Deformation approach to quantisation of field models

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Juillet 2015

IHES/M/15/13

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Talk given at XXIII International conference on Integrable systems & Quantum symmetries (23–27 *June* 2015, *CVUT Prague, Czech Republic*)

Abstract. Associativity-preserving deformation quantisation $\times \mapsto \star_\hbar$ via Kontsevich's summation over weighted graphs is lifted from the algebras of functions on finite-dimensional Poisson manifolds to the algebras of local functionals within the variational Poisson geometry of gauge fields over the space-time.

Summary. Starting from a Poisson bi-vector $\mathcal{P}$ on a given finite-dimensional Poisson manifold $(N, \mathcal{P})$, Kontsevich's graph summation formula[40] yields the explicit deformation $\times \mapsto \star_\hbar$ of commutative product $\times$ in the algebra $C^\infty(N)$ of smooth functions. The new operation $\star_\hbar$ on the space $C^\infty(N)[[\hbar]]$ of power series is specified by the Poisson structure on N: namely, $f \star_\hbar g = f \times g + \text{const} \cdot \hbar \{f, g\}_\mathcal{P} + o(\hbar)$ such that all the bi-differential terms at higher powers of the formal parameter $\hbar$ are completely determined by the Poisson bracket $\{\,,\,\}_\mathcal{P}$ in the leading deformation term. (In the context of fields and strings, the constant is set to $\boldsymbol{i}/2$ so that the parameter $\hbar$ is the usual Planck constant.) The deformed product $\star_\hbar$ is no longer commutative if $\mathcal{P} \neq 0$ but it always stays associative, $\bigl(f \star_\hbar g\bigr) \star_\hbar h \doteq f \star_\hbar \bigl(g \star_\hbar h\bigr)$ all $f,\ g,\ h \in C^\infty(N)[[\hbar]]$, by virtue of bi-vector's property $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ to be Poisson.

In this paper we extend the Poisson set-up and graph summation technique in the deformation $\times \mapsto \star_\hbar$ to the jet-space (super)geometry of N-valued fields $\phi \in \Gamma(\pi)$ over a base manifold M in their bundle π and, secondly, of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$ that encode the Poisson brackets $\{\,,\,\}_{\boldsymbol{\mathcal{P}}}$ on the space of local functionals taking $\Gamma(\pi) \to \Bbbk$. We explain why an extension of Kontsevich's graph technique[40] is possible and how it is done by using the geometry of iterated variations[32]. For instance, we derive the variational analogue of associative Moyal's $\star$-product, $f \star g = (f)\ \exp\bigl(\overleftarrow{\partial_i} \cdot \hbar \mathcal{P}^{ij} \cdot \overrightarrow{\partial_j}\bigr)\ (g)$, in the case when the coefficients $\mathcal{P}^{ij}$ of bi-vector $\mathcal{P}$ are constant (hence the identity $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ holds trivially). By using several well-known examples of variational Poisson bi-vectors $\boldsymbol{\mathcal{P}}$, we illustrate the construction of each bi-differential term in $\star_\hbar$ in the general case, i.e., for Hamiltonian total differential operators with coefficients depending on the fields ϕ and their derivatives; we then verify that the noncommutative quantised product $\star_\hbar$ is associative by virtue of $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] = 0$. We

Date: July 14, 2015.

1991 *Mathematics Subject Classification.* 53D55, 58E30, 81S10; secondary 53D17, 58Z05, 70S20.

Key words and phrases. Deformation quantisation, star-product, gauge fields, variational Poisson bracket, associativity.

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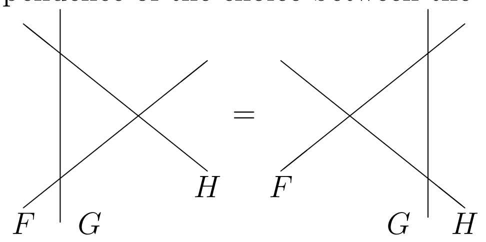
conclude that the existing instruments for calculation of variational Poisson structures do in fact specify points in the moduli spaces of deformation quantisations for field theory models.

This text is written for both mathematicians and physicists. A mathematician will discover here a realisation of renowned algebraic result[40] — now, in the geometry of (gauge) fields that extends the Poisson set-up for which Kontsevich's construction was conceived originally. As usual, solution of the problem at hand in a wider setting sheds new light on the classical objects, structures, and their logic. At the same time, a physicist will become acquainted with a working tool for regular quantisation of field models; we phrase this technique in terms of a step-by-step algorithm. As a by-product, the intrinsic self-regularisation in another, Batalin–Vilkovisky's approach to quantisation of gauge systems was achieved along the way to the new strategy[32]. Mathematical physicists would read this paper as a possible next step towards the proper language to describe fundamental interactions in the quantised space-time[cf. [35]].

Contents

1. Introduction

1.1. The aim of this paper is to develop a tool for regular deformation quantisation $(\mathcal{A},\times)\mapsto\bigl(\mathcal{A}[[\hbar]],\star_\hbar\bigr)$ of field theory models. The commutative associative unital algebras $(\mathcal{A},\times)$ of local functionals (or *observables*) equipped with variational Poisson structures $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ are the input data of quantisation algorithm; the noncommutative associative products $\star_\hbar$ in unital algebras $\mathcal{A}[[\hbar]]$ are the output. The associativity of (quantum) functionals' multiplications is preserved in the course of deformation $\times\mapsto\star_\hbar$; this is important, letting the mathematical apparatus work in the models of scattering that exhibits its output's independence of the choice between the two scenarios,



Established in experiment, this property is rendered by the *triangle equation*,[64]

$$(F\star_\hbar G)\star_\hbar H = F\star_\hbar (G\star_\hbar H);$$

a variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ marks the interaction channel at each star-product $\star_\hbar$ in the above equality's left- and right-hand sides.

Its concept going back to Weyl–Groenewold[28] and Moyal[52], the problem of associativity-preserving deformation quantisation $\times\mapsto\star_\hbar$ of commutative product $\times$ in the algebras $C^\infty(N)$ of functions on smooth finite-dimensional symplectic manifolds (N^n,ω) was considered by Bayen–Flato–Frønsdal–Lichnerowicz–Sternheimer[4]. Further progress within the symplectic picture was made by De Wilde–Lecomte[17] and independently, Fedosov[22]. To tackle the deformation quantisation problem in the case of finite-dimensional Poisson geometries $\bigl(N^n,\{\cdot,\cdot\}_{\mathcal{P}}\bigr)$ – that is, in absence of Darboux lemma which guarantees the existence of canonical coordinates on a chart $U_\alpha\subseteq N^n$ in the symplectic case – Kontsevich developed the graph complex technique[43, 44]; it yields the explicit construction of each term in the series $\times\mapsto\star_\hbar$, see [40, 42]. We recall this approach and analyse some of its features in section 2 below.[1] Specifically, the sum over a suitable set of weighted oriented graphs (see §2.1 below) determines on $N^n\ni\boldsymbol{u}$ a star-product $\star_\hbar$ which (i) contains a given Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ in the deformation's leading term at $\hbar^1$ and which (ii) is associative modulo the Jacobi identity for $\{\cdot,\cdot\}_{\mathcal{P}}$,

$$\bigl(f\star_\hbar g\bigr)\star_\hbar h - f\star_\hbar\bigl(g\star_\hbar h\bigr) = \Diamond\,\bigl(\{\{f,g\}_{\mathcal{P}},h\}_{\mathcal{P}}+\text{c. p.}\bigr),\qquad f,g,h\in C^\infty(N^n). \quad (1)$$

The key distinction between associativity mechanisms for the Darboux-symplectic and Poisson cases is a possibility of the star-product's self-action on non-constant coefficients $P^{ij}(\boldsymbol{u})$ of the bracket $\{\cdot,\cdot\}_{\mathcal{P}}$. It is readily seen that whenever those coefficients are constant, the graph summation formula for $\star_\hbar$ then yields the Moyal star-product[52],

$$\star_\hbar\Big|_{\boldsymbol{u}=(u^1,\ldots,u^n)} = \exp\Bigl(\frac{\overleftarrow{\partial}}{\partial v^i}\Big|_{v^i=u^i}\cdot\hbar P^{ij}(\boldsymbol{u})\cdot\frac{\overrightarrow{\partial}}{\partial w^j}\Big|_{w^j=u^j}\Bigr). \quad (2)$$

[1]The functionality of Kontsevich's algorithm for deformation quantisation $\times\mapsto\star_\hbar$ relies on the Formality statement[40], which in turn refers to universal facts about all associative algebras[41, 61].

This formula's geometric extension to the infinite-dimensional space of N^n-valued physical fields over a given m-dimensional manifold M^m will be obtained in §3.3.4, see Eq. (13) on p. 26 below.

Valid in finite-dimensional set-up, the result of [40] was at once known to be not working in the infinite dimension. For instance, it could not be applied to field-theoretic models — should one attempt to assemble such geometries via the limiting procedure by first taking infinitely many "zero-dimensional field theories" over the discrete topological space $M^0 = \bigcup_{\boldsymbol{x}\in M^m}\{\boldsymbol{x}\}$. In fact, not only the geometry of N^n-valued physical fields as n internal degrees of freedom attached at every base point $\boldsymbol{x} \in M^m$ is infinite-dimensional if $m > 0$ but also the mathematical apparatus to encode it becomes substantially more complex, cf. [32, 35]. Many elements of differential calculus were equally well known to be fragile in the course of transition from finite-dimensional geometry of N^n to the infinite jet spaces $J^\infty(\pi)$ for the bundles π of N^n-valued fields over M^m, or to the infinite jet spaces of maps $J^\infty(M^m \to N^n)$, cf. [39] vs [62] and [36] vs [2] or contrast [32] vs [59], [35] vs [45], and [40] vs this paper.[2]

1.2. The task of regular quantisation of (gauge) field models is one of the main problems in mathematical physics. Apart from the practised regularisation schemes within the second quantisation of fields[7] in quantum field theory[8] (QFT), there exist other working alternatives to the concept of deformation quantisation for gauge systems — not necessarily requiring that the physical fields themselves be quantised in advance. Let us name several such techniques.[3]

The BRST method[5], furthered by Batalin–Vilkovisky (BV) in [3], captures the gauge degrees of freedom in Euler–Lagrange's equations of motion for the physical fields $\phi \in \Gamma(\pi)$ — by building some auxiliary vector (super)bundles π_{BV} over the bundle π; the (anti)fields and (anti)ghosts are new fibre coordinates in π_{BV}. The action functional $S\colon \Gamma(\pi) \to \Bbbk$ of a gauge model at hand is extended to the full BV-action functional $\boldsymbol{S}_{\text{BV}}\colon \Gamma(\pi_{\text{BV}}) \to \Bbbk$, its density incorporating the entire BV-zoo so that the classical master-equation $[\![\boldsymbol{S}_{\text{BV}}, \boldsymbol{S}_{\text{BV}}]\!] = 0$ is satisfied. All the BV-objects from that supergeometry are then tensored with spaces $\Bbbk[[\hbar]]$ of formal power series in the Planck constant $\hbar$ so that finally, the quantum BV-action $\boldsymbol{S}^\hbar_{\text{BV}}$ satisfies the quantum master-equation $-\boldsymbol{i}\hbar\,\Delta\boldsymbol{S}^\hbar_{\text{BV}} + \frac{1}{2}[\![\boldsymbol{S}^\hbar_{\text{BV}}, \boldsymbol{S}^\hbar_{\text{BV}}]\!] = 0$. This equation is derived from the normalisation $\langle 1\rangle = 1$ of translation-invariant functional measure in the Feynman path integral with standard weight factor $\exp\bigl(\boldsymbol{i}\boldsymbol{S}^\hbar_{\text{BV}}/\hbar\bigr)$. In the context of present paper it is important that the formalism's *quantum observables* $\mathcal{O}^\hbar\colon \Gamma(\pi_{\text{BV}}) \to \Bbbk[[\hbar]]$ are local functionals

[2]The geometric *correspondence principle* is an heuristic, subject to revision set of rules which track the various patterns and hint analogies between the algebraic structures on finite-dimensional Poisson manifolds $\bigl(N^n, \{\cdot,\cdot\}_{\mathcal{P}}\bigr)$ and such structures' namesakes over the infinite jet spaces $J^\infty(\pi)$, see [46]. By making such correspondence work in the reverse direction, we shall not only construct the star-products $\star_\hbar$ of local functionals $\Gamma(\pi) \to \Bbbk$ defined at sections ϕ of the bundles π but also point out their well-defined reductions in the classical set-up (see §2.3).

[3]The concept of *quantisation* in field models is not identically the same as the idea of making (only) the fields *noncommutative* (e.g., see [12] with applications to the $U(1) \times SU(2) \times SU(3)$ Standard Model of particle physics). For physical fields over a given commutative space-time manifold M^m, the presence or absence of requirement that the calculus of observables be (graded-)commutative is one of the quantisation features still not an axiom.

which appear through the virtual shifts $\boldsymbol{S}^{\hbar}_{\text{BV}} + \epsilon\mathcal{O}^{\hbar} + \bar{o}(\epsilon)$ of quantum BV-action $\boldsymbol{S}^{\hbar}_{\text{BV}}$, hence satisfying the cocycle condition $\boldsymbol{\Omega}^{\hbar}(\mathcal{O}^{\hbar}) = -\boldsymbol{i}\hbar\,\Delta\mathcal{O}^{\hbar} + [\![\boldsymbol{S}^{\hbar}_{\text{BV}}, \mathcal{O}^{\hbar}]\!] = 0$ in the quantum BV-cohomology (see [26, 29] and [32, 36]). Summarising, it is the integral functionals – and local functionals as sums of such building blocks' associative products – which carry the information about gauge model's objects and their interactions.

The problem of intrinsic regularisation for the BV-Laplacian Δ, that structure parent to variational Schouten bracket $[\![\cdot,\cdot]\!]$, was long standing in the mathematical physics literature (see [29, §15.1] or [27] and [13, 63]); it was solved in [32]. With this paper we explore how far one advances and which other results, beyond the intrinsic regularisation of Batalin–Vilkovisky formalism, have become attainable by using the geometry of iterated variations.[4]

Let us recall that the Batalin–Vilkovisky formalism is a working instrument in Cattaneo–Felder's approach[10] to Kontsevich's deformation quantisation on $\bigl(N^n, \{\cdot,\cdot\}_{\mathcal{P}}\bigr)$ — now, via Feynman path integral calculation of correlation functions in an elegant Poisson σ-model[30, 58].

The flavour of scattering diagrams can be felt in many other methods to quantise the variational Poisson dynamics. Namely, V. G. Kac *et al.*[15, 16] employ the Poisson vertex algebras (specific to $\dim(M^m) = 1$). Earlier results belong to Pogrebkov[55]. The school of B. A. Dubrovin[20, 21] does the dispersion expansion (converting all the objects to formal power series in the scaling parameter). Another direction – at the level of Lie algebras underlying the integrability – is the BRST-quantisation procedure applied to variational Poisson structures; this direction of research is based on the works of Reshetikhin *et al.*[23, 56].

The issue of BRST- and BV-quantisation of zero-curvature representations for kinematically integrable – via the inverse scattering transform (IST) – field models, including KdV-type Hamiltonian hierarchies, was approached in [36]. In fact, the realisation of Lie-algebraic structures over such integrable systems in terms of homological vector fields $\boldsymbol{Q}$ and classical master-action functional $\boldsymbol{S}$ patterns upon the concept of [2], where similar σ-models and Maurer–Cartan's flat connection equation $\mathcal{E}_{\text{MC}}$ were considered for finite-dimensional, non-variational underlying geometries. Euler–Lagrange at $m = 3$, the zero-curvature equation $\mathcal{E}_{\text{MC}}$ for Lie algebra-valued connections itself is the starting point for construction of BV-superbundle over the infinite prolongation $\mathcal{E}^{\infty}_{\text{MC}}$, the BV-quantisation to follow. At arbitrary m where the Maurer–Cartan system $\mathcal{E}_{\text{MC}}$ is not necessarily Euler–Lagrange, the BV-quantisation method works formally for the classical master-equation, which the model's BV-action $\boldsymbol{S}$ does satisfy.[5]

[4]In fact, the properties of parity-odd structures Δ and $[\![\cdot,\cdot]\!]$ do not exploit the $\mathbb{Z}_2$-graded commutativity of Batalin–Vilkovisky geometry; for they are still valid[35] within the variational formal noncommutative symplectic geometry, which is a quantum field-theoretic extension of Kontsevich's calculus of cyclic words[45]. This time, the graded-commutative but not associative language describes the set-up of integral and local functionals that take paths along a (quasi)crystal tiling $M^m = \overline{\bigcup_{i\in\mathcal{I}} \Delta^m_i}$ to the unital algebras of cyclic words describing closed walks; we refer to [35] for details.

[5]On the one hand, it remains an intriguing open question how the new deformation parameters, arising from the BV-quantisation of Maurer–Cartan's zero-curvature equations $\mathcal{E}_{\text{MC}}$, correspond to the usual Planck constant $\hbar$ that shows up in the output of BV-quantisation of the underlying kinematically

1.3. The rest of this paper is structured as follows. In the next section we review and examine (the breadth of applicability for) Kontsevich's concept of deformation quantisation in its original phrasing[40] for finite-dimensional Poisson manifolds $\bigl(N^n, \{\cdot,\cdot\}_{\mathcal{P}}\bigr)$: the deformation $\times \mapsto \star_\hbar$ is approached via summation over a class of weighted oriented graphs. In section 3 we proceed with finite-dimensional fibre bundles π – in particular, affine bundles – over smooth manifolds M^m; the deformations' infinitesimal parts are now specified by the variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ for algebras of local functionals taking $\Gamma(\pi) \to \Bbbk$. To extend the deformation quantisation technique to such set-up, we let the elements of Gel'fand's calculus of singular linear integral operators and geometry of iterated variations enter the game in §3.2. Each graph in Kontsevich's summation formula now encodes a local variational differential operator. We then inspect in §3.3 the geometric mechanism through which the new star-products are associative; Taking Weyl–Groenewold–Moyal's star-product (2) as prototype, we illustrate the algorithm of variational deformation quantisation by presenting in §3.3.4 this structure's proper analogue $\star_\hbar$ for the class of Hamiltonian total differential operators $\bigl\| P^{ij}_\tau(\boldsymbol{x})\, \mathrm{d}^{|\tau|}/\mathrm{d}\boldsymbol{x}^\tau \bigr\|^{j=1\ldots n}_{i=1\ldots n}$ whose coefficients do not depend on physical field portraits $\boldsymbol{u} = \phi(\boldsymbol{x})$. In §3.3.5 we indicate a channel for the associativity to leak at order $O(\hbar^{\geqslant 3})$. This effect was altogether suppressed in the seminal picture of [40]; originally invisible, it can appear only in the framework of fibre bundles π over the base manifold M^m of positive dimension m. In the concluding section (see §4.2 on p. 33) we address the deformation quantisation technique in physical terms.

In retrospect, our argument reveals why the Virasoro and other W-algebra structures[6, 23] do so often arise from the deformation markers $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ – via the Fourier transform, then paving a way to Yangians[9, 51] – in the various schemes for quantisation of field models.

