalytic space such that the natural map $\overline{\mathbb{R}} \to \mathbb{P}^1(\mathbb{R})$ is a subanalytic map.

Notation 5.2.1. We will consider the bordered space

$$\mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}).$$

We denote by $\pi\colon \mathbb{R}_\infty \to \mathbb{R}$ the projection.

Note that $\mathbb{R}_\infty$ is isomorphic to $(\mathbb{R}, \mathbb{P}^1(\mathbb{R}))$ as a bordered space.

Consider the morphisms of bordered spaces

$$\begin{aligned} a\colon \mathbb{R}_\infty &\to \mathbb{R}_\infty, \\ \mu, q_1, q_2\colon \mathbb{R}_\infty \times \mathbb{R}_\infty &\to \mathbb{R}_\infty, \end{aligned} \tag{5.2.1}$$

where $a(t) = -t$, $\mu(t_1, t_2) = t_1 + t_2$ and q_1, q_2 are the natural projections. For a good topological space M, we will use the same notations for the associated morphisms

$$\begin{aligned} a\colon M \times \mathbb{R}_\infty &\to M \times \mathbb{R}_\infty, \\ \mu, q_1, q_2\colon M \times \mathbb{R}_\infty \times \mathbb{R}_\infty &\to M \times \mathbb{R}_\infty. \end{aligned}$$

We also use the natural morphisms

$$\begin{tikzcd} M \times \mathbb{R}_\infty \arrow[rr, "j"] \arrow[dr, "\pi"'] & & M \times \overline{\mathbb{R}} \arrow[dl, "\overline{\pi}"] \\ & M. & \end{tikzcd}$$

Definition 5.2.2. The functors

$$\begin{aligned} \overset{+}{\otimes}\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) &\to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}), \\ \mathscr{I}hom^+\colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) &\to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}), \end{aligned}$$

are defined by

$$\begin{aligned} K_1 \overset{+}{\otimes} K_2 &= \mathrm{R}\mu_{!!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2), \\ \mathscr{I}hom^+(K_1, K_2) &= \mathrm{R}q_{1*}\mathrm{R}\mathscr{I}hom\,(q_2^{-1}K_1, \mu^!\,K_2). \end{aligned}$$

Although we work now on $M\times\overline{\mathbb{R}}$, we keep the same notations as in (5.1.1) and one sets

$$\mathbf{k}_{\{t\geq 0\}} = \mathbf{k}_{\{(x,t)\in M\times\overline{\mathbb{R}}\,;\,t\in\mathbb{R},\ t\geq 0\}}, \tag{5.2.2}$$

and we use similar notation for $\mathbf{k}_{\{t=0\}}$, $\mathbf{k}_{\{t>0\}}$, $\mathbf{k}_{\{t\leq 0\}}$, $\mathbf{k}_{\{t<0\}}$, $\mathbf{k}_{\{t\neq 0\}}$ and $\mathbf{k}_{\{t=a\}}$, etc. These are sheaves on $M\times\overline{\mathbb{R}}$ whose stalk vanishes at points of $M\times(\overline{\mathbb{R}}\setminus\mathbb{R})$. We also regard them as objects of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$.

Lemma 5.2.3. *For $K\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ there are isomorphisms*

$$\mathbf{k}_{\{t=0\}}\overset{+}{\otimes}K\simeq K\simeq \mathscr{I}hom^{+}(\mathbf{k}_{\{t=0\}},K).$$

More generally, for $a\in\mathbb{R}$, we have

$$\mathbf{k}_{\{t=a\}}\overset{+}{\otimes}K\simeq \mathrm{R}\mu_{a*}K\simeq \mathscr{I}hom^{+}(\mathbf{k}_{\{t=-a\}},K),$$

where $\mu_a\colon M\times\mathbb{R}_\infty\to M\times\mathbb{R}_\infty$ is the morphism induced by the translation $t\mapsto t+a$.

Corollary 5.2.4. *The category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ has a structure of commutative tensor category with $\overset{+}{\otimes}$ as tensor product and $\mathbf{k}_{\{t=0\}}$ as unit object.*

As seen in (5.2.3) below, the functor $\mathscr{I}hom^{+}$ is the inner hom of the tensor category $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$.

Lemma 5.2.5. *For $K_1,K_2,K_3\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ one has*

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})}(K_1\overset{+}{\otimes}K_2,\ K_3)\\
&\qquad\simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})}\big(K_1,\ \mathscr{I}hom^{+}(K_2,K_3)\big),\\
&\mathscr{I}hom^{+}(K_1\overset{+}{\otimes}K_2,\ K_3)\simeq \mathscr{I}hom^{+}\big(K_1,\ \mathscr{I}hom^{+}(K_2,K_3)\big),\\
&\mathrm{R}\pi_*\mathrm{R}\mathscr{I}hom\,(K_1\overset{+}{\otimes}K_2,K_3)\simeq \mathrm{R}\pi_*\mathrm{R}\mathscr{I}hom\,(K_1,\mathscr{I}hom^{+}(K_2,K_3)).
\end{aligned}\tag{5.2.3}$$

The following lemmas are used to define the category of enhanced ind-sheaves.

Lemma 5.2.6. *For $K_1,K_2\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ and $L\in\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ one has*

$$\begin{aligned}
\pi^{-1}L\otimes(K_1\overset{+}{\otimes}K_2)&\simeq(\pi^{-1}L\otimes K_1)\overset{+}{\otimes}K_2,\\
\mathrm{R}\mathscr{I}hom\,(\pi^{-1}L,\mathscr{I}hom^{+}(K_1,K_2))&\simeq \mathscr{I}hom^{+}(\pi^{-1}L\otimes K_1,K_2)\\
&\simeq \mathscr{I}hom^{+}(K_1,\mathrm{R}\mathscr{I}hom\,(\pi^{-1}L,K_2)).
\end{aligned}$$

Lemma 5.2.7. *For $K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ and $L \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ one has*

$$\begin{aligned}
\pi^{-1}L \otimes K &\simeq (\pi^{-1}L \otimes \mathbf{k}_{\{t=0\}}) \overset{+}{\otimes} K, \\
\mathrm{R}\mathscr{I}hom\,(\pi^{-1}L, K) &\simeq \mathscr{I}hom^{+}(\pi^{-1}L \otimes \mathbf{k}_{\{t=0\}}, K), \\
a^{-1}\,\mathrm{R}\mathscr{I}hom\,(K, \pi^{!}\,L) &\simeq \mathscr{I}hom^{+}(K, \mathbf{k}_{\{t=0\}} \otimes \pi^{-1}L).
\end{aligned}$$

Lemma 5.2.8. *For $K_1, K_2 \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ there are isomorphisms*

$$\begin{aligned}
\mathrm{R}\pi_{!!}(K_1 \overset{+}{\otimes} K_2) &\simeq \mathrm{R}\pi_{!!}K_1 \otimes \mathrm{R}\pi_{!!}K_2, \\
\mathrm{R}\pi_{*}\mathscr{I}hom^{+}(K_1, K_2) &\simeq \mathrm{R}\mathscr{I}hom\,(\mathrm{R}\pi_{!!}K_1, \mathrm{R}\pi_{*}K_2).
\end{aligned}$$

Corollary 5.2.9. *For any $K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$, one has*

$$\begin{aligned}
&\mathrm{R}\pi_{!!}(\mathbf{k}_{t\geq 0} \overset{+}{\otimes} K) \simeq 0, \\
&\mathrm{R}\pi_{*}\mathscr{I}hom^{+}(\mathbf{k}_{t\geq 0}, K) \simeq 0.
\end{aligned}$$

Lemma 5.2.10. *For $K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$ and $L \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ one has*

$$\begin{aligned}
(\pi^{-1}L) \overset{+}{\otimes} K &\simeq \pi^{-1}(L \otimes \mathrm{R}\pi_{!!}K), \\
\mathscr{I}hom^{+}(\pi^{-1}L, K) &\simeq \pi^{!}\,\mathrm{R}\mathscr{I}hom\,(L, \mathrm{R}\pi_{*}K), \\
\mathscr{I}hom^{+}(K, \pi^{!}\,L) &\simeq \pi^{!}\,\mathrm{R}\mathscr{I}hom\,(\mathrm{R}\pi_{!!}K, L).
\end{aligned}$$

Proposition 5.2.11. *For $K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$, one has a distinguished triangle*

$$\pi^{-1}L \to \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \to \mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}}, K) \xrightarrow{+1}$$

with $L = \mathrm{R}\pi_{*}(\mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K) \simeq \mathrm{R}\pi_{!!}\mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}}, K)$.

5.3 Enhanced indsheaves

Definition 5.3.1. Consider the full triangulated subcategories of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$

$$\begin{aligned}
\mathrm{IC}_{\mathrm{t}^{*}\leq 0} &= \{K; \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \simeq 0\} \\
&= \{K; \mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}}, K) \simeq 0\}, \\
\mathrm{IC}_{\mathrm{t}^{*}\geq 0} &= \{K; \mathbf{k}_{\{t\leq 0\}} \overset{+}{\otimes} K \simeq 0\} \\
&= \{K; \mathscr{I}hom^{+}(\mathbf{k}_{\{t\leq 0\}}, K) \simeq 0\}, \\
\mathrm{IC}_{\mathrm{t}^{*}=0} &= \mathrm{IC}_{\mathrm{t}^{*}\leq 0} \cap \mathrm{IC}_{\mathrm{t}^{*}\geq 0}.
\end{aligned}$$

Consider also the corresponding quotient categories

$$\begin{aligned}\mathrm{E}^{\mathrm{b}}_{\pm}(\mathrm{I}\mathbf{k}_M) &= \mathrm{IC}_{\pm\mathrm{t}^*\geq 0}/\mathrm{IC}_{\mathrm{t}^*=0},\\ \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &= \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})/\mathrm{IC}_{\mathrm{t}^*=0}.\end{aligned}$$

They are triangulated categories. One calls $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ the triangulated category of enhanced indsheaves.

One defines similarly the categories $\mathrm{E}(\mathrm{I}\mathbf{k}_M)$, $\mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M)$ and $\mathrm{E}^{-}(\mathrm{I}\mathbf{k}_M)$.

One also defines the category $\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$ as

$$\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M) = \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}})/\left\{K\,;\,\pi^{-1}\mathrm{R}\pi_*K \xrightarrow{\sim} K\right\}. \tag{5.3.1}$$

Then $\mathrm{E}^{\mathrm{b}}(\mathbf{k}_M)$ is a full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M_\infty})$.

Proposition 5.3.2. *There are equivalences of triangulated categories*

$$\begin{aligned}\mathrm{E}^{\mathrm{b}}_{\pm}(\mathrm{I}\mathbf{k}_M) &\simeq \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})/\mathrm{IC}_{\pm\mathrm{t}^*\leq 0},\\ \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\simeq \mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)\oplus\mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M).\end{aligned}$$

This follows from Proposition 5.3.4 below.

Remark 5.3.3. The categories $\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)$ are the analogue of Tamarkin's construction in the framework of indsheaves.

