Periodic subvarieties of a projective variety under the action of a maximal rank abelian group of positive entropy

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PERIODIC SUBVARIETIES OF A PROJECTIVE VARIETY UNDER THE ACTION OF A MAXIMAL RANK ABELIAN GROUP OF POSITIVE ENTROPY

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ABSTRACT. We determine positive-dimensional G-periodic proper subvarieties of an n-dimensional projective variety X under the action of an abelian group G of maximal rank $n-1$ and of positive entropy. The motivation of the paper is to understand the obstruction for X to be G-equivariant birational to the quotient variety of an abelian variety modulo the action of a finite group.

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1. INTRODUCTION

We work over the field $\mathbb{C}$ of complex numbers. Let X be a normal projective variety of dimension $n \geq 2$. Denote by $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ the *Néron–Severi group*, i.e., the (finitely generated) abelian group of Cartier divisors modulo algebraic equivalence. The rank of its free part is called the *Picard number* of X. For a field $\mathbb{F} = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$, $\mathrm{NS}_{\mathbb{F}}(X)$ stands for $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{F}$. The first dynamical degree of an automorphism $g \in \mathrm{Aut}(X)$ is defined as the *spectral radius* of its natural pullback action g^* on $\mathrm{NS}_{\mathbb{C}}(X)$:

$$d_1(g) := \rho\big(g^*|_{\mathrm{NS}_{\mathbb{C}}(X)}\big) := \max\big\{|\lambda| : \lambda \text{ is an eigenvalue of } g^*|_{\mathrm{NS}_{\mathbb{C}}(X)}\big\}.$$

Such $g \in \mathrm{Aut}(X)$ is said to be of *positive entropy* (resp. *null-entropy*), if $d_1(g) > 1$ (resp. $d_1(g) = 1$). By the fundamental work of Gromov and Yomdin, the above definition is equivalent to the original definition in the dynamical system of holomorphisms on compact Kähler manifolds, and $d_1(g)$ of $g \in \mathrm{Aut}(X)$ depends only on the birational model of X. See [13, 22] and also [24, Lemmas 2.2 and 2.6], or Lemmas 2.1 and 2.8 below.

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Recall that a group H is *virtually solvable* (resp. *virtually free abelian*, ...), if a finite-index subgroup of H is solvable (resp. free abelian, ...).

Take a subgroup $G \le \mathrm{Aut}(X)$. Define the *null-entropy subset* of G as

$$N(G) := \big\{ g \in G : g \text{ is of null-entropy, i.e., } d_1(g) = 1 \big\}.$$

Such $G \le \mathrm{Aut}(X)$ is said to be of *positive entropy* (resp. *null-entropy*), if $N(G) = \{\mathrm{id}\}$ (resp. $N(G) = G$). It is known that either G contains a subgroup isomorphic to the non-abelian free group $\mathbb{Z} * \mathbb{Z}$, or G is virtually solvable. In the latter case or when $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable, there is a finite-index subgroup G_1 of G such that $N(G_1)$ is a normal subgroup of G_1 and $G_1/N(G_1)$ is a free abelian group of rank $r \le n-1$. We call this r the *dynamical rank* of G and denote it as $r = r(G)$, which is independent of the choice of the finite-index subgroup G_1 of G. See [5, 23] and references therein for details.

When the dynamical rank is maximal (i.e., $r = n-1$), Dinh–Sibony showed in [10] that the null-entropy subset $N(G)$ is a finite subgroup, assuming that G is commutative. In general, it is expected that $N(G)$ is finite except the case when X is an abelian variety. This has been confirmed recently in [9]. Note that there indeed exist examples of abelian varieties and their quasi-étale quotients admitting the action of commutative groups with maximal dynamical rank (cf. [10, Example 4.5] or [26, Example 1.7]).

The purpose of the paper is to understand the obstruction for X with the action of a maximal rank abelian group G of positive entropy, to be G-equivariant birational to a quasi-étale torus quotient. By virtue of [28] and [9], the remaining case we need to consider is the case when X is rationally connected. Our main results are Theorems 1.1, 1.2 (see their detailed versions: Theorems 4.1, 4.5, respectively) and Theorem 1.5 below.

By a *quasi-étale torus quotient*, we mean a quotient of an abelian variety T by a finite group F, which acts freely on T outside a codimension-2 subset of T. Note that such $T \to T/F$ is étale in codimension-1. A Zariski-closed subset Z of X is G*-periodic* if a finite-index subgroup of G set-theoretically stabilizes Z. A variety V of dimension d is *uniruled*, if there exists a dominant rational map $\mathbb{P}^1 \times W \dashrightarrow V$ for some variety W of dimension $d-1$. Note that being uniruled is a birational property.

Theorem 1.1. *Let X be a projective variety of dimension $n \ge 2$, and $G \le \mathrm{Aut}(X)$ such that the following conditions are satisfied.*

(i) *X has at worst $\mathbb{Q}$-factorial klt singularities.*
(ii) *$G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n-1$.*

Then after replacing G by a finite-index subgroup, the following assertions hold.

(1) *The union of all positive-dimensional G-periodic proper subvarieties of X is a Zariski-closed proper subset of X. Denote the irreducible decomposition of this union by $Z_1 \cup Z_2 \cup \cdots \cup Z_m$.*
(2) *Either Z_k is uniruled, or a finite-index subgroup of G fixes Z_k pointwise.*
(3) *If X has no G-periodic proper subvariety of positive dimension and $n \ge 3$, then X is equal to a quasi-étale torus quotient.*

(4) *The Picard number* $\rho(X) \geq n$. *If* $\rho(X) = n \geq 3$, *then* X *is* G*-equivariant birational to a quasi-étale torus quotient.*
(5) *Either* X *is* G*-equivariant birational to an abelian variety, or* X *has at most* $\rho(X) - n$ *of distinct* G*-periodic prime divisors.*

The assertions (1) and (3) of Theorem 1.1 follow from [28, Proposition 3.11] or Proposition 2.6, and [28, Theorem 2.4], respectively, with the help of [9, Theorem 4.1] or Proposition 2.2. We include them here for the convenience of the reader. Note that the condition (i) of Theorem 1.1 (or Theorem 4.1, Question 1.4) is not restrictive, since we can always take a G-equivariant resolution and even assume that X is smooth; its condition (ii) is birational in nature (see Proposition 2.2 and [28, Lemma 3.1]).

We remark that if the Picard number $\rho(X) > n^2$, then X is not equal to a quasi-étale torus quotient. Indeed, X is then not dominated by any abelian variety T via a generically finite surjective morphism. This is because the Picard number $\rho(T) \leq (\dim T)^2 = n^2$.

Theorem 1.2 below gives information about the pair (X, D), where D is a G-periodic non-uniruled prime divisor on X.

A variety X is *rationally connected* (resp. *rationally chain connected*) in the sense of Campana and Kollár–Miyaoka–Mori, if any two points on X are contained in an irreducible rational curve (resp. a chain of rational curves).

Theorem 1.2. *Let* X *be a normal projective variety of dimension* $n \geq 2$, *and* $G \leq \mathrm{Aut}(X)$ *such that the following conditions are satisfied.*

(i) $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ *is virtually solvable with maximal dynamical rank* $r(G) = n - 1$.
(ii) X *contains a* G*-periodic non-uniruled prime divisor* D.

Then after replacing G *by a finite-index subgroup, the following assertions hold.*

(1) X *is a rationally connected variety.*
(2) *Every* G*-periodic prime divisor, other than* D*, is uniruled.*
(3) *A finite-index subgroup of* G *fixes* D *pointwise.*

Furthermore, there is a G*-equivariant birational map* $X \dashrightarrow Y$, *which is isomorphic at the generic point of* D *with* $D_Y \subset Y$ *the strict transform of* D*, such that we have:*

(4) *Every positive-dimensional* G*-periodic proper subvariety of* Y *is contained in* D_Y.
(5) $K_Y + D_Y \sim_{\mathbb{Q}} 0$ *(*$\mathbb{Q}$*-linear equivalence); both* K_Y *and* D_Y *are* $\mathbb{Q}$*-Cartier; the pair* (Y, D_Y) *and hence* Y *both have at worst canonical singularities.*
(6) D_Y *has at worst canonical singularities and* $K_{D_Y} \sim_{\mathbb{Q}} 0$.

In dimension 2, Theorem 1.2 means that if X is a normal projective surface with an automorphism g of positive entropy and D is an irrational g-periodic curve, then X is a rational surface, D is an elliptic curve pointwise fixed by a power of g, and all other g-periodic curves are rational. See Lemma 3.7 and Remark 3.8 for an elementary treatment.

Remark 1.3. (1) In dimension 2, there is an example satisfying the conditions (i) and (ii) in Theorem 1.2. See [8, Theorem 2 or Example 3.3]. Indeed, in that example, X is a smooth rational surface and D is a smooth elliptic curve.

(2) Are there examples in higher dimensions satisfying the conditions (i) and (ii) in Theorem 1.2?

A positive answer to the question below roughly means that when $r(G) = n - 1$ is maximal, X is G-equivariant birational to a quasi-étale torus quotient if and only if X has no non-uniruled G-periodic prime divisor.

Question 1.4. Let X be a projective variety of dimension $n \geq 3$, and $G \leq \mathrm{Aut}(X)$ such that the following conditions are satisfied.

(i) X has at worst $\mathbb{Q}$-factorial klt singularities.
(ii) $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n - 1$.

Is it true that the following hold?

(1) Suppose that X does not have any G-periodic non-uniruled prime divisor. Then X is G-equivariant birational to a quasi-étale torus quotient.
(2) Suppose that X has a G-periodic non-uniruled prime divisor. Then X is not G-equivariant birational to a quasi-étale torus quotient.

The theorem below gives an affirmative answer to Question 1.4 (2), see also Proposition 4.4. The implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) below are proved in [28, Theorem 2.4]. We include them here for the convenience of the reader.

Theorem 1.5. *Let X be a projective variety of dimension $n \geq 3$, and $G \leq \mathrm{Aut}(X)$ such that $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n - 1$. Consider the following conditions:*

(1) *After replacing G by a finite-index subgroup, X is G-equivariant birational to a quasi-étale torus quotient X'.*
(2) *After replacing G by a finite-index subgroup, X is G-equivariant birational to a projective variety X' with only klt singularities, such that X' has no positive-dimensional G-periodic proper subvariety.*
(3) *After replacing G by a finite-index subgroup, X is G-equivariant birational to a projective variety X' with a G-periodic divisor D', such that (X', D') is $\mathbb{Q}$-factorial klt and $K_{X'} + D'$ is pseudo-effective.*
(4) *Every connected component of the union of positive-dimensional G-periodic proper subvarieties of X is rationally chain connected.*

Then the conditions (1), (2) *and* (3) *are equivalent, and imply the condition* (4).

