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# On the conservative dynamics of two-body systems at the fourth post-Newtonian approximation of general relativity

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The fourth post-Newtonian (4PN) two-body dynamics has been recently tackled by several different approaches: effective field theory, Arnowitt-Deser-Misner Hamiltonian, action-angle-Delaunay averaging, effective-one-body, gravitational self-force, first law of dynamics, and Fokker action. We review the achievements of these approaches and discuss the complementarity of their results. Our main conclusions are: (i) the results of the first complete derivation of the 4PN dynamics [T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **89**, 064058 (2014)] have been, *piecewise*, fully confirmed by several subsequent works; (ii) the results of the Delaunay-averaging technique [T. Damour, P. Jaranowski, and G. Schäfer, Phys. Rev. D **91**, 084024 (2015)] have been confirmed by several independent works; and (iii) several claims in a recent Fokker-action computation [L. Bernard *et al.*, arXiv:1512.02876v2 [gr-qc]] are incorrect, but can be corrected by the addition of a couple of *ambiguity parameters* linked to subtleties in the regularization of infra-red and ultra-violet divergences.

## I. INTRODUCTION

The general relativistic two-body problem has acquired a renewed importance in view of the impending prospect of detecting the gravitational wave signals emitted by inspiralling and coalescing compact binaries. After the successful completion, fifteen years ago, of the derivation of the third post-Newtonian (3PN) two-body dynamics [1] (see [2–5] for later rederivations), a natural challenge was to tackle the (conservative) fourth post-Newtonian (4PN) dynamics. Several partial steps in the derivation of the 4PN dynamics have been performed [6–14]<sup>1</sup>, and culminated in the recent first full derivation, within the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation of General Relativity, of the 4PN dynamics in Ref. [16], completed by an action-angle-Delaunay averaging study in Ref. [17]. A remarkable feature of the conservative 4PN dynamics of Ref. [16] is the presence of a *time-symmetric nonlocal-in-time* interaction, which is directly related to the 4PN-level tail-transported (retarded) interaction first discussed in Ref. [18] (see also Refs. [19, 20] for recent discussions).

In a recent preprint [21] Bernard *et al.* reported the computation of the Fokker action describing the 4PN dynamics in harmonic coordinates. They make several claims in (the version 2 of) their preprint, notably: (i) the presence of a *unique* ambiguity, of infra-red (IR) origin, parametrized by the parameter  $\alpha$ ; and (ii) the need

to choose the value  $\alpha = \alpha_{\text{B}^3\text{FM}} = 811/672$  to reproduce the analytically known [13] 4PN interaction energy for circular orbits. Moreover, they state that their 4PN dynamics disagrees with the one recently derived by us [14, 16, 17] and that part of the discrepancy comes from our treatment of the nonlocal contribution to the dynamics.

The first aims of the present paper will be: (1) to show that the claims, denoted (i) and (ii) above, of [21] are wrong; (2) to suggest that the discrepancies between their result and ours is due both to their incorrect evaluation of the conserved energy of the nonlocal 4PN dynamics and to the need to complete their result by at least one further contribution parametrized by an additional ambiguity parameter, denoted  $a$  below. The other aims of our paper will be: (3) to explain in more detail than in our original paper [17] the logical basis and consistency of our reduction of the nonlocal 4PN dynamics to a formally local action-angle Hamiltonian; and (4) to summarize the many independent results that have confirmed, piecemeal, all the elements of our 4PN dynamics [16, 17].

Regarding the last item, we wish to emphasize that, besides the straightforward PN (or, equivalently, effective-field-theory) calculations of the two-body dynamics, crucial information about the two-body dynamics has been acquired (as will be detailed below) through combining (in various ways) several other approaches: the effective-one-body (EOB) formalism [22–25], the first law of binary mechanics [26–28], and gravitational self-force (SF) theory, especially when combined with the Mano-Suzuki-Takasugi (MST) [29, 30] hypergeometric-expansion approach to Regge-Wheeler-Zerilli theory. We shall point out when needed the multi-way consistency checks between these approaches that have been obtained at the

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<sup>1</sup> See also [15] for the  $O(G^1)$  interaction, valid to *all* PN-orders.

4PN level.

Our notation for two-body systems will follow the one used in our previous works: the two masses are denoted  $m_1$ ,  $m_2$ , and we then denote  $M \equiv m_1 + m_2$ ,  $\mu \equiv m_1 m_2 / (m_1 + m_2)$ ,  $\nu \equiv \mu / M = m_1 m_2 / (m_1 + m_2)^2$ . We indicate powers of  $G$  and  $c$  when it is pedagogically useful, but we sometimes set  $G = c = 1$  when it is more convenient.

## II. EXTANT PIECEWISE CONFIRMATIONS OF THE 4PN ADM-DELAUNAY DYNAMICS AND LIMITS ON POSSIBLE DEVIATIONS

Let us start by recalling the structure of the 4PN results of Refs. [16, 17]. First, the ADM action derived in [16] has the following form (using the notation of [16])

$$S_{\text{DJS}} = \int \left[ \sum_a p_{ai} dx_a^i - H_{\text{DJS}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; s) dt \right] + S^{\text{nonloc}}(s), \quad (2.1)$$

where the *nonlocal* piece of the ADM action<sup>2</sup> is ( $I_{ij}$  denoting the quadrupole moment of the binary system; and  $I_{ij}^{(3)} \equiv d^3 I_{ij} / dt^3$ )

$$S^{\text{nonloc}}(s) = \frac{1}{5} \frac{G^2 M}{c^8} \text{Pf}_{2s/c} \iint \frac{dt dt'}{|t-t'|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t'), \quad (2.2)$$

(Pf  $\equiv$  Partie finie) and where the *local* piece of the ADM action has the structure

$$H_{\text{DJS}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; s) = H_{\text{DJS}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; r_{12}) + F[\mathbf{x}_a, \mathbf{p}_a] \ln \frac{r_{12}}{s}, \quad (2.3)$$

with

$$F[\mathbf{x}_a, \mathbf{p}_a] = \frac{2}{5} \frac{G^2 M}{c^8} (I_{ij}^{(3)})^2. \quad (2.4)$$

The length scale  $s$  entering the action above<sup>3</sup> is arbitrary because (as shown in [16]) the  $s$ -dependence of the non-local action (2.2) cancels against the  $s$ -dependence of the local action (2.3).

In our second paper, Ref. [17], we combined different techniques (which will be explained in detail below) for transforming the (center-of-mass-frame reduction of the)

nonlocal ADM action (2.1) into an equivalent ordinary action, say

$$S_{\text{DJS}'} = \int \left[ p_i dq^i - H_{\text{DJS}'}(q, p) dt \right], \quad (2.5)$$

where  $H_{\text{DJS}'}(q, p)$  is formally given by an expansion in (even) powers of  $q^i p_i$ . [Ref. [17] gave an exact formula for the action-angle version of  $S^{\text{nonloc}}$ , and the explicit form of the expansion of  $H_{\text{DJS}'}(q, p)$  in powers of  $q^i p_i$  through  $(q^i p_i)^6$ .] The ordinary action (2.5) is equivalent to (the center-of-mass-frame reduction of) (2.1) modulo some shifts of the phase-space coordinates that are implicitly defined by the Delaunay-like reduction procedure of [17]. These shifts do not affect the gauge-invariant observables deduced from either (2.1) or (2.5) which we shall focus on in the following. Note also in passing that the use of the EOB formalism in [17] is essentially a technical convenience, while the essential conceptual step used there is the reduction of a nonlocal action to a local form by using Delaunay-like averaging techniques (see below).

The aim of this Section is to summarize the current existing confirmations of the correctness of the 4PN actions (2.1) or (2.5). First, we wish to recall that the full Poincaré invariance of (2.1) (in a general frame) was explicitly checked in [16] (see also [14]). This is a highly nontrivial check because the ADM derivation of the action (contrary to the harmonic-coordinates Fokker one) is far from being manifestly Lorentz invariant.

To organize the other confirmations of (2.1) or (2.5), let us note that, in the center-of-mass frame, and when using suitably scaled<sup>4</sup> dynamical variables [e.g.  $r = r^{\text{phys}}/GM$ ,  $\mathbf{p} = \mathbf{p}_1/\mu = -\mathbf{p}_2/\mu$ ,  $S = S^{\text{phys}}/(GM\mu)$ ,  $\hat{H} = (H - Mc^2)/\mu$ ] the rescaled Hamiltonian  $\hat{H}$  has a polynomial structure in the symmetric mass ratio  $\nu$

$$\hat{H} = \hat{H}_0 + \nu \hat{H}_1 + \nu^2 \hat{H}_2 + \nu^3 \hat{H}_3 + \nu^4 \hat{H}_4, \quad (2.6)$$

where  $\hat{H}_0 = \frac{1}{2} \mathbf{p}^2 - 1/r + c^{-2}(\dots) + \dots + O(c^{-10})$  describes the 4PN dynamics of a test particle in the field generated by the mass  $M = m_1 + m_2$ .

The contribution  $\nu \hat{H}_1$  to the 4PN dynamics describes the 4PN approximation to the first-order self-force (1SF) dynamical effects. Over the recent years, many works have been devoted to both the numerical and the analytical computation of 1SF dynamical effects. To compare these results to the predictions following from our 4PN dynamics (2.1) or (2.5) we need *bridges* between PN results and SF results. Two such PN-SF bridges have been particularly useful over the last years: the EOB formalism [22–25], and the first law of binary mechanics [26–28].

The first example of PN-EOB-SF bridge was the derivation of the functional relation between the periastron precession of small-eccentricity orbits and the radial potentials entering the EOB Hamiltonian [6]. At

<sup>2</sup> In Ref. [16] the necessity of adding the nonlocal contribution  $S^{\text{nonloc}}(s)$  was derived by combining the structure of the IR divergence of the local ADM action [14] with the known existence of a 4PN-level, long-range tail-transported interaction [18]. Prompted by an argument in [21], we discuss in Appendix A the (limited) extent to which  $S^{\text{nonloc}}(s)$  can be (formally) directly derived from the ADM action.

<sup>3</sup> For clarity, we do not use here the formulation where  $s$  is replaced by  $r_{12} = |\mathbf{x}_1 - \mathbf{x}_2|$ .

<sup>4</sup> Beware that we will often oscillate between using scaled or unscaled dynamical variables.

the 1SF level, this relation (see Eqs. (5.21)–(5.25) in [6]) yields the precession function  $\rho(u)$  as a linear combination of  $a(u), a'(u), a''(u)$  and  $\bar{d}(u)$ , where  $a(u)$  is the 1SF correction to the main EOB radial potential  $A(u; \nu) = 1 - 2u + \nu a(u) + O(\nu^2)$  (which generalizes

$A^{\text{Schwarzschild}}(u) = 1 - 2GM/c^2 r \equiv 1 - 2u$ ), and  $\bar{d}(u)$  the 1SF correction to the second EOB radial potential  $\bar{D}(u) \equiv (A(u)B(u))^{-1} = 1 + \nu \bar{d}(u) + O(\nu^2)$ . Let us apply this relation to the 4PN-level values of the EOB potentials parametrizing the 4PN dynamics (2.5), as derived in [17], namely (adding the third EOB potential  $Q$ ):

$$A(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right) \nu u^4 + \left(\left(\frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128}{5}\gamma_E + \frac{256}{5}\ln 2\right)\nu + \left(\frac{41\pi^2}{32} - \frac{221}{6}\right)\nu^2 + \frac{64}{5}\nu \ln u\right) u^5, \quad (2.7a)$$

$$\bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2) u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_E - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu \ln u\right) u^4, \quad (2.7b)$$

$$\hat{Q}(\mathbf{r}', \mathbf{p}') = \left(2(4 - 3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right) u^3\right) (\mathbf{n}' \cdot \mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right) u^2 (\mathbf{n}' \cdot \mathbf{p}')^6 + O[\nu u (\mathbf{n}' \cdot \mathbf{p}')^8]. \quad (2.7c)$$

This yields

$$\rho(x) = 14x^2 + \left(\frac{397}{2} - \frac{123}{16}\pi^2\right) x^3 + \left(\frac{58265}{1536}\pi^2 - \frac{215729}{180} + \frac{5024}{15}\gamma_E + \frac{1184}{15}\ln 2 + \frac{2916}{5}\ln 3 + \frac{2512}{15}\ln x\right) x^4 + O(x^5 \ln x). \quad (2.8)$$

The 4PN-level contribution to the precession function  $\rho(x)$  is of the form  $\rho_{4\text{PN}}(x) = (\rho_4^c + \rho_4^{\text{ln}} \ln x) x^4$ , with the rational logarithmic coefficient  $\rho_4^{\text{ln}} = \frac{2512}{15}$  [8], and the transcendental non-logarithmic 4PN coefficient

$$\rho_4^{c,\text{DJS}} = \frac{58265}{1536}\pi^2 - \frac{215729}{180} + \frac{5024}{15}\gamma_E + \frac{1184}{15}\ln 2 + \frac{2916}{5}\ln 3. \quad (2.9)$$

The *analytical* values of the 4PN-level functions  $A$ ,  $\bar{D}$ ,  $Q$  have been recently independently confirmed by two self-force computations (based on the recently derived eccentric-extension of the first law [28]), see Refs. [31] and [32]. [Note also that the coefficients of higher powers of  $\mathbf{n}' \cdot \mathbf{p}'$  in  $Q$  were recently provided: the powers 8 and 10 in [32]; and the powers 12 to 20 in [33].]

