Perturbative quantum field theory meets number theory

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Dedicated to the memory of Raymond Stora,
mentor and friend.

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Abstract

Feynman amplitudes are being expressed in terms of a well structured family of special functions and a denumerable set of numbers - *periods*, studied by algebraic geometers and number theorists. The periods appear as residues of the poles of regularized primitively divergent multidimensional integrals. In low orders of perturbation theory (up to six loops in the massless φ^4 theory) the family of hyperlogarithms and multiple zeta values (MZVs) serves the job. The hyperlogarithms form a double shuffle differential graded Hopf algebra. Its subalgebra of single valued multiple polylogarithms describes a large class of euclidean Feynman amplitudes. As the grading of the double shuffle algebra of MZVs is only conjectural, mathematicians are introducing an abstract graded Hopf algebra of *motivic zeta values* whose weight spaces have dimensions majorizing (hopefully equal to) the dimensions of the corresponding spaces of real MZVs.

The present expository notes provide an updated version of 2014's lectures on this subject presented by the author to a mixed audience of mathematicians and theoretical physicists in Sofia and in Madrid.

[1]Extended version of a talk at the 2014 ICMAT Research Trimester "Multiple Zeta Values, Multiple Polylogarithms, and Quantum Field Theory", Madrid.

Contents

1 Introduction

++ In the period preceding the start of the Large Hadron Collider (LHC) at CERN the "theoretical theorists" indulged into physically inspired speculations. That produced (occasionally) interesting mathematical insights but the contact of the resulting activity with real physics, as much as it existed at all, mainly came through its impact on quantum field theory (QFT). When LHC began working at full swing the major part of the theory which does have true applications in particle physics turned out to be good old perturbative QFT - as it used to be over sixty years ago with quantum electrodynamics. There is a difference, however. Half a century ago the dominating view still was that QFT is "plagued with divergences" and that renormalization merely "hides the difficulties under the carpet". In the words of Freeman Dyson [D72] perturbative QFT was an issue for divorce between mathematics and physics. The work of Stueckelberg, Bogolubov, Epstein and Glaser, Stora and others gradually made it clear (in the period 1950-1980, although it took quite a bit longer to get generally acknowledged) that perturbative renormalization can be neatly formulated as a problem of extension of distributions, originally defined for non-coinciding arguments in position space. A parallel development, due to Stueckelberg and Petermann, Gell-Mann and Low, Bogolubov and Shirkov (see [BS]), culminating in the work of Kenneth Wilson, the renormalization group, became a tool to study QFT - well beyond keeping track of renormalization ambiguities. (The authors of [FHS] have felt the need, even in 2012 - the year of the final confirmation of the Standard Model through the discovery of the Higgs boson - to appeal to fellow theorists "to stop worrying [about divergences] and love QFT".) It was however a newer development, pioneered by David Broadhurst that led to an unlikely confluence between particle physics and number theory (see e.g. [BK] and references to earlier work cited there). In a nutshell, renormalization consists in subtracting a pole term whose residue is an interesting number - a *period* in the sense of [KZ] - associated with the corresponding Feynman amplitude, independent of the ambiguities inherent to the renormalization procedure. These numbers also appear in the renormalization group beta-function [S97, GGV] and, somewhat mysteriously, in the successive approximation of such an all important physical quantity as the anomalous magnetic moment of the electron (see [Sch] as well as Eq. (3.18) below). More generally, for rational ratios of invariants and masses, euclidean Feynman amplitudes are periods [BW]. Theorists are trying to reduce the evaluation of Feynman amplitudes to an expansion with rational coefficients in a basis of transcendental functions and numbers (see [ABDG, D] and references therein). Thanks to the rich algebraic structure of the resulting class of functions, this development did not make mathematically minded theorists redundant - substituted by computer programmers.

The present lecture provides an introductory survey of the double shuffle and Hopf algebra of hyperlogarithms and of the associated multiple zeta values and illustrates their applications to QFT on simple examples of evaluating massless Feynman amplitudes in the position space picture. We note by passing that this picture is advantageous for exhibiting the causal factorization principle of Epstein-Glaser [EG, NST] and it allows an extension to a curved space-time (see [BF, HW, DF, deMH]). It is also preferable from computational point of view when dealing with off-shell massless amplitudes (see [S14, DDEHPS, T14, T15]). On the other hand, on shell scattering amplitudes are studied for good reasons in momentum space. Moreover, the pioneering work of Bloch, Kreimer and others [BEK, BlK, BrS, BS13, Bl15] that displayed the link between Feynman amplitudes, algebraic geometry and number theory is using the "graph polynomial" in the Schwinger (or Feynman) momentum space α-representation. The equivalence of the definition of *quantum periods* in the different pictures is established in [Sch].

We begin in Sect. 2 with a brief introduction to position space renormalization highlighting the role of "Feynman periods". We point out in Sect. 2.3 that (primitive) 4-point functions in the φ^4 theory are conformally invariant and can be expressed as functions of a complex variable z (that appears subsequently as the argument of multiple polylogarithms). Sect. 3 is devoted to the double shuffle algebra of hyperlogarithms including the Knizhnik-Zamolodchikov equation for their generating function $L(z)$. The definition of monodromy of $L(z)$ (3.13) involves the "Drinfeld associator" - the generating series of multiple zeta values (MZVs) whose formal and motivic generalizations are surveyed in Sect. 4. We give, in particular, a pedestrian summary of Brown's derivation of the Hilbert-Poincaré series of the dimensions of weight spaces of motivic zeta values and formulate the more refined Broadhurst-Kreimer's conjecture. The Hopf algebra of MZVs is extended at the end of Sect. 4 to a comodule structure of a quotient Hopf algebra of multiple polylogarithms. In Sect. 5 we review Brown's theory [B04, B] of single valued hyperlogarithms and end up with a couple of illustrative applications. An appendix is devoted to a brief historical survey, including a glimpse into the life and work of Leonhard Euler with whom originates to a large extent the theory of MZVs and polylogarithms.

2 Residues of primitively divergent amplitudes

2.1 Periods in position space renormalization

A position space Feynman integrand $G(\vec{x})$ in a massless QFT is a rational homogeneous function of $\vec{x} \in \mathbb{R}^{\mathbb{N}}$. If G corresponds to a connected graph with $V(\geq 2)$ vertices then, in a four-dimensional (4D) space-time, $N = 4(V-1)$. The integrand is *convergent* if it is locally integrable everywhere so that it defines a homogeneous distribution on $\mathbb{R}^{\mathbb{N}}$. G is *superficially divergent* if it gives rise to a homogeneous density in $\mathbb{R}^N$ of non-positive degree:

$$G(\lambda\vec{x})\, d^N\lambda x = \lambda^{-\kappa} G(\vec{x})\, d^N x\ , \quad \kappa \geq 0\ , \quad \vec{x} \in \mathbb{R}^N \quad (\lambda > 0)\,; \tag{2.1}$$

κ is called the (superficial) *degree of divergence.* In a scalar QFT with massless propagators a connected graph with a set $\mathcal{L}$ of internal lines gives rise to a Feynman amplitude that is a multiple of the product

$$G(\vec{x}) = \prod_{(i,j)\in\mathcal{L}} \frac{1}{x_{ij}^2},\ x_{ij} = x_i - x_j,\ x^2 = \sum_\alpha x^\alpha x_\alpha. \tag{2.2}$$

If G is superficially divergent (i.e. if $\kappa = 2L - N \geq 0$ where L is the number of lines in $\mathcal{L}$) then it is *divergent* - that is, it does not admit a homogeneous extension as a distribution on $\mathbb{R}^N$. (For more general spin-tensor fields whose propagators have polynomial numerators a superficially divergent amplitude may, in fact, turn out to be convergent - see Sect. 5.2 of [NST].) A divergent amplitude is *primitively divergent* if it defines a homogeneous distribution away from the *small diagonal* ($x_i = x_j$ for all i, j). The following proposition (Theorem 2.3 of [NST12]) serves as a definition of both the *residue ResG* and of a *renormalized* (primitively divergent) *amplitude* $G^\rho(\vec{x})$.

Proposition 2.1. *If $G(\vec{x})$ (2.2) is primitively divergent then for any smooth norm $\rho(\vec{x})$ on $\mathbb{R}^N$ there exists a distribution ResG such that*

$$[\rho(\vec{x})]^\epsilon G(\vec{x}) - \frac{1}{\epsilon}(Res\, G)(\vec{x}) = G^\rho(\vec{x}) + O(\epsilon).\ suppResG = \{0\}. \tag{2.3}$$

Here G^ρ is a distribution valued extension of $G(\vec{x})$ to $\mathbb{R}^N$. Its calculation is reduced to the case $\kappa = 0$ of a logarithmically divergent graph by using the *identity*

$$(Res\,G)(\vec{x}) = \frac{(-1)^\kappa}{\kappa!}\,\partial_{i_1}...\partial_{i_\kappa} Res\,(x^{i_1}...x^{i_\kappa} G)(\vec{x}) \tag{2.4}$$

where summation is assumed (from 1 to N) over the repeated indices $i_1, ..., i_\kappa$*. If* G *is homogeneous of degree* $-N$ *then*

$$(Res\,G)(\vec{x}) = res\,(G)\,\delta(\vec{x}) \quad (\text{for } \partial_i(x^i G) = 0)\,. \tag{2.5}$$

Here the numerical residue $res\,G$ *is given by an integral over the hypersurface* $\Sigma_\rho = \{\vec{x}|\,\rho(\vec{x}) = 1\}$*:*

$$res\,G = \frac{1}{\pi^{N/2}} \int_{\Sigma_\rho} G(\vec{x}) \sum_{i=1}^{N} (-1)^{i-1} x^i dx^1 \wedge ...\hat{dx^i}... \wedge dx^N, \tag{2.6}$$

(a hat over an argument meaning, as usual, that this argument is omitted). The residue $res\,G$ *is independent of the (transverse to the dilation) surface* Σ_ρ *since the form in the integrant is closed in the projective space* $\mathbb{P}^{N-1}$.