The following proposition generalizes a well-known result on surface — there are only finitely many g-periodic curves if g is an automorphism of positive entropy on a projective surface. We prove a result of this type up to dimension 3 in the present paper. Naturally, we would like to know whether it is still true in higher dimensions.

Proposition 1.6. *Let X be a projective variety of dimension $n \le 3$, and $G \le \mathrm{Aut}(X)$ such that the following conditions are satisfied.*

(i) *X has at worst $\mathbb{Q}$-factorial klt singularities.*
(ii) *X is not birational to an abelian variety.*
(iii) *$G = \langle g_1, \ldots, g_{n-1} \rangle \simeq \mathbb{Z}^{\oplus n-1}$ is of positive entropy.*

Then for any non-trivial $g \in G$, there are at most $\rho(X) - n$ of g-periodic prime divisors.

2. PRELIMINARY RESULTS

Notation. We refer to [18] for the standard definitions, notations and terminologies in birational geometry. For instance, see [18, Definitions 2.34 and 2.37] for the definitions of *canonical* singularity, Kawamata log terminal singularity (*klt*), divisorial log terminal singularity (*dlt*), and log canonical singularity (*lc*).

Let X be a normal projective variety. X is called $\mathbb{Q}$-*factorial*, if every integral Weil divisor M on X is $\mathbb{Q}$-Cartier, i.e., sM is a Cartier divisor for some integer $s \ge 1$.

Let M be an $\mathbb{R}$-Cartier divisor (an $\mathbb{R}$-linear combination of Cartier integral divisors) on X. We call M is *nef*, if the intersection $M \cdot C \ge 0$ for every irreducible curve C on X. Denote by $\mathrm{Nef}(X)$ the closed cone of all nef $\mathbb{R}$-Cartier divisors on X. We call M is *pseudo-effective*, if it is contained in the closure of the cone of all effective $\mathbb{R}$-divisors on X.

For a birational map $f : X \dashrightarrow Y$, which is isomorphic at the generic point of a subvariety B, define the *strict transform* $B_Y \subset Y$ as the Zariski-closure of the image of $B \cap \mathrm{dom}(f)$ under the restriction $f|_{\mathrm{dom}(f)}$ of f to the domain $\mathrm{dom}(f)$ of the map f.

For an automorphism g of X, we use $g|_X$ to emphasize that g acts on X. For a g-invariant subspace V of some cohomology space $H^*(X, \mathbb{C})$, we use $g^*|_V$ to denote the natural pullback action of g^* on V. The *spectral radius* $\rho\big(g^*|_V\big)$ is the maximal absolute value of all eigenvalues of $g^*|_V$ as a linear transformation on V.

The result below shows that our notion of the first dynamical degree of an automorphism as in the introduction is equivalent to the same one on its equivariant resolution, and hence equivalent to the usual definition in the dynamical system (see Lemma 2.8).

Lemma 2.1 (cf. [24, Lemma 2.6] or [21, Lemma A.8])**.** *Let X and Y be two normal projective varieties of dimension $n \ge 2$, and $f : X \to Y$ a g-equivariant generically finite surjective morphism. Then we have $d_1(g|_X) = d_1(g|_Y)$. In particular, $g|_X$ is of positive entropy (resp. null-entropy) if and only if so is $g|_Y$.*

Proof. We follow the proof of [24, Lemma 2.6]. Let $W \to X \to Y$ be a g-equivariant resolution. By using the Lefschetz hyperplane theorem (on W), we reduce to the surface case. Then both $d_1(g|_X)$ and $d_1(g|_Y)$ are equal to $d_1(g|_W)$. □

Consider the following hypotheses. We note that the natural map $G|_{\mathrm{NS}_{\mathbb{R}}(X)} \to G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is an isomorphism, for the comparison with the same hypothesis in [28].

Hyp(A). Let X be a normal projective variety of dimension $n \geq 2$, and $G \leq \mathrm{Aut}(X)$ such that the representation $G^* := G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is isomorphic to $\mathbb{Z}^{\oplus n-1}$, and every element of $G^* \setminus \{\mathrm{id}\}$ is of positive entropy.

Hyp(A'). Let X be a normal projective variety of dimension $n \geq 2$, and $G \leq \mathrm{Aut}(X)$ such that $G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n - 1$.

Obviously, Hyp(A) implies Hyp(A'). The converse is also true up to finite-index, by the following proposition.

Proposition 2.2. *Suppose that* (X, G) *satisfies* Hyp(A')*. Then, replacing* G *by a finite-index subgroup, the null-entropy subset* $N(G)$ *of* G *is a (necessarily normal) subgroup of* G *and virtually contained in the identity connected component* $\mathrm{Aut}_0(X)$ *of* $\mathrm{Aut}(X)$*, i.e.,*

$$\big[N(G) : N(G) \cap \mathrm{Aut}_0(X)\big] < \infty.$$

In particular, the pair (X, G) *with* G *replaced by a finite-index subgroup, satisfies* Hyp(A).

Proof. Let $\pi : \widetilde{X} \to X$ be an $\mathrm{Aut}(X)$-equivariant resolution of X due to Hironaka. Replacing G by a finite-index subgroup, we may assume that $G|_{\mathrm{NS}_{\mathbb{C}}(\widetilde{X})}$ is solvable and has connected Zariski-closure in $\mathrm{GL}\big(\mathrm{NS}_{\mathbb{C}}(\widetilde{X})\big)$. On the other hand, for any $g \in G$, we have $d_1(g|_{\widetilde{X}}) = d_1(g|_X)$ by Lemma 2.1. Thus, if we identify $G|_{\widetilde{X}}$ with $G|_X$, via the natural map π, then

$$N(G)|_{\widetilde{X}} = N(G)|_X = N(G|_X) = N(G|_{\widetilde{X}}),$$

where the second equality holds by definition. By [9, Theorem 4.1 (1)], we know that $N(G)|_{\widetilde{X}}$ is virtually contained in $\mathrm{Aut}_0(\widetilde{X})$. Hence $N(G)|_X$ is virtually contained in $\mathrm{Aut}_0(X)$, since the $\mathrm{Aut}(X)$-equivariant birational morphism $\widetilde{X} \to X$ induces an isomorphism $\mathrm{Aut}_0(\widetilde{X}) \to \mathrm{Aut}_0(X)$. Therefore, $N(G)|_{\mathrm{NS}_{\mathbb{C}}(X)} = N(G)|_{\mathrm{NS}_{\mathbb{C}}(\widetilde{X})}$ is finite, since the continuous part $\mathrm{Aut}_0(\widetilde{X})$ acts trivially on the lattice $\mathrm{NS}(\widetilde{X})$ (modulo torsion), and hence acts trivially on $\mathrm{NS}_{\mathbb{C}}(\widetilde{X})$. Now as in [28, Lemma 3.1], replacing G by a finite-index subgroup, we have $G|_{\mathrm{NS}_{\mathbb{C}}(\widetilde{X})} \simeq G|_{\widetilde{X}}/N(G|_{\widetilde{X}}) \simeq \mathbb{Z}^{\oplus n-1}$, and also $G|_{\mathrm{NS}_{\mathbb{C}}(X)} \simeq \mathbb{Z}^{\oplus n-1}$. □

Let X be a normal projective variety of dimension $n \geq 2$, and $G \leq \mathrm{Aut}(X)$. Denote the union of all positive-dimensional G-periodic proper subvarieties of X by $\mathrm{Per}_+(X, G)$, i.e.,

$$\mathrm{Per}_+(X, G) := \bigcup_{Y \text{ is } G\text{-periodic}} Y,$$

where Y runs over all positive-dimensional G-periodic proper subvarieties of X.

The result below follows from the equivariance assumption.

Lemma 2.3. *Let* $f : X_1 \to X_2$ *be a* G*-equivariant generically finite surjective morphism. Then we have the following relation:*

$$\mathrm{Per}_+(X_1, G) = f^{-1}\big(\mathrm{Per}_+(X_2, G)\big),$$

where f^{-1} *denotes the set-theoretical inverse.* □

In the rest of this section, we prepare some results under Hyp(A). First note that if X is smooth, a *quasi-nef sequence* with $1 \le k \le n$

$$0 \neq L_1 \cdots L_k \in \overline{L_1 \cdots L_{k-1} \cdot \mathrm{Nef}(X)} \subseteq H^{k,k}(X, \mathbb{R})$$

was constructed in [23, §2.7]. Here as in [28, Lemma 3.4], we give a more general form for mildly singular variety X. Besides, we introduce a nef and big $\mathbb{R}$-Cartier divisor A, which plays an important role in running the Log Minimal Model Program (LMMP for short) with scaling (cf. [3, Corollary 1.4.2] or [2, Theorem 1.9 (i)]).

Lemma 2.4. *Suppose that (X, G) satisfies Hyp(A). Then there are nef $\mathbb{R}$-Cartier divisors L_i for $1 \le i \le n$ with $L_1 \cdots L_n \neq 0$, such that for any $g \in G$,*

$$g^* L_i = \exp \chi_i(g) L_i$$

for some characters $\chi_i : G \to (\mathbb{R}, +)$, and the group homomorphism

$$\varphi : G \to \big(\mathbb{R}^{\oplus n-1}, +\big), \quad g \mapsto \big(\chi_1(g), \ldots, \chi_{n-1}(g)\big)$$

has image a spanning (discrete) lattice of $\big(\mathbb{R}^{\oplus n-1}, +\big)$ and satisfies the following:

$$\mathrm{Ker}\, \varphi = N(G), \quad G^* \simeq G/N(G) \xrightarrow{\sim} \mathrm{Im}\, \varphi \simeq \mathbb{Z}^{\oplus n-1}. \tag{†}$$

In particular,

$$A := \sum_{i=1}^{n} L_i$$

is a nef and big $\mathbb{R}$-Cartier divisor.

Proof. Let $\pi : \widetilde{X} \to X$ be a G-equivariant resolution of X due to Hironaka. We follow the proof of [10, Theorem 4.3], and consider the action of G on the pullback $\pi^* \mathrm{Nef}(X)$ of the nef cone $\mathrm{Nef}(X) \subset \mathrm{NS}_{\mathbb{R}}(X)$ (instead of the Kähler cone $\mathcal{K}(\widetilde{X}) \subset H^{1,1}(\widetilde{X}, \mathbb{R})$ there). Then there are nef $\mathbb{R}$-Cartier divisors $\pi^* L_i$ with $1 \le i \le n$ on $\widetilde{X}$ as common eigenvectors of G acting on $\pi^* \mathrm{NS}_{\mathbb{R}}(X)$, i.e., $g^*(\pi^* L_i) = \exp \chi_i(g) \pi^* L_i$, such that $\chi_1 + \cdots + \chi_n = 0$ and the induced homomorphism φ satisfies (†). By taking a pushforward, these L_i satisfy $g^* L_i = \exp \chi_i(g) L_i$. For details, see [28, Lemma 3.4] or [27, proof of Theorems 1.2 and 2.2, p. 137].