Remark 1.1 (on Feynman path integral approach to the Kontsevich quantisation formula). Cattaneo and Felder[10] recognised Kontsevich's graph summation formula $\times \mapsto \star_\hbar$ as Feynman path integral calculation of correlation functions in a topological bosonic open string model by Ikeda[30, 58]; let us remember that an appropriate Batalin–Vilkovisky build-up over that gauge model on a disc with non-empty boundary is attached to but still does not embed into the Poisson geometry to-quantise. Not only do all the graphs Γ re-appear in the intermediate diagrams which combine the copies of Poisson structure $\{\cdot,\cdot\}_{\mathcal{P}}$ but all the respective weights $w(\Gamma) \in \mathbb{R}$ are also reproduced (see [10] for details and further comments on a path integral phrasing of the quasi-isomorphism in Formality statement[40, 41, 61]).

We shall discuss the variational, field-theoretic extension of Ikeda's σ-model in a subsequent publication. In the meantime it can be argued that the mathematics of open strings in it *is* the mechanism under which field geometries get quantised.[6]

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integrable Euler–Lagrange system for the physical fields $\phi \in \Gamma(\pi)$. On the other hand, the BV-deformation concept in [36] can be parallelled with the quantum IST method by Drinfel'd[18] and with the construction of quantum groups[49].

[6]Kontsevich ponders in [40] whether the deformation quantisation is natural for quantum mechanics — topological open string theory seeming more relavant at that time. Our present argument hints that these two paradigms could in fact be two non-excluding realisations of the one principle.

2. Kontsevich's deformation quantisation on finite-dimensional Poisson manifolds

In this section we recall the graph technique[40] for deformation quantisation $\times \mapsto \star_\hbar$ on finite-dimensional smooth Poisson manifolds $\bigl(N^n, \{\cdot,\cdot\}_{\mathcal{P}}\bigr)$.

2.1. Let $U_\alpha \subseteq N^n$ be a chart. Suppose first that the space $C^\infty(U_\alpha)[[\hbar]] \ni f, g$ of power series in an indeterminate $\hbar$ is equipped with a $\Bbbk[[\hbar]]$-linear star-product $\star_\hbar$ which deforms the usual, $\Bbbk[[\hbar]]$-linear commutative associative product $\times$, defined pointwise over U_α, via[7] $f \star_\hbar g = f \times g + \hbar B_1(f,g) + \bar{o}(\hbar)$. Assume further that the deformed product $\star_\hbar$ remains associative up to $\bar{o}(\hbar^2)$ for all arguments from $C^\infty(U_\alpha) \hookrightarrow C^\infty(U_\alpha)[[\hbar]]$: whatever be $f, g, h \in C^\infty(U_\alpha)$, one has that

$$(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h) = \bar{o}(\hbar^2).$$

By definition, put

$$\{f,g\}_\star := \frac{f \star_\hbar g - g \star_\hbar f}{\hbar}\bigg|_{\hbar:=0}, \qquad f, g \in C^\infty(U_\alpha).$$

It is readily seen that the bi-linear skew-symmetric bi-derivation $\{\cdot,\cdot\}_\star$ satisfies the Jacobi identity,

$$\{\{f,g\}_\star, h\}_\star + \{\{g,h\}_\star, f\}_\star + \{\{h,f\}_\star, g\}_\star = 0, \qquad f, g, h \in C^\infty(U_\alpha),$$

that is, acting on each of its arguments by the Leibniz rule, the structure $\{\cdot,\cdot\}_\star$ is a Poisson bracket on $U_\alpha \subseteq N^n$. This tells us that the left-hand side of Jacobi's identity for $\{\cdot,\cdot\}_\star$ is an obstruction to the $\bar{o}(\hbar^2)$-associativity of star-product $\star_\hbar$ in $C^\infty(U_\alpha)[[\hbar]]$.

Example 2.1. Let $(p,q) \in \Bbbk^2$ and f, g be functions in the variables p and q. Consider the associative star-product

$$(f \star g)(p,q;\hbar) = f\big|_{(p,q)} \exp\Bigl(\frac{\overleftarrow{\partial}}{\partial p} \cdot \hbar \cdot \frac{\overrightarrow{\partial}}{\partial q}\Bigr) g\big|_{(p,q)}.$$

By construction, we have that

$$\{f,g\}_\star = (f)\overleftarrow{\partial}/\partial p \cdot \overrightarrow{\partial}/\partial q(g) - (g)\overleftarrow{\partial}/\partial p \cdot \overrightarrow{\partial}/\partial q(f),$$

which is the two functions' Poisson bracket referred to the canonical Darboux coordinates p and q. Because the *symmetric* part $B_1^+(\cdot,\cdot)$ can always be gauged out from the star-product $f \star g = f \times g + \hbar B_1(f,g) + \bar{o}(\hbar)$, there would remain only the skew-symmetric term in the right-hand side of the formula

$$\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} = \frac{1}{2}\Bigl(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} + \frac{\partial g}{\partial p} \cdot \frac{\partial f}{\partial q}\Bigr) + \frac{1}{2}\Bigl(\frac{\partial f}{\partial p} \cdot \frac{\partial g}{\partial q} - \frac{\partial g}{\partial p} \cdot \frac{\partial f}{\partial q}\Bigr).$$

From now on, we shall always assume at once that $B_1(\cdot,\cdot)$ is skew-symmetric.

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[7] By assumption, the leading deformation term $\hbar B_1(\cdot,\cdot)$ in $\star_\hbar$ is a bi-derivation, hence same are its symmetric and skew-symmetric parts, $B_1^+(f,g) = \frac{1}{2}(B_1(f,g) + B_1(g,f))$ and $B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$, respectively. The symmetric part B_1^+ of a given B_1 might not be vanishing identically *ab initio* but it then can be trivialised – at the expense of using the gauge transformations $f \mapsto f + \hbar D_1(f) + \bar{o}(\hbar)$, $g \mapsto g + \hbar D_1(g) + \bar{o}(\hbar)$ of its arguments (see [40] for further analysis of the properties which the 2-cocycle B_1 does have in Hochschild's cohomology complex, and for the rôle and use of that complex in the original proof of Formality statement relating the spaces of polyvectors and polydifferential operators on N^n).

Now let us consider this geometric set-up from a diametrally opposite perspective. Namely, suppose that on N^n, a Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ is given in advance. Can then the commutative associative multiplication $\times$ in the algebra $C^\infty(U_\alpha) \ni f, g$ be deformed to an associative (possibly, modulo $\bar{o}(\hbar^2)$) star-product $\star_\hbar$ such that the formal power series $f \star_\hbar g = f \times g + \hbar \cdot \{f,g\}_{\mathcal{P}} + \ldots$ is well defined on the coordinate domain $U_\alpha \subseteq \mathbb{R}^n$? More specifically, the bi-linear, not necessarily commutative star-product $\star_\hbar = \times + \hbar\{\cdot,\cdot\}_{\mathcal{P}} + \sum_{k>1} \hbar^k B_k(\cdot,\cdot)$ must satisfy the four axioms:

(1) it is associative,

$$(f \star_\hbar g) \star_\hbar h \doteq f \star_\hbar (g \star_\hbar h) \quad \bmod \{\{f,g\}_{\mathcal{P}}, h\}_{\mathcal{P}} + \text{c. p.} = 0, \qquad f,g,h \in C^\infty(U_\alpha), \quad (1')$$

modulo the property of bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ on U_α to be Poisson;

(2) the unit function $1 \in C^\infty(U_\alpha)$ remains the neutral element for $\star_\hbar$; whatever $f \in C^\infty(U_\alpha)$, one has that $f \star_\hbar 1 = f = 1 \star_\hbar f$;

(3) each term $B_k(\cdot,\cdot)$, including the skew-symmetric Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}} = B_1(\cdot,\cdot)$ to start with at $\hbar$, is a bi-linear differential operator of bounded order;

(4) the product $\star_\hbar$ is (let to be) $\Bbbk[[\hbar]]$-linear over $C^\infty(U_\alpha)[[\hbar]]$.

The answer to the above question of star-product's existence is affirmative. Kontsevich proves[40] that this is always possible; his proof is constructive (cf. [10] and [41, 61]).

The graph technique is a convenient way to encode the bi-differential terms $B_k(\cdot,\cdot)$ in perturbation series $\star_\hbar$; simultaneously, it is a key to the logic of pictures in the graph complex-based proof of Formality statement (see [40, 42, 43, 44]). By construction, every term in $B_k(f,g)$ at $\hbar^k$, $k \geqslant 0$ is encoded by a graph Γ with $k+2$ vertices, of which two vertices contain the respective arguments f and g and each of the remaining k vertices is a source for two oriented edges (in total, there are k such "forks" with $2k$ arrows in every such graph Γ); neither tadpoles nor multiple edges are permitted (cf. [10]). The two edges issued from a vertex are *ordered*, so that the precedent and antecedent edges correspond to the first and second indexes of the Poisson bi-vectors $\mathcal{P}$.

To encode multi-vectors in a standard way, consider the parity-odd neighbour $\Pi T^* N^n$ of cotangent bundle to the manifold N^n and denote by $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ the n-tuple of $\mathbb{Z}_2$-parity odd fibre coordinates over a chart $U_\alpha \subseteq N^n$. Whenever $\{u^i, u^j\}_{\mathcal{P}}(\boldsymbol{u}) = P^{ij}(\boldsymbol{u})$ at $\boldsymbol{u} = (u^1, \ldots, u^n) \in U_\alpha$, construct the bi-vector $\mathcal{P} = \frac{1}{2}\langle \xi_i P^{ij}\big|_{\boldsymbol{u}} \xi_j \rangle \in \Gamma(\bigwedge^2 TU_\alpha)$; we recall that Poisson bi-vectors satisfy the classical master-equation $[\![\mathcal{P},\mathcal{P}]\!] = 0$, see p. 11.

Now, install a copy of the given Poisson bi-vector $\mathcal{P}$ at each of the k forks' tops, and superscrbe a summation index running from 1 to $n = \dim N^n$ at every edge of the graph Γ; the precedence-antecedence relation between the edges associates the indexes they carry with the respective indexes in bi-vector's copy.

Convention. The correspondence between indexed oriented edges and analytic expressions occurring in $B_k(\cdot,\cdot)$ is established in Fig. 1 (cf. footnote 10 on p. 9). The expressions corresponding to different connectivity components of a graph Γ are multiplied by using the original product $\times$.

Because other arrows may stick into the vertices connected by the edge $\xrightarrow{i}$ in Fig. 1, the objects Obj_{tail} and Obj_{head} contained there can be derivatives (with respect to u^α's) of the bi-vector $\mathcal{P}$ or, specifically to Obj_{head} but never possible to Obj_{tail}, star-product's arguments f and g. On the same grounds, because there is another arrow issued from

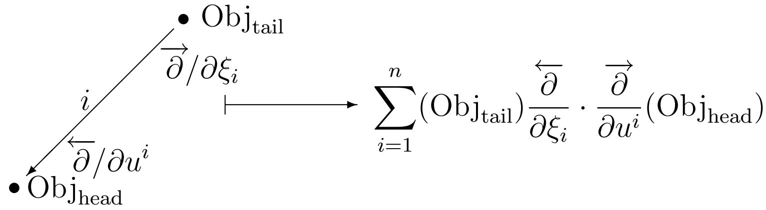


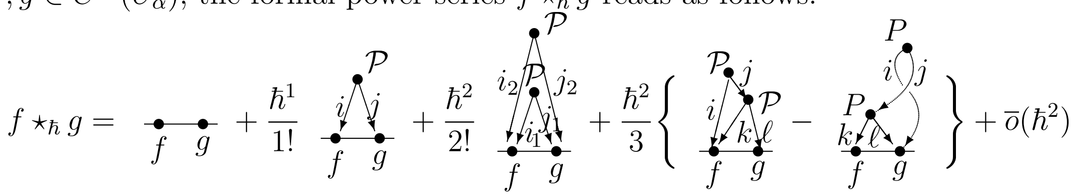
FIGURE 1

the vertex with Obj_{tail}, the formula encoded by a graph Γ does in fact not depend on any of the auxiliary, parity-odd variables ξ_j.

Every graph is accompained with its weight $w(\Gamma) \in \mathbb{R}$; these numbers are obtained by calculating certain explicitly given integrals over configuration spaces of k points – in fact, the graphs' vertices containing $\mathcal{P}$ – on the Lobachevsky plane (in its Poincaré model in the upper half-plane), see [40]. Cattaneo and Felder[10] recall how the graphs (not only these; for tadpoles are also admissible) and their weights $w(\Gamma)$ arise through Feynman diagrams and path integral calculations in a known Poisson σ-model[30, 58].

Let us remember that such weights $w(\Gamma)$ *exist*, providing a solution of associativity equation ($1'$). In fact, by this paper we ponder *what* this existence result, proven by Kontsevich, does actually give at the level of local variational differential operators[8] that enlarge the bare contraction of indexes in Fig. 1.

Example 2.2. In a fixed system of local coordinates $(u^1, \ldots, u^n)$ on $U_\alpha \subseteq N^n$, for a given Poisson bi-vector $\mathcal{P}\big|_{\boldsymbol{u}} = \frac{1}{2}\langle \xi_i P^{ij}(\boldsymbol{u})\xi_j\rangle$, and for any Hamiltonian functions $f, g \in C^\infty(U_\alpha)$, the formal power series $f \star_\hbar g$ reads as follows:[9]



(3)

This sum of weighted graphs is realised by the formula[10]

$$f \star_\hbar g = f \times g + \frac{1}{1!}(f)\frac{\overleftarrow{\partial}}{\partial u^i} \cdot \hbar P^{ij} \cdot \frac{\overrightarrow{\partial}}{\partial u^j}(g) + \frac{1}{2!}(f)\begin{bmatrix} \frac{\overleftarrow{\partial}}{\partial u^{i_1}} \cdot \hbar P^{i_1 j_1} \cdot \frac{\overrightarrow{\partial}}{\partial u^{j_1}} \\ \frac{\overleftarrow{\partial}}{\partial u^{i_2}} \cdot \hbar P^{i_2 j_2} \cdot \frac{\overrightarrow{\partial}}{\partial u^{j_2}} \end{bmatrix}(g) +$$
$$+\frac{1}{3}\left\{(f)\frac{\overleftarrow{\partial}}{\partial u^i}\frac{\overleftarrow{\partial}}{\partial u^k} \cdot \hbar P^{ij} \cdot \frac{\overrightarrow{\partial}}{\partial u^j}(\hbar P^{k\ell}) \cdot \frac{\overrightarrow{\partial}}{\partial u^\ell}(g) - (f)\frac{\overleftarrow{\partial}}{\partial u^k} \cdot (\hbar P^{k\ell})\frac{\overleftarrow{\partial}}{\partial u^j} \cdot \hbar P^{ij} \cdot \frac{\overrightarrow{\partial}}{\partial u^i}(g)\right\} + \bar{o}(\hbar^2).$$

The values of (derivatives of) both Hamiltonian functions and coefficients of the Poisson bi-vector $\mathcal{P}$ are calculated at $\boldsymbol{u} \in U_\alpha \subseteq N^n$ in the right-hand side of the above equality.

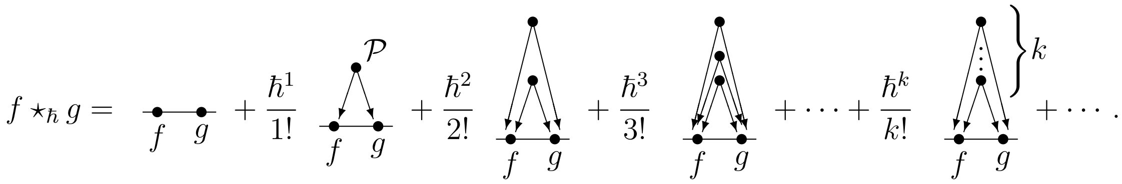
[8] As usual, to understand what the symbol "$\cdot$" really stands for in the right-hand side of correspondence in Fig. 1 means to extend Kontsevich's graph technique to field models (cf. [32, 35]).

[9] The precedence-antecedence of edges is determinated here by the ordering of indexes $i \prec j$, $i_1 \prec j_1$, $i_2 \prec j_2$, and $k \prec \ell$.

[10] Note that a graph itself suggests the easiest-to-read way to write down the respective differential operator's formula; this inscription of derivatives along the edges will be particularly convenient in the variational set-up of §3, see p. 19 below.

2.2. Where is the Jacobi identity for $\{\cdot,\cdot\}_{\mathcal{P}}$ hidden in the associator $(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h)$? To recognise it, let us consider first a well-known example where the Jacobi identity itself holds trivially.

Example 2.3 (Moyal–Weyl–Groenewold)**.** Suppose that in a fixed coordinate system $(u^1,\ldots,u^n)$ on $U_\alpha \subseteq N^n$, all coefficients P^{ij} of the Poisson bi-vector are constant. In effect, the graphs with at least one arrow ariving at a vertex containing $\mathcal{P}$ make no contribution to the star-product $\star_\hbar$. The only contributing graphs are portrayed in this figure,



These graphs are such that their weights in the power series combine it to Moyal's exponent,

$$(f \star_\hbar g)(\boldsymbol{u};\hbar) = \left[(f(\boldsymbol{u})) \exp\left(\frac{\overleftarrow{\partial}}{\partial u^i} \cdot \hbar P^{ij} \cdot \frac{\overrightarrow{\partial}}{\partial v^j} \right) (g(\boldsymbol{v})) \right]\Big|_{\boldsymbol{u}=\boldsymbol{v}}. \qquad (2)$$

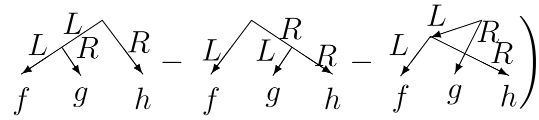
Here we accept that the use of every next copy of the bi-vector $\mathcal{P}$ creates a new pair of summation indexes. The introduction of two identical copies, $\boldsymbol{u} \in U_\alpha$ and $\boldsymbol{v} \in U_\alpha$, of the geometry in which the Hamiltonians f and g are defined reveals an idea that will be used heavily in what follows. For instance, the associativity of Moyal's star-product is established by the *a posteriori* congruence mechanism (cf. [11]). Indeed, from the identity $\big(f(u)\cdot g(v)\big|_{u=v}\big)\frac{\overleftarrow{\partial}}{\partial u} = \Big[\big(f(u)\cdot g(v)\big)\Big(\frac{\overleftarrow{\partial}}{\partial u}+\frac{\overleftarrow{\partial}}{\partial v}\Big)\Big]\Big|_{u=v}$ we infer that

$$\begin{aligned}
&\big((f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h)\big)(\boldsymbol{u};\hbar) = \\
&\quad = \Big[\Big[(f|_{\boldsymbol{u}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial u^i}\hbar P^{ij}\frac{\overrightarrow{\partial}}{\partial v^j}\Big)(g|_{\boldsymbol{v}})\Big]\Big|_{\boldsymbol{u}=\boldsymbol{v}} \exp\Big(\frac{\overleftarrow{\partial}}{\partial u^k}\hbar P^{k\ell}\frac{\overrightarrow{\partial}}{\partial w^\ell}\Big)(h|_{\boldsymbol{w}})\Big]\Big|_{\boldsymbol{u}=\boldsymbol{w}} - \\
&\quad - \Big[(f|_{\boldsymbol{u}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial u^i}\hbar P^{ij}\frac{\overrightarrow{\partial}}{\partial v^j}\Big)\Big[(g|_{\boldsymbol{v}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial v^k}\hbar P^{k\ell}\frac{\overrightarrow{\partial}}{\partial w^\ell}\Big)(h|_{\boldsymbol{w}})\Big]\Big|_{\boldsymbol{v}=\boldsymbol{w}}\Big]\Big|_{\boldsymbol{u}=\boldsymbol{v}} = \\
&= \Big[(f|_{\boldsymbol{u}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial u^i}\hbar P^{ij}\frac{\overrightarrow{\partial}}{\partial v^j}\Big)(g|_{\boldsymbol{v}}) \exp\Big(\Big(\frac{\overleftarrow{\partial}}{\partial u^k}+\frac{\overleftarrow{\partial}}{\partial v^k}\Big)\cdot \hbar P^{k\ell}\frac{\overrightarrow{\partial}}{\partial w^\ell}\Big)(h|_{\boldsymbol{w}})\Big]\Big|_{\boldsymbol{u}=\boldsymbol{v}=\boldsymbol{w}} - \\
&- \Big[(f|_{\boldsymbol{u}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial u^i}\hbar P^{ij}\cdot\Big(\frac{\overrightarrow{\partial}}{\partial v^j}+\frac{\overrightarrow{\partial}}{\partial w^j}\Big)\Big)\Big[(g|_{\boldsymbol{v}}) \exp\Big(\frac{\overleftarrow{\partial}}{\partial v^k}\hbar P^{k\ell}\frac{\overrightarrow{\partial}}{\partial w^\ell}\Big)(h|_{\boldsymbol{w}})\Big]\Big|_{\boldsymbol{u}=\boldsymbol{v}=\boldsymbol{w}} \equiv 0,
\end{aligned}$$

which is due to the Baker–Campbell–Hausdorff formula for the exponent of sums of *commuting* derivatives, and by having indexes relabelled.