In addition to analytical confirmations, there are also numerical self-force confirmations of the 4PN-level values of  $\rho(x)$  and  $\rho_4^{c,\text{DJS}}$  based on direct dynamical computations of the precession of slightly eccentric orbits.

The first numerical SF determination of the precession function  $\rho(u)$  was made in [34]. In particular, the latter work confirmed the value  $\rho_4^{\text{ln}} = \frac{2512}{15}$  and derived an estimate of the value of the non-logarithmic coefficient, namely

$$\rho_4^{c,\text{num}[34]} = 69_{-4}^{+7}. \quad (2.10)$$

We were informed by Maarten van de Meent that he has very recently obtained a much more accurate determination of  $\rho_4^c$  with *preliminary* results yielding [35]

$$\rho_4^{c,\text{num}[35]} = 64.640566(2), \quad (2.11)$$

where the number in parentheses indicates a preliminary estimate of the uncertainty on the last digit. Note that both numerical estimates of  $\rho_4^c$  assume the analytical values of the 4PN and 5PN logarithmic contributions to  $\rho(u)$ . [There is no doubt about the (1SF) 4PN and 5PN logarithmic contributions to the dynamics: the 4PN ones can be straightforwardly deduced from the tail-related

4PN logarithmic term written in Eq. (6.39) of [18] (equivalent to the  $F[\mathbf{x}_a, \mathbf{p}_a] \ln \frac{r_{12}}{s}$  logarithmic contribution in (2.3)), while the 5PN ones are straightforwardly derivable from the higher-tail results of Refs. [7] and [17] (Sec. IXA).] Both numerical SF results (2.10), (2.11) confirm, within their respective error bars, the numerical value

$$\rho_4^{c,\text{DJS}} = 64.64056\,47571\,19378\,19014\,84255 \dots \quad (2.12)$$

of the result, (2.9), predicted by our 4PN dynamics. The numerical result (2.10) differs from the analytical value (2.9) by

$$\Delta\rho_4^c \equiv \rho_4^{c,\text{num}[34]} - \rho_4^{c,\text{DJS}} = 4_{-4}^{+7}, \quad (2.13)$$

while the more recent numerical value (2.11) differs from the analytical value (2.9) only by

$$\Delta\rho_4^c \equiv \rho_4^{c,\text{num}[35]} - \rho_4^{c,\text{DJS}} = (1 \pm 2) \times 10^{-6}. \quad (2.14)$$

What is especially important in such a numerical check is that we are talking here about a direct dynamical check of the 4PN dynamics derived in [16] and [17]. Indeed, no use is made here of the first-law-of-mechanics bridge to go from SF computations of the Detweiler-Barack-Sago redshift invariant to 4PN dynamical functions. The SF computations done in [34] and [35] directly estimate the effect of the nonlocal gravitational self-force  $F^\mu$  on slightly eccentric orbits to extract the precession function  $\rho(u)$  and, then, its PN expansion coefficients. One has therefore here a direct confirmation of the way Refs. [16, 17] computed (via well established EOB results [6]) the precession effect of the nonlocal 4PN dynamics.

Let us now summarize the numerical SF confirmations of the other 1SF predictions one can draw from the 4PN results (2.7). The 4PN-level coefficients of the three EOB potentials  $A$ ,  $\bar{D}$  and  $Q$  (the latter being considered at  $O(p_r^4)$ , i.e. at the level of the fourth power of the eccentricity) are of the form  $A_{4\text{PN}}(u) = \nu(a_5^c + a_5^{\text{ln}} \ln u + \nu a_5') u^5$ ,  $\bar{D}_{4\text{PN}}(u) = \nu(\bar{d}_4^c + \bar{d}_4^{\text{ln}} \ln u + \nu \bar{d}_4') u^4$ , and  $Q_{4\text{PN}}(u, p_r) = \nu q_{4,3}(\nu) u^3 p_r^4 + \nu q_{6,2}(\nu) u^2 p_r^6 + O(p_r^8)$ , with  $q_{4,3}(\nu) = q_{4,3} + \nu q'_{4,3} + \nu^2 q''_{4,3}$  and  $q_{6,2}(\nu) = q_{6,2} + \nu q'_{6,2} + \nu^2 q''_{6,2}$ . Among the 4PN coefficients, the 1SF contributions are the ones at order  $O(\nu)$ , i.e. the unprimed ones in our notation, namely

$$a_5^c = \frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128}{5}\gamma_E + \frac{256}{5} \ln 2, \quad (2.15a)$$

$$a_5^{\text{ln}} = \frac{64}{5}, \quad (2.15b)$$

$$\begin{aligned} \bar{d}_4^c = & -\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_E \\ & - \frac{6496}{15} \ln 2 + \frac{2916}{5} \ln 3, \end{aligned} \quad (2.15c)$$

$$\bar{d}_4^{\text{ln}} = \frac{592}{15}, \quad (2.15d)$$

$$q_{4,3} = -\frac{5308}{15} + \frac{496256}{45} \ln 2 - \frac{33048}{5} \ln 3, \quad (2.15e)$$

$$\begin{aligned} q_{6,2} = & -\frac{827}{3} - \frac{2358912}{25} \ln 2 \\ & + \frac{1399437}{50} \ln 3 + \frac{390625}{18} \ln 5. \end{aligned} \quad (2.15f)$$

The numerical values of the non-logarithmic coefficients are

$$a_5^c = 23.50338\,92426\,03436\,23875\,76146 \dots, \quad (2.16a)$$

$$\bar{d}_4^c = 221.57199\,11921\,48164\,02323\,37716 \dots, \quad (2.16b)$$

$$q_{4,3} = 28.71104\,42849\,55949\,75749\,69412 \dots, \quad (2.16c)$$

$$q_{6,2} = -2.78300\,76369\,52232\,48902\,84545 \dots. \quad (2.16d)$$

All these 1SF-order coefficients have been independently checked, either analytically, or numerically, by various SF computations (which used the first law of binary dynamics as a bridge). We have already mentioned above the theoretical consensus on the 4PN logarithmic contributions  $a_5^{\text{ln}}$  and  $\bar{d}_4^{\text{ln}}$ . [Note the absence of any logarithmic contribution to  $Q_{4\text{PN}}(u, p_r)$ .] The non-logarithmic contribution  $a_5^c$  to the main EOB radial potential  $A(u)$  is actually not a deep check of the ADM results (2.1), (2.5) because Ref. [16] used the analytical SF result of Bini and Damour [13] to calibrate their single IR ambiguity constant  $C$ . Let us, however, note two things. First, all the transcendental contributions to  $a_5^c$  have been reproduced by the local ADM computation [12, 14], the needed calibration of  $C$  using only a rational shift. Second, the analytical determination of  $a_5^c$  in Ref. [13] (which was preceded by accurate numerical estimations [7, 9]) has been later analytically confirmed in [36] and [37], as well as numerically confirmed by extremely high accuracy self-force results [38].

The non-logarithmic 1SF 4PN contribution  $\bar{d}_4^c$  to the second EOB potential  $\bar{D}(u)$  has been fully confirmed (again using the first law of mechanics) by recent SF works, both analytically and numerically. As already mentioned, direct analytical checks have been obtained in Refs. [31], and [32]. In addition, Ref. [31] has pointed out that the recent numerical SF results of Ref. [39] provide a numerical check on the value  $\bar{d}_4^c$  at the accuracy level  $\pm 0.05$ . [The latter error level takes into account the fact that the  $a_5$  terms have been fully confirmed.]

The 1SF 4PN contributions to  $Q(r, p_r)$  have also been recently confirmed, both analytically [32] and numerically, with an uncertainty  $\delta q_{4,3} = \pm 4$  [39] (see also the recent numerical determination of the coefficient  $q_4(u)$  of  $\nu p_r^4$  in the EOB  $Q$  potential [40]).

Let us finally note that the terms proportional to higher powers of  $\nu$  in (2.6) are much less sensitive than the terms of order  $\nu$  to subtle regularization ambiguities. Actually, as shown in Refs. [10, 12] the regularization subtleties are in direct correspondence with the power of  $\nu$ . The terms of order  $\nu^3$  and  $\nu^4$  are not ambiguous at all (and have been independently derived, in the effective-field-theory approach, for the relation between energy and orbital frequency for circular orbits, in Ref.

[11]), while the terms of order  $\nu^2$  are delicate, but can be unambiguously derived when using dimensional regularization for treating the UV divergences [12]. Actually, as explicitly shown in [21] (see also next section), the recent harmonic-coordinates Fokker-action computation of Ref. [21] agrees (modulo some contact transformation) with the action (2.1) for all powers of  $\nu$ , *except* for the first power. We think that this is related to the fact that the most delicate IR effects are linked with the nonlocal contribution (2.2), which is easily seen to be purely of order  $\nu^1$  (in  $S/M\mu$ ). [This is also explicitly displayed in Eqs. (7.5)–(7.7) of [17].] As we shall further discuss below, physical effects mixing local and nonlocal effects, and thereby being sensitive to IR divergences, are very delicate to determine unambiguously. We can, however, conclude that the non IR-sensitive part of the results of [21] provide an independent confirmation of all the terms in the action (2.1) which are of order  $\nu^n$  with  $n \geq 2$ .

Summarizing this Section: Many independent SF results have confirmed all the (IR-sensitive) terms *linear in  $\nu$* , while the other PN calculations at the 4PN level, Refs. [11] and, especially, Ref. [21], have confirmed all the terms *nonlinear in  $\nu$*  (i.e.  $\propto \nu^2, \nu^3$ , and  $\nu^4$ ). We conclude that all the results of [16, 17] have been (piecewise) confirmed.

### III. INCOMPATIBILITIES BETWEEN THE 4PN FOKKER ACTION RESULTS OF [21] AND SELF-FORCE RESULTS

The simplest way to compare the results of the recent Fokker action computation [21] to self-force data is to compute the SF effects induced by the *difference* between the dynamics of Ref. [21], and that of Refs. [16, 17]. This difference has been worked out in [21].

Reference [21] obtained, after applying a suitable contact transformation (alluded to, though not explicitly presented, at the beginning of their Sec. V B) to their original harmonic-coordinates result an action of the same form as the ADM results (2.1), namely

$$S_{\text{B}^3\text{FM}} = \int \left[ \sum_a p_{ai} dx_a^i - H_{\text{B}^3\text{FM}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; s) dt \right] + S^{\text{nonloc}}(s), \quad (3.1)$$

with the same nonlocal (or “tail”) action<sup>5</sup>, but with a different local Hamiltonian.

$$H_{\text{B}^3\text{FM}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; s) = H_{\text{B}^3\text{FM}}^{\text{loc}}(\mathbf{x}_a, \mathbf{p}_a; r_{12}) + F[\mathbf{x}_a, \mathbf{p}_a] \ln \frac{r_{12}}{s}, \quad (3.2)$$

<sup>5</sup> We use here the fact that, at the 4PN level,  $M_{\text{B}^3\text{FM}} = M_{\text{DJS}} + O(1/c^2)$  with  $M_{\text{DJS}} \equiv M^{\text{here}} = m_1 + m_2$ .

[Ref. [21] does not explicitly display the  $\ln s$  dependence of the local Hamiltonian but agrees with [16] on the cancellation of that dependence.] For simplicity, we do not discuss here the issue of the order-reduction of the derivatives of  $\mathbf{x}_a$  and  $\mathbf{p}_a$  entering (via  $I_{ij}^{(3)}$ ) both the nonlocal action (2.2) and the coefficient  $F$  of the logarithm, Eq. (2.4). Indeed, there is complete agreement, at the level of the action, between [16] and [21] for what concerns the nonlocal piece of the action. [As said in [16], the order reduction of  $I_{ij}^{(3)}$  (i.e. its on-shell replacement by a local function of  $\mathbf{x}_a(t)$  and  $\mathbf{p}_a(t)$ ) entails a suitable nonlocal shift of the dynamical variables (which was explicated in Eqs. (5.14), (5.15) of [21]).] The important issue is the difference between the two local Hamiltonians which, according to Eq. (5.19) of [21], is

$$H_{\text{B}^3\text{FM}}^{\text{loc}} - H_{\text{DJS}}^{\text{loc}} = \frac{G^4 M m_1^2 m_2^2}{c^8 r_{12}^4} \times \left[ a_{\text{B}^3\text{FM}} \left( \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \right)^2 + b_{\text{B}^3\text{FM}} \left( \frac{\mathbf{p}_1}{m_1} - \frac{\mathbf{p}_2}{m_2} \right)^2 + c_{\text{B}^3\text{FM}} \frac{GM}{r_{12}} \right], \quad (3.3)$$

with

$$(a, b, c)_{\text{B}^3\text{FM}} = \left( \frac{1429}{315}, \frac{826}{315}, \frac{902}{315} \right). \quad (3.4)$$

Before discussing further the origin of the discrepancy [i.e. the fact that  $(a, b, c)_{\text{B}^3\text{FM}} \neq (0, 0, 0)$ ], we wish to emphasize two things: (i) the discrepant terms in Eq. (3.3) represent only three (Galileo-invariant) terms among the hundreds (exactly 219) of contributions to the (non center-of-mass) two-body Hamiltonian, as displayed in the Appendix of [16], or in Sec. VIII E of [14]; and (ii) the three discrepant terms (3.3) (when expressed in the center of mass, and in reduced variables, i.e. in the sense of (2.6)) are *linear* in the symmetric mass ratio  $\nu$ , i.e. there are of 1SF order.

