Three-body problem in 3D space: ground state, (quasi)-exact-solvability

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Abstract

We study aspects of the quantum and classical dynamics of a 3-body system in 3D space with interaction depending only on mutual distances. The study is restricted to solutions in the space of relative motion which are functions of mutual distances only. It is shown that the ground state (and some other states) in the quantum case and the planar trajectories in the classical case are of this type. The quantum (and classical) system for which these states are eigenstates is found and its Hamiltonian is constructed. It corresponds to a three-dimensional quantum particle moving in a curved space with special metric. The kinetic energy of the system has a hidden $sl(4, R)$ Lie (Poisson) algebra structure, alternatively, the hidden algebra $h^{(3)}$ typical for the H_3 Calogero model. We find an exactly solvable three-body generalized harmonic oscillator-type potential as well as a quasi-exactly-solvable three-body sextic polynomial type potential.

INTRODUCTION

The Hamiltonian for 3-body quantum system of 3-dimensional particles with translation-invariant potential, which depends on relative distances between particles only, is of the form,

$$\mathcal{H} \ = \ -\sum_{i=1}^{3} \Delta_i^{(3)} \ + \ V(r_{12},\, r_{13},\, r_{23}) \ , \qquad (1)$$

with coordinate vector of ith particle $\mathbf{r}_i \equiv \mathbf{r}_i^{(3)} = (x_{i,1}\,, x_{i,2}\,, x_{i,3})$, where

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| \ , \qquad (2)$$

is the (relative) distance between particles i and j. The number of relative distances is equal to the number of edges of the triangle formed by taking the body positions as vertices. We call this triangle the *triangle of interaction*. Here, $\Delta_i^{(3)}$ is the 3-dimensional Laplacian,

$$\Delta_i^{(d)} \ = \ \frac{\partial^2}{\partial \mathbf{r}_i \partial \mathbf{r}_i} \ ,$$

associated with the ith body. For simplicity all masses are assumed to be equal: $m_i = m = 1/2$. The configuration space for $\mathcal{H}$ is $\mathbf{R}^9$. The center-of-mass motion described by vectorial coordinate

$$\mathbf{R}_{_0} \ = \ \frac{1}{\sqrt{3}} \sum_{k=1}^{3} \mathbf{r}_{_k} \ ,$$

can be separated out; this motion is described by a 3-dimensional plane wave.

The spectral problem is formulated in the space of relative motion $\mathbf{R}_r \equiv \mathbf{R}^6$; it is of the form,

$$\mathcal{H}_r\, \Psi(x) \ \equiv \ \bigg(-\Delta_r^{(6)} + V(r_{12},\, r_{13},\, r_{23}) \bigg)\, \Psi(x) \ = \ E\Psi(x) \ , \ \Psi \in L_2(\mathbf{R}_r) \ , \qquad (3)$$

where $\Delta_r^{(6)}$ is the flat-space Laplacian in the space of relative motion. If the space of relative motion $\mathbf{R}_r$ is parameterized by two, 3-dimensional vectorial Jacobi coordinates

$$\mathbf{r}_j^{(F)} \ = \ \frac{1}{\sqrt{j(j+1)}} \sum_{k=1}^{j} k\, (\mathbf{r}_{k+1} - \mathbf{r}_k) \ , \qquad j = 1, 2 \ ,$$

the flat-space 6-dimensional Laplacian in the space of relative motion becomes diagonal

$$\Delta_r^{(6)} \ = \ \frac{\partial^2}{\partial \mathbf{r}_i^{(F)} \partial \mathbf{r}_i^{(F)}} \ . \qquad (4)$$

Observation:

There exists a family of the eigenstates of the Hamiltonian (1), including the ground state, which depends on three relative distances $\{r_{ij}\}$ only .

Our primary goal is to find the differential operator in the space of relative distances $\{r_{ij}\}$ for which these states are eigenstates. In other words, to find a differential equation depending only on $\{r_{ij}\}$ for which these states are solutions. This implies a study of the evolution of the triangle of interaction.

I. GENERALITIES

As a first step let us change variables in the space of relative motion $\mathbf{R}_r : (\mathbf{r}_j^{(F)}) \leftrightarrow (r_{ij}, \Omega)$, where the number of (independent) relative distances r_{ij} is equal to 3 and Ω is a collection of three angular variables. Thus, we split $\mathbf{R}_r$ into a sum of the space of relative distances $\tilde{\mathbf{R}}$ and a space parameterized by angular variables, essentially those on the sphere S^3. There are known several ways to introduce variables in $\mathbf{R}_r$: the perimetric coordinates by Hylleraas [1], the scalar products of vectorial Jacobi coordinates $\mathbf{r}_j^{(F)}$ [2] and the relative (mutual) distances r_{ij} (see e.g. [3]). We follow the last one. In turn, the angular variables are introduced as the two Euler angles on the S^2 sphere defining the normal to the interaction plane (triangle) and the azimuthal angle of rotation of the interaction triangle around its barycenter, see e.g. [2].

A key observation is that in new coordinates (r_{ij}, Ω) the flat-space Laplace operator (the kinetic energy operator) in the space of relative motion $\mathbf{R}_r$ takes the form of the sum of two the second-order differential operators

$$\Delta_r^{(6)} \ = \ \Delta_R(r_{ij}) + \tilde{\Delta}(r_{ij}, \Omega, \partial_\Omega) \ , \tag{5}$$

where the first operator depends on relative distances *only*, while the second operator depends on angular derivatives in such a way that it annihilates any angle-independent function,

$$\tilde{\Delta}(r_{ij}, \Omega, \partial_\Omega)\, \Psi(r_{ij}) \ = \ 0 \ .$$

If we look for angle-independent solutions of (3), the decomposition (5) reduces the general spectral problem (3) to a particular spectral problem

$$\tilde{\mathcal{H}}_R\, \Psi(r_{ij}) \equiv \left(-\Delta_R(r_{ij}) + V(r_{12},\, r_{13},\, r_{23}) \right) \Psi(r_{ij}) = E\Psi(r_{ij}) \ , \ \Psi \in L_2(\tilde{\mathbf{R}}) \ , \tag{6}$$

where $\tilde{\mathbf{R}}$ is the space of relative distances. Surprisingly, one can find the gauge factor $\Gamma(r_{ij})$ such that the operator $\Delta_R(r_{ij})$ takes the form of the Schrödinger operator,

$$\Gamma^{-1}\,\Delta_R\,(r_{ij})\,\Gamma \;=\; \Delta_{LB}(r_{ij}) - \tilde{V}(r_{ij}) \;\equiv -\tilde{H}_R\ , \tag{7}$$

where Δ_{LB} is the Laplace-Beltrami operator with contravariant metric g^{ij}, in general, on some non-flat, (non-constant curvature) manifold. It makes sense of the kinetic energy. Here $\tilde{V}(r_{ij})$ is the effective potential. The potential $\tilde{V}$ becomes singular at the boundary of the configuration space, where the determinant $D = \det g^{ij}$ vanishes. The operator $\tilde{H}_R$ is Hermitian with measure $D^{-\frac{1}{2}}$. Eventually, we arrive at the spectral problem for the Hamiltonian

$$H_R \;=\; -\Delta_{LB}(r_{ij}) + V(r_{ij}) + \tilde{V}(r_{ij})\ . \tag{8}$$

Following the *de-quantization* procedure of replacement of the quantum momentum (derivative) by the classical momentum

$$-i\,\partial \;\rightarrow\; p\ ,$$

one can get a classical analogue of (8),

$$H_R^{(c)} \;=\; g^{ij}p_ip_j + V(r_{ij}) + \tilde{V}(r_{ij})\ . \tag{9}$$

It describes the motion of 3-dimensional rigid body with tensor of inertia $(g^{ij})^{-1}$.

