O-minimal flows on abelian varieties

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Abstract. Let $A$ be an abelian variety over $\mathbb{C}$ of dimension $n$ and $\pi : \mathbb{C}^n \rightarrow A$ be the complex uniformisation. Let $X$ be an unbounded subset of $\mathbb{C}^n$ definable in a suitable o-minimal structure. We give a description of the Zariski closure of $\pi(X)$.

1. Introduction.

Let $A$ be a complex abelian variety of dimension $n$. Write $A = \mathbb{C}^n/\Lambda$ where $\Lambda \subset \mathbb{C}^n$ is a lattice and let $\pi : \mathbb{C}^n \rightarrow A$ be the uniformisation map.

A subvariety $V$ of $A$ is called weakly special if $V = P + B$ where $P$ is a point of $A$ and $B$ is an abelian subvariety. The abelian Ax-Lindemann-Weierstrass theorem is the following.

Theorem 1.1. Let $Y$ be a complex algebraic subset of $\mathbb{C}^n$. The components of the Zariski closure of $\pi(Y)$ are weakly special subvarieties.

This theorem is due to Ax (see [1] and [2]) and plays an important role in the new proof by Pila and Zannier of the Manin-Mumford conjecture [7]. Note that the paper [7] provides a different proof of the abelian Ax-Lindemann-Weierstrass theorem. For a proof close in spirit to the contents of this paper, see Section 9 of [5]. In reality, in this statement, $Y$ can be taken to be only semialgebraic ($\mathbb{C}^n$ being identified with $\mathbb{R}^{2n}$).

The aim of this paper is to investigate the Zariski closure of the sets $\pi(X)$ where $X$ is definable in an o-minimal structure which is a much wider class of objects. We refer to the book [11] for the notion of a set definable in an o-minimal structure, in particular the structures $\mathbb{R}_{an}$ and $\mathbb{R}_{an,\exp}$. Just recall that $\mathbb{R}_{an}$ is the o-minimal structure generated by the restricted analytic functions and $\mathbb{R}_{an,\exp}$ is additionally generated by the graph of the real exponential. For a subset $\Sigma$ of $A$, we denote by $\text{Zar}(\Sigma)$ its Zariski closure.

To be able to prove anything, we will need to make certain additional assumptions. Firstly, the set $X$ will be assumed to be unbounded. The necessity of this condition can be demonstrated by the following...
example. Let $F$ be a connected bounded fundamental domain for the action of $\Lambda$ on $\mathbb{C}^n$. The restriction of $\pi$ to $F$ is definable in $\mathbb{R}^n$. Let $V$ be any algebraic subvariety of $A$ and let $\tilde{V} = \pi^{-1}(V) \cap F$. Then $V$ is definable in $\mathbb{R}^n$ and $\text{Zar}(\pi(\tilde{V})) = V$.

However, when $X$ is an unbounded real analytic manifold, we prove the following.

**Theorem 1.2.** Let $X$ be an unbounded real analytic manifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in an o-minimal structure which is an extension of $\mathbb{R}^n$.

Let $V = \text{Zar}(\pi(X))$. For any point $P$ of $\pi(X)$ there is a positive dimensional abelian subvariety $B_P$ of $A$ such that $P + B_P$ is contained in $V$.

In particular, $V$ contains a Zariski dense set of weakly special subvarieties.

To investigate general definable sets $X$, we will also impose some mild restrictions on the o-minimal structure. Let $S$ be an o-minimal structure over $\mathbb{R}$, containing $\mathbb{R}^n$ and whose definable sets admit an analytic stratification (as defined in [11], Chapter 3). This condition holds for most ‘usual’ o-minimal structures, for example $\mathbb{R}^n$ and $\mathbb{R}^n,\exp$. We fix such a structure $S$ and in what follows and by definable, we will mean ‘definable in $S$’.