We note that N is even, in fact, divisible by 4, so that $\mathbb{P}^{N-1}$ is orientable.

The functional $res\,G$ is a *period* according to the definition of [KZ, M-S]. Such residues are often called "Feynman" or "quantum" periods in the present context (see e.g. [Sch]). The same numbers appear in the expansion of the renormalization group beta function (see [S97, GGV]).

The convention of accompanying the 4D volume d^4x by a π^{-2} ($2\pi^2$ being the volume of the unit sphere $\mathbb{S}^3$ in four dimensions), reflected in the prefactor, goes back at least to Broadhurst and is adopted in [Sch, BrS]; it yields rational residues for one- and two-loop graphs. For graphs with three or higher *number of loops* $\ell\,(= h_1$, the *first Betti number* of the graph) one encounters, in general, multiple zeta values of overall weight not exceeding $2\ell - 3$ (cf. [BK, Sch, S14]). If we denote by L and V the numbers of internal lines and vertices of a connected graph then $\ell = L - V + 1 (= V - 1$ for a connected 4-point graph in the φ^4 theory). With the above choice of the 4D volume form the only residues at three, four and five loops (in the φ^4 theory) are integer multiples of $\zeta(3), \zeta(5)$ and $\zeta(7)$, respectively. The first double zeta value, $\zeta(3,5)$, appears at six loops (with a rational coefficient) (see the census in [Sch]). All *known* residues were (up to 2013) rational linear combinations of multiple zeta values (MZVs) [BK, Sch]. A seven loop graph was recently demonstrated [P, B14] to involve *multiple Deligne values* - i.e., values of *hyperlogarithms* at sixth roots of unity.

The definition of a period is deceptively simple: a complex number is a period if its real and imaginary parts can be written as absolutely convergent integrals of rational functions with rational coefficients in domains given by polynomial inequalities with rational coefficients. The set $\mathbb{P}$ of all periods would not change if we replace everywhere in the definition "rational" by "algebraic". If we denote by $\bar{\mathbb{Q}}$ the *field of algebraic numbers* (the inverse of an algebraic number being also algebraic) then we would have the inclusions

$$\mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathbb{P} \subset \mathbb{C}. \tag{2.7}$$

The periods form a ring (they can be added and multiplied) but the inverse of a period needs not be a period. Feynman amplitudes in an arbitrary (relativistic, local) QFT can be normalized in such a way that the only numerical coefficient to powers of coupling constants and ratios of dimensional parameters that appear are periods [BW]. The set of all periods is still countable although it contains infinitely many transcendent numbers. A useful criterion for transcendence is given by the *Hermite-Lindemann theorem: if z is a non-zero complex number then either z or e^z is transcendental.* It follows that $e(= e^1)$ is transcendental and so is π as $e^{i\pi} = -1$ and i is algebraic. Furthermore, the natural logarithm of an algebraic number different from 0 and 1 is transcendental. Examples of periods include the transcendentals

$$\pi = \iint_{x^2+y^2\leq 1} dxdy, \;\ln n = \int_1^n \frac{dx}{x}, \; n = 2, 3, ..., \tag{2.8}$$

as well as the values of iterated integrals, to be introduced in Sect. 3, at algebraic arguments. They include both the classical MZVs as well as the above mentioned multiple Deligne values. The basis e of natural logarithms, the Euler constant $\gamma = -\Gamma'(1)$, as well as $\ln(\ln n), \ln(\ln(\ln n))$, ..., and $1/\pi$ are believed (but not proven) not to be periods.

2.2 Vacuum completion of 4-point graphs in φ^4

In the important special case of the φ^4 theory (in four space-time dimensions) the definition of residue admits an elegant generalization which also simplifies its practical calculation. Following Schnetz [Sch, S14] we associate to each 4-point graph Γ of the φ^4 theory a *completed vacuum graph* $\bar{\Gamma}$, obtained from Γ by joining all four external lines in a new vertex "at infinity". An n-vertex *4-regular vacuum graph* - having four edges incident with each vertex and no tadpole loops - gives rise to n 4-point graphs (with $(n-1)$ vertices each) corresponding to the n possible choices of the vertex at infinity. The introduction of such completed graphs is justified by the following result (see Proposition 2.6 and Theorem 2.7 of [Sch] as well as Sect. 3.1 of [T15]).

Theorem 2.2. *A 4-regular vacuum graph $\bar{\Gamma}$ with at least three vertices is said to be completed primitive if the only way to split it by a four edge cut is by splitting off one vertex. A 4-point Feynman amplitude corresponding to* *a connected 4-regular graph* Γ *is primitively divergent iff its completion* $\bar{\Gamma}$ *is completed primitive. All 4-point graphs with the same primitive completion have the same residue.*

The period of a completed primitive graph $\bar{\Gamma}$ *is equal to the residue of each 4-point graph* $\Gamma = \bar{\Gamma} - v$ *(obtained from* $\bar{\Gamma}$ *by cutting off an arbitrary vertex* v*). The resulting common period can be evaluated from* $\bar{\Gamma}$ *by choosing arbitrarily three vertices* $\{0, e\,(s.t.\, e^2 = 1), \infty\}$*, setting all propagators corresponding to edges of the type* (x_i, ∞) *equal to 1 and integrating over the remaining* $n-2$ *vertices of* Γ *(*$n = V(\Gamma)$*):*

$$Per(\bar{\Gamma}) \equiv res(\Gamma) = \int \Gamma(e, x_2, ..., x_{n-1}, 0) \prod_{i=2}^{n-1} \frac{d^4 x_i}{\pi^2} \ . \tag{2.9}$$

The proof uses the conformal invariance of residues in the φ^4-theory.

There are infinitely many primitively divergent 4-point graphs (while there is a single primitive 2-point graph - corresponding to the self-energy amplitude $(x_{12}^2)^{-3}$). A remarkable sequence of ℓ-loop graphs ($\ell \geq 3$) with four external lines, the *zig-zag graphs*, can be characterized by their n-point vacuum completions $\bar{\Gamma}_n$, $n = \ell + 2$ as follows. $\bar{\Gamma}_n$ admits a closed *Hamiltonian cycle* that passes through all vertices in consecutive order such that each vertex i is also connected with $i \pm 2\ (mod\, n)$. These graphs were conjectured by Broadhurst and Kreimer [BK] in 1995 and proven by Brown and Schnetz [BS12] to have residues

$$\begin{aligned} Per(\bar{\Gamma}_{\ell+2}) &= \frac{4 - 4^{3-\ell}}{\ell} \binom{2\ell-2}{\ell-1} \zeta(2\ell-3) \quad \text{for } \ell = 3, 5, ... ; \\ &= \frac{4}{\ell} \binom{2\ell-2}{\ell-1} \zeta(2\ell-3) \quad \text{for } \ell = 4, 6, ... \end{aligned} \tag{2.10}$$

We note that the periods for $\ell = 3, 4$ also belong to the wheel with ℓ spokes series and are given by $\binom{2\ell-2}{\ell-1}\zeta(2\ell-3)$ (cf. (5.9) below).