The discrepancy Eq. (3.3) therefore implies 1SF-detectable effects away from all the 1SF checks of the ADM dynamics (2.1) reviewed in the previous Section. Let us now quantify the corresponding 1SF differences. To do that it is convenient to transcribe the 4PN-level Hamiltonian difference (3.3) in terms of corresponding differences in the three EOB potentials  $A, \bar{D}, Q$ . [We recall in passing that the EOB parametrization of the dynamics in terms of  $A, \bar{D}, Q$  is completely gauge-fixed, and therefore directly linked to gauge-invariant quantities.]

Considering, for more generality, a 4PN-level Hamiltonian difference with general coefficients  $(a, b, c)$  in Eq. (3.3), the corresponding additional contributions to  $A, \bar{D}, Q$  are easily found to be

$$\delta^{a,b,c} A = (2b + 2c) \nu u^5, \quad (3.5a)$$

$$\delta^{a,b,c}\bar{D} = (2a + 8b)\nu u^4, \quad (3.5b)$$

$$\delta^{a,b,c}Q = 0. \quad (3.5c)$$

Note in passing that those contributions are invariant under the ‘‘gauge transformation’’

$$\delta^g(a, b, c) = g(4, -1, 1), \quad (3.6)$$

which corresponds to the most general canonical transformation respecting the structure (3.3) (with generating function  $\propto g\nu p_r/r^3$ ). Inserting the values (3.4) in Eqs. (3.5) leads to changes in  $A$  and  $\bar{D}$  (with respect to the values computed in [17])

$$\delta_{4\text{PN}}^{\text{B}^3\text{FM}} A = \frac{384}{35}\nu u^5, \quad (3.7)$$

$$\delta_{4\text{PN}}^{\text{B}^3\text{FM}} \bar{D} = \frac{9466}{315}\nu u^4. \quad (3.8)$$

In other words, the only 1SF 4PN-level coefficients that are different are  $a_5^c$  and  $\bar{d}_4^c$  with

$$\delta^{\text{B}^3\text{FM}} a_5^c = \frac{384}{35} \approx 10.97143, \quad (3.9)$$

$$\delta^{\text{B}^3\text{FM}} \bar{d}_4^c = \frac{9466}{315} \approx 30.05079. \quad (3.10)$$

Such large, 4PN-level deviations away from the results of [17] are in violent contradiction with the many SF confirmations of the ADM 4PN dynamics reviewed in the previous section. They have been obtained here by using the Delaunay-like reduction used in [17] to convert the nonlocal dynamics (2.1) into the (formally) local one (2.5). The authors of [21] express doubts about some aspects of the results of [17]. We shall address their concerns in the following Sections and conclude that their concerns are unsubstantiated. Therefore we conclude that taking the results of [21] at face value does lead to the large changes (3.9), (3.10) above, and are therefore strictly incompatible with extant SF knowledge.

We wish to go further and point out an even more blatant contradiction with SF tests of periastron precession. First, we note (using, e.g. Eq. (8.3) in [17]) that the changes above entail a corresponding change in the 4PN-level precession coefficient  $\rho_4^c$  given by  $\delta\rho_4^c = 10\delta a_5^c + \delta\bar{d}_4^c$ , i.e.

$$\delta^{\text{B}^3\text{FM}} \rho_4^c = (2a + 28b + 20c)_{\text{B}^3\text{FM}}, \quad (3.11)$$

i.e.

$$\delta^{\text{B}^3\text{FM}} \rho_4^c = \frac{44026}{315} \approx 139.7650794. \quad (3.12)$$

As we recalled above, SF computations [34, 35] of the precession function  $\rho(u)$  [6], and of its PN coefficients, does not rely on the first law of binary mechanics, but involve direct computations of the additional precession induced by the nonlocal self-force. The difference (3.12) is therefore (assuming the correctness of the Delaunay

reduction of [17]) a direct dynamical consequence of the Fokker-action result of [21]. It would be interesting to confirm this result by a purely dynamical computation of the (nonlocal) 4PN precession. We note here that the difference is already excluded by the ‘‘old’’ result (2.13), and even more so (by  $10^8$  standard deviations!) by the recent one (2.14).

#### IV. LOGICAL BASIS OF THE ACTION-ANGLE DELAUNAY-LIKE METHOD OF [17]

Independently of the above SF confirmations of the 4PN results of [16, 17], let us reassess the logical basis and the consistency of the methodology used in our Delaunay-EOB derivation, and indicate, by contrast, where, in our opinion, lie the flaws of Ref. [21] that have led to the non zero values (3.4), which are incompatible with many SF results. Our framework is the action-angle formulation of (planar) Hamiltonian dynamics. In [17] we used the standard Delaunay notation (modulo the use of calligraphic letters  $L \rightarrow \mathcal{L}$ ,  $G \rightarrow \mathcal{G}$ ). Namely,  $(\mathcal{L}, \ell; \mathcal{G}, g)$  are the two planar action-angle canonical pairs:  $\mathcal{L} = \sqrt{a}$  is conjugate to the mean anomaly angle  $\ell$ , while  $\mathcal{G} = \sqrt{a(1-e^2)}$  is conjugate to the argument of the periastron  $g = \omega$ . In order to better exhibit the meaning (and consistency) of our approach in the circular limit, we shall use here the combination of Delaunay variables introduced by Poincaré, namely, the two action-angle pairs  $(\Lambda, \lambda; I_r, \varpi)$  where (using suitably scaled variables as in [17])

$$\Lambda = \mathcal{L} = I_r + I_\varphi, \quad (4.1a)$$

$$\lambda = \ell + g, \quad (4.1b)$$

$$I_r = \mathcal{L} - \mathcal{G} = \mathcal{L} - I_\varphi, \quad (4.1c)$$

$$\varpi = -g = -\omega. \quad (4.1d)$$

Here,  $I_\varphi = \mathcal{G}$  is the angular momentum ( $I_\varphi = \frac{1}{2\pi} \oint p_\varphi d\varphi = p_\varphi$ ) and  $I_r$  the radial action ( $I_r = \frac{1}{2\pi} \oint p_r dr$ ). The sum  $\Lambda = I_r + I_\varphi$  is conjugate to the mean longitude  $\lambda = \ell + g$ , while  $I_r$  is conjugate to (minus) the argument of the periastron  $\varpi = -\omega$ . The latter (surprising) minus sign is necessary to have  $dL \wedge d\ell + dG \wedge dg = d\Lambda \wedge d\lambda + dI_r \wedge d\varpi$  with  $I_r = \mathcal{L} - \mathcal{G} = \sqrt{a} - \sqrt{a(1-e^2)} > 0$ . In the circular limit,  $\lambda$  becomes the usual polar angle  $\varphi$  in the orbital plane and  $I_r \rightarrow 0$ , so that the action variable  $\Lambda$  becomes equal to the angular momentum  $I_\varphi = p_\varphi$ . In this limit, the general Hamilton equation  $\dot{\lambda} = \partial H / \partial \Lambda$  gives back the usual circular link between the orbital frequency  $\Omega = \dot{\varphi}$  and the derivative of the energy with respect to the angular momentum  $p_\varphi$ . However, when dealing with non-circular orbits, and tackling the nonlocal action (2.2), it is important to work with the clearly defined canonical action-angle pairs  $(\Lambda, \lambda; I_r, \varpi)$ . In principle, it would be better, when discussing the circular limit, to replace the second pair  $(I_r, \varpi)$  by the associated Poincaré variables

$\xi = \sqrt{2I_r} \cos \varpi$ ,  $\eta = \sqrt{2I_r} \sin \varpi$ , because they are canonical ( $d\xi \wedge d\eta = dI_r \wedge d\varpi$ ) and regular in the circular limit (while  $\varpi$  becomes ill-defined as  $I_r \rightarrow 0$ ). Keeping in mind such an additional change of variables, we shall, however, find simpler to express our methodology in terms of  $I_r$  and  $\varpi$ .

The basic methodology we used in [17] for dealing with the nonlocal 4PN dynamics, Eqs. (2.1), (2.2), consists of four steps: (a) we reexpress the action (2.1) as a nonlocal action in the action-angle variables  $(\Lambda, \lambda; I_r, \varpi)$ ; (b) we expand it (formally to infinite order) in powers of the eccentricity, i.e. in powers of  $\sqrt{I_r}$ ; (c) we “order-reduce” the nonlocal dependence on the action-angle variables by using the on-shell equations of motion<sup>6</sup>; and (d) we eliminate, à la Delaunay, the periodic terms in the order-reduced Hamiltonian by a canonical transformation of the action-angle variables. After these four steps, we end up with an Hamiltonian which is an ordinary (local) function of the (transformed) action variables alone.

In order not to get distracted by irrelevant technicalities, let us illustrate this methodology [which was applied in [17] to the full 4PN action (2.1)–(2.2)] on a simpler toy example which contains some of the key ingredients of the action (2.1)–(2.2), namely the action

$$S_{\text{toy}} = \int [\mathbf{p}(t) \cdot d\mathbf{r}(t) - H_{\text{toy}}(t)dt], \quad (4.2)$$

$$H_{\text{toy}}(t) = \frac{1}{2}\mathbf{p}(t)^2 - \frac{1}{r(t)} + \varepsilon \int_{-\infty}^{+\infty} dt' \mu(t' - t) \frac{\mathbf{r}(t') \cdot \mathbf{r}(t)}{|\mathbf{r}(t')|^3 |\mathbf{r}(t)|^3}, \quad (4.3)$$

where  $\mu(\tau)$  is an even function of the time difference  $\tau \equiv t' - t$ . In the real case (2.1)–(2.2),  $\mu(\tau) = 1/|\tau|$  with an additional partie finie Pf prescription. The subtleties linked to the Pf prescription and to the slow decay of  $1/|\tau|$  for  $|\tau| \rightarrow +\infty$  are only of technical nature. For instance, the oscillatory nature of the integrand  $I_{ij}^{(3)} I_{ij}^{(3)}$  (mimicked as  $x'_i x_i / (r'^3 r^3)$  in our toy example) ensures the large  $|\tau|$  convergence<sup>7</sup> of the nonlocal integral (2.2) without having to assume that  $I_{ij}^{(3)}$  tends to zero when  $|\tau| \rightarrow +\infty$ . [Indeed, the fact that  $-x_i/r^3$  is the on-shell value of the second time-derivative of  $x_i$ , and  $I_{ij}^{(3)}$  the third time-derivative of  $I_{ij}$ , ensure that their large-time averages both vanish for conservative bound motions. Note that our toy model is the “electromagnetic” (dipolar) analog of the gravitational (quadrupolar) tail action.] For simplicity, as we wish here to emphasize issues of principle without getting bogged down by secondary technical issues, we will not specify the weight

<sup>6</sup> As discussed in detail below, this is equivalent to applying suitable nonlocal shifts of the phase-space variables.

<sup>7</sup> We have convergence both at  $\tau \rightarrow +\infty$  and  $\tau \rightarrow -\infty$ , without having absolute convergence.

$\mu(\tau)$  used in our toy model, but proceed as if it were a smooth, integrable *even* function of  $\tau$ .

The first two steps, (a) and (b), of our procedure yield (when considered, for illustration, at linear order in the eccentricity  $e$ )

$$H_{\text{toy}}(t) = -\frac{1}{2\Lambda^2} + \varepsilon \int_{-\infty}^{+\infty} d\tau \mu(\tau) \frac{1}{\Lambda^4 \Lambda'^4} \left[ \cos(\lambda' - \lambda) + 2e \cos(2\lambda - \lambda' + \varpi) + 2e' \cos(2\lambda' - \lambda + \varpi') + O(e^2) \right]. \quad (4.4)$$

Here and below  $f'$  denotes the variable  $f$  taken at time  $t' = t + \tau$ , while  $f$  denotes its value at time  $t$ .

Step (c) consists in writing, for any variable  $f$ , its value  $f'$  at the shifted time  $t'$  in terms of canonical variables at time  $t$  and of an integral over intermediate times involving the rhs of the (Hamiltonian) equations of motion. For a nonlocal action, the latter read  $S_\lambda(t) = 0$ ,  $S_\Lambda(t) = 0$ , etc., with

$$S_\lambda(t) \equiv \frac{\delta S}{\delta \lambda(t)} \equiv -\dot{\Lambda}(t) - \frac{\delta H}{\delta \lambda(t)}, \quad (4.5a)$$

$$S_\Lambda(t) \equiv \frac{\delta S}{\delta \Lambda(t)} \equiv \dot{\lambda}(t) - \frac{\delta H}{\delta \Lambda(t)}, \quad \text{etc.}, \quad (4.5b)$$

where  $\delta/\delta f(t)$  denotes a *functional* derivative (acting on the action  $S$  or on  $\int dt H^{\text{nonlocal}}(t)$ ). In the normal case of local actions [44, 45], the use of the identities (4.5a), (4.5b) is the main tool allowing one to show how the replacement of the equations of motion within an action is equivalent to a suitable shift of the dynamical variables (a “field redefinition”). In the case of nonlocal actions, one must integrate the identities (4.5a), (4.5b) before replacing  $\lambda' \equiv \lambda(t+\tau)$ ,  $\Lambda' \equiv \Lambda(t+\tau)$ ,  $\dots$  in the nonlocal piece of the action. In this integration, the terms  $S_\lambda(t)$ ,  $S_\Lambda(t)$ ,  $\dots$  are treated as additional terms, which would be zero on-shell, but which are now considered as “source terms” on the rhs of the usual Hamilton equations of motion,  $\dot{\lambda}(t) - \delta H/\delta \Lambda(t) = S_\Lambda$ ,  $\dots$ . In view of the well-known possibility of neglecting “double-zero terms”, it is enough to work *linearly* in these (simple-zero) source terms. Moreover, as we are working within a PN-expanded scheme (and as the nonlocality enters only at order  $\varepsilon = 1/c^8$ ), it is actually enough (modulo terms of order  $O(c^{-2}\varepsilon)$ ), for the purpose of replacing  $\lambda'$ ,  $\Lambda'$ ,  $\dots$  in the nonlocal piece of the action, to compute them as solutions of the (sourced) *Newtonian-level* equations of motion, i.e. solutions of the differential equations (4.5a), (4.5b) with  $\delta H/\delta \Lambda, \dots \rightarrow \delta H_0/\delta \Lambda, \dots$ , where  $H_0 = -1/(2\Lambda^2)$ . The explicit form of these sourced equations of motion are

$$\dot{\Lambda}(t) = -S_\lambda(t), \quad (4.6a)$$

$$\dot{\lambda}(t) = \frac{1}{\Lambda^3(t)} + S_\Lambda(t), \quad \text{etc.} \quad (4.6b)$$