The Hamiltonians (8), (9) are the main objects of study of this paper.

II. THREE-BODY CASE: CONCRETE RESULTS

After straightforward calculations the operator $\Delta_R(r_{ij})$ in decomposition (5) is found to be

$$\Delta_R(r_{ij}) \;=\; \Big[\, 2\,(\partial^2_{r_{12}} + \partial^2_{r_{23}} + \partial^2_{r_{13}}) + \frac{4}{r_{12}}\,\partial_{r_{12}} + \frac{4}{r_{23}}\,\partial_{r_{23}} + \frac{4}{r_{13}}\,\partial_{r_{13}} \tag{10}$$
$$+\frac{r_{12}^2 - r_{13}^2 + r_{23}^2}{r_{12}r_{23}}\,\partial_{r_{12}}\partial_{r_{23}} + \frac{r_{12}^2 + r_{13}^2 - r_{23}^2}{r_{12}r_{13}}\,\partial_{r_{12}}\partial_{r_{13}} + \frac{r_{13}^2 + r_{23}^2 - r_{12}^2}{r_{13}r_{23}}\,\partial_{r_{23}}\partial_{r_{13}}\,\Big]\ ,$$

cf. e.g. [3]. It does not depend on the choice of the angular variables Ω. Its configuration space is

$$0 < r_{12}, r_{13}, r_{23} < \infty, \quad r_{23} < \ r_{12} + r_{13}, \quad r_{13} < r_{12} + r_{23}, \quad r_{12} < r_{13} + r_{23}. \tag{11}$$

In the space with Cartesian coordinates $(x, y, z) = (r_{12}, r_{13}, r_{23})$ the configuration space lies in the first octant and is the interior of the inverted tetrahedral-shaped object with base at infinity, vertex at the origin and edges $(t, t, 2t)$, $(t, 2t, t)$ and $(2t, t, t)$, $0 \leq t < \infty$.

Formally, the operator (10) is invariant under reflections $Z_2 \oplus Z_2 \oplus Z_2$,

$$r_{12} \to -r_{12}\ , \qquad r_{13} \Leftrightarrow -r_{13}\ , \qquad r_{23} \Leftrightarrow -r_{23}\ ,$$

and w.r.t. S_3-group action. If we introduce new variables,

$$r_{12}^2 \;=\; \rho_{12}\ ,\ r_{13}^2 \;=\; \rho_{13}\ ,\ r_{23}^2 \;=\; \rho_{23}\ , \tag{12}$$

the operator (10) becomes algebraic,

$$\begin{aligned}\Delta_R(\rho_{ij}) \;&=\; 4(\rho_{12}\partial^2_{\rho_{12}} + \rho_{13}\partial^2_{\rho_{13}} + \rho_{23}\partial^2_{\rho_{23}}) + 6(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{23}}) + \\ 2\bigg((\rho_{12} + \rho_{13} - \rho_{23})\partial_{\rho_{12}}\partial_{\rho_{13}} &+ (\rho_{12} + \rho_{23} - \rho_{13})\partial_{\rho_{12}}\partial_{\rho_{23}} + (\rho_{13} + \rho_{23} - \rho_{12})\partial_{\rho_{13}}\partial_{\rho_{23}}\bigg)\ .\end{aligned} \tag{13}$$

From (11) and (12) it follows that the corresponding configuration space in ρ variables is given by the conditions

$$0 < \rho_{12}, \rho_{13}, \rho_{23} < \infty,\ \rho_{23} < (\sqrt{\rho_{12}} + \sqrt{\rho_{13}})^2,\ \rho_{13} < (\sqrt{\rho_{12}} + \sqrt{\rho_{23}})^2,\ \rho_{12} < (\sqrt{\rho_{13}} + \sqrt{\rho_{23}})^2.$$

We remark that

$$\rho_{12}^2 + \rho_{13}^2 + \rho_{23}^2 - 2\rho_{12}\rho_{13} - 2\rho_{12}\rho_{23} - 2\rho_{13}\rho_{23} \;<\; 0\ , \tag{14}$$

because the left-hand side (l.h.s.) is equal to

$$-(r_{12} + r_{13} - r_{23})(r_{12} + r_{23} - r_{13})(r_{13} + r_{23} - r_{12})(r_{12} + r_{13} + r_{23})$$

and conditions (11) should hold. Therefore, l.h.s. is proportional to the square of the area of the triangle of interaction $S^2_{\triangle}$.

The associated contravariant metric for the operator $\Delta_R(\rho_{ij})$ defined by coefficients in front of second derivatives is remarkably simple

$$g^{\mu\nu}(\rho) \;=\; \left|\begin{array}{ccc} 4\rho_{12} & \rho_{12} + \rho_{13} - \rho_{23} & \rho_{12} + \rho_{23} - \rho_{13} \\ & & \\ \rho_{12} + \rho_{13} - \rho_{23} & 4\rho_{13} & \rho_{13} + \rho_{23} - \rho_{12} \\ & & \\ \rho_{12} + \rho_{23} - \rho_{13} & \rho_{13} + \rho_{23} - \rho_{12} & 4\rho_{23} \end{array}\right|\ , \tag{15}$$

it is linear in ρ-coordinates(!) with factorized determinant

$$\det g^{\mu\nu} \;=\; -6\,(\rho_{12}+\rho_{13}+\rho_{23})\left(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}-2\rho_{12}\rho_{23}-2\rho_{13}\rho_{23}\right)\equiv D>0\ , \tag{16}$$

and is positive definite. It is worth noting a remarkable factorization property of the determinant

$$D \;=\; -\,6\,(r_{12}^2+r_{13}^2+r_{23}^2)\ \times$$

$$(r_{12}+r_{13}-r_{23})(r_{12}+r_{23}-r_{13})(r_{13}+r_{23}-r_{12})(r_{12}+r_{13}+r_{23}) \;=$$

$$=\; 96\,P\ S^2_{\triangle}\ ,$$

where $P=r_{12}^2+r_{13}^2+r_{23}^2$ - the sum of squared of sides of the interaction triangle.

