On the proof of the homology conjecture for monomial non-unital algebras

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Abstract

We consider the bar complex of a monomial non-unital associative algebra $A = k\langle X\rangle/(w_1, ..., w_t)$. For any fixed monomial $w = x_1..x_n \in A$ one can define certain subcomplex of the Bar complex of A. It was conjectured in [3] that homology of this complex is at most one. We prove here this conjecture, and describe the place where this nontrivial homology appears in terms of length of the Dyck path associated to a given monomial in $w \in A$.

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1 Homologies of the subcomplex of the bar complex, defined by a monomial, are at most one-dimensional

Let A be a monomial algebra without a unit. We fix a presentation of A by generators and monomial relations $w_1, ..., w_t$: $A = k\langle X\rangle/(w_1, ..., w_t)$, and suppose that monomials $w_1, ..., w_t$ have the property that no monomial is a submonomial of another one, and this set of monomials does not contain any generator from the set X. For introduction to monomial algebras, their structural and homological properties on can refer to [1]. Note that algebras we consider here are non-unital, which makes it possible for Bar complex to have nontrivial homologies.

Consider the following subcomplex B_w of the bar complex, associated to a **monomial** $w = x_1..x_n$.

$$0 \longrightarrow B_k = \{x_1 \otimes ... \otimes x_n\} \xrightarrow{D} B_{k-1} = \{x_1 \otimes ... \otimes x_i x_{i+1} \otimes ... \otimes x_n\} \xrightarrow{D} ... \xrightarrow{D} B_1 = \{x_1 ... x_n\} \longrightarrow 0$$

Clearly, $B_i \subset A^{\otimes j}$ and these subcomplexes of a bar complex, for all words w in the monomial algebra A form a spectral sequence. Consider $B = \oplus B_i$ as a graded linear space. We have a linear map $D : B \longrightarrow B$, satisfying $D^2 = 0$.

First we will prove quite surprising conjecture, constituting one of two puzzles in [3], which states that homologies of the defined above subcomplex of a bar complex are at most one-dimensional.

The next step would be to find the place in the complex, where nonzero homology appears, and express the result in combinatorial terms related to the monomial algebra data, namely to the **length** of a **Dyck path** defined by the word in a **monomial algebra.**

Theorem 1.1. *The full homology of the defined above subcomplex B_w of the bar complex of the non-unital monomial algebra A is at most one-dimensional:*

$$\dim H_\bullet(B_w) = \dim \mathtt{ker}\, D/\mathtt{im}\, D \in \{0,1\}$$

Proof. (induction by n)

Basis: if the word w is empty, the complex is zero and $H_w = 0$. If $w = x$ is a letter, the complex is $0 \longrightarrow k \longrightarrow 0$, and $H_w = k$ is one-dimensional. If $w = xy$ then in case $xy \neq 0$ in A, the complex is $0 \longrightarrow (x \otimes y)k \longrightarrow xyk \longrightarrow 0$. It is exact, $H_w = 0$. In case $xy = 0$, the complex is $0 \longrightarrow (x \otimes y)k \longrightarrow 0$, and $H_w = k$.

We need to prove: $\dim \mathtt{ker}\, D/\mathtt{im}\, D \in \{0,1\}$.

Since $\dim \mathtt{im}\, D = \dim B - \dim \mathtt{ker}\, D$, it is the same as

$$\dim \mathtt{ker}\, D - \dim \mathtt{im}\, D =$$

$$2\dim \mathtt{ker}\, D - \dim B \in \{0,1\}$$

Thus, $\dim H_\bullet(B_w, D) = 2\dim \mathtt{ker}\, D - \dim B$.

Let us split the vector space B as $B = E \oplus F$, where $E = \{x_1 \otimes ...\}_k$ and $F = \{x_1...\}_k$ are subspases spanned by those monomials where first letter is followed directly by a tensor symbol, and by those where the tensor appearing later (or does not appear at all). Consider the linear map $J : E \longrightarrow F$, defined by $x_1 \otimes u \mapsto x_1 u$.