Viewing these equations as differential equations with respect to  $t' = t + \tau$  (or  $\tau$ ) for the unknowns  $\Lambda' \equiv \Lambda(t')$  and

$\lambda' \equiv \lambda(t')$ , and imposing the initial conditions at  $t' = t$  (i.e.  $\tau = 0$ ),  $\Lambda'(t' = t) = \Lambda$ ,  $\lambda(t' = t) = \lambda$ , yields as unique solution (to linear order in the source terms)

$$\Lambda' = \Lambda - \int_t^{t'} dt_1 S_\Lambda(t_1), \quad (4.7a)$$

$$\begin{aligned} \lambda' = \lambda + \frac{t' - t}{\Lambda^3} + \frac{3}{\Lambda^4} \int_t^{t'} dt_1 (t' - t_1) S_\lambda(t_1) \\ + \int_t^{t'} dt_1 S_\Lambda(t_1). \end{aligned} \quad (4.7b)$$

[We do not assume here  $t < t'$  but use the convention that  $\int_t^{t'} = -\int_{t'}^t$ .] When replacing the identities (4.7) in a nonlocal action  $S = \int dt dt' \mu(t' - t) \mathcal{S}(\Lambda, \Lambda', \lambda' - \lambda, \dots)$  (keeping only the terms linear in the source terms, which is allowed modulo “double-zero” terms), the extra terms involving the source contributions will have the form  $\int dt_1 [\xi_\Lambda(t_1) S_\Lambda(t_1) + \xi_\lambda(t_1) S_\lambda(t_1) + \dots]$ , where the quantities  $\xi_\Lambda(t_1), \xi_\lambda(t_1), \dots$  are given by integrals over  $t$  and  $t'$  of a function of dynamical variables:  $\xi_\Lambda(t_1) = \iint dt dt' \mu(t' - t) \mathcal{A}(t' - t, t_1 - t, \Lambda(t), \lambda(t)), \dots$ . [Note that, because of the inequalities  $t < t_1 < t'$  or  $t' < t_1 < t$ , for given values of  $t_1$  and  $\tau = t' - t$ ,  $t$  and  $t'$  both range over a bounded interval.] The extra terms involving the source contributions can then be “field-redefined away” by corresponding shifts of the dynamical variables:  $\delta\Lambda(t_1) = \xi_\Lambda(t_1), \delta\lambda(t_1) = \xi_\lambda(t_1), \dots$ . [These shifts have, in general, nonlocal structures of the type  $\delta\Lambda(t_1) = \int dt K_\Lambda(t - t_1, \Lambda(t), \lambda(t)), \delta\lambda(t_1) = \int dt K_\lambda(t - t_1, \Lambda(t), \lambda(t)), \text{ etc.}$ ]

After performing these (nonlocal) shifts, we are left with an action obtained by replacing  $\lambda', \Lambda', \dots$  by the rhs of (4.7) in which one has set to zero the terms  $S_\lambda(t_1), S_\Lambda(t_1), \dots$ . In other words, the use of these shifts justifies the naive replacement of  $\lambda', \Lambda', \dots$  by the solution of the (Newtonian) equations of motion, namely

$$\lambda' = \lambda + \frac{\tau}{\Lambda^3}, \quad (4.8a)$$

$$\Lambda' = \Lambda. \quad (4.8b)$$

Note that the very simple time structure of the unperturbed solution in action-angle variables plays a very useful role in our Delaunay-based reduction procedure.

With the replacements (4.8a), (4.8b) [i.e. after steps (a), (b), and (c)] we have an Hamiltonian of the form

$$\begin{aligned} H'_{\text{toy}}(t) = -\frac{1}{2\Lambda^2} + \varepsilon \int_{-\infty}^{+\infty} d\tau \mu(\tau) \frac{1}{\Lambda^8} \left\{ \cos\left(\frac{\tau}{\Lambda^3}\right) \right. \\ \left. + 2e \left[ \cos\left(\frac{\tau}{\Lambda^3} - \lambda - \varpi\right) \right. \right. \\ \left. \left. + \cos\left(\frac{2\tau}{\Lambda^3} + \lambda + \varpi\right) \right] + O(e^2) \right\}, \end{aligned} \quad (4.9)$$

where we recall that  $e$  is the function of  $I_r/\Lambda$  defined so that  $1 - \sqrt{1 - e^2} = I_r/\Lambda$ , i.e.

$$e = \sqrt{1 - \left(1 - \frac{I_r}{\Lambda}\right)^2}. \quad (4.10)$$

At this stage, Eq. (4.9) defines an ordinary Hamiltonian, expressed in terms of action-angle variables  $(\Lambda, \lambda; I_r, \varpi)$ . [Strictly speaking, the variables  $\Lambda, \lambda$ , etc. at the stage of Eq. (4.9) differ from the ones entering the original action by the shifts  $\xi_\Lambda, \xi_\lambda$ , etc. mentioned above. They should be denoted as  $\Lambda^{\text{shifted}}, \lambda^{\text{shifted}}$ , etc., but, for simplicity, we do not indicate the shifted nature of  $\Lambda, \lambda, \dots$ ]

As we are now dealing with an ordinary, local Hamiltonian, our fourth step (d), i.e. the Delaunay elimination of periodic terms by suitable canonical transformations is standard. For instance, a generating function proportional to

$$\varepsilon \int_{-\infty}^{+\infty} d\tau \mu(\tau) \frac{1}{\Lambda^5} \sin\left(\frac{\tau}{\Lambda^3} - \lambda - \varpi\right) \quad (4.11)$$

will eliminate the contribution  $+2e \cos(\tau/\Lambda^3 - \lambda - \varpi)$  on the rhs of Eq. (4.9).

After completing step (d) (formally to all orders in the eccentricity) we end up with an ordinary (local) Hamiltonian that depends *only on (shifted) action variables*. E.g., for our toy model

$$\begin{aligned} H''_{\text{toy}}(\Lambda, I_r) = -\frac{1}{2\Lambda^2} + \varepsilon \int_{-\infty}^{+\infty} d\tau \mu(\tau) \frac{1}{\Lambda^8} \\ \times \left[ \cos\left(\frac{\tau}{\Lambda^3}\right) + O(e^2) \right]. \end{aligned} \quad (4.12)$$

Let us emphasize that our procedure allows one, in particular, to construct (formally to *all* orders in eccentricity) a *strictly conserved energy* that consistently includes the nonlocal tails at the 4PN level. [The nonlocal effects are recognizable through their integral nature, e.g. the integral  $\varepsilon \int_{-\infty}^{+\infty} d\tau \mu(\tau) (\dots)$  in (4.12).] Namely, our final, Delaunay Hamiltonian is strictly conserved. This shows that the contrary statement below Eq. (5.17) in [21] is incorrect (at least when allowing for an expansion in powers of  $I_r/\Lambda \sim e^2$ ).

In addition, our construction also gives two conserved action variables:  $\Lambda$  and  $I_r$ . These conserved quantities are such that the difference,  $\Lambda - I_r$  is conserved and reduces to the usual orbital angular momentum  $I_\varphi = \frac{1}{2\pi} \oint p_\varphi d\varphi = p_\varphi$  in the local case. We can then consider that  $I_\varphi := \Lambda - I_r$  defines a *strictly conserved angular momentum* that consistently includes the 4PN nonlocal dynamical effects. [Indeed, one easily checks that the rotational symmetry  $\varphi' = \varphi + \text{const}$  remains a symmetry at all stages of our construction so that  $I_\varphi$  is indeed the Noether conserved quantity associated with this symmetry.]

Let us also emphasize another consistency feature of our final (reduced, local, Delaunay) Hamiltonian. As the above detailed derivation clearly shows, our final Hamiltonian action (in terms of  $\Lambda^{\text{shifted}}$ ,  $\lambda^{\text{shifted}}$ , ... but without explicitly indicating the shifts) is simply

$$S'' = \int [\Lambda d\lambda + I_r d\varpi - H''(\Lambda, I_r) dt], \quad (4.13)$$

and the corresponding equations of motion are the usual ones:

$$\dot{\lambda} = \frac{\partial H''(\Lambda, I_r)}{\partial \Lambda}, \quad \dot{\varpi} = \frac{\partial H''(\Lambda, I_r)}{\partial I_r}, \quad \dot{\Lambda} = 0, \quad \dot{I}_r = 0. \quad (4.14)$$

We said above that, in the circular limit, the mean anomaly  $\lambda = \ell + g = \ell + \omega$  reduces to a usual polar angle  $\varphi$ . [Indeed, this is true in the Newtonian case, and, as all the Delaunay shifts are periodic in the unique *fast* angular variable  $\lambda$ , we have that  $\varphi = \lambda +$  terms periodic in  $\lambda$  that vanish with  $e$ .] Therefore our derivation shows, among other things, that the orbital frequency  $\Omega (= \dot{\lambda})$  of a circular ( $I_r = 0$ ) binary (with conserved angular momentum  $\Lambda = I_\varphi$ ) is given by the usual formula

$$\Omega = \dot{\lambda} = \left. \frac{\partial H''(\Lambda, I_r)}{\partial \Lambda} \right|_{I_r=0} = \left. \frac{\partial H''(I_r + I_\varphi, I_r)}{\partial I_\varphi} \right|_{I_r=0}. \quad (4.15)$$

In particular, this means, in our toy example, that the circular angular frequency is of the form

$$\begin{aligned} \Omega_{\text{toy}} &= \frac{\partial H''_{\text{toy}}(\Lambda, 0)}{\partial \Lambda} \\ &= \frac{1}{\Lambda^3} + \varepsilon \frac{\partial}{\partial \Lambda} \left( \int_{-\infty}^{+\infty} d\tau \mu(\tau) \frac{1}{\Lambda^8} \cos\left(\frac{\tau}{\Lambda^3}\right) \right). \end{aligned} \quad (4.16)$$

One sees that the computation of the rhs will involve two type of contributions coming from the tail term: a “normal” contribution where the  $\Lambda$ -differentiation of the second term acts on the prefactor  $1/\Lambda^8$  of the tail integral, and a “new” contribution where  $\partial/\partial\Lambda$  acts on the argument of  $\cos(\tau/\Lambda^3)$ .

We recall that, in the full 4PN case, the analog of our final result (4.12) (say, for simplicity, at the  $O(e^0)$  level) is the one discussed in Sec. V of [16], which leads to (with  $J$  now denoting  $I_\varphi \equiv \Lambda - I_r$ )

$$\begin{aligned} E_{4\text{PN}}^{\text{circ}}(J) &= H_{\text{DJS}}^{\text{loc}}(\mathbf{r}, \mathbf{p}; s) + F(\mathbf{r}, \mathbf{p}) \ln \frac{r}{s} \\ &\quad - \frac{1}{5} \frac{G^2 M}{c^8} \text{Pf}_{2s/c} \int_{-\infty}^{+\infty} \frac{d\tau}{|\tau|} f(\tau), \end{aligned} \quad (4.17)$$

where  $f(\tau) := [I_{ij}^{(3)}(t+\tau) I_{ij}^{(3)}(t)]^{\text{circ}}$ , as in Sec. V of [16]. [In the circular limit  $f(\tau)$  does not depend on  $t$ .] The explicit value of the last, tail term<sup>8</sup> is given, in view of

<sup>8</sup> Here, we find convenient to work with a tail term defined with

Eqs. (5.7)–(5.9) in [16] by

$$\begin{aligned} [H^{\text{tail}(s)}]^{\text{circ}}(J) &= \frac{64}{5} \frac{G^2 M}{c^8} (\mu \Omega^3 r_{12}^2)^2 \ln \left( 4e^{\gamma_E} \frac{\Omega(J)s}{c} \right) \\ &= \frac{64}{5} \mu c^2 \frac{\nu}{j^{10}} \ln \left( \frac{4e^{\gamma_E} \hat{s}}{j^3} \right), \end{aligned} \quad (4.18)$$

where  $j \equiv cJ/(GM\mu)$ , and  $\hat{s} \equiv s/m$  with  $m \equiv GM/c^2$ .

If we alternatively decide to define the tail contribution to the reduced Hamiltonian by incorporating the term  $F \ln(r/s)$  in it, we must simply replace the scale  $s$  by  $r = r_{12}$ , so that (using the fact that along Newtonian circular orbits  $r^{\text{circ}} = mj^2$ )

$$[H^{\text{tail}(s=r_{12})}]^{\text{circ}}(J) = \frac{64}{5} \mu c^2 \frac{\nu}{j^{10}} \ln \left( \frac{4e^{\gamma_E}}{j} \right). \quad (4.19)$$

Evidently, we must keep in mind that the meaning of the two quantities (4.18) and (4.19) is different and that their  $J$ -derivative will differ by a term involving the  $J$ -derivative of  $r^{\text{circ}} = mj^2$ . But the latter difference has a conceptually trivial origin (in view of the *local* nature of the  $F \ln(r/s)$  term in the Hamiltonian) and does not interfere with the issue we wish to emphasize here, namely the existence, in the orbital frequency,  $\Omega^{\text{circ}} = dH^{\text{circ}}/dJ$ , of a contribution coming from differentiating the  $J$ -dependence of the argument  $4e^{\gamma_E} \Omega(J)s/c$  of the logarithm in the first (or second) Eq. (4.18).