The determinant can rewritten in terms of elementary symmetric polynomials $\sigma_{1,2}$,

$$\begin{aligned}
\tau_1=\sigma_1(\rho_{12},\,\rho_{13},\,\rho_{23}) &\;=\; \rho_{12}+\rho_{13}+\rho_{23}\ ,\\
\tau_2=\sigma_2(\rho_{12},\,\rho_{13},\,\rho_{23}) &\;=\; \rho_{12}\,\rho_{13}+\rho_{12}\,\rho_{23}+\rho_{13}\,\rho_{23}\ ,\\
\tau_3=\sigma_3(\rho_{12},\,\rho_{13},\,\rho_{23}) &\;=\; \rho_{12}\rho_{13}\rho_{23}\ ,
\end{aligned} \tag{17}$$

which are invariant w.r.t. S_3-group action, as follows,

$$D \;=\; 6\,\tau_1\,\left(4\tau_2-\tau_1^2\right)\ . \tag{18}$$

When $\det g^{\mu\nu}=0$, hence, either $\tau_1=0$, or $\tau_1^2=4\tau_2$ - it defines the boundary of the configuration space, see (14).

It can be shown that there exists the 1st order symmetry operator

$$L_1 \;=\; (\rho_{13}-\rho_{23})\partial_{\rho_{12}}+(\rho_{23}-\rho_{12})\partial_{\rho_{13}}+(\rho_{12}-\rho_{13})\partial_{\rho_{23}}\ , \tag{19}$$

for the operator (13),

$$[\Delta_R(\rho_{ij})\ ,\ L_1]=0\ .$$

Here, L_1 is an algebraic operator, which is anti-invariant under the S_3-group action. The existence of the symmetry L_1 implies that in the space of relative distances one variable can be separated out in (13).

Set

$$w_1 \;=\; \rho_{12}+\rho_{13}+\rho_{23}\quad ,\quad w_2 \;=\; 2\sqrt{\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-\rho_{12}\rho_{13}-\rho_{12}\rho_{23}-\rho_{13}\rho_{23}}\quad , \tag{20}$$

where $w_2 = 2\sqrt{(\tau_1^2 - 3\tau_2)}$ as well, which are invariant under the action of L_1, and

$$w_3 = \frac{\sqrt{3}}{9}\Bigg(\mathrm{sgn}\,(\rho_{23}-\rho_{13})\arcsin(\frac{2\rho_{12}-\rho_{23}-\rho_{13}}{w_2}) + \mathrm{sgn}(\rho_{13}-\rho_{12})\ \arcsin(\frac{2\rho_{23}-\rho_{13}-\rho_{12}}{w_2})$$

$$+\mathrm{sgn}(\rho_{12}-\rho_{23})\ \arcsin(\frac{2\rho_{13}-\rho_{23}-\rho_{12}}{w_2}) - \frac{3\pi}{4}\Bigg)\,, \tag{21}$$

with $\mathrm{sgn}(x) = \frac{x}{|x|}$ for nonzero x. These coordinates are invariant under a cyclic permutation of the indices on the ρ_{jk}: $1 \to 2 \to 3 \to 1$. Under a transposition of exactly two indices, see e.g. (12), (3), we see that w_1, w_2 remain invariant, and $w_3 \to -w_3 - \frac{\sqrt{3}\pi}{6}$. Expressions for w_3 vary, depending on which of the 6 non-overlapping regions of $(\rho_{12}, \rho_{13}, \rho_{23})$ space we choose to evaluate them:

1.

$$(a):\ \rho_{23} > \rho_{13} > \rho_{12}\ ,\quad (b):\ \rho_{13} > \rho_{12} > \rho_{23}\ ,\quad (c):\ \rho_{12} > \rho_{23} > \rho_{13}\ ,$$

2.

$$(d):\ \rho_{13} > \rho_{23} > \rho_{12}\ ,\quad (e):\ \rho_{12} > \rho_{13} > \rho_{23}\ ,\quad (f):\ \rho_{23} > \rho_{12} > \rho_{13}\ ,$$

The regions in class 1 are related by cyclic permutations, as are the regions in class 2. We map between regions by a transposition. Thus it is enough to evaluate w_3 in the region $(a):\ \rho_{23} > \rho_{13} > \rho_{12}$. The other 5 expressions will then follow from the permutation symmetries. In this case we have

$$(a):\ w_3 = -\frac{\sqrt{3}}{9}\arcsin\Bigg[\frac{2\sqrt{2}}{w_2^3}((2-\sqrt{3})\rho_{13} - \rho_{23} + (\sqrt{3}-1)\rho_{12})\times$$

$$(2\rho_{23} - (1+\sqrt{3})\rho_{13} + (\sqrt{3}-1)\rho_{12})((2+\sqrt{3})\rho_{12} - (1+\sqrt{3})\rho_{13} - \rho_{23})\Bigg]\,.$$

(The special cases where exactly two of the ρ_{jk} are equal can be obtained from these results by continuity. Here, w_3 is a single-valued differentiable function of $\rho_{12}, \rho_{13}, \rho_{23}$ everywhere in the physical domain (configuration space), except for the points $\rho_{12} = \rho_{13} = \rho_{23}$ where it is undefined.)

In these coordinates, the operators (19) and (13) take the form

$$\begin{aligned} L_1(w)\ &= \partial_{w_3}\ , \\ \Delta_R(w)\ &=\ 6\,w_1\partial^2_{w_1}\ +\ 6\,w_1\partial^2_{w_2} + 2\,\frac{w_1}{w_2^2}\partial^2_{w_3}\ +\ 12\,w_2\partial^2_{w_1w_2}\ +\ 18\,\partial_{w_1} \\ &\quad +\ 6\,\frac{w_1}{w_2}\partial_{w_2}\ , \end{aligned}$$

respectively. It is evident that for the w_3-independent potential

$$V(w_1,\, w_2; w_3)\ =\ g(w_1,\, w_2)\ ,$$

the operator L_1 is still an integral, where g is an arbitrary function.

Both operators (13) and (19) are $sl(4,\mathbf{R})$-Lie algebraic - they can be rewritten in terms of the generators of the maximal affine subalgebra b_4 of the algebra $sl(4,\mathbf{R})$, see e.g. [4, 5]

$$\begin{aligned}
\mathcal{J}_i^- &= \frac{\partial}{\partial u_i}\ , \qquad\qquad i=1,2,3\ ,\\
\mathcal{J}_{ij}{}^0 &= u_i\frac{\partial}{\partial u_j}\ , \qquad i,j=1,2,3\ ,\\
\mathcal{J}^0(N) &= \sum_{i=1}^{3} u_i\frac{\partial}{\partial u_i} - N\,,
\end{aligned}\tag{22}$$

$$\mathcal{J}_i^+(N) = u_i\mathcal{J}^0(N)\ =\ u_i\left(\sum_{j=1}^{3} u_j\frac{\partial}{\partial u_j} - N\right)\ ,\quad i=1,2,3\ ,\tag{23}$$

where N is parameter and

$$u_1\equiv\rho_{12}\ ,\qquad u_2\equiv\rho_{13}\ ,\qquad u_3\equiv\rho_{23}\ .$$

