Decorated super-Teichmueller space

R. C. PENNER and Anton M. ZEITLIN

DECORATED SUPER-TEICHMÜLLER SPACE

R. C. PENNER AND ANTON M. ZEITLIN

ABSTRACT. We introduce coordinates for a principal bundle $S\tilde{T}(F)$ over the super Teichmüller space $ST(F)$ of a surface F with $s \geq 1$ punctures that extend the lambda length coordinates on the decorated bundle $\tilde{T}(F) = T(F) \times \mathbb{R}_+^s$ over the usual Teichmüller space $T(F)$. In effect, the action of a Fuchsian subgroup of $PSL(2, \mathbb{R})$ on Minkowski space $\mathbb{R}^{2,1}$ is replaced by the action of a super Fuchsian subgroup of $OSp(1|2)$ on the super Minkowski space $\mathbb{R}^{2,1|2}$, where $OSp(1|2)$ denotes the orthosymplectic Lie supergroup, and the lambda lengths are extended by fermionic invariants of suitable triples of isotropic vectors in $\mathbb{R}^{2,1|2}$. As in the bosonic case, there is the analogue of the Ptolemy transformation now on both even and odd coordinates as well as an invariant even two-form on $S\tilde{T}(F)$ generalizing the Weil-Petersson Kähler form. This finally solves a problem posed in Yuri Ivanovitch Manin's Moscow seminar some thirty years ago to find the super analogue of decorated Teichmüller theory and provides a natural geometric interpretation in $\mathbb{R}^{2,1|2}$ for the super moduli of $S\tilde{T}(F)$.

CONTENTS

INTRODUCTION

Let $F = F_g^s$ be a connected orientable surface of genus $g \geq 0$ with $s \geq 1$ punctures and negative Euler characteristic $2 - 2g - s < 0$ in order that $F = \mathcal{U}/\Gamma$ is uniformized by a Fuchsian group Γ. Namely, let $\mathcal{U} = \{z = x + iy \in \mathbb{C} : y > 0\}$

denote the upper half plane with its Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$ and projective matrix group $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm I$ of oriented isometries, where I denotes the identity matrix; there is then an injective representation $\rho : \pi_1 \to PSL(2,\mathbb{R})$ of the fundamental group $\pi_1 = \pi_1(F)$, which is a free group of rank $2g + s - 1$, onto a discrete subgroup $\Gamma < PSL(2,\mathbb{R})$ so that non-trivial loops about punctures are represented by parabolic transformations, namely, those with absolute trace equal to two. See [1, 11, 21] for example.

The *Teichmüller space* of F is

$$T(F) \;=\; \mathrm{Hom}'(\pi_1, PSL(2,\mathbb{R}))/PSL(2,\mathbb{R}),$$

where the prime indicates Fuchsian representations as just defined and the action of $PSL(2,\mathbb{R})$ on Hom' is by conjugation. The *super-Teichmüller space* of F as already formulated in the context of representation theory and moduli spaces by Bryant and Hodgkin [5, 12] (see also [9]) is

$$ST(F) \;=\; \mathrm{Hom}'(\pi_1, OSp(1|2))/OSp(1|2),$$

where the corresponding super Fuchsian representations comprising Hom' are defined to be those whose projection $\pi_1 \to OSp(1|2) \to SL(2,\mathbb{R}) \to PSL(2,\mathbb{R})$ are Fuchsian, where $OSp(1|2)$ denotes the orthosymplectic group of $(2|1)$-by-$(2|1)$ dimensional super matrices with its canonical projection $OSp(1|2) \to SL(2,\mathbb{R})$, cf. [15] or Appendix I, and the action on Hom' is again by conjugation. The similarities are evident. In particular, the *mapping class group* $MC(F)$ of homotopy classes of orientation-preserving homeomorphisms of F acts on $T(F)$ and $ST(F)$ in the natural way.

Consider a graph $\tau \subset F$ embedded in F as a deformation retract also called a *spine* of F. The *valence* of a vertex of τ is the number of half-edges incident upon it, where a half-edge is defined as a complementary component to an interior point of the edge, and τ is said to be *trivalent* if each vertex has valence exactly three. An orientation on F induces the counter clockwise ordering on the half edges of τ incident on each fixed vertex thus giving the abstract graph τ the structure of a *fatgraph* sometimes also called a ribbon graph. There is a combinatorial move on trivalent fatgraph spines $\tau \subset F$ called a *flip* as illustrated in Figure 1, where one contracts an edge of τ with distinct endpoints and then expands the resulting 4-valent vertex in the unique distinct manner in order to produce another trivalent fatgraph spine. This leads to the so-called *Ptolemy groupoid* of F whose objects are homotopy classes of trivalent fatgraph spines in F and whose morphisms are compositions of flips.

As we shall recall in the next introductory section dedicated to the bosonic case, finite compositions of flips act transitively on homotopy classes of trivalent fatgraph spines. It follows that flips generate $MC(F)$ in the sense that if $\tau \subset F$ is a trivalent fatgraph spine and $\varphi \in MC(F)$, then there is a sequence $\varphi(\tau) = \tau_1 - \tau_2 - \cdots - \tau_n = \tau$ of trivalent fatgraph spines of F where any consecutive pair differ by a flip.
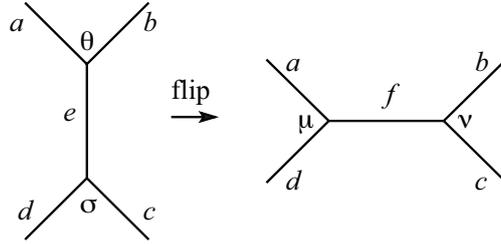
FIGURE 1. A flip on a trivalent fatgraph with notation subsequently explained

In fact [24], the components of $ST(F)$ are naturally indexed by the set $\Omega(F)$ of spin structures on F, and here is a basic difference from the bosonic case: the super-Teichmüller space is disconnected with the $MC(F)$-action permuting components. There are a number of equivalent formulations of spin structure, and we shall rely upon several of them at various junctures. Milnor's elegant formulation of a spin structure on F is a class in the mod two first cohomology of the unit tangent bundle of F which is non-zero on the fiber class; see [16, 13]. More combinatorial formulations from the literature which we shall require are as follows:

• The description [13] due to Johnson in terms of *quadratic forms* $\mathcal{Q}(F)$ on $H_1 = H_1(F; \mathbb{Z}_2)$, i.e., functions $q : H_1 \to \mathbb{Z}_2$ which are quadratic for the intersection pairing $\cdot : H_1 \otimes H_1 \to \mathbb{Z}_2$ in the sense that $q(a+b) = q(a) + q(b) + a \cdot b$ if $a, b \in H_1$.

• Cimasoni and Reshetikhin [7, 8] formulate spin structures using [13] in terms of so-called Kastelyn orientations and dimer configurations on the one-skeleton of a suitable CW decomposition of F as we shall explain in detail later.

• A spin structure on a uniformized surface $F = \mathcal{U}/\Gamma$ is determined by a lift $\tilde{\rho} : \pi_1 \to SL(2, \mathbb{R})$ of $\rho : \pi_1 \to PSL_2(\mathbb{R})$, and Natanzon [18] computes in terms of the quadratic form q that trace $\tilde{\rho}(\gamma) > 0$ if and only if $q([\gamma]) \neq 0$, where $[\gamma] \in H_1$ here and in the sequel is the image of $\gamma \in \pi_1$ under the mod two Hurewicz map.

Our first main result gives yet another combinatorial formulation of spin structures on F in terms of the equivalence classes $\mathcal{O}(\tau)$ of all orientations on a trivalent fatgraph spine $\tau \subset F$, where the equivalence relation is generated by reversing the orientation of each edge incident on some fixed vertex, with the added bonus of a computable evolution under flips.

Theorem A. *Fix any trivalent fatgraph spine $\tau \subset F = F_g^s$. Then $\mathcal{O}(\tau)$ and $\mathcal{Q}(F)$ are isomorphic as affine $H^1(F; \mathbb{Z}_2)$-spaces. Moreover, the action of $MC(F)$ on $\mathcal{Q}(F)$ or $\Omega(F)$ lifts to the action of the Ptolemy groupoid on $\mathcal{O}(\tau)$ illustrated in Figure 2.*

A spin structure on F manifest for instance as a quadratic form $q \in \mathcal{Q}(F)$ distinguishes two types of punctures as follows. If γ_p is a simple loop about the puncture
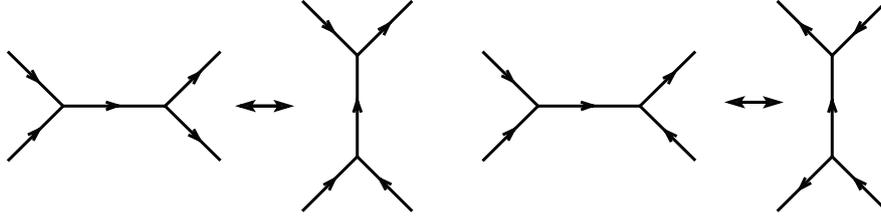
FIGURE 2. Flip transformation for oriented fatgraphs

p of F, then p is called a *Neveu-Schwarz* (NS) puncture if $q([\gamma_p]) = 0$ and is otherwise called a *Ramond* (R) puncture. The dimension of a component of $ST(F)$ depends on the types of punctures, and each component of $ST(F_g^s)$ is in fact [24] a superball of dimension $(6g - 6 + 2s | 4g - 4 + 2n_{NS} + n_R)$, where n_{NS} and n_R are the respective numbers of NS and R punctures. It is not hard to see that n_R must be even since the sum of the $\mathbb{Z}_2$-monodromies about the punctures must agree with the trivial monodromy about a small inessential curve in the surface, and we furthermore evidently have $s = n_{NS} + n_R$.

In the bosonic case [19, 21], there is a principal $\mathbb{R}_+^s$-bundle

$$\tilde{T}(F) = T(F) \times \mathbb{R}_+^s \to T(F)$$

called the *decorated Teichmüller space* of $F = F_g^s$ recalled in some detail in the next introductory section. Letting $\mathbb{R}_+^{p|q}$ here and in the sequel denote the subspace of $\mathbb{R}^{p,q}$ whose even coordinates have positive body, there is likewise a principal $\mathbb{R}_+^{s|n_R}$-bundle over each component of $ST(F)$ with n_R Ramond punctures called the *decorated super-Teichmüller space* and written simply

$$S\tilde{T}(F) \to ST(F),$$

and the action of $MC(F)$ lifts to $S\tilde{T}(F)$ as we shall see. Here we combine the main results from Sections 8 and 9 into one result:

Omnibus Theorem B. *Fix a surface $F = F_g^s$ of genus $g \geq 0$ with $s \geq 1$ punctures, where $2g - 2 + s > 0$, let $\tau \subset F$ be some trivalent fatgraph spine and suppose that ω is an orientation on the edges of τ whose class in $\mathcal{O}(\tau)$ determines the component C of $S\tilde{T}(F)$. Then there are global affine coordinates on C, one even coordinate called a λ-length for each edge and one odd coordinate called a μ-invariant for each vertex of τ, the latter of which are taken modulo an overall change of sign denoted by $\mathbb{Z}_2$; that is, λ-lengths and μ-invariants establish a real-analytic homeomorphism*

$$C \to \mathbb{R}_+^{6g-6+3s|4g-4+2s}/\mathbb{Z}_2.$$

These coordinates are natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{\Gamma} \in S\tilde{T}(F)$, then $\tilde{\varphi}(\tilde{\Gamma})$ is determined by the orientation and coordinates on edges and vertices of $\varphi(\tau)$ induced by φ from the orientation ω, the λ-lengths and

μ-invariants on τ.

Orientations on fatgraph spines evolve under flips in accordance with the previous theorem, and the following super Ptolemy transformation *further describes the evolution of λ-lengths and μ-invariants in the notation of Figure 1 where nearby Roman letters denote λ-lengths, nearby Greek letters denote μ-invariants and $\chi = \frac{ac}{bd}$ denotes the* cross-ratio

$$ef = (ac + bd)\Big(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big),$$

$$\nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1+\chi}}.$$

Finally, there is an even 2-form on $S\tilde{T}(F)$ which is invariant under super Ptolemy transformations, namely,

$$\omega = \sum_v d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2,$$

where the sum is over all vertices v of τ where the consecutive half edges incident on v in clockwise order have induced λ-lengths a, b, c and θ is the μ-invariant of v.

In order to further explain the coordinates and give an intrinsic meaning to the decorated spaces, we shall recall and extend the bosonic case in the next section.

The super-Teichmüller space is of interest on its own as one of the higher Teichmüller theories associated with the simplest supergroup extension of $PSL(2, \mathbb{R})$, namely, the orthosymplectic group $OSp(1|2)$ which is however special among supergroups in that its invariant bilinear form is non-degenerate. Nevertheless, this should be the first step in a general approach to higher super Teichmüller theory as well as its quantization. Notice that the symplectic or corresponding Poisson structure of Theorem B has constant coefficients opening the possibility of canonical quantization as in the bosonic case [6, 14].

Furthermore just as for λ-lengths in the bosonic case, our coordinates on decorated super-Teichmüller space provide a computationally effective description of super moduli geometry. Another key reason for interest in the super-Teichmüller space is that it is a cornerstone of superstring perturbation theory, and the geometry of the supersymmetric moduli uncovered here is evidently more involved than its bosonic counterpart. Moreover, the well-known relationship [23] of Teichmüller theory with (2+1)-dimensional gravity may suggest that the super-Teichmüller theory as probed here may play an analogous role for $(2 + 1)$-dimensional supergravity.

We finally mention that prior to our work, there was a PhD thesis [3], presumably to be continued in [4], where the author provided other coordinates on a version of super-Teichmüller space via a different treatment of spin structures and based upon

quite a different approach effectively using so-called shear coordinates instead of λ-lengths without our deep connection to super Minkowski geometry described in the next several sections.

Bosonic background and beyond

The decorated Teichmüller space $\tilde{T}(F)$ is intimately connected to the geometry of Minkowski space $\mathbb{R}^{2,1}$, namely, $\mathbb{R}^3$ imbued with the quadratic form $z^2 - x^2 - y^2$ and its corresponding bilinear pairing $< (x,y,z), (x',y',z') >= zz' - xx' - yy'$ or equivalently the quadratic form $x_1 x_2 - y^2$ in the variables $x_1 = z - x$ and $x_2 = z + x$. The upper sheet $\mathbb{H} = \{(x,y,z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = 1 \text{ and } z > 0\}$ of the hyperboloid with the induced metric gives a model for the hyperbolic plane [1]. Furthermore [19, 21], the positive light cone $L^+ = \{(x,y,z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = 0 \text{ and } z > 0\}$ parametrizes horocycles (that is, those curves with geodesic curvature unity) in the sense that $h(u) = \{w \in \mathbb{H} :< v, w >= 1\}$ establishes a bijection[1] between all $u \in L^+$ and the collection of all horocycles $h(u) \subset \mathbb{H}$, and moreover, this identification is geometrically natural in the sense that $\frac{1}{2} \log < u, v >$ is the signed hyperbolic distance between $h(u)$ and $h(v)$. This invariant $\sqrt{< u, v >}$ of a pair $\{h(u), h(v)\}$ of horocycles is called the *lambda length*, and these are the basic coordinates on $\tilde{T}(F)$.