As clearly pointed out already in [16], the computation of the tail contribution to the (circular, Delaunay) function  $H(J) = E(J)$  “involves the evaluation of the non-local piece  $[H_{4\text{PN}}^{\text{tail}(s)}]$  along circular motion (without any differentiation)”. This lack of any differentiation in the evaluation of  $[H^{\text{tail}}]^{\text{circ}}(J)$  distinguishes it among the other gauge-invariant functions one can associate with the sequence of circular orbits, and rendered very clear<sup>9</sup> that this necessarily led to the correct value of the function  $[H^{\text{tail}}]^{\text{circ}}(J)$ . Our discussion above has confirmed the consistency of this result and has shown that, indeed, when computing the orbital frequency it is correct to  $J$ -differentiate the argument of the tail log.

Let us now show that part of the difference between our results and those of [21] has its origin in the fact that Bernard *et al.* [21] effectively treat the orbital frequency  $\Omega$  entering the tail contribution (4.18) as a constant under differentiation (instead of treating  $\Omega$  as a function of the angular momentum  $J$ ). To see this, let us start by deriving a useful general result about the functional relation

some given, constant scale  $s$ . Such a scale is not a dynamical variable and is not affected by the  $\Lambda$ -differentiation above, i.e. the  $f$ -differentiation in the present circular case.

<sup>9</sup> In view, e.g., of the result of Ref. [46] about the consistency of the order-reduction of higher-order Hamiltonians, see also Sec. III in [17] and below.

between energy and orbital frequency,  $E(\Omega)$ , or equivalently the function  $E(x)$  (where  $x \equiv (GM\Omega/c^3)^{2/3}$ ). Let us first consider the case an ordinary Hamiltonian dynamics of the type (with  $J \equiv p_\varphi$ )

$$H^{\text{ord}}(r, p_r, J) = H_0(r, p_r, J) + \varepsilon H_1(r, p_r, J). \quad (4.20)$$

When considering the sequence of circular orbits, it is well known<sup>10</sup> that, modulo  $O(\varepsilon^2)$ , one can replace from the start the radius  $r$  on the rhs by its *unperturbed* circular value as a function of  $J$ :  $r_0^{\text{circ}}(J)$ , defined as the solution of  $\partial H_0(r, p_r = 0, J)/\partial r = 0$  for a given  $J$ . [We are assuming here that the  $p_r$  dependence of  $H^{\text{ord}}$  starts at order  $p_r^2$  so that one can consistently set  $p_r = 0$  before considering the  $J$  dependence.] This yields for the circular-reduced dependence of  $H^{\text{ord}}$  as a function of  $J$ :

$$H_{\text{circ}}^{\text{ord}}(J) = H_0^{\text{circ}}(J) + \varepsilon H_1^{\text{circ}}(J) + O(\varepsilon^2), \quad (4.21)$$

where  $H_0^{\text{circ}}(J) = H_0(r_0^{\text{circ}}(J), p_r = 0, J)$  and  $H_1^{\text{circ}}(J) = H_1(r_0^{\text{circ}}(J), p_r = 0, J)$ . Note that the  $O(\varepsilon)$  contribution to  $H_{\text{circ}}^{\text{ord}}(J)$  is directly equal to the zeroth-order-circular value of the original  $O(\varepsilon)$  contribution to the Hamiltonian (4.20) [without any contribution from  $\varepsilon r_1^{\text{circ}}(J)$ , and without any differentiation]. Furthermore, the on-shell vanishing (along exact circular orbits) of  $\partial H^{\text{ord}}/\partial r$  also ensures that the circular-reduced  $J$ -dependence of the orbital frequency  $\Omega = \partial H^{\text{ord}}(r, p_r, J)/\partial J$  is simply given by  $J$ -differentiation of  $H_{\text{circ}}^{\text{ord}}(J)$ :

$$\Omega^{\text{circ}}(J) = \frac{dH_{\text{circ}}^{\text{ord}}(J)}{dJ} = \Omega_0^{\text{circ}}(J) + \varepsilon \frac{dH_1^{\text{circ}}(J)}{dJ} + O(\varepsilon^2), \quad (4.22)$$

where  $\Omega_0^{\text{circ}}(J) = dH_0^{\text{circ}}(J)/dJ$  is its unperturbed value.

From the two results (4.21), (4.22), it is easy to derive the perturbed value of the function  $E^\Omega(\Omega) \equiv H_{\text{circ}}^{\text{ord}}(J^{\text{circ}}(\Omega))$ , where  $J^{\text{circ}}(\Omega)$  is the inverse of the function  $\Omega^{\text{circ}}(J)$ . One gets

$$E^\Omega(\Omega) = E_0^\Omega(\Omega) + \varepsilon E_1^\Omega(\Omega) + O(\varepsilon^2), \quad (4.23)$$

with

$$E_1^\Omega(\Omega) = \left[ H_1^{\text{circ}}(J) - \frac{dE_0^\Omega}{d\Omega} \frac{dH_1^{\text{circ}}(J)}{dJ} \right]_{J=J_0^{\text{circ}}(\Omega)}, \quad (4.24)$$

where  $J_0^{\text{circ}}(\Omega)$  is the inverse of the function  $\Omega_0^{\text{circ}}(J)$ .

Let us apply the convenient result (4.24) to the  $r_{12}$ -scale tail perturbation written as

$$\varepsilon H_1^{\text{tail}}(r_{12}) = \frac{64}{5} \mu c^2 \frac{\nu}{j^{10}} \ln \left( \frac{4e^{7E} \Omega r_0^{\text{circ}}(J)}{c} \right), \quad (4.25)$$

where we leave open for the moment the functional dependence of  $\Omega$  within the tail logarithm. In our case

$E_0^\Omega = -\frac{1}{2} \mu c^2 \widehat{\Omega}^{2/3}$  and  $j_0^{\text{circ}}(\Omega) = \widehat{\Omega}^{-1/3}$  (where  $\widehat{\Omega} \equiv m\Omega$ ), so that the general result (4.24) yields the simple expression

$$E_1^\Omega(\Omega) = \left( 1 + \frac{1}{3} j \frac{d}{dj} \right) H_1^{\text{circ}}(j), \quad (4.26)$$

which now involves a crucial  $j$ -derivative.

Applying Eq. (4.26) to Eq. (4.25) yields

$$E_{r_{12}\text{-tail}}^\Omega(\Omega) = -\frac{448}{15} \mu c^2 \nu x^{10} \left[ \ln \left( \frac{4e^{7E} \Omega r_0^{\text{circ}}(j)}{c} \right) - \frac{1}{7} \frac{d \ln \Omega}{d \ln j} - \frac{1}{7} \frac{d \ln r_0^{\text{circ}}(j)}{d \ln j} \right], \quad (4.27)$$

where (on-shell)  $\ln \left( \frac{4e^{7E} \Omega r_0^{\text{circ}}(j)}{c} \right) = \ln(4e^{7E} x^{\frac{1}{2}})$ . As  $r_0^{\text{circ}}(j) = mj^2$ , the contribution of the last term<sup>11</sup> in the bracket is  $-2/7$ , which corresponds to the result claimed in [21] [see Eq. (5.30) there; note that in Eq. (4.27) we factored 448/15, instead of the 224/15 they factor in their Eq. (5.30), to better exhibit the role of  $\ln(\Omega r_0)$ ]. By contrast, our Delaunay-based method led us to have  $\widehat{\Omega} = \widehat{\Omega}_0^{\text{circ}}(j) = j^{-3}$  in Eq. (4.25) (see notably Eq. (4.11) in [17]), which yields the additional term

$$-\frac{1}{7} \frac{d \ln \Omega_0^{\text{circ}}(j)}{d \ln j} = +\frac{3}{7}, \quad (4.28)$$

so that the bracket in (4.27) becomes

$$\ln \left( \frac{4e^{7E} \Omega r_0}{c} \right) + \frac{3}{7} - \frac{2}{7} = \ln \left( \frac{4e^{7E} \Omega r_0}{c} \right) + \frac{1}{7}. \quad (4.29)$$

The shift  $+\frac{3}{7}$ , Eq. (4.28), connecting the result  $\ln \left( \frac{4e^{7E} \Omega r_0}{c} \right) - \frac{2}{7}$  of [21] to our result [16, 17], namely  $\ln \left( \frac{4e^{7E} \Omega r_0}{c} \right) + \frac{1}{7}$ , Eq. (4.29), is the basis of the claim of Ref. [21] that our value of the constant  $C$  should be changed into  $C + \frac{3}{7}$  [see Eq. (5.31) in [21]].

We disagree with this conclusion of [21] because of the Delaunay-based logic, recalled above, that unescapably leads to the result (4.29) of [16, 17]. However, while preparing this work, we understood what was the logical basis of the claim of [21] that  $\Omega$  should be treated as a constant. See next section.

## V. SUBTLITIES IN THE DEFINITION OF THE ENERGY IN PRESENCE OF NONLOCAL EFFECTS

In this Section we discuss subtle aspects of the order-reduction of nonlocal actions which are automatically incorporated within our approach, but which can easily

<sup>10</sup> The on-shell vanishing of  $\partial H^{\text{ord}}/\partial r$  ensures that when replacing the exact solution  $r_{\text{exact}}^{\text{circ}}(J) = r_0^{\text{circ}}(J) + \varepsilon r_1^{\text{circ}}(J)$ , the  $\varepsilon r_1^{\text{circ}}(J)$  perturbation will only contribute at order  $O(\varepsilon^2)$ .

<sup>11</sup> Note, in passing, that if we were considering the  $s$ -scaled tail (with a fixed scale  $s$ ) the latter term would be  $-\frac{1}{7} d \ln s / d \ln j = 0$ .

lead to apparent paradoxes and/or errors when trying to define a conserved energy along circular orbits.

It will be convenient to use an even simpler toy model than the one used above to explain these subtleties and establish the need of including the nonzero value (4.28) of the  $j$ -variation of the frequency  $\Omega$  entering the tail logarithm (4.25). Let us start (as above) with a nonlocal interaction described (in Poincaré-Delaunay variables) by a toy action of the form

$$S = \int [\Lambda d\lambda + I_r d\varpi - H^{\text{nonloc}}(t) dt], \quad (5.1)$$

with

$$H^{\text{nonloc}}(t) = -\frac{1}{2\Lambda^2} + \varepsilon \int d\tau \mu(\tau) \Lambda^m \Lambda^n [\cos(\lambda' - \lambda) + O(e)], \quad (5.2)$$

where, as above,  $\Lambda \equiv \Lambda(t)$ ,  $\Lambda' \equiv \Lambda(t') \equiv \Lambda(t + \tau)$ , etc.

Above, we explained how to completely reduce the nonlocality of this action (i.e. the dependence of  $H^{\text{nonloc}}(t)$  on dynamical variables at time  $t' \neq t$ ) to a purely local dependence. In this reduction the term  $\int d\tau \mu(\tau) \Lambda^m \Lambda^n \cos(\lambda' - \lambda)$  was replaced by  $\int d\tau \mu(\tau) \Lambda^{2n} \cos(\Omega(\Lambda)\tau)$  with  $\Omega(\Lambda) = 1/\Lambda^3$ . And we have then seen the crucial role played by the  $\Lambda$  dependence of  $\Omega$  (corresponding to the  $J$  dependence discussed in the previous section).

In order to better see the crucial transition from the original  $\cos(\lambda(t') - \lambda(t))$  (where there is no apparent  $\Lambda$  dependence) to the final  $\cos(\Omega(\Lambda)(t' - t))$ , we shall study a simple toy model where the original oscillatory factor  $\cos(\lambda(t') - \lambda(t)) \equiv \cos(\frac{\lambda(t') - \lambda(t)}{t' - t}(t' - t))$  is replaced by  $\cos(\dot{\lambda}(t' - t))$  with  $\dot{\lambda} \equiv d\lambda/dt$ .

A motivation for considering such a simplified dynamics is as follows. If one were formally considering a nonlocality kernel  $\mu(\tau)$  which decays sufficiently fast for large  $|\tau|$ 's (say  $\mu(\tau) = \exp[-\frac{1}{2}(\frac{\tau}{\sigma})^2]/\sqrt{2\pi\sigma^2}$ ), one could formally replace in any nonlocal Hamiltonian  $H^{\text{nonloc}} = \int d\tau \mu(\tau) f(q, q', p, p')$ , the time-shifted variables  $q' = q(t + \tau)$  and  $p' = p(t + \tau)$  by their Taylor expansions,  $q' = q + \tau\dot{q} + \frac{1}{2}\tau^2\ddot{q} + \dots$ , etc. This then replaces the original nonlocality by a generalized Ostrogradski Hamiltonian depending on the infinite tower of derivatives of the phase-space variables:  $H^{\text{ostro}}(q, p; \dot{q}, \ddot{q}, \ddot{\ddot{q}}, \dots)$ . The theory of the order-reduction of such *quasi-local*, Ostrogradski-type Hamiltonians has been abundantly treated in the literature, particularly in the context of the PN-expanded dynamics of binary systems (see, e.g., Refs. [44–47]). By considerations that would be too long to explain here, one can then see that for nonlocalities of the Delaunay-type  $\int d\tau \mu(\tau) \Lambda^m \Lambda^n \cos(\lambda' - \lambda)$ , in which one Taylor-expands the time-shifted variables, e.g.  $\lambda' = \lambda(t + \tau) = \lambda + \tau\dot{\lambda} + \frac{1}{2}\tau^2\ddot{\lambda} + \dots$ ,  $\Lambda' = \Lambda(t + \tau) = \Lambda + \tau\dot{\Lambda} + \frac{1}{2}\tau^2\ddot{\Lambda} + \dots$ , the crucially delicate nonlocality is only contained in the term  $\tau\dot{\lambda}$ . The other ones are essentially equivalent to double-zero terms [because of some

special structure of the relevant Delaunay-like Hamiltonians, notably a  $\mathbb{Z}_2$  symmetry transformation combining time reversal ( $t \rightarrow -t$ ,  $t' \rightarrow -t'$ ,  $\tau \rightarrow -\tau$ ) and angle reversals ( $\lambda \rightarrow -\lambda$ ,  $\varpi \rightarrow -\varpi$ ), and keeping the action variables fixed].