Next we introduce the notion of *essential Zariski closure*. Let $X$ be an unbounded definable set as before. For $R > 0$, let $B(0, R)$ be the open unit ball of centre 0 and radius $R$. The behavior of the set $\pi(X \cap B(0, R))$ when $R \to \infty$ is what we call an o-minimal flow. We show that for $R$ large enough, the Zariski closure of the set $\pi(X \setminus (X \cap B(0, R)))$ is constant. We call this Zariski closure, the essential Zariski closure of $\pi(X)$ and denote it by $\text{Zaress}(\pi(X))$.

For an abelian subvariety $B$ of $A$, write $V_B \subset \mathbb{C}^n$ the tangent space to $B$ at the origin and $p_B$ the projection $\mathbb{C}^n \to V_B$.

We prove the following:

**Theorem 1.3.** Let $X$ be an unbounded definable set of $\mathbb{C}^n$. Let $V$ be $\text{Zaress}(\pi(X))$.

For each point $P$, in $\pi(X)$, there exists a positive dimensional abelian subvariety $B_P$ of $A$ such that $P + B_P$ is contained in $V$.

In particular, $V$ contains a Zariski dense set of weakly special subvarieties.

We prove a characterisation of subvarieties of an abelian variety containing a Zariski dense set of weakly special subvarieties (see proposition 4.1) and deduce from theorem 1.3 the following.
Theorem 1.4. Assume that $X$ is a definable subset of $\mathbb{C}^n$ such that for all abelian subvarieties $B$ of $A$, $p_B(X)$ is unbounded. Then components of $\text{Zaress}(\pi(X))$ are weakly special.

The strategy of the proof of the theorem 1.2 relies on the theory of o-minimality and Pila-Wilkie counting theorem. Let $X$ be as in the statement and $V$ be the Zariski closure of $\pi(X)$. Using a suitable definable set and applying Pila-Wilkie theorem, we show that there exists a positive dimensional semi-algebraic set $W \subset \mathbb{C}^n = \mathbb{R}^{2n}$ such that $X + W$ is contained in $\pi(\square) V$. Applying the Ax-Lindemann-Weierstrass theorem, we then show that for any $P$ of $\pi(X)$, there exists a weakly special subvariety $P + B_P \subset V$.

Finally, we would like to point out one possible application of our theorem. Recall the following theorem of Bloch-Ochiai (see Chapter 9 of [3]) which is proved using Nevanlinna theory.

Theorem 1.5. Let $A$ be an abelian variety and $f : \mathbb{C} \to A$ be a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.

In some cases our theorem 1.4 implies theorem 1.5.

Consider for example $A = \mathbb{C}^n/\Lambda$ (where $\Lambda$ is a lattice such that $A$ is a simple abelian variety) and $f : \mathbb{C} \to A$ given by $f(z) = (z, \ldots, z, e^z, \ldots, e^z)$ with $s$ factors of $z$ and $r$ times of $e^z$ with $r + s = n$. Then consider the set $X \subset \mathbb{C}^n$ given by

$$X = \{(x + iy, \ldots, x + iy, e^{x+iy}, \ldots, e^{x+iy}) : x \in \mathbb{R}, y \in [0, 2\pi]\}.$$ 

Clearly $X$ is unbounded and definable in $\mathbb{R}_{an,exp}$ and its image in $A$ is contained in $f(\mathbb{C})$. By theorem 1.4, the Zariski closure of $f(\mathbb{C})$ is $A$ (since $A$ is simple).

It is not however always possible to “extract” such a definable unbounded set $X$ from $f(\mathbb{C})$ as the example of $(e^z, e^{iz}) \subset \mathbb{C}^2$ shows. Indeed, in this example, for any subset $Y \subset \mathbb{C}$ such that $f(Y)$ is definable, both the real and imaginary parts of $z \in Y$ must be bounded.

Another (counter)-example is the following. Define the iterated exponential function $\exp_n(x)$ by $\exp_1 = \exp$ and $\exp_n = \exp \circ \exp_{n-1}$. By Proposition 9.10 of [4], a definable function is bounded by $\exp_n(x^m)$ for some $n, m$. Therefore a graph of a function which ‘grows faster’ than any $\exp_n$ will not satisfy the assumptions of our theorems.