2.3 Primitive conformal amplitudes

Each primitively divergent Feynman amplitude $G(x_1, ..., x_4)$ defines a conformally covariant (locally integrable) function away from the small diagonal $x_1 = ... = x_4$. On the other hand, every four points, $x_1, ..., x_4$, can be confined by a conformal transformation to a 2-plane (for instance by sending a point to infinity and another to the origin). Then we can represent each *euclidean* point x_i by a complex number z_i so that

$$x_{ij}^2 = |z_{ij}|^2 = (z_i - z_j)(\bar{z}_i - \bar{z}_j). \tag{2.11}$$

To make the correspondence between 4-vectors x and complex numbers z explicit we fix a unit vector e and let n be a variable unit vector parametrizing a 2-sphere orthogonal to e. Then any euclidean 4-vector x can be written (in spherical coordinates) in the form:

$$x = r(cos\rho\, e + sin\rho\, n)\,,\ e^2 = 1 = n^2\,,\ en = 0\,,\ r \geq 0\,,\ 0 \leq \rho \leq \pi\ . \tag{2.12}$$

In these coordinates the 4D volume element takes the form

$$d^4x = r^3 dr sin^2\rho\, d\rho\, d^2n\ , \quad \int_{\mathbb{S}^2} d^2n = 4\pi\ . \tag{2.13}$$

We associate with the vector x (2.12) a complex number z such that:

$$z = re^{i\rho} \to x^2 (= r^2) = z\bar{z}\ , \quad (x-e)^2 = |1-z|^2 = (1-z)(1-\bar{z}) \tag{2.14}$$

$$\int_{n\in\mathbb{S}^2} \frac{d^4x}{\pi^2} = |z-\bar{z}|^2 \frac{d^2z}{\pi}\ , \int_{\mathbb{S}^2} \delta(x) d^4x = \delta(z) d^2z\ . \tag{2.15}$$

For a graph with four distinct external vertices in the φ^4 theory the amplitude (integrated over the internal vertices) has scale dimension 12 (in mass or inverse length units) and can be written in the form:

$$G(x_1, ..., x_4) = \frac{g(u,v)}{\prod_{i<j} x_{ij}^2} = \frac{F(z)}{\prod_{i<j} |z_{ij}|^2} \tag{2.16}$$

where the indices run in the range $1 \leq i < j \leq 4$, the (positive real) variables u, v, and (the complex) z are conformally invariant crossratios:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z}\ , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = |1-z|^2\ , \quad z = \frac{z_{12} z_{34}}{z_{13} z_{24}}\ . \tag{2.17}$$

The crossratios z and $\bar{z}$ are the simplest realizations of the argument z of the hyperlogarithmic functions introduced in the next section. They also appear (as a consequence of the so called *dual conformal invariance* [DHSS, DHKS]) in the expressions of momentum space integrals like

$$T(p_1^2, p_2^2, p_3^2) = \int \frac{d^4k}{\pi^2 k^2 (p_1+k)^2 (k-p_3)^2} = \frac{F(z)}{p_3^2} \tag{2.18}$$

where $p_1 + p_2 + p_3 = 0$, $\frac{p_1^2}{p_3^2} = z\bar{z}$, $\frac{p_2^2}{p_3^2} = |1-z|^2$ (see Eqs. (5-9) of [D]).

3 Double shuffle algebra of hyperlogarithms

The story of polylogarithms begins with the dilogarithmic function (see the inspired and inspiring lecture [Z] as well as the brief historical survey in the Appendix). Here we shall start instead with the modern general notion of a hyperlogarithm [B, B09] whose physical applications are surveyed in [P, D].

Let $\sigma_0 = 0, \sigma_1, ..., \sigma_N$ be distinct complex numbers corresponding to an alphabet $X = \{e_0, ..., e_N\}$. Let X^* be the set of words w in this alphabet including the empty word $\emptyset$. The hyperlogarithm $L_w(z)$ is an iterated integral [C, B09] defined recursively in any dense simply connected open subset U of the punctured complex plane $D = \mathbb{C} \setminus \Sigma$, $\Sigma = \{\sigma_0, ..., \sigma_N\}$ by the differential equations[2]

$$\frac{d}{dz} L_{w\sigma}(z) = \frac{L_w(z)}{z-\sigma}\,, \quad \sigma \in \Sigma\,, \quad L_\emptyset = 1\,, \tag{3.1}$$

and the initial condition

$$L_w(0) = 0 \quad \text{for} \quad w \neq 0^n (= 0 \dots 0), \quad L_{0^n}(z) = \frac{(\ln z)^n}{n!}. \tag{3.2}$$

Denoting by σ^n a word of n consecutive σ's we find, for $\sigma \neq 0$,

$$L_{\sigma^n}(z) = \frac{(\ln(1-\frac{z}{\sigma}))^n}{n!}. \tag{3.3}$$

There is a correspondence between iterated integrals and multiple power series:

$$(-1)^d L_{\sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) = Li_{n_1,\dots,n_d}(\frac{\sigma_2}{\sigma_1}, ..., \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}) \tag{3.4}$$

where $Li_{n_1,\dots,n_d}$ is given by the d-fold series

$$Li_{n_1,\dots,n_d}(z_1, ..., z_d) = \sum_{1 \leq k_1 < \dots < k_d} \frac{z_1^{k_1} \dots z_d^{k_d}}{k_1^{n_1} \dots k_d^{n_d}}. \tag{3.5}$$

More generally, we have

$$(-1)^d L_{0^{n_0}\sigma_1 0^{n_1-1} \dots \sigma_d 0^{n_d-1}}(z) =$$

$$\sum_{\substack{k_0 \geq 0\; k_i \geq n_i, 1 \leq i \leq d \\ k_0 + \dots + k_d = n_0 + \dots + n_d}} (-1)^{k_0+n_0} \prod_{i=1}^{d} \binom{k_i - 1}{n_i - 1} L_{0^{k_0}}(z)\, Li_{k_1 - k_r}(\frac{\sigma_2}{\sigma_1}, ..., \frac{\sigma_d}{\sigma_{d-1}}, \frac{z}{\sigma_d}). \tag{3.6}$$

[2] We use following [B11, S14] concatenation to the right. Other authors, [B14, D], use the opposite convention. This also concerns the definition of coproduct (4.27) (4.29) below.

In particular, $L_{01}(z) = Li_2(z) - \ln z Li_1(z) = Li_2(z) + \ln z \ln(1-z)$. The number of letters $|w| = n_0 + ... + n_d$ of a word w defines its *weight*, while the number d of non zero letters is its *depth*. We observe that the product $L_w L_{w'}$ of two hyperlogarithms of weights $|w|, |w'|$ and depths d, d' can be expanded in hyperlogarithms of weight $|w| + |w'|$ and depth $d + d'$ (as the product of simplices can be expanded into a sum of higher dimensional simplices). This observation can be formalized as follows. The set X^* of words can be equipped with a commutative *shuffle product* $w \sqcup\!\sqcup w'$ defined recursively by

$$\emptyset \sqcup\!\sqcup w = w (= w \sqcup\!\sqcup \emptyset), \quad au \sqcup\!\sqcup bv = a(u \sqcup\!\sqcup bv) + b(au \sqcup\!\sqcup v) \tag{3.7}$$

where u, v, w are (arbitrary) words while a, b are letters (note that the empty word $\emptyset$ is *not* a letter). We denote by

$$\mathcal{O}_\Sigma = \mathbb{C}\big[z, \Big(\frac{1}{z - \sigma_i}\Big)_{i=1,\dots,N}\big] \tag{3.8}$$

the ring of regular functions on D. Extending by $\mathcal{O}_\Sigma$ linearity the correspondence $w \to L_w$ one proves that it defines a homomorphism of shuffle algebras $\mathcal{O}_\Sigma \otimes \mathbb{C}(X) \to \mathcal{L}_\Sigma$ where $\mathcal{L}_\Sigma$ is the $\mathcal{O}_\Sigma$ span of $L_w, w \in X^*$. The commutativity of the shuffle product is made obvious by the identity

$$L_{u \sqcup\!\sqcup v} = L_u L_v (= L_v L_u). \tag{3.9}$$

It is easy to verify, in particular, that the dilogarithm $Li_2(z) (= -L_{10}(z))$ given by (3.5) for $d = 1, n_1 = 2$ disappears from the shuffle product:

$$L_{0 \sqcup\!\sqcup 1}(z) = L_{01}(z) + L_{10}(z) = L_0(z) L_1(z). \tag{3.10}$$

If the shuffle relations are suggested by the expansion of products of iterated integrals, the product of series expansions of type (3.5) suggests the (also commutative) *stuffle product*. Rather than giving a cumbersome general definition we shall illustrate the rule on the simple example of the product of depth one and depth two factors (cf. [D]):

$$\begin{aligned} Li_{n_1,n_2}(z_1, z_2) Li_{n_3}(z_3) &= Li_{n_1,n_2,n_3}(z_1, z_2, z_3) + \\ &Li_{n_1,n_3,n_2}(z_1, z_3, z_2) + Li_{n_3,n_1,n_2}(z_3, z_1, z_2) + \\ &Li_{n_1,n_2+n_3}(z_1, z_2 z_3) + Li_{n_1+n_3,n_2}(z_1 z_3, z_2). \end{aligned} \tag{3.11}$$

We observe the that the multiple polylogarithms of one variable (with $z_1 = ... = z_{d-1} = 1$ considered in [B04, S14] span a shuffle but not a stuffle algebra.

As seen from the above example the stuffle product also respects the weight but (in contrast to the shuffle product) only filters the depth (the depth of each term in the right hand side does not exceed the sum of depths of the factors in the left hand side (which is three in Eq. (3.11)).