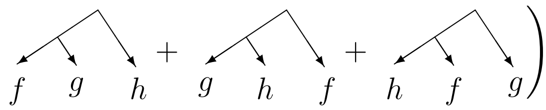
On the other hand, whenever the coefficients $P^{ij}(\boldsymbol{u})$ are not constant on the domain $U_\alpha \ni \boldsymbol{u}$, the classical master-equation[11] $[\![\mathcal{P},\mathcal{P}]\!] = 0$ is a nontrivial constraint for the bi-vector $\mathcal{P}$. Likewise, the (process of) composition of terms in the power series $(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h)$ is also nontrivial; for the content of parentheses, $f \star_\hbar g$ or $g \star_\hbar h$, becomes one of the arguments in the star-products which are taken last. Effectively, each arrow acting on these arguments runs over all the vertices in $f \star_\hbar g$ or $g \star_\hbar h$ obeying the Leibniz rule. Therefore, a graph that stands at the end of day in the associator can be reached by following several scenarios in $\star_\hbar \circ \star_\hbar$. (Viewed as an infinite triangular system of bi-linear algebraic equations for the weights $w(\Gamma)$, the associativity of $\star_\hbar$ highlights their nontrivial combinatorics, see [41].) It is instructive to draw all the weighted graphs in $(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h)$ modulo $\bar{o}(\hbar^2)$; after the cancellations, there remains only

$$\frac{2}{3} \cdot \Bigg(\quad - \quad - \quad \Bigg) + \bar{o}(\hbar^2);$$



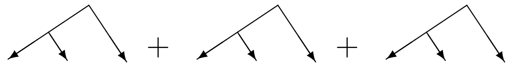
the edge relation is Left $\prec$ Right. Swapping the Left $\rightleftarrows$ Right "legs" of outer and, resp., inner Poisson bi-vectors in the second and third terms, we recover the standard expression of Jacobi identity for $\{\cdot,\cdot\}_\mathcal{P}$:

$$(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h) = \frac{2}{3} \cdot \mathbf{1} \Bigg(\quad + \quad + \quad \Bigg) + \bar{o}(\hbar^2).$$



The boldface unit in the right-hand side is a zero-order differential operator. By tracking the associativity mechanism up to $\bar{o}(\hbar^3)$ and higher-order approximations, one recovers the higher-power (with respect to $\hbar$), higher-order (with respect to $\partial/\partial u^i$) components of the linear differential operator $\diamondsuit$ in the factorisation

$$(f \star_\hbar g) \star_\hbar h - f \star_\hbar (g \star_\hbar h) = \diamondsuit \Big(\quad + \quad + \quad \Big). \tag{1}$$



Let us remember that graph equality (1) *is* the definition of operator $\diamondsuit$.

2.3. The deformation $\times \mapsto \star_\hbar$ was developed in [40, §2–6] by using a given system of coordinates $\boldsymbol{u}$ on *affine* manifolds $U_\alpha \subseteq \mathbb{R}^n$. To extend this procedure from the locally linear portrait of affine submanifolds to the generic set-up of smooth n-dimensional Poisson manifolds N^n and to quantisation $(\mathcal{A}, \times) \mapsto (\mathcal{A}[[\hbar]], \star_\hbar)$ in the algebras of smooth functions on them, Gel'fand–Kazhdan's formal geometry[24] was employed in [40, §7]. One of the features of such approach is that the objects – constructed over *infinitesimal* neighbourhoods of points $\boldsymbol{u}_\alpha \in N^n$ – often lose the property to have unique, well-defined values in $\Bbbk$ at points other than the neighbourhoods markers $\boldsymbol{u}_\alpha$. Let us now analyse why Gel'fand–Kazhdan's formal geometry is an admissible still not the only possible way to make the deformation quantisation consistent on smooth finite-dimensional Poisson manifolds.

Namely, let us understand that by construction, the quantised product $f \star_\hbar g$ of two functions $f, g \in C^\infty(N^n)$ itself is a *scalar*, that is, a power series belonging

[11] The Jacobi identity for Poisson bracket $\{\cdot,\cdot\}_\mathcal{P}$ is equivalent to the zero-value condition $[\![\mathcal{P},\mathcal{P}]\!](f,g,h) = 0$ for all Hamiltonians f, g, h; the tri-vector $[\![\mathcal{P},\mathcal{P}]\!]$ is viewed here as a tri-linear totally antisymmetric mapping and we denote by $[\![\cdot,\cdot]\!]$ the *Schouten bracket* (i.e., parity-odd Poisson bracket): in coordinates, one proves that $[\![\mathcal{P},\mathcal{P}]\!] \stackrel{\text{Th.}}{=} (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial u^i} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_i}(\mathcal{P}) - (\mathcal{P})\frac{\overleftarrow{\partial}}{\partial \xi_i} \cdot \frac{\overrightarrow{\partial}}{\partial u^i}(\mathcal{P})$.

to $C^\infty(N^n)[[\hbar]]$ and taking points $\boldsymbol{u} \in N^n$ to $\Bbbk[[\hbar]]$. A scalar at hand is produced from the objects –such as $f, g \in C^\infty(N^n)$ and $\{\cdot,\cdot\}_{\mathcal{P}} \in \Gamma(\bigwedge^2 TN)$– that were obtained by using given systems of local coordinates on the charts $U_\alpha \subseteq N^n$. Now, every reparametrisation $\boldsymbol{u} = \boldsymbol{u}(\widetilde{\boldsymbol{u}})$ is plugged into this newly built scalar — replacing all occurrences of $\boldsymbol{u}$ (or $\partial/\partial\boldsymbol{u}$) in it. In other words, already known at every point $\boldsymbol{u} \in N^n$, the scalar $(f \star_\hbar g)(\boldsymbol{u};\hbar)$ is then taken *as is*[12] — instead of being re-derived anew for every new shape of its components f, g and $\{\cdot,\cdot\}_{\mathcal{P}}$, now referred to the newly parametrised domains U_β.

This argument shows that under a change of coordinates, every polydifferential operator encoded by a Kontsevich's graph in (3) is transformed in such a way that the operator's output is a scalar.[13] For instance, such is the tensorial transformation law for coefficients of the Poisson bi-vector: the Jacobian matrices in it stem from the differentials of two Hamiltonians,

$$\begin{aligned}\{f,g\}_{\mathcal{P}}(\boldsymbol{u}) &= \bigl(f\big|_{\boldsymbol{u}}\bigr)\frac{\overleftarrow{\partial}}{\partial u^i}\cdot P^{ij}\big|_{\boldsymbol{u}}\cdot\frac{\overrightarrow{\partial}}{\partial u^j}\bigl(g\big|_{\boldsymbol{u}}\bigr) = \\ &= \Bigl[\bigl(f\big|_{\boldsymbol{u}(\widetilde{\boldsymbol{u}})}\bigr)\frac{\overleftarrow{\partial}}{\partial\widetilde{u}^\alpha}\cdot\frac{\partial\widetilde{u}^\alpha}{\partial u^i}\Bigr]\cdot P^{ij}\big|_{\boldsymbol{u}(\widetilde{\boldsymbol{u}})}\cdot\Bigl[\frac{\partial\widetilde{u}^\beta}{\partial u^j}\cdot\frac{\overrightarrow{\partial}}{\partial\widetilde{u}^\beta}\bigl(g\big|_{\boldsymbol{u}(\widetilde{\boldsymbol{u}})}\bigr)\Bigr] = \\ &= \bigl(f\big|_{\widetilde{\boldsymbol{u}}}\bigr)\frac{\overleftarrow{\partial}}{\partial\widetilde{u}^\alpha}\cdot\widetilde{P}^{\alpha\beta}\big|_{\widetilde{\boldsymbol{u}}}\cdot\frac{\overrightarrow{\partial}}{\partial\widetilde{u}^\beta}\bigl(g\big|_{\widetilde{\boldsymbol{u}}}\bigr) = \{f,g\}_{\widetilde{\mathcal{P}}}(\widetilde{\boldsymbol{u}}).\end{aligned}$$

[12]The notion of scalar function can be interpreted as follows: at each point of its domain set, a sticker is attached. The inscription upon every such sticker reads the function's value –in the target set– at that point. Note that this construction is robust with respect to arbitrary bijective transformations (i.e., not necessarily continuous ones!) of the domain set. In consequence, the definition of scalar functions as association of one and only one value with every argument is invariant under arbitrary reparametrisations of points in the function's domain.

[13]The same idea is used to define the notion of *covariant* (or *absolute*) *differential* ∇ and affine connection $\{\Gamma^k_{ij}\}$ on smooth n-dimensional real manifolds. Namely, one first postulates that in *Cartesian* coordinates on the *vector* space $\mathbb{R}^n$, the coefficients $(\nabla T)^{\vec{\imath}}_{\vec{\jmath},\alpha}$ of ∇T are produced from the respective components $T^{\vec{\imath}}_{\vec{\jmath}}$ of every tensor T by taking their derivatives $\partial/\partial u^\alpha(T^{\vec{\imath}}_{\vec{\jmath}})$ along the n coordinate directions $u^1, \ldots, u^n$. Into every other coordinate system on (a chart in) the space $\mathbb{R}^n$, the output is transformed by using the postulate that ∇T itself is a tensor (whenever T was). The arising commutative diagram determines the non-tensorial reparametrisation rule for Christoffel symbols. Next, one postulates that this rule *is* the transformation law for all given collections $\{\Gamma^k_{ij}\}$ of Christoffel symbols on a given smooth n-dimensional manifold N^n (i.e., not necessarily carrying the global vector space structure of $\mathbb{R}^n$). The commutative diagram by using which these rules were introduced guarantees that the covariant differential of a tensor is again a tensor[1].

Note that, expressed in a whatever system of local coordinates, the restriction of absolute (or covariant) differential ∇ to the space of differentiable scalar functions does coincide with the usual de Rham differential d. But this does not imply that this would still be the case for tensors of positive rank; for acting on their components, the covariant differential is transformed in a way different from d, i.e., not in the way it is transformed whenever a given tensor's coefficient itself is treated as a "function".

Likewise, Poisson brackets are reparametrised in a known way (namely, so that these binary operations' output is scalar). But this does not imply that such transformation of a stand-alone Poisson bracket would re-appear and dictate the reparametrisation at every vertex and edge of each Kontsevich's graph — whenever a graph be more complex than the Λ-picture for $\{\cdot,\cdot\}_{\mathcal{P}}$. This does also not imply that after reparametrisation, the operator's output would still be assembled by using the newly parametrised Poisson bracket $\{\cdot,\cdot\}_{\mathcal{P}}$ yet following the old scenario that associates the operators to graphs.

Still let us remember that, even if they are nominally incorporated into the new coefficients $\widetilde{P}^{\alpha\beta}(\widetilde{\boldsymbol{u}})$ of the Poisson bracket $\{\cdot,\cdot\}_{\widetilde{\mathcal{P}}}$, the Jacobians $\partial\widetilde{\boldsymbol{u}}/\partial\boldsymbol{u}$ stay near f and g on the arrows which were issued – from the vertex containing the Poisson bi-vector $\mathcal{P}$ – towards the two vertices containing the Hamiltonians. This understanding will play a crucial rôle in the next section where the Jacobians $\partial\widetilde{\boldsymbol{u}}_\tau/\partial\boldsymbol{u}_\sigma$, as soon as they are accumulated from one or several edges arriving at a graph's vertex, will be followed by the total derivatives $(-\mathrm{d}/\mathrm{d}\boldsymbol{x})^\sigma$ along the base manifold M^m. (Currently, we have $m = 0$ and we set $M^0 = \{\mathrm{pt}\}$ and $\sigma = \varnothing$.)

In conclusion, the postulate that $f \star_\hbar g$ is a scalar extends Kontsevich's deformation quantisation from affine manifolds $U_\alpha \subseteq \mathbb{R}^n$ to all smooth n-dimensional manifolds $N^n \supseteq U_\alpha$ endowed with Poisson structures $\{\cdot,\cdot\}_{\mathcal{P}}$.

In the next section we lift Kontsevich's graph technique: namely, from the quantisation $\times \mapsto \star_\hbar$ of smooth functions' product $\times$ for the finite-dimensional Poisson set-up $\bigl(N^n, \{\cdot,\cdot\}_{\mathcal{P}}\bigr)$ to the deformation of product of local functionals — in the geometry of N^n-valued physical fields over the base manifold M^m. In particular, we shall consider the *affine* bundles π with prototype fibres N^n so that reparametrisations of the gauge field variables $\boldsymbol{u}$ occur pointwise over $M^m \ni \boldsymbol{x}$. We shall analyse the construction and behaviour of local variational operators which are encoded by Kontsevich's graphs, now containing at each vertex a copy of the variational Poisson structure $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ over the jet space $J^\infty(\pi)$.

It is Gel'fand's formalism of singular linear integral operators[25] that becomes our working language.

3. Deformation quantisation $\times \mapsto \star_\hbar$ in the algebras of local functionals for field models

In this section we develop a working instrument for regular quantisation in field theory models. This procedure invokes a deformation $\times \mapsto \star_\hbar$ of the product in such models' algebras of local functionals $\Gamma(\pi) \to \Bbbk$ that take physical field configurations to numbers (here we let $\Bbbk = \mathbb{R}$). We now describe the full set-up, codify the method's implementation rules, and comment on the properties of its output. The geometric and analytic grounds – which the extension of finite-dimensional concept from §2 is based on – is a combination of the standard geometry of partial differential equations[53] and more functional-analytic Gel'fand's calculus[25] of singular linear integral operators supported on the diagonal.[14]

3.1. Field model geometry. To extend the geometry of §2 to the geometry of physical fields, let us list the ingredients of fibre bundle set-up (in particular, gauge fields are sections of the respective *affine* bundles). In retrospect, the construction in §2 can be viewed as a special case of such "bundles" over a point M^0.

Let $\bigl(M^m, \text{dvol}(\cdot)\bigr)$ be an m-dimensional oriented real manifold equipped with a volume element.[15] For simplicity, let the manifold M^m be smooth, so that C^∞-smooth reparametrisations $\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}(\boldsymbol{x})$ of local coordinates on it are allowed.

Let $\pi\colon E^{m+n} \to M^m$ be a smooth n-dimensional fibre bundle over the base M^m (in particular, affine[16] or vector bundle). Denote by $\boldsymbol{u} = (u^1, \ldots, u^n)$ the n-tuple of local coordinates on the fibre N^n for $1 \leqslant n < \infty$.

Denote by $J^\infty(\pi)$ the total space of the bundle π_∞ of infinite jets $j_\infty(\boldsymbol{s})(\cdot)$ for sections $\boldsymbol{s} \in \Gamma(\pi)$ of the bundle π over M^m; the infinite jet space $J^\infty(\pi)$ is the projective limit $\operatorname{proj\,lim}_{k\to+\infty} J^k(\pi)$ of the sequence of finite jet spaces $J^k(\pi)$,

$$M^m \xleftarrow{\pi} E^{m+n} = J^0(\pi) \leftarrow J^1(\pi) \leftarrow \ldots \leftarrow J^k(\pi) \leftarrow \ldots \leftarrow J^\infty(\pi).$$

It is readily seen – by using the chain rule – that smooth reparametrisations $\widetilde{\boldsymbol{x}}(\boldsymbol{x})$ of local coordinates on the base M^m induce linear transformations of smooth sections' derivatives up to positive order k for all $k > 0$; note that the forgetful maps $J^k(\pi) \to J^0(\pi)$ determine the vector bundle structures over the bundles $\pi\colon E^{m+n} \to M^m$ which we started with. By definition, we put $[\boldsymbol{u}]$ for an object's dependence on sections $\boldsymbol{s}$ and their derivatives up to arbitrarily large but still finite order.

Denote by $\bar{H}^m(\pi)$ the vector space of integral functionals $\Gamma(\pi) \to \Bbbk$ of form $F = \int f\bigl(\boldsymbol{x}_1, [\boldsymbol{u}]\bigr) \cdot \text{dvol}(\boldsymbol{x}_1)$ such that $F(\boldsymbol{s}) = \int_{M^m} f\bigl(\boldsymbol{x}_1, j_\infty(\boldsymbol{s})(\boldsymbol{x}_1)\bigr) \cdot \text{dvol}(\boldsymbol{x}_1)$. By brute force, introduce the multiplication $\times\colon F \otimes G \mapsto F \times G = \int f\bigl(\boldsymbol{x}_1, [\boldsymbol{u}]\bigr) \cdot \text{dvol}(\boldsymbol{x}_1) \times$

[14]Both domains are known from the vast literature; let us refer to the papers [32, 35] where they are brought together; an acquaintance with both texts strengthens the reader's positions.

[15]Not excluding the case where the volume element $\text{dvol}(\boldsymbol{x})$ can nontrivially depend on the jets $j_\infty(\phi)(\boldsymbol{x})$ of physical fields $\phi \in \Gamma(\pi)$ over the points $\boldsymbol{x} \in M^m$, let us nevertheless – for the sake of brevity – not write such admissible second argument in $\text{dvol}\bigl(\cdot, j_\infty(\phi)(\cdot)\bigr)$.

[16]The study of Lie algebra-valued gauge connections and their affine transformations under the action of gauge group refers to the fibres' affine structure; it is only the local portrait of fibres which will be exploited in what follows (cf. [35]).