If N is non-negative integer, a finite-dimensional representation space occurs,

$$\mathcal{P}_N^{(3)}\ =\ \langle u_1^{p_1}u_2^{p_2}u_3^{p_3}|\ 0\le p_1+p_2+p_3\le N\rangle\ .\tag{24}$$

Explicitly, these operators look as

$$\Delta_R^{(3)}(\mathcal{J})\ =\ 4(\,\mathcal{J}_{11}^0\,\mathcal{J}_1^- + \mathcal{J}_{22}^0\,\mathcal{J}_2^- + \mathcal{J}_{33}^0\,\mathcal{J}_3^-\,) + 6\,(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^-) + \tag{25}$$

$$2\left(\mathcal{J}_{11}^0\,(\mathcal{J}_2^- + \mathcal{J}_3^-) + \mathcal{J}_{22}^0\,(\mathcal{J}_1^- + \mathcal{J}_3^-) + \mathcal{J}_{33}^0\,(\mathcal{J}_1^- + \mathcal{J}_2^-) - \mathcal{J}_{31}^0\,\mathcal{J}_2^- - \mathcal{J}_{23}^0\,\mathcal{J}_1^- - \mathcal{J}_{12}^0\,\mathcal{J}_3^-\right)\ ,$$

and

$$L_1\ =\ \mathcal{J}_{21}^0\ -\ \mathcal{J}_{31}^0\ +\ \mathcal{J}_{32}^0\ -\mathcal{J}_{12}^0\ +\ \mathcal{J}_{13}^0\ -\ \mathcal{J}_{23}^0\ .\tag{26}$$

The remarkable property of the algebraic operator $\Delta_R(\rho_{ij})$ (13) is its gauge-equivalence to the Schrödinger operator. Making the gauge transformation with determinant (16), (18) as the factor,

$$\Gamma\ =\ D^{-1/4}\ \sim\ \frac{1}{\tau_1^{1/4}(4\tau_2-\tau_1^2)^{1/4}}\ ,$$

see also (17), we find that

$$\Gamma^{-1}\,\Delta_R(\rho_{ij})\,\Gamma\ =\ \Delta_{LB}(\rho_{ij}) - \tilde{V}\ ,\tag{27}$$

where the effective potential

$$\tilde{V}(\rho_{ij}) \;=\; \frac{9}{8\,(\rho_{12}+\rho_{13}+\rho_{23})} \;+\; \frac{(\rho_{12}+\rho_{13}+\rho_{23})}{2\,(\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}-2\rho_{12}\rho_{23}-2\rho_{13}\rho_{23})}\ .$$

Note that in r-coordinates

$$\frac{4}{(r_{12}+r_{13}-r_{23})(r_{12}+r_{23}-r_{13})(r_{13}+r_{23}-r_{12})} \;=$$

$$\frac{1}{r_{12}r_{13}r_{23}}\left(\frac{r_{23}}{r_{12}+r_{13}-r_{23}}+\frac{r_{13}}{r_{12}+r_{23}-r_{13}}+\frac{r_{12}}{r_{13}+r_{23}-r_{12}}+1\right),$$

and

$$\frac{1}{(r_{23}+r_{13}-r_{12})(r_{23}+r_{12}-r_{13})(r_{13}+r_{12}-r_{23})(r_{12}+r_{13}+r_{23})}$$

$$=\frac{1}{8r_{23}r_{13}(r_{23}+r_{13})}\left[\frac{1}{r_{23}+r_{13}-r_{12}}+\frac{1}{r_{23}+r_{13}+r_{12}}\right]$$

$$+\frac{1}{8r_{23}r_{13}r_{12}}\left[\frac{1}{r_{12}-r_{13}+r_{23}}+\frac{1}{r_{12}+r_{13}-r_{23}}\right]\ ,$$

thus, the effective potential can be written differently,

$$\tilde{V}(r_{ij}) \;=\; \frac{9}{8\,(r_{12}^2+r_{13}^2+r_{23}^2)}$$

$$+\ \frac{r_{12}^2+r_{13}^2+r_{23}^2}{16}\left[\frac{1}{r_{13}r_{23}(r_{13}+r_{23})}\left(\frac{1}{r_{13}+r_{23}-r_{12}}+\frac{1}{r_{12}+r_{13}+r_{23}}\right)\right.$$

$$\left.+\ \frac{1}{r_{12}r_{13}r_{23}}\left(\frac{1}{r_{12}+r_{23}-r_{13}}+\frac{1}{r_{12}+r_{13}-r_{23}}\right)\right]\ .$$

In turn,

$$\Delta_{LB}(\rho_{ij}) \;=\; 4(\rho_{12}\partial^2_{\rho_{12}}+\rho_{13}\partial^2_{\rho_{13}}+\rho_{23}\partial^2_{\rho_{23}})$$

$$+\ 2\Big((\rho_{12}+\rho_{13}-\rho_{23})\partial_{\rho_{12}}\partial_{\rho_{13}}\ +(\rho_{12}+\rho_{23}-\rho_{13})\partial_{\rho_{12}}\partial_{\rho_{23}}\ +(\rho_{13}+\rho_{23}-\rho_{12})\partial_{\rho_{13}}\partial_{\rho_{23}}\Big)$$

$$-3\left(\frac{\rho_{12}\partial_{\rho_{12}}+\rho_{13}\partial_{\rho_{13}}+\rho_{23}\partial_{\rho_{23}}}{\rho_{12}+\rho_{13}+\rho_{23}}\right)+4\,(\partial_{\rho_{12}}+\partial_{\rho_{23}}+\partial_{\rho_{13}})\ , \tag{28}$$

is the Laplace-Beltrami operator,

$$\Delta_{LB}(\rho_{ij}) \;=\; \sqrt{D}\,\partial_\mu\frac{1}{\sqrt{D}}\,g^{\mu\nu}\partial_\nu\ ,\quad \partial_\nu\equiv\frac{\partial}{\partial_{\rho_\nu}}\ ,$$

see (15), (16). Eventually, taking into account (27) we arrive at the Hamiltonian

$$\mathcal{H}_{rd}(r_{ij}) \;=\; -\Delta_{LB}(r_{ij})+\tilde{V}(r_{ij})+V(r_{12},\,r_{13},\,r_{23})\ , \tag{29}$$

in the space of relative distances, or

$$\mathcal{H}_{rd}(\rho_{ij}) \;=\; -\Delta_{LB}(\rho_{ij}) + \tilde{V}(\rho_{ij}) + V(\rho_{ij}) \ , \tag{30}$$

in ρ-space, see (12). The Hamiltonian (29), or (30) describes the three-dimensional quantum particle moving in the curved space with metric $g^{\mu\nu}$. The Ricci scalar, see e.g. [6], for this space is equal to

$$Rs \;=\; -\frac{41\,(\rho_{12}+\rho_{13}+\rho_{23})^2 - 84\,(\rho_{12}\,\rho_{13}+\rho_{12}\,\rho_{23}+\rho_{23}\,\rho_{13})}{12\,(\rho_{12}+\rho_{13}+\rho_{23})\left((\rho_{12}+\rho_{13}+\rho_{23})^2 - 4\,(\rho_{12}\,\rho_{13}+\rho_{12}\,\rho_{23}+\rho_{23}\,\rho_{13})\right)}$$
$$= \frac{-84\,\tau_2 \;+\; 41\,\tau_1{}^2}{12\,\tau_1(4\,\tau_2 \;-\; \tau_1^2)} \ .$$

It is singular at the boundary of the configuration space. The Cotton tensor, see e.g. [6], for this metric is nonzero, so the space is not conformally flat.