We conclude this introduction with an open question in the spirit of [10]. It concerns the topological closure of $\pi(X)$ rather than Zariski
closure. Recall from [10] that a real weakly special subvariety is defined to be a translate of a real subtorus of $A$ (hence not necessarily algebraic).

**Conjecture 1.6.** Let $X$ be as before be an unbounded definable real analytic manifold. We denote by $\overline{\pi(X)}$ the topological closure of $\pi(X)$.

There exists a real analytic submanifold $V$ of $A$ containing a dense subset of real weakly special subvarieties such that

$$\overline{\pi(X)} = \pi(X) \cup V.$$ 

In section 4, we prove a characterisation of subvarieties of abelian varieties containing a Zariski dense subset of weakly special subvarieties, namely that such a subvariety is a union of weakly special ones. We believe this result and our argument to be of independent interest.

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2. **Proof of theorem 1.2.**

In this section we assume that $X$ is an unbounded real analytic submanifold of $\mathbb{C}^n = \mathbb{R}^{2n}$ definable in some o-minimal structure which contains $\mathbb{R}^{an}$. Let $V$ be the Zariski closure of $\pi(X)$ in $A$.

2.1. **A definable set and point counting.** The contents of this section are essentially a reproduction of the arguments of Orr from Section 9 of [5] with slight adjustments. In this section we define a certain definable set associated with $X$ and, using Pila-Wilkie theorem, show that this set contains a positive dimensional semi-algebraic subset.

Choose a fundamental set $F$ for the action of $\Lambda$ on $\mathbb{C}^n$ such that $X \cap F$ is non-empty. We choose $F$ to be an open connected subset of $\mathbb{C}^n$ such that $F$ is compact and $\Lambda$-translates of $F$ cover $\mathbb{C}^n$. The set $F$ is an ‘open parallelepedid’.

Consider the definable set

$$\Sigma = \{ x \in \mathbb{C}^n : \text{dim}(X + x) \cap \tilde{V} = \text{dim}(X) \}.$$ 

The argument is exactly the same as in the proof of Lemma 9.3 of [5].
We have the following lemma:

**Lemma 2.1.** If \( \lambda \in \Lambda \) and \( X \cap (\mathcal{F} - \lambda) \neq \emptyset \), then \( \lambda \in \Sigma \).

**Proof.** From \( \Lambda \)-invariance of \( \pi^{-1}V + \lambda = \pi^{-1}V \), we see that for \( \lambda \) as in the statement (in particular for \( \lambda \in \Lambda \)), \( X + \lambda \subset \pi^{-1}V \).

It follows that \( (X + \lambda) \cap \tilde{V} = (X + \lambda) \cap \mathcal{F} \).

As \( \mathcal{F} - \lambda \) is an open subset of \( \mathbb{C}^n \), we see that \( \dim(X \cap (\mathcal{F} - \lambda)) = \dim(X) = \dim((X + \lambda) \cap \mathcal{F}) \).

The conclusion follows. \( \Box \)

Fix a basis \( \lambda_1, \ldots, \lambda_{2n} \) of \( \Lambda \). Then \( \Lambda \otimes \mathbb{Q} \) is identified with \( \mathbb{Q}^{2n} \). We define the height of an element \( \lambda = \sum a_i \lambda_i \in \Lambda \) (\( a_i \in \mathbb{Z} \)) as

\[
H(\lambda) = \max(|a_1|, \ldots, |a_{2n}|).
\]

This height thus coincides with the usual height on \( \mathbb{Q}^n \).