Proposition 5.3.4. *One has*

$$\begin{aligned}
{}^{\perp}\mathrm{IC}_{\pm\mathrm{t}^*\leq 0} &= \{K;\mathbf{k}_{\{\pm t\geq 0\}}\overset{+}{\otimes}K\xrightarrow{\sim}K\} = \{K;\mathbf{k}_{\{\pm t>0\}}\overset{+}{\otimes}K\simeq 0\},\\
\mathrm{IC}^{\perp}_{\pm\mathrm{t}^*\leq 0} &= \{K;K\xrightarrow{\sim}\mathscr{I}hom^{+}(\mathbf{k}_{\{\pm t\geq 0\}},K)\} = \{K;\mathscr{I}hom^{+}(\mathbf{k}_{\{\pm t>0\}},K)\simeq 0\},\\
{}^{\perp}\mathrm{IC}_{\mathrm{t}^*=0} &= \{K;(\mathbf{k}_{\{t\geq 0\}}\oplus\mathbf{k}_{\{t\leq 0\}})\overset{+}{\otimes}K\xrightarrow{\sim}K\}\\
&= \{K;\mathbf{k}_{M\times\mathbb{R}}\overset{+}{\otimes}K\simeq 0\} = \{K;\mathrm{R}\pi_{!!}K\simeq 0\},\\
\mathrm{IC}^{\perp}_{\mathrm{t}^*=0} &= \{K;\mathscr{I}hom^{+}(\mathbf{k}_{\{t\geq 0\}}\oplus\mathbf{k}_{\{t\leq 0\}},K)\xrightarrow{\sim}K\}\\
&= \{K;\mathscr{I}hom^{+}(\mathbf{k}_{M\times\mathbb{R}},K)\simeq 0\} = \{K;\mathrm{R}\pi_*K\simeq 0\},\\
{}^{\perp}\mathrm{IC}_{\mathrm{t}^*=0} &= {}^{\perp}\mathrm{IC}_{\mathrm{t}^*\geq 0}\oplus{}^{\perp}\mathrm{IC}_{\mathrm{t}^*\leq 0},\\
\mathrm{IC}^{\perp}_{\mathrm{t}^*=0} &= \mathrm{IC}^{\perp}_{\mathrm{t}^*\geq 0}\oplus\mathrm{IC}^{\perp}_{\mathrm{t}^*\leq 0}.
\end{aligned}$$

Moreover, one has the equivalences

$$\begin{aligned}
\mathrm{E}^{\mathrm{b}}_{\pm}(\mathrm{I}\mathbf{k}_M) &\xrightarrow{\sim} {}^{\perp}\mathrm{IC}_{\pm \mathrm{t}^* \le 0}, \quad \mathrm{K} \mapsto \mathbf{k}_{\{\pm \mathrm{t} \ge 0\}} \overset{+}{\otimes} \mathrm{K},\\
\mathrm{E}^{\mathrm{b}}_{\pm}(\mathrm{I}\mathbf{k}_M) &\xrightarrow{\sim} \mathrm{IC}^{\perp}_{\pm \mathrm{t}^* \le 0}, \quad \mathrm{K} \mapsto \mathscr{I}\mathrm{hom}^{+}(\mathbf{k}_{\{\pm \mathrm{t} \ge 0\}}, \mathrm{K}),\\
\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\xrightarrow{\sim} {}^{\perp}\mathrm{IC}_{\mathrm{t}^* = 0}, \quad \mathrm{K} \mapsto (\mathbf{k}_{\{\mathrm{t} \ge 0\}} \oplus \mathbf{k}_{\{\mathrm{t} \le 0\}}) \overset{+}{\otimes} \mathrm{K},\\
\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\xrightarrow{\sim} \mathrm{IC}^{\perp}_{\mathrm{t}^* = 0}, \quad \mathrm{K} \mapsto \mathscr{I}\mathrm{hom}^{+}(\mathbf{k}_{\{\mathrm{t} \ge 0\}} \oplus \mathbf{k}_{\{\mathrm{t} \le 0\}}, \mathrm{K}),
\end{aligned}$$

where the quasi-inverse functors are given by the quotient functors.

Also notice that

$$\begin{aligned}
\mathrm{IC}_{\mathrm{t}^*=0} &= \left\{K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty})\,;\, \pi^{-1}\mathrm{R}\pi_* K \xrightarrow{\sim} K\right\}\\
&= \left\{K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty})\,;\, K \xrightarrow{\sim} \pi^{!}\,\mathrm{R}\pi_{!!} K\right\}\\
&= \left\{K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty})\,;\, K \simeq \pi^{-1}L \text{ for some } L \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)\right\}.
\end{aligned}$$

Therefore,

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \simeq \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty}) / \{K \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty}); \pi^{-1}\mathrm{R}\pi_* K \xrightarrow{\sim} K\}. \tag{5.3.2}$$

These categories are illustrated as follows:

$$\begin{tikzcd}
 & & \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M \times \mathbb{R}_\infty}) & & \\
 & \mathrm{IC}_{\mathrm{t}^* \ge 0} \arrow[ur, "\mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M)"] & & \mathrm{IC}_{\mathrm{t}^* \le 0} \arrow[ul, "\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)"'] & \\
{}^{\perp}\mathrm{IC}_{\mathrm{t}^* \le 0} \arrow[ur, "\mathrm{IC}_{\mathrm{t}^*=0}"] & \mathrm{IC}^{\perp}_{\mathrm{t}^* \le 0} \arrow[u, "\mathrm{IC}_{\mathrm{t}^*=0}"] & \mathrm{IC}_{\mathrm{t}^*=0} \arrow[ul, "\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)"'] \arrow[ur, "\mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M)"] & {}^{\perp}\mathrm{IC}_{\mathrm{t}^* \ge 0} \arrow[u, "\mathrm{IC}_{\mathrm{t}^*=0}"] & \mathrm{IC}^{\perp}_{\mathrm{t}^* \ge 0} \arrow[ul, "\mathrm{IC}_{\mathrm{t}^*=0}"'] \\
 & & 0 \arrow[ull, "\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)"] \arrow[ul, "\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M)"'] \arrow[u] \arrow[ur, "\mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M)"] \arrow[urr, "\mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M)"'] & &
\end{tikzcd}$$

Here, $A \overset{C}{\longrightarrow} B$ or $A \xrightarrow[C]{} B$ means that $C \simeq B/A$.

Definition 5.3.5. One introduces the functors

$$\begin{aligned}
\mathrm{L}^{\mathrm{E}} &= (\mathbf{k}_{\{t \ge 0\}} \oplus \mathbf{k}_{\{t \le 0\}}) \overset{+}{\otimes} (\,\bullet\,), \quad \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to {}^{\perp}\mathrm{IC}_{\mathrm{t}^*=0} \subset \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M} \times \mathbb{R}_\infty}),\\
\mathrm{R}^{\mathrm{E}} &= \mathscr{I}hom^{+}(\mathbf{k}_{\{t \ge 0\}} \oplus \mathbf{k}_{\{t \le 0\}}, \bullet), \quad \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathrm{IC}^{\perp}_{\mathrm{t}^*=0} \subset \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{\mathrm{M} \times \mathbb{R}_\infty}).
\end{aligned}$$

The functors L^{E} and R^{E} are the left and right adjoint of the quotient functor $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$, and the two compositions

$$\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \overset{\mathrm{L}^{\mathrm{E}}}{\underset{\mathrm{R}^{\mathrm{E}}}{\rightrightarrows}} \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) \longrightarrow \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$$

are isomorphic to the identity.

Definition 5.3.6. One defines the hom-functor

$$\begin{aligned} &\mathscr{I}hom^{\mathrm{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{+}(\mathrm{I}\mathbf{k}_M) \\ &\qquad \mathscr{I}hom^{\mathrm{E}}(K_1, K_2) = \mathrm{R}\pi_*\mathrm{R}\mathscr{I}hom\,(\mathrm{L}^{\mathrm{E}}(K_1), \mathrm{R}^{\mathrm{E}}(K_2)), \end{aligned} \tag{5.3.3}$$

and one sets

$$\mathscr{H}om^{\mathrm{E}} = \alpha_M \circ \mathscr{I}hom^{\mathrm{E}} \;:\; \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathsf{D}^{+}(\mathbf{k}_M), \tag{5.3.4}$$

$$\mathrm{RHom}^{\mathrm{E}}(K_1, K_2) = \mathrm{R}\Gamma(M; \mathscr{H}om^{\mathrm{E}}(K_1, K_2)). \tag{5.3.5}$$

Note that

$$\begin{aligned} \mathscr{I}hom^{\mathrm{E}}(K_1, K_2) &\simeq \mathrm{R}\pi_*\mathrm{R}\mathscr{I}hom\,(\mathrm{L}^{\mathrm{E}}(K_1), \mathrm{L}^{\mathrm{E}}(K_2)) \\ &\simeq \mathrm{R}\pi_*\mathrm{R}\mathscr{I}hom\,(\mathrm{R}^{\mathrm{E}}(K_1), \mathrm{R}^{\mathrm{E}}(K_2)) \end{aligned}$$

and

$$\mathrm{Hom}_{\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M_\infty})}(K_1, K_2) \simeq H^0\big(\mathrm{RHom}^{\mathrm{E}}(K_1, K_2)\big).$$

5.4 Operations on enhanced indsheaves

By Lemma 5.2.10 the following definition is well posed.

Definition 5.4.1. The bifunctors

$$\begin{aligned} \overset{+}{\otimes} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M), \\ \mathscr{I}hom^{+} \colon \mathrm{E}^{-}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M) &\to \mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M) \end{aligned}$$

are those induced by the bifunctors $\overset{+}{\otimes}$ and $\mathscr{I}hom^{+}$ defined on $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$.

For any $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ there is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$

$$\mathbf{k}_{t\geq 0} \overset{+}{\otimes} K \xrightarrow{\sim} \mathscr{I}hom^{+}(\mathbf{k}_{t\geq 0}, K),$$

which follows from Proposition 5.2.11.

The bifunctor $\overset{+}{\otimes}$ gives $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ a structure of a commutative tensor category with $\mathbf{k}_{\{t=0\}}$ as a unit object. Moreover, $\mathscr{I}hom^{+}$ is the inner hom of the tensor category $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$:

Lemma 5.4.2. *For $K_1, K_2, K_3 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ there is an isomorphism*

$$\mathrm{Hom}_{\mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M)}(K_1 \overset{+}{\otimes} K_2, K_3) \simeq \mathrm{Hom}_{\mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M)}(K_1, \mathscr{I}hom^{+}(K_2, K_3)).$$

We have the following orthogonal relations:

$$\begin{aligned} &\mathrm{E}^{\mathrm{b}}_{+}(\mathrm{I}\mathbf{k}_M) \overset{+}{\otimes} \mathrm{E}^{\mathrm{b}}_{-}(\mathrm{I}\mathbf{k}_M) \simeq 0, \\ &\mathscr{I}hom^{+}(\mathrm{E}^{\mathrm{b}}_{\pm}(\mathrm{I}\mathbf{k}_M), \mathrm{E}^{\mathrm{b}}_{\mp}(\mathrm{I}\mathbf{k}_M)) \simeq 0. \end{aligned}$$

Definition 5.4.3. By Lemma 5.2.10 one gets functors

$$\begin{aligned} \pi^{-1}(\,\bullet\,) \otimes (\,\bullet\,) \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M), \\ \mathrm{R}\mathscr{I}hom\,(\pi^{-1}(\,\bullet\,), \bullet\,) \colon \mathsf{D}^{-}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M) &\to \mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M). \end{aligned}$$

Remark 5.4.4. The functor $\otimes$ *does not* factor through $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$, and the functor $\mathrm{R}\mathscr{I}hom$ *does not* factor through $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$.

Let $f \colon M \to N$ be a continuous map of good topological spaces. Denote by $\tilde{f} \colon M \times \mathbb{R}_\infty \to N \times \mathbb{R}_\infty$ the associated morphism. Then the composition of functors

$$\mathrm{R}\tilde{f}_{!!}, \mathrm{R}\tilde{f}_{*} \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{N\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N), \tag{5.4.1}$$

$$\tilde{f}^{-1}, \tilde{f}^{\,!} \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{N\times\mathbb{R}_\infty}) \to \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M), \tag{5.4.2}$$

factor through $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$, respectively.

Definition 5.4.5. One denotes by

$$\begin{aligned} \mathsf{E}f_{!!}, \mathsf{E}f_{*} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) &\to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N), \\ \mathsf{E}f^{-1}, \mathsf{E}f^{\,!} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N) &\to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M), \end{aligned}$$

the functors induced by (5.4.1) and (5.4.2), respectively.

Definition 5.4.6. For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$, we define their external tensor product by

$$K \overset{+}{\boxtimes} L = \mathsf{E}p_1^{-1}K \overset{+}{\otimes} \mathsf{E}p_2^{-1}L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times N}),$$

where p_1 and p_2 denote the projections from $M \times N$ to M and N, respectively.

Using Definition 5.3.5, for $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$ one has

$$\begin{aligned}
\mathsf{E}f_{!!}F &\simeq \mathrm{R}\tilde{f}_{!!}\mathrm{L}^{\mathrm{E}}(F) \simeq \mathrm{R}\tilde{f}_{!!}\mathrm{R}^{\mathrm{E}}(F),\\
\mathsf{E}f_{*}F &\simeq \mathrm{R}\tilde{f}_{*}\mathrm{L}^{\mathrm{E}}(F) \simeq \mathrm{R}\tilde{f}_{*}\mathrm{R}^{\mathrm{E}}(F),\\
\mathsf{E}f^{-1}G &\simeq \tilde{f}^{-1}\mathrm{L}^{\mathrm{E}}(G) \simeq \tilde{f}^{-1}\mathrm{R}^{\mathrm{E}}(G),\\
\mathsf{E}f^{!}G &\simeq \tilde{f}^{!}\,\mathrm{L}^{\mathrm{E}}(G) \simeq \tilde{f}^{!}\,\mathrm{R}^{\mathrm{E}}(G).
\end{aligned}$$

The above operations satisfy analogous properties as the external operations for indsheaves.