The kernel of this map is a linear span of monomials of the type $x_1 \otimes x_2...x_s...$, where $x_1x_2...x_s$ is a shortest beginning subword of w, which is zero in A, i.e. $x_1x_2...x_s = 0$, but $x_1x_2...x_{s-1} \neq 0$:

$$\mathtt{ker}\, J = \{x_1 \otimes x_2...x_s...\}_k$$

Denote by $L = \{x_2...x_s\}_k$, where $x_2...x_s...$ as above, so that $\mathtt{ker}\, J = x_1 \otimes L$.

First we calculate the images of differential D on elements from E and F.

Namely:

$$D(x_1 \otimes u) = x_1 u - x_1 \otimes du = J(x_1 \otimes u) - x_1 \otimes du$$

Here d is just the same differential as D, but acting on the word $x_2...x_n$.

$$D(x_1 u) = x_1 du = J(x_1 \otimes du)$$

Calculate now the $\mathtt{ker}\, D$. Let $\varepsilon \in B$, such that $D(\varepsilon) = 0$. We have $B = E \oplus F$, so ε is uniquely presented as:

$$\varepsilon = x_1 u + x_1 \otimes v$$

Applying D to elements from E and F as above, we have

$$0 = D(\varepsilon) = J(x_1 \otimes du) + J(x_1 \otimes v) - x_1 \otimes dv$$

Note that here $J(x_1 \otimes du) + J(x_1 \otimes v) \in F$ and $x_1 \otimes dv \in E$. Hence this is equivalent to $x_1 \otimes dv = 0$ and $J(x_1 \otimes (du+v)) = 0$. Which means $dv = 0$ and $x_1 \otimes (du+v) \in \mathtt{ker}\, J = x_1 \otimes L$. Again, the letter two conditions are equivalent to $dv = 0$ and $du + v \in L$, which in turn can be reformulated as $dv = 0$ and $v \in -du + L$. The latter two means $v \in -du + (L \cap \mathtt{ker}\, d)$.

Now,

$$\mathrm{dim}\mathtt{ker}\, D = \mathtt{dim}\, F + \mathtt{dim}\,(L \cap \mathtt{ker}\, d) = \mathtt{dim}\, E - \mathtt{dim}\, L + \mathtt{dim}\,(L \cap \mathtt{ker}\, d).$$

Thus

$$H_\bullet(B_w, D) = 2\mathtt{dim}\,\mathtt{ker}\, D - \mathtt{dim}\, B =$$

$$2\mathtt{dim}\, E - 2\mathtt{dim}\, L + 2\mathtt{dim}\,(L \cap \mathtt{ker}\, d) - \mathtt{dim}\, E - \mathtt{dim}\, E + \mathtt{dim}\, L =$$

$$2\mathtt{dim}\,(L \cap \mathtt{ker}\, d) - \mathtt{dim}\, L = H_\bullet(L_{w'}, d)$$

Here d is a usual bar differential on the subword of w, starting from x_2, $w' = ux_{s+1}...x_n$ and $L_{w'}$ is a bar subcomplex, defined by the word w'.

We got that $H_\bullet(B_w, D) = H_\bullet(L_{w'}, d)$, the letter by the inductive assumption is in $\{0, 1\}$, thus so is $H_\bullet(B_w, D)$.

□

Let us formulate one corollary, which will be used in section2, in the proof of theorem**??**

Corollary 1.2. *For any word w of noncommutative non-unital algebra A, the defined above subcomplex B_w of the bar complex is exact if and only if the dimension of B_w is even.*

Proof. For any complex B it is true, that its dimension is even or odd together with the dimension of its homology. Indeed,

$$H_\bullet(B) = \mathtt{ker}\, D/\mathtt{im}\, D, \quad B = \mathtt{ker}\, D + \mathtt{im}\, D,$$

so,

$$\dim H_\bullet(B) = \dim \mathtt{ker}\, D - \dim \mathtt{im}\, D, \quad \dim B = \dim \mathtt{ker}\, D + \dim \mathtt{im}\, D,$$

thus, $\dim H_\bullet(B) - \dim B = -2\dim \mathtt{im}\, D$ is even.