Independently of this motivation, we shall simply use here, for pedagogical purposes, as a simple model of the nonlocal action (5.2) the following first-order Ostrogradski Hamiltonian

$$H_{\text{toy}}^{\text{ostro}}(\lambda, \Lambda, \varpi, I_r; \dot{\lambda}, \dot{\varpi}) = -\frac{1}{2\Lambda^2} + \varepsilon \int d\tau \mu(\tau) \Lambda^{2n} [\cos(\dot{\lambda}\tau) + O(e)], \quad (5.3)$$

where the  $\dot{\varpi}$  dependence only occur within the  $O(e)$  remainder.

Note that this first-order Ostrogradski Hamiltonian has a *quasi-local* (but *not* local) structure in that it depends not only on the phase-space variables  $(q, p)$ , but also on some of their (first) derivatives (here a dependence on the two  $\dot{q}$ 's). Though the “nonlocality” associated with the presence of these derivatives in the Hamiltonian (5.3) might seem trivial, it will actually display the origin of the discrepancy between [21] and us, and show why the claims of [21] are unfounded.

In Ref. [46] we have investigated in detail Ostrogradski Hamiltonians depending on  $\dot{q}$ ,  $\dot{p}$  in addition to  $q$ ,  $p$ . Our main results were the following. First, when considering a general  $H^{\text{ostro}}(q, p; \dot{q}, \dot{p})$  [with action  $S_{\text{ostro}} = \int (p\dot{q} - H^{\text{ostro}})dt$ ], there exists a *conserved energy* canonically associated with such an Ostrogradski Hamiltonian, which is *not* given by the Hamiltonian but rather by the following (Noether-theorem-deduced) quantity

$$\mathcal{E}^{\text{cons}}(q, p; \dot{q}, \dot{p}) = H^{\text{ostro}}(q, p; \dot{q}, \dot{p}) - \dot{q} \frac{\partial H^{\text{ostro}}}{\partial \dot{q}} - \dot{p} \frac{\partial H^{\text{ostro}}}{\partial \dot{p}}. \quad (5.4)$$

Second, when considering the case of interest here where

$$H^{\text{ostro}}(q, p; \dot{q}, \dot{p}) = H_0(q, p) + \varepsilon H_1(q, p; \dot{q}, \dot{p}), \quad (5.5)$$

one can reduce [modulo  $O(\varepsilon^2)$ ]  $H^{\text{ostro}}$  to an *ordinary* Hamiltonian  $H^{\text{ordin}}(q, p')$  [with action  $S^{\text{ordin}} = \int (p'dq' - H^{\text{ordin}}dt)$ ] by means of the following shifts

$$q' = q + \frac{\partial H^{\text{ostro}}}{\partial \dot{p}}, \quad p' = p - \frac{\partial H^{\text{ostro}}}{\partial \dot{q}}. \quad (5.6)$$

In addition the *function*  $H^{\text{ordin}}(q, p)$  is obtained by the naive (“incorrect”) order-reducing of  $H^{\text{ostro}}(q, p; \dot{q}, \dot{p})$ , i.e. by inserting the lower-order equations of motion in  $H^{\text{ostro}}$ ,

$$H^{\text{ordin}}(q, p) = H^{\text{ostro}} \left( q, p; \frac{\delta H_0}{\delta p}, -\frac{\delta H_0}{\delta q} \right) + O(\varepsilon^2). \quad (5.7)$$

And, finally, the (Noether) conserved energy (expressed in terms of the original variables) is (numerically) equal to the (order-reduced) ordinary Hamiltonian  $H^{\text{ordin}}$  [expressed in terms of the corresponding *shifted* variables  $q'$ ,  $p'$  of Eq. (5.6)]:

$$\mathcal{E}^{\text{cons}}(q, p; \dot{q}, \dot{p}) = H^{\text{ordin}}(q', p') + O(\varepsilon^2). \quad (5.8)$$

Let us now apply the general results (5.4)–(5.8) (recalled from [46]) to our toy model (5.3). We first note the existence of a (Noetherian) conserved energy, namely

$$\begin{aligned} \mathcal{E}_{\text{toy}}^{\text{cons}}(\lambda, \Lambda; \dot{\lambda}, \dot{\varpi}) &= H_{\text{toy}}^{\text{ostro}}(\lambda, \Lambda; \dot{\lambda}, \dot{\varpi}) \\ &\quad - \dot{\lambda} \frac{\partial H_{\text{toy}}^{\text{ostro}}}{\partial \dot{\lambda}} - \dot{\varpi} \frac{\partial H_{\text{toy}}^{\text{ostro}}}{\partial \dot{\varpi}}. \end{aligned} \quad (5.9)$$

In the circular limit the last term  $-\dot{\varpi} \partial H_{\text{toy}}^{\text{ostro}} / \partial \dot{\varpi}$  vanishes because of the vanishing eccentricity<sup>12</sup>. On the other hand the penultimate term *does not* vanish in the circular limit because it involves a  $\tau$ -even integrand:

$$\frac{\partial}{\partial \dot{\lambda}} \int d\tau \mu(\tau) \cos(\dot{\lambda}\tau) = - \int d\tau \mu(\tau) \tau \sin(\dot{\lambda}\tau), \quad (5.10)$$

so that

$$\begin{aligned} \mathcal{E}_{\text{circ}}^{\text{cons}}(\Lambda, \dot{\lambda}) &= -\frac{1}{2\Lambda^2} \\ &\quad + \varepsilon \Lambda^{2n} \int d\tau \mu(\tau) \left[ \cos(\dot{\lambda}\tau) + \dot{\lambda}\tau \sin(\dot{\lambda}\tau) \right]. \end{aligned} \quad (5.11)$$

In the circular limit the extra term  $-\dot{\lambda} \partial H / \partial \dot{\lambda}$  is independent of time and cannot therefore be guessed simply by looking at the equations of motion. This is probably part of the problem that Bernard *et al.* [21] had. They indeed use as conserved circular energy the circular-reduced Hamiltonian itself, i.e., in our toy model, simply  $H_{\text{toy}}^{\text{ostro}}(\Lambda, \dot{\lambda})$ , which differs from the “good” (Noetherian) conserved energy (5.9), (5.11).

Another useful outcome of the results (5.4)–(5.8) recalled above is that, while there is no need to shift the angular variables  $\lambda, \varpi$  (because  $\delta\lambda = \partial H^{\text{ostro}} / \partial \dot{\Lambda} = 0$ ,  $\delta\varpi = \partial H^{\text{ostro}} / \partial \dot{I}_r = 0$ ), one must shift the action variables if one wishes to have an order-reduced dynamics. The first fact says in particular that, in the circular limit, the orbital angular frequency  $\dot{\lambda} = \dot{\lambda}'$  is unambiguous and is given either by the (functional) derivative

$$\begin{aligned} \dot{\lambda} &= \frac{\delta H_{\text{toy}}^{\text{ostro}}(\Lambda, \dot{\lambda})}{\delta \Lambda} \\ &= \frac{1}{2\Lambda^3} + 2n\varepsilon \Lambda^{2n-1} \int d\tau \mu(\tau) \cos(\dot{\lambda}\tau), \end{aligned} \quad (5.12)$$

in which the frequency  $\dot{\lambda}$  in the argument of the cosine is treated as a constant, or [up to  $O(\varepsilon^2)$  corrections] by the (ordinary) derivative of the order-reduced, ordinary Hamiltonian

$$\begin{aligned} H_{\text{toy}}^{\text{ordin}}(\lambda', \varpi', \Lambda', I'_r) &= -\frac{1}{2\Lambda'^2} \\ &\quad + \varepsilon \Lambda'^{2n} \int d\tau \mu(\tau) \left[ \cos\left(\frac{\tau}{\Lambda'^3}\right) + O(e) \right], \end{aligned} \quad (5.13)$$

which yields (in the circular limit) an orbital frequency given by

$$\begin{aligned} \dot{\lambda}' &= \frac{\partial H_{\text{toy}}^{\text{ordin}}}{\partial \Lambda'} \\ &= \frac{1}{\Lambda'^3} + \varepsilon \frac{\partial}{\partial \Lambda'} \left[ \Lambda'^{2n} \int d\tau \mu(\tau) \cos\left(\frac{\tau}{\Lambda'^3}\right) \right]. \end{aligned} \quad (5.14)$$

Contrary to Eq. (5.12) the  $\Lambda'$ -derivative on the rhs of Eq. (5.14) now acts both on the prefactor  $\Lambda'^{2n}$  and on the argument of the cosine. Note that in the real 4PN problem the power  $2n$  is equal to  $-10$  and the weight function  $\mu(\tau)$  is either  $\text{Pf}_{2s/c}|\tau|^{-1}$  or  $\text{Pf}_{2r_{12}/c}|\tau|^{-1}$  (as one chooses). Using, say, the latter ( $r_{12}$ -) choice, and, e.g., Eq. (5.8) of [16], one has then

$$\varepsilon \Lambda^{-10} \int_{-\infty}^{+\infty} d\tau \mu(\tau) \cos(\dot{\lambda}\tau) = -\varepsilon \Lambda^{-10} \ln \left( \frac{4e^{\gamma_E} \dot{\lambda} r_{12}}{c} \right), \quad (5.15)$$

which shows that the difference between the two calculations (5.12) versus (5.14) parallels what we explained above about the difference between the prescription used by Ref. [21] (treating  $\dot{\lambda}$  as a constant within the argument of the tail logarithm) and the prescription used by us [16, 17], which corresponds to an order-reduced, ordinary (Delaunay) Hamiltonian.

The detailed reasoning we just made shows that the calculation (5.14) [involving the ordinary, order-reduced Hamiltonian (5.13)] is not only self-consistent (as we already argued above) but, most importantly, is consistent with the *nonlocal* dynamics of the original Ostrogradski Hamiltonian (5.3), under the following two conditions: (i) the conserved energy associated with the nonlocal description (5.3) should not be equated (contrary to the prescription used in [21]), when considering the circular limit, with the value of  $H^{\text{ostro}}$ , but rather with Eqs. (5.9), (5.11), and (ii) the variables  $\lambda, \Lambda, \varpi, I_r$  entering the (weakly) nonlocal dynamics (5.3) cannot be equated with the variables  $\lambda', \Lambda', \varpi', I'_r$  entering the corresponding order-reduced ordinary dynamics (5.13). More precisely, the link between the two sets of variables is  $\lambda' = \lambda$ ,  $\varpi' = \varpi$  but

$$\Lambda' = \Lambda - \frac{\partial H_{\text{toy}}^{\text{ostro}}}{\partial \dot{\lambda}} = \Lambda + \varepsilon \Lambda'^{2n} \int d\tau \mu(\tau) \tau \sin(\dot{\lambda}\tau), \quad (5.16a)$$

$$I'_r = I_r - \frac{\partial H_{\text{toy}}^{\text{ostro}}}{\partial \dot{\varpi}} = I_r + O(e). \quad (5.16b)$$

<sup>12</sup> Here, one should use regular Poincaré variables to rigorously show that the  $O(e)$  terms in (5.3) do not contribute when  $e \rightarrow 0$ .

It is easily checked that the shift (5.16a) ensures the consistency between the two (different) functions  $\Lambda \rightarrow \tilde{\lambda}$ , Eq. (5.12), and  $\Lambda' \rightarrow \tilde{\lambda}' = \tilde{\lambda}$ , Eq. (5.14). It is moreover checked that the conserved energy  $\mathcal{E}_{\text{circ}}^{\text{cons}}(\Lambda, \tilde{\lambda})$ , Eq. (5.11), becomes numerically equal to the ordinary Hamiltonian-energy (5.13) when also taking into account the shift (5.16a).

Summarizing so far: we have explicitly shown, by two different approaches (one of them being the one used in [16, 17], the other being as close as possible to the one used in [21]), that the correct ( $r_{12}$ -scaled) tail contribution to the (Noetherian) conserved energy (along circular orbits) expressed in terms of the orbital frequency is given by Eqs. (4.27), (4.29), in agreement with our results [16, 17].