Proposition 5.4.7. *Let $f\colon M \to N$ be a continuous map of good topological spaces.*

(i) *The functor $\mathsf{E}f_{!!}$ is left adjoint to $\mathsf{E}f^{!}$.*

(ii) *The functor $\mathsf{E}f^{-1}$ is left adjoint to $\mathsf{E}f_{*}$.*

Proposition 5.4.8. *Given two continuous maps of good topological spaces $L \xrightarrow{g} M \xrightarrow{f} N$, one has*

$$\mathsf{E}(f\circ g)_{!!} \simeq \mathsf{E}f_{!!}\circ \mathsf{E}g_{!!}, \qquad \mathsf{E}(f\circ g)_{*} \simeq \mathsf{E}f_{*}\circ \mathsf{E}g_{*}$$

and

$$\mathsf{E}(f\circ g)^{-1} \simeq \mathsf{E}g^{-1}\circ \mathsf{E}f^{-1}, \qquad \mathsf{E}(f\circ g)^{!} \simeq \mathsf{E}g^{!}\circ \mathsf{E}f^{!}.$$

Proposition 5.4.9. *Let $f\colon M \to N$ be a continuous map of good topological spaces. For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $L, L_1, L_2 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$, one has isomorphisms*

$$\begin{aligned}
\mathsf{E}f_{!!}(\mathsf{E}f^{-1}L \overset{+}{\otimes} K) &\simeq L \overset{+}{\otimes} \mathsf{E}f_{!!}K,\\
\mathsf{E}f^{-1}(L_1 \overset{+}{\otimes} L_2) &\simeq \mathsf{E}f^{-1}L_1 \overset{+}{\otimes} \mathsf{E}f^{-1}L_2,\\
\mathscr{I}hom^{+}(L, \mathsf{E}f_{*}K) &\simeq \mathsf{E}f_{*}\mathscr{I}hom^{+}(\mathsf{E}f^{-1}L, K),\\
\mathscr{I}hom^{+}(\mathsf{E}f_{!!}K, L) &\simeq \mathsf{E}f_{*}\mathscr{I}hom^{+}(K, \mathsf{E}f^{!}L),\\
\mathsf{E}f^{!}\mathscr{I}hom^{+}(L_1, L_2) &\simeq \mathscr{I}hom^{+}(\mathsf{E}f^{-1}L_1, \mathsf{E}f^{!}L_2),
\end{aligned}$$

and a morphism

$$\mathsf{E} f^{-1}\mathscr{I}hom^{+}(L_1, L_2) \to \mathscr{I}hom^{+}(\mathsf{E} f^{-1}L_1, \mathsf{E} f^{-1}L_2).$$

Proposition 5.4.10. *Consider a Cartesian diagram of good topological spaces*

$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\square"] & N' \arrow[d, "g"] \\ M \arrow[r, "f"] & N. \end{tikzcd}$$

Then there are isomorphisms in the category of functors from $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ *to* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{N'})$*:*

$$\mathsf{E} g^{-1}\mathsf{E} f_{!!} \simeq \mathsf{E} f'_{!!}\mathsf{E} g'^{-1}, \qquad \mathsf{E} g^{!}\mathsf{E} f_{*} \simeq \mathsf{E} f'_{*}\mathsf{E} g'^{!}.$$

Lemma 5.4.11. *For* $f\colon M \to N$ *a morphism of good topological spaces,* $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ *and* $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$*, one has*

$$\mathrm{R}f_{*}\mathscr{H}om^{\mathrm{E}}(K, \mathsf{E} f^{!}L) \simeq \mathscr{H}om^{\mathrm{E}}(\mathsf{E} f_{!!}K, L),$$
$$\mathrm{R}f_{*}\mathscr{H}om^{\mathrm{E}}(\mathsf{E} f^{-1}L, K) \simeq \mathscr{H}om^{\mathrm{E}}(L, \mathsf{E} f_{*}K).$$

Remark 5.4.12. Let $f\colon M \to N$ be a morphism of good topological spaces and $L_1, L_2 \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$. Since α and $f^{!}$ do not commute in general, the isomorphism $f^{!}\mathscr{H}om^{\mathrm{E}}(L_1, L_2) \simeq \mathscr{H}om^{\mathrm{E}}(\mathsf{E} f^{-1}L_1, \mathsf{E} f^{!}L_2)$ does not hold in general.

5.5 Stable objects

The notion of stable object which will be introduced below is related to the notion of torsion object from [Ta08] (see also [GS12, §5]).

Notation 5.5.1. Consider the indsheaves on $M \times \overline{\mathbb{R}}$

$$\mathbf{k}_{\{t \gg 0\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\, \mathbf{k}_{\{t \geq a\}}, \qquad \mathbf{k}_{\{t < *\}} := \text{“}\varinjlim_{a\to+\infty}\text{”}\, \mathbf{k}_{\{t < a\}}.$$

We regard them as objects of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$.

There is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$

$$\mathbf{k}_{M\times\mathbb{R}} \to \mathbf{k}_{\{t\gg 0\}} \to \mathbf{k}_{\{t<*\}}[1] \xrightarrow{+1}$$

and there are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$

$$\mathbf{k}_{\{t\geq -a\}} \overset{+}{\otimes} \mathbf{k}_{\{t\gg 0\}} \xrightarrow{\sim} \mathbf{k}_{\{t\gg 0\}} \xrightarrow{\sim} \mathbf{k}_{\{t\geq a\}} \overset{+}{\otimes} \mathbf{k}_{\{t\gg 0\}} \qquad (a \in \mathbb{R}_{\geq 0}).$$

Notation 5.5.2. Denote by $\mathbf{k}_M^{\mathrm{E}}$ the object of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ associated with $\mathbf{k}_{\{t\gg 0\}} \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$. More generally, for $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, set

$$F^{\mathrm{E}} := \mathbf{k}_M^{\mathrm{E}} \otimes \pi^{-1}F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M).$$

Note that

$$\mathrm{L}^{\mathrm{E}}(\mathbf{k}_M^{\mathrm{E}}) \simeq \mathbf{k}_{\{t\gg 0\}}, \quad \mathrm{R}^{\mathrm{E}}(\mathbf{k}_M^{\mathrm{E}}) \simeq \mathbf{k}_{\{t<*\}}[1].$$

Proposition 5.5.3. *Let $K \in \mathrm{E}_+^{\mathrm{b}}(M)$ (equivalently, $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $K \simeq \mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K$). Then the following conditions are equivalent.*

(a) $\mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \xrightarrow{\sim} \mathbf{k}_{\{t\geq a\}} \overset{+}{\otimes} K$ *for any* $a \geq 0$,

(b) $\mathscr{I}hom^+(\mathbf{k}_{\{t\geq a\}}, K) \xrightarrow{\sim} \mathscr{I}hom^+(\mathbf{k}_{\{t\geq 0\}}, K)$ *for any* $a \geq 0$,

(c) $\mathbf{k}_{\{t\geq 0\}} \overset{+}{\otimes} K \xrightarrow{\sim} \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} K$,

(d) $\mathscr{I}hom^+(\mathbf{k}_M^{\mathrm{E}}, K) \xrightarrow{\sim} \mathscr{I}hom^+(\mathbf{k}_{\{t\geq 0\}}, K)$,

(e) $K \simeq \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} L$ *for some* $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$,

(f) $K \simeq \mathscr{I}hom^+(\mathbf{k}_M^{\mathrm{E}}, L)$ *for some* $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$.

Definition 5.5.4. A *stable* object is an object of $\mathrm{E}_+^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ that satisfies the equivalent conditions in Proposition 5.5.3.

Lemma 5.5.5. *For $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ and $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$, there is an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$*

$$\mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} \mathscr{I}hom^+(F, K) \xrightarrow{\sim} \mathscr{I}hom^+(F, \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} K).$$

Corollary 5.5.6. *For $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ and $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, we have*

$$\mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} \mathrm{R}\mathscr{I}hom\,(\pi^{-1}F, K) \simeq \mathrm{R}\mathscr{I}hom\,(\pi^{-1}F, \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} K).$$

Proposition 5.5.7. *Let $f\colon M \to N$ be a continuous map of good topological spaces. Then, the functors $\mathsf{E}f_{!!}$, $\mathsf{E}f^{-1}$ and $\mathsf{E}f^!$ send stable objects to stable objects. More precisely, we have:*

(i) *For* $K \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ *one has*

$$\mathsf{E}f_{!!}(\mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} K) \simeq \mathbf{k}_N^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f_{!!}K.$$

(ii) *For* $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_N)$ *one has*

$$\mathsf{E}f^{-1}(\mathbf{k}_N^{\mathrm{E}} \overset{+}{\otimes} L) \simeq \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f^{-1}L,$$
$$\mathsf{E}f^{!}(\mathbf{k}_N^{\mathrm{E}} \overset{+}{\otimes} L) \simeq \mathbf{k}_M^{\mathrm{E}} \overset{+}{\otimes} \mathsf{E}f^{!}L.$$

Definition 5.5.8. One defines the functors

$$(5.5.1)\qquad \begin{aligned} &e_M, \epsilon_M \colon \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M_\infty}) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M_\infty}),\\ &\qquad e_M(F) = \mathbf{k}_{M_\infty}^{\mathrm{E}} \otimes \pi^{-1}F, \quad \epsilon_M(F) = \mathbf{k}_{\{t\geq 0\}} \otimes \pi^{-1}F. \end{aligned}$$

Note that

$$e_M(F) \simeq \mathbf{k}_{M_\infty}^{\mathrm{E}} \overset{+}{\otimes} \epsilon_M(F).$$

Proposition 5.5.9. *The functors* e_M *and* ϵ_M *are fully faithful.*

Definition 5.5.10. We define the duality functor

$$\mathrm{D}_M^{\mathrm{E}} \colon \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)^{\mathrm{op}}, \qquad K \mapsto \mathscr{I}hom^{+}(K, \omega_M^{\mathsf{E}}),$$

where $\omega_M^{\mathsf{E}} := \mathbf{k}_M^{\mathsf{E}} \otimes \pi^{-1}\omega_M$.

The functor $\mathrm{D}_M^{\mathrm{E}}$ is related to the usual duality functor for sheaves by the formula:

$$(5.5.2)\qquad \mathrm{D}_M^{\mathrm{E}}(\mathbf{k}_M^{\mathsf{E}} \overset{+}{\otimes} F) \simeq \mathbf{k}_M^{\mathsf{E}} \overset{+}{\otimes} a^{-1}\mathrm{D}_{M\times\mathbb{R}}F \quad \text{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M),$$

where $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ and a is the involution of $M \times \mathbb{R}$ given by $(x,t) \mapsto (x,-t)$.

5.6 Constructible enhanced indsheaves

In this subsection, we assume that M is a subanalytic space. Recall the natural morphism

$$j_M \colon M \times \mathbb{R}_\infty \to M \times \overline{\mathbb{R}},$$

and the category $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$.

Definition 5.6.1. We denote by $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ whose objects F are such that $\mathrm{R}j_{M!}F$ is $\mathbb{R}$-constructible.

We regard $\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{M\times\mathbb{R}_\infty})$ as a full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_{M\times\mathbb{R}_\infty})$.

Definition 5.6.2. One says that an object $K\in\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ is $\mathbb{R}$-*constructible* if for any relatively compact subanalytic open subset $U\subset M$ there exists an isomorphism

$$\pi^{-1}\mathbf{k}_U\otimes K\simeq\mathbf{k}_M^{\mathsf{E}}\overset{+}{\otimes}F\text{ for some }F\in\mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_{M\times\mathbb{R}_\infty}).$$

One denotes by $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$ the full subcategory of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ consisting of $\mathbb{R}$-constructible objects.

Clearly, $\mathbb{R}$-constructible objects of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ are stable. One proves that:

Theorem 5.6.3. (i) *The category $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$ is triangulated.*

(ii) *The property for $K\in\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbf{k}_M)$ of being $\mathbb{R}$-constructible is a local property over M.*

(iii) *The functors $\overset{+}{\otimes}$ and $\mathscr{I}hom^+$ when restricted to $\mathbb{R}$-constructible objects give $\mathbb{R}$-constructible objects.*

(iv) *For $K\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$, $\mathrm{D}^{\mathrm{E}}_MK\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$ and $\mathrm{D}^{\mathrm{E}}_M\circ\mathrm{D}^{\mathrm{E}}_MK\simeq K$.*

(v) *For $K_1,K_2\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$, $\mathrm{D}^{\mathrm{E}}_M\mathscr{I}hom^+(K_1,K_2)\simeq K_1\overset{+}{\otimes}\mathrm{D}^{\mathrm{E}}_MK_2$.*

(vi) *Let $f\colon M\to N$ be a morphism of subanalytic spaces.*

(a) *If $G\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_N)$, then $\mathsf{E}f^{-1}G$ and $\mathsf{E}f^{!}G$ belong to $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$,*

(b) *For $F\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_M)$, $\mathsf{E}f_{!!}F\in\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbf{k}_N)$ as soon as $\mathrm{supp}^{\mathrm{E}}(F):=\pi_M\big(\mathrm{supp}(\mathrm{R}^{\mathrm{E}}F)\big)$ is proper over N.*

5.7 Enhanced indsheaves with ring action

Let $\mathcal{A}$ be a sheaf of $\mathbf{k}$-algebras on M. For $\dagger=\ ,\mathrm{b},+,-$, we define

$$\mathsf{D}^{\dagger}(\mathrm{I}(\pi^{-1}\mathcal{A})):=\mathsf{D}^{\dagger}(\mathrm{I}(\overline{\pi}^{-1}\mathcal{A}))/\mathsf{D}^{\dagger}(\mathrm{I}((\overline{\pi}^{-1}\mathcal{A})|_{M\times(\overline{\mathbb{R}}\setminus\mathbb{R})})),$$

where $\overline{\pi}\colon M\times\overline{\mathbb{R}}\to M$ is the projection. Then we set

$$\mathrm{E}^{\dagger}(\mathrm{I}\mathcal{A})=\mathsf{D}^{\dagger}(\mathrm{I}(\pi^{-1}\mathcal{A}))/\left\{K\in\mathsf{D}^{\dagger}(\mathrm{I}(\pi^{-1}\mathcal{A}))\,;\,K\simeq\pi^{-1}L\text{ for some }L\in\mathsf{D}^{\dagger}(\mathrm{I}\mathcal{A})\right\}.$$

We call objects of $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A})$ enhanced indsheaves with $\mathcal{A}$-action.