$\int g(\boldsymbol{x}_2, [\boldsymbol{u}]) \cdot \mathrm{dvol}(\boldsymbol{x}_2) = \iint f(\boldsymbol{x}_1, [\boldsymbol{u}]) \times g(\boldsymbol{x}_2, [\boldsymbol{u}]) \cdot \mathrm{dvol}(\boldsymbol{x}_1)\, \mathrm{dvol}(\boldsymbol{x}_2) \colon \Gamma(\pi) \to \Bbbk$ for $G = \int g(\boldsymbol{x}_2, [\boldsymbol{u}]) \cdot \mathrm{dvol}(\boldsymbol{x}_2)$. This yields the algebra $\overline{\mathfrak{M}}^m(\pi)$ of *local functionals*.[17]

Remark 3.1. From what follows it will be readily seen that functionals F_1 and F_2 such that $F_1 - F_2 \colon \Gamma(\pi) \to 0 \in \Bbbk$ can still contribute differently to the tails of quantisation series beyond the leading deformation term (cf. footnotes 29 and 30 on p. 29). This effect of nontrivial synonyms of zero also shows up in other implementations of the geometry of iterated variations, e.g., in the Batalin–Vilkovisky formalism (see [32]).

Referring only to the fibre's local portrait but not to its global organisation, we introduce the $\mathbb{Z}_2$-parity odd coordinates[18] $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ in the reversed-parity cotangent spaces $\Pi T^*_{(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x}))} N^n$ to the fibres $N^n \simeq \pi^{-1}(\boldsymbol{x})$ of the bundle π, see [35, §2.1] and [32, §2.1].

For *vector* spaces $N^n = \mathbb{R}^n$, the vector space isomorphism $T_{(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x}))} N^n \simeq N^n$ reduces the construction of Kupershmidt's variational cotangent bundle[47] over $\pi_\infty \colon J^\infty(\pi) \to M^m$ to the handy Whitney sum $J^\infty(\pi \times_M \Pi\widehat{\pi})$.

Convention. The notation $\pi \times_M \Pi\widehat{\pi}$ will be used in what follows to avoid an agglomeration of formulae; for the case of affine bundle π already impels the construction of horizontal jet bundle $\overline{J^\infty_{\pi_\infty}}(\Pi T\pi)$ over the space $J^\infty(\pi)$.

The variational bi-vectors $\boldsymbol{\mathcal{P}} \in \bar{H}^m(\pi \times_M \Pi\widehat{\pi})$ are integral functionals of form

$$\boldsymbol{\mathcal{P}} = \tfrac{1}{2} \int \langle \boldsymbol{\xi} \cdot A\big|_{(\boldsymbol{x},[\boldsymbol{u}])}(\boldsymbol{\xi}) \rangle = \tfrac{1}{2} \int \xi_i\, P^{ij}_\tau(\boldsymbol{x}, [\boldsymbol{u}])\, \xi_{j,\tau} \cdot \mathrm{dvol}(\boldsymbol{x}),$$

where the linear total differential operators $A = \big\| P^{ij}_\tau \cdot \big(\tfrac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\big)^\tau \big\|^{j=1,\ldots,n}_{i=1,\ldots,n}$ are skew-adjoint (to make the object $\boldsymbol{\mathcal{P}}$ well defined); for all multi-indexes τ, the parity-odd symbols $\xi_{j,\varnothing} = \xi_j$, ξ_{j,x^k}, $\xi_{j,x^k x^\ell}$, $\ldots$, $\xi_{j,\tau}$, $\ldots$ are the respective jet fibre coordinates.

The construction of variational k-vectors with $k \geqslant 0$ is alike. Due to the introduction of parity-odd variables $\boldsymbol{\xi}$ as canonical conjugates of the n-tuples $\boldsymbol{u}$, the vector space of all variational multivectors is naturally endowed with the parity-odd variational Poisson bracket, or variational *Schouten bracket* $[\![\cdot,\cdot]\!]$. Its construction – as descendent structure with respect to the Batalin–Vilkovisky Laplacian Δ – was recalled in [32, 35]; for consistency, we shall discuss the composition of $[\![\cdot,\cdot]\!]$ in what follows (see p. 17 below).

The variational bi-vectors $\boldsymbol{\mathcal{P}}$ are called *Poisson* if they satisfy the classical master-equation $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] \cong 0$. The horizontal cohomology class equivalence $\cong 0$ means, in particular, that the variational tri-vector $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!]$, viewed as an integral functional, takes $\Gamma(\pi \times_M \Pi\widehat{\pi}) \to 0 \in \Bbbk$.

---

[17] We recall that in the (graded-)commutative set-up one has that $\big(F \overset{\overline{\mathfrak{M}}^m(\pi)}{\times} G\big)(\boldsymbol{s}) = F(\boldsymbol{s}) \overset{\Bbbk}{\times} G(\boldsymbol{s})$ but a known mechanism destroys this algebra homomorphism in a larger setting of formal noncommutative variational symplectic geometry and its calculus of cyclic words ([35], cf. [45, 54]).

[18] Introduced by hand, the parity-odd fibre variables $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$ are not physical; much of the formalism is spent on expressing the fact that *observable* objects do in retrospect not depend on a choice of sections $\boldsymbol{s}^\dagger$ for the parity-odd subbundle of their geometry, see [32, §3.1] and, contained therein, references to classical works on Feynman path integral in the Batalin–Vilkovisky geometry and on the Schwinger–Dyson condition (cf. [10]).

Every variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ induces the respective variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}\colon \bar{H}^m(\pi)\times\bar{H}^m(\pi)\to\bar{H}^m(\pi)$ on the space of integral functionals $\Gamma(\pi)\to\Bbbk$. The axiomatic construction of $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is explained in Definition 1 on p. 19; it is the *derived bracket* $[\![[\![\boldsymbol{\mathcal{P}},\cdot]\!],\cdot]\!]$ of two Hamiltonians (see [35, §3]).[19]

Remark 3.2. The value of $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ in $\bar{H}^m(\pi)$ at two integral functionals does not depend on a choice of representatives for the two arguments and for the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}\in\bar{H}^m\bigl(\pi\times_M\Pi\widehat{\pi}\bigr)$, taken modulo the integral functionals that map all sections of the respective (super)bundle to $0\in\Bbbk$. This is no longer necessarily so for the higher-order terms, beyond the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ at $\hbar^1$, in expansions (3).

The bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is extended, via the Leibniz rule, from the vector space $\bar{H}^m(\pi)$ of integral functionals H_1, H_2, $\ldots$ to the Poisson structure on the algebra $\overline{\mathfrak{M}}^m(\pi)$ of (sums of) such functionals' formal products $H_1\times\ldots\times H_\ell\colon\Gamma(\pi)\to\Bbbk$. This is done by using the graded Leibniz rule that extends the variational Schouten bracket $[\![\cdot,\cdot]\!]$ from the vector space $\bar{H}^m\bigl(\pi\times_M\Pi\widehat{\pi}\bigr)$ of variational multivectors to the differential graded Lie algebra $\overline{\mathfrak{M}}^m\bigl(\pi\times_M\Pi\widehat{\pi}\bigr)$, which obviously contains the vector space $\overline{\mathfrak{M}}^m(\pi)$ as its zero-grading component.

Let us remember that every integral functional –e.g., taken as a building block in a local functional– does carry its own integration variable which runs through that integral functional's own copy of the base M^m for the respective (super)bundle. For a given field model over $\bigl(M^m,\ \mathrm{dvol}(\cdot)\bigr)$, the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}=\frac{1}{2}\int\xi_i\,P^{ij}_\tau(\boldsymbol{x},[\boldsymbol{u}])\,\bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\bigr)^\tau(\xi_j)\cdot\mathrm{dvol}(\boldsymbol{x})$ and two Hamiltonians, $F=\int f(\boldsymbol{x}_1,[\boldsymbol{u}])\cdot\mathrm{dvol}(\boldsymbol{x}_1)$ and $G=\int g(\boldsymbol{x}_2,[\boldsymbol{u}])\cdot\mathrm{dvol}(\boldsymbol{x}_2)$, are integral functionals defined at sections of the bundles $\pi\times_M\Pi\widehat{\pi}$ and π, respectively. In total, these three objects carry *three* copies of the given volume element $\mathrm{dvol}(\cdot)$ on M^m. On the other hand, the variational Poisson bracket $\{F,G\}_{\boldsymbol{\mathcal{P}}}$ of F and G with respect to $\boldsymbol{\mathcal{P}}$ is an integral functional $\Gamma(\pi)\to\Bbbk$ that carries *one* copy of the volume element. Why and where to have the two copies of $\mathrm{dvol}(\cdot)$ gone ? The answer to this innocent question was a key to the intrinsic regularisation in Batalin–Vilkovisky's approach to quantisation of gauge systems. Now, the same argument helps us to lift Kontsevich's graph technique of deformation quantisation to the field set-up.

3.2. Elements of the geometry of iterated variations.

3.2.1. Let $(\boldsymbol{s},\boldsymbol{s}^\dagger)$ be a two-component section of the Whitney sum $\pi\times_M\Pi\hat{\pi}$ of bundles. Suppose that this section undergoes an infinitesimal shift along the direction

$$(\delta\boldsymbol{s},\delta\boldsymbol{s}^\dagger)\bigl(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x}),\boldsymbol{s}^\dagger(\boldsymbol{x})\bigr)=\sum_{i=1}^{n}\Bigl(\delta\boldsymbol{s}^i(\boldsymbol{x})\cdot\vec{e}_i(\boldsymbol{x})+\delta\boldsymbol{s}^\dagger_i(\boldsymbol{x})\cdot\vec{e}^{\,\dagger,i}(\boldsymbol{x})\Bigr),$$

which we decompose with respect to the adapted basis $(\vec{e}_i,\vec{e}^{\,\dagger,i})$ in the tangent space $T_{(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x}))}\pi^{-1}(\boldsymbol{x})\oplus T_{\boldsymbol{s}^\dagger(\boldsymbol{x})}T^*_{(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x}))}\pi^{-1}(\boldsymbol{x})$. At their attachment point, the vectors $\vec{e}_i$ and $\vec{e}^{\,\dagger,j}$ are – by definition – tangent to the respective coordinate lines for variables u^i and ξ_j. By construction, these vectors $\vec{e}_i$ and $\vec{e}^{\,\dagger,j}$ are dual; at every i running from 1 to n, the

[19]Note that an attempt to modify the volume element $\mathrm{dvol}(\cdot)$ on M^m can affect the output of $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ due to the variational symplectic geometry engaged in its construction (e.g., via $[\![\cdot,\cdot]\!]$).

two *ordered* couplings of (co)vectors attached over $\boldsymbol{x} \in M^m$ at the fibres' points with coordinates $\boldsymbol{s}(\boldsymbol{x})$ and $\boldsymbol{s}^\dagger(\boldsymbol{x})$ are

$$\langle \overset{\text{first}}{\vec{e}_i}\,,\ \overset{\text{second}}{\vec{e}^{\,\dagger,i}} \rangle = +1 \qquad \text{and} \qquad \langle \overset{\text{first}}{\vec{e}^{\,\dagger,i}},\ \overset{\text{second}}{\vec{e}_i} \rangle = -1. \tag{4}$$

Likewise, the coefficients $\delta s^i(\,\cdot\,, \boldsymbol{s}(\,\cdot\,))$ and $\delta s^\dagger_i(\,\cdot\,, \boldsymbol{s}(\,\cdot\,), \boldsymbol{s}^\dagger(\,\cdot\,))$ of the virtual shifts along the i^{th} coordinate lines u^i and ξ_i are normalised via

$$\delta s^i(\boldsymbol{x}, \boldsymbol{s}(\boldsymbol{x})) \cdot \delta s^\dagger_i(\boldsymbol{x}, \boldsymbol{s}(\boldsymbol{x}), \boldsymbol{s}^\dagger(\boldsymbol{x})) \equiv 1 \qquad \text{(no summation!)} \tag{5}$$

over all internal points $\boldsymbol{x} \in \operatorname{supp} \delta s^i \subseteq M^m$ (see footnote 18). It is precisely this mathematical construction in terms of which the physical idea of fields as degrees of freedom attached at every point of the space-time is expressed.

The directed variations $\overrightarrow{\delta \boldsymbol{s}}$ and $\overrightarrow{\delta \boldsymbol{s}^\dagger}$, as well as $\overleftarrow{\delta \boldsymbol{s}}$ and $\overleftarrow{\delta \boldsymbol{s}^\dagger}$, are singular linear integral operators supported, due to (4), on the diagonal. Each variation contains n copies of Dirac's $\boldsymbol{\delta}$-distribution weighted by the respective coefficients δs^i and $\delta s^\dagger_i$. We have that

$$\begin{aligned}
\overrightarrow{\delta \boldsymbol{s}} &= \int \mathrm{d}\boldsymbol{y} \left\langle (\delta s^i)\Bigl(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{y}}\Bigr)^{\sigma}(\boldsymbol{y}) \cdot \underrightarrow{\overset{\text{first}}{\vec{e}_i(\boldsymbol{y})} \mid \overset{\text{second}}{\vec{e}^{\,\dagger,i}(\,\cdot\,)}} \right\rangle \frac{\overrightarrow{\partial}}{\partial u^i_\sigma}, \\
\overrightarrow{\delta \boldsymbol{s}^\dagger} &= \int \mathrm{d}\boldsymbol{y} \left\langle (\delta s^\dagger_i)\Bigl(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{y}}\Bigr)^{\sigma}(\boldsymbol{y}) \cdot \underrightarrow{(-\overset{\text{first}}{\vec{e}^{\,\dagger,i}})(\boldsymbol{y}) \mid \overset{\text{second}}{\vec{e}_i(\,\cdot\,)}} \right\rangle \frac{\overrightarrow{\partial}}{\partial \xi_{i,\sigma}}, \\
\overleftarrow{\delta \boldsymbol{s}} &= \int \mathrm{d}\boldsymbol{y}\, \frac{\overleftarrow{\partial}}{\partial u^i_\sigma} \left\langle \underleftarrow{\overset{\text{second}}{\vec{e}^{\,\dagger,i}(\,\cdot\,)} \mid \overset{\text{first}}{\vec{e}_i(\boldsymbol{y})}} \cdot \Bigl(\frac{\overrightarrow{\partial}}{\partial \boldsymbol{y}}\Bigr)^{\sigma}(\delta s^i)(\boldsymbol{y}) \right\rangle, \\
\overleftarrow{\delta \boldsymbol{s}^\dagger} &= \int \mathrm{d}\boldsymbol{y}\, \frac{\overleftarrow{\partial}}{\partial \xi_{i,\sigma}} \left\langle \underleftarrow{\overset{\text{second}}{\vec{e}_i(\,\cdot\,)} \mid (-\overset{\text{first}}{\vec{e}^{\,\dagger,i}})(\boldsymbol{y})} \cdot \Bigl(\frac{\overrightarrow{\partial}}{\partial \boldsymbol{y}}\Bigr)^{\sigma}(\delta s^\dagger_i)(\boldsymbol{y}) \right\rangle,
\end{aligned}$$

see [32, §2.2–3] for details; for brevity, the indication of fibre points for given $\boldsymbol{s}(\cdot)$ and $\boldsymbol{s}^\dagger(\cdot)$ is omitted in such formulae. Whenever acting on the spaces of local functionals, which were discussed in the preceding section, these linear operators yield those functionals' responses to infinitesimal shifts of their arguments. i. e., of the sections at which the functionals are evaluated.

Remark 3.3. By convention, the differentials of functionals' densities are expanded with respect to the bases $\vec{e}_i$, $\vec{e}^{\,\dagger,j}$ in the fibres tangent spaces; the plus or minus signs in the sections' shifts are chosen in such a way that the couplings always evaluate to $+1$.

3.2.2. Let us now explain how the edges in Kontsevich's graphs get oriented; in fact, this mechanism is unseparable from the integration by parts, which we re-address in the succeding section. Geometrically, every edge is realised by the linking of variations – with respect to the canonical conjugate variables u^i and ξ_i – of objects that are contained in the two vertices. Such edge's orientation is the ordering $\delta \boldsymbol{s}^\dagger \prec \delta \boldsymbol{s}$ of singular linear integral operators; initially, they act as shown in Fig. 2. Every edge in an oriented graph Γ contributes to the summand (which Γ encodes in the star-product $\star_\hbar$) by the linking of variations and by the linking of differentials of objects contained in the vertices. Novel with respect to the classical set-up of §2, the variations $\delta \boldsymbol{s}$ and $\delta \boldsymbol{s}^\dagger$ are brought into the picture in order to properly handle the derivatives – previously, non-existent – along the base manifold M^m. At the end of the day, the

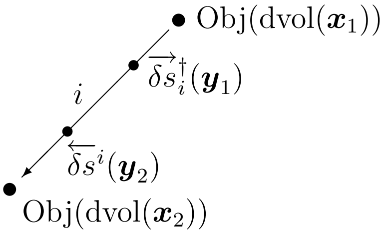


FIGURE 2. Indexed by i, this summand appeared in the operation $\bullet$ in [40]; here we extend it to the set-up of fields by letting the points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{y}_1$, and $\boldsymbol{y}_2$ run through the integration domain M^m of positive dimension m.

linking of normalised variations yields the singular linear integral operators that act via multiplication by ± 1. In turn, the linking of objects converts one of them into a singular linear integral operator such that the (co)vectors contained in it act on their duals, resulting in the multiples ∓ 1. Let us analyse this construction[32, 35] in more detail, also keeping track of the on-the-diagonal integration by parts that convert derivatives along one copy of the integration domain M^m into $(-1)\times$ derivatives with respect to the same variables, now referred to another copy of the base.[20]

As it was shown in [32], the derivatives are transported along the edge to arrow's head according to the scenarios drawn in Fig. 3; each derivative is referred to the copy

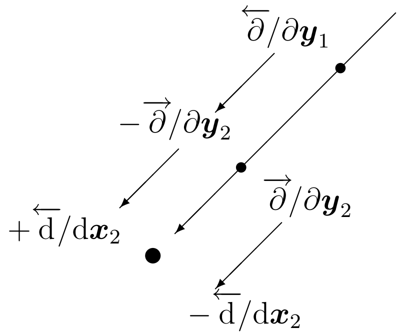


FIGURE 3. Each on-the-diagonal push of a derivative along the edge creates an extra minus sign.

of base manifold M^m over which the object or structure it acts on is defined.

The singular linear integral operators $\delta\boldsymbol{s}$ and $\delta\boldsymbol{s}^\dagger$ act not only on the spaces of local functionals but also on elements of their native space of singular distributions. At the same time, regular integral functionals – like $\boldsymbol{\mathcal{P}}$ and F or G, which are contained

[20]The definition of total derivative $\mathrm{d}/\mathrm{d}x$, which is

$$\Bigl(j_\infty(s)^*\Bigl(\Bigl(\frac{\mathrm{d}}{\mathrm{d}x}f\Bigr)(x,[u])\Bigr)\Bigr)(x_0) \stackrel{\text{def}}{=} \Bigl(\frac{\partial}{\partial x}\Bigl(j_\infty(s)^*\Bigl(f(x,[u])\Bigr)\Bigr)\Bigr)(x_0),$$

explains why the partial derivatives $\partial/\partial x$ reshape into $\mathrm{d}/\mathrm{d}x$ as soon as they arrive to the graph's vertices and there, they act on the objects f which are defined over jet bundles and which are evaluated at the infinite jets $j_\infty(s)$ of sections s.

in vertices of the graph Γ at hand, – themselves can discard the copy of volume elements $\mathrm{dvol}(\,\cdot\,)$ which they are equipped with and by this, reshape into singular integral operators.