Note that A is nef by its definition. Then it is big because

$$A^n = (L_1 + \cdots + L_n)^n \ge L_1 \cdots L_n > 0.$$

The latter inequality follows from [10, Lemma 4.4]. More precisely, that lemma implies that $L_1 \cdots L_n$ is nonzero and hence positive since these L_i are nef. □

For a nef $\mathbb{R}$-Cartier divisor L on a projective variety X, define the *null locus* of L as

$$\mathrm{Null}(L) := \bigcup_{L|_Z \text{ is not big}} Z,$$

where Z runs over all positive-dimensional proper subvarieties of X. Note that $L|_Z$ is nef, so it is not big if and only if $L^{\dim Z} \cdot Z = 0$.

Lemma 2.5 (cf. [28, Lemma 3.9]). *Suppose that (X, G) satisfies Hyp(A). Then*

$$\mathrm{Per}_+(X, G) = \mathrm{Null}(A),$$

and it is a Zariski-closed proper subset of X, where A is constructed in Lemma 2.4. In particular, A is ample if and only if every G-periodic proper subvariety of X is a point.

Below is the key proposition in [28] which was used to prove [28, Theorem 2.4]. Note that we do *not* need the pseudo-effectivity of $K_X + D$ or $\dim X \geq 3$.

Proposition 2.6 (cf. [28, Proposition 3.11]). *Suppose that (X, G) satisfies Hyp(A). Assume that for some effective $\mathbb{R}$-divisor D whose irreducible components are G-periodic, the pair (X, D) has at worst $\mathbb{Q}$-factorial klt singularities. Let $A = \sum L_i$ be the nef and big $\mathbb{R}$-Cartier divisor as in Lemma 2.4. Replacing G by a finite-index subgroup and A by a large multiple, the following are true.*

(1) *There is a sequence $\tau_s \circ \cdots \circ \tau_0$ of G-equivariant birational maps:*

$$X = X_0 \overset{\tau_0}{\dashrightarrow} X_1 \overset{\tau_1}{\dashrightarrow} \cdots \overset{\tau_{s-1}}{\dashrightarrow} X_s \xrightarrow{\tau_s} X_{s+1} = Y \qquad (\star)$$

such that each $\tau_j : X_j \dashrightarrow X_{j+1}$ for $0 \leq j < s$ is either a divisorial contraction of a $(K_{X_j} + D_j)$-negative extremal ray or a $(K_{X_j} + D_j)$-flip; the $\tau_s : X_s \to X_{s+1} = Y$ is a birational morphism such that $K_{X_s} + D_s = \tau_s^*(K_Y + D_Y)$ is $\mathbb{R}$-Cartier; here $D_i \subset X_i$ for $0 \leq i \leq s+1$ is the direct image of D and $D_Y := D_{s+1}$.

(2) *For $0 \leq i \leq s+1$, the direct image A_i of A on X_i is a nef and big $\mathbb{R}$-Cartier divisor.*

(3) *For $0 \leq i \leq s+1$, the pair $(X_i, D_i + A_i)$ and hence the pair (X_i, D_i) have at worst klt singularities; X_j is $\mathbb{Q}$-factorial for $0 \leq j \leq s$.*

(4) *$K_Y + D_Y + A_Y$ is an ample $\mathbb{R}$-Cartier divisor, where $A_Y := A_{s+1}$.*

(5) *For $0 \leq i \leq s+1$, the union of all positive-dimensional G-periodic proper subvarieties of each X_i is a Zariski-closed proper subset of X_i. Further, $A_i|_Z \equiv 0$ (numerical equivalence) for every positive-dimensional G-periodic proper subvariety Z of X_i.*

(6) *For $0 \leq i \leq s+1$, the induced action of G on each X_i is biregular. Further, each (X_i, G) also satisfies Hyp(A).*

Note that if (X, D) is only a dlt pair, one has the following proposition (but need $K_X + D$ to be pseudo-effective). The main idea is to apply Proposition 2.6 to the klt pair $\big(X, (1-\varepsilon)D\big)$ for some $0 < \varepsilon \ll 1$.

Proposition 2.7 (cf. [28, Proposition 2.6]). *Suppose that (X, G) satisfies Hyp(A). Suppose further that for some effective $\mathbb{R}$-divisor D whose irreducible components are G-periodic, the pair (X, D) has at worst $\mathbb{Q}$-factorial dlt singularities, and $K_X + D$ is a pseudo-effective divisor. Then there is a birational map $X \dashrightarrow Y$ such that:*

(1) *Y is a normal projective variety. The map $X \dashrightarrow Y$ is surjective in codimension-1. Replacing G by a finite-index subgroup, the induced action of G on Y is biregular.*

(2) *The pair (Y, D_Y) has only log canonical singularities and $K_Y + D_Y \sim_{\mathbb{Q}} 0$, where D_Y is the direct image of D.*

(3) *Every G-periodic positive-dimensional proper subvariety of Y is contained in the support of D_Y.*

For a Kähler manifold X, the *first dynamical degree* $d_1(g)$ of a surjective endomorphism g of X can be equivalently defined as the spectral radius of the pullback action g^* on $H^{1,1}(X,\mathbb{R})$ (cf. [21, §A.2]).

Lemma 2.8.

(1) *Let X be a compact Kähler manifold of dimension n, and g a surjective endomorphism of X. Let V be a g-invariant subspace of $H^{1,1}(X,\mathbb{R})$ containing a Kähler current B. Then $d_1(g)$ equals the spectral radius $\rho\big(g^*|_V\big)$.*

(2) *Suppose that X is a smooth projective variety and g is a surjective endomorphism of X. Then $\rho\big(g^*|_{H^{1,1}(X,\mathbb{R})}\big) = \rho\big(g^*|_{\mathrm{NS}_{\mathbb{R}}(X)}\big)$. So the two definitions (preceding this lemma or in the introduction) of the first dynamical degree for endomorphisms or automorphisms coincide for smooth projective varieties.*

Proof. (1) It suffices to show that $d_1(g) \le \rho\big(g^*|_V\big)$. Let $\mathcal{P}$ be the closed cone in $H^{1,1}(X,\mathbb{R})$ consisting of classes of positive closed $(1,1)$-currents, and $\mathcal{C} := \mathcal{P} \cap V$. Note that $\mathcal{P}$ is a strictly convex cone preserved by the pullback action g^*, so is $\mathcal{C}$. Replacing V by the subspace spanned by $\mathcal{C}$, we may assume that $V = \mathcal{C} + (-\mathcal{C})$. Take an interior point $B_1 \in \mathcal{C}$. Then $B' := B_1 + \varepsilon B$ is still contained in the interior of $\mathcal{C}$ (also in the interior of $\mathcal{P}$) for sufficiently small $\varepsilon > 0$. Fix a Kähler class ω of X. We can define a linear form $\chi : H^{1,1}(X,\mathbb{R}) \to \mathbb{R}$ by $\chi(\xi) = \int_X \xi \smile \omega^{n-1}$. Note that for a non-trivial class T in $\mathcal{P}$, one has $\chi(T) > 0$ (cf. [21, Lemmas A.3 and A.4]). So by applying [21, Proposition A.2] to the triplets $\big(H^{1,1}(X,\mathbb{R}), \mathcal{P}, B'\big)$ and $(V, \mathcal{C}, B')$, we obtain the following

$$d_1(g) = \lim_{m\to\infty} \chi\big((g^m)^* B'\big)^{\frac{1}{m}} = \rho\big(g^*|_V\big).$$

Note that in the proof above we have replaced V by a subspace, so we actually prove that $d_1(g) \le \rho\big(g^*|_V\big)$. This proves the assertion (1).

(2) In this case, $\mathrm{NS}_{\mathbb{R}}(X)$ is a g-invariant subspace of $H^{1,1}(X,\mathbb{R})$ containing an ample divisor, whose first Chern class induces a Kähler class. So the assertion (2) follows from the first one. This proves Lemma 2.8. □

Under Hyp(A), the rank of the Néron–Severi group has the following lower bound (see also [10, Theorem 4.3]).

Lemma 2.9. *Suppose that (X, G) satisfies Hyp(A). Then we have:*

(1) *The Picard number $\rho(X) \ge n$.*

(2) *Assume the existence of an $\mathbb{R}$-Cartier non-trivial divisor M such that $g^*M \equiv M$ for any $g \in G$. Then $\rho(X) \ge n+1$.*

(3) *If $\rho(X) = n$ and K_X is $\mathbb{Q}$-Cartier, then $K_X \equiv 0$.*

Proof. (1) We use the notations as in Lemma 2.4. We first claim that L_i for $1 \le i \le n-1$ are linearly independent in $\mathrm{NS}_{\mathbb{R}}(X)$. Indeed, suppose that $\sum_{i=1}^{n-1} a_i L_i = 0$ for some real numbers

a_i. Acting on this equality by g^p for an element $g \in G$ of positive entropy, we have

$$0 = \sum_{i=1}^{n-1} a_i (g^p)^* L_i = \sum_{i=1}^{n-1} a_i \exp \chi_i(g^p) L_i = \sum_{i=1}^{n-1} a_i \exp \big(p\chi_i(g) \big) L_i.$$

Then there are two characters χ_{j_1} and χ_{j_2} (depending on g) such that $\chi_{j_1}(g) = \chi_{j_2}(g)$ (using Vandermonde determinant). Therefore, the spanning lattice $\varphi(G)$ of $\big(\mathbb{R}^{\oplus n-1}, +\big)$ is contained in a finite union of hyperplanes. This is a contradiction (cf. Lemma 2.4). Thus the claim holds.