It is convenient both here and in the sequel to consider not only trivalent fatgraph spines of $F = F_g^s$ but also their duals: an *ideal triangulation* Δ of F is a maximal family of arcs embedded in F with endpoints at the punctures, which are here regarded as distinguished points of F, where no two arcs in Δ are properly homotopic or intersect except perhaps at their endpoints. By maximality, each complementary region to Δ in F is an ideal triangle, and these meet along their frontiers in F. Construct a trivalent fatgraph spine $\tau = \tau(\Delta)$ of F with one vertex for each complementary region and one edge for each arc in Δ required to connect the vertices corresponding to regions on either side of the arc. This construction evidently establishes a bijection between homotopy classes of trivalent fatgraph spines and homotopy classes of ideal triangulations of F together with a natural identification between their edges. The dual of a flip, also called a flip, is the removal of an edge from Δ with distinct triangles on either side followed by its replacement by the other diagonal of the quadrilateral formed by their union.

A key point of the decorated space in the bosonic case is that the fiber $\mathbb{R}_+^s$ of $\tilde{T}(F) = T(F) \times \mathbb{R}_+^s$ is identified with all s-tuples of (lengths of) horocycles in F with one horocycle about each puncture; this is precisely the sense in which a usual hyperbolic structure in $T(F)$ is *decorated with a horocycle at each puncture* in $\tilde{T}(F)$. The $MC(F)$-action on $T(F)$ thus lifts to $\tilde{T}(F)$ by permuting lengths. An arc connecting punctures in a decorated hyperbolic surface has a well-defined lambda length computable in the surface itself or equivalently in $\mathbb{R}^{2,1}$ as just discussed.

---

[1]In fact, any positive constant will suffice here, and $2^{-\frac{1}{2}}$ is the more natural choice; cf. [19, 21].

Omnibus Theorem C. [19, 20, 21] *Fix a surface $F = F_g^s$ of genus $g \geq 0$ with $s \geq 1$ punctures, where $2g - 2 + s > 0$, and let Δ be a homotopy class of ideal triangulation or equivalently of a trivalent fatgraph spine of F. Then the assignment of lambda lengths to the arcs in Δ establishes a real-analytic homeomorphism $\tilde{T}(F) \to \mathbb{R}_+^\Delta$. Moreover, the lambda length $\lambda(\alpha; \tilde{\Gamma})$ of a (homotopy class of) arc α in F connecting punctures for $\tilde{\Gamma} \in \tilde{T}(F)$ is natural in the sense that if $\varphi \in MC(F)$ has induced action $\tilde{\varphi}$ on $\tilde{T}(F)$, then $\lambda(\alpha; \tilde{\Gamma}) = \lambda(\varphi(\alpha); \tilde{\varphi}(\Gamma))$.*

The Ptolemy transformation $ef = ac + bd$ describes the evolution of lambda lengths under flips in the notation of Figure 1.

The Weil-Petersson Kähler form on $M(F)$ pulls back to the Ptolemy-invariant form

$$\omega = 2 \sum_v d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a,$$

where the sum is over all complementary triangles to Δ with consecutive half edges in clockwise order having induced lambda lengths a, b, c.

A convex hull construction in $\mathbb{R}^{2|1}$ gives rise to a real-analytic $MC(F)$-invariant ideal cell decomposition of $\tilde{T}(F)/\mathbb{R}_+$ itself where there is one open simplex together with certain of its faces for each homotopy class of decompositions of F into ideal polygons and the face relation is generated by removal of arcs.

Thus in the equivalent formalism of trivalent fatgraph spines as opposed to ideal triangulations, Theorem B extends all but the last paragraph of Theorem C from $\tilde{T}(F)$ to $S\tilde{T}(F)$, and the proof of the former provides a paradigm for the proof of the latter. Notice that by this last paragraph of Theorem C, we may connect interior points of simplices for any two ideal triangulations of F by a path in $\tilde{T}(F)/\mathbb{R}_+$, which can be perturbed to general position with respect to the codimension-one faces of the cell decomposition. Since crossing these faces corresponds to flips, it follows that any two (homotopy classes of) ideal triangulations of F are related by a finite sequence of flips, and dually we recover Whitehead's result [10]:

Corollary D *For any surface $F = F_g^s$ with $2g - 2 + s > 0$, finite sequences of flips act transitively on homotopy classes of trivalent fatgraph spines in F.*

Moreover, the codimension-two faces analogously give rise to a presentation of the Ptolemy groupoid of F, cf. [19, 21].

We next discuss the proof of the first part of Theorem C, namely, the construction of lambda length coordinates on $\tilde{T}(F)$ which brings us more deeply into Minkowski space. The topological universal cover $\tilde{F}$ of F may be identified with upper half space $\mathcal{U}$ or equivalently via the Cayley transform $z \mapsto \frac{z-i}{z+1}$ with the unit disk $\mathbb{D}$ supporting its Poincaré metric and ideal boundary the circle $\mathbb{S}^1$ at infinity. Central projection

of $\mathbb{H}$ from $(0, 0, -1) \in \mathbb{R}^{2,1}$ to the disk at height zero establishes an isometry of $\mathbb{H}$ and $\mathbb{D}$ which continuously extends to the projection $L^+ \to \mathbb{S}^1$ mapping $u \in L^+$ to the center of the horocycle $h(u)$.

An ideal triangulation Δ of F lifts to an ideal triangulation $\tilde{\Delta}$ of $\tilde{F}$, and the collection of ideal points of $\tilde{\Delta}_\infty \subset \mathbb{S}^1$ is invariant under homotopy of Δ in F. In order to define a point of $\tilde{T}(F)$, we must determine a Fuchsian representation $\rho : \pi_1 \to SO_+(2, 1) \approx PSL(2, \mathbb{R})$ in the component $SO_+(2, 1)$ of the identity of the Minkowski isometry group $SO(2, 1)$, corresponding to the underlying point in $T(F)$, together with a lift $\ell : \tilde{\Delta}_\infty \to L^+$, corresponding via affine duality to the decoration and realizing the lambda lengths in the obvious sense, which is π_1-equivariant with respect to our constructed representation $\rho : \pi_1 \to SO_+(1, 2)$, namely, we have $\ell(\gamma(p)) = \rho(\gamma)(\ell(p))$, for all $p \in \tilde{\Delta}_\infty$ and $\gamma \in \pi_1$. This construction of ℓ and ρ from lambda lengths is performed recursively as we shall recall in proving Theorem B.

In our current case, there is an embedding of $OSp(1|2)$ into the super Lorentz group of the super Minkowski space $\mathbb{R}^{2,1|2}$ with pairing $x_1 x_2 - y^2 + 2\phi\theta$ as described in the next section. There is again a positive light cone $\hat{L}^+$ comprised of isotropic vectors so that the bodies of x_1 and x_2 are non negative. However, here is another fundamental distinction between the bosonic case and the general case treated here: whereas the action of $SO_+(2, 1)$ on L^+ is transitive, an $OSp(1|2)$-orbit of positive isotropic vectors is determined by a fermionic invariant $\pm\xi$ up to sign. Though most of the computations of this paper can be completed in the general setting, the super Teichmüller theory seems to require taking this fermion label $\pm\xi = 0$. It is the special light cone $\hat{L}_0^+ \subset \hat{L}^+$ consisting of those positive isotropic vectors with vanishing fermion label $\xi = 0$ that provide the analogue of L^+ for us here. Again, we define the λ-length of a pair of points in $\hat{L}_0^+$ to be the square root of their inner product and prove that this is the unique invariant of the $OSp(1|2)$-orbit of a pair of linearly independent points in $\hat{L}_0^+$.

We shall again recursively define a mapping $\ell : \tilde{\Delta}_\infty \to \hat{L}_0^+$ which realizes λ-lengths in the obvious sense and is π_1-equivariant with respect to the representation $\hat{\rho} : \pi_1 \to OSp(1|2)$ that we construct. Here is yet another fundamental distinction between the bosonic case and the general case: whereas $SO_+(2, 1)$ acts transitively on triples of rays in L^+ which are consistent with the positive orientation on $\mathbb{R}^{2,1}$, an $OSp(1|2)$ orbit of a triple in $\hat{L}_0^+$ whose underlying bosonic vectors in L^+ have this property is again determined by a fermion invariant $\pm\mu$ up to sign. Manin [15] had already observed this basic phenomenon hence our term μ-invariant for the odd invariants associated to vertices of a fatgraph spine in Theorem B which come from consistent choices of signs and are, like λ-lengths, realized by the mapping ℓ in the obvious sense. Consistency here is given by an explicit relationship on signs of μ-invariants for adjacent triangles in $\tilde{\Delta}$.

There are thus three basic differences here from the bosonic case: the failure of transitivity of the $OSp(1|2)$-action on points and on triples in $\hat{L}_0^+$ already mentioned and the further fact that the identification of L^+ with the space of horocycles in $\mathbb{H}$

has no known analogue in the general case. Thus, the decorated super-Teichmüller space $S\tilde{T}(F)$ *can only be defined here* as the space of $OSp(1|2)$ orbits of those maps $\ell : \tilde{\Delta}_\infty \to \hat{L}_0^+$ that are π_1-equivariant for some super Fuchsian representation with no intrinsic interpretation of *super horocycle* for the decoration beyond the analogous but un-illuminating affine dual in $\mathbb{R}^{2,1|2}$ of a point in $\hat{L}_0^+$. Indeed, in addition to the research frontiers discussed at the end of the previous section that the current work presumably illuminates, so also first glimpses of *super hyperbolic geometry* are hopefully to be gleaned here.

1. The hyperboloid, light cone and $OSp(1|2)$-action

The supergroup $OSp(1|2)$ is defined as follows. The group elements are square $(2|1) \times (2|1)$ supermatrices with superdeterminant equal to 1 which satisfy the relation

$$(1) \qquad\qquad g^{st} J g = J,$$

where the superscript st denotes the super transpose and $J = \left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$. We refer the reader to Appendix I for more information about $OSp(1|2)$ including the definition of super transpose (or see (13) below) and our sign conventions for products of supermatrices. The useful property

$$(2) \qquad\qquad g^{st} = J g^{-1} J^{-1}$$

is of course a direct consequence of (1). We are interested in the adjoint action of $OSp(1|2)$ and to this end consider its even element

$$(3) \qquad\qquad N_0 = -yh + x_1 X_- - x_2 X_+ + \phi v_- - \theta v_+.$$

We claim that the adjoint action $N_0 \mapsto g^{-1} N_0 g$ of $g \in OSp(1|2)$ on N_0 leaves invariant $x_1 x_2 - y^2 + 2\phi\theta$ since it is proportional to the quadratic form arising from the Killing form of $OSp(1|2)$ applied to N_0.

We shall prove this differently as follows. Observe that in the 3-dimensional representation of $OSp(1|2)$ owing to the property (2), the element $M_c = JN_c$, where $N_c = N_0 + cI$ for any fixed constant c, transforms as

$$(4) \qquad\qquad N_c \mapsto g^{-1} N_c g \quad \Rightarrow \quad M_c \mapsto g^{st} M_c g$$

under the adjoint action. In particular when c is invertible, this implies that the superdeterminant of

$$(5) \qquad\qquad M_c = \begin{pmatrix} x_1 & y - c & \phi \\ y + c & x_2 & \theta \\ -\phi & -\theta & c \end{pmatrix}$$

is preserved under the action of $OSp(1|2)$ sending M_c to $g^{st} M_c g$. It is not hard to calculate that

$$(6) \qquad\qquad sdet(M_c) = \frac{1}{c}(x_1 x_2 - y^2 + 2\phi\theta + c^2),$$

and so $x_1 x_2 - y^2 + 2\phi\theta$ is invariant under the action of $OSp(1|2)$ as was claimed. The following proposition therefore holds.

Proposition 1.1. *The formula $M_c \mapsto g^{st} M_c g$, for fixed but arbitrary c, gives the action of the $OSp(1|2)$ subgroup of the full Lorentz supergroup of the superspace $\mathbb{R}^{2,1|2}$ with the Minkowski pairing defined by the quadratic form $x_1 x_2 - y^2 + 2\phi\theta$.*

To be entirely explicit, the pairing of two vectors $A = (x_1, x_2, y, \phi, \theta)$ and $A' = (x_1', x_2', y', \phi', \theta')$ in $\mathbb{R}^{2,1|2}$ is given by

$$(7) \qquad \langle A, A' \rangle = \frac{1}{2}(x_1 x_2' + x_1' x_2) - yy' + \phi\theta' + \phi'\theta.$$

In keeping with [19, 21], we shall henceforth refer to the square root of such an inner product as a λ-*length*.

Two surfaces of special importance for us in the following are the *(super) hyperboloid* $\hat{\mathbb{H}}$ consisting of points $A \in \mathbb{R}^{2,1|2}$ satisfying the condition $\langle A, A \rangle = 1$ corresponding to $c = 1$ in equation (5), where the bodies of the x_1- and x_2-coordinates of A are non-negative, and most especially the *(positive super) light cone* $\hat{L}^+$ consisting of points $B \in \mathbb{R}^{2,1|2}$ satisfying $\langle B, B \rangle = 0$ and corresponding to $c = 0$, where again the bodies of x_1- and x_2-coordinates are non-negative.

A standard superspace which however plays a subsidiary role for us, the complex *superplane* $\mathbb{C}^{1|1}$ consists of pairs (z, η), and its subspace the *super upper half-plane* $\hat{\mathcal{U}}$ is comprised of those points (z, η) such that the body of the real part of z is non-negative. It is well known [9, 24] that $OSp(1|2)$ acts on $\hat{\mathcal{U}}$ by means of super-conformal transformations

$$z \to \frac{az+b}{cz+d} + \eta\frac{\gamma z + \delta}{(cz+d)^2},$$

$$(8) \qquad \eta \to \frac{\gamma z + \delta}{cz+d} + \eta\frac{1 + \frac{1}{2}\delta\gamma}{cz+d},$$

Another direct analogue of the standard bosonic case, we have

Theorem 1.2. *The expressions*

$$(9) \qquad \eta = \frac{\theta}{x_2}(1 + iy) - i\phi, \quad z = \frac{i - y - i\phi\theta}{x_2}$$

define an $OSp(1|2)$-equivariant monomorphism from the hyperboloid $\hat{\mathbb{H}}$ onto the super half-plane $\hat{\mathcal{U}}$.