Here is an example: the edge $\boldsymbol{\mathcal{P}} \xrightarrow{i} F$ encodes the formula

$$\iint \mathrm{d}\boldsymbol{x}_1 \cdot \Bigl(\tfrac{1}{2}\xi_\alpha P^{\alpha\beta}_\sigma\bigr|_{(\boldsymbol{x}_1,[\boldsymbol{u}])}\xi_{\beta,\sigma}\Bigr)\frac{\overleftarrow{\partial}}{\partial\xi_{i,\tau}}$$
$$\Bigl\langle \overset{\text{first}}{\underline{\underline{\vec{e}_i(\boldsymbol{x}_1)}}}\Bigr| \iint \mathrm{d}\boldsymbol{y}_1\mathrm{d}\boldsymbol{y}_2 \Bigl\langle \overset{\text{first}}{\underline{(-\vec{e}^{\,\dagger,i})(\boldsymbol{y}_1)}}\cdot \delta s_i^\dagger(\boldsymbol{y}_1)|\delta s^i(\boldsymbol{y}_2)\cdot \overset{\text{second}}{\underline{\vec{e}_i(\boldsymbol{y}_2)}}\Bigr\rangle \Bigl|\overset{\text{second}}{\underline{\underline{\vec{e}^{\,\dagger,i}(\boldsymbol{x}_2)}}}\Bigr\rangle$$
$$\ulcorner\Bigl(+\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^\tau\Bigl(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^{\sigma_2}\urcorner\frac{\overrightarrow{\partial}}{\partial u^i_{\sigma_2}}(f)(\boldsymbol{x}_2,[\boldsymbol{u}])\cdot\mathrm{dvol}(\boldsymbol{x}_2).$$

The singular distributions wright the diagonal $\boldsymbol{x}_1 = \boldsymbol{y}_1 = \boldsymbol{y}_2 = \boldsymbol{x}_2$; both couplings evaluate to $+1$, and normalisation (5) makes the edge's cargo invisible (indeed, it contributes via multiplication by $+1$).

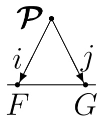
The three singular operators can be directed in the opposite way, against the edge's orientation – along which the derivatives are transported in any case. This would keep the volume element at the arrow's tail. For example, consider the edge $\boldsymbol{\mathcal{P}} \xrightarrow{j} G$, which yields the formula

$$\iint \mathrm{dvol}(\boldsymbol{x}_1)\mathrm{d}\boldsymbol{x}_2 \cdot \Bigl(\tfrac{1}{2}\xi_\alpha P^{\alpha\beta}_\sigma\bigr|_{(\boldsymbol{x}_1,[\boldsymbol{u}])}\xi_{\beta,\sigma}\Bigr)\frac{\overleftarrow{\partial}}{\partial\xi_{j,\tau}}$$
$$\Bigl\langle \overset{\text{second}}{\underline{\underline{\vec{e}_j(\boldsymbol{x}_1)}}}\Bigr| \iint \mathrm{d}\boldsymbol{y}_1\mathrm{d}\boldsymbol{y}_2 \Bigl\langle \overset{\text{second}}{\underline{(-\vec{e}^{\,\dagger,j})(\boldsymbol{y}_1)}}\cdot \delta s_j^\dagger(\boldsymbol{y}_j)|\delta s^j(\boldsymbol{y}_2)\cdot \overset{\text{first}}{\underline{\vec{e}_j(\boldsymbol{y}_2)}}\Bigr\rangle \Bigl|\overset{\text{first}}{\underline{\underline{\vec{e}^{\,\dagger,j}(\boldsymbol{x}_2)}}}\Bigr\rangle$$
$$\ulcorner\Bigl(+\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^\tau\Bigl(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^{\sigma_2}\urcorner\frac{\overrightarrow{\partial}}{\partial u^j_{\sigma_2}}(g)(\boldsymbol{x}_2,[\boldsymbol{u}]).$$

In this case, both couplings evaluate to -1 by (4) still their values' product is $+1$; the diagonal-making and normalisation mechanism remains the same as before.

Summarising, the ordering in (4) is the only mechanism that creates sign factors. Let us repeat that the direction in which the operators act along the edge does not necessarily coincide with that edge's orientation in the graph Γ; that arrow specifies the direction to transport the derivatives by using the integrations by parts.

Definition 1. The *variational Poisson bracket* $\{F,G\}_{\boldsymbol{\mathcal{P}}}$ of two integral functionals F and G with respect to a given variational Poisson bi-vector $\boldsymbol{\mathcal{P}}$ is the graph



. (6)

By using two pairs of normalised variations and by letting the volume element stay in the vertex containing G, we realise the geometry of singular distributions encoded by this picture via the formula

$$\iiint \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x} (f|_{(\boldsymbol{x}_1,[\boldsymbol{u}])}) \frac{\overleftarrow{\partial}}{\partial u^i_{\sigma_1}} \ulcorner \Bigl(-\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Bigr)^{\sigma_1} \Bigl(+\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Bigr)^{\tau_1} \urcorner$$

$$\Bigl\langle \underbrace{\vec{e}^{\,\dagger,i}(\boldsymbol{x}_1)}_{}^{\text{first}} | \iint \mathrm{d}\boldsymbol{y}_1 \mathrm{d}\boldsymbol{y}_2 \Bigl\langle \overset{\text{first}}{\vec{e}^{\,i}}(\boldsymbol{y}_2) \cdot \delta s^i(\boldsymbol{y}_2) | \delta s^\dagger_i(\boldsymbol{y}_1) \cdot \overset{\text{second}}{(-\vec{e}^{\,\dagger,i})}(\boldsymbol{y}_1) \Bigr\rangle \, | \overset{\text{second}}{\underline{\vec{e}_i(\boldsymbol{x})}} \Bigr\rangle \cdot$$

$$\cdot \frac{\overrightarrow{\partial}}{\partial \xi_{i,\tau_1}} \Bigl(\tfrac{1}{2} \xi_\alpha P^{\alpha\beta}_\zeta \Bigr|_{(\boldsymbol{x},[\boldsymbol{u}])} \xi_{\beta,\zeta} \Bigr) \frac{\overleftarrow{\partial}}{\partial \xi_{j,\tau_2}} \cdot$$

$$\cdot \Bigl\langle \overset{\text{first}}{\underline{\underline{\vec{e}_j(\boldsymbol{x})}}} | \iint \mathrm{d}\boldsymbol{z}_1 \mathrm{d}\boldsymbol{z}_2 \Bigl\langle \overset{\text{first}}{(-\vec{e}^{\,\dagger,j})}(\boldsymbol{z}_1) \cdot \delta s^\dagger_j(\boldsymbol{z}_1) | \delta s^j(\boldsymbol{z}_2) \cdot \overset{\text{second}}{\vec{e}_j}(\boldsymbol{z}_2) \Bigr\rangle \, | \overset{\text{second}}{\underline{\underline{\vec{e}^{\,\dagger,j}(\boldsymbol{x}_2)}}} \Bigr\rangle$$

$$\ulcorner \Bigl(+\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^{\tau_2} \Bigl(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Bigr)^{\sigma_2} \urcorner \frac{\overrightarrow{\partial}}{\partial u^j_{\sigma_2}} (g|_{(\boldsymbol{x}_2,[\boldsymbol{u}])}) \cdot \mathrm{dvol}(\boldsymbol{x}_2).$$

The two pairs of couplings evaluate to $\underline{(-1)} \cdot (-1) \times (+1) \cdot \underline{\underline{(+1)}} = +1$. The algorithm's output is therefore perfectly familiar, for it yields the equality

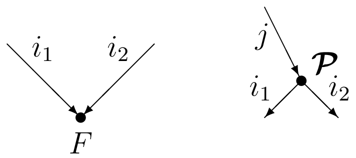
$$\{F, G\}_{\boldsymbol{\mathcal{P}}} = \frac{1}{2} \int \Bigl\langle \frac{\delta F}{\delta \boldsymbol{u}} \cdot \overrightarrow{A}\Bigl(\frac{\delta G}{\delta \boldsymbol{u}}\Bigr) \Bigr\rangle - \frac{1}{2} \int \Bigl\langle \Bigl(\frac{\delta F}{\delta \boldsymbol{u}}\Bigr) \overleftarrow{A} \cdot \frac{\delta G}{\delta \boldsymbol{u}} \Bigr\rangle, \tag{7}$$

where A is the Hamiltonian operator built into the variational Poisson bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2} \int \langle \boldsymbol{\xi} \cdot \overrightarrow{A}(\boldsymbol{\xi}) \rangle$. Because the operator A is skew-adjoint, one could now integrate by parts, obtaining an even shorter expression,

$$\cong \int \Bigl\langle \frac{\delta F}{\delta \boldsymbol{u}} \cdot \overrightarrow{A}\Bigl(\frac{\delta G}{\delta \boldsymbol{u}}\Bigr) \Bigr\rangle.$$

Still let it be remembered that it is the Λ-graph in (6) that does define the variational Poisson bracket — whereas this handy short formula is its remote consequence. Indeed, much information has been lost in the course of evaluations and, especially, in the course of transporting the – now, total – derivatives to their final positions in the variational derivatives $\delta/\delta \boldsymbol{u}$.

3.2.3. When the variational Poisson bracket of two given functionals is assembled by Definition 1 – to be evaluated at a section $\boldsymbol{s} \in \Gamma(\pi)$ of the bundle of physical fields, – the total derivatives $\mathrm{d}/\mathrm{d}\boldsymbol{x}$ follow immediately the partial derivatives $\partial/\partial u^i_\sigma$ in the construction of variational derivatives $\delta/\delta \boldsymbol{u}$. Such inseparability of the horizontal and vertical derivations referring to their own geometries M^m and N^n, respectively, is typical for one-step reasonings (like the production of Euler–Lagrange's equations of motion from a given action functional). However, a necessity to iterate the virtual shifts of sections $\boldsymbol{s} \in \Gamma(\pi)$ reveals the conceptual difficulty of classical jet bundle geometry (e. g., this is acknowledged in [29, §15.1]). This point is readily seen by using graphs beyond the Λ-graph for $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ in Kontsevich's expansion (3). Indeed, consider a vertex where two or more arrows arrive — or a vertex that contains $\boldsymbol{\mathcal{P}}$ (so that two partial derivatives, $\overrightarrow{\partial}/\partial \xi_{i_1,\tau_1}$ and $\overleftarrow{\partial}/\partial \xi_{i_2,\tau_2}$, retro-act on its content) and that serves as the head for another arrow (hence, bringing the partial derivative $\partial/\partial u^i_\sigma$ followed by $(-\mathrm{d}/\mathrm{d}\boldsymbol{x})^\sigma$ and possibly, by the total derivative(s) $(+\mathrm{d}/\mathrm{d}\boldsymbol{x})^\tau$ specified by that arrow's tail), see (8):



(8)

In which consecutive order are those partial and total derivatives, related to different edges, applied to the content of a vertex?

Furthermore, we concluded section 2 by interpreting the mechanism of associativity – modulo the Jacobi identity – for $\star_\hbar$ as cancellation of similar terms, that is, of those weighted graphs Γ which are achieved by combining different pairs of weighted graphs in the two consecutive star-products. The mandatory analytic expressions' independence of a scenario to attain them prescribes that the application of total derivatives $\mathrm{d}/\mathrm{d}\boldsymbol{x}_\ell$ in the *inner* star-products in $(F\star_\hbar G)\star_\hbar H - F\star_\hbar(G\star_\hbar H)$ must be delayed – until all the partial derivatives $\partial/\partial\boldsymbol{u}_\sigma$ would have finished acting in the *outer* star-products. In other words, whenever the associativity is actually being verified, the inspection may not be interrupted half-way!

Les mariages se font au ciel et se consomment sur la terre. This saying expresses the main idea in the geometry of iterated variations[32, 33, 35]: the vertical derivations $\partial/\partial u^i_\sigma$ and (the lifts $\mathrm{d}/\mathrm{d}\boldsymbol{x}_\ell$ of) horizontal derivations $\partial/\partial\boldsymbol{x}_\ell$ are performed at different stages. First, the vertical derivations $\partial/\partial\boldsymbol{u}_\sigma$ along N^n, together with their counterparts $\partial/\partial\xi_\tau$ from the parity-odd symplectic duals, frame the edges of entire graph Γ. In the meantime, the derivatives along the base M^m are stored inside the variations $\delta\boldsymbol{s}$ by using $\partial/\partial\boldsymbol{y}_k$. At the end of the day, all the horizontal derivatives $(\pm\partial/\partial\boldsymbol{y}_k)^\sigma$ are channelled from δs^i to $(\mp\partial/\partial\boldsymbol{x}_\ell)^\sigma$, finally acting on the objects which are targets of $\partial/\partial u^i_\sigma$.

Remark 3.4. We emphasize that by the definition of total derivative (see footnote 20 on p. 18), the derivatives $\partial/\partial\boldsymbol{y}_k(\delta s^i)$ of virtual shifts δs^i for sections $u^i = s^i(\boldsymbol{x}_\ell)$ reshape, under integration by parts, into the derivatives $-\partial/\partial\boldsymbol{x}_\ell(s^i)$ of those sections — not affecting the synthetic, parity-odd variables $\xi_{j,\zeta}$ which constitute the fibres of another bundle. Consequently, the *total* derivatives $(+\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x}_\ell)^\tau\circ(-\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x}_\ell)^\sigma$ refer only to the jet space $J^\infty(\pi)$ where they act on the respective fibre variables $\boldsymbol{u}_\sigma$ in a vertice's content.

Example 3.1. The first graph in (8) corresponds to the formula

$$\bigl(F|_{(\boldsymbol{x}_1,[\boldsymbol{u}])}\bigr)\frac{\overleftarrow{\partial}}{\partial u^{i_1}_{\sigma_1}}\frac{\overleftarrow{\partial}}{\partial u^{i_2}_{\sigma_2}}\ulcorner\Bigl(-\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Bigr)^{\sigma_1\cup\sigma_2}\circ\Bigl(+\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Bigr)^{\tau_1\cup\tau_2}\urcorner, \tag{9}$$

where the multi-indexes τ_1 and τ_2 arrive from the respective arrow tails.

The second graph in Fig. 8 contributes with the expression

$$\ulcorner\Bigl(+\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_k}\Bigr)^{\tau_1}\urcorner\frac{\overrightarrow{\partial}}{\partial\xi_{i_1,\tau_1}}\int\tfrac{1}{2}\xi_\alpha\Bigl\{\ulcorner\Bigl(+\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\tau}\Bigl(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\sigma}\urcorner\frac{\overrightarrow{\partial}}{\partial u^j_\sigma}\bigl(P^{\alpha\beta}_\zeta|_{(\boldsymbol{x},[\boldsymbol{u}])}\bigr)\Bigr\}\xi_{\beta,\zeta}\,\mathrm{dvol}(\boldsymbol{x})$$
$$\frac{\overleftarrow{\partial}}{\partial\xi_{i_2,\tau_2}}\ulcorner\Bigl(+\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_\ell}\Bigr)^{\tau_2}\urcorner,$$

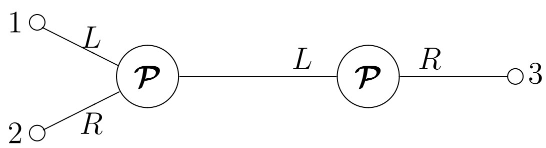
where the multi-index τ arrives from the tail of arrow labelled by j and where the copies of base M^m for objects at the heads of arrows labelled using i_1 and i_2 are indexed using k and ℓ, respectively.

To indicate the delayed arrival of total derivatives to their final places (where we write them at once), let us embrace these operators by using $\lceil\ldots\rceil$ in all formulas (e. g., see Definition 1 on p. 19 above).

3.2.4. Summarising, the local portrait of oriented edges around every vertex in a given graph Γ determines the vertex-incoming partial derivatives with respect to variables $\boldsymbol{u}_\sigma$, in-coming graded partial derivatives with respect to the parity-odd variables $\boldsymbol{\xi}_\tau$, and (powers of) delayed $(\pm 1)\times$ total derivatives. All these derivations act on the object contained in the vertex at hand, that is, on either the Hamiltonian density or structural constants $P^{\alpha\beta}_{\zeta}\bigl(\boldsymbol{x},[\boldsymbol{u}]\bigr)$ of the variational Poisson bracket; note that in the both cases, the arguments are referred to the geometry of $J^\infty(\pi)$, hence those objects are entirely expressed in terms of the geometry of physical fields.

Globally, each graph Γ in Kontsevich's summation formula now determines a singular linear integral operator that acts on a local functional — which is contained in one of the oriented graph's terminal vertices (or sinks). It is obvious that there remain no letters $\boldsymbol{\xi}$, neither in such operators themselves nor in the objects they produce.

The Jacobi identity is the sum of three graphs,



, (10)

that sum taken over the three cyclic permutations of the content of vertices 1, 2, and 3.

3.3. The sought-for associativity of $\star_\hbar$. Why it leaks.

3.3.1. The parity-odd letters $\xi_i \equiv \xi_{i,\varnothing}$ and their jet fibre descendants $\xi_{i,\tau}$ with $|\tau| > 0$ were introduced to let the multi-indexes τ be stored.[21] In the variational Poisson bivector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\int \xi_{i,\varnothing}\cdot P^{ij}_{\tau}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\cdot\bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\bigr)^{\tau}(\xi_j)\cdot \mathrm{dvol}(\boldsymbol{x})$, the symbols $\xi_{i,\varnothing}$ and $\xi_{j,\tau}$ carry the derivatives $(\overleftarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\tau$ and $(\overrightarrow{\mathrm{d}}/\mathrm{d}\boldsymbol{x})^\tau$ by which variational Poisson bracket (7) acts on the two Hamiltonians' variations $\boldsymbol{p}_\alpha = \delta H_\alpha/\delta\boldsymbol{u}$. Indeed, we have that

$$\{H_1,H_2\}_{\boldsymbol{\mathcal{P}}} \cong \int\Bigl\{p_{1,i}\cdot P^{ij}_{\tau}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigl(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\tau}(p_{2,j}) - (p_{1,i})\Bigl(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\Bigr)^{\tau}\cdot P^{ij}_{\tau}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\cdot p_{2,j}\Bigr\}\cdot\mathrm{dvol}(\boldsymbol{x}).$$

To have the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ glued over the entire $J^\infty(\pi)$ from its construction over coordinate charts for that infinite jet bundle, let us inspect how its coefficients $P^{ij}_{\tau}(\boldsymbol{x},[\boldsymbol{u}])$ transform under a change of (in)dependent variables.

[21] We recall from footnote 20 that the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ *is* the primary object over which $\boldsymbol{\mathcal{P}}$ is built. The presence of its coefficients $P^{ij}_{\tau}\bigl(\boldsymbol{x},[\boldsymbol{u}]\bigr)$ in the terms of $\star_\hbar$ encoded by Kontsevich's graphs Γ is such that neither the virtual shifts $\delta\boldsymbol{s}$ for variables $\boldsymbol{u}_\sigma$ in P^{ij}_{τ} nor integrations by parts, which were discussed in §3.2, produce any total derivatives $\mathrm{d}/\mathrm{d}\boldsymbol{x}$ falling on the formal variables $\boldsymbol{\xi}_\tau$.