Next we only need to show that L_n is not a linear combination of those L_i with $i < n$. This can be seen by the construction of such L_n (cf. [10, proof of Theorem 4.3]). In fact, there is an $f \in G$ of positive entropy such that the coordinates of $\varphi(f)$ in $\mathbb{R}^{\oplus n-1}$ are all strictly negative, and hence $f^* L_n = d_1(f) L_n$ (cf. Lemma 2.8). Suppose that $L_n = \sum_{i=1}^{n-1} b_i L_i$ for some real numbers b_i. Let f act on both sides. Then we have

$$f^* L_n = \sum_{i=1}^{n-1} b_i f^* L_i = \sum_{i=1}^{n-1} b_i \exp \chi_i(f) L_i.$$

On the other hand, $f^* L_n = d_1(f) L_n$. Hence we have

$$d_1(f) \sum_{i=1}^{n-1} b_i L_i = \sum_{i=1}^{n-1} b_i \exp \chi_i(f) L_i,$$

which implies that $\big(d_1(f) - \exp \chi_i(f)\big) b_i = 0$ for any i. It follows that all b_i vanish, since $d_1(f) > 1 > \exp \chi_i(f)$. Hence $L_1, \dots, L_n$ are linearly independent, so $\rho(X) \geq n$.

(2) By the assertion (1), it suffices to show that the numerical equivalence class $[M]$ ($\neq 0$) is not a linear combination of the classes of $L_1, \dots, L_n$ in $\mathrm{NS}_{\mathbb{R}}(X)$. Suppose to the contrary that $M \equiv \sum_{k=1}^{n} c_k L_k$ for some real numbers c_k. Letting the f as in the assertion (1) act on both sides, we have

$$\sum_{k=1}^{n} c_k L_k \equiv M \equiv f^* M \equiv \sum_{k=1}^{n} c_k f^* L_k = \sum_{k=1}^{n-1} c_k \exp \chi_k(f) L_k + c_n d_1(f) L_n,$$

which implies that $c_k = 0$ for all k, because $d_1(f) > 1 > \exp \chi_k(f)$. Hence $M \equiv 0$, which is a contradiction.

(3) It follows from the assertion (2) by taking $M = K_X$. □

Proposition 2.10. *Suppose that (X, G) satisfies Hyp(A) and X has at worst $\mathbb{Q}$-factorial klt singularities. Suppose further that the irregularity $q(X) = 0$. Let $B_1, \dots, B_s$ be distinct G-periodic prime divisors on X. Then $B_1, \dots, B_s$ are linearly independent in $\mathrm{NS}_{\mathbb{Q}}(X)$ with $s \leq \rho(X) - n$. In particular, $\rho(X) \geq n$, and the equality holds true only when $K_X \equiv 0$.*

Proof. Replacing G by a finite-index subgroup, we may assume that all of B_i have been stabilized by G. Suppose to the contrary that these B_i are linearly dependent in $\mathrm{NS}_{\mathbb{Q}}(X)$. Then we have $\sum_{i=i}^{s} a_i B_i \equiv 0$ in $\mathrm{NS}_{\mathbb{Q}}(X)$ for some integers a_i, not all zero. After rearranging the order of B_i, we may assume that $E_1 := \sum_{i=1}^{s_1} a_i B_i \equiv \sum_{j=s_1+1}^{s_2} b_j B_j =: E_2$, where $a_i, b_j = -a_j$

are positive integers. Since $q(X) = 0$ by assumption, we have $E_1 \sim E_2$ (linear equivalence) after replacing E_i by some multiples. Hence the Iitaka D-dimension $\kappa := \kappa(X, E_1) \geq 1$.

Replacing E_1 by some mE_1, we may assume that the map $\Phi_{|E_1|} : X \dashrightarrow \mathbb{P}H^0\big(X, \mathcal{O}_X(E_1)\big)$ gives rise to the Iitaka fibration associated to E_1, so that its image has dimension equal to κ. Take a G-equivariant resolution $\pi : \widetilde{X} \to X$ such that the linear system $|\pi^*E_1|$ equals $|M| + F$, where M is base point free, F is the fixed component of $|\pi^*E_1|$, and both of their divisor classes are G-stable. Now the rational map $\Phi_{|E_1|} : X \dashrightarrow \mathbb{P}H^0\big(X, \mathcal{O}_X(E_1)\big)$ is birational to the G-equivariant morphism $\Phi_{|M|} : \widetilde{X} \to Y \subset \mathbb{P}H^0\big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(M)\big)$ with $\dim Y = \kappa$.

If $\kappa = n$, then M is a nef and big divisor. So by [24, Lemma 2.23], G is virtually contained in $\mathrm{Aut}_0(\widetilde{X})$ and hence is of null-entropy on $\widetilde{X}$, and also on X (cf. Lemma 2.1). This contradicts that the dynamical rank $r(G) = n - 1 \geq 1$. Thus we have $1 \leq \kappa \leq n - 1$. In other words, $\Phi_{|M|}$ is a non-trivial G-equivariant fibration with general fibres of dimension $n - \kappa \in \{1, \ldots, n-1\}$. Then by [23, Lemma 2.10], the dynamical rank $r(G) \leq n - 2$, which contradicts Hyp(A). So we have proved the linearly independence of these B_i.

We continue using the notations as in Lemmas 2.4 and 2.9. By the argument similar to the proof of Lemma 2.9, we can show that $L_1, \ldots, L_n, B_1, \ldots, B_s$ are linearly independent in $\mathrm{NS}_{\mathbb{Q}}(X)$. Thus we have $n + s \leq \rho(X)$. This ends the proof of Proposition 2.10. ☐

The following lemma generalizes a fact, which asserts that every effective divisor on an abelian variety is indeed nef.

Lemma 2.11. *Suppose that $\pi : T \to X$ is a finite surjective morphism between normal projective varieties. Suppose further that T satisfies one of the following conditions.*

(i) *T has at worst klt singularities and contains no rational curve; $K_T \sim_{\mathbb{Q}} 0$.*
(ii) *T is an abelian variety.*

Then we have:

(1) *Every pseudo-effective $\mathbb{R}$-Cartier divisor on X is nef.*
(2) *Every big $\mathbb{R}$-Cartier divisor on X is ample.*

Proof. Since π is finite and by the projection formula, an $\mathbb{R}$-Cartier divisor D on X is pseudo-effective, big, nef or ample if and only if so is π^*D. Thus we only need to prove this lemma for $X = T$. Further, we may assume that T satisfies the condition (i), since the condition (ii) implies the condition (i). By the Kodaira lemma, which states that every big $\mathbb{R}$-divisor is the sum of an ample $\mathbb{Q}$-divisor and an effective $\mathbb{R}$-divisor (cf. [20, Lemma 3.16]), it suffices to prove the assertion (1). Since the cone of all pseudo-effective $\mathbb{R}$-Cartier divisors on T is the closure of the cone of all effective $\mathbb{R}$-Cartier divisors on T in $\mathrm{NS}_{\mathbb{R}}(T)$ and the nef cone $\mathrm{Nef}(T)$ is closed, we only need to show that every effective $\mathbb{R}$-Cartier divisor on T is nef. For this it suffices to show that every effective Cartier divisor on T is nef. Suppose to the contrary that some effective Cartier divisor D on T is not nef. By [18, Corollary 2.35], $(T, \varepsilon D)$ is klt for all sufficiently small rational number $\varepsilon > 0$. Now $K_T + \varepsilon D \sim_{\mathbb{Q}} \varepsilon D$ is not nef. Therefore, applying the Cone

Theorem in MMP to $(T, \varepsilon D)$ (cf. [18, Theorem 3.7]), we obtain an extremal rational curve on T, which contradicts the condition (i). This proves Lemma 2.11. □

The following result proves the implication (1) $\Longrightarrow$ (2) in Theorem 1.5.

Lemma 2.12. *Let X be a quasi-étale torus quotient T/F for some abelian variety T and a finite group F acting freely outside a codimension-2 subset of T, and $G \le \mathrm{Aut}(X)$ such that (X, G) satisfies Hyp(A). Then X has no positive-dimensional G-periodic proper subvariety.*

Proof. Let $\widetilde{T} \to X$ be the Galois covering (or minimal split covering in the Beauville's sense; see [1, §3]) corresponding to the unique maximal lattice L in $\pi_1(X \setminus \mathrm{Sing}\, X)$ such that $\widetilde{T}$ is an abelian variety. Then there exists a group $\widetilde{G}$ (which is the lifting of G) acting faithfully on $\widetilde{T}$, such that $G = \widetilde{G}/F$. See also [27, §2.15]. Note that the action of G on X can be identified with a not necessarily faithful action of $\widetilde{G}$ on X (with finite kernel). Replacing $\widetilde{G}$ by a finite-index subgroup, we may assume that the new $\widetilde{G}$ acts faithfully on both $\widetilde{T}$ and X (cf. [27, Lemma 2.4]), and both $(\widetilde{T}, \widetilde{G})$ and $(X, \widetilde{G})$ satisfy Hyp(A) (cf. [28, Lemma 3.1]). By Lemma 2.11, the nef and big $\mathbb{R}$-Cartier divisor $\widetilde{A}$ on $\widetilde{T}$ as constructed in Lemma 2.4, is ample. Hence every $\widetilde{G}$-periodic proper subvariety of $\widetilde{T}$ is a point (see Lemma 2.5). The same holds for X by Lemma 2.3. □

3. Some general results from birational geometry

In this section, we prepare some general results which will be used in the section 4 to prove the main theorems. They should be of interest in their own right.

We first quote the following result, which will be frequently used in the sequel of the paper.

Lemma 3.1 (cf. [15, Corollary 1.5])**.** *Let (X, Δ) be a dlt pair for some effective $\mathbb{Q}$-divisor Δ and $\phi : W \to X$ a birational projective morphism. Denote by $\mathrm{Exc}\, \phi$ the exceptional locus of ϕ, i.e., the subset of W along which ϕ is not an isomorphism. Then we have:*

1. *Every fibre of ϕ is rationally chain connected.*
2. *Every connected component of $\mathrm{Exc}\, \phi$ is rationally chain connected.*
3. *Every irreducible component of $\mathrm{Exc}\, \phi$ is uniruled. In particular, if D is a non-uniruled prime divisor on W, then the image of D on X is still a divisor.*

Below is an easy fact, but we give the proof for the convenience of the reader.