Proof. The easiest way to prove this statement is to consider the infinitesimal actions of the corresponding generators described in Appendix I. For example, the transformation $M_c \mapsto (\exp(\alpha v_+))^{st} M_c \exp(\alpha v_+)$ amounts to the infinitesimal

$$\delta_\alpha y = -\alpha\phi, \quad \delta_\alpha\phi = \alpha x_1, \quad \delta_\alpha x_1 = 0,$$

$$(10) \qquad \delta_\alpha x_2 = -2\alpha\theta, \quad \delta_\alpha\theta = \alpha y,$$

and therefore

$$(11) \qquad \delta_\alpha z = \alpha\left(\frac{2\theta}{x_2^2}(i-y) + \frac{1}{x_2}(\phi - ix_1\theta + i\phi y)\right).$$

Meanwhile, we have

$$\eta z = i\phi\frac{i-y}{x_2} - i\theta\frac{(y-i)^2}{x_2^2}$$

$$= \frac{i\phi}{x_2} + \frac{\phi}{x_2} - \frac{i\theta y^2}{x_2^2} - \frac{2\theta y}{x_2^2} + \frac{i\theta}{x_2^2}$$

$$(12) \qquad = \frac{i\phi y}{x_2} + \frac{\phi}{x_2} - \frac{ix_1}{x_2} - \frac{2\theta y}{x_2^2} + \frac{2i\theta}{x_2^2},$$

where we have used the relation $x_1 x_2 - y^2 + 2\phi\theta = 1$ in the last line. It follows that $\delta_\alpha z = \alpha\eta z$ as required, and one can similarly show that $\delta_\alpha\eta = -\alpha z$. Thus, this corresponds to the superconformal transformation. We leave it for the reader to complete the proof for the other four infinitesimal transformations corresponding to the generators discussed in Appendix I. ∎

Again just to be entirely explicit in the context of relevant subsequent calculations, the light cone is described by

$$\hat{L}^+ = \left\{ \begin{pmatrix} x_1 & y & \phi \\ y & x_2 & \theta \\ -\phi & -\theta & 0 \end{pmatrix} : x_1 x_2 - y^2 + 2\phi\theta = 0 \text{ and } x_1, x_2 \text{ have non} - \text{negative bodies} \right\},$$

and the action of the supermatrix $g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \in OSp(1|2)$ on $A \in \hat{L}^+$ is given by

$$(13) \quad g \cdot A = g^{st}Ag = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}\begin{pmatrix} x_1 & y & \phi \\ y & x_2 & \theta \\ -\phi & -\theta & 0 \end{pmatrix}\begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix},$$

where these products of supermatrices have the signs[2] explained in Appendix I. This is entirely analogous to the bosonic action [19, 21] of $PSL(2,\mathbb{R}) \approx SO_+(2,1)$ on the light cone in $\mathbb{R}^{2,1}$ given by the change of basis for binary symmetric bilinear forms.

There is the particular element of $OSp(1|2)$ given by $g^r = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ that is of special significance. The supermatrix g^r generates the center of $OSp(1|2)$, and its explicit action on any $A = (x_1, x_2, y, \phi, \theta)$ is given by $g^r \cdot A = (x_1, x_2, y, -\phi, -\theta)$. Thus, g^r simply changes the signs of the fermions and will henceforth be referred to as *(fermionic) reflection*.

---

[2]Namely, $\begin{pmatrix} a_1 & b_1 & \alpha_1 \\ c_1 & d_1 & \beta_1 \\ \gamma_1 & \delta_1 & f_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 & \alpha_2 \\ c_2 & d_2 & \beta_2 \\ \gamma_2 & \delta_2 & f_2 \end{pmatrix} = \begin{pmatrix} a_1a_2+b_1c_2-\alpha_1\gamma_2 & a_1b_2+b_1d_2-\alpha_1\delta_2 & a_1\alpha_2+b_1\beta_2+\alpha_1 f_2 \\ c_1a_2+d_1c_2-\beta_1\gamma_2 & c_1b_2+d_1d_2-\beta_1\delta_2 & c_1\alpha_2+d_1\beta_2+\beta_1 f_2 \\ \gamma_1a_2+\delta_1c_2+f_1\gamma_2 & \gamma_1b_2+\delta_1d_2+f_1\delta_2 & -\gamma_1\alpha_2-\delta_1\beta_2+f_1f_2 \end{pmatrix}$.

2. Orbits of $OSp(1|2)$ in the light cone

We next show that $OSp(1|2)$ does not act transitively on the light cone $\hat{L}^+$, and in fact, the moduli space of orbits is homeomorphic to the space $\mathbb{R}^{0|1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by the change of sign of fermions. To begin, we normalize with respect to the subgroup $SL(2, \mathbb{R}) < OSp(1|2)$.

Lemma 2.1. *For each vector $A \in \hat{L}^+$, there is some $g \in SL(2, \mathbb{R}) < OSp(1|2)$ so that $g \cdot A = t(1, 1, 1 + \phi\psi, \phi, \psi)$, where t has positive body.*

Proof. Consider an arbitrary vector $(x_1, x_2, y, \rho, \lambda) \in \hat{L}^+$. Since one of x_1 or x_2 is invertible, we can apply an element of the $SL(2, \mathbb{R})$ subgroup to transform to a vector $(x_1', x_2', y', \rho', \lambda')$ where both of x_1', x_2' are invertible and the body of y' is positive. We can subsequently apply a diagonal matrix from the $SL(2, \mathbb{R})$ subgroup in order that the resulting vector $(x_1'', x_2'', y'', \rho'', \lambda'')$ satisfies $x_1'' = x_2''$ and hence has the required form. ∎

The next result provides the classification of orbits, namely, we can reduce all degrees of freedom to a single fermion modulo sign via the action of $OSp(1|2)$.

Proposition 2.2. *Every vector in the light cone can be put into the form*

$$e_\theta = (1, 0, 0, 0, \theta) \in \hat{L}^+$$

via an $OSp(1|2)$ transformation. The only solutions to the equation $e_{\theta'} = g \cdot e_\theta$, where $g \in OSp(1|2)$, are given by $\theta' = \pm\theta$.

Proof. According to the previous lemma, in order to prove the first part, we may assume that our specified vector is of the form $A_{\phi\psi}^t = t(1, 1, 1 + \phi\psi, \phi, \psi)$, where t has positive body. A direct computation then shows that the matrix

$$(14) \qquad g_{\phi,\psi}^t = \begin{pmatrix} 0 & -\sqrt{t} & 0 \\ \frac{1}{\sqrt{t}} & \sqrt{t}(1 + \phi\psi) & -\psi \\ 0 & \sqrt{t}\psi & 1 \end{pmatrix}$$

achieves the required expression $g_{\phi,\psi}^t \cdot A_{\phi\psi}^t = e_\theta$, where in fact $\theta = t\sqrt{t}(\psi - \phi)$ and $\sqrt{t}$ is likewise taken with positive body.

The second part is proven by explicitly solving the equation $e_{\theta'} = g \cdot e_\theta$ as follows. Consider an arbitrary element

$$(15) \qquad g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \in OSp(1|2).$$

The vector $g \cdot e_\theta = (x_1, x_2, y, \rho, \lambda)$ is characterized by the identities

$$\begin{aligned} x_1 &= a^2 + 2\gamma\theta c, \quad \rho = a\alpha + \gamma\theta\beta + c\theta f, \\ x_2 &= b^2 + 2\delta\theta d, \quad \lambda = b\alpha + \delta\theta\beta + d\theta f, \\ (16) \qquad y &= ab + \gamma\theta d - c\theta\delta. \end{aligned}$$

Thus, if $g \cdot e_\theta = e_{\theta'}$, then using $y = 0$ as well as the constraints on the entries of g given in Appendix I, we find that $\rho = 0$, $\lambda = \theta'$ imply

$$(17) \qquad\qquad \alpha = -\frac{c}{a}\theta, \quad a\theta' = \theta,$$

and then $x_1 = 1$, $x_2 = 0$ imply

$$(18) \qquad\qquad a = \pm(1 \mp c\beta\theta), \quad b = \pm\theta\beta.$$

It follows that $\theta' = \pm\theta$ as was claimed. ∎

The next result follows immediately.

Corollary 2.3. *The moduli space of orbits of the $OSp(1|2)$ action on the light cone is given by $\mathbb{R}^{0|1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ reflects the sign of the fermion.*

The explicit solution to the equation $e_\theta = g \cdot e_\theta$ given in Proposition 2.2 yields the following Corollary which will be of utility in the sequel.

Corollary 2.4. *For $\theta \neq 0$, an element g^s of the stabilizer subgroup of e_θ in $OSp(1|2)$ necessarily has the form*

$$(19) \qquad g^s = \begin{pmatrix} 1 + c\theta\beta & \theta\beta & -c\theta \\ c & 1 + c\theta\beta & \beta \\ \beta + c^2\theta & c\theta & 1 + c\beta\theta \end{pmatrix},$$

where c, θ are free parameters with c even and θ odd. Moreover for $\theta = 0$, the stabilizer of e_0 has one component for Ramond punctures given by the expression $g^s = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & \beta \\ \beta & 0 & 1 \end{pmatrix}$ above as well as a second component for Neveu-Schwarz punctures given by its composition $g^r g^s = \begin{pmatrix} -1 & 0 & 0 \\ -c & -1 & -\beta \\ \beta & 0 & 1 \end{pmatrix}$ with the fermionic reflection g^r.

Given a point $A \in \hat{L}^+$, the fermion $\pm\theta$ (defined up to an overall sign) so that A and e_θ lie in the same $OSp(1|2)$-orbit is called the *fermion label* of A, and it admits the following simple expression.

Proposition 2.5. *If $A = (x_1, x_2, y, \rho, \lambda) \in \hat{L}^+$, then the fermion label of A is given by $\pm\theta = \sqrt{x_1}\lambda - \frac{y}{\sqrt{x_1}}\rho$ if x_1 is invertible and by $\pm\theta = \sqrt{x_2}\rho - \frac{y}{\sqrt{x_2}}\lambda$ if x_2 is invertible.*

Here and throughout since x_1- and x_2-coordinates have non-negative body on the positive light cone, there are well-defined square roots $\sqrt{x_1}$ and $\sqrt{x_2}$ also with non-negative body. The fermionic reflection on λ, ρ thus changes the sign of θ here.

Proof. The result follows from direct calculation starting from the formulas (16) using the constraints on entries of $OSp(1|2)$ in Appendix I as we explicate in the

case where x_1 is invertible. We have

$$\sqrt{x_1}\lambda = (a + \frac{\gamma\theta c}{a})(b\alpha + \delta\theta\beta + d\theta f)$$

$$(20) \qquad = ab\alpha - ad\alpha\theta\beta + ad\theta f + cb\theta\alpha\beta.$$

At the same time, we have

$$(21) \qquad \frac{y}{\sqrt{x_1}} = (ab + \gamma\theta d - c\theta\delta)\frac{1}{a}(1 - \frac{\gamma\theta c}{a^2}) = b + \frac{\gamma\theta}{a^2} - \frac{c\theta\delta}{a},$$

and therefore

$$\frac{y}{\sqrt{x_1}}\rho = (b + \frac{\gamma\theta}{a^2} - \frac{c\theta\delta}{a})(a\alpha + \gamma\theta\beta + c\theta f)$$

$$(22) \qquad = ab\alpha - cb\alpha\theta\beta + bc\theta f - ad\theta\beta\alpha.$$

It follows that

$$(23) \qquad \sqrt{x_1}\lambda - \frac{y}{\sqrt{x_1}}\rho = (ad - bc)\theta f = \theta,$$

where we again use the constraints on elements of $OSp(1|2)$ from Appendix I. ∎

The $OSp(1|2)$-orbit of $e_0 \in \hat{L}^+$ will play a special role for us here. We shall denote it $\hat{L}_0^+ = OSp(1,2) \cdot e_0 \subset \hat{L}^+$ and refer to it as the *special light cone*.

Corollary 2.6. *The special light cone $\hat{L}_0^+$ is isomorphic to superprojective space $\mathbb{R}P^{1|1}$ with the action of $OSp(1|2)$ given by superconformal transformations. Provided $x_2 \neq 0$, the natural correspondence between $(x_1, x_2, y, \phi, \psi) \in \hat{L}_0^+$ and $(z, \eta) \in \mathbb{R}P^{1|1}$ is given by:*

$$(24) \qquad z = \frac{-y}{x_2}, \quad \eta = \frac{\psi}{x_2}.$$

The proof follows along the lines of Theorem 1.2.

3. Orbits of isotropic independent pairs and positive triples

This section provides abstract identifications for the spaces of $OSp(1|2)$-orbits of linearly independent ordered pairs and certain triples of points in the special light cone $\hat{L}_0^+$. Specifically, we shall say that an ordered triple ABC of points in $\hat{L}_0^+$ is *positive* provided A, B, C are linearly independent and the underlying triple of bosonic vectors in the usual light cone $L^+ \subset \mathbb{R}^{2,1}$ in this order provides a positively oriented basis for $\mathbb{R}^3$ with its usual orientation. In fact, the latter moduli space of positive triples in the light cone plays a key role in the sequel, and several further parametrizations of it are derived in the next section.

The moduli space of all ordered pairs of linearly independent vectors (i.e., with non-vanishing Minkowski inner product or equivalently vectors lying in distinct rays) in the special light cone is described by

Lemma 3.1. *There is a unique $OSp(1|2)$-invariant of two linearly independent vectors $A, B \in \hat{L}_0^+$, and it is given by the pairing $\langle A, B \rangle$.*

Proof. According to Proposition 2.2, by applying an appropriate element of $OSp(1|2)$ to both points, we may assume $A = (1, 0, 0, 0, 0)$ and $B = (x_1, x_2, y, \phi, \eta) \in \hat{L}_0^+$. Note that x_2 must be invertible, for otherwise the vectors are not linearly independent in the super-sense, namely, there exist a, b with non-zero bodies so that $aA + bB$ has zero body.

We may apply a transformation g^s of the form (19) thus stabilizing A and mapping B to $g^s \cdot B = (\tilde{x}_1, \tilde{x}_2, \tilde{y}, \tilde{\phi}, \tilde{\eta}) \in \hat{L}_0^+$, where

$$\tilde{y} = y + cx_2 + \beta\eta, \quad \tilde{x}_2 = x_2,$$
(25)
$$\tilde{\eta} = \beta x_2 + \eta, \quad \tilde{\phi} = \beta y + c\beta x_2 + c\eta + \phi.$$

We wish to normalize so that this vector takes the form $s(0, 1, 0, 0, 0)$ and thus impose the two further conditions $\tilde{\eta} = 0$ and $\tilde{y} = 0$. This implies

(26)
$$\beta = -x_2^{-1}\eta \text{ and } c = -\frac{y}{x_2},$$

which gives $s = x_2 = \langle A, B \rangle$ as required. ∎

Corollary 3.2. *The moduli space of $OSp(1|2)$-orbits of ordered pairs of vectors in the special light cone lying in distinct rays is given by $\mathbb{R}_+$.*

Turning now to ordered triples of linearly independent vectors in $\hat{L}_0^+$, the underlying bosonic triple may provide either a positively or a negatively oriented basis of $\mathbb{R}^{2,1}$, and we have already christened the former case a positive triple. The well-known three-effectiveness of the action of $PSL(2, \mathbb{R})$ on positively oriented (i.e., correctly cyclically ordered) triples of points in the circle at infinity of hyperbolic space fails in our current context of $\mathbb{R}^{2,1|2}$ because there is one additional odd degree of freedom which cannot be fixed. We next compute the moduli space of orbits of positive triples to be $\mathbb{R}_+^{3|1}$ modulo the fermionic reflection and postpone the further discussion of this interesting additional parameter to the next section.