First, we consider the *affine* reparametrisations $\widetilde{\boldsymbol{u}} = J(\boldsymbol{x})\cdot\boldsymbol{u} + \overrightarrow{\Delta}(\boldsymbol{x})$ of fibre variables in the bundle π of gauge fields.

To track the transformation rule for variational derivatives,

$$\frac{\delta H(\boldsymbol{x},[\widetilde{\boldsymbol{u}}])}{\delta\widetilde{u}^\alpha}\cdot\delta\widetilde{u}^\alpha = \widetilde{p}_\alpha(\boldsymbol{x},[\widetilde{\boldsymbol{u}}])\cdot\delta\widetilde{u}^\alpha = p_\beta(\boldsymbol{x},[\boldsymbol{u}])\cdot\delta u^\beta = \frac{\delta H(\boldsymbol{x},[\boldsymbol{u}])}{\delta u^\beta}\cdot\delta u^\beta,$$

let us apply the chain rule

$$\dot{H}\left(\boldsymbol{x},[\widetilde{\boldsymbol{u}}(\boldsymbol{x},[\boldsymbol{u}])]\right) = \ell_H^{(\widetilde{\boldsymbol{u}})}(\dot{\widetilde{\boldsymbol{u}}}) = \left(\ell_H^{(\widetilde{\boldsymbol{u}})}\circ\ell_{\widetilde{\boldsymbol{u}}}^{(\boldsymbol{u})}\right)(\dot{\boldsymbol{u}})$$

for the Fréchet derivatives, or *linearisations*[53]; we then use the standard fact $\delta H/\delta\boldsymbol{u} = \left(\ell_H^{(\boldsymbol{u})}\right)^\dagger(1)$, whence[22]

$$\frac{\delta H}{\delta\boldsymbol{u}} = \left(\ell_{\widetilde{\boldsymbol{u}}}^{(\boldsymbol{u})}\right)^\dagger\left(\frac{\delta H}{\delta\widetilde{\boldsymbol{u}}}\right) \qquad\text{and}\qquad p_\beta(\boldsymbol{x},[\boldsymbol{u}]) = \frac{\partial\widetilde{u}^\alpha}{\partial u^\beta}(\boldsymbol{x})\cdot\widetilde{p}_\alpha(\boldsymbol{x},[\widetilde{\boldsymbol{u}}]). \tag{11}$$

Therefore,

$$\begin{aligned} p_{1,\beta}\cdot\left[P_\tau^{\beta\gamma}\cdot\left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\right](p_{2,\gamma}) - (p_{1,\beta})\left[\left(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\cdot P_\tau^{\beta\gamma}\right]\cdot p_{2,\gamma} &= \\ = \widetilde{p}_{1,\alpha}\cdot\frac{\partial\widetilde{u}^\alpha}{\partial u^\beta}\,P_\tau^{\beta\gamma}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\cdot\left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\left(\frac{\partial\widetilde{u}^\delta}{\partial u^\gamma}\cdot\widetilde{p}_{2,\delta}\right) &- \\ -\left(\widetilde{p}_{1,\alpha}\cdot\frac{\partial\widetilde{u}^\alpha}{\partial u^\beta}\right)\left(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\cdot P_\tau^{\beta\gamma}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\frac{\partial\widetilde{u}^\delta}{\partial u^\gamma}\cdot\widetilde{p}_{2,\delta}&\ . \end{aligned}$$

This formula yields the reparametrisation rule for Hamiltonian operators on $J^\infty(\pi)$ for affine bundles π,

$$\begin{aligned} \sum_{|\tau|\geqslant 0} P_\tau^{\beta\gamma}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau \longmapsto \sum_{|\sigma|\geqslant 0}\widetilde{P}_\sigma^{\alpha\delta}\Big|_{(\boldsymbol{x},[\widetilde{\boldsymbol{u}}])}\left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\sigma &= \\ = \sum_{|\tau|\geqslant 0}\frac{\partial\widetilde{u}^\alpha}{\partial u^\beta}(\boldsymbol{x})\cdot P_\tau^{\beta\gamma}\big|_{(\boldsymbol{x}[\boldsymbol{u}])}\cdot\left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\circ\frac{\partial\widetilde{u}^\delta}{\partial u^\gamma}(\boldsymbol{x})&\ . \end{aligned}$$

In brief, one has that $\sum\limits_\sigma\widetilde{P}_\sigma\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\sigma = \sum\limits_\tau J^T\cdot P_\tau\cdot\left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\tau\circ J$ for $\widetilde{\boldsymbol{u}} = J(\boldsymbol{x})\cdot\boldsymbol{u} + \vec{\Delta}(\boldsymbol{x})$.

We remember also that under arbitrary smooth reparametrisations $\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}(\boldsymbol{x})$ of local coordinates on the base M^m, the derivatives $(\mathrm{d}/\mathrm{d}\boldsymbol{x})^\sigma$ and jet variables u_σ^i with $|\sigma| = k > 0$ are reparametrised *linearly* — whenever expressed via the derivatives $(\mathrm{d}/\mathrm{d}\widetilde{\boldsymbol{x}})^\tau$ and jet variables $\widetilde{u}_\tau^i$ with $0 < |\tau| \leqslant k$; this is a standard exercise on the use of chain rule.[23]

[22]A bit counterintuitive, this "tensor calculus" lemma, which we thus prove, tells us that under base point-dependent reparametrisations in the fibres of π, its sections' derivatives obey those changes' prolongations — but the prolongations are such that under the many integrations by parts, the tedious chain rules for positive-order derivatives cancel out!

[23]Let us emphasize that this fact holds for all kinds of fibre bundles π, i.e., not necessarily affine.

Corollary 1 (anti-Darboux lemma)**.** Initially not contained in the coefficients $P^{ij}_{\tau}(\boldsymbol{x})$ of a variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ referred to a given coordinate system on a chart within $J^\infty(\pi)$, the dependent variables u^k and higher jet variables u^k_σ with $|\sigma|>0$ may not appear in that bracket's coefficients — now referred to any other system of local coordinates on a chart in the affine bundle π of gauge fields $\phi\in\Gamma(\pi)$.

Likewise, initially present in the coefficients $P^{ij}_{\tau}\bigl(\boldsymbol{x},[\boldsymbol{u}]\bigr)$, the jet fibre variables $u^i\equiv u^i_\varnothing$ or u^i_σ with $|\sigma|>0$ cannot be entirely eliminated –except for a complement to open dense subset– from $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ in the course of local coordinate reparametrisations in the affine bundle π.

In other words, the "Darboux shape" $P^{ij}_{\tau}(\boldsymbol{x})\cdot\bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\bigr)^\tau$ of Hamiltonian operators is either forever preserved or never achieved by using the *affine* reparametrisations $\widetilde{\boldsymbol{u}}=J(\boldsymbol{x})\cdot\boldsymbol{u}+\vec{\Delta}(\boldsymbol{x})$.

3.3.2. A much larger transformation group acts on the coefficients of variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ whenever the bundle π of physical fields is a smooth fibre bundle over M^m, so that the changes $\widetilde{\boldsymbol{x}}=\widetilde{\boldsymbol{x}}(\boldsymbol{x})$, $\widetilde{\boldsymbol{u}}=\widetilde{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{u})$ of local coordinates on the respective charts in the base M^m and fibres N^n then lift to the fibres of finite jet bundles $J^k(\pi)$ over M^m. Arguing as above, we deduce that the coefficients of variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ obey, in particular, the transformation law

$$\widetilde{P}_\tau\big|_{(\boldsymbol{x},[\widetilde{\boldsymbol{u}}])}\Bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\Bigr)^\tau=\overleftarrow{\bigl(\ell^{(\boldsymbol{u})}_{\widetilde{\boldsymbol{u}}}\bigr)^\dagger}\cdot P_\tau\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\Bigr)^\tau\circ\overrightarrow{\bigl(\ell^{(\boldsymbol{u})}_{\widetilde{\boldsymbol{u}}}\bigr)^\dagger}.$$

Following from (11), this formula shows that in earnest, the two adjoint linearisations of a smooth change $\widetilde{\boldsymbol{u}}=\widetilde{\boldsymbol{u}}(\boldsymbol{x},\boldsymbol{u})$ for fibre variables in π do stay on the respective edges of Λ-graph in (6), so that the output of $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ is the *scalar* density of the arguments' bracket (cf. §2.3).

3.3.3. The jet bundle geometry of field models furthers this idea. Namely, let us consider the jet space morphisms $\boldsymbol{w}^{(\infty)}\colon J^\infty(\pi)\to J^\infty(\widetilde{\pi})$ specified by Miura-type substitutions $\boldsymbol{w}=\boldsymbol{w}(\boldsymbol{x},[\boldsymbol{u}])\colon\Gamma(\pi_{(\infty)})\to\Gamma(\widetilde{\pi})$ of positive differential order. We see that Hamiltonian operators factorise via[24]

$$A\big|_{(\boldsymbol{x},[\boldsymbol{w}])}=\overrightarrow{\ell}^{(\boldsymbol{u})}_{\boldsymbol{w}}\circ B\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\circ\overrightarrow{\ell}^{(\boldsymbol{u})\,\dagger}_{\boldsymbol{w}},\tag{12}$$

where $\{\cdot,\cdot\}_{\frac{1}{2}\int\langle\boldsymbol{\chi},A(\boldsymbol{\chi})\rangle}$ is the variational Poisson bracket *induced* for functionals $H[\boldsymbol{w}]\colon\Gamma(\widetilde{\pi})\to\Bbbk$ from a *given* variational Poisson structure $\{\cdot,\cdot\}_{\frac{1}{2}\int\langle\boldsymbol{\xi}\cdot B(\boldsymbol{\xi})\rangle}$ for the pull-backs $H\bigl[\boldsymbol{w}[\boldsymbol{u}]\bigr]\colon\Gamma(\pi)\to\Bbbk$.

[24]Integrating by parts,

$$\tfrac{1}{2}\int\Bigl\langle(\boldsymbol{\xi})\Bigl(\overleftarrow{\ell}^{(\boldsymbol{u})}_{\boldsymbol{w}}\Bigr)^\dagger\cdot\Bigl(P_\tau\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\Bigr)^\tau\circ\overrightarrow{\ell}^{(\boldsymbol{u})\,\dagger}_{\boldsymbol{w}}\Bigr)(\boldsymbol{\xi})\Bigr\rangle\cong$$

$$\cong\tfrac{1}{2}\int\Bigl\langle\boldsymbol{\xi}\cdot\Bigl(\overrightarrow{\ell}^{(\boldsymbol{u})}_{\boldsymbol{w}}\circ P_\tau\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigl(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\Bigr)^\tau\overrightarrow{\ell}^{(\boldsymbol{u})\,\dagger}_{\boldsymbol{w}}\Bigr)(\boldsymbol{\xi})\Bigr\rangle=\tfrac{1}{2}\int\langle\boldsymbol{\xi},\overrightarrow{A}(\boldsymbol{\xi})\rangle\ ,$$

we construct the Hamiltonian differential operator that takes variational covectors to (the generating sections of) evolutionary vector fields.

For a given operator A over $J^\infty(\widetilde{\pi})$, its factorisation problem can be very hard. Solutions $\bigl(\pi,\ \boldsymbol{w}(\boldsymbol{x},[\boldsymbol{u}]),\ B\bigr)$ are "good" if the coefficients of differential operator B, referred to $(\boldsymbol{x},u^j_\sigma)$, do not contain the jet variables u^j_σ explicitly (that is, B is Darboux-canonical in the sense of Corollary 1). This technique correlates senior Poisson structures $\{\cdot,\cdot\}_{\frac{1}{2}\int\langle\boldsymbol{\chi},A_{i+1}(\boldsymbol{\chi})\rangle}$ for multi-Hamiltonian hierarchies with the junior Hamiltonian operators B_i for the respective modified hierarchies of completely integrable PDE systems (see [31, 38] and references therein).

Example 3.2 (root systems)**.** Consider the Korteweg–de Vries equation

$$w_t = -\tfrac{1}{2}w_{xxx} + 6ww_x = \bigl(-\tfrac{1}{2}D_x^3 + 2wD_x + 2D_x\circ w\bigr)\left(\frac{\delta}{\delta w}\int \tfrac{1}{2}w^2\mathrm{d}x\right)$$

realised by using its second, field-dependent variational Poisson structure (after the Fourier transform, the Hamiltonian operator $\hat{A}_2^{\mathrm{KdV}}$ encodes the Virasoro algebra, cf. [6]). Consider the Miura substitution[50] $w=\frac{1}{2}(u_x^2-u_{xx})$; let us explain in advance that the conserved current $w\,\mathrm{d}x = \frac{1}{2}(u_x^2-u_{xx})\,\mathrm{d}x$ stems – via the First Noether theorem – from Noether's symmetry $\varphi_1 = u_x$ of the action $\mathcal{L} = \iint\bigl(\frac{1}{2}u_xu_y + \frac{1}{2}e^{2u}\bigr)\,\mathrm{d}x\wedge\mathrm{d}y$ for the Liouville equation $\mathcal{E}_{\mathrm{Liou}} = \{u_{xy}=\exp(2u)\}$.

The mapping $w = w\bigl(x,[u]\bigr)$ is thus determined by the *integral* $w\in\ker\frac{\mathrm{d}}{\mathrm{d}y}\Bigr|_{\mathcal{E}_{\mathrm{Liou}}}$; we recall that the coefficient "2" in the right-hand side of $u_{xy}=\exp(2u)$ is the only entry of the Cartan matrix $K=\|2\|_1^1$ for Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

By definition, put $\vartheta=\frac{1}{2}u_x$ so that $\ell^{(u)}_\vartheta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ is the first Hamiltonian operator $\hat{B}_1^{\mathrm{mKdV}}$ of modified KdV hierarchy and so that $w=2\vartheta^2-\vartheta_x$. Denote by $\square = 4\vartheta + D_x = 2u_x + D_x$ the adjoint $(\ell^{(\vartheta)}_w)^\dagger$ of linearisation $\ell^{(\vartheta)}_w = 4\vartheta - D_x$. By using the chain rules $\delta/\delta\vartheta = (\ell^{(\vartheta)}_w)^\dagger\circ\delta/\delta w$ and $\delta/\delta u = (\ell^{(u)}_\vartheta)^\dagger\circ(\ell^{(\vartheta)}_w)^\dagger\circ\delta/\delta w$, we cast the (potential) modified KdV equations,

$$u_t = -\tfrac{1}{2}u_{xxx} + U_x^3 = \square(w), \qquad \vartheta_t = -\tfrac{1}{2}\vartheta_{xxx} + 12\vartheta^2\vartheta_x,$$

into their canonical De Donder–Weyl's representation[14]

$$u_t = \frac{\delta H[w[\vartheta]]}{\delta\vartheta}, \qquad \vartheta_t = -\frac{\delta H[w[\vartheta[u]]]}{\delta u} \qquad \text{with } H=\int\tfrac{1}{2}w^2\,\mathrm{d}x.$$

Clearly, we then recover the KdV evolution

$$w_t = \bigl(\ell^{(\vartheta)}_w(\vartheta_t)\bigr)[w] = \bigl(-\tfrac{1}{2}D_x^3 + 4wD_x + 2w_x\bigr)\bigl(\delta H\bigl(x,[w]\bigr)/\delta w\bigr).$$

This factorisation pattern,

$$\hat{A}_2^{\mathrm{KdV}} = \ell^{(\vartheta)}_w\circ\ell^{(u)}_\vartheta\circ\bigl(\ell^{(\vartheta)}_w\bigr)^\dagger,$$

is common to all the root systems of ranks $r\geqslant 1$, that is, for the (modified) Drinfel'd–Sokolov hierarchies[19]. It is seen that the hierarchy for respective analogue of potential modified KdV equation for $\boldsymbol{u}$ constitutes the maximal commutative subalgebra in the Lie algebra of Noether symmetries for Leznov–Saveliev's nonperiodic 2D Toda chains[48] $u^i_{xy} = \exp\Bigl(\sum\limits_{j=1}^{r}\frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}\cdot u^j\Bigr)$. The algorithm for construction of r integrals[65] $w^1,\ \ldots,$

w^r is known from [60], see [37] for an illustration. De Donder–Weyl's formalism[14] furthers the approach: the variables $\vartheta_1, \ldots, \vartheta_r$ are the canonical conjugate *momenta*, $\vartheta_i = \partial L/\partial u^i_y$, for the genuine *coordinates* $u^1, \ldots, u^r$ satisfying the 2D Toda equations. The Lagrangian density is $L = \frac{1}{2}\kappa_{ij}u^i_x u^j_y + \langle a_i, \exp(K^i_j u^j)\rangle$, where each row of the Cartan matrix $K = \|K^i_j\|$ is symmetrised to $\kappa = \|a_i \cdot K^i_j\|^{j=1,\ldots,r}_{i=1,\ldots,r}$ by using the root lengths, $a_i := 1/\langle\alpha_i, \alpha_i\rangle$ at every i. Consequently, the junior variational Poisson structure for the modified Drinfel'd–Sokolov hierarchy is

$$\widehat{B}_1 = \left\| \frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle\langle\alpha_j, \alpha_j\rangle} \frac{\mathrm{d}}{\mathrm{d}x} \right\|^{j=1,\ldots,r}_{i=1,\ldots,r}$$

for every root system $\alpha_1, \ldots, \alpha_r$.

Having thus factorised the higher, field-dependent variational Poisson structures through the junior variational Poisson structures whose coefficients do not depend explicitly on the new fields $\phi \in \Gamma(\pi)$, we reduce the large deformation quantisation problem for functionals $F[\boldsymbol{w}]$, $G[\boldsymbol{w}]$, $H[\boldsymbol{w}]\colon \Gamma(\widetilde{\pi}) \to \Bbbk$ to a computationally much simpler Moyal–Groenewold–Weyl's case of the same functionals $F\big[\boldsymbol{w}[\boldsymbol{u}]\big]$, $G\big[\boldsymbol{w}[\boldsymbol{u}]\big]$, $H\big[\boldsymbol{w}[\boldsymbol{u}]\big]\colon \Gamma(\pi) \to \Bbbk$, now referred to the bundle π of (potential) modified hierarchies.