Lemma 3.2. *Let X be a normal projective variety and D a Weil $\mathbb{Q}$-divisor. If D is $\mathbb{R}$-Cartier, then it is $\mathbb{Q}$-Cartier.*

Proof. Since D is $\mathbb{R}$-Cartier, we may write $D = \sum_{i=1}^n r_i D_i$ for some $r_i \in \mathbb{R}$ and some Cartier integral divisors D_i. On the other hand, since D is a Weil $\mathbb{Q}$-divisor, $D = \sum_{j=1}^m b_j P_j$ for some $b_j \in \mathbb{Q}$ and some prime Weil divisors P_j. Write $D_i = \sum_{j=1}^m a_{ij} P_j$, where $a_{ij} \in \mathbb{Z}$. So we have

$$D = \sum_{i=1}^n r_i \sum_{j=1}^m a_{ij} P_j = \sum_{j=1}^m \sum_{i=1}^n r_i a_{ij} P_j.$$

Hence

$$b_j = \sum_{i=1}^{n} r_i a_{ij},\ 1 \le j \le m,$$

i.e.,

$$\mathbf{A} \cdot \mathbf{r} = \mathbf{b},$$

where $\mathbf{A} = (a_{ij})^\intercal$ is an integral $m \times n$ matrix, $\mathbf{r} = (r_1, \dots, r_n)^\intercal \in \mathbb{R}^n$ and $\mathbf{b} = (b_1, \dots, b_m)^\intercal \in \mathbb{Q}^m$. In other words, $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$ has one real solution $\mathbf{x} = \mathbf{r}$. So it has at least one rational solution $(q_1, \dots, q_n)^\intercal \in \mathbb{Q}^n$, since both $\mathbf{A}$ and $\mathbf{b}$ are rational. Now $D = \sum_{i=1}^n q_i D_i$. Thus D is $\mathbb{Q}$-Cartier. □

It is well known that the birational automorphism group of a projective variety of general type is finite. Below is a similar result.

Lemma 3.3. *Let X be a non-uniruled normal projective variety, and $G \le \mathrm{Aut}(X)$ such that the linear equivalence class of an ample divisor H is G-periodic. Then G is finite.*

Proof. Replacing H by a large multiple, we may assume that H is very ample and the linear system $|H|$ defines a closed embedding into some projective space $\mathbb{P}H^0\big(X, \mathcal{O}_X(H)\big) \simeq \mathbb{P}^N$. Identify X with its image. Replacing G by a finite-index subgroup, we may assume that G itself stabilizes the linear equivalence class of H. Thus the above embedding is G-equivariant. So G is contained in $\mathrm{Aut}(\mathbb{P}^N, X)$, the Zariski-closed subgroup of $\mathrm{Aut}(\mathbb{P}^N)$ stabilizing X. Suppose to the contrary that G is not finite. Then the linear algebraic group $\mathrm{Aut}(\mathbb{P}^N, X)$ contains a 1-dimensional linear algebraic group G_a or G_m. Thus the orbit of a general point is a rational curve. But our X is non-uniruled. This is a contradiction. Hence G is finite. □

We give a criterion for $K_X + D$ to be pseudo-effective. See [19, Theorem 1.4 or 3.7] for a more general form.

Lemma 3.4. *Let X be a rationally connected normal projective variety, and D a non-uniruled prime divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Then $K_X + D$ is pseudo-effective.*

Proof. Take a log resolution $\widetilde{X} \to X$ for the pair (X, D), and denote by $\widetilde{D}$ the proper transform of D. Note that the pushforward of a pseudo-effective divisor is still pseudo-effective. Hence we may replace the pair (X, D) by $(\widetilde{X}, \widetilde{D})$, and assume that it is $\mathbb{Q}$-factorial dlt now.

Suppose to the contrary that $K_X + D$ is not pseudo-effective. We follow the proof of [19, Theorem 3.7]. After running a $(K_X + D)$-MMP with an ample scaling, we reach a Fano fibration $h : W \to Y$ as follows (cf. [3, Corollary 1.3.3])

$$\begin{array}{ccccccccccc} X = X_0 & \overset{f_0}{\dashrightarrow} & X_1 & \overset{f_1}{\dashrightarrow} & \cdots & \overset{f_{m-2}}{\dashrightarrow} & X_{m-1} & \overset{f_{m-1}}{\dashrightarrow} & X_m =: W \ . \\ & & & & & & & & \big\downarrow h \\ & & & & & & & & Y \end{array}$$

Note that each f_i above is either a divisorial contraction of a $(K_{X_i} + D_i)$-negative extremal ray or a $(K_{X_i} + D_i)$-flip, where $D_i \subset X_i$ is the direct image of D. So (X_i, D_i) is still $\mathbb{Q}$-factorial and dlt (cf. [18, Corollary 3.44]). Thus the direct (birational) image D_W on W of D is still a non-uniruled prime divisor because so is D (see Lemma 3.1). The argument in [19, proof of Theorem 3.7] says that then $h : W \to Y$ is a $\mathbb{P}^1$-fibration with D_W a cross-section. Hence D_W is birational to Y via the restriction map $h|_{D_W}$. Since X is rationally connected, so are each X_i and the h-image Y of W. Thus D_W is rationally connected and hence uniruled. This is a contradiction. So the lemma is proved. □

The following two lemmas are sufficient conditions to have canonical singularities.

Lemma 3.5 (cf. [16, Lemma 2.4])**.** *Let X be a non-uniruled normal projective variety of dimension n, and D an effective Weil $\mathbb{R}$-divisor such that $K_X + D$ is $\mathbb{R}$-Cartier and $K_X + D \equiv 0$ (numerical equivalence). Then $D = 0$ and X has at worst canonical singularities.*

Proof. We follow the proof of [16, Lemma 2.4]. Take a log resolution $\pi : \widetilde{X} \to X$ for the pair (X, D), and denote the proper transform of D by $\widetilde{D}$. Note that $\widetilde{X}$ is also non-uniruled. Then by [4, Theorem 2.6], $K_{\widetilde{X}}$ is pseudo-effective, and hence admits a Zariski σ-decomposition $K_{\widetilde{X}} = P + N$, where the $\mathbb{R}$-divisors P and N are the movable part and the negative part, respectively (cf. [20, Ch. III, §1.b]). On the other hand, we have

$$K_{\widetilde{X}} + \widetilde{D} = \pi^*(K_X + D) + E_1 - E_2 \equiv E_1 - E_2, \tag{‡}$$

where E_1 and E_2 are effective π-exceptional divisors and have no common component. Thus it follows that

$$E_1 \equiv P + N + \widetilde{D} + E_2.$$

Since $E_1 - (N + \widetilde{D} + E_2)$ is numerically equivalent to the movable divisor P, we have $N + \widetilde{D} + E_2$ larger than or equal to the negative part of the Zariski σ-decomposition of E_1, while the latter is just E_1 itself (cf. [20, Ch. III, Proposition 1.14]). Namely, $N + \widetilde{D} + E_2 \geq E_1$. Take a general ample divisor H on $\widetilde{X}$. Then

$$0 = H^{n-1} \cdot (P + N + \widetilde{D} + E_2 - E_1) = H^{n-1} \cdot P + H^{n-1} \cdot (N + \widetilde{D} + E_2 - E_1) \geq 0.$$

Hence $0 = H^{n-1} \cdot P = H^{n-1} \cdot (N + \widetilde{D} + E_2 - E_1)$. Moreover, since $N + \widetilde{D} + E_2 - E_1$ is effective, by the Nakai–Moishezon criterion for ampleness, it is zero, i.e., $E_1 = N + \widetilde{D} + E_2$. Now since E_1 and E_2 have no common component, we have $E_2 = 0$. Hence $E_1 = N + \widetilde{D}$. Since $\widetilde{D}$ is the π-proper transform of D and E_1 is π-exceptional, $\widetilde{D} = 0$. Thus $D = 0$ too.

The second part follows from the equality (‡) (with $E_2 = 0$ now) by definition. □

Lemma 3.6. *Let X be a rationally connected normal projective variety, and D a non-uniruled prime divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier and $K_X + D \equiv 0$. Then (X, D) has only canonical (and hence dlt) singularities. In particular, the prime divisor D itself as a variety is normal.*

Proof. Take a log resolution $\pi : \widetilde{X} \to X$ for the pair (X, D), and denote the proper transform of D by $\widetilde{D}$. Note that $\widetilde{X}$ is still rationally connected and $\widetilde{D}$ is non-uniruled. So $K_{\widetilde{X}} + \widetilde{D}$ is pseudo-effective by Lemma 3.4, and hence admits a Zariski σ-decomposition $K_{\widetilde{X}} + \widetilde{D} = P + N$ as in Lemma 3.5. On the other hand, we have

$$K_{\widetilde{X}} + \widetilde{D} = \pi^*(K_X + D) + E_1 - E_2 \equiv E_1 - E_2,$$

where E_1 and E_2 are effective π-exceptional divisors and have no common component. Then it follows that $E_1 \equiv P + N + E_2$. Now the same argument as in Lemma 3.5 implies that $E_2 = 0$, and hence (X, D) has only canonical singularities by definition. The final assertion comes from [18, Proposition 5.51 or Corollary 5.52]. □

When X is a surface, we have a more specific description of X and a periodic curve C.

Lemma 3.7. *Let X be a normal projective surface with an automorphism g of positive entropy, and C a g-periodic curve. Then either X is a rational surface, or C is a rational curve.*

Proof. Replacing X by a g-equivariant resolution, we may assume that X is smooth. Since X admits an automorphism of positive entropy, by [6, Proposition 1], either X is a rational surface, or it has Kodaira dimension $\kappa(X) = 0$.

Thus we have only to consider (and rule out) the case where $\kappa(X) = 0$ and C is irrational. Let $X \to X^m$ be the smooth blowdown to the (unique smooth) minimal model of X. Note that the strict transform C^m of C is still a curve by Lemma 3.1, and g descends to an automorphism on X^m. So we may replace (X, C) by (X^m, C^m), and assume that X is minimal. Hence $K_X \sim_{\mathbb{Q}} 0$. More precisely, X is either a $K3$ surface, or an Enriques surface, or an abelian surface (cf. [6, Proposition 1]).

Replacing g by some power, we may assume that g stabilizes the curve C. The generalized Perron–Frobenius theorem due to Birkhoff says that $(g^{\pm 1})^* L_{g^{\pm 1}} = d_1(g^{\pm 1}) L_{g^{\pm 1}}$ for some nonzero nef divisors $L_{g^{\pm 1}}$. Then $A := L_g + L_{g^{-1}}$ is nef and also big since $A^2 \geq L_g \cdot L_{g^{-1}} > 0$. It is perpendicular to C because $d_1(g^{\pm 1}) > 1$. Indeed,

$$L_{g^{\pm 1}} \cdot C = (g^{\pm 1})^*(L_{g^{\pm 1}} \cdot C) = (g^{\pm 1})^* L_{g^{\pm 1}} \cdot (g^{\pm 1})^* C = d_1(g^{\pm 1}) L_{g^{\pm 1}} \cdot C. \qquad (*)$$

It follows that $L_{g^{\pm 1}} \cdot C = 0$, and hence $A \cdot C = 0$. Thus $C^2 < 0$, since $A^2 > 0$ and by the Hodge index theorem.