Making the de-quantization of (30) we arrive at a three-dimensional classical system which is characterized by the Hamiltonian,

$$\mathcal{H}^{(c)}_{rd}(\rho_{ij}) \;=\; g^{\mu\nu}(\rho_{ij})\,P_i\,P_j + \tilde{V}(\rho_{ij}) + V(\rho_{ij}) \ , \tag{31}$$

where $P_i\,P_j$, $i,j=1,2,3$ are classical momenta in ρ-space and $g^{\mu\nu}(\rho_{ij})$ is given by (15). Here the underlying manifold (zero-potential case) admits an $so(3)$ algebra of constants of the motion linear in the momenta, i.e., Killing vectors. Thus, the free Hamilton-Jacobi equation is integrable. However, it admits no separable coordinate system.

A. (Quasi)-exact-solvability

Let us take the function

$$\Psi_0(\rho_{12},\,\rho_{13},\,\rho_{23}) \;=\; \tau_1^{1/4}(4\,\tau_2-\tau_1^2)^{\frac{\gamma}{2}}\,e^{-\omega\,\tau_1-\frac{A}{2}\,\tau_1^2} \ , \tag{32}$$

where $\gamma,\,\omega>0$ and $A\geq 0$ are constants and τ's are given by (17), and seek the potential for which this (32) is the ground state function for the Hamiltonian $\mathcal{H}_r(\rho_{ij})$, see (30). This potential can be found immediately by calculating the ratio

$$\frac{\Delta_{LB}(\rho_{ij})\Psi_0}{\Psi_0} \;=\; V_0 - E_0\,.$$

The result is

$$V_0(\tau_1,\,\tau_2) \;=\; \frac{9}{8\tau_1} + \gamma(\gamma-1)\left(\frac{2\tau_1}{4\tau_2-\tau_1^2}\right) \;+$$

$$6\,\omega^2\,\tau_1\;+\;6\,A\,\tau_1\,(2\,\omega\,\tau_1\;-\;2\gamma-3)\;+\;6\,A^2\tau_1^3\ , \tag{33}$$

with the energy of the ground state

$$E_0\;=\;12\,\omega\,(1+\,\gamma)\ . \tag{34}$$

Now, let us take the Hamiltonian $\mathcal{H}_{rd,0}\equiv -\Delta_{LB}+V_0$, see (30), with potential (33), subtract E_0 (34) and make the gauge rotation with Ψ_0 (32). As the result we obtain the $sl(4,\mathbf{R})$-Lie-algebraic operator with additional potential ΔV_N, [4, 5]

$$\Psi_0^{-1}\,(-\Delta_{LB}+V_0-E_0)\,\Psi_0\;=\;-\Delta_R(\mathcal{J})\;+\;2(1-2\,\gamma)\,(\mathcal{J}_1^-+\mathcal{J}_2^-+\mathcal{J}_3^-)\;+$$
$$12\,\omega\,(\mathcal{J}_{11}^0+\mathcal{J}_{22}^0+\mathcal{J}_{33}^0)+12\,A\,\left(\mathcal{J}_1^+(N)+\mathcal{J}_2^+(N)+\mathcal{J}_3^+(N)\right)\;+\;\Delta V_N \tag{35}$$
$$\equiv h^{(qes)}(J)\;+\;\Delta V_N\ ,$$

see (25), where

$$\Delta V_N\;=\;12\,A\,N\tau_1\ .$$

It is evident that for integer N the operator $h(J)$ has a finite-dimensional invariant subspace $\mathcal{P}_N^{(3)}$, (24), with $\dim\mathcal{P}_N^{(3)}\sim N^3$ at large N. Finally, we arrive at the quasi-exactly-solvable Hamiltonian in the space of relative distances:

$$\mathcal{H}_{rd,qes}(\rho_{ij})\;=\;-\Delta_{LB}(\rho_{ij})+V_N^{(qes)}(\rho_{ij})\ , \tag{36}$$

cf.(8), where

$$V^{(qes,N)}(\tau_1,\,\tau_2)\;=\;\frac{9}{8\tau_1}+\,\gamma(\gamma-1)\left(\frac{2\tau_1}{4\tau_2-\tau_1^2}\right)\;+$$
$$+\;6\,\omega^2\,\tau_1\;+\;6\,A\,\tau_1\,(2\,\omega\,\tau_1\;-\;2\gamma\;-\;2N\;-\;3)\;+\;6\,A^2\tau_1^3\ . \tag{37}$$

For this potential $\sim N^3$ eigenstates can be found by algebraic means. They have the factorized form of the polynomial multiplied by Ψ_0 (32),

$$\mathrm{Pol}_N(\rho_{12},\rho_{13},\rho_{23})\ \Psi_0(\tau_1,\tau_2)\ .$$

(Note that for given N we can always choose appropriate values of γ such that the boundary terms vanish for polynomials in the invariant subspace vanish and the Hamiltonian (36) acts as a self-adjoint operator.) These polynomials are the eigenfunctions of the quasi-exactly-solvable algebraic operator

$$h^{(qes)}(\rho)\;= \tag{38}$$

$$-4(\rho_{12}\partial^2_{\rho_{12}} + \rho_{13}\partial^2_{\rho_{13}} + \rho_{23}\partial^2_{\rho_{23}})$$

$$-2\left((\rho_{12}+\rho_{13}-\rho_{23})\partial_{\rho_{12}}\partial_{\rho_{13}} + (\rho_{12}+\rho_{23}-\rho_{13})\partial_{\rho_{12}}\partial_{\rho_{23}} + (\rho_{13}+\rho_{23}-\rho_{12})\partial_{\rho_{13}}\partial_{\rho_{23}}\right)$$

$$+2(1-2\,\gamma)(\partial_{\rho_{12}} + \partial_{\rho_{13}} + \partial_{\rho_{23}}) + 12\,\omega(\rho_{12}\partial_{\rho_{12}} + \rho_{13}\partial_{\rho_{13}} + \rho_{23}\partial_{\rho_{23}})$$

$$-12\,A\,(\rho_{12}+\rho_{13}+\rho_{23})(\rho_{12}\,\partial_{\rho_{12}} + \rho_{13}\,\partial_{\rho_{13}} + \rho_{23}\,\partial_{\rho_{23}} - N)$$