We can define also the functors

$$\overset{+}{\otimes}_{\beta\mathcal{A}} : \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A}^{\mathrm{op}}) \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A}) \to \mathrm{E}^{-}(\mathrm{I}\mathbf{k}_M),$$
$$\mathscr{I}hom^{+}_{\beta\mathcal{A}} : \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A}) \to \mathrm{E}^{+}(\mathrm{I}\mathbf{k}_M),$$

which satisfy similar properties to $\overset{+}{\otimes}$ and $\mathscr{I}hom^{+}$.

Similarly we can define

$$\overset{\mathrm{L}}{\otimes}_{\mathcal{A}} : \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A}^{\mathrm{op}}) \times \mathsf{D}^{\mathrm{b}}(\mathcal{A}) \to \mathrm{E}^{-}(\mathrm{I}\mathbf{k}_M),$$
$$\mathrm{R}\mathscr{H}om_{\mathcal{A}} : \mathsf{D}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}} \times \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathcal{A}) \to \mathrm{E}^{-}(\mathrm{I}\mathbf{k}_M).$$

If X is a complex manifold and $\mathcal{A} = \mathscr{D}_X$, we can define

$$\overset{\mathrm{D}}{\otimes} : \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X) \times \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{E}^{-}(\mathrm{I}\mathscr{D}_X).$$

6 Holonomic D-modules

6.1 Exponential D-modules

Let X be a complex analytic manifold, $Y \subset X$ a complex analytic hypersurface and set $U = X \setminus Y$. For $\varphi \in \mathcal{O}_X(*Y)$, one sets

$$\mathscr{D}_X \mathrm{e}^{\varphi} = \mathscr{D}_X / \{P\,;\, P\mathrm{e}^{\varphi} = 0 \text{ on } U\},$$
$$\mathscr{E}^{\varphi}_{U|X} = \mathscr{D}_X \mathrm{e}^{\varphi}(*Y).$$

Hence $\mathscr{D}_X \mathrm{e}^{\varphi}$ is a $\mathscr{D}_X$-submodule of $\mathscr{E}^{\varphi}_{U|X}$, and $\mathscr{D}_X \mathrm{e}^{\varphi}$ as well as $\mathscr{E}^{\varphi}_{U|X}$ are holonomic $\mathscr{D}_X$-modules. Note that $\mathscr{E}^{\varphi}_{U|X}$ is isomorphic to $\mathcal{O}_X(*Y)$ as an $\mathcal{O}_X$-module, and the connection $\mathcal{O}_X(*Y) \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*Y)$ is given by $u \mapsto du + ud\varphi$.

For $c \in \mathbb{R}$, set for short

$$\{\mathrm{Re}\,\varphi < c\} := \{x \in U; \mathrm{Re}\,\varphi(x) < c\} \subset X.$$

Notation 6.1.1. One sets

$$\mathbb{C}_{\{\mathrm{Re}\,\varphi < *\}} := \text{“}\varinjlim_{c \to +\infty}\text{”}\, \mathbb{C}_{\{\mathrm{Re}\,\varphi < c\}} \in \mathrm{I}\mathbb{C}_{\mathrm{X}},$$
$$E^{\varphi}_{U|X} := \mathrm{R}\mathscr{I}hom\,(\mathbb{C}_U, \mathbb{C}_{\{\mathrm{Re}\,\varphi < *\}}) \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}).$$

For example, denoting by $z \in \mathbb{C} \subset \mathsf{P}$ the affine coordinate of the complex projective line, one has

$$
H^j E^z_{\mathbb{C}|\mathsf{P}} \simeq \begin{cases} \mathbb{C}_{\{\operatorname{Re} z<*\}} & \text{for } j=0, \\ \mathbb{C}_{\{\infty\}} & \text{for } j=1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1.1}
$$

The next result (see [DK13, Prop. 6.2.2]) generalizes [KS03, Proposition 7.3] in which the case $X=\mathbb{C}$ and $\varphi(z)=1/z$ was treated (see §3.5).

Proposition 6.1.2. *Let $Y \subset X$ be a closed complex analytic hypersurface, and set $U = X \setminus Y$. For $\varphi \in \mathcal{O}_X(*Y)$, there is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$*

$$
\mathcal{DR}^{\mathrm{t}}_X(\mathscr{E}^{-\varphi}_{U|X}) \simeq E^{\varphi}_{U|X}[d_X].
$$

6.2 Enhanced tempered holomorphic functions

Consider first a real analytic manifold M and the natural morphism of bordered spaces

$$
j\colon M \times \mathbb{R}_\infty \to M \times \mathbb{P}^1(\mathbb{R}).
$$

Definition 6.2.1. One sets $\mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{R}_\infty} := j^{\,!}\,\mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{P}^1(\mathbb{R})}$ and one denotes by $\mathcal{D}b^{\mathsf{T}}_M \in \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}\times\mathbb{R}_\infty})$ the complex, concentrated in degree -1 and 0:

$$
\mathcal{D}b^{\mathsf{T}}_M := \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{R}_\infty} \xrightarrow{\partial_t - 1} \mathcal{D}b^{\mathrm{t}}_{M\times\mathbb{R}_\infty}.
$$

Note that $H^k(\mathcal{D}b^{\mathsf{T}}_M) = 0$ for $k \neq -1$.

Proposition 6.2.2. *There are isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{M}\times\mathbb{R}_\infty})$*

$$
\begin{aligned}
\mathcal{D}b^{\mathsf{T}}_M &\xrightarrow{\sim} \mathscr{I}hom^+(\mathbb{C}_{\{t\geq 0\}}, \mathcal{D}b^{\mathsf{T}}_M) \\
&\xleftarrow{\sim} \mathscr{I}hom^+(\mathbb{C}_{\{t\geq a\}}, \mathcal{D}b^{\mathsf{T}}_M) \quad \textit{for any } a \geq 0.
\end{aligned}
$$

Moreover, denoting by $\iota\colon M\times\mathbb{R} \to M\times\mathbb{R}_\infty$ the natural morphism, one has the isomorphism $\iota^{-1}\mathcal{D}b^{\mathsf{T}}_M \simeq \iota^{-1}\pi^{-1}\mathcal{D}b^{\mathrm{t}}_M\,[1]$ and therefore:

$$
\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t=0\}}, \mathcal{D}b^{\mathsf{T}}_M) \simeq \mathcal{D}b^{\mathrm{t}}_M. \tag{6.2.1}
$$

Now let X be again a complex manifold.

Definition 6.2.3. One sets

$$\begin{aligned}\mathcal{O}_X^{\mathsf{E}} &= \mathrm{R}\mathscr{H}om_{\pi^{-1}\mathscr{D}_{X^c}}(\pi^{-1}\mathcal{O}_{X^c}, \mathcal{D}b^{\mathsf{T}}_{X_{\mathbb{R}}}) \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X),\\ \Omega_X^{\mathsf{E}} &= \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\mathsf{E}}.\end{aligned}$$

One calls $\mathcal{O}_X^{\mathsf{E}}$ the *enhanced indsheaf of tempered holomorphic functions.*

Remark 6.2.4. When $X = \mathrm{pt}$, then $\mathcal{O}_X^{\mathsf{E}} \simeq \mathbb{C}_X^{\mathsf{E}}$.

Applying Proposition 6.2.2, we get

Proposition 6.2.5. *There are isomorphisms in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$

$$\begin{aligned}\mathcal{O}_X^{\mathsf{E}} &\xrightarrow{\sim} \mathscr{I}hom^{+}(\mathbb{C}_{\{t\geq 0\}}, \mathcal{O}_X^{\mathsf{E}})\\ &\xleftarrow{\sim} \mathscr{I}hom^{+}(\mathbb{C}_{\{t\geq a\}}, \mathcal{O}_X^{\mathsf{E}}) \quad \text{for any } a \geq 0.\end{aligned}$$

In particular, $\mathcal{O}_X^{\mathsf{E}}$ *is a stable object in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$.

As a consequence of Proposition 6.2.5 and Proposition 5.5.3, we get the following result.

Corollary 6.2.6. *There are isomorphisms in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$

$$\mathcal{O}_X^{\mathsf{E}} \simeq \mathscr{I}hom^{+}(\mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}}) \simeq \mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} \mathcal{O}_X^{\mathsf{E}}.$$

Then, using the isomorphisms

$$\begin{aligned}\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}}) &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_X^{\mathsf{E}}, \mathscr{I}hom^{+}(\mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}}))\\ &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} \mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}})\\ &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t=0\}} \overset{+}{\otimes} \mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}})\\ &\simeq \mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_{\{t=0\}}, \mathcal{O}_X^{\mathsf{E}})\end{aligned}$$

and (6.2.1), one gets the isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X)$:

$$\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}_X^{\mathsf{E}}, \mathcal{O}_X^{\mathsf{E}}) \simeq \mathcal{O}_X^{\mathsf{t}}. \tag{6.2.2}$$

6.3 Enhanced De Rham and Sol functors

For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$, set

$$\begin{aligned} \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}) &= \Omega^{\mathsf{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M}, \\ \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}) &= \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{O}^{\mathsf{E}}_X). \end{aligned}$$

We get functors

$$\begin{aligned} \mathcal{DR}^{\mathsf{E}}_X &\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X) \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X), \\ \mathcal{S}ol^{\mathsf{E}}_X &\colon \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)^{\mathrm{op}} \to \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X). \end{aligned}$$

Note that

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}) \simeq \mathcal{DR}^{\mathsf{E}}_X(\mathbb{D}_X\mathscr{M})[-d_X] \quad \text{for } \mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X).$$

By using Proposition 6.1.2, one can calculate explicitly $\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})$ when $\mathscr{M}$ is an exponential D-module.

Proposition 6.3.1. *Let $Y \subset X$ be a closed complex analytic hypersurface, and set $U = X \setminus Y$. For $\varphi \in \mathcal{O}_X(*Y)$, there are isomorphisms*

$$\begin{aligned} \mathcal{DR}^{\mathsf{E}}_X(\mathscr{E}^{\varphi}_{U|X}) &\simeq \mathrm{R}\mathscr{I}hom\,(\pi^{-1}\mathbb{C}_U, \text{“}\varinjlim_{c\to\infty}\text{”}\, \mathbb{C}_{\{t \geq \operatorname{Re}\varphi + c\}}) \\ &\simeq \mathbb{C}^{\mathsf{E}}_X \overset{+}{\otimes} \mathrm{R}\mathscr{I}hom\,(\pi^{-1}\mathbb{C}_U, \mathbb{C}_{\{t = \operatorname{Re}\varphi\}}). \end{aligned}$$

The next results are easy consequences of Theorem 2.2.1, Corollary 2.2.2, Corollary 2.2.6 and Corollary 2.2.7.