Lemma 3.3. *Let $\zeta^b \zeta^e \zeta^a$ be a positive triple in the special light cone. Then there is $g \in OSp(1|2)$, which is unique up to composition with the fermionic reflection, and unique even r, s, t, which have positive bodies, and odd ϕ so that*

$$g \cdot \zeta^e = t(1, 1, 1, \phi, \phi), \; g \cdot \zeta^b = r(0, 1, 0, 0, 0), \; g \cdot \zeta^a = s(1, 0, 0, 0, 0).$$

Proof. First, one can put ζ^a into the form $(1, 0, 0, 0)$ by means of $OSp(1|2)$ according to Lemma 2.2. Second, ζ^b can be put in the form $(0, f, 0, 0, 0)$ using the stabilizer of $(1, 0, 0, 0, 0)$ as in the proof of Lemma 3.1. Finally by a diagonal matrix whose entries have positive bodies, one can put ζ^e into the form where its x_1- and x_2-coordinates agree. Since $\zeta^b \zeta^e \zeta^a$ is a positive triple, the y-coordinate of ζ^e after these transformations has positive body, and so ζ^e indeed transforms to a vector with the required form $t(1, 1, 1, \phi, \phi)$ while the other two transformed vectors evidently

also have the desired forms $\zeta^b = r(0, 1, 0, 0, 0)$, $\zeta^a = s(1, 0, 0, 0, 0)$. The asserted uniqueness up to reflection of the transformation $g \in OSp(1|2)$ thus constructed follows from Corollary 2.4. ∎

Just to be clear, let us note that there are thus precisely two lifts to $\hat{L}_0^+$ of a positive triple T with abstract coordinates $(r, s, t, \theta) \in \mathbb{R}_+^{3|1}$ given by $g \cdot T$ (where the y-coordinate of ζ^e has positive body in the notation of the lemma) and its image $g^r g \cdot T$ under the fermionic reflection g^r. In any case, the invariants r, s, t have positive body and only the signs of the fermions change. As a parenthetical point of notation, we mention that the labeling abe is used here rather than the more natural abc in order to accommodate later conventions.

Corollary 3.4. *The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by $(r, s, t, \phi) \in \mathbb{R}_+^{3|1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts by fermionic reflection.*

In fact, the even invariants r, s, t of a triple, which we shall call *normalization coefficients*, are the direct analogues of the reciprocal h-lengths from [19, 21], and they can be nicely computed in terms of the λ-lengths

$$a^2 = <\zeta^b, \zeta^e>, \ b^2 = <\zeta^a, \zeta^e>, \ e^2 = <\zeta^a, \zeta^b> .$$

Lemma 3.5. *The normalization coefficients r, s, t in Lemma 3.3 are given by*

$$(27) \qquad\qquad r = \sqrt{2}\, \frac{ea}{b}, \quad s = \sqrt{2}\, \frac{be}{a}, \quad t = \sqrt{2}\, \frac{ab}{e}.$$

We have found that a positive triple of points in the special light cone naturally determines three even and one odd invariant, namely, the λ-lengths a, b, e or the normalization coeffcients r, s, t and the fermion label ϕ given in the previous lemma. It appears that the definition of the odd parameter ϕ depends on a choice of member of the positive triple, and we next discuss how to eliminate this dependence.

In fact given a positive triple ABC, there is a canonical cube root of unity in $OSp(1|2)$ called the *ABC prime transformation* which cyclically permutes its members $(A, B, C) \mapsto (B, C, A)$, and we shall compute this transformation on the coordinates of Lemma 3.3 explicitly and denote it $(r, s, t, \phi) \mapsto (r', s', t', \phi')$ where the λ-lengths are of course also cyclically permuted in the natural way. The choice-free way to express the additional odd degree of freedom is to average our ansatz ϕ over this prime transformation and define the *Manin invariant*[3] or simply the *μ-invariant* to be

$$(28) \qquad\qquad \theta = \tfrac{1}{3}(\phi + \phi' + \phi'').$$

---

[3] In fact, Manin [15] introduced the odd "pseudo-invariant" $\pm\theta$ of a triple of points in $\mathbb{R}^{1|1}$ capturing the basic non-transitivity of the $OSp(1|2)$ action there, and by making choices (of spin structure among other conventions) one can [3] lift this to a signed expression θ, namely, our Manin invariant.

In practice, μ-invariants and λ-lengths give a parametrization of decorated super-Teichmüller space as we shall see.

To complete our definition of the μ-invariant, it therefore remains to compute the prime transformation. To this end, we alter notation slightly setting $A = \zeta^b$, $B = \zeta^e$, $C = \zeta^a$ as illustrated in Figure 3.
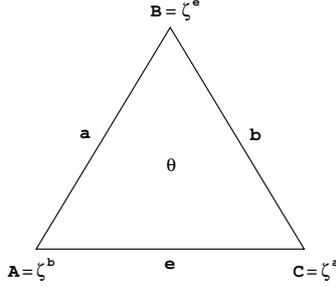

FIGURE 3. Parametrization of a positive triple in the special light cone by three λ-lengths a, b, e on edges and one μ-invariant θ.

Given three points

$$\begin{aligned}
A &= r(0, 1, 0, 0, 0), \\
B &= t(1, 1, 1, \phi, \phi), \\
C &= s(1, 0, 0, 0, 0)
\end{aligned}$$

(29)

in the special light cone $\hat{L}_0^+$, where r, s, t have positive body, we claim there exists a group element $g_{A,B,C}^{A',B',C'}$ from $OSp(1|2)$ respectively transforming the points A, B, C into

$$\begin{aligned}
A' &= t'(1, 1, 1, \phi', \phi'), \\
B' &= s'(1, 0, 0, 0, 0), \\
C' &= r'(0, 1, 0, 0, 0),
\end{aligned}$$

(30)

and in fact, there are two possible ways to construct such a transformation (again unique up to fermionic reflection). However, there is a unique transformation whose third power

(31)
$$g_{A'',B'',C''}^{A''',B''',C'''} \, g_{A',B',C'}^{A'',B'',C''} \, g_{A,B,C}^{A',B',C'} = 1$$

is equal to identity, and it turns out that its action on ϕ is rather simple.

Proposition 3.6. *The group element $g_{A,B,C}^{A',B',C'}$ acts so that $\phi'' = \phi' = \phi$, and this is the unique group element $g_{A,B,C}^{A',B',C'} \in OSp(1|2)$ satisfying the identity (31).*

Proof. We first use the group element $g^t_{\phi\psi}$ in (14) in order to transform B into $(1, 0, 0, 0, 0)$. The effects of this transformation on the other points produces

$$g^t_{\phi\phi} : C \mapsto st(0, 1, 0, 0, 0),$$

(32)
$$g^t_{\phi\phi} : A \mapsto r(t^{-1}, t, 1, -\frac{\phi}{\sqrt{t}}, -\sqrt{t}\phi).$$

Finally, acting by the diagonal matrix $g^d = \begin{pmatrix} \sqrt{t} & 0 & 0 \\ 0 & \sqrt{t^{-1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ we obtain the desired result by composing with the fermionic reflection g^f

(33)
$$g^{A',B',C'}_{A,B,C} = g^t_{\phi,\phi} g^d g^f = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & -\phi \\ 0 & -\phi & 1 \end{pmatrix}$$

so that $t' = r$, $r' = s$, $s' = t$. $\blacksquare$

Corollary 3.7. *The μ-invariant $\theta \equiv \phi$ of a positive triple ABC is invariant under the ABC prime transformation.*

As follows directly from this plus the discussion of the previous section, we have

Theorem 3.8. *The moduli space of $OSp(1|2)$-orbits of positive triples in the light cone is given by equivalence classes under fermionic reflection of three even λ-lengths a, b, e with positive bodies plus the μ-invariant θ defined up to fermionic reflection.*

4. Basic calculation and Ptolemy transformations

This section is dedicated to a computation called the "basic calculation" giving a parametrization of the moduli space of $OSp(1|2)$-orbits of four-tuples $ABCD$ in the special light cone $\hat{L}^+_0$ comprised of two positive triples CBA and DCA of points in terms of five λ-lengths and two μ-invariants. In effect, one positive triple CBA is put into the canonical position of Lemma 3.3 using certain of the putative parameters, and the remaining ones are then used to completely and uniquely determine the fourth point $D = (x_1, x_2, -y, \rho, \lambda)$, where y has non-negative body; it is precisely here that compatibility of signs of adjacent μ-invariants arises. This basic calculation is the critical ingredient both for our global coordinates and for the Ptolemy transformation just as in [19, 21]. In fact provided only that the triple CBA is in standard position, the standard position of DCA is easily calculated in terms of the coordinates of its vertices as follows.

Lemma 4.1. *The triple*

$$A = r(0, 1, 0, 0, 0), \; C = s(1, 0, 0, 0, 0), \; D = (x_1, x_2, -y, \rho, \lambda), \; \text{for } y \geq 0,$$

can be transformed by an element of $OSp(1|2)$, which is itself uniquely determined up to fermionic reflection, into the triple

$$\hat{A} = \hat{s}(1, 0, 0, 0, 0), \; \hat{C} = \hat{r}(0, 1, 0, 0, 0)), \; \hat{D} = \hat{t}(1, 1, 1, \sigma, \sigma),$$

where

$$\hat{s} = \sqrt{\frac{x_1}{x_2}}r, \quad \hat{r} = \sqrt{\frac{x_2}{x_1}}s, \quad \hat{t} = \sqrt{x_1 x_2},$$

(34)
$$\sigma = -\sqrt[4]{\frac{x_1}{x_2}}\frac{\lambda}{\sqrt{x_1 x_2}} = \sqrt[4]{\frac{x_2}{x_1}}\frac{\rho}{\sqrt{x_1 x_2}},$$

Proof. Act first by $\left(\begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to interchange the roles of A and C while mapping D to a point with non-negative y-coordinate and then act by $\left(\begin{smallmatrix} \sqrt[4]{x_1/x_2} & 0 & 0 \\ 0 & \sqrt[4]{x_2/x_1} & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ to realize equality of the first two coordinates. The composition of these two elements of $OSp(1|2)$ achieves the desired result in accordance with Lemma 3.3. ∎

Note that the normalization factors admit the uniform expressions

$$t = \sqrt{2}\frac{ab}{e}, \quad r = \sqrt{2}\frac{ea}{b}, \quad s = \sqrt{2}\frac{be}{a},$$

(35)
$$\hat{t} = \sqrt{2}\frac{cd}{e}, \quad \hat{r} = \sqrt{2}\frac{ec}{d}, \quad \hat{s} = \sqrt{2}\frac{de}{c}.$$

This transformation in $OSp(1|2)$ which maps DCA to standard position if BAC is in standard position is called the *switch transformation* of the quadruple $ABCD$ though in fact it depends only upon DCA according to the formula for it given above.
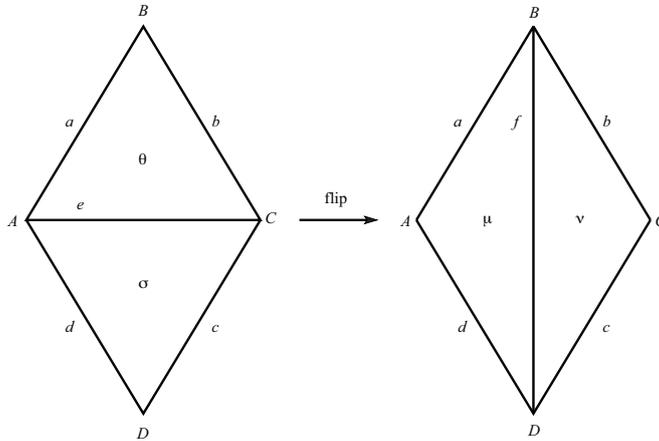

FIGURE 4. Standard notation for λ-lengths and μ-invariants near an edge before (left) and after (right) a flip

Now turning to the basic calculation itself, let $A = \zeta^b, B = \zeta^e, C = \zeta^a$ be points in the light cone as in Lemma 3.3 with λ-lengths a, b, e and μ-invariant θ. Up to fermionic reflection, the $OSp(1|2)$-orbit of the positive triple CBA is then uniquely

determined by these parameters according to Lemma 3.3. The basic calculation aims to compute $D = (x_1, x_2, -y, \rho, \lambda) \in \hat{L}_0^+$, where y has non-negative body, from the data

$$(36) \qquad \sigma = -\sqrt[4]{\frac{x_1}{x_2}} \frac{\lambda}{\sqrt{x_1 x_2}} = \sqrt[4]{\frac{x_2}{x_1}} \frac{\rho}{\sqrt{x_1 x_2}}, \quad \langle A, D \rangle = d^2, \quad \langle D, C \rangle = c^2$$

depicted in Figure 4, and the following arises directly from equations (27) and (36).

Proposition 4.2. *We have the expressions*

$$x_1 = \frac{\hat{t}\hat{s}}{r} = \sqrt{2}\frac{d^2 b}{ea}, \quad x_2 = \frac{\hat{r}\hat{t}}{s} = \sqrt{2}\frac{c^2 a}{eb},$$

$$y = \sqrt{x_1 x_2} = \sqrt{2}\frac{cd}{e},$$

$$(37) \qquad \lambda = -\sqrt{2}\frac{cd}{e}\sqrt{\frac{ac}{db}}\,\sigma, \quad \rho = \sqrt{2}\frac{cd}{e}\sqrt{\frac{db}{ac}}\,\sigma$$

solving the basic calculation.

Introducing the cross-ratio $\chi = \frac{ac}{db}$, this expression of the basic calculation can be concisely written

$$x_1 = \sqrt{2}\frac{cd}{e}\chi^{-1}, \quad x_2 = \sqrt{2}\frac{cd}{e}\chi, \quad y = \sqrt{2}\frac{cd}{e},$$

$$(38) \qquad \lambda = -\sqrt{2}\frac{cd}{e}\sqrt{\chi}\,\sigma, \quad \rho = \sqrt{2}\frac{cd}{e}\sqrt{\chi^{-1}}\,\sigma.$$

The utility of this version of the basic calculation for computing the Ptolemy transformation is already evident from the following proposition.