On the other hand, by the arithmetic genus formula, we have

$$0 > C^2 = (K_X + C) \cdot C = 2p_a(C) - 2 \geq 0,$$

since C is irrational. This is a contradiction. Lemma 3.7 is proved. □

Remark 3.8. Suppose X is a smooth projective rational surface with an automorphism g of positive entropy. Then $K_X^2 < 0$. Indeed, since $g^* K_X \sim K_X$, we have $A \cdot K_X = 0$ as calculated in the (*) of the lemma above with C replaced by K_X. Hence either $K_X \equiv 0$, or $K_X^2 < 0$. Since X is a smooth rational surface, K_X is not numerically trivial, so $K_X^2 < 0$.

If C is a g-periodic curve on X, then the arithmetic genus $p_a(C) \leq 1$. Otherwise, the Riemann–Roch theorem and the Serre duality imply that

$$h^0\big(X, \mathcal{O}_X(K_X + C)\big) \geq \chi(\mathcal{O}_X) + \frac{1}{2} C \cdot (K_X + C) = p_a(C) \geq 2.$$

So the nef part of the Zariski decomposition of $K_X + C$ is nonzero and g-invariant, contradicting $d_1(g) > 1$.

We end the section with the following rigidity result for the proof of Proposition 1.6. It follows from [18, Lemma 1.6] and [7, Proposition 1.14 or Lemma 1.15].

Lemma 3.9 (Rigidity Lemma). *Let $f : X \to Y$ be a projective surjective morphism of normal varieties. Suppose that all fibres of f are connected and of the same dimension. Let $f' : X \to Y'$ be another projective morphism of varieties such that f' contracts one fibre $f^{-1}(y_0)$ of f for some $y_0 \in Y$. Then there is a unique morphism $\pi : Y \to Y'$ such that $f' = \pi \circ f$.*

4. Generalizations of Theorems 1.1 and 1.2, and their proofs

Theorem 1.1 will follow from the more general form below. For a projective variety V, we take a resolution $\widetilde{V} \to V$ and define the *albanese map*

$$\mathrm{alb}_V : V \dashrightarrow \mathrm{Alb}(V) := \mathrm{Alb}(\widetilde{V})$$

as the natural composition $V \dashrightarrow \widetilde{V} \xrightarrow{\mathrm{alb}_{\widetilde{V}}} \mathrm{Alb}(\widetilde{V})$. It is known that alb_V is a well-defined morphism when V has at worst rational singularities.

Theorem 4.1. *Let X be a projective variety of dimension $n \geq 2$, and $G \leq \mathrm{Aut}(X)$ such that the following conditions are satisfied.*

(i) *X has at worst $\mathbb{Q}$-factorial klt singularities.*
(ii) *$G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n - 1$.*

Then after replacing G by a finite-index subgroup, the following assertions hold.

(1) *The union $\mathrm{Per}_+(X, G)$ of all positive-dimensional G-periodic proper subvarieties of X is a Zariski-closed proper subset of X.*
(2) *Let $\mathrm{Per}_+(X, G) = Z_1 \cup Z_2 \cup \cdots \cup Z_m$ be the irreducible decomposition. Then either Z_k is uniruled, or a finite-index subgroup of G fixes Z_k pointwise.*
(3) *If X has no G-periodic proper subvariety of positive dimension and $n \geq 3$, then X is equal to a quasi-étale torus quotient T/F for some abelian variety T and a finite group F whose action on T is free outside a finite subset of T, such that the action of G lifts to an action of a group $\widetilde{G}$ on T with $\widetilde{G}/\mathrm{Gal}(T/X) \simeq G$. Moreover, every G- (resp. $\widetilde{G}$-) periodic proper subvariety of X (resp. T) is a point.*
(4) *The Picard number $\rho(X) \geq n$. If $\rho(X) = n \geq 3$, then X is G-equivariant birational to a quasi-étale torus quotient.*
(5) *Either the albanese map $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is an $\mathrm{Aut}(X)$- (and hence G-) equivariant surjective birational morphism, or X has at most $\rho(X) - n$ of distinct G-periodic prime divisors.*

Before proving Theorem 4.1 we prepare the following two lemmas.

Lemma 4.2. *Suppose that we have the sequence* ($\star$) *of G-equivariant birational maps as in Proposition 2.6. Then we have the following relations among the* $\mathrm{Per}_+(X_i, G)$.

(1) *For a divisorial contraction τ_i with $0 \le i < s$ and for the birational morphism τ_i with $i = s$, we have*

$$\mathrm{Per}_+(X_i, G) = \tau_i^{-1}\big(\mathrm{Per}_+(X_{i+1}, G)\big).$$

Moreover, the exceptional locus $\mathrm{Exc}\,\tau_i$ *is an irreducible divisor and uniruled.*

(2) *If τ_i is a flip for some $0 \le i < s$:*

$$\begin{array}{ccc} X_i & \overset{\tau_i}{\dashrightarrow} & X_{i+1} = X_i^+ \\ & {}_{f}\searrow \quad \swarrow_{f^+} & \\ & V_i & \end{array},$$

then there is a Zariski-closed subset $\Delta_i \subset V_i$ such that

$$\mathrm{Exc}(f) = f^{-1}(\Delta_i) \text{ and } \mathrm{Exc}(f^+) = (f^+)^{-1}(\Delta_i).$$

Further,

$$\mathrm{Per}_+(X_i, G) = f^{-1}\big(\mathrm{Per}_+(V_i, G)\big) \text{ and } \mathrm{Per}_+(X_{i+1}, G) = (f^+)^{-1}\big(\mathrm{Per}_+(V_i, G)\big).$$

Every irreducible component of the flipping locus $\mathrm{Exc}\,f$ *or the flipped locus* $\mathrm{Exc}\,f^+$ *is uniruled.*

Proof. (1) The first assertion follows directly from Lemma 2.3. For the second one, we know that every (X_i, A_i) is klt (so is dlt) by Proposition 2.6 (3). Then it follows from Lemma 3.1 (3) that $\mathrm{Exc}\,\tau_i$, known to be an irreducible divisor, is uniruled.

(2) The first assertion follows from the uniqueness of the flip (cf. [18, Lemma 6.2 and Corollary 6.7]). Now the second assertion follows, using also the G-equivariance of the morphisms f and f^+ and Lemma 2.3.

Hence we still have to prove the last assertion. We assume that f is a contraction of $(K_{X_i}+A_i)$-negative extremal ray $\mathbb{R}_{\ge 0}[\ell]$. Choose a suitable ample divisor H such that

$$(K_{X_i} + A_i + \varepsilon H) \cdot \ell = 0 \text{ and } (X_i, A_i + \varepsilon H) \text{ is still klt}$$

for some $0 < \varepsilon \ll 1$. By the Cone Theorem in MMP (cf. [18] or [12, Theorem 1.1]), there is an $\mathbb{R}$-Cartier divisor Θ_i on V_i such that

$$K_{X_i} + A_i + \varepsilon H = f^*\Theta_i.$$

By the projection formula, $\Theta_i = K_{V_i} + f_*A_i + \varepsilon f_*H$. Then $(V_i, f_*A_i + \varepsilon f_*H)$ is a klt pair. So Lemma 3.1 (3) implies the last assertion. We have proved Lemma 4.2. □

The following lemma tells us the relationship among the irreducible components of these $\mathrm{Per}_+(X_i, G)$. We will also use this lemma to prove Theorem 4.5 (2) later.

Lemma 4.3. *Under the assumption of Lemma 4.2, for any $0 \le i \le s$, every non-uniruled irreducible component of* $\mathrm{Per}_+(X_i, G)$ *is* G*-equivariant birational to some irreducible component of* $\mathrm{Per}_+(X_{i+1}, G)$ *by* τ_i*, which is then isomorphic at the generic point of that irreducible component.*

Proof. We use the same notation as in Lemma 4.2. Let Z^i be any non-unruled irreducible component of $\mathrm{Per}_+(X_i, G)$.

If τ_i is a divisorial contraction for some $0 \le i < s$ or τ_s, by Lemma 4.2 (1) above, Z^i is not contained in the exceptional locus of τ_i. Hence Z^i is G-equivariant birational to its strict transform in X_{i+1}, and the latter is also an irreducible component of $\mathrm{Per}_+(X_{i+1}, G)$.

If τ_i is a flip for some $0 \le i < s$, by Lemma 4.2 (2), Z^i is not contained in the exceptional locus of $f : X_i \to V_i$. Hence Z^i is G-equivariant birational to its strict transform in V_i, and the latter one is G-equivariant birational to its strict transform in X_{i+1} via the map $f^+ : X_{i+1} \to V_i$. This last one in X_{i+1} is also the strict transform of Z^i via the birational map $X_i \dashrightarrow X_{i+1}$, and hence an irreducible component of $\mathrm{Per}_+(X_{i+1}, G)$. In the above argument, we use the fact that both exceptional loci of $f : X_i \to V_i$ and $f^+ : X_{i+1} \to V_i$ lie over the same Zariski-closed subset $\Delta_i \subset V_i$. $\square$

Proof of Theorem 4.1. By Proposition 2.2, replacing G by a finite-index subgroup, we may assume that (X, G) satisfies Hyp(A). So we can apply Proposition 2.6 by choosing $D = 0$. Then our assertion (1) is just Proposition 2.6 (5). The assertion (3) follows from [28, Theorem 2.4] (under the condition (iii) there).

Proof of Assertion (2). We are going to prove this assertion by the backward induction on the index i of X_i. We will use the sequence ($\star$) of G-equivariant birational maps as in Proposition 2.6 with $D = 0$, and recall that for $0 \le i \le s + 1$, let A_i (an $\mathbb{R}$-Cartier divisor) denote the direct image of A on X_i, respectively. By Proposition 2.6 (5), we know that $A_i|_Z \equiv 0$ for every positive-dimensional G-periodic proper subvariety Z of X_i. Replacing G by a finite-index subgroup, we may assume that G stabilizes every irreducible component of $\mathrm{Per}_+(X_i, G)$.

Let Z be an irreducible component of $\mathrm{Per}_+(Y, G)$. By Proposition 2.6 (4) (with $D = 0$ always in the current theorem), we know that $K_Y + A_Y$ is an ample $\mathbb{R}$-Cartier divisor on Y. Then $K_Y|_Z$ is also an ample $\mathbb{R}$-Cartier divisor on Z since $A_Y|_Z \equiv 0$. Assume further that Z is non-uniruled. Then by Lemma 3.3 applied to $H := K_Y|_Z$, we know that $G|_Z$ is finite. Hence a finite-index subgroup of G fixes Z pointwise. So the assertion (2) holds true on Y.