It is convenient to rewrite the definition of hyperlogarithms in terms of a formal series $L(z)$ with values in the (free) tensor algebra $\mathbb{C}(X)$ (the complex vector space generated by all words in X^*) which satisfies the *Knizhnik-Zamoldchikov equation*:

$$L(z) := \sum_w L_w(z)w, \; \frac{d}{dz} L(z) = L(z) \sum_{i=0}^{N} \frac{e_i}{z - \sigma_i}. \tag{3.12}$$

One assigns weight -1 to e_σ, so that $L(z)$ carries weight zero. If the index of the hyperlogarithm L_w is expressed by its (potential) singularities σ_i the word w which multiplies it in the series (3.12) should be written in terms of the corresponding (noncommuting) symbols e_i (thus justifying the apparent doubling of notation). In the special case when the alphabet X consists of just two letters e_0, e_1 corresponding to $\sigma_0 = 0, \sigma_1 = 1$, $L(z)$ is the generating function of the *classical multipolylogarithms* while its value at $z = 1, Z := L(1)$ is the generating function of MZVs. In these notations the *monodromy* of L around the points 0 and 1 is given by

$$\mathcal{M}_0 \, L(z) = e^{2\pi i e_0} \, L(z) \, , \quad \mathcal{M}_1 \, L(z) = Z \, e^{2\pi i e_1} \, Z^{-1} L(z), \; Z = \sum_w \zeta_w w, \tag{3.13}$$

so that $\mathcal{M}_0 L_{0^n}(z) = L_{0^n}(z) + 2\pi i L_{0^{(n-1)}}(z)$, $\mathcal{M}_1 Li_n(z) = Li_n(z) - 2\pi i L_{0^{(n-1)}}(z)$. The first relation (3.13) follows from the fact that $L(z)$ is the unique solution of the Knizhnik-Zamolodchikov equation obeying the "initial" condition

$$L(z) = e^{e_0 \ln z} h_0(z), \; h_0(0) = 1, \tag{3.14}$$

$h_0(z)$ being a formal power series in the words in X^* that is holomorphic in z in the neighborhood of $z = 0$. The second relation (3.13) is implied by the fact that there exists a counterpart $h_1(z)$ of h_0, holomorphic around $z = 1$ and satisfying $h_1(1) = 1$ such that

$$L(z) = Z \, e^{e_1 \ln(1-z)} \, h_1(z) \tag{3.15}$$

(see [S14] or Appendix C of [T14]). This construction can be viewed as a special case (corresponding to $N = 1$) of a monodromy representation of the fundamental group of the punctured plane D (studied in Sect. 6 of [B]).

The next simplest case, N=2, including the square roots of unity ± 1, has been considered by physicists [RV] under the name *harmonic polylogarithms.* The value of the function

$$L_{-10^{n-1}}(z) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{z^k}{k^n} = -Li_n(-z) \tag{3.16}$$

at $z = 1$ is the *Euler phi function* [Ay] (alias, Dirichlet eta function)

$$\phi(n) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^n} = (1 - 2^{1-2^{1-n}})\zeta(n),\ \phi(1) = \ln(2), \tag{3.17}$$

a special case of the Dirichlet *L-functions* [S]. It is remarkable that the anomalous magnetic moment of the electron, the most precisely measured quantity in physics, is expressed in terms of eta values:

$$\frac{g-2}{2} = \frac{1}{2}\frac{\alpha}{\pi} + (\phi(3) - 6\phi(1)\phi(2) + \frac{197}{2^4 3^2})(\frac{\alpha}{\pi})^2 + ... \tag{3.18}$$

(see [Sch] where also the next (α^3-)contribution is expressed in terms of (multiple) eta values). The same weight three combination, $\phi(3) - 6\phi(1)\phi(2)$, appears in the second order of the Lamb shift calculation (see [LPR]).

Remark 3.1 The repeated application of the recursive differential equations (3.1) leads to $dL_\sigma(z) = \frac{dz}{z-\sigma}$ ($d1 = 0$). Brown [B] calls such differential equations *unipotent* and proves that the double shuffle algebra L_Σ is a *differential graded* (by the weight) *algebra.*

The weight of consecutive terms in the expansion of $L(z)$ (3.15) is the sum of the weights of hyperlogarithms and the zeta factors. It is thus natural to begin the study of multiple polylogarithms with the algebra of MZVs.

4 Formal multizeta values

4.1 Shuffle regularized MZVs

We now turn to the alphabet X of two letters e_0, e_1 corresponding to $\sigma_0 = 0, \sigma_1 = 1$ and restrict the multiple polylogarithm (3.5) to a single variable:

$$Li_{n_1,...,n_d}(z) = \sum_{1 \le k_1 < ... < k_d} \frac{z^{k_d}}{k_1^{n_1} ... k_d^{n_d}}.$$

The MZV $\zeta(n_1, ..., n_d)$ is then defined as its value at 1 whenever the corresponding series converges. Using also (3.4) we can write:

$$(-1)^d \zeta_{10^{n_1-1}...10^{n_d-1}} = \zeta(n_1, ..., n_d) = \sum_{1 \le k_1 < ... < k_d} \frac{1}{k_1^{n_1} ... k_d^{n_d}} \text{ for } n_d > 1. \quad (4.1)$$

The convergent MZVs of a given weight satisfy a number of shuffle and stuffle identities. Looking for instance at the shuffle (sh) and the stuffle (st) products of two $-\zeta_{10} = \zeta(2)$ we find:

$$sh : \zeta_{10}^2 = 4\zeta_{1100} + 2\zeta_{1010} (= 4\zeta(1,3) + 2\zeta(2,2)); \; st : \zeta(2)^2 = 2\zeta(2,2) + \zeta(4);$$
$$\text{hence } \zeta(4) = 4\zeta(1,3) = \zeta(2)^2 - 2\zeta(2,2). \quad (4.2)$$

There are no non-zero convergent words of weight 1 and hence no shuffle or stuffle relations of weight 3. On the other hand, already Euler has discovered the relation: $\zeta(1,2) = \zeta(3)$. Thus shuffle and stuffle relations among convergent words do not exhaust all known relations among MZVs of a given weight. Introducing the divergent zeta values which correspond to $n_d = 1$ we observe that they cancel in the difference between the shuffle and stuffle products $u \sqcup\!\sqcup v - u * v$ of divergent words. For instance, at weight 3 we have

$$\zeta((1) \sqcup\!\sqcup (2)) = 2\zeta(1,2) + \zeta(2,1); \; \zeta((1) * (2)) = \zeta(1,2) + \zeta(3) + \zeta(2,1). \quad (4.3)$$

Extending the homomorphism $w \to \zeta(w)$ as a homomorphism of both the shuffle and the stuffle algebras to divergent words, assuming, in particular, that $\zeta((1) \sqcup\!\sqcup (2)) = \zeta((1) * (2)) = \zeta(1)\zeta(2)$ and taking the difference of the two equations (4.3) we observe that all divergent zeta's cancel and we recover the above Euler's identity.

Remark 4.1 In fact, the shuffle product is naturally defined (as we did in (3.7)) in the two-letter alphabet $\{0, 1\}$ used as lower indices, while the stuffle product has a simple formulation in the infinite alphabet of all positive integers, appearing (in parentheses) as arguments of zeta. Eq. (4.1) provides the translation between the two:

$$\vec{n} = (n_1, ..., n_d) \leftrightarrow (-1)^n \rho(\vec{n}) \text{ for } \rho(\vec{n}) = 10^{n_1-1}...10^{n_d-1}. \quad (4.4)$$

Using this correspondence one obtains, in particular, the first relation (4.3).

It is useful to introduce shuffle regularized MZVs using the following result (see Lemma 2.2 of [B11]).

Proposition 4.1. There is a unique way to define a set of real numbers $I(a_0; a_1, ..., a_n; a_{n+1})$ *for any* $a_i \in \{0, 1\}$*, such that*

(i) $I(0; 1, a_2, ...a_{n-1}, 0; 1) = (-1)^d \zeta(n_1, ..., n_d)$ *for* $\rho(\vec{n}) = (1, a_2, ..., a_{n-1}, 0)$;

(ii) $I(a_0; a_1; a_2) = 0, I(a_0, a_1) = 1$ *for all* $a_0, a_1, a_2 \in \{0, 1\}$;

(iii) $I(a_0; a_1, ..., a_r; a_{n+1}) I(a_0; a_{r+1}, ..., a_{r+s}) = \sum_{\sigma \in \Sigma(r,s)} I(a_0; a_{\sigma(1)}, ..., a_{\sigma(r+s)}; a_{n+1})\ (r + s = n)$;

(iv) $I(a; a_1, ..., a_n; a) = 0$ *for* $n > 0$;

(v) $I(a_0; a_1, ..., a_n; a_{n+1}) = (-1)^n I(a_{n+1}; a_n, ..., a_1; a_0)$;

(vi) $I(a_0; a_1, ..., a_n; a_{n+1}) = I(1 - a_{n+1}; 1 - a_n, ..., 1 - a_1; 1 - a_0)$. (4.5)

Here $\Sigma(r, s)$ is the set of permutations of the indices $(1, ..., n)$ preserving the order of the first r and the last s among them; Eq. (v) is the reverse of path formula, while (vi) expresses functoriality with respect to the map $t \to 1-t$. Eq. $\zeta(n_1, ..., n_d) = (-1)^d I(0; \rho(\vec{n}); 1)$ then defines the *shuffle regularized zeta values for all* $n_d \geq 1$. Condition (ii) implies, in particular, $\zeta(1) = 0$.