Proposition 4.3. *In the notation above with $f^2 = \langle A, D \rangle$, we have*

$$(39) \qquad ef = (ac + bd)\left(1 + \frac{\sigma\theta}{\chi^{\frac{1}{2}} + \chi^{-\frac{1}{2}}}\right).$$

Proof. Simply expand the expression

$$f^2 = \frac{t}{2}(x_1 + x_2) + ty + t\rho\theta + t\theta\lambda$$

$$(40) \qquad = \frac{d^2 b^2}{e^2} + \frac{a^2 c^2}{e^2} + 2\frac{abcd}{e^2} + 2\frac{abcd}{e^2}\left(\sqrt{\frac{db}{ac}} + \sqrt{\frac{ac}{db}}\right)\sigma\theta$$

in λ-lengths. This is equivalent to

$$(41) \qquad e^2 f^2 = (ac + bd)^2 + 2abcd(\chi^{\frac{1}{2}} + \chi^{-\frac{1}{2}})\sigma\theta,$$

and the desired formula for f follows upon taking square roots.   ∎

Now let us reproduce the same four points $A, B, C, D \in \hat{L}_0^+$ using the parameters of adjacent positive triples CBD and ADB. To do so, let us move the positive triple CBD to standard position

$$
\begin{aligned}
B &= \tilde{s}(1,0,0,0,0), \quad C = \tilde{r}(0,1,0,0,0), \\
D &= \tilde{t}(1,1,1,\tilde{\phi},\tilde{\phi}).
\end{aligned}
\tag{42}
$$

To this end, we first apply the stabilizer of $s(1,0,0,0,0)$, namely, the matrix

$$
g = \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & \zeta \\ \zeta & 0 & 1 \end{pmatrix},
\tag{43}
$$

as in Corollary 2.4, in order to transform $t(1,1,1,\theta,\theta) \to \bar{r}(0,1,0,0,0)$, where the corresponding c and ζ are given by $c = -1$ and $\zeta = -\theta$ yielding $\bar{r} = t$. Applying this transformation to $D = (x_1, x_2, -y, \rho, \lambda) \in \hat{L}_0^+$, we obtain a new vector $(\hat{x}_1, \hat{x}_2, -\hat{y}, \hat{\rho}, \hat{\lambda}) \in \hat{L}_0^+$, where

$$
\begin{aligned}
\hat{x}_1 &= x_1 + x_2 + 2y - 2\theta(\rho - \lambda), \\
\hat{x}_2 &= x_2, \\
\hat{\rho} &= \theta y + \theta x_2 - \lambda + \rho, \\
\hat{\lambda} &= -\theta x_2 + \lambda.
\end{aligned}
\tag{44}
$$

Applying $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ followed by the diagonal matrix $\begin{pmatrix} \sqrt[4]{\frac{\hat{x}_1}{\hat{x}_2}} & 0 & 0 \\ 0 & \sqrt[4]{\frac{\hat{x}_2}{\hat{x}_1}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to each of B, C, D yields:

$$
\tilde{t} = \sqrt{\hat{x}_1 \hat{x}_2}, \quad \tilde{r} = t\sqrt{\frac{\hat{x}_1}{\hat{x}_2}}, \quad \tilde{s} = s\sqrt{\frac{\hat{x}_2}{\hat{x}_1}}
$$

$$
\tilde{\phi} = -\frac{\hat{\lambda}}{\sqrt{\hat{x}_1 \hat{x}_2}}\left(\sqrt[4]{\frac{\hat{x}_1}{\hat{x}_2}}\right) = \frac{\hat{\rho}}{\sqrt{\hat{x}_1 \hat{x}_2}}\left(\sqrt[4]{\frac{\hat{x}_2}{\hat{x}_1}}\right)
\tag{45}
$$

for the parameters in equation (42). Thus, the μ-invariant of the positive triple BDC is given by $\tilde{\phi}$ as this is invariant under the BDC prime transformation.

Theorem 4.4. *The μ-invariants for the positive triples BDC and DBA depend only on the μ-invariants θ, σ and the cross-ratio χ and are given by*

$$
\nu = \frac{\theta\sqrt{\chi} + \sigma}{\sqrt{1 + \chi}},
\tag{46}
$$

$$
\mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1 + \chi}}.
\tag{47}
$$

Proof. Equation (46) follows from the direct calculation

$$(48) \qquad \nu = -\frac{\hat{\lambda}}{\sqrt{\hat{x}_1\hat{x}_2}}\Big(\sqrt[4]{\frac{\hat{x}_1}{\hat{x}_2}}\Big) = \frac{\theta x_2 - \lambda}{\chi^{\frac{3}{4}}(\sqrt{2}\frac{cd}{e})\sqrt{\sqrt{\chi}+\sqrt{\chi}^{-1}}} = \frac{\theta\sqrt{\chi}+\sigma}{\sqrt{1+\chi}}.$$

In order to write μ in terms of the same function f^{BDC}, we must transform $D, A, B \in \hat{L}_0^+$ as follows

$$D \mapsto \check{t}(1,1,1,\check{\phi},\check{\phi}), \quad A \mapsto \check{s}(1,0,0,0,0), \quad B \mapsto (\tilde{x}_1, \tilde{x}_2, -\tilde{y}, \check{\rho}, \check{\lambda}),$$

and this can be achieved by first applying the matrices given in Lemma 4.1 so that

$$\check{\rho} = -t\sqrt[4]{\frac{x_1}{x_2}}\theta, \quad \check{\lambda} = t\sqrt[4]{\frac{x_2}{x_1}}\theta,$$

$$\check{\phi} = -\sqrt[4]{\frac{x_1}{x_2}}\frac{\lambda}{\sqrt{x_1 x_2}} = \sqrt[4]{\frac{x_2}{x_1}}\frac{\rho}{\sqrt{x_1 x_2}}.$$

It follows that

$$(49) \qquad \mu = f^{BDC}(\check{\chi}, \check{\theta}, \check{\sigma}),$$

where the checked arguments of μ are related to checked variables from (49) as before and $\check{\chi} = \chi, \check{\theta} = \sigma, \check{\sigma} = -\theta$. It follows that $\mu = f^{DBA}(\chi, \theta, \sigma) = f^{BDC}(\chi, \sigma, -\theta)$ as required. ∎

As a direct corollary to the proof, we have

Corollary 4.5. *Consider the odd Ptolemy transformation on ordered pairs $(\theta, \sigma) \mapsto (\nu, \mu)$ together with the corresponding action on λ-lengths and apply it twice to the quadrilateral $ABCD$, with BAC in standard position. Then the effect is the switch transformation of $ABCD$ described in Lemma 4.1. Specifically after this transformation, DCA is in standard position with $D \mapsto \hat{t}(1,1,1,\sigma,\sigma)$ while the image of B is determined not by θ but rather by $-\theta$.*

5. Spin surfaces and orientations on fatgraphs

In this section, we relate the collection of spin structures on a punctured surface to orientations on any trivalent fatgraph spine of the surface. To begin, we recall results of Cimasoni-Reshetikhin from [7, 8].

The *boundary* of a one-dimensional CW complex $\mathcal{G}$ is its set $\partial\mathcal{G}$ of vertices of valence one. $\mathcal{G}$ is a *surface graph with boundary* for some compact oriented surface Σ with boundary $\partial\Sigma$ if $\mathcal{G}$ is embedded in Σ with $\mathcal{G} \cap \partial\Sigma = \partial\mathcal{G}$ so that $\bar{\mathcal{G}} = \mathcal{G} \cup \partial\Sigma$ is the 1-skeleton of a cellular decomposition of Σ. A *dimer configuration* on $\mathcal{G}$ is a choice of certain of its edges called *dimers* such that each vertex not in $\partial\mathcal{G}$ has exactly one incident dimer while vertices in $\partial\mathcal{G}$ may or may not have an incident dimer, the specification of which is regarded as an *a priori* boundary condition on the dimer configuration.

Given an orientation K on the edges of $\bar{\mathcal{G}}$ and a closed oriented edge curve C in it, we denote by n_C^K the number of edges counted with multiplicity where the orientation of C disagrees with that of K. A *Kasteleyn orientation* on $\bar{\mathcal{G}}$ is an orientation K on the edges of the 1-skeleton $\bar{\mathcal{G}}$ so that $n_{\partial f}^K = 1 \pmod 2$ for each face f of Σ. Define an equivalence relation $K_1 \sim K_2$ between two Kasteleyn orientations K_1, K_2 on $\mathcal{G}$ generated by altering the orientation on every edge incident on some fixed vertex, which is called a *Kastelyn reflection*, and let $\mathcal{K}(\mathcal{G})$ denote the set of equivalence classes. There is furthermore a cochain $\theta_{K_1,K_2} \in C^1(\Sigma; \mathbb{Z}_2)$ defined by setting $\theta_{K_1,K_2}(e) = 1$ if and only if K_1 and K_2 disagree on e, for e an edge of $\mathcal{G}$.

Theorem 5.1 (Corollary 1 of [7]). *In fact, $\theta_{K_1,K_2} \in Z^1(\Sigma; \mathbb{Z}_2)$ is a cocycle which is a coboundary if and only if $K_1 \sim K_2$. Furthermore, the set of equivalence classes of Kasteleyn orientations is an affine $H^1(\Sigma; \mathbb{Z}_2)$-space.*

Seminal work of Dennis Johnson [13] mentioned before identifies as affine $H^1(\Sigma; \mathbb{Z}_2)$-spaces the collection $\Omega(F)$ of all spin structures on a surface Σ with the collection $\mathcal{Q}(\Sigma)$ of all quadratic forms $q : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying $q(a+b) = q(a)+q(b)+a\cdot b$ with respect to the homology intersection pairing $a \cdot b$ of $a, b \in H_1(\Sigma; \mathbb{Z}_2)$. In fact, Kasteleyn orientations on a surface graph for Σ and quadratic functions on $H_1(\Sigma; \mathbb{Z}_2)$ are also isomorphic as affine $H^1(\Sigma; \mathbb{Z}_2)$-spaces, and the correspondence is given via an explicit construction relative to a fixed dimer D as follows.

Theorem 5.2 (Theorem 2.2 of [8]). *Fix a dimer configuration D on a surface graph with boundary $\mathcal{G}$ for the surface Σ and let $\alpha \in H_1(\Sigma; \mathbb{Z}_2)$ be represented by oriented closed curves $C_1, \ldots, C_m \in \bar{\mathcal{G}}$. If K is a Kasteleyn orientation on $\mathcal{G}$, then the function $q_D^K : H_1(\Sigma; \mathbb{Z}_2) \to \mathbb{Z}_2$ given by*

$$(50) \qquad q_D^K(\alpha) = \sum_{i<j} C_i \cdot C_j + \sum_{n=1}^{m}(1 + n_{C_i}^K + \ell_{C_i}^D) \pmod 2$$

is a well-defined quadratic form, where ℓ_C^D is the number of edges of D sticking out to the left of C, and n_C^K as before is the number of edges counted with multiplicity where the orientation of C disagrees with that of K. Moreover, for each fixed dimer D, this establishes an isomorphism $\mathcal{K}(\mathcal{G}) \approx \mathcal{Q}(\Sigma)$ as affine $H^1(\Sigma; \mathbb{Z}_2)$-spaces.

Together with [13], this establishes an isomorphism of affine $H^1(\Sigma; \mathbb{Z}_2)$-spaces between the collection $\Omega(\Sigma)$ of spin structures on Σ and $\mathcal{K}(\mathcal{G})$ for any surface graph $\mathcal{G}$ with boundary for Σ with respect to a fixed dimer configuration.

Now, given a trivalent fatgraph spine τ for $F = F_g^s$, we shall build an appropriate surface graph with boundary for a surface embedded in F as a deformation retract. Construct a CW decomposition of this compact so-called skinny surface $\Sigma = \Sigma(\tau) \subset F$ with boundary taking one hexagon H_v for each vertex v and one rectangle R_e for each edge e of τ glued together in the natural way in F as in Figure 5. There is a canonical surface graph $\mathcal{G} = \mathcal{G}_\tau$ for Σ comprised of the common boundaries of these hexagonal and rectangular regions as also illustrated in the figure by bold line
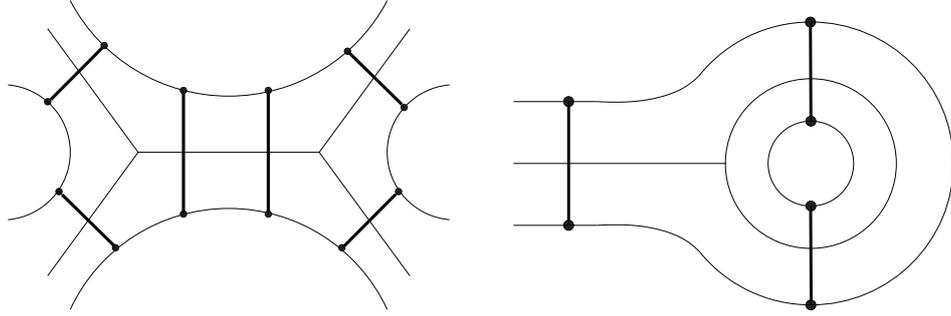
FIGURE 5. Cellular decomposition via fatgraphs

segments, two such segments for each edge of τ. There is also a canonical dimer $\mathcal{D} = \mathcal{G}$ given by exactly this same set of segments, so each vertex in $\partial\mathcal{G}$ has an incident dimer.

A hexagon has the two special Kastelyn orientations that are either outgoing or incoming at each vertex, and these are related by reversal of orientation of each edge. Furthermore identifying H_v with this abstract hexagon, for some vertex v of τ, there is a unique such orientation which agrees with the one induced from the counter-clockwise orientation of $\mathcal{G}$ with H_v on the left as illustrated on the top of Figure 6. Thus, any Kastelyn orientation on $\bar{\mathcal{G}}$ can be modified by Kastelyn reflections to agree with this one on H_v for each vertex v of τ, i.e., the special Kastelyn orientations saturate the equivalence classes.
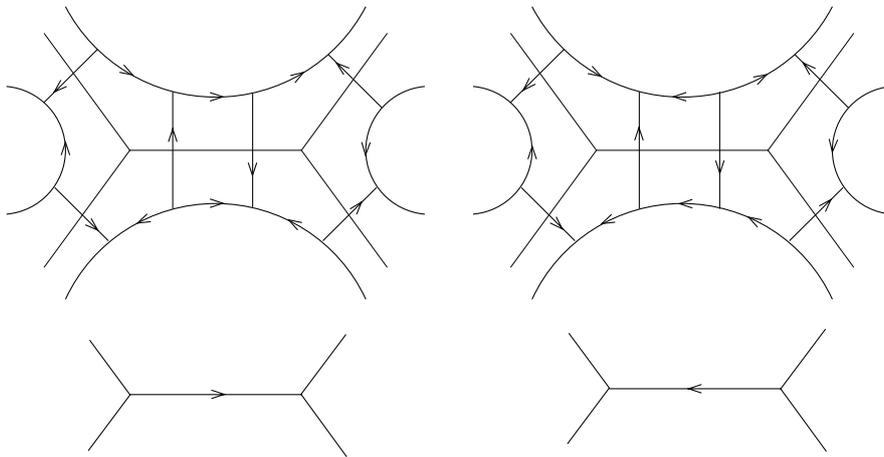
FIGURE 6. Extension of special Kastelyn orientations on hexagons to special ones on $\bar{\mathcal{G}}$ on the top and orientation on fatgraph τ from special Kasteleyn orientations on $\bar{\mathcal{G}}$ on the bottom

Suppose that K is such a special Kastelyn orientation on $\bar{\mathcal{G}}$ that thus agrees with the special hexagonal ones near each vertex of τ oriented as already discussed.

There are exactly two methods that K can extend as a Kastelyn orientation to a rectangle R_e as illustrated on the top in Figure 6, and these are naturally in bijective correspondence with orientations on the edges of τ themselves insofar as they are parallel as illustrated on the bottom of Figure 6.