## VI. INCOMPATIBILITY BETWEEN THE C-MODIFIED FOKKER-ACTION RESULT OF [21] AND SELF-FORCE RESULTS

An immediate consequence of the result of the previous Section is that the determination in Ref. [21] of the ratio between the IR cutoff scale  $r_0$  and the scale  $s_0 = r_0' e^{-11/12}$  entering the computation of the tail-transported nearzone metric perturbation [18] is incorrect. In other words, in terms of the notation of [16] [and of Eqs. (2.3)–(2.4) above] one should add to the Hamiltonian of [21] an extra term  $\Delta C F[\mathbf{x}_a, \mathbf{p}_a]$  with

$$\Delta C = -\frac{3}{7}. \quad (6.1)$$

If we understand correctly the notation used in [21], their (ambiguity) parameter  $\alpha$  [in their Eq. (3.9)] is equivalent to our (ambiguity) parameter  $-C$  modulo a fixed, additive constant linked to the difference between the two schemes. It would then mean that the correct value of their  $\alpha$  should be

$$\alpha_{\text{B}^3\text{FM}}^{\text{new}} = \alpha_{\text{B}^3\text{FM}} + \frac{3}{7} = \frac{811}{672} + \frac{3}{7} = \frac{157}{96}, \quad (6.2)$$

instead of the value, here denoted  $\alpha_{\text{B}^3\text{FM}}$ , in their Eq. (4.30). [Note that the denominator of  $\alpha_{\text{B}^3\text{FM}}^{\text{new}}$  is simpler than that of  $\alpha_{\text{B}^3\text{FM}}$ : namely  $96 = 2^5 \times 3$  while  $672 = 7 \times 96 = 2^5 \times 3 \times 7$ .]

Modifying the Hamiltonian of [21] by  $+\Delta C F$  then modifies the difference  $H_{\text{B}^3\text{FM}}^{\text{new}} - H_{\text{DJS}}$  into an expression of the same form as the rhs of Eq. (3.3) but with the following new values of the three coefficients ( $a, b, c$ ):

$$(a, b, c)_{\text{B}^3\text{FM}}^{\text{new}} = (a, b, c)_{\text{B}^3\text{FM}} + \Delta C \frac{16}{15} (-11, 12, 0). \quad (6.3)$$

Inserting  $\Delta C = -\frac{3}{7}$  from Eq. (6.1), and using Eq. (3.4) above, yields

$$(a, b, c)_{\text{B}^3\text{FM}}^{\text{new}} = \frac{1}{315} (3013, -902, 902). \quad (6.4)$$

The latter result is equivalent [modulo a gauge transformation, Eq. (3.6), with  $g = 288/315$ ] to the rhs of Eq. (5.34b) in [21], but with the different meaning that it is the Hamiltonian of Ref. [21] on the lhs which has to be shifted instead of  $H_{\text{DJS}}$ .

Inserting Eq. (6.4) into our general formulas (3.5) then lead to the following additional contributions (compared to our results [17]) to the EOB potentials  $A$  and  $\bar{D}$  entailed by the above modified version of the Hamiltonian of Ref. [21]:

$$\delta^{\text{new}} A = 0, \quad (6.5)$$

$$\delta^{\text{new}} \bar{D} = -\frac{34}{9} \nu u^4. \quad (6.6)$$

The first result, Eq. (6.5), shows that our prescription (6.1) for modifying the result of [21] now yields a value of the EOB  $A$  potential at 4PN in agreement with our result [16, 17] (and with the many existing SF computations of  $A$  recalled above). On the other hand, we see on Eq. (6.6) that, even after our correction, the 4PN contribution to the EOB  $\bar{D}$  potential predicted by the Fokker-action, harmonic-gauge computation of [21] significantly differs from our result [17].

In terms of the gauge-invariant 4PN-level parameters  $a_5^c$ ,  $\bar{d}_4^c$ ,  $\rho_4^c$  introduced above, the results (3.9), (3.10), (3.12) above are now replaced by

$$\delta^{\text{B}^3\text{FM}'} a_5^c = 0, \quad (6.7)$$

$$\delta^{\text{B}^3\text{FM}'} \bar{d}_4^c = -\frac{34}{9} = -3.77777 \dots, \quad (6.8)$$

$$\delta^{\text{B}^3\text{FM}'} \rho_4^c = -\frac{34}{9} = -3.77777 \dots. \quad (6.9)$$

As we already said above, there are many (analytical and numerical) SF results showing that the 4PN coefficients  $\bar{d}_4^c$  or  $\rho_4^c$  nicely agrees with the ADM result [17]. (Let us, in particular, recall the recent *analytical* confirmations [32, 43].) Numerical tests making use of the eccentric first-law [28] yield a limit  $|\delta \bar{d}_4^c|^{\text{num}} < 0.05$  on any possible deviation  $\delta \bar{d}_4^c$  away from our analytical result (see discussion and references, in [43] and above). As for direct dynamical limits on the 4PN-level coefficient of the precession function  $\rho(u)$  we have displayed them in Eqs. (2.13), (2.14) above. While the old limit (2.13) [34] would be compatible with the modified result (6.9), the recent limit (2.14) [35] is in violent disagreement with (6.9).

## VII. SUGGESTION FOR ADDING MORE IR AMBIGUITY PARAMETERS IN [21]

In view of the strong disagreement between the B3FM – DJS discrepancies (6.8), (6.9) remaining after making use of the sole IR ambiguity parameter  $C = -\alpha + \text{cst.}$ , we suspect that the Fokker action derived in [21] has to be

modified not only by the shift (6.1) of  $C$  (or  $-\alpha$ ), but by suitable shifts of further *ambiguity parameters* entering their present computation.

An independent argument suggesting the presence of more ambiguities in their calculations than the mere  $CF[\mathbf{r}, \mathbf{p}]$  ambiguity (which is also present in our ADM calculation, and was only fixed by using the analytical result of [13]) is presented next.

The core of the calculation of [21] is the evaluation of the (local) Fokker Lagrangian [their Eq. (4.20)]

$$L_g = \int d^3x \mathcal{L}_g \sim \int d^3x c^4 [\bar{h} \square \bar{h} + \bar{h} \partial \bar{h} \partial \bar{h} + \dots + \bar{h} \bar{h} \bar{h} \bar{h} \partial \bar{h} \partial \bar{h}], \quad (7.1)$$

in which one inserts the formal, near zone PN expansion of the metric variables, computed in harmonic coordinates. Both the Fokker Lagrangian density  $\mathcal{L}_g$  and the (time-symmetric) harmonic metric are formally Poincaré covariant. One would then expect the Lagrangian  $L_g$  to vary, under Poincaré transformations, as the time component of a four-vector. In particular, as all the ingredients of the calculation are *manifestly* invariant under spatial translations, one would expect the Lagrangian  $L_g$  to be manifestly translation invariant, and therefore to only depend on the relative position  $\mathbf{y}_{12} = \mathbf{y}_1 - \mathbf{y}_2$  of the two particles. However, Bernard *et al.* [21] state that the raw result of the computation (7.1) is *not* translation invariant, but depends on the individual positions  $\mathbf{y}_A$  of the particles. They found that the offending terms can be removed by some shifts  $\xi_A$  of the positions  $\mathbf{y}_A$  so as to yield a manifestly Poincaré covariant (modulo a time derivative) Lagrangian, say  $L_g^{\text{shifted}}$ . An inspection of the needed shifts (in their Appendix C) reveals a very large number of offending non translation invariant terms which are mostly connected with IR divergences [ $\propto \ln(r_{12}/r_0)$ ]. Moreover, there are terms in the shifts  $\xi_A$  which combine both IR and UV divergences [ $\propto \ln(r_{12}/r'_1)$ ]. This large zoology of non translationally invariant IR-divergent terms; this mixing of IR and UV divergences, together with the use of a method which uses Hadamard regularization as an intermediate step for UV divergences (before “correcting” it by extra dimensional-regularization terms); and finally, the regularization of IR divergences by means of a single (non-dimensionally-regularized) analytic regulator [ $\propto (r/r_0)^B$ ], strongly suggest to us that there are more ambiguities in the final result for  $L_g$  than the  $CF[\mathbf{r}, \mathbf{p}]$  one.

We therefore suggest that Bernard *et al.* [21] should (similarly to what happened at the 3PN level when using only Hadamard-type regularization [46, 48–50]) acknowledge the presence of further ambiguities in their result, parametrized by new ambiguity parameters. We shall assume here that these ambiguities affect the three discrepant terms (3.3) and only them.

In view of the unavoidable presence of the  $CF[\mathbf{r}, \mathbf{p}]$  ambiguity, and of the possibility of making the gauge transformation (3.6), the minimal number of extra am-

biguity parameters that need to be introduced is *one*. For instance, we could introduce a single ambiguity parametrized by adding to the result of [21] a term in the Hamiltonian of the form

$$\delta^{\Delta a} H = +\Delta a \frac{G^4 M m_1^2 m_2^2}{c^8 r_{12}^4} \left( \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_1)}{m_1} - \frac{(\mathbf{n}_{12} \cdot \mathbf{p}_2)}{m_2} \right)^2, \quad (7.2)$$

which has the effect of changing  $(a, b, c)$  into  $(a + \Delta a, b, c)$ . Such a change does not affect the EOB  $A$  potential [which is  $\propto 2(b+c)$ , Eq. (3.5a)], but affects the EOB  $\bar{D}$  potential [which is  $\propto 2(a+4b)$ , Eq. (3.5b)]. We conclude that one must add to their result in addition to  $\Delta CF$ , with  $\Delta C = -3/7$ , Eq. (3.3), the extra contribution (7.2) with

$$\Delta a = +\frac{17}{9}. \quad (7.3)$$

It is also possible that part of the ambiguities in their calculation are parametrized by a transformation of the type (3.6), but such a transformation has no gauge-invariant impact on the dynamics as it is induced by a canonical transformation with generating function  $\propto \nu \mathbf{n}_{12} \cdot (\mathbf{p}_1/m_1 - \mathbf{p}_2/m_2) r_{12}^{-3}$  (corresponding to changes of  $\mathbf{r}_1$  and  $\mathbf{r}_2$  in a purely radial direction). [Note, however, that even such “gauge ambiguities” will affect the computation of the radiative quadrupole moment of the system, see below.]

## VIII. CONCLUSIONS

Let us summarize our main conclusions concerning the conservative dynamics of a binary system at the 4PN approximation.

- We have reviewed the numerous confirmations of the correctness of the 4PN ADM dynamics [14, 16, 17]. When decomposed in powers of the symmetric mass ratio  $\nu$ , all its elements have been confirmed, piecewise, by independent computations. In particular, the crucial linear-in- $\nu$  terms (at the first-self-force level) have been confirmed, both analytically and numerically, by several self-force results.
- By contrast we have computed the periastron precession induced by the recently reported Fokker-action 4PN dynamics [21] and found it to be in violent contradiction with existent direct numerical self-force computations [35]: compare Eq. (3.12) to Eq. (2.14).
- After reviewing the logical basis and the consistency of the ADM-Delaunay-EOB derivation of [16, 17], we pointed out two different flaws in the Fokker-action harmonic-coordinates computation of [21]: (i) their computation of the functional link, along circular orbits, between the energy and the orbital frequency involves a correct

treatment on nonlocal (tail) effects but a flawed assumption about the value of the physically relevant conserved energy; (ii) after correcting for the previous issue, there remains a violent incompatibility between their theoretically predicted periastron precession and recent dynamical self-force computations: compare Eq. (6.9) to Eq. (2.14).

- We suggest that the result of [21] must be corrected by adding (at least) two infra-red ambiguity parameters to the Hamiltonian form of their result: (1)  $\Delta C = -\Delta\alpha = -\frac{3}{7}$ , (6.1); and (2)  $\Delta a = +\frac{17}{9}$ , (7.2), (7.3) (modulo a gauge transformation, Eq. (3.6)). We, however, emphasize that our suggested corrections represent only two rather minor adjustments among hundreds of terms that agree between two very difficult (four-loop level!) independent calculations, using different methods and different gauges.
- Before attempting a 4PN-level computation of gravitational wave emission (in the more convenient harmonic coordinates), it will be necessary to confirm, within the framework of [21], the presence, and value, of the couple of ambiguities (notably  $\Delta a$  and  $g$ ) mentioned above.

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#### Appendix A: Tail Hamiltonian in the ADM formalism

Reference [16] showed the necessity of including in the ADM action a tail-related contribution  $-\int dt H^{\text{tail sym}(s)}$ , with

$$H_{4\text{PN}}^{\text{tail sym}(s)}(t) = -\frac{1}{5} \frac{G^2 M}{c^8} I_{ij}^{(3)}(t) \times \text{Pf}_{2s/c} \int_{-\infty}^{+\infty} \frac{dv}{|v|} I_{ij}^{(3)}(t+v), \quad (\text{A1})$$

by recalling the existence (first shown by Blanchet and Damour [18]) of a fundamental breakdown of the standard PN scheme at the 4PN-level. Indeed, when computing *conservative* effects, as is appropriate both in the computation of the reduced ADM action, and in that of the Fokker action, the standard PN scheme is based on a formal near-zone expansion of the time-symmetric gravitational propagator, say

$$\square_{\text{sym}}^{-1} = \left( \Delta - \frac{1}{c^2} \partial_t^2 \right)^{-1}$$

$$= \left( \Delta^{-1} + \frac{1}{c^2} \Delta^{-2} \partial_t^2 + \dots \right) \delta(t-t'). \quad (\text{A2})$$

At the 4PN-level the *near-zone* solution generated by using the PN expansion (A2) is incomplete (as a near-zone solution), and must be completed by adding to it a 4PN-level *homogeneous* solution of the (linearized) field equations (regular within the source) which is a *nonlocal-in-time* functional of the dynamical configuration of the system (related to tail effects).