**Proposition 2.2.** There exists \( T_0 \geq 0 \) such that for all \( T \geq T_0 \),

\[
|\{x \in \Sigma \cap \Lambda : H(x) \leq T\}| \geq T/2.
\]

**Proof.** This is essentially Lemma 9.1 of [5].

The first observation is that if \( x_1 \) and \( x_2 \) are two points of \( \Lambda \) such that \( X \cap (\mathcal{F} - x_1) \) and \( X \cap (\mathcal{F} - x_2) \) are both non-empty, then \( \Sigma \cap \Lambda \) contains at least one point of height \( h \) for every \( h \) between \( H(x_1) \) and \( H(x_2) \).

Note that \( X \) is path-wise connected in the Euclidean topology. Let \( C \) be a path from a point in \( X \cap (\mathcal{F} - x_2) \) to a point in \( X \cap (\mathcal{F} - x_2) \).

When \( C \) crosses over from \( \mathcal{F} - u_1 \) to an adjacent domain \( \mathcal{F} - u_2 \), the heights of \( u_1 \) and \( u_2 \) change by at most one.

It follows that for any \( h \) between \( H(x_1) \) and \( H(x_2) \), there is a \( u \in \Lambda \) of height \( \leq h \) such that \( X \cap (\mathcal{F} - u) \) is not empty. This \( u \) belongs to \( \Sigma \cap X \).

By assumption \( X \) is unbounded. Thus as \( x \) varies in \( \Lambda \) such that \( X \cap \mathcal{F} - x \) is non-empty, \( h(x) \) goes to infinity.

It follows that there is an \( h_0 \) such that for any \( h > h_0, \Sigma \cap \Lambda \) contains at least one point of height \( h \).

Take \( T_0 = 2h_0 \). \( \Box \)

We now use the following theorem of Pila and Wilkie ([6], Theorem 1.8).

For a definable subset \( \Theta \subset \mathbb{R}^n \), we define \( \Theta^{alg} \) to be the union of all positive dimensional semi-algebraic subsets contained in \( \Theta \). We define \( \Theta^{tr} \) to be \( \Theta \setminus \Theta^{alg} \).
Theorem 2.3 (Pila-Wilkie). Let $\Theta$ be a subset of $\mathbb{R}^n$ definable in an o-minimal structure. Let $\epsilon > 0$. There exists a constant $c = c(\Theta, \epsilon)$ such that for any $T \geq 0$,
\[
|\{x \in \Theta : H(x) \leq T\}| \geq c T^\epsilon.
\]
From Proposition 2.2 it now follows that $\Sigma^{\text{alg}} \cap \Lambda$ is not empty.

Let $W$ be a connected positive dimensional semi-algebraic subset contained in $\Sigma$. For each $w \in W$, $\dim((X + w) \cap \tilde{V}) = \dim(X)$ and hence an analytic component of $(X + w) \cap F$ is contained in $\pi^{-1}V$. By analytic continuation, we see that $X + w \subset \pi^{-1}V$. We have proved:

Proposition 2.4. With the notations and assumptions of this section, there exists a positive dimensional semialgebraic subset $W$ such that $X + W \subset \pi^{-1}V$.

2.2. Final argument. We use the following lemma whose proof can for example be found in [5], Lemma 8.1.

Lemma 2.5. Let $Z$ be a connected complex analytic subset of $\mathbb{C}^g$. Let $X$ be a connected irreducible semialgebraic set contained in $Z$. Then there is a complex algebraic variety $Y$ such that $X \subset Y \subset Z$.

By proposition 2.4 and the above lemma, we see that for any $x \in X$, there exists a positive dimensional complex algebraic subset $Y_x$ containing $X$ and contained in $\pi^{-1}(V)$. By the abelian Ax-Lindemann-Weierstrass theorem 1.1, the Zariski closure of $\pi(Y_x)$ is a union of weakly special subvarieties of $V$. Therefore, $V$ contains a subvariety of the form $P + B_P$ where $P = \pi(x)$ and $B_P$ is a positive dimensional abelian subvariety of $A$. This finishes the proof of theorem 1.2.