Conversely, suppose that ω is an orientation on τ and let K_ω denote the associated special Kastelyn orientation on $\bar{\mathcal{G}}$ defined to be parallel on each R_e and special on each H_v as before. In the same spirit as Kastelyn reflection, we define a *fatgraph reflection* at a vertex v of τ to reverse the orientations of ω on every edge of τ incident on v, consider the equivalence relation $\omega_1 \sim \omega_2$ thus generated on the set of all orientations on τ and let $\mathcal{O}(\tau)$ denote the set of all equivalence classes. One easily checks that the fatgraph reflection at v is given precisely by the composition of Kastelyn reflections at the six vertices of H_v, so $\omega_1 \sim \omega_2$ implies $K_{\omega_1} \sim K_{\omega_2}$. Conversely, in order that the orientation on an edge of some H_v is invariant, we must perform Kastelyn reflection either at both or neither of its endpoints, and it follows that $K_{\omega_1} \sim K_{\omega_2}$ also implies $\omega_1 \sim \omega_2$. Furthermore, given orientations ω_1, ω_2 on τ, there is the analogous cochain $\theta_{\omega_1,\omega_2} \in C^1(\Sigma; \mathbb{Z}_2)$ taking a non-zero value on an edge of τ if and only if ω_1 and ω_2 disagree on the edge, and we clearly have $\theta_{\omega_1,\omega_2} \equiv \theta_{K_{\omega_1},K_{\omega_2}}$. Summarizing, we have:

Proposition 5.3. *For each orientation ω on the edges of a fatgraph τ, there exists a unique special Kastelyn orientation K_ω on $\bar{\mathcal{G}}_\tau$, and this induces an isomorphism $\mathcal{K}(\mathcal{G}_\tau) \approx \mathcal{O}(\tau)$ of equivalence classes under reflection as affine $H^1(F_g^s; \mathbb{Z}_2)$-spaces.*

As follows directly from Theorem 5.2 and Proposition 5.3, we have

Theorem 5.4. *Let τ be a fatgraph spine in the surface $F = F_g^s$ with corresponding surface graph $\mathcal{G}_\tau$ for $\Sigma \subset F$. Then the formula (50) with the canonical dimer configuration $\mathcal{D}$ establishes an isomorphism $\mathcal{O}(\tau) \approx \mathcal{Q}(\Sigma)$ as affine $H^1(F; \mathbb{Z}_2)$-spaces, and indeed also $\mathcal{Q}(\Sigma) \approx \mathcal{Q}(F)$ since $\Sigma \subset F$ is a homotopy equivalence.*

It remains for us here only to compute the effect that flipping a fatgraph edge has on an orientation class:

Lemma 5.5. *Suppose that τ_1 is a trivalent fatgraph spine for F and that τ_2 arises by flipping an edge of τ_1. There is a unique bijection $\mathcal{O}(\tau_1) \to \mathcal{O}(\tau_2)$ covering the identity map of $\mathcal{Q}(F)$, and it is described by Figure 2.*

Proof. Consider a neighborhood of the edge of τ_1 upon which the flip is performed depicted in Figure 7 where there are illustrated six distinct oriented paths in F denoted $\alpha, \beta, \gamma, \delta, \epsilon, \phi$, which may be completed to closed oriented curves in F and contribute to the value of the quadratic form. A tedious computation given in Appendix II compares formula (50) before and after the flip for these six paths and determines that there is the unique evolution of orientation class from τ_1 to τ_2 illustrated in Figure 2 that leaves invariant these contributions. ∎
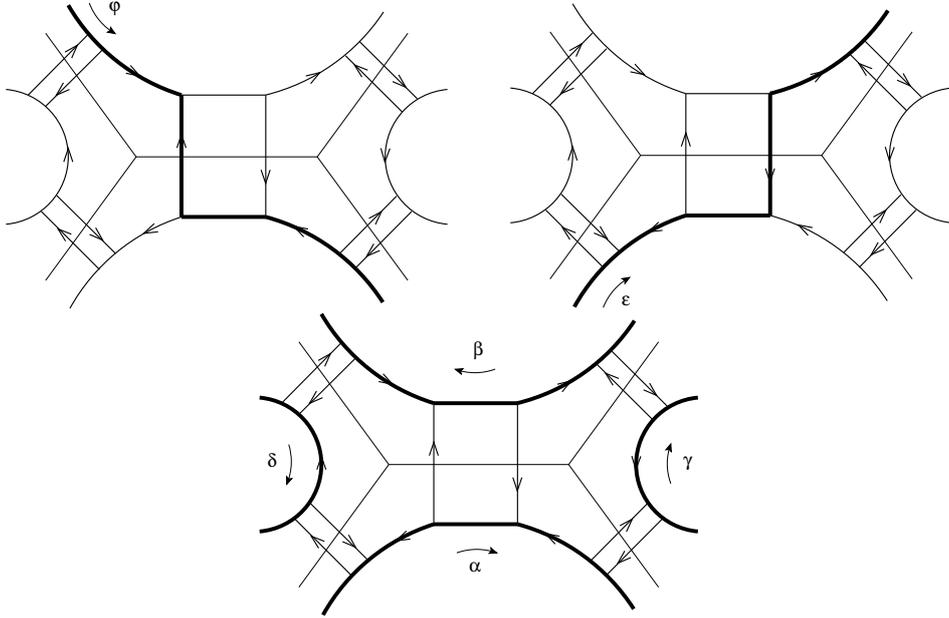
FIGURE 7. The oriented paths $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ on $\bar{\mathcal{G}}_{\tau_1}$ are indicated by bold lines

Theorem 5.6. *Let τ be a trivalent fatgraph spine of $F = F_g^s$. Then $\mathcal{O}(\tau)$ is isomorphic to the set $\Omega(F)$ of spin structures on F as affine $H^1(F; \mathbb{Z}_2)$-spaces. Moreover, the action of $MC(F)$ on $\Omega(F)$ lifts naturally to the action of the Ptolemy groupoid on $\mathcal{O}(\tau)$ illustrated in Figure 2.*

By naturality here, we mean that if $\varphi \in MC(F)$, then there is a finite sequence $F \supset \varphi(\tau) = \tau_1 \to \tau_2 \to \cdots \to \tau_n = \tau$ where τ_{i+1} arises from τ_i by a flip, for $i = 1, \ldots, n-1$. Suppose that ω is an orientation on τ inducing via the identification φ the orientation ω_1 on τ_1 and moreover that the evolution of orientation illustrated in Figure 2 serially induces from ω_1 the orientation $\omega' = \omega_{n+1}$ on $\tau_{n+1} = \tau$. The spin structure on F associated to ω maps to that of ω' under the action of φ on $\Omega(F)$ by construction.

6. Coordinates on decorated super-Teichmüller space

Equivalent to the choice of trivalent fatgraph spine $\tau \subset F$ is the specification of its dual ideal triangulation Δ of F. An orientation ω on τ induces an orientation on Δ by requiring that the oriented edge of Δ occurs clockwise from its dual oriented edge in τ near their point of intersection here using an orientation of the surface F. Dual to the fatgraph reflection at a vertex is the change of orientation on each edge in the frontier of a triangle complementary to Δ. We consider the lift of Δ to an ideal triangulation $\tilde{\Delta}$ of the universal cover $\tilde{F} \to F$. Fixing a base point in F, the fundamental group $\pi_1 = \pi_1(F)$ acts as deck transformations on $\tilde{F}$ leaving

invariant $\tilde{\Delta}$. Of course $\tilde{F}$ is *topologically equivalent* to $\mathbb{D}$, and a hyperbolic metric on F further determines a *metric equivalence*. In any case, we may consider the collection of ideal vertices $\tilde{\Delta}_\infty \subset \mathbb{S}^1$ of all the arcs in $\tilde{\Delta}$ as an abstract set.

Theorem 6.1. *Fix a surface $F = F_g^s$ of genus $g \geq 0$ with $s \geq 1$ punctures, where $2g-2+s > 0$, and let Δ be some ideal triangulation of F whose lift $\tilde{\Delta}$ to the universal cover $\pi : \tilde{F} \to F$ has ideal vertices $\tilde{\Delta}_\infty$. Suppose that ω is an orientation class on the arcs in Δ corresponding to a specified spin structure, and assign to each edge of Δ an even coordinate and to each triangle complementary to Δ an odd coordinate where the latter are taken modulo an overall sign. Then there is a function*

$$\ell : \tilde{\Delta}_\infty \to \hat{L}_0^+$$

uniquely determined up to post-composition with an element of $OSp(1|2)$ so that

    i) *if $a, b \in \tilde{\Delta}_\infty$ span an edge in $\tilde{\Delta}$, then the coordinate on this edge is given by the λ-length $\sqrt{<\ell(a), \ell(b)>}$;*

    ii) *if $c, b, a \in \tilde{\Delta}_\infty$ span a triangle complementary to $\tilde{\Delta}$ and occur in the positive order in $\mathbb{S}^1$, then up to a sign the coordinate of this triangle is given by the μ-invariant of the positive triple $\ell(c)\ell(b)\ell(a)$; moreover, if $d, c, a \in \tilde{\Delta}_\infty$ likewise occur in the positive order and span a triangle, then the coordinates of these respective triangles are given up to an overall sign by the μ-invariants of the positive triples $\ell(c)\ell(b)\ell(a)$ and $\ell(d)\ell(c)\ell(a)$ as related by Proposition 4.2.*

Furthermore, there is a representation $\hat{\rho} : \pi_1 = \pi_1(F) \to OSp(1|2)$ with respect to which ℓ is π_1-equivariant in the sense that $\hat{\rho}(\gamma)(\ell(a)) = \ell(\gamma(a))$ for each $\gamma \in \pi_1$ and $a \in \tilde{\Delta}_\infty$ so that $\pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \to SL(2, \mathbb{R}) \to PSL(2, \mathbb{R})$ is a Fuchsian representation whose lift $\pi_1 \xrightarrow{\hat{\rho}} OSp(1|2) \to SL(2, \mathbb{R})$ agrees with the specified spin structure. Moreover $\hat{\rho}$ is uniquely determined up to conjugacy by an element of $OSp(1|2)$.

Proof. The argument closely follows the bosonic case [19, 21] using the putative coordinates to recursively construct the mapping ℓ and finally the representation $\hat{\rho}$. Let us first consider the case that the trivalent fatgraph spine $\tau \subset F$ dual to Δ is bipartite, namely, there is a two-coloring of the vertices of τ so that the endpoints of any edge of τ have different colors. An example of a bipartite fatgraph spine is illustrated in Figure 8 for each topological type of punctured surface.

Dually letting $\mathcal{T}$ denote the collection of triangles complementary to Δ in F, there is a function $\delta : \mathcal{T} \to \{\pm 1\}$ so that for any two triangles $t_1, t_2 \in \mathcal{T}$ sharing an edge, we must have $\delta(t_1) \cdot \delta(t_2) = -1$. Notice that in this bipartite case there are exactly two distinct such functions differing by an overall sign. The coordinates on Δ lift to coordinates on $\tilde{\Delta}$ in the natural way assigning to each edge $e \in \tilde{\Delta}$ the even coordinate of $\pi(e)$ and to each complementary triangle $t \in \tilde{\mathcal{T}} =$
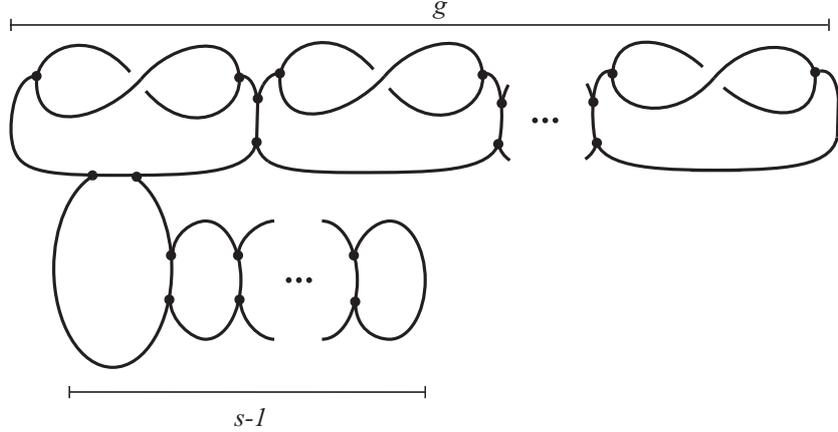
FIGURE 8. A bipartite fatgraph spine in F_g^s for any $g \geq 0$ and $s \geq 1$

{complementary triangles to $\tilde{\Delta}$} the odd coordinate θ_t of $\pi(t)$. The function δ likewise lifts to the eponymous function $\delta : \tilde{\mathcal{T}} \to \{\pm 1\}$ satisfying $\delta(t) = \delta(\pi(t))$.

Choose a distinguished triangle-edge pair or equivalently a distinguished oriented edge of $\tilde{\Delta}$ which determines the triangle $t \in \tilde{\mathcal{T}}$ lying to its left in the oriented surface $\tilde{F}$. According to Lemma 3.3, there is a positive triple ABC in the special light cone realizing the putative λ-lengths and μ-invariant θ_t so that the point B is opposite to the distinguished oriented edge, and ABC is furthermore uniquely determined up to fermionic reflection g^r. Choosing a particular representative ABC or $g^r ABC$ at this stage gauge fixes the $\mathbb{Z}_2$-action and thus determines well-defined signs on all of the fermionic coordinates $\theta_{\bar{t}}$ for $\bar{t} \in \tilde{\mathcal{T}}$. (In the special case when the fermionic coordinate on t vanishes, we have $g^r ABC = ABC$ and can still gauge fix at this point to determine the signs on all of the non-vanishing fermionic coordinates.) This lift to a positive triple in $\hat{L}_0^+$ of the vertices of t constitutes the basis step of our recursive construction of the mapping ℓ.

In order to lift the triangle $t' \in \tilde{\mathcal{T}}$ to the right of the distinguished oriented edge with its specified fermionic coordinate $\theta_{t'}$, we shall rely on the basic calculation formulated as Proposition 4.2 to lift the vertex of t' distinct from those of t to a point $D \in \hat{L}_0^+$, however, the coordinates employed to determine D are the specified putative λ-lengths and the δ-modified μ-invariant $\delta(t') \cdot \theta_{t'}$ rather than $\theta_{t'}$ itself. Conjugating by the ABC prime transformation and its square, we can likewise uniquely lift to $\hat{L}_0^+$ the vertices of the other two triangles in $\tilde{\mathcal{T}}$ adjacent to t.

Continue recursively in this way moving each triangle in $\tilde{\mathcal{T}}$ already lifted into standard position with its specified μ-invariant using $OSp(1|2)$ and then employing the putative λ-lengths and δ-modified μ-invariants to lift the next adjacent triangle in $\tilde{\mathcal{T}}$ using Proposition 4.2 in order to finally uniquely lift all of $\tilde{\Delta}_\infty$ and completely define $\ell : \tilde{\Delta}_\infty \to \hat{L}_0^+$ in keeping with requirements i) and ii) in the statement of the

theorem. There are actually only finitely many values of coordinates employed in this construction since there are only finitely many arcs in Δ and triangles in $\mathcal{T}$. In particular, the bodies of λ-lengths are uniformly bounded above and below. As in the pure even case [19, 21], it follows from this bounded geometry that the bosonic projection $\tilde{\Delta}_\infty \overset{\ell}{\dashrightarrow} \hat{L}_0^+ \dashrightarrow L^+$ induces a mapping from the ideal triangulation of $\tilde{F}$ onto a tessellation of all of $\mathbb{D}$.