[Let us note in passing that we disagree on some formal aspects of the definition of a Fokker action in [21]. Indeed, the standard definition (starting with [51] and going up to modern treatments based on the diagrammatic representation of the effective action [52]) systematically uses an iterative solution of the field equations based on the *time-symmetric* (half-retarded-half-advanced) Green's function. The 4PN ADM computation explicitly uses the time-symmetric version of the *TT* degrees of freedom (see Sec. IV of [14]). By contrast, it seems that [21] formally works with the nonconservative dynamics of a binary system interacting via the *retarded* Green's function. They then argue that the terms associated with radiation damping only contribute time-derivative terms to the action they compute. We understand technically why this works out formally, however, we object to such a conceptual framework.]

Reference [18] showed that in a suitable coordinate gauge the additional, 4PN-level, tail-related homogeneous metric perturbation  $h_{\mu\nu}^{\text{nonloc } 4\text{PN}}$  contains only a quadrupolar time-time component of the form

$$h_{00}^{\text{nonloc } 4\text{PN}} = \frac{1}{2} x^i x^j H_{ij}^{\text{nonloc}}(t), \quad (\text{A3})$$

with the nonlocal function of time  $H_{ij}^{\text{nonloc}}(t)$  given by

$$H_{ij}^{\text{nonloc}}(t) = -\frac{8}{5} \frac{G^2 M}{c^{10}} \text{Pf}_{2s/c} \int_{-\infty}^{+\infty} \frac{d\tau}{|\tau|} I_{ij}^{(6)}(t+\tau). \quad (\text{A4})$$

The pure, quasi-Newtonian, nature of the additional (nonlocal) metric perturbation (A3) allowed us, in [16], by considering the corresponding additional term in the equations of motion of the system, to: (i) derive the corresponding additional (time-symmetric) nonlocal contribution (A1) to the action (modulo a time derivative); and, (ii) to check the consistency with the purely *retarded* nature of the nonlocality in the equations of motion when adding the (time-antisymmetric) radiation-reaction force  $\mathcal{F}_{4\text{PN}}^{\text{rad reac}}$ .

Reference [21] formally showed how to recover the nonlocal contribution (A1) from the harmonic-gauge Fokker action, without considering the equations of motion, and without using the simplifying features of the special gauge leading to (A3) only. We wish here both to show how the same result can be achieved within the ADM formalism and why the procedure advocated in [21] is, in our opinion, exceedingly formal and actually ill-defined because of its sensitivity to nonconvergent integrations by parts.

In the ADM formalism, before eliminating the TT variables, the Routhian<sup>13</sup> [48] reads (in the following we often use without warning the ADM convention  $16\pi G = 1$ )

$$R[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}] = \int d^3x \left[ \frac{1}{4} (\partial_k h_{ij}^{\text{TT}})^2 - \frac{1}{4c^2} (\dot{h}_{ij}^{\text{TT}})^2 - \frac{1}{4} h_{ij}^{\text{TT}} J_{ij} + \dots \right], \quad (\text{A5})$$

where the leading-order source term for  $h_{ij}^{\text{TT}}$  reads

$$J_{ij} = 2 \sum_a \frac{p_{ai} p_{aj}}{m_a} \delta(\mathbf{x} - \mathbf{x}_a) + \frac{1}{2} \partial_i \phi \partial_j \phi + \dots, \quad (\text{A6})$$

and where the ellipsis denote higher-order terms that we shall not explicitly need here. Note that the longitudinal field degrees of freedom,  $\phi$  and  $\tilde{\pi}^{ij}$ , are viewed as being expressed in terms of the other degrees of freedom ( $\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \dot{h}_{ij}^{\text{TT}}$ ) by using the constraints (which are elliptic in nature).

The key point is that the ADM analog of the decomposition recalled above,  $g_{\mu\nu}^{\text{sym}} = g_{\mu\nu}^{\text{loc PN}} + h_{\mu\nu}^{\text{nonloc}}$ , reads

$$h_{ij}^{\text{TT sym}} = h_{ij}^{\text{TT loc PN}} + h_{ij}^{\text{TT nonloc}}. \quad (\text{A7})$$

Here,  $h_{ij}^{\text{TT loc PN}}$  is the usual, near-zone, time-symmetric PN-expanded formally local solution of the field equation for  $h_{ij}^{\text{TT}}$ , namely

$$\square h_{ij}^{\text{TT}} = -\frac{1}{2} \delta_{ij}^{\text{TTkl}} J_{kl}, \quad (\text{A8})$$

as taken into account in the usual ADM version of the PN expansion [14, 48]. On the other hand, the additional, tail-transported term  $h_{ij}^{\text{TT nonloc}}$  is, again, when viewed in the near zone, a *homogeneous* solution of the (linearized) field equations for  $h_{ij}^{\text{TT}}$ . The latter nonlocal, tail-transported homogeneous solution is directly deducible from the results of Schäfer [53] who studied radiation and tail effects, within the ADM framework, both in the wave zone and in the near zone. The Green's functions in the latter paper [see Eq. (22) there] yield the following tail-modified radiation-reaction contribution to the (usual, retarded) solution,

$$h_{ij}^{\text{TT reac}}(t) = -\frac{4G}{5c^5} \left[ \mathcal{J}_{ij}^{(1)}(t) + \frac{4GM}{c^3} \int_0^\infty d\tau \ln\left(\frac{c\tau}{2s}\right) \mathcal{J}_{ij}^{(3)}(t-\tau) \right], \quad (\text{A9})$$

where

$$\mathcal{J}_{ij}(t) = \int d^3x J_{(ij)}(t, \mathbf{x}) \quad (\text{A10})$$

is the trace-free part of the spatial integral of the source (A6) of the field equation (A8) for  $h_{ij}^{\text{TT}}$ . Note that the tail-modified reactive contribution (A9) to  $h_{ij}^{\text{TT}}$  is simply a function of time. This is the anti-Newtonian version of the gauge (A3), namely  $h_{ij} = F_{ij}(t)$  instead of  $h_{00} = \frac{1}{2} F_{ij}^{(2)} x^i x^j$ , with the same linearized curvature

$$R_{0i0j}^{\text{lin}} = -\frac{1}{2} (\partial_{ij} h_{00} + \partial_{00} h_{ij} - \partial_{0i} h_{0j} - \partial_{0j} h_{0i}).$$

The conservative, time-symmetric part of the tail-transported contribution to (A9) is

$$\begin{aligned} h_{ij}^{\text{TT nonloc}} &= \\ &+ \frac{8G^2 M}{5c^8} \int_0^{+\infty} d\tau \ln\left(\frac{c|\tau|}{2s}\right) \left[ \mathcal{J}_{ij}^{(3)}(t+\tau) - \mathcal{J}_{ij}^{(3)}(t-\tau) \right] \\ &= -\frac{8G^2 M}{5c^8} \text{Pf}_{2s/c} \int_{-\infty}^{+\infty} \frac{d\tau}{|\tau|} \mathcal{J}_{ij}^{(2)}(t+\tau). \end{aligned} \quad (\text{A11})$$

The elimination of the  $h_{ij}^{\text{TT}}$  field consists in replacing the complete solution (A7) of the TT field equation (A6) in the Routhian (A5). At leading order, the effect of  $h_{ij}^{\text{TT nonloc}}$  is linear in  $h_{ij}^{\text{TT nonloc}}$  and should a priori come from the first and third terms in Eq. (A5). Such a standard computation yields

$$\begin{aligned} [R^{\text{nonloc}}]_{\text{standard}} &= \int d^3x \left[ \frac{1}{4} \partial_k h_{ij}^{\text{TT nonloc}} \partial_k h_{ij}^{\text{TT loc PN}} \right. \\ &+ \frac{1}{4} \partial_k h_{ij}^{\text{TT loc PN}} \partial_k h_{ij}^{\text{TT nonloc}} \\ &\left. - \frac{1}{4} h_{ij}^{\text{TT nonloc}} J_{ij} \right], \end{aligned} \quad (\text{A12})$$

where we exhibited the two separate (though trivially equal) contributions coming from the (spatial) kinetic term  $\frac{1}{4} (\nabla h)^2$  in (A5).

However, the explicit expression (A11) of the nonlocal contribution to  $h_{ij}^{\text{TT}}$  shows, as already mentioned, that  $h_{ij}^{\text{TT nonloc}}$  is spatially independent, being a mere function of time. As a consequence the first two (equal) terms on the rhs of (A12) give simply a zero contribution so that this standard computation yields

$$[R^{\text{nonloc}}]_{\text{standard}} = \int d^3x \left[ -\frac{1}{4} h_{ij}^{\text{TT nonloc}} J_{ij} \right]. \quad (\text{A13})$$

Actually, this answer is incorrect, because it does not agree with the one (securely based on the equations of motion) that we used in [16].

A formal way out of this contradiction was advocated, and used, in [21]. It consists in replacing the standard kinetic terms  $(\partial h)^2$  in (A5) by  $-h \square h$ . If we apply this formal procedure to the spatial gradient terms in (A5) (which are the only relevant ones at this order), i.e.,  $\frac{1}{4} (\partial_k h_{ij}^{\text{TT}})^2 \rightarrow -\frac{1}{4} h_{ij}^{\text{TT}} \Delta h_{ij}^{\text{TT}}$ , the result of replacing the

<sup>13</sup> The Routh functional is a Hamilton function for the bodies but a Lagrange functional for the field degrees of freedom.

complete solution (A7) in the Routhian now yields, instead of (A12), the new expression

$$[R^{\text{nonloc}}]_{\text{nonstandard}} = \int d^3x \left[ -\frac{1}{4} h_{ij}^{\text{TT nonloc}} \Delta h_{ij}^{\text{TT loc PN}} - \frac{1}{4} h_{ij}^{\text{TT loc PN}} \Delta h_{ij}^{\text{TT nonloc}} - \frac{1}{4} h_{ij}^{\text{TT nonloc}} J_{ij} \right]. \quad (\text{A14})$$

Formally, this is identical to (A12) modulo a (spatial) integration by parts. However, if we use the fact that  $h_{ij}^{\text{TT nonloc}}$  is simply a function of time only, the second term on the rhs now yields a vanishing contribution, while the first one yields (upon using the field equation for  $h_{ij}^{\text{TT loc PN}}$ )

$$[R^{\text{nonloc}}]_{\text{nonstandard}} = \int d^3x \left[ +\frac{1}{8} h_{ij}^{\text{TT nonloc}} \delta_{ij}^{\text{TT}kl} J_{kl} - \frac{1}{4} h_{ij}^{\text{TT nonloc}} J_{ij} \right]. \quad (\text{A15})$$

If we now formally integrate by parts the spatially nonlocal projections described by the TT projection kernel and decide that the spatial derivatives they contain yield zero on  $h_{ij}^{\text{TT nonloc}}(t)$ , we further get

$$[R^{\text{nonloc}}]_{\text{nonstandard}}^{\text{formal by parts}} = \int d^3x \left[ +\frac{1}{8} h_{ij}^{\text{TT nonloc}} J_{ij} - \frac{1}{4} h_{ij}^{\text{TT nonloc}} J_{ij} \right] = \int d^3x \left[ -\frac{1}{8} h_{ij}^{\text{TT nonloc}} J_{ij} \right]. \quad (\text{A16})$$

As we see, we now have *half* the value (A13) given by using the standard kinetic terms. And this second result now agrees with the result given in our previous work. Indeed, the last result (A16) only involves the spatial integral  $\mathcal{J}_{ij}$ , Eq. (A10), of  $J_{ij}$ . Inserting also the expression

(A11) of  $h_{ij}^{\text{TT nonloc}}$  as a nonlocal integral of  $\mathcal{J}_{ij}^{(2)}$  yields

$$[R^{\text{nonloc}}]_{\text{nonstandard}} = +\frac{G^2 M}{5c^8} \mathcal{J}_{ij}(t) \times \text{Pf}_{2s/c} \int_{-\infty}^{+\infty} \frac{d\tau}{|\tau|} \mathcal{J}_{ij}^{(2)}(t+\tau), \quad (\text{A17})$$

so that the corresponding nonlocal contribution to the action reads (modulo an integration by parts with respect to time)

$$- \int R^{\text{nonloc}}_{\text{nonstandard}} dt = +\frac{G^2 M}{5c^8} \times \text{Pf}_{2s/c} \iint \frac{dt dt'}{|t' - t|} \mathcal{J}_{ij}^{(1)}(t') \mathcal{J}_{ij}^{(1)}(t). \quad (\text{A18})$$

Here, we note that the integral  $\mathcal{J}_{ij}$  of  $J_{ij}$ , Eq. (A10), is the following function of positions and momenta:

$$\mathcal{J}_{ij} = 2 \sum_a \frac{p_a \langle i p_{aj} \rangle}{m_a} - 2Gm_1 m_2 \frac{r_{12}^{(i} r_{12}^{j)}}{r_{12}^3}, \quad (\text{A19})$$

which is simply the on-shell value of the second time derivative, say  $\hat{I}_{ij}^{(2)}$  (in the notation used in [21]), of the quadrupole moment  $I_{ij} = \sum_a m_a x_a^{(i} x_a^{j)}$ . We thereby see that, modulo the eventual order reduction of the extra time derivative in  $\mathcal{J}_{ij}^{(1)} = \frac{d}{dt} \hat{I}_{ij}^{(2)}$ , the result (A18) agrees with the nonlocal action of [16], as well as with the equivalent action of [21]. However, our explicit calculation above has given us two new results: (1) A reduced-action calculation similar to the one performed in harmonic coordinates in [21] can be done within the ADM formalism; and (2) such a calculation has, at best, a formal value because it is very sensitive to ill controlled IR effects (spatial divergences) and can easily be conducted so as to give a wrong result, as exhibited in Eqs. (A12) and (A13), based on the standard  $(\partial h)^2$  action.

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