In fact, it suffices to add a condition involving multiplication by the divergent word (1),

$$\zeta((1) ⧢ w - (1) * w) = 0 \text{ for all convergent words } w, \tag{4.6}$$

to the shuffle and stuffle relations among convergent words in order to obtain all known relations among MZVs of a given weight. For $w = (n), n \geq 2$ (a word of depth 1), Eq. (4.6) gives

$$\zeta((1) ⧢ (n) - (1) * (n)) = \sum_{i=1}^{n-1} \zeta(i, n + 1 - i) - \zeta(n + 1) = 0 \tag{4.7}$$

(a relation known to Euler). The discovery (and the proof) that

$$\zeta(2n) = -\frac{B_{2n}}{2(2n)!}(2\pi i)^{2n},\ B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42},\ (-1)^{n-1} B_{2n} \in \mathbb{Q}_{>0}, \tag{4.8}$$

where B_n are the (Jacob) Bernoulli numbers, was among the first that made Euler famous (see Appendix). Nothing is known about the transcendentality of $\zeta(n)$ (or of $\frac{\zeta(n)}{\pi^n}$) for odd n. We introduce following Leila Schneps [S11] the notion of the $\mathbb{Q}$-algebra $\mathcal{FZ}$ of *formal MZVs* ζ^f which satisfy the relations:

$$\zeta^f(1) = 0,\ \zeta^f(u)\zeta^f(v) = \zeta^f(u ⧢ v) = \zeta^f(u * v),\ \zeta^f((1) ⧢ w - (1) * w) = 0. \tag{4.9}$$

The algebra $\mathcal{FZ} = \bigoplus_n \mathcal{FZ}_n$ is *weight graded* and

$$\mathcal{FZ}_0 = \mathbb{Q}, \mathcal{FZ}_1 = \{0\}, \mathcal{FZ}_2 = \langle\zeta(2)\rangle, \mathcal{FZ}_3 = \langle\zeta(3)\rangle, \mathcal{FZ}_4 = \langle\zeta(4)\rangle,$$
$$\mathcal{FZ}_5 = \langle\zeta(5), \zeta(2)\zeta(3)\rangle, \mathcal{FZ}_6 = \langle\zeta(2)^3, \zeta(3)^2\rangle,$$
$$\mathcal{FZ}_7 = \langle\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)\rangle, \quad (4.10)$$

where $< x, y, ... >$ is the $\mathbb{Q}$ vector space spanned by $x, y, ...$ (and we have replaced ζ^f by ζ in the right hand side for short). Clearly, there is a surjection $\zeta^f \to \zeta$ of $\mathcal{FZ}$ onto $\mathcal{Z}$. **The main conjecture in the theory of MZVs** is that *this surjection is an isomorphism of graded algebras.* This is a strong conjecture. If true it would imply that there is no linear relation among MZVs of different weights over the rationals. Actually, a less obvious statement is valid: such an isomorphism would imply that all MZVs are transcendental. Indeed, if a non-zero multiple zeta value is algebraic, then expanding out its minimal polynomial according to the shuffle relation $\zeta(u)\zeta(v) = \zeta(u \sqcup\!\sqcup v)$ (starting with $\zeta^2(w)$) would give a linear combination of multiple zetas in different weights equal to zero, contradicting the weight grading. In fact, we only know that there are infinitely many linearly independent over $\mathbb{Q}$ odd zeta values (Ball and Rivoal, 2001) and that $\zeta(3)$ is irrational (Apéry, 1978). From now on, we shall follow the physicists' practice to treat this conjecture as true and to omit the f's in the notation for (formal) MZVs.

Examples: E1. In order to see that the space $\mathcal{Z}_4$ of weight four zeta values is 1-dimensional we should add to Eqs. (4.2) the relation (4.7) for $n = 3$ and its depth three counterpart:

$$\zeta((1)\sqcup\!\sqcup(1,2) - (1)*(1,2)) = \zeta(1,1,2) - \zeta(1,3) - \zeta(2,2) (= \zeta(1,1,2) - \zeta(4)) = 0. \quad (4.11)$$

This allows to express all zeta values of weight four as (positive) integer multiples of $\zeta(1,3)$ (see Eq. (B.8) of [T14]).

E2. The shuffle and the stuffle products corresponding to $\zeta(2)\zeta(3)$ give two relations which combined with (4.7) for $n = 4$ allow to express the three double zeta values of weight five in terms of simple ones:

$$\zeta(1,4) = 2\zeta(5) - \zeta(2)\zeta(3),\ \zeta(2,3) = 3\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5),$$
$$\zeta(3,2) = \frac{9}{2}\zeta(5) - 2\zeta(2)\zeta(3). \quad (4.12)$$

In general, the number of convergent words of weight n and depth d in the alphabet $\{0,1\}$ is $\binom{n-2}{d}$, so their number at weight n is 2^{n-2}. As it follows from Eq. (4.10) the number of relations also grows fast: there are six relations among the eight MZVs at weight five; 14 such relations at weight six, 29, at weight seven. One first needs a double zeta value, say $\zeta(3,5)$, in order to write a basis (of four elements) at weight eight (there being 60 relations among the 2^6 elements of $\mathcal{FZ}_8$). Taking the identities among (formal) zeta values into account we can write the generating series Z of MZV (also called *Drinfeld's associator*) in terms of multiple commutators of e_0, e_1:

$$Z = 1 + \zeta(2)[e_0, e_1] + \zeta(3)[[e_0, e_1], e_0 + e_1] + ... \tag{4.13}$$

It is natural to ask what is the dimension d_n of the space $\mathcal{FZ}_n$ of (formal) MZVs of any given weight n and then to construct a basis of independent elements. These problems have only been solved for the so called *motivic MZV*. Here is a simple-minded substitute of their abstract construction.

4.2 Hopf algebra of motivic zeta values

Consider the *concatenation algebra*

$$\mathcal{C} = \mathbb{Q}\langle f_3, f_5, ...\rangle, \tag{4.14}$$

the free algebra over $\mathbb{Q}$ on the countable alphabet $\{f_3, f_5, ...\}$ (see Example 21 in [Wa]). *The algebra of motivic zeta values* is identified (non-canonically) with the algebra

$$\mathcal{C}[f_2] = \mathcal{C} \otimes_{\mathbb{Q}} \mathbb{Q}[f_2], \tag{4.15}$$

which plays an important role in the theory of mixed Tate motives (see Sect. 3 of [B11]).The algebra $\mathcal{C}[f_2]$ is graded by the weight (the sum of indices of f_i) and it is straightforward to compute the dimension d_n of the weight spaces $\mathcal{C}[f_2]_n$ for any n. Indeed, the generating (or *Hibert-Poincaré*) series for the dimensions $d_n^{\mathcal{C}}$ of the weight n subspace of $\mathcal{C}$ is given by

$$\sum_{n\geq 0} d_n^{\mathcal{C}} t^n = \frac{1}{1 - t^3 - t^5 - ...} = \frac{1 - t^2}{1 - t^2 - t^3} \tag{4.16}$$

while the corresponding series of the second factor $\mathbb{Q}[f_2]$ in (4.15) is $(1 - t^2)^{-1}$. Multiplying the two we obtain the dimensions d_n of the weight spaces conjectured by Don Zagier:

$$\sum_{n\geq 0} d_n t^n = \frac{1}{1 - t^2 - t^3}, \; d_0 = 1, d_1 = 0, d_2 = 1, d_{n+2} = d_n + d_{n-1}. \tag{4.17}$$

Here is a wonderful more detailed conjecture advanced by Broadhurst and Kreimer [BK] (1997); its motivic version is still occupying mathematicians.

Let $\mathcal{Z}_n^r$ be the linear span of $\zeta(n_1, ..., n_k)$, $n_1 + ... + n_k = n$, $k \le r$; we define $d_{n,r}$ as the dimension of the quotient space $\mathcal{Z}_n^r / \mathcal{Z}_n^{r-1}$. Broadhurst and Kreimer have advanced the following conjecture for the generating series of $d_{n,r}$ (based on experience with MZVs appearing in Feynman amplitudes):

$$D(X, Y) = \frac{1 + \mathcal{E}(X)Y}{1 - \mathcal{O}(X)Y + \mathcal{S}(X)Y^2(1 - Y^2)} = \sum d_{n,r} X^n Y^r. \qquad (4.18)$$

Here $\mathcal{E}(X)$ and $\mathcal{O}(X)$ generate series of even and odd powers of X,

$$\mathcal{E}(X) = \frac{X^2}{1 - X^2} = X^2 + X^4 + ..., \ \mathcal{O}(X) = \frac{X^3}{1 - X^2} = X^3 + X^5 + ..., \qquad (4.19)$$

while $\mathcal{S}(X)$ is the generating series for the dimensions of the spaces of *cusp modular forms* (see for background the physicists' oriented survey [Za]):

$$\mathcal{S}(X) = \frac{X^{12}}{(1 - X^4)(1 - X^6)}. \qquad (4.20)$$

Setting in (4.18) $Y = 1$ we recover the Zagier conjecture (4.17) (with $d_n = \sum_r d_{n,r}$), proven for motivic MZVs. The ansatz (4.18) can presently only be derived in the motivic case under additional assumptions (cf. [CGS]).