Theorem 6.3.2. *Let $f \colon X \to Y$ be a morphism of complex manifolds.*

(i) *There is an isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}(f^{-1}\mathscr{D}_Y))$

$$\mathsf{E} f^{\,!}\mathcal{O}^{\mathsf{E}}_Y[d_Y] \simeq \mathscr{D}_{Y\leftarrow X} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathsf{E}}_X[d_X].$$

(ii) *For any $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_Y)$ there is an isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X)$

$$\mathcal{DR}^{\mathsf{E}}_X(\mathrm{D}f^*\mathscr{N})[d_X] \simeq \mathsf{E} f^{\,!}\mathcal{DR}^{\mathsf{E}}_Y(\mathscr{N})[d_Y]. \tag{6.3.1}$$

(iii) *Let $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_X)$, and assume that $\operatorname{supp}\mathscr{M}$ is proper over Y. Then there are isomorphisms in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_Y)$

$$\begin{aligned} \mathcal{DR}^{\mathsf{E}}_Y(\mathrm{D}f_*\mathscr{M}) &\simeq \mathsf{E} f_{!!}\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}), \\ \mathrm{D}f_*(\mathcal{O}^{\mathsf{E}}_X \overset{\mathrm{D}}{\otimes} \mathscr{M}) &\simeq \mathcal{O}^{\mathsf{E}}_Y \overset{\mathrm{D}}{\otimes} \mathrm{D}f_*\mathscr{M}. \end{aligned}$$

6.4 Ordinary linear differential equations and Stokes phenomena

Let us recall the local theory of ordinary linear differential equations. Let $0 \in X \subset \mathbb{C}$ be an open neighborhood of $0 \in \mathbb{C}$ and let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module such that $\mathrm{SingSupp}(\mathscr{M}) \subset \{0\}$ and $\mathscr{M} \simeq \mathscr{M}(*\{0\})$. Then $\mathscr{M}$ is a locally free $\mathscr{O}_X(*\{0\})$-module of finite rank. Let us take a system of generators $(u_1, \dots, u_r)$ of $\mathscr{M}$ as an $\mathscr{O}_X(*\{0\})$-module on a neighborhood of 0. Then, setting $\vec{\boldsymbol{u}}$ the column vector consisting of these generators, we have

$$\frac{d}{dz}\vec{\boldsymbol{u}} = \boldsymbol{A}(z)\vec{\boldsymbol{u}}$$

for an $(r \times r)$-matrix $\boldsymbol{A}(z)$ whose components are in $\mathscr{O}_X(*\{0\})$. Then for any $\mathscr{D}_X$-module $\mathscr{L}$ such that $\mathscr{L} \simeq \mathscr{L}(*\{0\})$, we have

(6.4.1) $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{L}) = \{\vec{u} \in \mathscr{L}^r \,;\, \vec{u}$ satisfies equation (6.4.2) below$\}$

where we associate the homomorphism from $\mathscr{M}$ to $\mathscr{L}$ defined by $\vec{\boldsymbol{u}} \mapsto \vec{u}$ to $\vec{u}$. Here

$$\frac{d}{dz}\vec{u} = \boldsymbol{A}(z)\vec{u}. \tag{6.4.2}$$

Now we have the following results on the solutions of the ordinary linear differential equation (6.4.2).

(i) there exist linearly independent r formal (column) solutions $\widehat{\boldsymbol{u}}_j$ ($j = 1, \dots r$) of (6.4.2) with the form

$$\widehat{\boldsymbol{u}}_j = \mathrm{e}^{\varphi_j(z)} z^{\lambda_j} \sum_{k=0}^{r-1} a_{j,k}(z)(\log z)^k,$$

where $m \in \mathbb{Z}_{>0}$, $\varphi_j(z) \in z^{-1/m}\mathbb{C}[z^{-1/m}]$, $\lambda_j \in \mathbb{C}$, $a_{j,k}(z) = \sum_{n \in m^{-1}\mathbb{Z}_{\geq 0}} a_{j,k,n} z^n \in \mathbb{C}[[z^{1/m}]]^r$ with $a_{j,k,n} \in \mathbb{C}^r$.

(ii) for any $\theta_0 \in \mathbb{R}$ and each $j = 1, \dots, r$, there exist an angular neighborhood

$$D_{\theta_0} = \left\{ z = r\mathrm{e}^{i\theta} \,;\, |\theta - \theta_0| < \varepsilon \text{ and } 0 < r < \delta \right\}$$

for sufficiently small $\varepsilon, \delta > 0$ and holomorphic (column) solution $\boldsymbol{u}_j \in \mathscr{O}_X(D_{\theta_0})^r$ of (6.4.2) defined on D_{θ_0} such that

$$\boldsymbol{u}_j \sim \widehat{\boldsymbol{u}}_j,$$

in the sense that, for any $N > 0$, there exists $C > 0$ such that

$$|\boldsymbol{u}_j - \widehat{\boldsymbol{u}}_j^N| \leq C|\mathrm{e}^{\varphi_j(z)} z^{\lambda_j + N}| = C\mathrm{e}^{\mathrm{Re}(\varphi_j(z))}|z^{\lambda_j+N}|, \tag{6.4.3}$$

where $\widehat{\boldsymbol{u}}_j^N$ is the finite partial sum

$$\widehat{\boldsymbol{u}}_j^N = \mathrm{e}^{\varphi_j(z)} z^{\lambda_j} \sum_{k=0}^{r-1} \sum_{\substack{n \in m^{-1}\mathbb{Z}_{\geq 0}, \\ n \leq N}} a_{k,n}^j z^n (\log z)^k.$$

Here we choose branches of $z^{1/m}$ and $\log z$ on D_{θ_0}.

Note that a holomorphic solution $\boldsymbol{u}_j$ is not uniquely determined by the formal solution $\widehat{\boldsymbol{u}}_j$. In fact, $\boldsymbol{u}_j + \sum_k c_k \boldsymbol{u}_k$ also satisfies the same estimate (6.4.3) whenever

$$\mathrm{Re}(\varphi_k(z)) < \mathrm{Re}(\varphi_j(z)) \text{ on } D_{\theta_0} \text{ if } c_k \neq 0.$$

We can interpret these results as follows. Let $\varpi \colon \widetilde{X} \to X$ be the real blow up of X along $\{0\}$ defined in § 3.2. Then $\mathrm{e}^{-\varphi_j(z)}\boldsymbol{u}_j$ gives a section of $(\mathcal{A}_{\widetilde{X}})^r$ defined on a neighborhood of $\mathrm{e}^{i\theta_0} \in \varpi^{-1}(0)$. Define the $\mathscr{D}_{\widetilde{X}}^{\mathcal{A}}$-module

$$\mathscr{L}_j := \mathscr{D}_{\widetilde{X}}^{\mathcal{A}} \mathrm{e}^{\varphi_j(z)} = \mathscr{D}_{\widetilde{X}}^{\mathcal{A}} / \mathscr{D}_{\widetilde{X}}^{\mathcal{A}} \big(\frac{d}{dz} - \varphi_j'(z) \big).$$

Here we take a branch of φ_j on a domain and $\mathscr{L}_j$ is defined on such a domain.

Then $(\mathrm{e}^{-\varphi_j(z)}\boldsymbol{u}_j)\mathrm{e}^{\varphi_j} \in (\mathscr{L}_j)^r$ is a solution of equation (6.4.2), and hence (6.4.1) defines a morphism of $\mathscr{D}_{\widetilde{X}}^{\mathcal{A}}$-modules

$$\mathscr{M}^{\mathcal{A}} \to \mathscr{L}_j.$$

Collecting such a morphism for all j, we obtain an isomorphism defined on a neighborhood of $\mathrm{e}^{i\theta_0} \in \varpi^{-1}(0)$:

$$\mathscr{M}^{\mathcal{A}} \xrightarrow{\sim} \bigoplus_{j=1}^{r} \mathscr{L}_j. \tag{6.4.4}$$

Note that
$$\varpi_*\mathscr{M}^{\mathcal{A}} \simeq \mathscr{M}.$$

However, these isomorphisms (6.4.4) are not globally defined. That is, $\mathscr{M}^{\mathcal{A}}$ is only locally isomorphic to $\bigoplus_{j=1}^{r}\mathscr{L}_j$. We have

$$\mathscr{H}om_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathscr{L}_j,\mathscr{L}_{j'})|_{\varpi^{-1}(0)} \simeq \mathbb{C}_{U_{j,j'}} \subset \mathbb{C}_{\pi^{-1}(0)}, \tag{6.4.5}$$

where

$$U_{j,j'} = \left\{p \in \varpi^{-1}(0)\,;\begin{array}{l}\mathrm{Re}(\varphi_j(z)) \le \mathrm{Re}(\varphi_{j'}(z)) \text{ on } U\cap\widetilde{X}^{>0} \text{ for a neighbor-}\\ \text{hood } U \text{ of } p\end{array}\right\}.$$

Hence the isomorphism class of a $\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}|_{\varpi^{-1}(0)}$-module $\mathscr{L}$ locally isomorphic to $\bigoplus_{j=1}^{r}\mathscr{L}_j|_{\varpi^{-1}(0)}$ is determined by a topological data, which is called Stokes matrices.

Assuming that $m=1$, let us explian them more precisely. Let $\mathscr{L}$ be a $\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}|_{\varpi^{-1}(0)}$-module locally isomorphic to $\bigoplus_{j=1}^{r}\mathscr{L}_j|_{\varpi^{-1}(0)}$. We identify $\varpi^{-1}(0)$ with $\mathbb{R}/2\pi\mathbb{Z}$ by $\mathbb{R}/2\pi\mathbb{Z} \ni \theta \mapsto \mathrm{e}^{i\theta}$. Let us take $\{\theta_1,\dots,\theta_s\}$ such that $s \ge 2$, $\theta_0 < \theta_1 < \cdots < \theta_{s-1} < \theta_s$ and

$$\varpi^{-1}(0)\bigcap \bigcup_{\substack{1\le j,j'\le r,\\ \varphi_j\neq\varphi_{j'}}} \overline{\left\{z \in \widetilde{X}^{>0}\,; \mathrm{Re}\,\varphi_j(z) = \mathrm{Re}\,\varphi_{j'}(z)\right\}} \subset \{\theta_1,\dots,\theta_s\}.$$

Here we set $\theta_{k+ls} = \theta_k + 2\pi l$ for $1 \le k \le s$ and $l \in \mathbb{Z}$. Set $V_k = \{\theta\,; \theta_{k-1} < \theta < \theta_{k+1}\}$ and $W_k = \{\theta\,; \theta_k < \theta < \theta_{k+1}\} = V_k \cap V_{k+1}$. Then we have $\varpi^{-1}(0) = \bigcup_{1\le k\le s} V_k$.

By (6.4.5), any isomorphism $\mathscr{L} \xrightarrow{\sim} \bigoplus_{j=1}^{r}\mathscr{L}_j|_{\varpi^{-1}(0)}$ defined on a neighborhood of θ_k can be extended to an isomorphism defined on V_k. Therefore we have an isomorphism

$$\psi_k\colon \mathscr{L}|_{V_k} \xrightarrow{\sim} \bigoplus_{j=1}^{r}\mathscr{L}_j|_{V_k}.$$

Let us set

$$\xi_k = \psi_{k+1}\circ\psi_k^{-1}\colon \bigoplus_{j=1}^{r}\mathscr{L}_j|_{W_k} \xrightarrow{\sim} \bigoplus_{j=1}^{r}\mathscr{L}_j|_{W_k}.$$

Then $\mathscr{L}$ is obtained by patching $\bigoplus_{j=1}^{r} \mathscr{L}_j|_{V_k}$ by the ξ_k's. Each isomorphism ξ_k is given by the matrix $S_k = (s_{k;i',i})_{1\leq i,i'\leq r} \in \mathrm{GL}_r(\mathbb{C})$. Here $s_{k;i',i} \in \mathbb{C}$ is given by the morphism

$$\mathscr{L}_i|_{W_k} \rightarrowtail \bigoplus_{j=1}^{r} \mathscr{L}_j|_{W_k} \xrightarrow[\xi_k]{\sim} \bigoplus_{j=1}^{r} \mathscr{L}_j|_{W_k} \twoheadrightarrow \mathscr{L}_{i'}|_{W_k}$$

through

$$\Gamma\big(W_k; \mathscr{H}\!om_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathscr{L}_i, \mathscr{L}_{i'})|_{\varpi^{-1}(0)}\big) \simeq \Gamma\big(W_k; \mathbb{C}_{U_{i,i'}}\big) \subset \Gamma\big(W_k; \mathbb{C}_{\pi^{-1}(0)}\big) \simeq \mathbb{C}$$

due to (6.4.5). Hence, we have $s_{k;i',i} = 0$ if $W_k \not\subset U_{i,i'}$.

The matrices $\{S_k\}_{1\leq k\leq s}$ are called the *Stokes matrices*. Conversely, for a given family of matrices $\{S_k\}_{1\leq k\leq s}$, we can find a $\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}|_{\varpi^{-1}(0)}$-module $\mathscr{L}$ locally isomorphic to $\bigoplus_{j=1}^{r} \mathscr{L}_j|_{\varpi^{-1}(0)}$ by patching $\bigoplus_{j=1}^{r} \mathscr{L}_j|_{V_k}$ by $\{S_k\}_{1\leq k\leq s}$.

6.5 Normal form

The results in § 6.4 are generalized to higher dimensions by T. Mochizuki ([Mo09, Mo11]) and K. S. Kedlaya ([Ke10, Ke11]). In this subsection, we collect some of their results that we shall need.

Let X be a complex manifold and $D \subset X$ a normal crossing divisor. We shall use the notations introduced in § 3.2 and in particular the real blow up $\varpi\colon \widetilde{X} \to X$ and the notation $\mathscr{M}^{\mathcal{A}}$ of (3.2.11).