In this section we consider an unbounded definable set $X \subset \mathbb{C}^n$. We refer to section 8 of [4] for the definition of a real analytic cell. What is relevant to us is that a real analytic cell in $\mathbb{R}^n$ is a definable real analytic submanifold, definable-analytically isomorphic to $\mathbb{R}^m$ for some $m \leq n$. By Theorem 8.9 of [4], there is a finite number of analytic cells $X_1, \ldots, X_k$ such that $X$ is a disjoint union of the $X_k$.

Proposition 3.1. The essential closure $\text{Zar}(\pi(X))$ is the union of $\text{Zar}(\pi(X_i))$ where $X_i$s are the unbounded cells.

Proof. We start with a lemma.

Lemma 3.2. Let $Z$ be a real analytic manifold in $\mathbb{C}^n$ and $U \subset Z$ an open subset.
Then
\[ \text{Zar}(\pi(U)) = \text{Zar}(\pi(Z)) \]
In particular, if \( Z \) is an analytic unbounded submanifold of \( \mathbb{C}^n \), then
\[ \text{Zaress}(\pi(Z)) = \text{Zar}(\pi(Z)) \]

Proof. One inclusion is obvious.
Write \( \text{Zar}(\pi(U)) \subset \mathbb{P}^m \) for some \( m \) and let \( s \in H^0(\mathbb{P}^m, \mathcal{O}(l)) \) for \( l \geq 1 \) such that \( s \) is zero on \( \pi(U) \). Then \( s \circ \pi \) is zero on \( U \) and by analytic continuation \( s \circ \pi \) is zero on \( Z \). It follows that \( s \) is zero on \( \pi(Z) \), hence \( \text{Zar}(\pi(Z)) \subset \text{Zar}(\pi(U)) \).

\[ \square \]

Let \( X = X_1 \coprod \ldots \coprod X_k \) be a cell decomposition of \( X \). For \( R \) large enough, \( X \cap B(0, R) \) contains the union of all the bounded cells in the above decomposition.
We have
\[ \text{Zaress}(\pi(X)) = \bigcup_{\{i: X_i \text{ unbounded}\}} \text{Zaress}(\pi(X_i)). \]
By Lemma 3.2, for an unbounded cell \( X_i \),
\[ \text{Zaress}(\pi(X_i)) = \text{Zar}(\pi(X_i)). \]
The result follows.
\[ \square \]

4. Characterisation of subvarieties containing a dense set of weakly special subvarieties.

In this section we prove a proposition which we believe to be of independent interest.
Let \( A \) be an abelian variety and \( V \) a subvariety of \( A \). Define the stabiliser of \( V \) as
\[ \text{Stab}(V) = \{ P \in A : P + V = V \}. \]
Recall that for an abelian subvariety \( B \) of \( A \), there exists an abelian subvariety \( B' \) such that \( A = B + B' \) and \( B \cap B' \) is finite. We always refer to \( B \) and \( B' \) as above.

**Proposition 4.1.** Let \( V \) be an irreducible subvariety of \( A \).

(1) Assume \( \dim \text{Stab}(V) > 0 \).

Then there exists abelian subvarieties \( B \) and \( B' \) of \( A \) such that \( A = B + B' \) and \( V = B + V' \) where \( V' \) is a subvariety of \( B' \).
(2) Assume that Stab(V) is finite. Then the set of positive dimensional weakly special subvarieties contained in V is not Zariski dense.

(3) Assume again that Stab(V) is finite. Let Σ be the set of all positive dimensional weakly special subvarieties contained in V.

For an abelian subvariety B ⊂ A, denote by B' an abelian subvariety such that A = B + B'.

There exists a finite set B₁, . . . , B_r of abelian subvarieties of A and W₁, . . . , W_r of subvarieties of B_i such that

\[ \text{Zar}(\Sigma) = \bigcup_{i=1}^{r} B_i + W_i. \]

Proof. Assume \( \dim \text{Stab}(V) > 0 \) and let B be the neutral component of Stab(V).

Let B' be an abelian subvariety such that A = B + B' and let \( \psi : A \to A/B \) be the quotient. Let \( V' = \psi(B') \).