which is the quasi-exactly-solvable $sl(4,\,\mathbf{R})$-Lie-algebraic operator

$$h^{(qes)}(J) \;=\; -4(\,\mathcal{J}^0_{11}\,\mathcal{J}^-_1 + \mathcal{J}^0_{22}\,\mathcal{J}^-_2 + \mathcal{J}^0_{33}\,\mathcal{J}^-_3\,) - 6\,(\mathcal{J}^-_1 + \mathcal{J}^-_2 + \mathcal{J}^-_3) \tag{39}$$

$$-2\left(\mathcal{J}^0_{11}\,(\mathcal{J}^-_2 + \mathcal{J}^-_3) + \mathcal{J}^0_{22}\,(\mathcal{J}^-_1 + \mathcal{J}^-_3) + \mathcal{J}^0_{33}\,(\mathcal{J}^-_1 + \mathcal{J}^-_2) - \mathcal{J}^0_{31}\,\mathcal{J}^-_2 - \mathcal{J}^0_{23}\,\mathcal{J}^-_1 - \mathcal{J}^0_{12}\,\mathcal{J}^-_3\right)$$

$$+2\,(1-2\,\gamma)\,(\mathcal{J}^-_1 + \mathcal{J}^-_2 + \mathcal{J}^-_3) + 12\,\omega\,(\mathcal{J}^0_{11} + \mathcal{J}^0_{22} + \mathcal{J}^0_{33})$$

$$+12\,A\,(\,J^+_1(N) + J^+_2(N) + J^+_3(N)\,)\ ,$$

cf. (35).

As for the original problem (6) in the space of relative motion

$$\tilde{\mathcal{H}}_R\,\Psi(r_{ij}) \equiv \left(-\Delta_R(r_{ij}) + V(r_{ij})\right)\Psi(r_{ij}) \;=\; E\Psi(r_{ij})\ ,\ \ \Psi \in L_2(\tilde{\mathbf{R}})\ ,$$

the potential for which quasi-exactly-solvable, polynomial solutions occur of the form

$$\mathrm{Pol}_N(\rho_{12},\rho_{13},\rho_{23})\ \Gamma\ \Psi_0(\tau_1,\tau_2)\ ,$$

where $\Gamma \sim D^{-1/4}$, see (18), is given by

$$V^{(qes,N)}_{relative}(\tau) \;=\; \left(\gamma-\frac{1}{2}\right)^2\left(\frac{2\tau_1}{4\tau_2-\tau_1^2}\right) \;+$$

$$+\ 6\,\omega^2\,\tau_1 \;+\; 6\,A\,\tau_1\,(2\,\omega\,\tau_1 \;-\; 2\gamma \;-\; 2N \;-\; 3) \;+\; 6\,A^2\tau_1^3\ , \tag{40}$$

cf. (37); it does not depend on τ_3.

If the parameter A vanishes in (32), (37) and (35), (39) we will arrive at the exactly-solvable problem, where Ψ_0 (32) at $A=0$, plays the role of the ground state function,

$$\Psi_0(\rho_{12},\,\rho_{13},\,\rho_{23}) \;=\; \tau_1^{1/4}(4\,\tau_2-\tau_1^2)^{\frac{\gamma}{2}}\,e^{-\omega\,\tau_1}\ , \tag{41}$$

The $sl(4,\mathbf{R})$-Lie-algebraic operator (39) contains no raising generators $\{\mathcal{J}^+(N)\}$ and becomes

$$h^{(exact)} = -\Delta_R(\mathcal{J}) + 2(1-2\,\gamma)\,(\mathcal{J}^-_1 + \mathcal{J}^-_2 + \mathcal{J}^-_3) + 12\,\omega\,(\mathcal{J}^0_{11} + \mathcal{J}^0_{22} + \mathcal{J}^0_{33})\ ,$$

see (25), and, hence, preserves the infinite flag of finite-dimensional invariant subspaces $\mathcal{P}_N^{(3)}$ (24) at $N = 0, 1, 2 \ldots$. The potential (37) becomes

$$V^{(es)}(\tau_1,\, \tau_2)\ =\ \frac{9}{8\tau_1} + \gamma(\gamma-1)\left(\frac{2\tau_1}{4\tau_2 - \tau_1^2}\right)\ +\ 6\,\omega^2\,\tau_1\ = \tag{42}$$

$$= \frac{9}{8\,(\rho_{12}+\rho_{13}+\rho_{23})}\ +\ 6\omega^2\,(\rho_{12}+\rho_{13}+\rho_{23})$$

$$-\ \gamma(\gamma-1)\left(\frac{2(\rho_{12}+\rho_{13}+\rho_{13})}{\rho_{12}^2+\rho_{13}^2+\rho_{23}^2-2\rho_{12}\rho_{13}-2\rho_{12}\rho_{23}-2\rho_{13}\rho_{23}}\right)$$

$$= \frac{9}{8\,(r_{12}^2+r_{13}^2+r_{23}^2)}\ +\ 6\omega^2\,\left(r_{12}^2+r_{13}^2+r_{23}^2\right)$$

$$+\ \gamma(\gamma-1)\frac{r_{12}^2+r_{13}^2+r_{23}^2}{16}\left[\frac{1}{r_{13}r_{23}(r_{13}+r_{23})}\left(\frac{1}{r_{13}+r_{23}-r_{12}}+\frac{1}{r_{12}+r_{13}+r_{23}}\right)\right.$$

$$\left.+\frac{1}{r_{12}r_{13}r_{23}}\left(\frac{1}{r_{12}+r_{23}-r_{13}}+\frac{1}{r_{12}+r_{13}-r_{23}}\right)\right].$$

Eventually, we arrive at the exactly-solvable Hamiltonian in the space of relative distances

$$\mathcal{H}_{rd,es}(\rho_{ij})\ =\ -\Delta_{LB}(\rho_{ij}) + V^{(es)}(\rho_{ij})\ , \tag{43}$$

where the spectra of energies

$$E_{n_1,n_2,n_3}\ =\ 12\omega(n_1+n_2+n_3+\gamma+1)\ ,\quad n_1, n_2, n_3 = 0, 1, 2, \ldots$$

is equidistant. All eigenfunctions have the factorized form of a polynomial multiplied by Ψ_0 (41),

$$\text{Pol}_N(\rho_{12}, \rho_{13}, \rho_{23})\ \Psi_0(\tau_1, \tau_2)\ ,\quad N = 0, 1, \ldots\ .$$