3.3.4. Let $\boldsymbol{\mathcal{P}} = \frac{1}{2}\int\langle\boldsymbol{\xi} \cdot P_\tau|_{\boldsymbol{x}}(\boldsymbol{\xi})\rangle$ be a variational Poisson bi-vector such that in a given coordinate system, its coefficients $P^{ij}_\tau(\boldsymbol{x})$ do not depend explicitly on the fibre variables $\boldsymbol{u}_\sigma$ in the bundle π of physical fields. Let $F = \int f(\boldsymbol{x}_1, [\boldsymbol{u}]) \cdot \mathrm{dvol}(\boldsymbol{x}_1)$ and $G = \int g(\boldsymbol{x}_2, [\boldsymbol{v}]) \cdot \mathrm{dvol}(\boldsymbol{x}_2)$ be integral functionals, or *scalars*, referred to two identical copies of the jet space $J^\infty(\pi)$. The variational generalisation of Moyal–Groenewold–Weyl's star-product $\star_\hbar$ of F and G is the local functional $F \star_\hbar G$ – that is, itself a scalar – which is constructed from (3) for the variational Poisson bracket $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ by using the techniques from geometry of iterated variations and which is expressed by the formula

$$F \star_\hbar G \cong \int \Bigg(f\big|_{(\boldsymbol{x}_1,[\boldsymbol{u}])} \exp\bigg(\frac{\overleftarrow{\partial}}{\partial u^i_\sigma} \lceil \Big(-\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Big)^\sigma \Big(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_1}\Big)^\tau \rceil \cdot \frac{\overrightarrow{\partial}}{\partial \xi_{i,\tau}} \Big(\frac{\hbar}{2}\xi_\alpha P^{\alpha\beta}_\lambda(\boldsymbol{x})\xi_{\beta,\lambda}\Big) \frac{\overleftarrow{\partial}}{\partial \xi_{j,\zeta}} \cdot \\ \cdot \lceil \Big(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Big)^\zeta \Big(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}_2}\Big)^\chi \rceil \frac{\overrightarrow{\partial}}{\partial v^j_\chi} \bigg) g\big|_{(\boldsymbol{x}_2,[\boldsymbol{v}])} \Bigg) \Bigg|_{\substack{\boldsymbol{x}_1=\boldsymbol{x}=\boldsymbol{x}_2\\ [\boldsymbol{u}]=[\boldsymbol{v}]}} \cdot \mathrm{dvol}(\boldsymbol{x}). \quad (13)$$

The angular brackets $\lceil\ldots\rceil$ in (13) conclude total derivatives the action of which – in every term of the towered Λ-graph expansion of $\star_\hbar$ – antecedes[25] the action of partial

[25]Likewise, the action of total derivatives contained, e. g., in $F \star_\hbar G$ itself constituting a part of the object $(F \star_\hbar G) \star_\hbar H - F \star_\hbar (G \star_\hbar H)$ is also delayed until all the partial derivatives would have acted on f or g.

derivatives with respect to u^i_σ and v^j_χ. This implies that

$$F \star_\hbar G \cong F \times G + \frac{\hbar^1}{1!}\{F,G\}_{\boldsymbol{\mathcal{P}}} + \frac{\hbar^2}{2!}\int \Bigl(f\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigr) \frac{\overleftarrow{\partial}}{\partial u^{i_1}_{\sigma_1}} \frac{\overleftarrow{\partial}}{\partial u^{i_2}_{\sigma_2}} \left(-\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^{\sigma_1\cup\sigma_2} \left(\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^{\tau_1\cup\tau_2} \cdot$$
$$\cdot \frac{\overrightarrow{\partial}}{\partial \xi_{i_1,\tau_1}} \left(\tfrac{1}{2}\xi_{\alpha_1} P^{\alpha_1\beta_1}_{\lambda_1}(\boldsymbol{x})\xi_{\beta_1,\lambda_1}\right) \frac{\overleftarrow{\partial}}{\partial \xi_{j_1,\zeta_1}} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_{i_2,\tau_2}} \left(\tfrac{1}{2}\xi_{\alpha_2} P^{\alpha_2\beta_2}_{\lambda_2}(\boldsymbol{x})\xi_{\beta_2,\lambda_2}\right) \frac{\overleftarrow{\partial}}{\partial \xi_{j_2,\zeta_2}} \cdot$$
$$\cdot \left(\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^{\zeta_1\cup\zeta_2} \left(-\frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}\boldsymbol{x}}\right)^{\chi_1\cup\chi_2} \frac{\overrightarrow{\partial}}{\partial u^{j_2}_{\chi_2}} \frac{\overrightarrow{\partial}}{\partial u^{j_1}_{\chi_1}} \Bigl(g\big|_{(\boldsymbol{x},[\boldsymbol{u}])}\Bigr) \cdot \mathrm{dvol}(\boldsymbol{x}) + \bar{o}(\hbar^2).$$

Produced from the variational Poisson structure $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$, in any other system of coordinates on $J^\infty(\pi)$ the star-product $F \star_\hbar G$ is expressed by using the postulate that this local functional is a scalar indeed: the total derivatives $\mathrm{d}/\mathrm{d}\boldsymbol{x}$ and (the derivatives with respect to) jet fibre coordinates $\boldsymbol{u}_\sigma$ are replaced – via the chain rule – just where they are in the above formula.

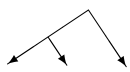
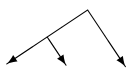
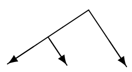
The associativity of (13) is proved in a standard way (see Example 2.3 on p. 10 and footnote 25 on the preceding page). The associator $(F \star_\hbar G) \star_\hbar H - F \star_\hbar (G \star_\hbar H)$ of three given integral functionals over $J^\infty(\pi)$ itself is an *integral* (sic!) functional whose density is identically zero at all points $(\boldsymbol{x},[\boldsymbol{u}])$ of $J^\infty(\pi)$ over M^m.

Remark 3.5. The analogous property of Jacobi identity for the variational Schouten bracket $[\![\cdot,\cdot]\!]$ is proven in [33, 35].

Formula (13) provides the deformation quantisation of first and, via factorisation by using the junior Poisson bracket for the modified system, of second variational Poisson structures for Drinfel'd–Sokolov hierarchies (e. g., for the Korteweg–de Vries equation that corresponds to the root system A_1 of Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ — or to the Virasoro algebra).

3.3.5. Let us now take generic variational Poisson brackets $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$, i. e., beyond Moyal's case $P^{ij}_\tau(\boldsymbol{x})$ of their coefficients' field independence. For instance, suppose that factorisation (12), reducing a given Hamiltonian operator $\widehat{A}_2$ on $J^\infty(\widetilde{\pi})$ to Moyal's case for $\widehat{B}_1$ on $J^\infty(\pi)$, is not yet known.

We recall that every weighted graph Γ in Kontsevich's summation formula for star-product $\star_\hbar$ determines a local variational operator whose behaviour under coordinate reparametrisations is governed by the postulate of scalar output for the operator's action on integral functionals. (In particular, the Hamiltonians F, G, H in the associator and the Jacobiator for $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$, containing F, G, and H, are integral functionals.) In terms of graphs, the associativity of $\star_\hbar$ is inferred from the factorisation

$(F \star_\hbar G) \star_\hbar H - F \star_\hbar (G \star_\hbar H) = \Diamond\Big($  $+$  $+$  $\Big)$. (1)

Every edge issued from the local variational operator $\Diamond$ proceeds over its argument's vertices by the Leibniz rule. The total derivatives $\pm\mathrm{d}/\mathrm{d}\boldsymbol{x}$, whose delayed application is previewed by the composition $\star_\hbar \circ \star_\hbar$ in the left-hand side and by the operator $\Diamond$ in the right-hand side of (1), antecede the partial derivatives $\partial/\partial\boldsymbol{u}_\sigma$. (All the total derivatives will act simultaneously and by their definition, that is, when their arguments are restricted by $j_\infty(\boldsymbol{s})(\,\cdot\,)$ to the jet of a section $\boldsymbol{s} \in \Gamma(\pi)$, see footnote 20 on p. 18.)[26] Let us emphasize that the argument of $\diamondsuit$ itself is a well-defined object, with its own partial and total derivatives going in proper order (see (10)).

The Jacobiator in (1) is the map $\Gamma(\pi) \to 0 \in \Bbbk$ but, in terms of [32, 35], it can be a *synonym of zero*: that (cohomologically trivial) integral functional's density – built from the Hamiltonians F, G, and H — is not necessarily vanishing over all points $\boldsymbol{x} \in M^m$ (which is in contrast with [33, 34]). Consequently, the argument of $\diamondsuit$ could entail a nontrivial output of that local variational operator.[27] Indeed, whenever two or more arrows arrive at a vertex in the argument of $\diamondsuit$, see (8), the order in which partial and then total derivatives go is (9). Therefore, even if the argument of $\diamondsuit$ is cohomologically trivial, the renowned lemma $\delta/\delta\boldsymbol{u} \circ \mathrm{d}_h \equiv 0$ is no longer applicable and the operator $\diamondsuit$ can produce a *nontrivial* integral functional.[28]

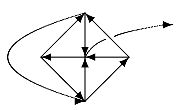
Let us estimate the left-hand side's order in $\hbar$ in (1), which can be *finite* through action of local variational operator $\diamondsuit$ on a synonym of zero. There are two Poisson structures in the Jacobiator $\cong 0$, giving $\hbar^2$; we recall from §2 that $\diamondsuit = \frac{2}{3} \cdot \mathbf{1} + \hbar^1 \cdot (\ldots)$, so that both sides of (1) are $\cong 0$ modulo $\bar{o}(\hbar^2)$. The edges connecting $\diamondsuit$ to its argument are issued from vertices in the graph that encodes the operator, yet those do contain the copies of $\hbar\boldsymbol{\mathcal{P}}$ by construction. Combined with the preceding paragraph, this argument reveals the mechanism for the associativity of star-product $\star_\hbar$ to leak as $\bar{o}(\hbar^2)$, that is, at $O(\hbar^3)$ at the least or at higher orders in $\hbar$. Note that this mechanism shows up in the variational geometry of field models; it was impossible to detect its existence within the classical set-up of [40], see §2 above.

3.4. Gauge freedom. Apart from the transformations $f \mapsto f + \hbar^1 D_1(f) + \bar{o}(\hbar)$, $g \mapsto G + \hbar^1 D_1(g) + \bar{o}(\hbar)$ that could have been used *ab initio* to make the term $B_1(\,\cdot\,,\,\cdot\,) = B_1^+(\,\cdot\,,\,\cdot\,) + B_1^-(\,\cdot\,,\,\cdot\,)$ in $f \star_\hbar g = f \times g + \hbar B_1(f,g) + \bar{o}(\hbar)$ skew-symmetric[40], there are many other gauge degrees of freedom hidden in the geometry of deformation quantisation. (For example, such are the shifts $\mathcal{P} \mapsto \mathcal{P} + \hbar\mathcal{P}_1 + \bar{o}(\hbar)$ in [40, §7] where the elements of formal geometry are engaged.) In this respect, the variational world of field models

[26] Jacobiator's property to be an *object* chops the accumulation of copies of the base manifold M^m and prevents their drain into the respective tuple of such integration domains inside $\diamondsuit$. The same encapsulation mechanism makes the quantum Batalin–Vilkovisky differential $\boldsymbol{\Omega}^\hbar$ a differential indeed, see [32, §3.1].

[27] This effect is typical in the Batalin–Vilkovisky formalism; examples illustrating the usefulness of synonyms of zero in a geometry of iterated variations are available, e. g., from [32] where we proved that the BV-Laplacian Δ *is* a graded derivation of the variational Schouten bracket $[\![\cdot,\cdot]\!]$.

[28] In the set-up of variational Poisson bi-vectors which are installed at every vertex of the graphs in $\diamondsuit$, there still is a possibility to have graphs with only *one* edge connecting the operator with its argument. For instance, consider the figure


$(\ldots)$.

It is readily seen that two edges are issued from every vertex of this graph; there are neither tadpoles nor multiple edges in it. Applying the local variational operator encoded by this graph to any cohomologically trivial integral functional, one obtains the integral functional $\Gamma(\pi) \to 0 \in \Bbbk$ the density of which is identically vanishing at all $\boldsymbol{x} \in M^m \hookrightarrow J^\infty(\pi)$.

is much more flexible than the finite-dimensional and rigid classical geometry. Let us compare the sets of admissible gauge transformations in the two pictures.

3.4.1. Enough has been said about the freedom in describing the models to-quantise by using

- local coordinates $\boldsymbol{x} \in V_\alpha \subseteq M^m$ on the base – and their arbitrary smooth reparametrisations $\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{x}}(\boldsymbol{x})$,
- variables $\boldsymbol{u}$ in the fibres $N^n \simeq \pi^{-1}(\boldsymbol{x})$ for the (affine) bundle π of physical fields — and such variables' (affine) reparametrisations $\widetilde{\boldsymbol{u}} = \widetilde{\boldsymbol{u}}(\boldsymbol{x}, \boldsymbol{u})$ performed pointwise over $M^m \ni \boldsymbol{x}$,
- jet fibre variables $\boldsymbol{u}_\sigma$ — and their reparametrisations, which are induced at $|\sigma| > 0$ from the former through the chain rules,
- and by using the (base-preserving) Miura-type, positive differential order substitutions $\widetilde{\boldsymbol{u}}\bigl(\boldsymbol{x}, [\boldsymbol{u}]\bigr)$.

The behaviour of local variational polydifferential operators, which are encoded by the graphs in expansion (3), is controlled by the postulate that every such operator's output is a scalar. (This is also true in the classical picture of [40]; that idea is a sample contribution to the finite-dimensional geometry from field-theoretic set-up: by finding proper variational analogues of classical objects, we shed more light on the rules of operation with such spectrally-reduced counterparts, $\mathrm{d}/\mathrm{d}\boldsymbol{x} := 0$ at $m = 0$.)

3.4.2. $F \mapsto F + \int \mathrm{d}_h(\Theta) \sim O(1)$ *in* $\hbar$. Viewed as functions $N^n \to \Bbbk$, to be multiplied by using $\times$, the objects $f, g, h \in C^\infty(N^n)$ have their values uniquely defined at all points $\boldsymbol{u} \in N^n$. Viewed as Hamiltonians, to be Poisson-multiplied by using $\{\cdot,\cdot\}_{\mathcal{P}}$, each "basement" f, g, or h of a graph's leg acquires a gauge degree of freedom, which is the shift by a global – over N^n – constant $\in \Bbbk$.

Viewed as functionals $\Gamma(\pi) \to \Bbbk$ that take configurations $\phi \in \Gamma(\pi)$ of N^n-valued physical fields over M^m to numbers, the integral objects $F, G, H \in \bar{H}^m(\pi) \hookrightarrow \overline{\mathfrak{M}}^m(\pi) \ni F \times G$ can be shifted by using the null functionals $Z \colon \Gamma(\pi) \to 0 \in \Bbbk$. Those can be of topological nature,[29] $Z \in H^m(\pi)$. We always quotient them out in this paper by taking the factorgroup $\bar{H}^m(\pi)/H^m(\pi)$. Secondly, the null integral functionals $\Gamma(\pi) \to 0 \in \Bbbk$ can mark the zero class $\int \mathrm{d}_h(\Theta) \cong \int 0 \in \bar{H}^m(\pi)$ in the senior horizontal cohomology group[30] for $J^\infty(\pi)$ over M^m. However, we recall about the effect which is produced by

[29] For instance, set $m = 1$, let $M^m := \mathbb{S}^1 \cup \mathbb{S}^1$, take the usual angle variables $\varphi_1, \varphi_2 \colon \mathbb{R}^1 \to \mathbb{S}^1$ on the two circles, and consider the null Lagrangian $\mathcal{L} = \int \mathrm{d}\varphi_1 - \int \mathrm{d}\varphi_2$ that takes *every* section of an affine bundle π over such M^1 to $2\pi - 2\pi = 0 \in \Bbbk$. Obviously, the cohomology class $\mathcal{L}$ in $H^1(\pi)$ is nonzero for the top-degree form $\mathrm{d}\varphi_1 - \mathrm{d}\varphi_2$; for it is only locally but not globally exact.

[30] The integrations by parts $\cong$ over M^m are nominally present in the construction of horizontal cohomolohgy groups $\bar{H}^i(\pi)$ for the jet space $J^\infty(\pi)$ over the bundle π of physical fields; referring to §3.3.5 (cf. [32, 34]), we advise supreme caution in doing that — if doing at all.

By default, let us technically assume that no boundary terms would ever appear from ∂M^m in any formulae. For that, either let the base M^m be a closed manifold (hence $\partial M^m = \varnothing$; for instance, take $M^1 = \mathbb{S}^1$) or the class $\Gamma(\pi)$ of *admissible* sections for the bundle π is such that they all vanish at ∂M^m together with a sufficient number of their one-side derivatives – enough for all the integrands under study to also vanish at ∂M^m. For instance, suppose that $M^1 = \mathbb{R}$ and all the field profiles $\boldsymbol{u} = \phi(\boldsymbol{x})$ rapidly decay towards the spatial infinity. Still let us warn the reader against idle, unmotivated integrations by parts.

synonyms of zero, that is, by elements $Z \cong \int 0$ with not everywhere vanishing densities. Higher-order local variational operators in expansion (3) can create cohomologically *nontrivial* terms from cohomologically trivial Hamiltonians $\int \mathrm{d}_h(\Xi), \int \mathrm{d}_h(\Theta), \int \mathrm{d}_h(\Upsilon)$, see §3.3.5. This shows that a gauge transformation $F \mapsto F + \int \mathrm{d}_h(\Xi)$, $G \mapsto G + \int \mathrm{d}_h(\Theta)$, $H \mapsto H + \int \mathrm{d}_h(\Upsilon)$ of order $O(1)$ with respect to $\hbar$, preserving the cohomology classes of Poisson brackets at $\sim \hbar^1$ in the star-products and nohow touching the functionals' values at sections $\phi \in \Gamma(\pi)$, does modify both the tails $\sim \hbar^{\geqslant 2}$ in the star-product expansions and associativity balance $\sim \hbar^{\geqslant 3}$ for $\star_\hbar$. This degree of gauge freedom could not be detected in the classical set-up of [40].

Conjecture 2. For completely integrable bi-Hamiltonian hierarchies $\{H_i^{(k)}, 1 \leqslant k \leqslant n, i \in \mathbb{N}; [\![\varepsilon \boldsymbol{\mathcal{P}}_1 + \boldsymbol{\mathcal{P}}_2, \varepsilon \boldsymbol{\mathcal{P}}_1 + \boldsymbol{\mathcal{P}}_2]\!] \cong 0\}$ of Korteweg–de Vries type, it is the classical Gardner deformations algorithm[50] that recursively generates the Poisson-commuting Hamiltonians' *densities* in such gauges that not only $H_i^{(\cdot)} \star_\hbar H_j^{(\cdot)} = H_i^{(\cdot)} \times H_j^{(\cdot)} + \bar{o}(\hbar^1)$ but moreover, $H_i^{(\cdot)} \star_\hbar H_j^{(\cdot)} = H_i^{(\cdot)} \times H_j^{(\cdot)}$ at all orders of perturbation.

3.4.3. $\boldsymbol{\mathcal{P}} + \int \mathrm{d}_h(\boldsymbol{\mathcal{Q}}) \sim O(1)$ *in* $\hbar$. In the classical picture $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$ of finite-dimensional Poisson manifolds, the Poisson bi-vectors $\mathcal{P} = \frac{1}{2}\langle \xi_i P^{ij}(\boldsymbol{u}) \xi_j \rangle \in \Gamma(\bigwedge^2 TN^n)$ are rigid solutions of the classical master-equation $[\![\mathcal{P}, \mathcal{P}]\!] = 0$. The Poisson brackets

$$\{\cdot,\cdot\} \colon C^\infty(N^n)/\mathbb{R} \times C^\infty(N^n)/\mathbb{R} \to C^\infty(N^n)$$

uniquely determine the coefficients P^{ij} at every $\boldsymbol{u} \in N^n$. In retrospect, the *Hamiltonian* $\{f, g\}_{\mathcal{P}}$ can be shifted by using a global (over N^n) constant $\in \Bbbk = \mathbb{R}$; the Hamiltonians f and g are determined each up to an overall constant. Under smooth reparametrisations $\boldsymbol{u} = \boldsymbol{u}(\widetilde{\boldsymbol{u}})$, the coefficients $P^{ij}(\boldsymbol{u})$ are transformed into $\widetilde{P}^{\alpha\beta}(\widetilde{\boldsymbol{u}})$ obeying the postulate of Poisson bracket's covariance, $\{f, g\}_{\mathcal{P}}(\boldsymbol{u}) = \{f, g\}_{\widetilde{\mathcal{P}}}(\widetilde{\boldsymbol{u}})$. The same principle plugs $\boldsymbol{u}(\widetilde{\boldsymbol{u}})$ for $\boldsymbol{u}$ (resp., expands $\partial/\partial\boldsymbol{u}$ via $\partial/\partial\widetilde{\boldsymbol{u}}$ through the chain rule) wherever appropriate in each higher-order term within (3); none of those terms is affected by the constant shifts of the arguments f and g. Summarising, the classical Poisson geometry of polydifferential operators and their scalar output is very rigid.