The concatenation algebra $\mathcal{C}$, identified with the quotient

$$\mathcal{C} = \mathcal{C}[f_2]/\mathbb{Q}[f_2], \qquad (4.21)$$

can be equipped with a *Hopf algebra structure* (with f_i as primitive elements) with the *deconcatenation coproduct* $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ given by

$$\Delta(f_{i_1} ... f_{i_r}) = 1 \otimes f_{i_1} ... f_{i_r} + f_{i_1} ... f_{i_r} \otimes 1 + \sum_{k=1}^{r-1} f_{i_1} ... f_{i_k} \otimes f_{i_{k+1}} ... f_{i_r}. \qquad (4.22)$$

This coproduct can be extended to the trivial comodule $\mathcal{C}[f_2]$ (4.15) by setting

$$\Delta : \mathcal{C}[f_2] \to \mathcal{C} \otimes \mathcal{C}[f_2], \ \Delta(f_2) = 1 \otimes f_2 \qquad (4.23)$$

(and assuming that f_2 commutes with f_{odd}). Remarkably, there appear to be a one-to-one (albeit non-canonical) correspondence between the bases of the weight spaces $\mathcal{Z}_n$ and $\mathcal{C}[f_2]_n$ as displayed in the following list ([B11], 3.4)

$$\begin{aligned}
&\langle\zeta(2)\rangle \leftrightarrow \langle f_2\rangle; \langle\zeta(3)\rangle \leftrightarrow \langle f_3\rangle; \langle\zeta(2)^2\rangle \leftrightarrow \langle f_2^2\rangle;\\
&\langle\zeta(5), \zeta(2)\zeta(3)\rangle \leftrightarrow \langle f_5, f_2 f_3 (= f_3 f_2)\rangle; \langle\zeta(2)^3, \zeta(3)^2\rangle \leftrightarrow \langle f_2^3, f_3 \sqcup\!\sqcup f_3\rangle;\\
&\langle\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)\rangle \leftrightarrow \langle f_7, f_2 f_5, f_2^2 f_3\rangle;\\
&\langle\zeta(2)^4, \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5), \zeta(3,5)\rangle \leftrightarrow \langle f_2^4, f_3 \sqcup\!\sqcup f_3 f_2, f_3 \sqcup\!\sqcup f_5, f_5 f_3\rangle. \quad (4.24)
\end{aligned}$$

There is a counterpart of Proposition 4.1 defining *motivic iterated integrals* whose Hopf algebra[3], [Gon], is non-canonically isomorphic to $\mathcal{C}[f_2]$. It allows to define a surjective *period map* $\mathcal{C}[f_2] \to \mathcal{Z}$ onto the algebra of real MZVs ([B11] Theorem 3.5). Since, on the other hand, $\mathcal{C}[f_2]$ satisfies the defining relations of the formal zeta values we have the surjections $\mathcal{FZ} \to \mathcal{C}[f_2] \to \mathcal{Z}$. Our main conjecture would then mean that the two (surjective) maps are also injective and thus define isomorphisms of graded algebras.. If true it would imply that the (infinite sequence of) numbers $\pi, \zeta(3), \zeta(5), ...$ are transcendental algebraically independent over the rationals (cf. [Wa]). It would also fix the dimension of the weight spaces $\mathcal{Z}_n$ to be equal to d_n (4.17). Presently, we only know that this is true for $n = 0, 1, 2, 3, 4$; in general, the above cited results prove that

$$dim\mathcal{Z}_n \leq d_n \ , \quad dim\mathcal{Z}_n = d_n \quad \text{for} \quad n \leq 4 \ . \qquad (4.25)$$

Remark 4.2 The validity of the above sharpened conjecture would imply, in particular, that $\zeta(2n+1)$ are primitive elements of the Hopf algebra of MZVs:

$$\Delta(\zeta(2n+1)) = \zeta(2n+1) \otimes 1 + 1 \otimes \zeta(2n+1). \qquad (4.26)$$

Eq. (4.8) precludes the possibility of extending this property to even zeta values. Indeed, it implies the relation $\zeta(2n) = b_n \zeta(2)^n, b_n = \frac{(24)^n |B_{2n}|}{2(2n)!}$ which is only compatible with the one-sided coproduct $\Delta\zeta(2) = 1 \otimes \zeta(2)$.

If for weights $n \leq 7$ one can express all MZVs in terms of (products of) simple zeta values (of depth one) the last equation (4.24) shows that for $n \geq 8$ this is no longer possible. Brown [B12] has established that the *Hoffman elements* $\zeta(n_1, ..., n_d)$ with $n_i \in \{2, 3\}$ form a basis of motivic zeta values for all n (see also [D12, Wa]).

The coproduct for MZV, described in Remark 4.2 extends to hyperlogarithms and can be formulated in terms of the regularized iterated integrals of

[3] Brown's definition which we follow differs from Goncharov's (adopted in [CGS]) in that the motivic $\zeta^m(2)$ is non-zero

Proposition 4.1. - see Theorem 3.8 of [B11] and Sect. 5.3 of [D]. Here we shall just reproduce the special case of the coproduct of a classical polylogarithm:

$$\Delta Li_n(z) = Li_n(z) \otimes 1 + \sum_{k=0|}^{n-1} \frac{(\ln z)^k}{k!} \otimes Li_{n-k}(z). \tag{4.27}$$

According to Remark 4.2 specializing to $z = 1$ in (4.27) for even n leads to a contradiction unless we factor the algebra of hyperlogarithms by $\zeta(2)$ or, better, by $\ln(-1) = i\pi (= \sqrt{-6\zeta(2)})$ setting

$$\mathcal{H} := \mathcal{L}_\Sigma / i\pi \mathcal{L}_\Sigma \quad \text{so that} \quad \mathcal{L}_\Sigma = \mathcal{H}[i\pi] \ . \tag{4.28}$$

The coaction Δ is then defined on the comodule $\mathcal{L}_\Sigma$ as follows:

$$\Delta : \mathcal{L}_\Sigma \to \mathcal{H} \otimes \mathcal{L}_\Sigma \ , \quad \Delta(i\pi) = 1 \otimes i\pi \ . \tag{4.29}$$

The asymmetry of the coproduct is also reflected in its relation to differentiation and to the discontinuity $disc_\sigma = \mathcal{M}_\sigma - 1$:

$$\Delta(\frac{\partial}{\partial z} F) = (\frac{\partial}{\partial z} \otimes id)\Delta F \ , \quad \Delta(disc_\sigma F) = (id \otimes disc_\sigma)\Delta F \ . \tag{4.30}$$

We leave it to the reader to verify that e.g. for $F = Li_2(z)$ both sides of (4.30) give the same result. This allows us, in particular, to consider $\mathcal{L}_\Sigma$ as a *differential graded Hopf algebra*.

5 Single-valued hyperlogarithms. Applications

Knowing the action of the monodromy M_{σ_i} around each singular point of a hyperlogarithm one can construct single valued hyperlogarithms in the tensor product of $\mathcal{L}_\Sigma$ with its complex conjugate [B]. We shall survey this construction for classical multiple polylogarithms $\mathcal{L}_\Sigma = \mathcal{L}_c$, defined as $\mathcal{O}$-linear combination of $L_w(z)$ for w, words in the "Morse alphabet" $X = \{e_0, e_1\} \leftrightarrow \{0, 1\}$, where $\mathcal{O} = \mathbb{C}[z, \frac{1}{z}, \frac{1}{z-1}]$. This case is spelled out in [B04, S14]. The tensor product $\bar{\mathcal{L}_c} \otimes \mathcal{L}_c$ contains functions of $(\bar{z}, z)$ transforming under arbitrary representations of the monodromy group (see Theorem 7.4 of [B]) including the trivial one, - i.e. the *single-valued multiple polylogarithms*

(SVMPs). We introduce an $\bar{\mathcal{O}}\mathcal{O}$ basis of homogeneous SVMPs $P_w(z)$ and will denote by

$$P_X(z) = \sum_{w \in X^*} P_w(z) w \tag{5.1}$$

its generating series. Their significance stems from the fact that a large class of euclidean Feynman amplitudes are single valued. The following theorem is a special case of Theorem 8.1 proven by Brown [B] (coinciding with Theorem 2.5 of [S14]).