Definition 6.5.1. We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has *a normal form* along D if
(i) $\mathscr{M} \simeq \mathscr{M}(*D)$,
(ii) $\mathrm{SingSupp}(\mathscr{M}) \subset D$,
(iii) for any $x \in \varpi^{-1}(D) \subset \widetilde{X}$, there exist an open neighborhood $U \subset X$ of $\varpi(x)$ and finitely many $\varphi_i \in \Gamma(U; \mathcal{O}_X(*D))$ such that

$$(\mathscr{M}^{\mathcal{A}})|_V \simeq \left(\bigoplus_i (\mathscr{E}^{\varphi_i}_{U\setminus D|U})^{\mathcal{A}}\right)\Bigg|_V$$

for some open neighborhood V of x with $V \subset \varpi^{-1}(U)$.

A ramification of X along D on a neighborhood U of $x \in D$ is a finite map

$$p\colon X' \to U$$

of the form $p(z') = ({z'_1}^{m_1}, \dots, {z'_r}^{m_r}, z'_{r+1}, \dots, z'_n)$ for some $(m_1, \dots, m_r) \in (\mathbb{Z}_{>0})^r$. Here $(z'_1, \dots, z'_n)$ is a local coordinate system on X', $(z_1, \dots, z_n)$ a local coordinate system on X such that $D = \{z_1 \cdots z_r = 0\}$.

Definition 6.5.2. We say that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ has *a quasi-normal form* along D if it satisfies (i) and (ii) in Definition 6.5.1, and if for any $x \in D$ there exists a ramification $p\colon X' \to U$ on a neighborhood U of x such that $\mathrm{D}p^*(\mathscr{M}|_U)$ has a normal form along $p^{-1}(D \cap U)$.

Remark 6.5.3. In the above definition, $\mathrm{D}p^*(\mathscr{M}|_U)$ as well as $\mathrm{D}p_*\mathrm{D}p^*(\mathscr{M}|_U)$ is concentrated in degree zero and $\mathscr{M}|_U$ is a direct summand of $\mathrm{D}p_*\mathrm{D}p^*(\mathscr{M}|_U)$.

The next result is an essential tool in the study of holonomic $\mathscr{D}$-module and is easily deduced from the fundamental work of Mochizuki [Mo09, Mo11] (see also Sabbah [Sa00] for preliminary results and see Kedlaya [Ke10, Ke11] for the analytic case).

Theorem 6.5.4. *Let X be a complex manifold, $\mathscr{M}$ a holonomic $\mathscr{D}_X$-module and $x \in X$. Then there exist an open neighborhood U of x, a closed analytic hypersurface $Y \subset U$, a complex manifold X' and a projective morphism $f\colon X' \to U$ such that*

(i) $\mathrm{SingSupp}(\mathscr{M}) \cap U \subset Y$,

(ii) *$D := f^{-1}(Y)$ is a normal crossing divisor of X',*

(iii) *f induces an isomorphism $X' \setminus D \to U \setminus Y$,*

(iv) *$(\mathrm{D}f^*\mathscr{M})(*D)$ has a quasi-normal form along D.*

Remark that, under assumption (iii), $(\mathrm{D}f^*\mathscr{M})(*D)$ is concentrated in degree zero.

Using Theorem 6.5.4, one easily deduces the next lemma.

Lemma 6.5.5. *Let $P_X(\mathscr{M})$ be a statement concerning a complex manifold X and a holonomic object $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Consider the following conditions.*

(a) *Let $X = \bigcup_{i \in I} U_i$ be an open covering. Then $P_X(\mathscr{M})$ is true if and only if $P_{U_i}(\mathscr{M}|_{U_i})$ is true for any $i \in I$.*

(b) *If $P_X(\mathscr{M})$ is true, then $P_X(\mathscr{M}[n])$ is true for any $n \in \mathbb{Z}$.*

(c) *Let $\mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \xrightarrow{+1}$ be a distinguished triangle in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. If $P_X(\mathscr{M}')$ and $P_X(\mathscr{M}'')$ are true, then $P_X(\mathscr{M})$ is true.*

(d) *Let $\mathscr{M}$ and $\mathscr{M}'$ be holonomic $\mathscr{D}_X$-modules. If $P_X(\mathscr{M} \oplus \mathscr{M}')$ is true, then $P_X(\mathscr{M})$ is true.*

(e) *Let $f\colon X \to Y$ be a projective morphism and $\mathscr{M}$ a good holonomic $\mathscr{D}_X$-module. If $P_X(\mathscr{M})$ is true, then $P_Y(\mathrm{D}f_!\mathscr{M})$ is true.*

(f) *If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module with a normal form along a normal crossing divisor of X, then $P_X(\mathscr{M})$ is true.*

If conditions (a)–(f) *are satisfied, then $P_X(\mathscr{M})$ is true for any complex manifold X and any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$.*

Sketch of the proof. The proof is similar to the regular case (Lemma 3.1.4).

We shall only prove here that $P_X(\mathscr{M})$ is true for any holonomic $\mathscr{D}_X$-module $\mathscr{M}$ which has a quasi-normal form along a normal crossing divisor D.

Let $p\colon X' \to U$ be as in Definition 6.5.2. Then $\mathrm{D}p^*(\mathscr{M}|_U)$ has a normal form along $p^{-1}(D \cap U)$. Hence $P_{X'}\big(\mathrm{D}p^*(\mathscr{M}|_U)\big)$ is true by hypothesis (f). Hence $P_U\big(\mathrm{D}p_!\mathrm{D}p^*(\mathscr{M}|_U)\big)$ is true by hypothesis (e). We have a chain of morphisms

$$\mathscr{M}|_U \to \mathrm{D}p_!\mathrm{D}p^*(\mathscr{M}|_U) \to \mathscr{M}|_U,$$

whose composition is equal to $m\,\mathrm{id}_{\mathscr{M}}$ where m is the number of the generic fiber of p. Hence $\mathscr{M}|_U$ is a direct summand of $\mathrm{D}p_!\mathrm{D}p^*(\mathscr{M}|_U)$. Then, hypothesis (d) implies that $P_U(\mathscr{M}|_U)$ is true. Q.E.D.

6.6 Enhanced de Rham functor on the real blow up

By Lemma 6.5.5, many statements on holonomic D-modules can be reduced to the normal form case. In order to investigate this case, we shall introduce the enhanced de Rham functor on the real blow up.

Let D be a normal crossing divisor of a complex manifold X and let $\varpi\colon \widetilde{X} \to X$ be the real blow up of X along D as in § 3.2. By a similar construction to $\mathcal{O}^{\mathsf{t}}_{\widetilde{X}}$, we can construct the enhanced indsheaf

$$\mathcal{O}^{\mathsf{E}}_{\widetilde{X}} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}^{\mathcal{A}}_{\widetilde{X}})$$

which satisfies

$$\mathcal{O}^{\mathsf{E}}_{\widetilde{X}} \simeq \mathsf{E}\varpi^{!}\mathcal{O}^{\mathsf{E}}_{X}(*D) \quad \text{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}(\varpi^{-1}\mathscr{D}_X)), \tag{6.6.1}$$

$$\mathsf{E}\varpi_{*}\mathcal{O}^{\mathsf{E}}_{\widetilde{X}} \simeq \mathcal{O}^{\mathsf{E}}_{X}(*D) \quad \text{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_X), \tag{6.6.2}$$

where

$$\mathcal{O}^{\mathsf{E}}_{X}(*D) := \mathcal{O}^{\mathsf{E}}_{X} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X(*D) \simeq \mathrm{R}\mathscr{I}hom\,(\pi^{-1}\mathbb{C}_{X\setminus D}, \mathcal{O}^{\mathsf{E}}_{X}).$$

We set $\Omega^{\mathsf{E}}_{\widetilde{X}} = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}^{\mathsf{E}}_{\widetilde{X}} \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}^{\mathcal{A}}_{\widetilde{X}})$.

Then, for $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}^{\mathcal{A}}_{\widetilde{X}})$, we define the enhanced de Rham functor on $\widetilde{X}$ by

$$\begin{aligned} \mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}(\mathscr{N}) &= \Omega^{\mathsf{E}}_{\widetilde{X}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}} \mathscr{N}, \\ \mathcal{S}ol^{\mathsf{E}}_{\widetilde{X}}(\mathscr{N}) &= \mathrm{R}\mathscr{H}om_{\mathscr{D}^{\mathcal{A}}_{\widetilde{X}}}(\mathscr{N}, \mathcal{O}^{\mathsf{E}}_{\widetilde{X}}). \end{aligned}$$

Then (6.6.1) and (6.6.2) imply that

$$\mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \simeq \mathsf{E}\varpi^{!}\mathcal{DR}^{\mathsf{E}}_{X}(\mathscr{M}(*D)) \quad \text{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\widetilde{\mathrm{X}}}), \tag{6.6.3}$$

$$\mathsf{E}\varpi_{*}\mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}}) \simeq \mathcal{DR}^{\mathsf{E}}_{X}(\mathscr{M}(*D)) \quad \text{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}}). \tag{6.6.4}$$

for any $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$.

6.7 Extended Riemann-Hilbert correspondence

The following theorem is the main theorem.

Theorem 6.7.1 (Generalized Riemann-Hilbert problem)**.** *There exists a canonical isomorphism functorial with respect to* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$:

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathsf{E}}_{X} \xrightarrow{\sim} \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_{X}(\mathscr{M}), \mathcal{O}^{\mathsf{E}}_{X}) \ \textit{in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}). \tag{6.7.1}$$

The idea of the proof is to reduce the problem to the case where $\mathscr{M}$ is an exponential D-module. However, in this case, we can treat $\mathcal{DR}^{\mathsf{E}}_{X}(\mathscr{M})$ by Proposition 6.3.1, but not $\mathcal{S}ol^{\mathsf{E}}_{X}(\mathscr{M})$. In order to calculate it, we need the commutativity of the enhanced de Rham functor and the duality functor (see Theorem 6.8.3 and its consequence Corollary 6.8.8 below).

Postponing the proof of Theorem 6.7.1, we shall first give its corollaries.

Corollary 6.7.2. *There exists a canonical isomorphism functorial with respect to* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$*:*

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{t}}_X \xrightarrow{\sim} \mathscr{I}hom^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathcal{O}^{\mathsf{E}}_X) \text{ in } \mathsf{D}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}). \tag{6.7.2}$$

Proof. Let us apply the functor $\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}^{\mathsf{E}}_X, \bullet)$ to the isomorphism (6.7.1). Since $\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}^{\mathsf{E}}_X, \mathcal{O}^{\mathsf{E}}_X) \simeq \mathcal{O}^{\mathrm{t}}_X$ by (6.2.2), we get

$$\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}^{\mathsf{E}}_X, \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathsf{E}}_X) \;\simeq\; \mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathrm{t}}_X.$$

On the other-hand, we have

$$\begin{aligned} &\mathscr{I}hom^{\mathrm{E}}(\mathbb{C}^{\mathsf{E}}_X, \mathscr{I}hom^{+}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathcal{O}^{\mathsf{E}}_X)) \\ &\qquad\qquad \simeq \mathscr{I}hom^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathscr{I}hom^{+}(\mathbb{C}^{\mathsf{E}}_X, \mathcal{O}^{\mathsf{E}}_X)) \\ &\qquad\qquad \simeq \mathscr{I}hom^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathcal{O}^{\mathsf{E}}_X). \end{aligned}$$

Q.E.D.

Corollary 6.7.3 (Enhanced Riemann-Hilbert theorem)**.** *There exists a canonical isomorphism functorial with respect to* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$*:*

$$\mathscr{M} \xrightarrow{\sim} \mathscr{H}om^{\mathrm{E}}(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}), \mathcal{O}^{\mathsf{E}}_X) \text{ in } \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}}). \tag{6.7.3}$$

Proof. Apply the functor α_X to (6.7.2). Q.E.D.

As a corollary, we cobtain

Theorem 6.7.4. *The functor*

$$\mathcal{DR}^{\mathsf{E}}_X : \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$$

is fully faithful.

Remark 6.7.5. Corollary 6.7.3 due to [DK13, Th. 9.6.1] is a natural formulation of the Riemann-Hilbert correspondence for irregular D-modules. Theorem 6.7.1 due to [KS14, Th. 4.5] is a generalization to the irregular case of Theorem 3.3.2 which is itself a generalization/reformulation of a theorem of J-E. Björk ([Bj93]).

6.8 Constructibility of solutions

Theorem 6.8.1. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. *Then* $\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})$ *and* $\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M})$ *belong to* $\mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$.

Sketch of the proof. Using Lemma 6.5.5, one reduces the proof to the case where $\mathscr{M}$ has a normal form along a normal crossing divisor D. Let $\varpi\colon \widetilde{X} \to X$ be the real blow up along D.