Taking in account theorem1.1, we see that since $\dim H_\bullet(B_w)$ can be either 0 or 1, the subcomplex B_w is exact if and only if the $\dim B$ is even. □

2 The exact position of nontrivial homology in the bar subcomplex of a monomial algebra

Let as above B_w be the subcomplex of the bar complex of monomial non-unital algebra A, associated to the monomial $w = x_1...x_n \in \langle X \rangle$.

$$0 \longrightarrow B_k = \{x_1 \otimes ... \otimes x_n\} \xrightarrow{D_k} B_{k-1} = \{x_1 \otimes ... \otimes x_i x_{i+1} \otimes ... \otimes x_n\} \xrightarrow{D_{k-1}} ... \xrightarrow{D_2} B_1 = \{x_1 \otimes ... \otimes x_n\} \longrightarrow 0$$

For any word $w = x_1...x_n$ and a fixed set of generating monomials (relations) $w_1, ..., w_t$ we can define a *Dyck path* as follows. Take the first (minimal) beginning subword $x_1...x_{d_1}$ of

$x_1...x_n$, which is zero in A, that is contains as a submonomial one of the monomials $w_1, ..., w_t$. In other words, $x_1...x_{d_1}$ contains as a subword (beginning) one of $w_1, ..., w_t$, but $x_1...x_d$ for any $d < d_1$ does not.

Then take minimal first subword $x_2...x_{d_2}$ of $x_2...x_n$, which is zero in A, etc. We get a sequence of numbers $d_1 \leqslant d_2 \leqslant ... \leqslant d_p$, called a **Dyck path**.

Dyck paths are remarkable combinatorial objects [2]. The number of Dyck paths of order p is a Catalan number:

$$C_n = frac1n + 1(\frac{2n}{n})$$

We say, that **Dyck path** $d_1 \leqslant d_2 \leqslant ... \leqslant d_p$ has **length** r if there are r different numbers in this sequence.

Theorem 2.1. *Let* B_w*:*

$$0 \xrightarrow{D_{k+1}} B_k \xrightarrow{D_k} B_{k-1} \xrightarrow{D_{k-1}} ... \xrightarrow{D_2} B_1 \xrightarrow{D_1} 0$$

be a non-exact complex, associated to the word $w = x_1...x_n$ *in a monomial algebra* A*, and* r *be the length of the corresponding Dyck path. Then the complex* B *has its nonzero homology on the place* $n - r$*.*

More precisely, if we denote

$H_1 = \mathtt{ker}\, D_k/\mathtt{im}\, D_{k+1},\ H_2 = \mathtt{ker}\, D_{k-1}/\mathtt{im}\, D_k, ...,\ H_i = \mathtt{ker}\, D_{k-i+1}/\mathtt{im}\, D_{k-i+2}, ...,\ H_k = \mathtt{ker}\, D_1/\mathtt{im}\, D_2,$

then

$$H_{n-r} = 1, H_i = 0 (i = 1, .., k, i \neq n - r).$$

Lemma 2.2. *If there exists a letter* x_i *in the monomial* $w = x_1...x_n$*, which is not present in the defining relations* $w_1, ..., w_t$*, then the complex associated to the monomial* $w = x_1...x_n$ *is exact.*

Proof. First, let's note the following, without loss of generality we can suppose that all letters in $w = x_1...x_n$ are different. Indeed, the defined above complex only depends on the positions, where the words $w_1, ..., w_t$ are appearing in the word w. Therefore, for an arbitrary word w and its subwords $w_1, ..., w_t$, one can take another word w', with the property that all its letters are different, and as a relations take subwords $w'_1, ..., w'_t$ of w', sitting on the same positions as $w_1, ..., w_t$ in w. Then the corresponding complexes would coincide:

$$B_{w,w_1,...,w_t} = B_{w',w'_1,...,w'_t}.$$

Thus, the study of a subclass of all monomial algebras, consisting of those algebras, where each generator appearing in relations at most once, is sufficient for the study of the properties of the bar complex.

Let us take a letter $x_i \in w$, which is not present at any relation $w_1, ..., w_t$. Suppose it is not the last letter in $w : x_i \neq x_n$. Otherwise we can suppose it is not the first letter $x_i \neq x_1$, since it can't appear twice in the word w.