By induction we assume that for any irreducible component Z^{i+1} of $\mathrm{Per}_+(X_{i+1}, G)$, either Z^{i+1} is uniruled, or a finite-index subgroup of G fixes Z^{i+1} pointwise. Now we choose any irreducible component Z^i of $\mathrm{Per}_+(X_i, G)$. Assume further that this Z^i is non-uniruled. Then by Lemma 4.3, Z^i is G-equivariant birational to its strict transform in X_{i+1} by τ_i, and the latter is also an irreducible component of $\mathrm{Per}_+(X_{i+1}, G)$. By the inductive hypothesis, a finite-index subgroup of G fixes that latter strict transform of Z^i, and then it also fixes Z^i pointwise. This proves the assertion (2).

Proof of Assertion (4). The first part of the assertion (4) has been proved by Lemma 2.9. If $\rho(X) = n$, the same lemma also tells us that K_X is numerically trivial. Then K_X is pesedo-effective. Thus the second part follows from [28, Theorem 2.4] (under the condition (ii) there).

Proof of Assertion (5). We first assume that the irregularity $q(X) > 0$. Take an $\mathrm{Aut}(X)$-equivariant resolution $\pi : \widetilde{X} \to X$. Then $q(\widetilde{X}) = q(X) > 0$ because X has only klt and hence rational singularities (cf. [18, Theorem 5.22]). By [23, Lemma 2.13], $\mathrm{alb}_{\widetilde{X}}$ is a (necessarily $\mathrm{Aut}(\widetilde{X})$-equivariant) surjective birational morphism. Hence the same holds for alb_X because X has only rational singularities. Next we assume that $q(X) = 0$. Suppose that X has s of G-periodic prime divisors $B_1, \ldots, B_s$. Then the upper bound of s has been given by Proposition 2.10. This proves the assertion (5).

We have completed the proof of Theorem 4.1. □

The proposition below gives an affirmative answer to Question 1.4 (2).

Proposition 4.4. *Suppose that (X, G) satisfies Hyp(A). Suppose further that X is G-equivariant birational to a quasi-étale torus quotient. Then we have:*

(1) *Every connected component Z_k of $\mathrm{Per}_+(X, G)$ (i.e., the union of all positive-dimensional G-periodic proper subvarieties of X) is rationally chain connected.*
(2) *Every irreducible component of $\mathrm{Per}_+(X, G)$ is uniruled. In particular, Question 1.4 (2) has a positive answer.*

Proof. Since the assertion (2) follows directly from the first one, we prove only the assertion (1). Suppose that X is G-equivariant birational to a quasi-étale torus quotient $Y := T/F$ for some abelian variety T and a finite group F (note that Y is klt). Since the image of a rationally chain connected Zariski-closed set is still rationally chain connected, we may replace $X \dashrightarrow Y$ by a G-equivariant resolution of indeterminacy and assume that $X \to Y$ is already a G-equivariant birational morphism (see Lemma 2.3 and [28, Lemma 3.1]). Note that the image of Z_k on Y is G-periodic and hence a point P by Lemma 2.12. By Zariski's main theorem, the inverse image on X of the point P on the normal variety Y is connected. This inverse of P is also G-periodic and contains Z_k, so it equals Z_k, since Z_k is a connected component of $\mathrm{Per}_+(X, G)$. Then by Lemma 3.1, Z_k is rationally chain connected. □

The following is a generalization of Theorem 1.2.

Theorem 4.5. *Let X be a projective variety of dimension $n \geq 2$, and $G \leq \mathrm{Aut}(X)$ such that the following conditions are satisfied.*

(i) *$G|_{\mathrm{NS}_{\mathbb{C}}(X)}$ is virtually solvable with maximal dynamical rank $r(G) = n - 1$.*
(ii) *X contains a G-periodic non-uniruled prime divisor D such that (X, D) has at worst $\mathbb{Q}$-factorial dlt singularities.*

Then after replacing G by a finite-index subgroup, we have:

(1) *X is rationally connected, and the Picard number $\rho(X) \geq n + 1$.*

(2) *Let $Z_1 \cup Z_2 \cup \cdots \cup Z_m$ be the irreducible decomposition of the union of all positive-dimensional G-periodic proper subvarieties of X, with $Z_1 = D$. Then for $k \geq 2$, Z_k is uniruled. In particular, every G-periodic prime divisor, other than D, is uniruled.*
(3) *A finite-index subgroup of G fixes D pointwise.*

Furthermore, there exists a surjective in codimension-1 G-equivariant birational map $X \dashrightarrow Y$ with D_Y the strict transform of D, such that the following are true.

(4) *Every positive-dimensional G-periodic proper subvariety of Y is contained in D_Y. In particular, the positive-dimensional part of $\operatorname{Sing} Y$ is contained in D_Y.*
(5) *$K_Y + D_Y \sim_{\mathbb{Q}} 0$; both K_Y and D_Y are $\mathbb{Q}$-Cartier; the pair (Y, D_Y) and hence Y both have at worst canonical singularities.*
(6) *D_Y has at worst canonical singularities and $K_{D_Y} \sim_{\mathbb{Q}} 0$.*
(7) *$-mD_Y|_{D_Y}$ is an ample Cartier divisor on D_Y for some integer $m > 0$.*

Remark. The order of our proof of this theorem is (1), (3), (4), (5), (6), (7) and (2).

Proof. The assumption implies that X is also klt, so the conditions of Theorem 4.1 are satisfied and we may apply Theorem 4.1. Replacing G by a finite-index subgroup, we may assume that D is stabilized by G and (X, G) satisfies Hyp(A) (see Proposition 2.2). By the affirmative answer to Question 1.4 (2) (see Proposition 4.4), X is not G-equivariant birational to a quasi-étale torus quotient.

Proof of Assertion (1). We first show that X is rationally connected. The surface case has been dealt with by Lemma 3.7, so we only consider the case $n \geq 3$. Suppose to the contrary that X is not rationally connected. Replacing G by a finite-index subgroup, X is G-equivariant birational to a quasi-étale torus quotient (cf. [28, Theorem 2.4]). This contradicts Proposition 4.4.

The second part follows from Proposition 2.10 with $B_1 := D$. The assertion (1) is proved.

The assertion (3) is a direct consequence of Theorem 4.1 (2), since D is a G-periodic non-uniruled prime divisor.

Proof of Assertion (4). By the assertion (1) above, we can apply Lemma 3.4 and say that $K_X + D$ is pseudo-effective. This in turn allows us to apply Proposition 2.7 to the dlt pair (X, D). Note that the G-equivariant birational map $X \dashrightarrow Y$ is originally constructed in Proposition 2.6 for the pair $(X, (1-\varepsilon)D)$ with $\varepsilon > 0$ sufficiently small. Then the assertion (4) comes directly from Proposition 2.7 (3).

Proof of Assertion (5). We first prepare the following for the proof of this assertion. Note that (X, D) is dlt, then (X, D_ε) is klt, where $D_\varepsilon := (1-\varepsilon)D$ for some $0 < \varepsilon \ll 1$ (cf. [18, Proposition 2.41]). So we can apply Proposition 2.6 to the klt pair (X, D_ε). Then there is a sequence $\tau_s \circ \cdots \circ \tau_0$ of G-equivariant birational maps:

$$X = X_0 \overset{\tau_0}{\dashrightarrow} X_1 \overset{\tau_1}{\dashrightarrow} \cdots \overset{\tau_{s-1}}{\dashrightarrow} X_s \xrightarrow{\tau_s} X_{s+1} = Y \qquad (\star\star)$$

such that each $\tau_j : X_j \dashrightarrow X_{j+1}$ for $0 \leq j < s$ is either a divisorial contraction of a $(K_{X_j} + D_{\varepsilon,j})$-negative extremal ray or a $(K_{X_j} + D_{\varepsilon,j})$-flip; the $\tau_s : X_s \to X_{s+1} = Y$ is a birational morphism such that

$$K_{X_s} + D_{\varepsilon,s} = \tau_s^*(K_Y + D_{\varepsilon,Y});$$

here $D_{\varepsilon,i} \subset X_i$ for $0 \le i \le s+1$ denotes the direct image of D_ε. It follows from [18, Corollaries 3.42 and 3.43] that each $(X_i, D_{\varepsilon,i})$ for $0 \le i \le s$ is klt. So $(Y, D_{\varepsilon,Y})$ is also klt. In particular, by Lemma 3.1, each $D_{\varepsilon,i}$ for $0 \le i \le s+1$ is indeed a divisor since D is non-uniruled.

Now the first part of the assertion (5), i.e., $K_Y + D_Y \sim_{\mathbb{Q}} 0$, follows from Proposition 2.7 (2).

By the first part we have proved and Proposition 2.6 (4), we know that

$$-\varepsilon D_Y + A_Y \sim_{\mathbb{Q}} K_Y + (1-\varepsilon)D_Y + A_Y$$

is an ample $\mathbb{R}$-Cartier divisor. Note also that A_Y is $\mathbb{R}$-Cartier by Proposition 2.6 (2), and then so is D_Y. Hence by Lemma 3.2, D_Y is $\mathbb{Q}$-Cartier, and then so is K_Y.

Note that Y is rationally connected (since so is X) and D_Y is a non-uniruled divisor. Hence $K_Y + D_Y \sim_{\mathbb{Q}} 0$ implies that (Y, D_Y) has only canonical singularities (and D_Y is a normal variety) by Lemma 3.6, so does Y (cf. [18, Corollary 2.35]). This proves the assertion (5).

Proof of Assertion (6). By the adjunction theorem for dlt pair (cf. [11, Proposition 3.9.2] or [17, §16 and §17]), there exists an effective divisor $\mathrm{Diff}_{D_Y}(0)$ on D_Y such that

$$K_{D_Y} + \mathrm{Diff}_{D_Y}(0) = (K_Y + D_Y)|_{D_Y} \sim_{\mathbb{Q}} 0.$$

Note that D_Y itself (as a variety) is non-uniruled and normal. Then by applying Lemma 3.5 to the pair $\big(D_Y, \mathrm{Diff}_{D_Y}(0)\big)$, we have $\mathrm{Diff}_{D_Y}(0) = 0$ and D_Y has at worst canonical singularities. Thus $K_{D_Y} \sim_{\mathbb{Q}} 0$. This proves the assertion (6).

Proof of Assertion (7). By the assertion (4) we have proved, every positive-dimensional G-periodic subvariety of Y is contained in D_Y, so $\mathrm{Per}_+(Y,G) = D_Y$. In particular, by Proposition 2.6 (5), we have $A_Y|_{D_Y} \equiv 0$. We already see in the proof of the assertion (5) that $-\varepsilon D_Y + A_Y$ is an ample $\mathbb{R}$-Cartier divisor, and then so is $(-\varepsilon D_Y + A_Y)|_{D_Y} \equiv -\varepsilon D_Y|_{D_Y}$. Note that by the assertion (5), D_Y is $\mathbb{Q}$-Cartier. The assertion (7) follows.