Then, $\mathscr{M}^{\mathcal{A}}$ is locally isomorphic to a direct sum of $(\mathscr{E}^{\varphi}_{U\setminus D|U})^{\mathcal{A}}$ with $\varphi \in \Gamma(U; \mathcal{O}_X(*D))$. Since $\mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}\big((\mathscr{E}^{\varphi}_{U\setminus D|U})^{\mathcal{A}}\big) \simeq \mathsf{E}\varpi^{!}\mathcal{DR}^{\mathsf{E}}_X(\mathscr{E}^{\varphi}_{U\setminus D|U})$ is $\mathbb{R}$-constructible by Proposition 6.3.1, $\mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}})$ is $\mathbb{R}$-constructible. Hence $\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}) \simeq \mathsf{E}\varpi_*\mathcal{DR}^{\mathsf{E}}_{\widetilde{X}}(\mathscr{M}^{\mathcal{A}})$ is $\mathbb{R}$-constructible. Q.E.D.

Lemma 6.8.2. *Let X_1 and X_2 be a pair of complex manifolds. Let* $\mathscr{M}_j \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{X_j})$ $(j = 1, 2)$. *Then we have a canonical isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_{X_1}(\mathscr{M}_1) \overset{+}{\boxtimes} \mathcal{DR}^{\mathsf{E}}_{X_2}(\mathscr{M}_2) \xrightarrow{\sim} \mathcal{DR}^{\mathsf{E}}_{X_1\times X_2}(\mathscr{M}_1 \overset{\mathrm{D}}{\boxtimes} \mathscr{M}_2). \tag{6.8.1}$$

Sketch of the proof. Using Lemma 6.5.5, one reduces the proof to the case where $\mathscr{M}_1$ and $\mathscr{M}_2$ are exponential $\mathscr{D}$-modules. In this case, the result follows from Proposition 6.3.1. Q.E.D.

Theorem 6.8.3. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. *Then, we have*

$$\mathcal{DR}^{\mathsf{E}}_X(\mathbb{D}_X\mathscr{M}) \simeq \mathrm{D}^{\mathsf{E}}_X\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}).$$

Note that we have $\mathcal{DR}^{\mathsf{E}}_X(\mathbb{D}_X\mathscr{M}) \simeq \mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M})\,[d_X]$.

Idea of the proof.

Let $\mathscr{T}$ be a monoidal category with $\mathbf{1}$ as a unit object. Recall that a pair of objects X and Y are dual if and only if there exist morphisms

$$X \otimes Y \xrightarrow{\varepsilon} \mathbf{1},$$
$$\mathbf{1} \xrightarrow{\eta} Y \otimes X$$

such that the composition

$$X \xrightarrow{X\otimes\eta} X \otimes Y \otimes X \xrightarrow{\varepsilon\otimes X} X$$

is equal to id_X and

$$Y \xrightarrow{\eta\otimes Y} Y\otimes X\otimes Y \xrightarrow{Y\otimes\varepsilon} Y$$

is equal to id_Y.

This criterion of duality has many variations.

Sheaf case: Let M be a real analytic manifold, and let F, $G \in \mathsf{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbf{k}_M)$. Denote by Δ_M the diagonal subset of $M\times M$.

Now F and G are dual to each other, i.e., $G \simeq \mathrm{D}_M F$, if and only if there exist morphisms

$$F\boxtimes G \xrightarrow{\varepsilon} \omega_{\Delta_M},$$
$$\mathbf{k}_{\Delta_M} \xrightarrow{\eta} G\boxtimes F$$

such that the composition

$$F\boxtimes\mathbf{k}_{\Delta_M} \xrightarrow{F\boxtimes\eta} F\boxtimes G\boxtimes F \xrightarrow{\varepsilon\boxtimes F} \omega_{\Delta_M}\boxtimes F$$

is equal to id_F via isomorphism (6.8.8) below and

$$\mathbf{k}_{\Delta_M}\boxtimes G \xrightarrow{\eta\boxtimes G} G\boxtimes F\boxtimes G \xrightarrow{G\boxtimes\varepsilon} G\boxtimes\omega_{\Delta_M}$$

is equal to id_G via isomorphism (6.8.9) below.

Enhanced indsheaf case: Let F and $G \in \mathrm{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathrm{I}\mathbf{k}_M)$. They are dual to each other, i.e., $G\simeq \mathrm{D}^{\mathrm{E}}_M F$, if and only if there exist morphisms

$$F\overset{+}{\boxtimes} G \xrightarrow{\varepsilon} \omega^{\mathsf{E}}_{\Delta_M},\qquad \mathbf{k}^{\mathsf{E}}_{\Delta_M} \xrightarrow{\eta} G\overset{+}{\boxtimes} F \tag{6.8.2}$$

such that the composition

$$F\overset{+}{\boxtimes}\mathbf{k}^{\mathsf{E}}_{\Delta_M} \xrightarrow{F\overset{+}{\boxtimes}\eta} F\overset{+}{\boxtimes}G\overset{+}{\boxtimes}F \xrightarrow{\varepsilon\overset{+}{\boxtimes}F} \omega^{\mathsf{E}}_{\Delta_M}\overset{+}{\boxtimes}F \tag{6.8.3}$$

is equal to id_F via the enhanced version of isomorphism (6.8.8) and

$$\mathbf{k}_{\mathsf{E}\Delta_M}\overset{+}{\boxtimes}G \xrightarrow{\eta\overset{+}{\boxtimes}G} G\overset{+}{\boxtimes}F\overset{+}{\boxtimes}G \xrightarrow{G\overset{+}{\boxtimes}\varepsilon} G\overset{+}{\boxtimes}\omega^{\mathsf{E}}_{\Delta_M} \tag{6.8.4}$$

is equal to id_G via the enhanced version of isomorphism (6.8.9).

Holonomic $\mathscr{D}$-module case: Let X be a complex manifold and let $\delta\colon X \hookrightarrow X\times X$ be the diagonal embedding. We set $\mathscr{B}_{\Delta_X} := \mathrm{D}\delta_*\mathcal{O}_X$.

Let $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. They are dual to each other, i.e., $\mathscr{N} \simeq \mathbb{D}_X\mathscr{M}$, if and only if there exist morphisms

$$\begin{gathered} \mathscr{M} \overset{\mathrm{D}}{\boxtimes} \mathscr{N} \xrightarrow{\varepsilon} \mathscr{B}_{\Delta_X}[d_X], \\ \mathscr{B}_{\Delta_X}[-d_X] \xrightarrow{\eta} \mathscr{N} \overset{\mathrm{D}}{\boxtimes} \mathscr{M} \end{gathered} \tag{6.8.5}$$

such that the composition

$$\mathscr{M} \overset{\mathrm{D}}{\boxtimes} \mathscr{B}_{\Delta_X}[-d_X] \xrightarrow{\mathscr{M}\overset{\mathrm{D}}{\boxtimes}\eta} \mathscr{M} \overset{\mathrm{D}}{\boxtimes} \mathscr{N} \overset{\mathrm{D}}{\boxtimes} \mathscr{M} \xrightarrow{\varepsilon\overset{\mathrm{D}}{\boxtimes}\mathscr{M}} \mathscr{B}_{\Delta_X}[d_X] \overset{\mathrm{D}}{\boxtimes} \mathscr{M} \tag{6.8.6}$$

is equal to $\mathrm{id}_{\mathscr{M}}$ via isomorphism (6.8.10) and

$$\mathscr{B}_{\Delta_X}[-d_X] \overset{\mathrm{D}}{\boxtimes} \mathscr{N} \xrightarrow{\eta\overset{\mathrm{D}}{\boxtimes}\mathscr{N}} \mathscr{N} \overset{\mathrm{D}}{\boxtimes} \mathscr{M} \overset{\mathrm{D}}{\boxtimes} \mathscr{M} \xrightarrow{\mathscr{N}\overset{\mathrm{D}}{\boxtimes}\varepsilon} \mathscr{N} \overset{\mathrm{D}}{\boxtimes} \mathscr{B}_{\Delta_X}[d_X] \tag{6.8.7}$$

is equal to $\mathrm{id}_{\mathscr{N}}$ via isomorphism (6.8.11).

Now we shall prove Theorem 6.8.3. Set $\mathscr{N} = \mathbb{D}_X\mathscr{M}$. Then we have morphisms as in (6.8.5) which satisfy the conditions that the compositions (6.8.6) and (6.8.7) are equal to $\mathrm{id}_{\mathscr{M}}$ and $\mathrm{id}_{\mathscr{N}}$, respectively. Now we shall apply the functor $\mathcal{DR}^{\mathsf{E}}$. Then we obtain morphisms as in (6.8.2) with $M = X_{\mathbb{R}}$, $\mathbf{k} = \mathbb{C}$, $F = \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})$ and $G = \mathcal{DR}^{\mathsf{E}}_X(\mathscr{N})$. Note that we have

$$\mathcal{DR}^{\mathsf{E}}_{X\times X}(\mathscr{B}_{\Delta_X}[-d_X]) \simeq \mathbb{C}^{\mathsf{E}}_{\Delta_X}, \quad \mathcal{DR}^{\mathsf{E}}_{X\times X}(\mathscr{B}_{\Delta_X}[d_X]) \simeq \omega^{\mathsf{E}}_{\Delta_X}.$$

By applying the functor $\mathcal{DR}^{\mathsf{E}}_{X\times X\times X}$, the morphisms in (6.8.6) and (6.8.7) are sent to (6.8.3) and (6.8.4). Hence the compositions (6.8.3) and (6.8.4) are equal to id_F and id_G, respectively. Thus we conclude that $G \simeq \mathrm{D}^{\mathsf{E}}_X F$. $\square$

Here is the lemma that we used in the course of the proof of Theorem 6.8.3.

Lemma 6.8.4. *Let M be a real manifold and let $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then we have the isomorphisms*

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times M})}(F \boxtimes \mathbf{k}_{\Delta_M},\ \omega_{\Delta_M} \boxtimes G) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)}(F, G), \tag{6.8.8}$$

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times M})}(\mathbf{k}_{\Delta_M} \boxtimes F,\ G \boxtimes \omega_{\Delta_M}) \simeq \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)}(F, G), \tag{6.8.9}$$

where $\Delta_M \subset M \times M$ is the diagonal subset.

Proof. Define the maps $p_{i_1,\dots,i_n}$ by $p_{i_1,\dots,i_n}(x_1,\dots,x_m) = (x_{i_1},\dots,x_{i_n})$. Then we have a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dr, phantom, "\square"] & M \times M \arrow[d, "p_{1,2,2}"] \\
M \times M \arrow[r, "p_{1,1,2}"'] \arrow[d, "p_2"] & M \times M \times M \\
M. &
\end{tikzcd}$$

In the sequel, we write for short Hom instead of $\operatorname{Hom}_{\mathbf{k}_N}$ with $N = M, M \times M \times M$. Then we have

$$\begin{aligned}
\operatorname{Hom}(F \boxtimes \mathbf{k}_{\Delta_M},\ \omega_{\Delta_M} \boxtimes G) &\simeq \operatorname{Hom}(\mathrm{R}p_{1,2,2!}(F \boxtimes \mathbf{k}_M), \mathrm{R}p_{1,1,2*}p_2^! G) \\
&\simeq \operatorname{Hom}(F \boxtimes \mathbf{k}_M, p_{1,2,2}^! \mathrm{R}p_{1,1,2*}p_2^! G) \\
&\simeq \operatorname{Hom}(F \boxtimes \mathbf{k}_M, \mathrm{R}\delta_* \delta^! p_2^! G) \\
&\simeq \operatorname{Hom}(\delta^{-1}(F \boxtimes \mathbf{k}_M), \delta^! p_2^! G) \\
&\simeq \operatorname{Hom}(F, G).
\end{aligned}$$

Q.E.D.

Similarly, we have the following D-module version. Here again, we write for short Hom instead of $\operatorname{Hom}_{\mathscr{D}_Y}$ with $Y = X, X \times X \times X$.

Lemma 6.8.5. *Let X be a complex manifold and let $\mathscr{M}, \mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Then we have the isomorphisms*

$$\operatorname{Hom}(\mathscr{M} \overset{\mathrm{D}}{\boxtimes} \mathscr{B}_{\Delta_X}[-d_X],\ \mathscr{B}_{\Delta_X}[d_X] \overset{\mathrm{D}}{\boxtimes} \mathscr{N}) \simeq \operatorname{Hom}(\mathscr{M}, \mathscr{N}), \tag{6.8.10}$$

$$\operatorname{Hom}(\mathscr{B}_{\Delta_M}[-d_X] \overset{\mathrm{D}}{\boxtimes} \mathscr{M},\ \mathscr{N} \overset{\mathrm{D}}{\boxtimes} \mathscr{B}_{\Delta_X}[d_X]) \simeq \operatorname{Hom}(\mathscr{M}, \mathscr{N}). \tag{6.8.11}$$

As applications of Theorem 6.8.3, we obtain the following corollaries.