The variational Poisson bi-vectors $\boldsymbol{\mathcal{P}} = \frac{1}{2}\int \xi_i P^{ij}_\tau(\boldsymbol{x}, [\boldsymbol{u}]) \left(\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{x}}\right)^\tau (\xi_j) \cdot \mathrm{dvol}(\boldsymbol{x})$, that is, solutions of the classical master-equation $[\![\boldsymbol{\mathcal{P}}, \boldsymbol{\mathcal{P}}]\!] \cong 0$ over $J^\infty(\pi)$, can be gauge-transformed to $\boldsymbol{\mathcal{P}}' = \boldsymbol{\mathcal{P}} + \int \mathrm{d}_h(\boldsymbol{\mathcal{Q}})$ by adding cohomologically trivial bi-vectors[31]

$$\int \mathrm{d}_h\big(\boldsymbol{\mathcal{Q}}\big|_{(\boldsymbol{x},[\boldsymbol{u}])}([\boldsymbol{\xi}], [\boldsymbol{\xi}])\big) \cong \int 0.$$

This degree of freedom has order $O(1)$ in $\hbar$; it was invisible in the classical setting of [40]. After such transformation, the variational Poisson bracket defined on p. 19 is

Strange though it may seem, almost all of the above is irrelevant in practice (so that indeed, much physics can be expressed by the boundary values at ∂M^m, cf. [57]). An overwhelming majority of integrations by parts which we addressed in §3.2 are performed over the supports $\operatorname{supp} \delta\boldsymbol{s}\big(\,\cdot\,, \boldsymbol{s}(\,\cdot\,)\big) \subseteq M^m$ of fields' $\boldsymbol{s} \in \Gamma(\pi)$ virtual excitations. Personifying the concept of physical fields, these variations must vanish at their supports' boundaries together with all their derivatives, i. e., $j_\infty(\delta\boldsymbol{s})|_{\partial(\operatorname{supp}\delta\boldsymbol{s})} = 0$, while such domains themselves can be taken arbitrarily small.

[31]In fact, by adding any cohomologically trivial variational multi-vectors $\int \mathrm{d}_h(\boldsymbol{\mathcal{Q}}|_{(\boldsymbol{x},[\boldsymbol{u}])}([\boldsymbol{\xi}], \ldots, [\boldsymbol{\xi}]))$, whence the homotopy theory is more natural in the *variational* set-up of field models in $J^\infty(\pi)$ rather than in the Poisson geometry $(N^n, \{\cdot,\cdot\}_{\mathcal{P}})$ of mechanical systems.

shifted by a cohomologically trivial term[32] (which is skew-symmetric with respect to the variational derivatives of Poisson bracket's arguments F and G; such trivial terms could vanish identically). The cohomologically trivial shifts $\boldsymbol{\mathcal{P}} \mapsto \boldsymbol{\mathcal{P}} + \int \mathrm{d}_h(\boldsymbol{\mathcal{Q}})$ of variational Poisson bi-vector can nevertheless modify the tail $\sim \hbar^{\geqslant 2}$ of expansion (3).

In retrospect, every term in (3) can be shifted – independently from the rest – by using a cohomologically trivial term; also, one could decide to integrate by parts in a term at hand. These shifts are not necessarily related to the shifts which those expansion terms do acquire from either the cohomology class-preserving shifts of the Hamiltonians or the shifts of variational Poisson bi-vector.

3.4.4. $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] \cong 0$. Likewise, the left-hand side of Jacobi identity in (1) – that is, the variational tri-vector's value $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!](F,G,H)$ at the three Hamiltonians F, G, and H from the associator $(F \star_\hbar G) \star_\hbar H - F \star_\hbar (G \star_\hbar H)$ in the left-hand side of (1) – itself can be a synonym of zero, then contributing with order $\sim \hbar^{\geqslant 3}$ terms to the leak of associativity (see §3.3.5). Note that neither the gauge shifts $F \mapsto \int \mathrm{d}_h(\Xi)$, etc., nor a gauge shift $\boldsymbol{\mathcal{P}} \mapsto \boldsymbol{\mathcal{P}}' = \boldsymbol{\mathcal{P}} + \int \mathrm{d}_h(\boldsymbol{\mathcal{Q}})$ would alter[33] the tri-vector $[\![\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{P}}]\!] = [\![\boldsymbol{\mathcal{P}}',\boldsymbol{\mathcal{P}}']\!]$. This proves that the order – moreover, the entire channel — of associativity leak is well defined for a given variational Poisson structure $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$; these brackets mark the deformation quantisation scenarios for field models.

[32]For instance, take the most renowned Hamiltonian operator $\hat{A}_1^{\mathrm{KdV}} = \mathrm{d}/\mathrm{d}x$ and construct the variational Poisson bi-vector $\boldsymbol{\mathcal{P}} = \frac{1}{2}\int \xi\xi_x \,\mathrm{d}x$. Now let $\boldsymbol{\mathcal{Q}}$ be quadratic in the variables ξ_σ; in the frames of maximal minimalism, put $\boldsymbol{\mathcal{Q}} := 2\xi\xi_x$. Consider the variational Poisson bi-vector $\boldsymbol{\mathcal{P}}' = \boldsymbol{\mathcal{P}} + \int \mathrm{d}_h(\boldsymbol{\mathcal{Q}}) = \frac{1}{2}\int(\xi\xi_x + \xi\xi_{xx})\,\mathrm{d}x$. For $p_1 = \delta F/\delta u$ and $p_2 = \delta G/\delta u$, we have that

$$\{F,G\}_{\boldsymbol{\mathcal{P}}'} = \tfrac{1}{2}\int \left(p_1 \cdot \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}(p_2) - (p_1)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x} \cdot p_2 \right) \cdot \mathrm{d}x + \tfrac{1}{2}\int \left(p_1 \cdot \frac{\overrightarrow{\mathrm{d}^2}}{\mathrm{d}x^2}(p_2) - (p_1)\frac{\overleftarrow{\mathrm{d}^2}}{\mathrm{d}x^2} \cdot p_2 \right) \cdot \mathrm{d}x,$$

where the second term is of course trivial, for

$$\int \left(p_1 \cdot \frac{\overrightarrow{\mathrm{d}}}{\mathrm{d}x}(p_2) - (p_1)\frac{\overleftarrow{\mathrm{d}}}{\mathrm{d}x} \cdot p_2 \right) \cdot \mathrm{d}x \cong \int 0$$

in the bundle's horizontal cohomology.

[33]This particular type of stability is common for the classical and variational geometries.

4. Conclusion

4.1. Beyond the first step. Realised in terms of graphs Γ and their weights $w(\Gamma)$, the deformation quantisation procedure[40] builds the star-product's associativity on the Jacobi identity for Poisson structure in deformation's leading order. This factorisation is encoded by formula (1) using the graphs. We established the rule that associates local variational polydifferential operators with those graphs and we analysed such operators' behaviour in the course of smooth reparametrisations for local coordinates $\boldsymbol{x}$ on the base manifold M^m and of smooth (in particular, *affine*) reparametrisations of the field variables $\boldsymbol{u}$ and their derivatives with respect to $\boldsymbol{x}$. This correspondence between variational polydifferential operators and graphs sheds more light on the classical picture of deformation quantisation[40], which was conceived for *affine* manifolds and which was then furthered in [40, §7] to the general case of smooth finite-dimensional manifolds by using Gel'fand–Kazhdan's formal geometry[24]. In this paper we proposed another approach. The alternative scheme is based on the natural property of such local variational polydifferential operators' output to be scalars. By taking this principle into account, one gains a clear view of the construction of objects in deformation quantisation theory, both in the full set-up of fields and in the reduced set-up of classical Poisson geometry. To perceive why it is this way of reasoning that works, we first realised the affine manifolds from classical picture by now taking them as fibres in the affine bundles for gauge fields; as soon as this was done, the dependence of fibre coordinate reparametrisations on points $\boldsymbol{x} \in M^m$ of the bundles' base manifolds prescribed both the postulate above and the choice of relevant mathematical apparatus. The new technique merges the standard concepts from variational Poisson geometry on jet spaces with Gel'fand's language of singular linear integral operators[25] and with the arising geometry of iterated variations[32]. Designed originally for the intrinsic regularisation of Batalin–Vilkovisky's method to quantise gauge systems, this technique provides a natural solution to the problem of lifting Kontsevich's quantisation procedure to the geometry of field theories. The deformation quantisation scheme that was known before constitutes a part of our present reasoning, now grasping not only the field values $(\phi^1, \ldots, \phi^m)(\boldsymbol{x})$ but also their variation along the $m > 0$ dimensions of underlying space-time $M^m \ni \boldsymbol{x} = (x^1, \ldots, x^m)$. The concept which we analysed in this paper yields an explicit algorithm for regular deformation quantisation of bi-Hamiltonian hierarchies of Korteweg–de Vries type[6, 19] and other relevant PDE systems, e. g., gauge field models. The algorithm's input data – variational Poisson structures $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ – mark points in the moduli spaces of deformation quantisations for all such field theory models.

Starting with the commutative associative unital algebras $(\mathcal{A}, \times)$ containing sums of integral functionals' formal products that take $\Gamma(\pi) \to \Bbbk$ in the bundle π of physical fields, we have outlined a geometric mechanism for construction of non-commutative associative unital algebras $\big(\mathcal{A}[[\hbar]], \star_\hbar\big)$ of quantum functionals and their star-products. For every such datum, the quantisation scheme continues in a natural way,

$$(\mathcal{A}, \times) \longmapsto \big(\mathcal{A}[[\hbar]], \star_\hbar\big) \longmapsto \big(\mathcal{A}[[\hbar]], \times_\hbar\big),$$

towards the commutative but not associative unital algebras $\big(\mathcal{A}[[\hbar]], \times_\hbar\big)$ of cyclic words and their topological pair-of-pants multiplication $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1$. Let us recall that the formal noncommutative variational symplectic geometry[35] itself is a lift – to the set-up of quantum field models – of the seminal finite-dimensional construction from the pioneering paper [45].

4.2. Discussion: possible physical sense. Over the space-time $\big(M^m, \text{dvol}(\,\cdot\,)\big)$ let us tower the (affine) bundle π of physical fields. Leaving aside the question what such fields *really* are and what could there be the actual mechanism of the volume element's dependence $\text{dvol}\big(\boldsymbol{x}, \boldsymbol{s}(\boldsymbol{x})\big)$ on the fields, let us accept that the fibre variables $\boldsymbol{u}$ satisfy their own equations of motion (e. g., Yang–Mills' or Einstein's equations $\mathcal{E}$ derived from an appropriate action functional), so that the class of admissible sections $\boldsymbol{u} = \boldsymbol{s}(\boldsymbol{x})$ is the solution set $\{\phi \in \Gamma(\pi)\}$.

Such model's *observables* are local functionals $\mathcal{O}\colon \Gamma(\pi) \to \Bbbk$ taking physical field configurations to numbers. These functionals $\mathcal{O}$ are either integral building blocks like $F_i = \int f_i(\boldsymbol{x}_i, [\boldsymbol{u}]) \cdot \text{dvol}(\boldsymbol{x}_i) \in \bar{H}^m(\pi)$ or they are (sums of) formal products $F_1 \cdot \ldots \cdot F_\ell \in \overline{\mathfrak{M}}^m(\pi)$ of such building blocks. It could well be that the virtual shifts $\mathcal{O} = \frac{\partial}{\partial\varepsilon}\big|_{\varepsilon=0} \big(\boldsymbol{S} + \varepsilon\mathcal{O} + \bar{o}(\varepsilon)\big)$ of the model's master-action $\boldsymbol{S}$ form the only admissible class of observables.

Every bundle geometry and in particular, every physical field dynamics described by the equations $\mathcal{E}^\infty \subseteq J^\infty(\pi)$ gives rise to the set of that geometry's variational Poisson structures $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$. We recall that this set is, generally speaking, not only bundle-dependent but also field equation-dependent[46].

A *particle* (possibly, itself necessarily an observable) is a local functional; *quantum particles* $\mathcal{O}^\hbar\colon \Gamma(\pi) \to \Bbbk[[\hbar]]$ yield Planck-constant expansions of their values at physical field configurations $\phi(\boldsymbol{x})$, $\boldsymbol{x} \in M^m$. By specifying the star-product $\star_\hbar$, each variational Poisson structure $\boldsymbol{\mathcal{P}}$ opens a channel of reactions $\mathcal{O}_1^\hbar \otimes \mathcal{O}_2^\hbar \mapsto \mathcal{O}_1^\hbar \star_\hbar \mathcal{O}_2^\hbar$ for quantum particles, see Fig. 4.

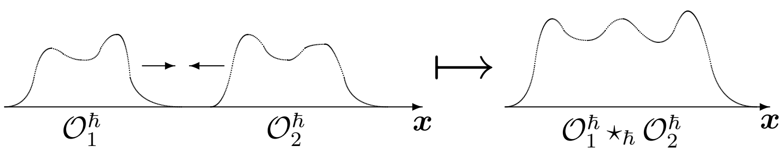
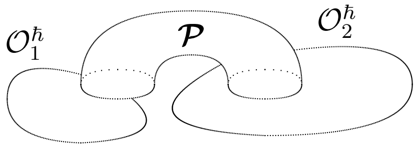


FIGURE 4. The observables – their densities $f(\boldsymbol{x}_1, [\boldsymbol{u}])$ and $g(\boldsymbol{x}_2, [\boldsymbol{v}])$ are drawn schematically — merge into one particle or produce a jet of such objects. The two particles' interaction in the space-time $M^m \ni \boldsymbol{x}$ contains the geometric mechanism of congruence $\boldsymbol{x}_1 = \boldsymbol{x}_2$, $[\boldsymbol{u}] = [\boldsymbol{v}]$ for the respective copies of bundle π where each functional is referred to. In section §3.2 we expressed that mechanism in terms of Gel'fand's language of singular linear integral operators supported on the diagonal — which *is* the congruence, or locality.

While distant, $\text{supp}(f) \cap \text{supp}(g) = \varnothing \subset M^m$, that is, not yet overlapping through a given template $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}(\boldsymbol{x},[\boldsymbol{u}])}$, the two functionals do in fact not interact: $\mathcal{O}_1^\hbar \star_\hbar \mathcal{O}_2^\hbar = \mathcal{O}_1^\hbar \times \mathcal{O}_2^\hbar$. But as soon as the dimensionful objects $\mathcal{O}_1^\hbar$ and $\mathcal{O}_2^\hbar$ approach each other to the extent that the interaction template $\boldsymbol{\mathcal{P}}(\boldsymbol{x}, [\boldsymbol{u}])$ intersects both particles' supports, see (14),



(14)

the leading deformation term in $\star_\hbar$ defined from $\boldsymbol{\mathcal{P}}$ contributes to the star-product $\mathcal{O}_1^\hbar \star_\hbar \mathcal{O}_2^\hbar$ at order $\hbar^1$. (By construction, the supports of higher order interaction terms in expansion (3) for $\star_\hbar$ either are entirely contained within or at most coincide with the support of $\{\cdot,\cdot\}_{\boldsymbol{\mathcal{P}}}$ in M^m.) The output $\mathcal{O}_1^\hbar \star_\hbar \mathcal{O}_2^\hbar$ is the product obtained from its arguments via a given reaction channel $\boldsymbol{\mathcal{P}}$.

We emphasize that it is not the physical fields (e.g., the metric tensor $g_{\mu\nu}(\boldsymbol{x})$ on the space-time fourfold $M^{3,1}$) but it is the local functionals $\mathcal{O}\colon \Gamma(\pi) \to \Bbbk$ which are quantised via deformation. The underlying field portrait manifests in the construction of local functionals through the volume element $\mathrm{dvol}(\,\cdot\,)$.

The idea of delayed integrations by parts (over $M^m \ni \boldsymbol{x}$), which is the key element in the geometry of iterated variations[32], is realised by having the scheduled but not yet performed integrations stored in the derivatives of virtual shifts $\delta\boldsymbol{s}\big(\boldsymbol{x},\boldsymbol{s}(\boldsymbol{x})\big)$ for values $\boldsymbol{s}(\boldsymbol{x})$ of the physical fields. In effect, by temporarily storing the spatial derivatives $\pm\mathrm{d}/\mathrm{d}\boldsymbol{x}$, such test shifts $\delta\boldsymbol{s}$ carry the information about the fields' non-constance over space $M^m \ni \boldsymbol{x}$. One might say that the intermediate components of a not yet completed geometric object $\mathcal{O}_1^\hbar \star_\hbar \mathcal{O}_1^\hbar$ are kept in a meta-stable state; for the derivations are hidden in the graphs' edges instead of showing up in the vertices. The integrations by parts are released from hold at the moment when the object is *observed*, that is, when it is evaluated at a given field portrait $\phi(\boldsymbol{x})$, $\boldsymbol{x} \in M^m$.

So much was pronounced in the past about the interference of fact and process of *observation* on a quantum system under study that we would now hardly add anything: indeed, one must know in advance and keep in mind *what* is going to be measured; for no process of measurement may be interrupted along the way. For instance — and unlike it was the case in classical picture of deformation quantisation over the "space-time" that was shrunk to a point $M^0 = \{\mathrm{pt}\}$, the star-product $\star_\hbar$ does not look associative until its associativity is *examined*. (Quite logically, one may not be sure in advance whether this structure is or is not associative — before the actual verification is performed.) The analytic technique of delayed integrations by parts can therefore be viewed as an attempt of mathematically rigorous phrasing of the rôle of observer in quantum measurements.

For consistency, let us recall from §3.3.5 that we did preview a possibility for $\star_\hbar$ to be not *exactly* associative, this property leaking at orders $\hbar^{\geqslant 3}$ for a class of models. One might wonder whether the associativity of local functionals' multiplication is exact or violated in the nature of all particles, all reaction channels, and all energy spectra.

To conclude this paper, we note that a proper, conceptually immediate lift of Cattaneo–Felder's construction[10] to the variational Poisson set-up of field models speaks in favour of the quantisation itself. In presence of the hidden, stringy space-times D^2 behind the infinite jets $j_\infty(\phi)$ of physical fields $\boldsymbol{x} \mapsto \phi(\boldsymbol{x})$, Kontsevich's graph summation formula (3) calculates Feynman path integrals, the values of which are model's correlation functions. This depicts the probabilistic character of events by using an apparently deterministic language of deformation quantisation.

Acknowledgements. This research was supported in part by JBI RUG project 103511 (Groningen, The Netherlands); a part of this research was done while the author was visiting at the IHÉS (Bures-sur-Yvette, France).

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