Then

\[ V = \{ B + x : x ∈ V \} = \{ B + x : x ∈ V' \} = B + V'. \]

This proves (1).

We will now prove (2). Assume that Stab(V) is finite. We start by reducing to the case where Stab(V) = \{0\}. Let \( A' = A/\text{Stab}(V) \) and let \( \phi : A \to A' \) be the quotient map and let \( V' = \phi(V) \). Note that \( \phi^{-1}(V') = V + \text{Stab}(V) = V \). We claim that Stab(V') = \{0\}. Let \( P ∈ \text{Stab}(V') \) and \( Q ∈ \phi^{-1}(P) \). We have

\[ \phi(Q + V) = P + V' = V' \]

It follows that \( Q + V ⊂ \phi^{-1}(V') = V \) and for dimension reasons \( Q + V = V \). Hence \( Q ∈ \text{Stab}(V) \) and \( P = \phi(Q) = 0. \)

As the conclusion of (2) holds for V if and only if it holds for V', we may therefore assume that Stab(V') = \{0\}.

For \( m > 1 \), consider the map

\[ \phi_m : V^m \to A^{m-1} \]

defined by

\[ \phi_m(x_1, . . . , x_m) = (x_1 - x_2, . . . , x_m - x_{m-1}) . \]

By [12], Lemma 3.1, there exists \( m > 1 \) such that the map \( \phi_m \) is a generic embedding.

Let \( P + B \) be a positive dimensional weakly special subvariety contained in V. Then \( \phi_m((P + B)^m) = B^{m-1} \). The map \( \phi_m \) is therefore not injective on \((P + B)^m\). Therefore V can not contain a Zariski dense
set of positive dimensional subvarieties of the form \( P + B \). This proves (2).

Let us now prove (3). Let \( \Sigma \) as in the statement, the set of all positive dimensional weakly special subvarieties contained in \( V \) and let \( W \) be a component of \( \text{Zar}(\Sigma) \). Then \( W \) contains a Zariski dense set of weakly special subvarieties and by (2), \( \text{Stab}(W) \) is positive dimensional. It follows from (1) that \( W = B + W' \) where \( B \) is an abelian subvariety of \( A \) and \( W' \) a subvariety of \( B' \). Since \( \text{Zar}(\Sigma) \) has finitely many components, the conclusion of (3) follows. \( \square \)

**Remark 4.2.** The geometric aspect of Lang’s conjecture predicts that given a variety of general type \( V \), the union subvarieties of \( W \) not of general type, is not Zariski dense. It is a known fact that a subvariety \( V \) of an abelian variety is of general type if and only if \( \text{Stab}(V) \) is finite. Therefore, our proposition 4.1 implies the geometric Lang’s conjecture for subvarieties of abelian varieties.

**Remark 4.3.** This proposition is an abelian analogue of the result of the first author (see [9]) in the hyperbolic case which is proved by completely different methods.

5. Proof Theorems 1.3 and 1.4.

In this section we deduce theorems from what precedes.

Let \( A \) and \( X \) be as in the assumptions of Theorem 1.3. Let \( V \) be a component of the essential Zariski closure of \( \pi(X) \).

In section 3 we have seen that \( \text{Zare}(\pi(X)) \) is a finite union of Zariski closures of sets of the form \( \pi(Y) \) where \( Y \) is an unbounded definable real analytic submanifold of \( \mathbb{C}^n \). Therefore, the conclusion of theorem 1.3 follows from theorem 1.2.

Let now \( X \) be as in 1.4. By theorem 1.3, \( V = \text{Zare}(X) \) contains a Zariski dense set of positive dimensional weakly special subvarieties. From proposition 4.1, we deduce that \( V \) is of the form \( V = B + V' \) where \( B \) is a positive dimensional abelian subvariety of \( A \) and \( V' \) is a subvariety of \( B' \). Reiterating the argument with \( B' \) and \( V' \), we conclude that components of \( V \) are weakly special.

**References**


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