The complex B is a span of tensors $B = span\{u_1 \otimes ... \otimes u_N\}$, where $u_j \neq 0$ in A. So, obviously, if x_i is not the last letter, the space B splits into two parts: span of those words where x_i is directly followed by a tensor symbol $\otimes$: $(x_i \otimes ...)$, or not: $(x_i ... \otimes ...)$. Denote them V_1 and V_2 respectively, so $B = span\{u_1 \otimes ... \otimes u_N\} = V_1 \otimes V_2$.

If we define a linear map $\varphi : V_1 \to V_2$ by $\varphi(\dots x_i \otimes u \otimes \dots) = \dots x_i u \otimes \dots$, we see that $\mathtt{ker}\,\varphi = 0$, since x_i is not contained in the relations, and it is clearly an onto map, so it is a bijection. It means $\dim B = 2\dim V_1$ is even, hence the complex is exact (corollary from the theorem1.1). $\square$

Proof of Theorem2.1. Let us define a sequence of complexes

$$B,\ L^{(1)},\ L^{(2)}, \dots, L^{(r-1)},\ L^{(r)}$$

inductively, using the definition of complex L from Section1. Namely, let us split B as $B = E \oplus F$, where

$$E = \{x_1 \otimes \dots\}_k, \qquad F = \{x_1 \dots\}_k,$$

and consider the linear map $J : E \to F$ defined by $x_1 \otimes w \mapsto x_1 w$. Then

$$\mathtt{ker}\, J = \{x_1 \otimes \underbrace{x_2 \dots x_s}_{u} \dots\}_k,$$

where $u = x_2 \dots x_s$ is the first (minimal) zero subword of $x_1 \dots x_n$ starting with x_1 (in other words, s is defined as a minimal number, such that $x_1 \dots x_s = 0$, but $x_1 \dots x_{s-1} \neq 0$).

Definition. We denote by $L(B)$ the linear space spanned by the monomials defining the $\mathtt{ker}\, J$:

$$L(B) = \{\underbrace{x_2 \dots x_s}_{u} \dots\}_k,$$

such that $\mathtt{ker}\, J = x_1 \otimes L$. As a complex $L = B_{ux_{s+1}\dots x_n}$ is a bar complex defined by the word $ux_{s+1} \dots x_n$.

Then we set $L^{(1)} = L(B)$ and continue constructing $L^{(2)} = L(L^{(1)})$ and so on. Each of the complexes $L^{(i+1)}$ is obtained as a quotient of $L^{(i)}$

Lemma 2.3. *The following diagram of complexes is commutative:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & B_k & \xrightarrow{D} & B_{k-1} & \longrightarrow & \dots & \longrightarrow & B_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & L_k^{(1)} & \xrightarrow{d^{(1)}} & L_{k-1}^{(1)} & \longrightarrow & \dots & \longrightarrow & L_1^{(1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & L_k^{(r-1)} & \xrightarrow{d^{(r-1)}} & L_{k-1}^{(r-1)} & \longrightarrow & \dots & \longrightarrow & L_1^{(r-1)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & L_k^{(r)} & \xrightarrow{d^{(r)}} & L_{k-1}^{(r)} & \longrightarrow & \dots & \longrightarrow & L_1^{(r)} & \longrightarrow & 0
\end{array}
$$

Proof. Consider the square of the diagram:

$$
\begin{array}{cccccc}
 & B_k & \xrightarrow{D_k} & B_{k-1} & \\
M_k & \downarrow & & \downarrow & M_{k-1} \\
 & L_k & \xrightarrow{d_k^{(1)}} & L_{k-1} &
\end{array}
$$

Here maps $M_k : B_k \longrightarrow L_k$ are defined by $x_1 \otimes x_2 ... x_{s_1} ... \mapsto u_1 ...$, where $u_1 = x_2 ... x_{s_1}$, all other monomials are mapped to zero.