Theorem 5.1 There exists a unique family of single-valued functions $P_w(z), w \in X^*, z \in \mathbb{C} \setminus \{0,1\}$ such that their generating function (5.1) satisfies the following Knizhnik-Zamolodchikov equations and initial condition:

$$\begin{aligned} &\partial P_X(z) = P_X(z)(\frac{e_0}{z} + \frac{e_1}{z-1}) \ , \ \partial := \frac{\partial}{\partial z} \ , \ \bar{\partial} := \frac{\partial}{\partial \bar{z}}, \\ &\bar{\partial} P_X(z) = (\frac{e_0}{\bar{z}} + \frac{e_1'}{1-\bar{z}}) P_X(z) \ , \ Z_{-e_0,-e_1'} e_1' Z^{-1}_{-e_0,-e_1'} = Z_{e_0,e_1} e_1 Z^{-1}_{e_0,e_1} \ , \\ &P_X(z) \sim e^{e_0 \ln(z\bar{z})} \quad \mathit{for} \quad z \sim 0 \ . \end{aligned} \tag{5.2}$$

The functions $P_w(z)$ are linearly independent over $\bar{\mathcal{O}}\mathcal{O}$ and satisfy the shuffle relations. Every element of their linear span has a primitive with respect to $\frac{\partial}{\partial z}$, and every single valued function $F(z) \in \bar{\mathcal{L}}_c\mathcal{L}_c$ can be written as a unique $\bar{\mathcal{O}}\mathcal{O}$-linear combination of $P_w(z)$.

The equation for e_1' is dictated by the expression for the monodromy of $L_w(z)$ (3.13) around $z=1$ and can be solved recursively in terms of elements of the Lie algebra over the *ring of zeta integers* $\mathbb{Z}[\mathcal{Z}]$, generated by e_0, e_1 and their multiple commutators (see Lemma 2.6 of [S14]). The result is:

$$e_1' = e_1 + 2\zeta(3)[[[e_0, e_1], e_1] \ , \ e_0 + e_1] + \zeta(5)(...) + ... , \tag{5.3}$$

where the parenthesis multiplying $\zeta(5)$ consists of eight bracket words of degree six in $\{e_0, e_1\}$. It follows, in particular, that $e_1' = e_1$ for words of weight not exceeding three or depth not exceeding one.

We proceed to constructing some simple examples of basic SVMPs. For words involving (repeatedly) a single letter we have

$$P_{0^n}(z) = \frac{(\ln \bar{z}z)^n}{n!} \ (P_w(0) = 0 \ \text{for} \ w \neq 0^n, w \neq \emptyset), \ P_{1^n}(z) = \frac{(\ln|1-z|^2)^n}{n!}. \tag{5.4}$$

The depth-one-weight-two SVMPs, which satisfy the differential equations

$$\partial P_{01} = \frac{P_0}{z-1} \ , \quad \bar{\partial} P_{01} = \frac{P_1}{\bar{z}} \ (P_{01}(0) = 0 = P_{10}(0)) \ ,$$
$$\partial P_{10} = \frac{P_1}{z} \ , \quad \bar{\partial} P_{10} = \frac{P_0}{\bar{z}-1} \ , \tag{5.5}$$

are given by

$$P_{01} = L_{10}(\bar{z}) + L_{01}(z) + L_0(\bar{z})L_1(z) = Li_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln(1-z),$$
$$P_{10} = L_{01}(\bar{z}) + L_{10}(z) + L_1(\bar{z})L_0(z) = Li_2(\bar{z}) - Li_2(z) + \ln \bar{z}z \ln(1-\bar{z}) \ . \tag{5.6}$$

They obey the shuffle relation $P_{01} + P_{10} = P_0 P_1$ so that the only new weight two function is their difference,

$$P_{01} - P_{10} = 2(Li_2(z) - Li_2(\bar{z}) + \ln \bar{z}z \ln \frac{1-z}{1-\bar{z}}) = 4iD(z) \ , \tag{5.7}$$

proportional to the *Bloch-Wigner dilogarithm* (see [Bl] as well as the stimulating survey [Z]), $D(z) = Im(Li_2(z) + \ln(1-z) \ln |z|)$. One can also write down depth-one SVMPs of arbitrary weight encountered in the expression $F_n(z)$ for the graphical function associated with the *wheel diagram with* $(n+1)$ *spokes*, first computed by Broadhurst in 1985 (for a modern treatment and references to earlier work - see [S14]):

$$F_n(z) = (-1)^n \frac{P_{0^{n-1}10^n}(z) - P_{0^n 10^{n-1}}(z)}{z - \bar{z}} =$$
$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{n} P_{0^{n-k}}(z) \frac{Li_{n+k}(z) - Li_{n+k}(\bar{z})}{z - \bar{z}} \ . \tag{5.8}$$

The period of the wheel amplitude is given by the limit of this expression for $z \to 1$

$$F_n(1) = \binom{2n}{n} Li_{2n-1}(1) = \binom{2n}{n} \zeta(2n-1). \tag{5.9}$$

Just like MZVs appear as values at $z = 1$ of multiple polylogarithms the values at one of SVMPs define *single-valued periods* [B13] which find applications in QFT (and in superstring theory - in the hands of Stephan Stieberger). Their generating function is

$$Z^{sv} = P_{e_0,e_1}(1) = 1 + 2\zeta(3)[e_0,[e_1,e_0]] + 2\zeta(5)(...) + ... \Rightarrow \zeta^{sv}(2) = 0. \tag{5.10}$$

The structure of a graded Hopf algebra of the family of hyperlogarithms allows to read off there symmetry properties from the simpler properties of ordinary logarithms, as illustrated in Example 25 of [D] which begins with a derivation of the inversion formula for the dilog: $Li_2\left(\frac{1}{x}\right) = i\pi \ln x - Li_2(x) - \frac{1}{2}\ln^2 x + 2\zeta(2)$. Remarkably, the SVMPs satisfy simpler symmetry relations under the permutation group $\mathcal{S}_3$ of Möbius transformations of $\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}$ that interchange the singular points (see Sect. 2.6 of [S14]). $\mathcal{S}_3$ is generated by two involutive transformations, $s_1 : z \to 1-z$, $s_2 : z \to \frac{1}{z}$ such that $s_1 s_2 : z \to \frac{z-1}{z}$, $(s_1 s_2)^3 = 1$. The formal power series $P_{e_0,e_1}(z)$ satisfies simple symmetry relations under s_1 and s_2 (cf. Lemma 2.17 of [S14]):

$$P_{e_0,e_1}(1-z) = P_{e_0,e_1}(1)P_{e_1,e_0}(z),\; P_{e_0,e_1}(\frac{1}{z}) = P_{e_0,-e_0-e_1}(1)P_{-e_0-e_1,e_1}(z). \tag{5.11}$$

According to (5.10) the first factor in the right hand side of (5.11) does not contribute to the transformation law of SVMPs of weight one and two; s_1 just permutes the indices 0 and 1 while $P_0(\frac{1}{z}) = -P_0(z), P_1(\frac{1}{z}) = P_1(z) - P_0(z)$ and

$$P_{01}(\frac{1}{z}) = P_{00}(z) - P_{01}(z),\; P_{10}(\frac{1}{z}) = P_{00}(z) - P_{10}(z) \Rightarrow D(\frac{1}{z}) = -D(z) \tag{5.12}$$

where D(z) is the Bloch-Wigner dilogarithm (5.7).

Finally we shall demonstrate as a simple illustration of the theory how one can calculate - without really integrating - the integral

$$I(x_1,x_2,x_3,x_4) = \int \frac{d^4x}{\pi^2} \prod_{i=1}^{4} \frac{1}{(x-x_i)^2} = \frac{f(u,v)}{x_{13}^2 x_{24}^2}, \tag{5.13}$$

where u, v are the crossratios (2.17). Using the conformal invariance of $f(u,v)$ we can set $x_1 \to \infty, x_2 = e\,(e^2 = 1), x_4 = 0; x_3^2 = \bar{z}z, (x_3 - e)^2 = |1-z|^2$ (cf. Sect. 2.3). Applying to the result the 4-dimensional Laplacian with respect to x_3 which acts on $F(z) = f(u,v)$ as $\frac{1}{4}\Delta_3 F(z) = \frac{1}{z-\bar{z}}\bar{\partial}\partial[(z-\bar{z})F(z)]$, and using the fact that the massless scalar propagator is the Green function of $-\Delta$ we obtain:

$$\bar{\partial}\partial[(z-\bar{z})F(z)] = \frac{\bar{z}-z}{\bar{z}z|1-z|^2} = \frac{1}{\bar{z}(z-1)} - \frac{1}{z(\bar{z}-1)}$$
$$\Rightarrow F(z) = \frac{P_{01}(z) - P_{10}(z)}{z-\bar{z}}. \tag{5.14}$$

Thus $F(z)$ is given by (5.8) for $n = 1$, $(z - \bar{z})F(z)$ being the only odd with respect to complex conjugation SVMP of weight two. We note that the odd denominator $z - \bar{z}$ also comes from the Jacobian J of the change of integration variables $\{x^\alpha\} \to \{D_i = (x - x_i)^2\}$, $\alpha, i = 1, ..., 4$ in (5.13):

$$I(x_1, ..., x_4) = \frac{1}{\pi^2} \int \frac{1}{J} \prod_{i=1}^{4} \frac{dD_i}{D_i},\ J = det\left(\frac{\partial D_i}{\partial x^\alpha}\right). \tag{5.15}$$

Indeed, at the singularity $D_i = 0$ we have

$$J|_{D_i=0} = 4x_{13}^2 x_{24}^2 \sqrt{2(u + v + uv) - 1 - u^2 - v^2} = 4x_{13}^2 x_{24}^2 \sqrt{-(z - \bar{z})^2}. \tag{5.16}$$

(More about the "d(log) forms and generalized unitarity cuts" the reader will find in Sect. 6 of [H].) Integrals of the type of (5.13) have been calculated long ago by more conventional methods [UD]. For an application of the present techniques to a (previously unknown) 3-loop correlator - see [DDEHPS].