Proof of Assertion (2). Suppose to the contrary that some Z_k with $k \ge 2$ is non-uniruled. Note that in our proof of the assertion (5), we applied Proposition 2.6 to the klt pair (X, D_ε) and produced a sequence ($\star\star$) of G-equivariant birational maps. So by Lemma 4.3, such Z_k is G-equivariant birational to some irreducible component of $\mathrm{Per}_+(Y,G)$ by $\tau_s \circ \cdots \circ \tau_0$, which is isomorphic at the generic point of Z_k. On the other hand, the assertion (4) says that $\mathrm{Per}_+(Y,G) = D_Y$ has only one irreducible component. So such Z_k is birational to D_Y. By the irreducibility of Z_k we know that Z_k coincides with $D = Z_1$, which is a contradiction. This ends the proof of the assertion (2).

We have completed the proof of Theorem 4.5. □

Remark 4.6. With the assumption and notation in Theorem 4.5, we have:

(1) Note that the positive-dimensional part of $\operatorname{Sing} Y$ is contained in D_Y by Theorem 4.5 (4). So we have

$$\dim(\operatorname{Sing} Y) \le \max\{0, \dim Y - 3\}.$$

Indeed, by Theorem 4.5 (5), (Y, D_Y) is a canonical pair. After $(\dim Y - 2)$-times hyperplane cutting as in [18, Corollary 5.18], we reach a canonical surface pair (S, D_S) (cf. [18, Lemma 5.17 (1)]). So by [18, Theorem 4.5], $D_S \cap \operatorname{Sing} S = \varnothing$, and hence Y is smooth at its codimension-2 points lying inside D_Y. This shows that $\dim(D_Y \cap \operatorname{Sing} Y) \le \max\{0, \dim Y - 3\}$.

(2) Suppose $\dim Y = 2$. Then Y is smooth in a neighbourhood of D_Y, and D_Y is a (smooth) elliptic curve, since D_Y is normal and $K_{D_Y} \sim_{\mathbb{Q}} 0$.

(3) Suppose $\dim Y = 3$. Then Y has at worst isolated singularities. Further, $K_{D_Y} \sim_{\mathbb{Q}} 0$ implies that D_Y is either a smooth abelian surface or hyperelliptic surface, or a normal $K3$ surface or Enriques surface with at worst Du Val singularities.

Proof of Theorem 1.2. Take a G-equivariant log resolution $\pi : \widetilde{X} \to X$ for the pair (X, D), and denote by $\widetilde{D}$ the proper transform of D. Note that $\widetilde{D}$ is still a G-periodic non-uniruled prime divisor. Replacing G by a finite-index subgroup, $(\widetilde{X}, G)$ satisfies the conditions (i) and (ii) of Theorem 4.5 (see Proposition 2.2 and [28, Lemma 3.1]). Thus the assertions (1) $\sim$ (6) in Theorem 4.5 holds for $\widetilde{X}$. This implies the corresponding assertions in Theorem 1.2, except the assertion (2). Suppose that X has a G-periodic prime divisor D_2 different from D. Then the π-proper transform $\widetilde{D}_2$ of D_2 is an irreducible component of $\operatorname{Per}_+(\widetilde{X}, G)$ different from $\widetilde{D}$, so it is uniruled by Theorem 4.5 (2). Hence D_2 is uniruled. This proves Theorem 1.2. □

Proof of Theorem 1.5. (1) $\Longrightarrow$ (2) is proved by Lemma 2.12.

(2) $\Longrightarrow$ (1) comes from [28, Theorem 2.4] or Theorem 4.1 (3).

(1) $\Longrightarrow$ (3) is true by letting $D' = 0$, and note that quotient singularities are $\mathbb{Q}$-factorial klt, and $K_{X'} \sim_{\mathbb{Q}} 0$.

(3) $\Longrightarrow$ (1) follows from [28, Theorem 2.4] (under the condition (ii) there).

(1) $\Longrightarrow$ (4) is just our Proposition 4.4 (1). □

Proof of Proposition 1.6. We may assume that the irregularity $q(X) = 0$ by the condition (ii) and [23, Theorem 1.2]. This also holds for any resolution of X because X has only klt and hence rational singularities (cf. [18, Theorem 5.22]). The surface case is well known. Actually, it follows from the Hodge index theorem and the fact that every g-periodic curve is perpendicular to the nef and big divisor $A := L_g + L_{g^{-1}}$ as in the proof of Lemma 3.7, where $L_{g^{\pm 1}}$ are the nef divisors corresponding to the first dynamical degree $d_1(g^{\pm 1})$ of $g^{\pm 1}$. So we still have to consider the case $n = 3$.

We only need to prove the claim that there are only finitely many g-periodic prime divisors D_j with $1 \le j \le k$ for some $k > 0$. Assuming this claim for the time being, for any $1 \le i \le n-1$, it follows from the commutativity of G that each $g_i(D_j)$ is also g-periodic. Therefore, for any j, we know that D_j is g_i-periodic for any i and hence G-periodic. Then by Proposition 2.10, $k \le \rho(X) - n$.

Suppose to the contrary that the above claim does not hold. Namely, there are infinitely many distinct g-periodic prime divisors D_j with $j \geq 1$. Let

$$\kappa := \kappa\big(X, \sum_{j=1}^{r} D_j\big) = \max\left\{\kappa\big(X, \sum_{j=1}^{t} D_j\big) : D_j \text{ is } g\text{-periodic}, t \geq 1\right\}$$

for some $r \geq 1$ and denote $E_0 := \sum_{j=1}^{r} D_j$. Replacing g by its power, we may assume that $g(D_j) = D_j$ for all $j \leq r$. As reasoned in Proposition 2.10 we have $\kappa \geq 1$.

For any $1 \leq i \leq n-1$, let $E_i := g_i^* E_0$. It is easy to see that E_i is also g-periodic since g commutes with each g_i, and hence $\kappa(X, E_i) = \kappa(X, E_0 + E_i) = \kappa$ by the maximality of κ. Replacing E_0 by some mE_0, we may assume that the dominant rational map

$$\Phi_{|E_i|} : X \dashrightarrow \Phi_{|E_i|}(X) \subseteq \mathbb{P}H^0\big(X, \mathcal{O}_X(E_i)\big)$$

is an Iitaka fibration associated to E_i and its image has dimension equal to κ for any $0 \leq i \leq n-1$. Take a g-equivariant resolution $\pi : X' \to X$ of $\operatorname{Sing} X$ and $\operatorname{Bs}(|E_i|)$, such that the linear system $|\pi^* E_i| = |M_i| + F_i$, where each M_i is base point free, F_i is the fixed component of $|\pi^* E_i|$, and their divisor classes are g-stable. Now the morphism $\Phi_{|M_i|}$ is birational to $\Phi_{|E_i|}$. Let $Y_i \to \Phi_{|M_i|}(X')$ be the normalization, and

$$\phi_i : X' \to Y_i$$

the induced morphism, which is an algebraic fibre space with connected fibres. Denote by A_i the ample divisor on Y_i such that $M_i = \phi_i^* A_i$. We have $\kappa(X', M_0 + M_i) = \kappa(X, E_0 + E_i) = \kappa$ by the maximality of κ. Thus the free divisor $M_0 + M_i$ is the pullback of some ample divisor on a variety of dimension κ, which implies that $(M_0 + M_i)^{\kappa+1} = 0$. In particular, $M_0^{\kappa} \cdot M_i = 0 = M_0 \cdot M_i^{\kappa}$.

We assert that $\kappa \leq n - 2 = 1$. Indeed, by blowing up Y_i and X' further, we may assume that Y_i is also smooth. Replacing ϕ_i by the new morphism, the new A_i on the new Y_i is only nef and big. Nevertheless, we obtain a g-equivariant fibration $\phi_i : X' \to Y_i$ of smooth varieties such that g preserves the nef and big divisor A_i on Y_i. It follows from [25, Lemma 2.5] that $\kappa \leq n - 2 = 1$, thus $\kappa = 1$ in the present case. (Remark: in what follows, the blowing up of Y_i is unnecessary, since Y_i is a normal and hence a smooth curve. In particular, the divisor A_i is still ample, and ϕ_i is flat and hence equidimensional; see [14, Proposition 9.7]. Indeed, the argument below works as long as ϕ_i is equidimensional.)

For $1 \leq i \leq n-1$, let C be any curve in a general fibre F_i of ϕ_i. Take general ample divisors H_j on X' containing C with $1 \leq j < n - \kappa$. Let $S := H_1 \cap \cdots \cap H_{n-\kappa-1}$. Then

$$0 \leq C \cdot M_0 = C \cdot M_0|_S \leq M_i^{\kappa}|_S \cdot M_0|_S = M_i^{\kappa} \cdot M_0 \cdot H_1 \cdots H_{n-\kappa-1} = 0.$$

Thus $A_0 \cdot (\phi_0)_* C = 0$ by the projection formula. So ϕ_0 contracts C (and hence the whole F_i) by the ampleness of A_0. Then by the Rigidity Lemma 3.9, $\phi_0 = t_i \circ \phi_i$ for some morphism $t_i : Y_i \to Y_0$. Interchanging the role of M_0 with M_i, we get another morphism $s_i : Y_0 \to Y_i$ such that $\phi_i = s_i \circ \phi_0$. Hence $\phi_i = s_i \circ t_i \circ \phi_i$. The surjectivity of ϕ_i then implies that $s_i \circ t_i = \mathrm{id}$. Similarly, $t_i \circ s_i = \mathrm{id}$. Thus s_i and t_i are isomorphisms and inverse to each other by the normality of Y_i. Therefore, we can write $M_i = \phi_i^* A_i = \phi_0^* B_i$ with $B_i := s_i^* A_i$ an ample divisor on Y_0.

Now the automorphism g_i on X descends to an isomorphism between the bases of the Iitaka fibrations $\Phi_{|E_0|}$ and $\Phi_{|E_i|}$, while the latter two are birational to $\Phi_{|M_0|}$ and $\Phi_{|M_i|}$, respectively. So g_i induces an isomorphism from (the normalization of) $\Phi_{|A_0|}(Y_0)$ to (the normalization of) $\Phi_{|B_i|}(Y_0)$, which is an automorphism of Y_0 now. Thus G acts on Y_0 bi-regularly. Replacing $X \dashrightarrow Y_0$ by a G-equivariant resolution X'' of the graph, we have a non-trivial G-equivariant fibration between two smooth projective varieties. Contradicts the maximal dynamical rank assumption on G (cf. [23, Lemma 2.10]). This ends the proof of Proposition 1.6. □

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