Then

$$D_k(x_1 \otimes x_2 ... x_{s_1} w_1 \otimes w_2 \otimes w_3 ...) =$$

$$x_1 x_2 ... x_{s_1} w_1 \otimes w_2 ... - x_1 \otimes x_2 ... x_{s_1} w_1 w_2 \otimes w_3 ... + ...$$

Applying M to this polynomial, we get

$$M_k(D_k)) = -u_1 w_1 w_2 \otimes w_3 ... + u_1 w_1 \otimes w_2 w_3 ... - ...$$

Now we do it the other way around:

$$M_k(x_1 \otimes x_2 ... x_{s_1} w_1 \otimes w_2 \otimes w_3 ...) = u_1 w_1 \otimes w_2 \otimes w_3 ...$$

$$d_k(M_k(x_1 \otimes x_2 ... x_{s_1} w_1 \otimes w_2 \otimes w_3 ...)) = d_k(u_1 w_1 \otimes w_2 \otimes w_3 ...) =$$

$$u_1 w_1 w_2 \otimes w_3 ... - u_1 w_1 \otimes w_2 w_3 ... + ...$$

So, we see that $M(D(v)) = -d(M(v))$ for any 'nontrivial' monomial $v = x_1 \otimes x_2 ... x_{s_1} ...$.

The same type of argument works in any other row of the diagram, just formulas for the maps should be substituted accordingly.

□

The n^{th} row of the diagram is defined by the n^{th} element of the Dyck path (counted without repetitions). Note that a number of spaces at the beginning of $L^{(i)}$ is zero. It is our first goal to calculate where the first non-zero space in each row appears. For $L^{(1)} = B_{u_1 x_{s_1+1} \dots x_n}$, where $u_1 = x_2 \dots x_{s_1}$ is defined as: $x_1 \dots x_{s_1}$ is the first (starting with x_1) zero subword $x_1 \dots x_{s_1}$ of $\omega = x_1 \dots x_n$. To get elements of this kind inside B (taking into account that each application of the differential cuts out one $\otimes$ symbol) we need $s_1 - 2$ steps. So, $L_i^{(1)}$ is non-zero after $s_1 - 2$ steps from the left (from $L_k^{(1)}$). Since $L^{(2)} = B_{u_2 x_{s_2+1} \dots x_n}$, where $u_2 = x_3 \dots x_{s_1+1} \dots x_{s_2}$ is defined by $x_2 \dots x_{s_2}$ being the first (starting with x_2) zero subword of $x_2 \dots x_n$, to get elements of this kind into B, we need to cut out $s_2 - s_1 - 1$ extra tensor symbols, so $L_i^{(2)}$ is non-zero after $s_1 - 2 + s_2 - s_1 - 1 = s_2 - 3$ steps. The number of steps, which will be added at the third row is $s_3 - s_2 - 1$, so $L_i^{(3)}$ is non-zero after $s_3 - 4$ steps and so on.

Now consider the step $L^{(r-1)}$: $L^{(r-1)} = B_{u_{r-1} x_{s_{r-1}+1} \dots x_n}$, where $u_{r-1} = x_r \dots x_{s_{r-1}}$ is defined as follows: $x_r \dots x_{s_{r-1}}$ is the first (starting with x_{r-1}) zero subword $x_{r-1} \dots x_{s_{r-1}}$ of $x_{r-1} \dots x_n$. $L^{(r-1)}$ is non-zero after $s_{r-1} - r$ steps and it continues as the Bar complex of the word $u_{r-1} x_{s_{r-1}+1} \dots x_n$ of length $n - (s_{r-1} + 1) + 2 = n - s_{r-1} + 1$. Since this is the last step in the Dyck path, the only relation we have in the word, is the whole word. Thus the complex is nearly the "free" complex with the exception that at the last term we have zero space instead of the one-dimensional space $\{u_{r-1} x_{s_{r-1}+1} \dots x_n\}_k$. Exactly at this last term, the homology is 1. So, to get to this place, where the nonzero homology appears, we need to do $M - 1$ more steps, where M is the length of the free complex on a word of length $n - s_{r-1} + 1$. It is easy to see that M for a word of length k is k. Hence we have non-zero homology at the place $n - s_{r-1} + 1 - 1 + s_{r-1} - r = n - r$ from the beginning of the complex B (from the left-hand side term B_k).

Note that in this construction we used that $s_1 < \dots < s_r$ and $s_r = n$. Otherwise the complex would be exact. Indeed, the fact that two neighboring numbers coincide: $s_{i-1} = s_i$ would mean that x_i does not appear in relations, hence according to Lemma 2.2 the complex is exact, which case is excluded from the statement of the theorem. Analogously, if $s_r \neq n$, then x_n does not appear in relations. $\square$

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