These polynomials are eigenfunctions of the exactly-solvable algebraic operator

$$h^{(exact)}(\rho)\ =\ -4(\rho_{12}\partial^2_{\rho_{12}}+\rho_{13}\partial^2_{\rho_{13}}+\rho_{23}\partial^2_{\rho_{23}})+(2-4\,\gamma)(\partial_{\rho_{12}}+\partial_{\rho_{13}}+\partial_{\rho_{23}})+12\,\omega(\rho_{12}\partial_{\rho_{12}}+\rho_{13}\partial_{\rho_{13}}+\rho_{23}\partial_{\rho_{23}})$$

$$-2(\rho_{12}+\rho_{13}-\rho_{23})\partial_{\rho_{12}}\partial_{\rho_{13}} - 2(\rho_{12}+\rho_{23}-\rho_{13})\partial_{\rho_{12}}\partial_{\rho_{23}} - 2(\rho_{13}+\rho_{23}-\rho_{12})\partial_{\rho_{13}}\partial_{\rho_{23}}\ , \tag{44}$$

or, equivalently, of the exactly-solvable $sl(4, \mathbf{R})$-Lie-algebraic operator

$$h^{(exact)}(J)\ =\ -4(\,\mathcal{J}_{11}^0\,\mathcal{J}_1^- + \mathcal{J}_{22}^0\,\mathcal{J}_2^- + \mathcal{J}_{33}^0\,\mathcal{J}_3^-\,) - 6\,(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^-)$$

$$-2\left(\mathcal{J}_{11}^0\,(\mathcal{J}_2^- + \mathcal{J}_3^-) + \mathcal{J}_{22}^0\,(\mathcal{J}_1^- + \mathcal{J}_3^-) + \mathcal{J}_{33}^0\,(\mathcal{J}_1^- + \mathcal{J}_2^-) - \mathcal{J}_{31}^0\,\mathcal{J}_2^- - \mathcal{J}_{23}^0\,\mathcal{J}_1^- - \mathcal{J}_{12}^0\,\mathcal{J}_3^-\right)$$

$$+2\,(1-2\,\gamma)\,(\mathcal{J}_1^- + \mathcal{J}_2^- + \mathcal{J}_3^-) + 12\,\omega\,(\mathcal{J}_{11}^0 + \mathcal{J}_{22}^0 + \mathcal{J}_{33}^0)\ . \tag{45}$$

Those polynomials are orthogonal w.r.t. Ψ_0^2, (32) at $A=0$, their domain is given by (14). Being written in variables $w_{1,2,3}$, see above, they are factorizable, $F(w_1,w_2)\,f(w_3)$. To the best of our knowledge these orthogonal polynomials have not been studied in literature.

The Hamiltonian with potential (42) can be considered as a three-dimensional generalization of the 3-body Calogero model [7], see also [8], [9], with loss of the property of pairwise interaction. Now the potential of interaction contains two- and three-body interaction terms. If $\gamma=0,1$ in (42) we arrive at the celebrated harmonic oscillator potential in the space of relative distances, see e.g. [10]. In turn, in the space of relative motion this potential contains no singular terms and becomes,

$$V \ = \ 6\omega^2\tau_1 \ = \ 6\omega^2(\rho_{12}+\rho_{13}+\rho_{23}) \ = \ 6\omega^2(r_{12}^2+\ r_{13}^2+\ r_{23}^2)\ ,$$

see [10].

The quasi-exactly-solvable $sl(4,\mathbf{R})$-Lie-algebraic operator $h^{(qes)}(J)$, (39) as well as the exactly-solvable operator as a degeneration at $A=0$, written originally in ρ variables (38) can be rewritten in τ variables (17). Surprisingly, this operator is algebraic (!) as well

$$\begin{aligned} h^{(qes)}(\tau) \ = \ & -6\,\tau_1\partial_1^2 - 2\tau_1(7\tau_2-\tau_1^2)\partial_2^2 - 2\tau_3(6\tau_2-\tau_1^2)\partial_3^2 -\ 24\,\tau_2\partial_{1,2}^2 - 36\tau_3\partial_{1,3}^2 - \\ & 2\,(4\tau_2^2+9\tau_1\tau_3-\tau_1^2\tau_2)\partial_{2,3}^2 - 18\partial_1 - 14\tau_1\partial_2 - 2(7\tau_2-\tau_1^2)\partial_3 + \\ & 2(1-2\gamma)(3\partial_1+2\tau_1\partial_2+\tau_2\partial_3) + 12\omega\,(\tau_1\partial_1+2\tau_2\partial_2+3\tau_3\partial_3) + \\ & 12\,A\tau_1(\tau_1\partial_1+2\tau_2\partial_2+3\tau_3\partial_3-N)\ . \end{aligned} \tag{46}$$

Evidently, it remains algebraic at $A=0$,

$$\begin{aligned} h^{(es)}(\tau) \ = \ & -6\,\tau_1\partial_1^2 - 2\tau_1(7\tau_2-\tau_1^2)\partial_2^2 - 2\tau_3(6\tau_2-\tau_1^2)\partial_3^2 -\ 24\,\tau_2\partial_{1,2}^2 - 36\tau_3\partial_{1,3}^2 - \\ & 2\,(4\tau_2^2+9\tau_1\tau_3-\tau_1^2\tau_2)\partial_{2,3}^2 - 18\partial_1 - 14\tau_1\partial_2 - 2(7\tau_2-\tau_1^2)\partial_3 + \\ & 2(1-2\gamma)(3\partial_1+2\tau_1\partial_2+\tau_2\partial_3) + 12\omega\,(\tau_1\partial_1+2\tau_2\partial_2+3\tau_3\partial_3)\ , \end{aligned} \tag{47}$$

becoming the exactly-solvable one.

It can be immediately checked that the quasi-exactly-solvable operator (46) has the finite-dimensional invariant subspace in polynomials,

$$\mathcal{P}_N^{(1,2,3)} \ = \ \langle \tau_1^{p_1}\tau_2^{p_2}\tau_3^{p_3}|\ 0\le p_1+2p_2+3p_3\le N\rangle\ , \tag{48}$$

cf. (24). This finite-dimensional space appears as a finite-dimensional representation space of the algebra of differential operators $h^{(3)}$ which was discovered in the relation with H_3 (non-crystallographic) rational Calogero model [11] as its hidden algebra.

The algebra $h^{(3)}$ is infinite-dimensional but finitely-generated, for discussion see [11]. Their generating elements can be split into two classes. The first class of generators (lowering and Cartan operators) act in $\mathcal{P}_N^{(1,2,3)}$ for any N and therefore they preserve the flag $\mathcal{P}^{(1,2,3)}$. The second class operators (raising operators) act on the space $\mathcal{P}_N^{(1,2,3)}$ only.