We claim that changing the distinguished oriented edge of $\tilde{\Delta}$ used to initiate this construction merely modifies the function ℓ by post-composition with a group element in $OSp(1|2)$. To see this, first notice that if we change orientation of the distinguished oriented edge, then t' will be in standard position with its μ-invariant $\theta_{t'}$, and the triangle t is then produced using the δ-modified μ-invariant $\delta(t) \cdot \theta_t$. Applying the switch transformation (and possibly overall fermionic reflection for the $\mathbb{Z}_2$ gauge group), based on the fact that $\delta(t) \cdot \delta(t') = -1$ together with Proposition 4.5, we recover the image of the previous construction under the switch transformation, which lies in $OSp(1|2)$, with the correct signs for fermionic parameters. Moreover, any change in the distinguished oriented edge can be achieved by means of a finite composition of prime transformations and at most one reversal of orientation. Since prime transformations are again group elements of $OSp(1|2)$, we conclude that our equivalence class of lifts under the action of $OSp(1|2)$ is independent of the distinguished oriented edge as required. Uniqueness of the mapping ℓ up to post-composition with an element of $OSp(1|2)$ follows by construction from the uniqueness statements in Lemma 3.3 and Proposition 4.2 completing the proof of the first part under the bipartite assumption.

From the lift $\tilde{\Delta}_\infty \overset{\ell}{\dashrightarrow} \hat{L}_0^+$, we shall presently construct the required representation $\hat{\rho} : \pi_1 \to \hat{\Gamma} < OSp(1|2)$. To this end, choose a connected fundamental domain $D \subset \tilde{F}$ for the action of π_1 comprised of a collection of triangles in $\tilde{\mathcal{T}}$ which contains the base triangle t specified earlier which began the recursive construction of ℓ. The frontier edges of D in $\tilde{F}$ arise in pairs c, c' together with an abstract identification $c' = \gamma(c)$ induced by some $\gamma \in \pi_1$, and we let $c'_i = \gamma_i(c_i)$ enumerate the collection of these edge pairings, for $i = 1, \ldots, 4g + 2s$, where the γ_i thus freely generate π_1. To determine the image $\hat{\rho}(\gamma_i) \in OSp(1|2)$ of $\gamma_i \in \pi_1$ in order to define the representation $\hat{\rho}$, let us further enumerate two triangles for each edge pairing, namely, the unique triangles $t_i \supset c_i$ and $t'_i \supset c'_i$ in $\tilde{\mathcal{T}}$ where $t_i \subset D$ and $t'_i \not\subset D$. Notice as before that the coordinates for t_i or t'_i have gauge-fixed signs from the basis step of the construction of ℓ, and the parameters on t_i lifted from F agree with those on t'_i since $\gamma_i(t_i) = t'_i$ implies $\pi(t_i) = \pi(t'_i)$.

Let $T_i = \ell(t_i)$ be the corresponding positive triple in $\hat{L}_0^+$. According to Lemma 3.3 again, there is a unique $h_i \in OSp(1|2)$ such that $T_i \cdot h_i = \bar{T}_i$, where $OSp(1|2)$ acts on the right and $\bar{T}_i$ indicates the triple of points in standard position realizing the λ-lengths and signed μ-invariant of T_i so that the fermion-dependent vertex is opposite to c_i. Let $h'_i \in OSp(1|2)$ be the analogous group element for T'_i. We must furthermore take account of the spin structure in accordance with [17, 18]

relating the sign of the trace of the bosonic reduction of $\hat{\rho}(\gamma)$ with the value of the corresponding quadratic form on the underlying mod two homology class $[\gamma] \in H_1(F; \mathbb{Z}_2)$, for $\gamma \in \pi_1$. In particular, notice that the fermionic reflection g^r has the central element $-I \in SL(2, \mathbb{R})$ as its bosonic reduction. Given the quadratic form $q \in \mathcal{Q}(F)$ corresponding via Theorem 5.4 to our specified orientation ω, we define

$$
\hat{\rho}(\gamma_i) = \left\{
\begin{array}{l}
h_i^{-1} \cdot g^r \cdot h_i', \\
\quad \text{if trace}(h_i^{-1} \cdot h_i') > 0 \ \& \ q([\gamma_i]) = 0 \text{ or trace}(h_i^{-1} \cdot h_i') < 0 \ \& \ q([\gamma_i]) = 1; \\
\\
h_i^{-1} \cdot h_i', \\
\quad \text{if trace}(h_i^{-1} \cdot h_i') < 0 \ \& \ q([\gamma_i]) = 0 \text{ or trace}(h_i^{-1} \cdot h_i') > 0 \ \& \ q([\gamma_i]) = 1.
\end{array}
\right.
$$

By construction, these group elements compose correctly so as to produce a representation, and since π_1 is a free group, there are no relations to confirm whence $\hat{\rho}$ is indeed a representation of π_1 in $OSp(1|2)$ onto a subgroup $\hat{\Gamma} = \hat{\rho}(\pi_1) < OSp(1|2)$. Also by construction, the projectivized bosonic reduction $\Gamma = \rho(\pi_1) < PSL(2, \mathbb{R})$ leaves invariant the tessellation of $\mathbb{D}$ discussed before, and an argument in [19, 21] going back to Poincaré thus proves that Γ is indeed a Fuchsian group uniformizing the punctured surface. Finally, the bosonic reduction $\tilde{\Gamma} = \tilde{\rho}(\pi_1) < SL(2, \mathbb{R})$ itself as a lift of $\Gamma < PSL(2, \mathbb{R})$ gives the correct spin structure on the underlying Riemann surface in keeping with [17, 18] by construction since multiplication by g^r alters the sign of the trace of the bosonic reduction.

This completes the required construction of a representation from the asserted parameters including the spin structure. The procedure described is clearly equivariant for the fermionic reflection under the initial choice of sign on the μ-invariant in the base triangle, and the distinction between pairs $t_i \subset D$ and $t_i' \not\subset D$ amounts only to replacing a generator by its inverse. There are thus two essential choices: the choice of base triangle t to begin the inductive construction as well as the fundamental domain D containing it. We must show that these choices are resolved by the overall conjugacy in the definition of the decorated super-Teichmüller space.

To this end, consider two such fundamental domains containing specified base triangles $t_i \subset D_i$ with corresponding mappings $\ell_i : \tilde{\Delta}_\infty \to \hat{L}_0^+$ and representations $\hat{\rho}_i : \pi_1 \to OSp(1|2)$, for $i = 1, 2$. There is a unique triangle $t_1' \subset D_2$ whose projection to F agrees with that of t_1 since D_2 is a fundamental domain for the action of π_1, whence t_1 and $\bar{t}_1'$ thus share the same invariants. By Lemma 3.3 again $\mathbb{Z}_2$ gauge-fixed as before, there is a unique $g \in OSp(1|2)$ so that $g(t_1) = t_1'$. By uniqueness in Lemma 3.3 and Proposition 4.2, we must then have $\ell_1 = \ell_2 \circ g$, and so g furthermore conjugates $\hat{\rho}_1$ to $\hat{\rho}_2$ as required. Notice that the trace of the bosonic reduction of an element of $OSp(1|2)$ is invariant under conjugacy, so the spin structure is left invariant by conjugation of representations.

We must finally extend the construction from bipartite to general trivalent fatgraph spines $\tau \subset F$. To this end according Corollary D, there is a finite sequence of flips starting from τ and ending with a bipartite fatgraph spine $\tau' \subset F$ such as

the one depicted in Figure 8 to which we may apply the construction just described based upon the coordinates on τ' induced from those on τ via super Ptolemy transformations in Proposition 4.3 and Theorem 4.4 as well as the orientation class on τ' induced from that on τ in Theorem 5.6 determining the spin structure. Since these super Ptolemy transformations are computed relative to a fixed configuration of points in $\hat{L}_0^+$, this gives a well-defined lift $\ell : \tilde{\Delta}_\infty \to \hat{L}_0^+$ for any trivalent fatgraph spine and hence a corresponding representation $\hat{\rho} : \pi_1 \to OSp(1|2)$ which is determined up to conjugacy since the mapping ℓ is determined up to post-composition with an element of $OSp(1|2)$. ∎

Corollary 6.2. *Fix a trivalent fatgraph spine $\tau \subset F$ for a surface $F = F_g^s$ of negative Euler characteristic and specify an orientation on the edges of τ determining the component C of $S\tilde{T}(F)$. Then λ-lengths on the edges together with μ-invariants on the vertices of τ, the latter taken modulo an overall sign, provide global coordinates on C. Moreover, these coordinates are natural in the sense that if $\varphi \in MC(F)$ and $\varphi(\tau) = \tau_1 - \tau_2 - \cdots - \tau_n = \tau$ is a sequence of trivalent fatgraph spines of F with consecutive ones related by a flip, then we identically induce coordinates and orientation class on $\varphi(\tau)$ from these data on τ using φ and perform the corresponding sequence of super Ptolemy transformations in Proposition 4.3 and Theorem 4.4 and evolution of orientation in Theorem 5.6 to induce new coordinates and orientation on τ itself. Then these induced coordinates and orientation class on τ describe the action of φ on $S\tilde{T}(F)$.*

Proof. This follows directly from the previous result and Theorem 5.6 together with the observation that λ-lengths and μ-invariants are defined intrinsically in $\mathbb{R}^{2,1|2}$. ∎

7. Shear coordinates and Ptolemy-invariant 2-form

First, let us recall in some detail the Ptolemy transformations studied in Section 4. There were two aspects to the calculation corresponding to even and odd, namely, the even Ptolemy transformation is a simple modification

$$(51) \qquad ef = (ac + bd)\Big(1 + \frac{\sigma\theta\sqrt{\chi}}{1+\chi}\Big)$$

of the standard pure even case $ef = ac + bd$, and the odd Ptolemy transformation again in the notation of Section 4 is given by

$$(52) \qquad \nu = \frac{\sigma + \theta\sqrt{\chi}}{\sqrt{1+\chi}}, \quad \mu = \frac{\sigma\sqrt{\chi} - \theta}{\sqrt{1+\chi}}.$$

Taken together (51) and (52) give expression to the *super Ptolemy transformation.*

These formulas in particular specialize to those of [3] where shear coordinates on the super-Teichmüller space are introduced. Namely as in the classical case, given an ideal triangulation Δ of the surface F, to each edge AC as in Figure 4 is

associated its cross ratio $\chi_e = \chi = \frac{ac}{bd}$, written again in terms of λ-lengths. Given the two positive triples CBA and DCA in the special light cone $\hat{L}_0^+$, denote by χ_a, χ_b, χ_c, χ_d these cross ratios associated to each respective edge a, b, c, d also illustrated in Figure 4 identifying an edge with its λ-length here for convenience; of course, each of these depends in turn on the λ-lengths of other nearby edges, for instance, $\chi_b = \frac{eg}{ah}$, where g, h are λ-lengths on the frontier edges of the other complementary triangle to Δ than CBA that contains CB. Nevertheless, under the even Ptolemy transformation, we find that $\chi_b = \frac{eg}{ah}$ transforms to

$$\chi_b \frac{ac}{ef} = \chi_b(1 + \chi^{-1})^{-1}\Big(1 + \sigma\theta(\chi^{\frac{1}{2}} + \chi^{-\frac{1}{2}})^{-1}\Big)^{-1} = \chi_b(1 + \chi^{-1} + \sigma\theta\sqrt{\chi^{-1}})^{-1},$$

and one similarly computes

$$\chi_a \mapsto \chi_a(1 + \chi + \sigma\theta\sqrt{\chi}), \quad \chi_c \mapsto \chi_c(1 + \chi + \sigma\theta\sqrt{\chi}),$$

(53) $$\chi_d \mapsto \chi_d(1 + \chi^{-1} + \sigma\theta\sqrt{\chi^{-1}})^{-1}.$$

In fact, these transformations together with equation (52) coincide with those in [3] up to a conventional inversion of the cross ratio coordinate (see Figure 2 in [3]).

In the pure even case [20, 21], the Kähler 2-form of the Weil-Petersson Hermitian metric on the Teichmüller space $T(F)$ or moduli space $M(F)$ of a punctured surface F was computed relative to the λ-lengths on any convenient triangulation Δ of F. Namely, it is given by

(54) $$\omega_\Delta = 2 \sum d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a,$$

where the sum is over all complementary triangles T to Δ in F with frontier edges of T occurring in the cyclic order a, b, c compatibly with the clockwise orientation on $T \subset F$. It is not difficult to check directly that this expression is invariant under the pure even Ptolemy transformation $ef = ac + bd$, and indeed in the 1980's before it was recognized as this particular 2-form, it was nevertheless already confirmed to be invariant under Ptolemy transformations and hence arise from some 2-form on the quotient moduli space $T(F)/MC(F)$. Much this same computation applies in the current case.

Theorem 7.1. *If Δ is an ideal triangulation of the punctured surface F, then consider the even 2-form on $S\widetilde{T}(F)$ given by*

$$\hat{\omega}_\Delta = \sum d\log a \wedge d\log b + d\log b \wedge d\log c + d\log c \wedge d\log a - (d\theta)^2,$$

where the sum is over all triangles whose consecutive edges in the clockwise ordering have λ-lengths a, b, c and μ-invariant θ. Then this 2-form is invariant under super Ptolemy transformations.

Notice that $\hat{\omega}_\Delta$ is manifestly invariant under the fermionic reflection changing the signs of all fermions, and we have dropped the pre factor 2 here compared to the

Kähler form so that if one converts the expression $\hat{\omega}_\Delta$ from λ-lengths to shear coordinates as described before, then the resulting 2-form is associated with the Poisson bracket given in [3].

Proof. Let us remind [2] the reader that on supermanifolds the de Rham operator anticommutes with odd constants, and in local coordiantes $(\{x_i\}, \{\theta_j\})$ is given by $d = dx_i \partial_{x_i} + d\theta_i \partial_{\theta_i}$, where the odd derivative acts from the left and the $d\theta_i$ are even.

Adopt the notation $\tilde{x} = d \log x = \frac{dx}{x}$ for any invertible expression x and compute as in the pure even case [20, 21] for a pair CBA and DCB of positive triples that the contribution to $\hat{\omega}_\Delta$ before the flip on AC in the notation of Figure 1 is given by

$$(55) \qquad \tilde{a}\tilde{b} + \tilde{b}\tilde{e} + \tilde{e}\tilde{a} + \tilde{e}\tilde{d} + \tilde{d}\tilde{c} + \tilde{c}\tilde{a} - (d\theta)^2 - (d\sigma)^2$$

and after the flip is given by

$$(56) \qquad \tilde{b}\tilde{c} + \tilde{c}\tilde{f} + \tilde{f}\tilde{b} + \tilde{f}\tilde{d} + \tilde{d}\tilde{a} + \tilde{a}\tilde{f} - (d\nu)^2 - (d\mu)^2.$$

We must show that these two expressions coincide.