Proposition 6.8.6. *Let $f\colon X \to Y$ be a morphism of complex manifolds. Then, for any $\mathscr{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_Y)$,*

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathrm{D}f^*\mathscr{N}) \simeq \mathsf{E}f^{-1}\mathcal{S}ol^{\mathsf{E}}_Y(\mathscr{N}). \tag{6.8.12}$$

Proof. We have

$$\begin{aligned}\mathcal{S}ol^{\mathsf{E}}_X(\mathrm{D}f^*\mathscr{N}) &\simeq \mathrm{D}^{\mathrm{E}}_X\mathcal{DR}^{\mathsf{E}}_X(\mathrm{D}f^*\mathscr{N})[-d_X]\\ &\simeq \mathrm{D}^{\mathrm{E}}_X\mathsf{E}f^!\mathcal{DR}^{\mathsf{E}}_Y(\mathscr{N})[-d_Y]\\ &\simeq \mathsf{E}f^{-1}\mathrm{D}^{\mathrm{E}}_X\mathcal{DR}^{\mathsf{E}}_Y(\mathscr{N})[-d_Y]\\ &\simeq \mathsf{E}f^{-1}\mathcal{S}ol^{\mathsf{E}}_Y(\mathscr{N}).\end{aligned}$$

Q.E.D.

Corollary 6.8.7. *Let X be a complex manifold and $\mathscr{M},\mathscr{N}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$. Then we have*

$$\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M}\overset{\mathrm{D}}{\otimes}\mathscr{N})\simeq\mathsf{E}\delta^!\big(\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})\overset{+}{\boxtimes}\mathcal{DR}^{\mathsf{E}}_X(\mathscr{N})\big)[d_X], \tag{6.8.13}$$

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}\overset{\mathrm{D}}{\otimes}\mathscr{N})\simeq\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M})\overset{+}{\otimes}\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{N}), \tag{6.8.14}$$

where $\delta\colon X\to X\times X$ is the diagonal embedding.

Proof. Since $\mathscr{M}\overset{\mathrm{D}}{\otimes}\mathscr{N}\simeq\mathrm{D}\delta^*(\mathscr{M}\overset{\mathrm{D}}{\boxtimes}\mathscr{N})$, it is enough to apply (6.3.1) and (6.8.12). Q.E.D.

Corollary 6.8.8. *For a closed hypersurface Y of a complex manifold X and $\mathscr{M}\in\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{O}_X)$, we have*

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}(*Y))\simeq\pi^{-1}\mathbb{C}_{X\setminus Y}\otimes\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{M}).$$

Proof. It follows from Theorem 6.8.3 and isomorphisms

$$\mathcal{DR}^{\mathsf{E}}(\mathscr{M}(*Y))\simeq\mathrm{R}\mathscr{I}hom\big(\pi^{-1}\mathbb{C}_{X\setminus Y},\mathcal{DR}^{\mathsf{E}}(\mathscr{M})\big)$$

and

$$\mathrm{D}^{\mathrm{E}}_X\big(\mathrm{R}\mathscr{I}hom\,(\pi^{-1}\mathbb{C}_{X\setminus Y},\mathcal{DR}^{\mathsf{E}}(\mathscr{M})\big)\simeq\pi^{-1}\mathbb{C}_{X\setminus Y}\otimes\mathrm{D}^{\mathrm{E}}_X\big(\mathcal{DR}^{\mathsf{E}}(\mathscr{M})\big)$$

(see Theorem 5.6.3 (v)). Q.E.D.

Corollary 6.8.9. *For a closed hypersurface Y of a complex manifold X and $\varphi\in\mathcal{O}_X(*Y)$, we have*

$$\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{E}^{\varphi}_{X\setminus Y|X})\simeq\mathbb{C}^{\mathsf{E}}_X\overset{+}{\otimes}\mathbb{C}_{\{t=-\operatorname{Re}\varphi\}}.$$

This follows from Proposition 6.3.1, (5.5.2) and Theorem 6.8.3, because $\mathscr{E}^{\varphi}_{X\setminus Y|X}\simeq\Big(\mathbb{D}_X\mathscr{E}^{-\varphi}_{X\setminus Y|X}\Big)(*D)$.

6.9 Sketch of the proof of Theorem 6.7.1

The proof is parallel with the one of Theorem 3.3.2.

First we shall construct a morphism (6.7.1). We have a canonical morphism

$$\mathcal{O}_X^{\mathsf{E}} \overset{+}{\otimes}_{\beta\mathcal{O}_X} \mathcal{O}_X^{\mathsf{E}} \longrightarrow \mathcal{O}_X^{\mathsf{E}}.$$

Hence we have

$$(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X^{\mathsf{E}}) \overset{+}{\otimes} \mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}) \longrightarrow \mathcal{O}_X^{\mathsf{E}} \overset{+}{\otimes}_{\beta\mathcal{O}_X} \mathcal{O}_X^{\mathsf{E}} \longrightarrow \mathcal{O}_X^{\mathsf{E}}$$

which induces a morphism

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X^{\mathsf{E}} \longrightarrow \mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathcal{O}_X^{\mathsf{E}}). \tag{6.9.1}$$

In order to see that it is an isomorphism, we shall apply Lemma 6.5.5.

We shall only check property (f) of this lemma. Hence, we are reduced to the case where $\mathscr{M}$ has a normal form along a normal crossing divisor D. Then we have $\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}) \simeq \varpi^{-1}\mathbb{C}_{X\setminus D} \otimes \mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M})$ by Corollary 6.8.8, which implies that

$$\mathscr{I}hom^{+}(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathcal{O}_X^{\mathsf{E}}) \simeq \mathscr{I}hom^{+}\big(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathcal{O}_X^{\mathsf{E}}(*D)\big).$$

Let $\varpi\colon \widetilde{X} \to X$ be the real blow-up of X along D. Then we have

$$\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathcal{O}_X^{\mathsf{E}} \simeq \mathsf{E}\varpi_{*}(\mathscr{M}^{\mathcal{A}} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}^{\mathsf{E}})$$

and

$$\mathscr{I}hom^{+}\big(\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}), \mathcal{O}_X^{\mathsf{E}}(*D)\big) \simeq \mathsf{E}\varpi_{*}\mathscr{I}hom^{+}(\mathcal{S}ol_{\widetilde{X}}^{\mathsf{E}}(\mathscr{M}^{\mathcal{A}}), \mathcal{O}_{\widetilde{X}}^{\mathsf{E}}).$$

Hence it is enough to show that

$$\mathscr{M}^{\mathcal{A}} \overset{\mathrm{L}}{\otimes}_{\mathcal{A}_{\widetilde{X}}} \mathcal{O}_{\widetilde{X}}^{\mathsf{E}} \longrightarrow \mathscr{I}hom^{+}(\mathcal{S}ol_{\widetilde{X}}^{\mathsf{E}}(\mathscr{M}^{\mathcal{A}}), \mathcal{O}_{\widetilde{X}}^{\mathsf{E}}) \tag{6.9.2}$$

is an isomorphism.

Since the question is local and $\mathscr{M}^{\mathcal{A}}$ is locally isomorphic to a direct sum of exponential D-modules $\big(\mathscr{E}_{X\setminus D|X}^{\varphi}\big)^{\mathcal{A}}$ with $\varphi \in \mathcal{O}_X(*D)$, we may assume that $\mathscr{M}^{\mathcal{A}} = \big(\mathscr{E}_{X\setminus D|X}^{\varphi}\big)^{\mathcal{A}}$. Since (6.9.2) is the image of (6.9.1) by the functor $\mathsf{E}\varpi^{!}$, it is enough to show that (6.9.1) is an isomorphism when $\mathscr{M} = \mathscr{E}_{X\setminus D|X}^{\varphi}$.

In this case, Corollary 6.8.9 implies that

$$\mathcal{S}ol_X^{\mathsf{E}}(\mathscr{M}) \simeq \mathbb{C}_X^{\mathsf{E}} \overset{+}{\otimes} \mathbb{C}_{\{t=-\operatorname{Re}\varphi\}},$$

and we can easily see that (6.9.1) is an isomorphism. □

6.10 Integral transform with irregular kernels

Theorem 6.10.1. *Let X be a complex manifold and let $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_X)$ and $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}(\mathscr{D}_X)$. There is a natural isomorphism*

$$\mathcal{DR}^{\mathsf{E}}_X(\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathscr{M}) \simeq \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})).$$

Proof. By Theorem 6.7.1, we have an isomorphism in $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathscr{D}_{\mathrm{X}})$:

$$\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathsf{E}}_X \xrightarrow{\sim} \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathcal{O}^{\mathsf{E}}_X). \tag{6.10.1}$$

Let us apply $\mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \bullet$ to both sides of (6.10.1). We have

$$\begin{aligned} \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} (\mathscr{L} \overset{\mathrm{D}}{\otimes} \mathcal{O}^{\mathsf{E}}_X) &\simeq (\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L})^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathsf{E}}_X \\ &\simeq \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}), \end{aligned}$$

and

$$\begin{aligned} \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathcal{O}^{\mathsf{E}}_X) &\simeq \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathscr{M}^{\mathrm{r}} \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathcal{O}^{\mathsf{E}}_X) \\ &\simeq \mathscr{I}hom^+(\mathcal{S}ol^{\mathsf{E}}_X(\mathscr{L}), \mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})). \end{aligned}$$

(We do not give the proof of the first isomorphism and refer to [KS14, Lem. 3.1.2].) Q.E.D.

Consider morphisms of complex manifolds

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Notation 6.10.2. (i) For $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_S)$ recall that one sets

$$\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L} := \mathrm{D}g_*(\mathrm{D}f^*\mathscr{M} \overset{\mathrm{D}}{\otimes} \mathscr{L}).$$

(ii) For $L \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{S}})$, $F \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{X}})$ and $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_{\mathrm{Y}})$ one sets

$$\begin{aligned} &L \overset{\mathrm{E}}{\circ} G := \mathsf{E}f_{!!}(L \overset{+}{\otimes} \mathsf{E}g^{-1}G), \\ &\Phi^{\mathrm{E}}_L(G) = L \overset{\mathrm{E}}{\circ} G, \quad \Psi^{\mathrm{E}}_L(F) = \mathsf{E}g_*\mathscr{I}hom^+(L, \mathsf{E}f^{\,!}F). \end{aligned} \tag{6.10.2}$$

Note that we have a pair of adjoint functors

$$\Phi^{\mathrm{E}}_L : \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_Y) \rightleftarrows \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_X) : \Psi^{\mathrm{E}}_L . \tag{6.10.3}$$

Theorem 6.10.3. *Let* $\mathscr{M} \in \mathsf{D}^{\mathrm{b}}_{\text{q-good}}(\mathscr{D}_X)$, $\mathscr{L} \in \mathsf{D}^{\mathrm{b}}_{\text{g-hol}}(\mathscr{D}_S) := \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_S) \cap \mathsf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathscr{D}_S)$ *and let* $L := \mathcal{S}ol^{\mathsf{E}}_S(\mathscr{L})$. *Assume that* $f^{-1}\operatorname{supp}(\mathscr{M}) \cap \operatorname{supp}(\mathscr{L})$ *is proper over* Y. *Then there is a natural isomorphism in* $\mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_Y)$:

$$\Psi^{\mathrm{E}}_L\big(\mathcal{DR}^{\mathsf{E}}_X(\mathscr{M})\big)\,[d_X - d_S] \simeq \mathcal{DR}^{\mathsf{E}}_Y(\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L}).$$

Proof. The proof goes as in the regular case (Theorem 3.4.2) by using Theorems 6.3.2 and 6.10.1. Q.E.D.

Corollary 6.10.4. *In the situation of* Theorem 6.10.3*, let* $G \in \mathrm{E}^{\mathrm{b}}(\mathrm{I}\mathbb{C}_Y)$. *Then there is a natural isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbb{C})$

$$\begin{aligned} &\mathrm{RHom}^{\mathrm{E}}(L \overset{\mathrm{E}}{\circ} G, \Omega^{\mathsf{E}}_X \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_X} \mathscr{M})\,[d_X - d_S] \\ &\qquad\qquad \simeq \mathrm{RHom}^{\mathrm{E}}(G, \Omega^{\mathsf{E}}_Y \overset{\mathrm{L}}{\otimes}_{\mathscr{D}_Y} (\mathscr{M} \overset{\mathrm{D}}{\circ} \mathscr{L})). \end{aligned}$$

Proof. This follows from Theorem 6.10.3 and the adjunction (6.10.3). Q.E.D.

Note that Corollary 6.10.4 is a generalisation of [KS01, Th.7.4.12] to not necessarily regular holonomic $\mathscr{D}$-modules.

Remark 6.10.5. In [KS14] we apply Theorem 6.10.3 to the study of the Laplace transform which allows us to generalize results of [KS97] to non conic situations.

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