Let us introduce the following notation for the derivatives:

$$\partial_i \equiv \frac{\partial}{\partial \tau_i}\ , \quad \partial_{ij} \equiv \frac{\partial^2}{\partial \tau_i \partial \tau_j}\ , \quad \partial_{ijk} \equiv \frac{\partial^3}{\partial \tau_i \partial \tau_j \partial \tau_k}\ .$$

The first class of generating elements consist of the 22 generators where 13 of them are the first order operators

$$\begin{aligned}
&T_0^{(1)} = \partial_1\,, \quad T_0^{(2)} = \partial_2\,, \qquad T_0^{(3)} = \partial_3\,,\\
&T_1^{(1)} = \tau_1\partial_1\,, \quad T_2^{(2)} = \tau_2\partial_2\,, \qquad T_3^{(3)} = \tau_3\partial_3\,,\\
&T_1^{(3)} = \tau_1\partial_3\,, \quad T_{11}^{(3)} = \tau_1^2\partial_3\,, \qquad T_{111}^{(3)} = \tau_1^3\partial_3\,,\\
&T_1^{(2)} = \tau_1\partial_2\,, \quad T_{11}^{(2)} = \tau_1^2\partial_2\,, \qquad T_2^{(3)} = \tau_2\partial_3\,,\\
&\qquad\qquad T_{12}^{(3)} = \tau_1\tau_2\partial_3\ ,
\end{aligned} \tag{49}$$

the 6 are of the second order

$$\begin{aligned}
&T_2^{(11)} = \tau_2\partial_{11}\,, \quad T_{22}^{(13)} = \tau_2^2\partial_{13}\,, \quad T_{222}^{(33)} = \tau_2^3\partial_{33}\,,\\
&T_3^{(12)} = \tau_3\partial_{12}\,, \quad T_3^{(22)} = \tau_3\partial_{22}\,, \quad T_{13}^{(22)} = \tau_1\tau_3\partial_{22}\ ,
\end{aligned} \tag{50}$$

and 2 are of the third order

$$T_3^{(111)} = \tau_3\partial_{111}\,, \quad T_{33}^{(222)} = \tau_3^2\partial_{222}\ . \tag{51}$$

The generators of the second class consist of 8 operators where 1 of them is of the first order

$$T_1^{+} = \tau_1 T_0\ , \tag{52}$$

4 are of the second order

$$T_{2,-1}^{+} = \tau_2\partial_1 T_0\,, \quad T_{3,-2}^{+} = \tau_3\partial_2 T_0\,, \quad T_{22,-3}^{+} = \tau_2^2\partial_3 T_0\,, \quad T_2^{+} = \tau_2 T_0(T_0+1)\ , \tag{53}$$

and 3 are of the third order

$$T^{+}_{3,-11} = \tau_3 \partial_{11} T_0\ ,\quad T^{+}_{3,-1} = \tau_3 \partial_1 T_0 (T_0+1)\ ,\quad T^{+}_{3} = \tau_3 T_0 (T_0+1)(T_0+2)\ , \tag{54}$$

where we have introduced the diagonal operator (the Euler-Cartan generator)

$$T_0 = \tau_1 \partial_1 + 2\tau_2 \partial_2 + 3\tau_3 \partial_3 - N\ . \tag{55}$$

for a convenience. In fact, this operator is the identity operator, it is of the zeroth order and, hence, it belongs to the first class.

It is not surprising that the algebraic operator $h^{(qes)}(\tau)$ (46) can be rewritten in terms of generators of the $h^{(3)}$-algebra,

$$\begin{aligned} h^{(qes)}(T) \ =\ -\Big[& 6\,T^{(1)}_1\,T^{(1)}_0 + 2\,(7\,T^{(2)}_2 - T^{(2)}_{11})\,T^{(2)}_1 + T^{(3)}_3(6\,T^{(3)}_2 - T^{(3)}_{11}) \\ & + T^{(1)}_0\,(24\,T^{(2)}_2 + 36\,T^{(3)}_3) + 2\,(4\,T^{(3)}_2\,T^{(2)}_2 + 9\,T^{(2)}_1\,T^{(3)}_3 -\ T^{(3)}_{11}\,T^{(2)}_2) \\ & + 2\,(9\,T^{(1)}_0 + 7\,T^{(2)}_1) + 2\,(7\,T^{(3)}_2 - T^{(3)}_{11})\Big] \\ & +2\,(1-2\,\gamma)\,(T^{(3)}_2 + 2\,T^{(2)}_1 + 3\,T^{(1)}_0) + 12\,\omega\,(J_0+N) + 12\,A\,J^{+}_1\ , \end{aligned} \tag{56}$$

as well as the algebraic operator $h^{(es)}(\tau)$ (47), which occurs at $A = 0$, can be rewritten in terms of generators of the $h^{(3)}$-algebra,

$$\begin{aligned} h^{(es)}(T) \ =\ -\Big[& 6\,T^{(1)}_1\,T^{(1)}_0 + 2\,(7\,T^{(2)}_2 - T^{(2)}_{11})\,T^{(2)}_1 + T^{(3)}_3(6\,T^{(3)}_2 - T^{(3)}_{11}) \\ & + T^{(1)}_0\,(24\,T^{(2)}_2 + 36\,T^{(3)}_3) + 2\,(4\,T^{(3)}_2\,T^{(2)}_2 + 9\,T^{(2)}_1\,T^{(3)}_3 -\ T^{(3)}_{11}\,T^{(2)}_2) \\ & + 2\,(9\,T^{(1)}_0 + 7\,T^{(2)}_1) + 2\,(7\,T^{(3)}_2 - T^{(3)}_{11})\Big] \\ & +2\,(1-2\,\gamma)\,(T^{(3)}_2 + 2\,T^{(2)}_1 + 3\,T^{(1)}_0) + 12\,\omega\,J_0\ , \end{aligned} \tag{57}$$

where without a loss of generality we put $N = 0$.

CONCLUSIONS

In this paper we found the Schrödinger type equation in the space $\tilde{\mathbf{R}}$ of relative distances $\{r_{ij}\}$,

$$\mathcal{H}_{rd}\Psi(r_{12},\,r_{13},\,r_{23}) \ =\ E\Psi(r_{12},\,r_{13},\,r_{23})\ ,\ \mathcal{H}_{rd} \ =\ -\Delta_{LB}(r_{ij}) + V(r_{12},\,r_{13},\,r_{23})\ , \tag{58}$$

where the Laplace-Beltrami operator Δ_{LB}, see e.g. (28), makes sense as the kinetic energy of a three-dimensional particle in curved space with metric (15). This equation describes angle-independent solutions of the original 3-body problem (1), including the ground state. Hence, finding the ground state involves the solution of the differential equation in three variables, contrary to the original six-dimensional Schrödinger equation of the relative motion. Since the Hamiltonian $\mathcal{H}_r$ is Hermitian, the variational method can be employed with only three-dimensional integrals involved.

The gauge-rotated Laplace-Beltrami operator, with determinant of the metric D raised to a certain degree as the gauge factor, appears as the algebraic operator both in the variables which are squares of relative distances and which are the elementary symmetric polynomials in squares of relative distances as arguments. The former algebraic operator has the hidden algebra $sl(4, \mathbf{R})$, while latter one has the hidden algebra $h^{(3)}$, thus, becoming Lie-algebraic operators. Both operators can be extended to (quasi)-exactly-solvable operators. Interestingly, both (quasi)-exactly-solvable operators lead to the *same* (quasi)-exactly-solvable potentials in the space of relative distances.

The above formalism admits a natural generalization to the case of arbitrary $d > 1$ dimensional bodies. The Laplace-Beltrami operator remains unchanged, the effective potential (27) is changed but not dramatically. It will be presented elsewhere.

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