To this end, notice that if $U = \frac{\sigma\theta\sqrt{\chi}}{1+\chi} = \frac{\sigma\theta}{\chi^{\frac{1}{2}} + \chi^{-\frac{1}{2}}}$, then

$$\tilde{e} + \tilde{f} = \frac{1}{ac + bd}(ac(\tilde{a} + \tilde{c}) + bd(\tilde{b} + \tilde{d})) + dU,$$

$$(57)$$

and we find

$$\tilde{b}\tilde{c} + \tilde{c}\tilde{f} + \tilde{f}\tilde{b} + \tilde{f}\tilde{d} + \tilde{d}\tilde{a} + \tilde{a}\tilde{f} = \tilde{b}\tilde{c} + \tilde{d}\tilde{a} + \tilde{f}(\tilde{b} + \tilde{d} - \tilde{a} - \tilde{c})$$
$$= \tilde{b}\tilde{c} + \tilde{d}\tilde{a} - \tilde{e}(\tilde{b} + \tilde{d} - \tilde{a} - \tilde{c})$$
$$+ \frac{ac}{ac + bd}(\tilde{a} + \tilde{c})(\tilde{b} + \tilde{d}) + \frac{bd}{ac + bd}(\tilde{a} + \tilde{c})(\tilde{b} + \tilde{d}) - dU\tilde{\chi}$$
$$= \frac{ac}{ac + bd}(\tilde{a} + \tilde{c})(\tilde{b} + \tilde{d}) + \frac{bd}{ac + bd}(\tilde{a} + \tilde{c})(\tilde{b} + \tilde{d}) - dU\tilde{\chi},$$

where

$$(58) \qquad dU\tilde{\chi} = \frac{d(\sigma\theta)d\chi}{(1 + \chi)\sqrt{\chi}}.$$

Meanwhile, the super Ptolemy transformation gives

$$(d\mu)^2 + (d\nu)^2 = d\sigma^2 + d\theta^2 + 2\frac{d\sigma}{\sqrt{1+\chi}}d\Big(\frac{\sqrt{\chi}}{\sqrt{1+\chi}}\Big)\theta + 2\frac{\sqrt{\chi}d\theta}{\sqrt{1+\chi}}d\Big(\frac{1}{\sqrt{1+\chi}}\Big)\sigma$$
$$= -2\frac{d\sigma\sqrt{\chi}}{\sqrt{1+\chi}}d\Big(\frac{1}{\sqrt{1+\chi}}\Big)\theta - 2\frac{d\theta}{\sqrt{1+\chi}}d\Big(\frac{\sqrt{\chi}}{\sqrt{1+\chi}}\Big)\sigma$$
$$= d\sigma^2 + d\theta^2 + \frac{d(\theta\sigma)d\chi}{(1+\chi)\sqrt{\chi}}$$

which coincides with equation (58) as required. ∎

APPENDIX I. $OSp(1|2)$: NOTATION AND CONVENTIONS

In this appendix, we provide basic information concerning the Lie supergroup $OSp(1|2)$ and its Lie superalgebra. Our cursory treatment here is presumably sufficient for the purposes of the text, and we refer the interested reader to [2, 15] for further details about general Lie superalgebras and supergroups.

Let us first introduce certain conventions (which differ from those in [3]). Given a Lie superalgebra $\mathfrak{g}$, one can consider its Grassmann envelope, namely, the Lie superagebra $\mathfrak{g}(S) = S \otimes \mathfrak{g}$ for some Grassmann algebra S with decomposition $S = S_0 \oplus S_1$ into even and odd elements. It follows that $\mathfrak{g}(S)$ is both a right and left S-module, i.e., $s \otimes T = (-1)^{|s||T|}(1 \otimes T)(s \otimes 1)$ if $s \in S$ and $T \in \mathfrak{g}$ are homogeneous elements of respective degrees $|s|$ and $|T|$. This rule allows one to construct a representation of the corresponding Lie superalgebra $\mathfrak{g}(S)$ in the space $S \otimes \mathbb{R}^{m|n}$ from a given representation of $\mathfrak{g}$ in $\mathbb{R}^{m|n}$. One can then produce a representation of the corresponding Lie group $G(S)$ by exponentiating pure even elements from $\mathfrak{g}(S)$ in $S \otimes \mathbb{R}^{m|n}$.

When writing a super matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ representing the action of $G(S)$ or $\mathfrak{g}(S)$ as elements of $S \otimes End(\mathbb{R}^{m|n})$ on $S \times \mathbb{R}^{m|n}$, notice that for pure even supermatrices, the composition rule is given by

$$(59) \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' - BC' & AB' + BD' \\ CA' + DC' & DD' - CB' \end{pmatrix},$$

where the products on the right hand side are the usual products of (super)matrices.

The usual (super)matrix multiplication (without the minus signs above) is recovered upon replacing B with $-B$. This difference in sign is related to the fact that one typically considers the action of group elements on $\mathbb{R}_S^{m|n} = S_0^{\times m} \times S_1^{\times n}$, which can be identified with the space of even elements in $S \otimes \mathbb{R}^{m|n}$, and the extra minus sign in front of B comes from that isomorphism. However, throughout this paper we keep the above convention for multiplication of superalgebras (with the extra signs) since it gives a cleaner relationship with the representation of the original Lie superalgebra .

Another essential ingredient is the superdeterminant or Berezinian of an even supermatrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with D invertible which is defined as

$$(60) \qquad sdet(M) = (det D)^{-1}(A + BD^{-1}C),$$

where the products of matrices are standard matrix products, and we have the nonstandard plus sign $A + BD^{-1}C$ in the definition reflecting our conventions.

Now, the Lie superalgebra $OSp(1|2)$ has three even $h, X_{\pm}$ and two odd generators $v_{\pm}$ satisfying the commutation relations

$$(61) \qquad [h, v_{\pm}] = \pm v_{\pm}, \quad [v_{\pm}, v_{\pm}] = \mp 2X_{\pm}, \quad [v_+, v_-] = h.$$

The corresponding realization via $(2|1) \times (2|1)$ supermatrices is given by

$$v_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad v_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the corresponding supergroup $OSp(1|2)$ can be faithfully realized as $(2|1) \times (2|1)$ supermatrices g with $sdet$ equal to one obeying the relation

(62)
$$g^{st} J g = J,$$

where

(63)
$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and where the supertranspose g^{st} of g is given by

(64)
$$g = \begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ \gamma & \delta & f \end{pmatrix} \quad \text{implies} \quad g^{st} = \begin{pmatrix} a & c & \gamma \\ b & d & \delta \\ -\alpha & -\beta & f \end{pmatrix}.$$

This leads to the following system of constraints on the entries of g:

(65)
$$f = 1 + \alpha\beta, \quad ad - bc = f^{-1}, \quad d\gamma - c\delta = \beta$$
$$\delta = b\beta - d\alpha, \quad \gamma = a\beta - c\alpha, \quad \alpha = b\gamma - a\delta.$$

In particular, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ describes the canonical inclusion $SL(2,\mathbb{R}) < OSp(1|2)$, and the body of the upper-left 2×2 block conversely gives the bosonic reduction in $SL(2,\mathbb{R})$ of a matrix in $OSp(1|2)$.

Appendix II. Flip action on oriented fatgraphs

In this appendix, we prove the last part of Theorem 5.6 and compute the evolution of orientations under flips illustrated in Figure 2. We must verify that the contributions of the arcs $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$ illustrated in Figure 7 to a quadratic form via Theorem 5.2 with the canonical dimer $\mathcal{D}$ for the specified orientations of edges is the same before and after the flip. To that end for the purpose of explicit computation, let us fix the representative arcs φ, ε as edge-paths before and after the flip as illustrated in Figure 9. Representatives edge-paths of the other arcs $\alpha, \beta, \gamma, \delta$ are taken to lie entirely in the boundary.

Now in the notation of Theorem 5.2 for the canonical dimer $\mathcal{D}$, we find $\ell_\alpha^{\mathcal{D}} = \ell_\beta^{\mathcal{D}} = \ell_\gamma^{\mathcal{D}} = \ell_\delta^{\mathcal{D}} = 0 \pmod 2$ both before and after the flip, $\ell_\varphi^{\mathcal{D}} = \ell_\varepsilon^{\mathcal{D}} = 1 \pmod 2$ before the flip and $\ell_\varphi^{\mathcal{D}} = 1 \pmod 2$, $\ell_\varepsilon^{\mathcal{D}} = 0 \pmod 2$ after the flip. The numbers n_C^K of Theorem 5.2 are tabulated in Figure 10 in the eight cases where the interior edge runs from left to right with the northwest leaf pointing towards it as may always be arranged by fatgraph reflections on the two vertices.
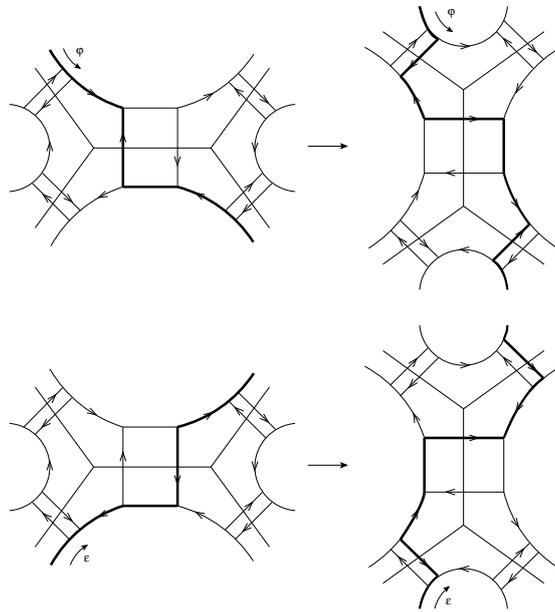
FIGURE 9. Arcs φ and ε before and after the flip

Uniqueness of this solution for the orientations after the flip is obvious since changing orientation on anything other than all leaves simultaneously evidently changes certain of the numbers n_C^K. Moreover, changing the orientation on each leaf is a composition of the fatgraph reflections at the two interior vertices.

Finally, enumerating the eight cases in Figure 10 in the manner $\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{smallmatrix}$, one finds by inspection that cases 1, 3, 4, 8 are identical as are cases 2, 5, 6, 7 thus leading to the two cases illustrated in Figure 2. ∎
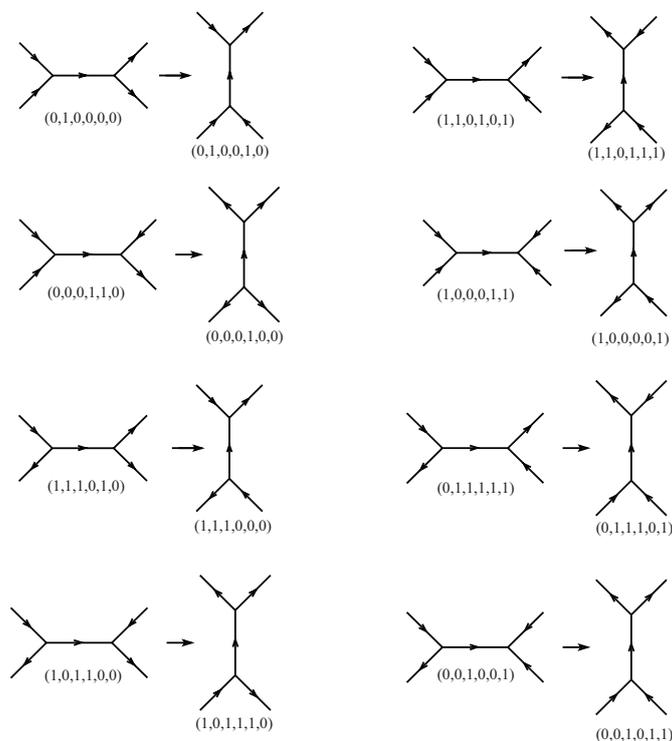
FIGURE 10. All possible salient cases of orientation before the flip together with resulting orientation after. The numbers n_C^K of Theorem 5.2 are tabulated as a vector $(n_\alpha^K, n_\beta^K, n_\gamma^K, n_\delta^K, n_\varepsilon^K, n_\varphi^K)$ in each case.

REFERENCES

[1] A. Beardon, The Geometry of Discrete Groups, Springer, 1983.

[2] F. A. Berezin, *Introduction to superanalysis*, Springer, 1987.

[3] F. Bouschbacher, *Shear coordinates on the super-Teichmüller space*, PhD thesis, Strasbourg, 2014, https://tel.archives-ouvertes.fr/tel-00835500v1/document

[4] F. Bouschbacher, V. Fock, F. Constantino, *Shear coordinates on the super-Teichmüller space*, to appear.

[5] P. Bryant, L. Hodgkin, *Nielsen's theorem and super-Teichmüllerspace*, Ann. Inst. H. Poincare Phys. Theor., **58** (3) (1993) 247-265.

[6] L. Chekhov, V. V. Fock, *Quantum Teichmüller space*, Theor.Math.Phys. **120** (1999) 1245-1259; Teor. Mat. Fiz. **120** (1999) 511-528, arXiv:math/9908165.

[7] D. Cimasoni, N. Reshetikhin, *Dimers on surface graphs and spin structures. I*, Comm. Math. Phys. **275** (2007), 187-208.

[8] D. Cimasoni, N. Reshetikhin, *Dimers on surface graphs and spin structures. II*, Comm. Math. Phys. **281** (2008), 445-468.

[9] L. Crane, J. Rabin, *Super Riemann surfaces: uniformization and Teichmüller theory*, Comm. Math. Phys. **113** (1998) 601-623.

[10] A. Fathi, F. Laudenbach, V. Poenaru, Travaux de Thurston sur les surfaces, *Asterisque* **66-67** (1979)

[11] L. Ford, Automorphic Functions, American Mathematical Society (1951).

[12] L. Hodgkin, *Super Teichmüller Spaces: Puctures and Elliptic Points*, Lett. Math. Phys. **15** (1988) 159-163.

[13] D. Johnson, *Spin structures and quadratic forms on surfaces*, J. London Math. Soc. **22** (1980) 365-373.

[14] R. Kashaev, *Quantization of Teichmüller spaces and the quantum dilogarithm*, Lett. Math. Phys. **43** (1998) 105-115.

[15] Yu. I. Manin, *Topics in noncommutative geometry*, Princeton University Press, 1991.

[16] J. Milnor, Remarks concerning spin manifolds in *Differential and Combinatorial Topology*, Princeton, 1965.

[17] S.M. Natanzon, *Moduli of Riemann surfaces, Hurwitz-type spaces, and their superanalogues*, Russian Math. Surveys **54** 1 (1999) 61-117.

[18] S.M. Natanzon, *Moduli of Riemann Surfaces, Real Algebraic Curves, and Their Superanalogs*, Transl. Math. Monographs **225**, American Mathematical Society, 2004.

[19] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys. **113** (1987), 299-339.

[20] —, *Weil-Petersson volumes*, J. Diff. Geom. **35** (1992), 559-608.

[21] —, *Decorated Teichmüller theory*, European Mathematical Society, 2012.

[22] K. Strebel, *Quadratic Differentials*, Ergebnisse der Mathematik und ihrer Grenzgebiete **5** Springer-Verlag, Berlin, 1984

[23] H. Verlinde, *Conformal field theory, two-dimensional quantum gravity and quantization of Teichmüller space*, Nucl. Phys. B **337** (1990) 652-680.

[24] E. Witten, *Notes On Super Riemann Surfaces And Their Moduli*, arXiv:1209.2459.

R. C. Penner,
Institut des Hautes Études Scientifiques
Le Bois-Marie 35, route de Chartres
91440 Bures-sur-Yvette, France
rpenner@ihes.fr

Departments of Mathematics and Physics
California Institute of Technology
Pasadena, CA 91125 USA
rpenner@caltech.edu

Anton M. Zeitlin,
Department of Mathematics,
Columbia University,
Room 509, MC 4406,
2990 Broadway,
New York, NY 10027,

IPME RAS, V.O. Bolshoj pr., 61, 199178,
St. Petersburg
zeitlin@math.columbia.edu,
http://math.columbia.edu/∼zeitlin
http://www.ipme.ru/zam.html