6 Outlook

Multidimensional Feynman integrals give rise to a family of functions and numbers with the structure of a differential graded double shuffle Hopf algebra. It is displayed most readily for conformally invariant position space amplitudes in a massless QFT.

The dimensions of weight spaces of MZVs (which exhaust the Feynman periods up to six loops in the massless φ^4 theory) do not exceed - and are conjectured to coincide with - their motivic counterparts studied by Francis Brown [B11]. Values of hyperlogarithms at sixth roots of unity first appear at seven loops. For the two-loop sunrise integral with massive propagators one encounters multiple elliptic polylogarithms [BV, ABW, BKV].

The interplay between algebraic geometry, number theory and perturbative QFT is a young and vigorous subject and our survey is far from complete. We have not touched upon the application of cluster algebras to multileg on-shell Feynman amplitudes - see [GGSVV] for a remarkable first step in this direction. As hyperlogarithms and associated numbers do not suffice for expressing massive and higher order Feynman amplitudes, mathematicians are exploring their generalizations [BL, Brown, B15] and physicists are closely following this development [ABW, BS]. For the connections of MZVs with other parts of mathematics (including the Grothendieck-Teichmüller Lie algebra, mixed Tate motives and modular forms) - see [S11].

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Appendix. Historical Notes

Leonhard Euler (1707-1783)

Just as Archimedes (287-212) dominated the mathematics of the 3d century BC so, 2000 years later, Euler is dominating the mathematics of 18th century. Born in the family of the Protestant minister of the parish church in Riehen, a suburb of the free (Swiss) city of Basel (see [G] where pictures of the church and of the parish residence are reproduced), he entered the University of Basel at the age of 13 to study theology. His mathematics professor, Leibniz's student Johann Bernoulli (1667-1748 - who had inherited the Basel chair of his brother Jacob, 1654-1705), offered to give private consultations at his home to the diligent boy on Saturday afternoons. At the age of twenty Euler accepted a call to the Academy of Sciences of St. Petersburg (founded a few years earlier by the czar Peter I, the Great) where two of Johann's sons, Daniel and Niklaus Bernoulli, were already active. Contrary to most other foreign members he mastered quickly the Russian language, both in writing and speaking. It is during this very active first St. Petersburg period (along with major work on mechanics, music theory, and naval architecture) that Euler first became interested (around 1729) in the "Basel problem" - the problem of finding what we would now call (after Riemann) $\zeta(2)(=\sum_1^\infty 1/n^2)$. (It is actually a problem which a young professor in Bologna, Pietro Mengoli, successor of the great Cavalieri, posed in 1644/1650 [Ay] and which excited the brothers-rivals Jacob and Johann Bernoulli in Basel.) Euler started by devising efficient approximation for calculating the (slowly convergent) series for $\zeta(2)$. As Weil [W] puts it "as with most of the questions that ever attracted his attention, he never abandoned it, soon making a number of fundamental contributions ...". In 1731 the 24-year-old Euler introduced the "Euler-Mascheroni constant" (see [La]):

$$\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k} - \ln n) = \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n}(= 0.5772...) \ . \qquad (A.1)$$

He then discovered the so-called Euler-MacLaurin formula and introduced for the first time the Bernoulli numbers into the subject [W]. Next came, in 1735, his sensational discovery of the formula $\zeta(2) = \frac{\pi^2}{6}$, based on a bold application of the theory of algebraic equations to the transcendental equation $1 - sinx = 0$. This was soon followed by by the calculation of $\zeta(m)$ for $m = 4, 6$, etc. In the same period Euler calculated $\zeta(3)$ up to ten significant digits and convinced himself that it is not a rational multiple of π^3 (with a small denominator) and found on the way the identity $\zeta(1, 2) = \zeta(3)$ [D12]. Peeling off consecutive prime factors from $\zeta(s)$, starting with two (see [G]),

$$(1 - 2^{-s})\zeta(s) = 1 + 3^{-s} + 5^{-s} + ... \, ,$$

Euler discovered in 1737 the fabulous product formula,

$$\prod_p (1 - p^{-s})\zeta(s) = 1 \ . \qquad (A.2)$$

It was during the subsequent *Berlin period* (1741-1766, invited by Frederick II) that Euler conjectured, in 1749, the functional equation for the zeta function that became, 110 years later, the basis of Riemann's great 1859 paper, [W]. Euler's work on number theory was done, as Fermat's a century earlier, against a background of contempt towards the field by the majority of mathematicians. He was not deterred. As he once observed "one may see how closely and wonderfully infinitesimal analysis is related not only to ordinary analysis but even to the theory of numbers, however repugnant the latter may seem to that higher kind of calculus" (see [We] Chapter III Sect. V). Euler's "defense of Christianity" of 1747, as Weil ([We] Chapter III, Sect. II) puts it, "did nothing to ingratiate its author with the would be philosopher-king Frederick." Disgusted by the superficial (but fashionable at the court of Frederick) anticlerical Voltaire, Euler took the opportunity offered to him by Empress Catherine II (the Great) to return to St. Petersburg where he spent the last (most productive!) period of his life (1766-1783).

Polylogarithms and multiple zeta values

The study of polylogarithms has started with the dilogarithm function. Its integral representation (that serves as an analytic continuation of the series)

$$Li_2(z) \; (= \sum_{n=1}^{\infty} \frac{z^n}{n^2}) = -\int_0^z \frac{\ln(1-t)}{t} dt \tag{A.3}$$

first appears in 1696 in a letter of Leibniz (1646-1716) to Johann Bernoulli. As already noted, Euler started playing with the corresponding series around 1729. According to Maximon [M] (who is taking care to establish the priority of British mathematicians) the first study of the properties of the integral (A.3) for complex z belongs to John Landen (1719-1790) whose memoir appears in the *Phil. Trans. R. Soc. Lond.* in 1760, albeit most authors credit for this Euler (who wrote on the subject later, in 1768). The first comprehensive study of the dilog was given in the book of Spence of 1809 (see [A, M]) whose results are usually attributed to later work of Abel. The name *Euler dilogarithm* was only introduced by Hill (1828). The two-variable, five-term relation

$$\begin{aligned} &Li_2(x) + Li_2(y) + Li_2(\frac{1-x}{1-xy}) + Li_2(1-xy) + Li_2(\frac{1-y}{1-xy}) = \\ &\frac{\pi^2}{6} - \ln x \ln(1-x) - \ln y \ln(1-y) + \ln(\frac{1-x}{1-xy}) \ln(\frac{1-y}{1-xy}) \end{aligned} \tag{A.4}$$

was discovered and rediscovered by Spence (1809), Abel (1827), Hill (1828), Kummer (1840), Schaeffer (1846) (see [Z]). The oriented (odd under permutations of the vertices) volume of the ideal tetrahedron in hyperbolic space is expressed in terms of the Bloch-Wigner function (5.7): $\tilde{D}(z_1, ..., z_4) := D(\frac{z_{12}z_{34}}{z_{13}z_{24}})$. It has been found by Lobachevsky in 1836 (for a review see [Mil]). An early (1955) reference, in which Clausen's (1832) dilogarithm appears in a fourth order calculation in QFT, is [KS]. Two years later A. Petermann and (independently) C.A. Sommerfield uncovered $\zeta(3)$ in a calculation of the electron magnetic moment (see the lively review [St] where important later work by Laporta and Remiddi [LR] is also surveyed). It reappeared in another calculation in perturbative quantum electrodynamics in the mid 1960's [R]. The modern notations and a survey of dilogarithmic identities and their polylogarithmic generalizations are given by an electrical engineer [L] (for a nice informative review of his book - see [A]). The *harmonic polylogarithms* (with singularities at the three roots of the equation $x^3 = x$) are surveyed in [RV]. David Broadhurst was a pioneer in the systematic study of MZV in QFT (see [BK] and references to earlier work cited there as well as his popular talk [B10] in which he shares his enthusiasm with beautiful numbers - like $\zeta(3)$ - appearing in various branches of physics). The resurgence of polylogarithms in pure mathematics, anticipated by 19th century work of Kummer and Poincaré and a 20 century contribution by Lappo-Danilevsky, was prepared by the work of Chen [C, B09] on iterated path integrals. The coproduct of hyperlogarithms was written down by Goncharov [Gon] as a planar decorated version of Connes-Kreimer's Hopf algebra of rooted trees [CK]. One of a number of recent conferences dedicated to this topic had the telling title *Polylogarithms as a Bridge between Number Theory and Particle Physics* (see the notes [Zh] which contain a historical survey with a bibliography of some 394 entries). Recent developments and perspectives are surveyed in Francis Brown's lecture [Br14] at the 2014 Intentional Congress of Mathematicians as well as in the lectures ([Bl15, CGS]) in these proceedings.

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